Locally Boolean Domains and Universal Models for Infinitary Sequential Languages

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Abstract

In the first part of this Thesis we develop the theory of locally boolean domains and bistable maps (as introduced in [Lai05b]) and show that the category of locally boolean domains and bistable maps is equivalent to the category of Curien-Lamarche games and observably sequential functions (cf. [CCF94]). Further we show that the category of locally boolean domains has inverse limits of ω-chains of embedding/projection pairs.

In the second part we consider the category of locally boolean domains and bistable maps as model for functional programming languages: in [Lai05a] J. Laird has shown that an infinitary sequential extension of the functional core language PCF has a fully abstract model in the category of locally boolean domains. We introduce an extension SPCF_∞ of his language by recursive types and show that it is universal for its model in locally boolean domains. Finally we consider an infinitary target language CPS_∞ for the CPS translation of [RS98] and show that it is universal for a model in locally boolean domains which is constructed like Dana Scott's D_∞ where $D = \mathsf{O} = \{\bot, \top\}$.

Zusammenfassung

Im ersten Teil dieser Arbeit wird die Theorie lokal boolescher Bereiche und bistabiler Abbildungen (siehe [Lai05b]) entwickelt. Es wird gezeigt, dass die Kategorie lokal boolescher Bereiche und bistabiler Abbildungen zur Kategorie von Curien-Lamarche Spielen und beobachtbar sequenzieller Funktionen äquivalent ist. Weiterhin zeigen wir, dass die Kategorie lokal boolescher Bereiche und bistabiler Abbildungen inverse Limiten von ω-Ketten von Einbettungs-/Projektionspaaren besitzt.

Im zweiten Teil der Arbeit betrachten wir die Kategorie lokal boolescher Bereiche und bistabiler Abbildungen als Modell für funktionale Programmiersprachen: in [Lai05a] hat J. Laird gezeigt, dass es in der Kategorie lokal boolescher Bereiche ein voll abstraktes Modell für eine infinitäre, sequentielle Erweiterung der funktionalen Kernsprache PCF gibt. Wir definieren SPCF_∞, eine Erweiterung von Lairds Sprache um rekursive Typen, und zeigen, dass diese Sprache universell bezüglich ihres Modells in der Kategorie lokal boolescher Bereiche ist. Schließlich betrachten wir für die CPS Übersetzung aus [RS98] eine infinitäre Zielsprache CPS_∞ und zeigen, dass sie universell bezüglich ihres Modells in der Kategorie lokal boolescher Bereiche ist, welches wie Dana Scotts D_∞ mit $D = \mathsf{O} = \{\perp, \top\}$ konstruiert ist.

Erklärung

Hiermit versichere ich, dass ich diese Dissertation selbständig verfasst und nur die angegebenen Hilfsmittel verwendet habe.

Tobias Löw

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1 Introduction

The aim of this thesis is to show that the category **LBD** of locally boolean domains and bistable maps (as introduced by J. Laird in [Lai05b]) is equivalent to the category **OSA** of Curien-Lamarche games and observably sequential maps (as introduced by R. Cartwright, P.L. Curien and M. Felleisen in [CCF94]). Further we introduce the language SPCF_∞, a sequential extension of PCF by recursive types, error elements and a **catch**-construct, and show that it is universal for its model in **LBD**. Finally we consider an infinitary target language CPS_∞ for the CPS translation (of [RS98]) and show that CPS_∞ is universal for a model in **LBD** which is constructed like Dana Scott's D_∞ where $D = \mathsf{O} = \{\bot, \top\}$.

1.1 Sequentiality and Full Abstraction

The investigation of sequential functional programming languages started end of the 1960ies when D. Scott introduced the language LCF (Logic of Computable Functions) for reasoning about computable functionals of higher type. This paper was finally published as [Sco93] but circulated for a long time as an unpublished but most influencing technical report. In [Plo77] G. Plotkin first gave a detailed meta-mathematical analysis of PCF (Programming Computable Functions), the functional kernel language underlying the logical calculus LCF.

The language PCF is simply typed λ-calculus extended by a base type of natural numbers, some basic arithmetic operations, a conditional and fixpoint combinators for expressing general recursion. In [Plo77] Plotkin formulated an operational semantics for PCF as a term rewriting systems constrained by a leftmost-outermost reduction-strategy which is sequential in the sense that each PCF term t contains a unique subterm t' that has to be reduced in the next step of evaluation.

Having an operational and denotational semantics for PCF there arises the question how these two semantics should be related. Obviously, reduction preserves the denotation of terms. In [Plo77] he proved *computational adequacy*, i.e. that a closed term t of base type reduces to a numeral $\underline{n}$ whenever $[\![t]\!] = \underline{n}$. Thus for closed terms of base type their denotational semantics coincides with their operational semantics. Two (closed) terms t_1 and t_2 can be used interchangeably iff for all contexts $C[-]$ of base type $C[t_1]$ and $C[t_2]$ have the same meaning. Such terms are called *observationally equivalent*. Obviously, if two terms have the same denotational semantics then they are also observationally equivalent. A model is called *fully abstract* iff denotational equality coincides with observational equivalence. Already in [Sco93] D. Scott observed that his domain model lacks full abstraction because of the **parallel-or** function which is continuous but not sequentially computable. In [Plo77] Plotkin showed that Scott's domain model is fully abstract for the extension of PCF with **parallel-or**. (If one further adds a *continuous existential quantifier* then the denotable elements of the Scott model are precisely the computable ones as also shown in [Plo77].)

In [Mil77] R. Milner constructed a fully abstract model as the ideal completion of a quotient by observational equivalence of those PCF-terms which denote finite elements. Moreover, he showed that all order extensional fully abstract domain (i.e. cpo-enriched)models of PCF are isomorphic. However, since Milner's model is a (kind of) term model it does not give rise to a syntax-free characterisation of sequentiality. Since that a lot of people have tried to overcome this unsatisfying situation by suggesting different approaches to a syntax-free semantical characterisation of PCF sequentiality.

First in [Ber78, Ber79] G. Berry introduced his stable domains as a model for PCF which excludes the incriminated **parallel-or** but nevertheless contains functions which are not sequential in the sense of Milner-Vuillemin [Mil77, Vui74] providing a satisfying characterisation of sequentiality for first order functions. In [KP93] G. Kahn and G. Plotkin introduced so-called "concrete domains" allowing them to define a notion of sequentiality à la Milner-Vuillemin for functions between them. A disadvantage of their approach was that the underlying model is not cartesian closed anymore. This defect was remedied in [BC82] by G. Berry and P.-L. Curien albeit where they introduced a category **SA** of *sequential data structures* and *sequential algorithms*. But this model is not well-pointed since sequential algorithms may be different although they are extensionally equal, i.e. behave the same way for all arguments (e.g. "left" and "right" version of addition etc.).

In a long range attack Bucciarelli and Ehrhard finally managed to characterise the extensional collapse of **SA** in [BE91, Ehr96] as the category **SS** of strongly stable functions between strongly stable domains. The category **SS** is still not order extensional since it validates e.g. $\mathsf{O} \times \mathsf{O} \cong 2_\bot$ and thus not fully abstract for PCF. Nevertheless, it captures a more liberal notion of sequentiality which was studied thoroughly in [Lon02] and also lies at the heart of our investigations in this Thesis.

In the early 1990ies F. Lamarche and P.-L. Curien came up with a reformulation of the relevant part of **SA** in terms of games and strategies [Lam92, Cur94]. They restricted concrete data structures to so-called filiform ones (every datum can be constructed in only one way) which can be described as very simple games (with 2 player and no winning) and reformulated sequential algorithms as strategies for these games. (We write **SA** also for this slightly more restrictive category.) In [CCF94] (see also [CF92, AC98]) R. Cartwright, P.-L. Curien and M. Felleisen showed that an extension of **SA** with non-recuperable error elements gives rise to a fully abstract model **OSA** (observably sequential algorithms) for SPCF an extension of PCF with error elements and control operators **catch**.

This was the starting point for the flourishing field of Game Semantics. Abramsky, Malacaria and Jagadeesan in [AJM00] and Hyland and Ong [HO00] (see also Nickau [Nic94]) came up with sophisticated games models capturing PCF definability without being (order) extensional. It was shown by R. Loader in [Loa01] that already for finitary PCF (booleans instead of natural numbers as basic data type) observational equivalence is not decidable. Hence PCF sequentiality cannot be characterised effectively and thus there cannot exist a simple characterisation of the fully abstract model for finitary PCF.

Later on game semantics was extended to more complicated non-functional languages where quotients can be obtained more easily. For languages with store observational equality coincides with equality of strategies [AM97, AHM98]. In Laird's Thesis [Lai98] it was shown that PCF_μ, i.e. PCF extended with continuations, has a fully abstract model in **SA** (and that **SA** is the quotient of the model for PCF_μ given by innocent, but not necessarily well-bracketed strategies à la [HO00].

1.2 Locally boolean domains

Thus (observably) sequential algorithms have turned out as an important semantic model capturing a notion of sequentiality more liberal than PCF definability. Moreover, this model is wellpointed, i.e. extensional, in presence of error elements as shown in [CCF94]. Thus, there should be a presentation of **OSA** where functions are not given by algorithms but rather as continuous functions preserving some structure. This structure was identified by J. Laird around 2002 culminating in his notion of locally boolean domain [Lai05b]. He started from G. Berry's notion of bidomain [Ber78, Ber79] (domains with an extensional and a stable order) for which he could show in [Lai03b] that they give rise to a fully abstract model for unary PCF, i.e. PCF over base type O with basic operation $\wedge : \mathsf{O} \times \mathsf{O} \to \mathsf{O}$.

He further observed that $\wedge$ can be eliminated by requiring that functions are also "costable", i.e. preserve binary suprema of elements which are bounded from below w.r.t. a costable order. Instead of dealing with three different orders Laird showed that it suffices to consider the extensional and the bistable order which is the intersection of the stable and costable order. In [Lai03a] he then proved that one obtains a universal model for the language SPCF+, i.e. SPCF extended by countable sums and products, in the category **BB** of bistable biorders and monotone and bistable (i.e. preserving binary infima and suprema of bistably bounded elements) functions.

As bistable biorders are far more general than observably sequential algorithms in [Lai05b] (see also [Str04, Cur05] J. Laird identified a full subcategory **LBD** of **BB** which is equivalent to **OSA**, namely so-called locally boolean domains where the bistable structure is derived from an involution operation (w.r.t. the extensional order).

1.3 Overview of this thesis

In chapter 2 we give a detailed exposition of the theory of locally boolean domains. Based on the work of J. Laird in [Lai05b] and an unpublished note [Str04] of T. Streicher we define a locally boolean order as a partial ordered set $(D, \sqsubseteq)$ equipped with an involution $\neg : D \to D$ where infima and suprema of certain bounded pairs have to exist. After introducing a stable order $\leq_s$ and a costable order $\leq_c$ on D we define a locally boolean domain as a complete locally boolean order where finite elements w.r.t. $\leq_s$ are also compact w.r.t. $\sqsubseteq$ and each element is the supremum of the finite primes stably below it. We prove a lot of (sometimes fairly technical) lemmas that are useful later on. The key observations are that a locally boolean domain is a dI-domain w.r.t. $\leq_s$, and that the prime elements of a locally boolean domain form a tree w.r.t. $\leq_s$. We introduce the notion of bistable map between locally boolean domains, i.e. Scott-continuous functions that preserve infima of stably upper bounded and suprema of costably lower bounded pairs.

In chapter 3 we give a quick recap of Curien-Lamarche games with error elements. We show how to construct a locally boolean domain from a Curien-Lamarche game and vice versa. After characterising the bistable maps between locally boolean domains as those functions that are sequential in the sense of Milner-Vuillemin [Mil77, Vui74, KP93] and error propagating we establish an equivalence between the category **LBD** and the category **OSA** of Curien-Lamarche games and observably sequential maps/algorithms. Finally we analyse the structure of exponentials in the category **LBD** and show that **LBD** is cpo-enriched w.r.t. to the extensional order and w.r.t. to the stable order.

In chapter 4 we show that **LBD** is closed under basic categorical constructions like products, biliftings and sums. Next we show that inverse limits of ω-chains of embedding/projection pairs (w.r.t. $\leq_s$) exist in **LBD** and are constructed as usual. Finally, adapting a result of J. Longley in [Lon02] we show that every countably based locally boolean domain appears as retract of $\mathsf{U} = [\mathsf{N} \rightarrow \mathsf{N}]$ where N are the bilifted natural numbers, i.e. that U is a universal object for countably based locally boolean domains.

In chapter 5 we introduce the language SPCF_∞, an infinitary version of SPCF as considered in [CCF94]. More explicitly, it is obtained from simply typed λ-calculus by adding (countably) infinite sums and products, error elements, a control operator **catch** and recursive types.[1] Using evaluation contexts (in order to formalise the behaviour of the control operator **catch**) we present a call-by-name operational semantics for SPCF_∞. In the second part of this chapter we show that the category **LBD** gives rise to a computationally adequate model for SPCF_∞ and that SPCF_∞ is universal for this model. Recursive types in SPCF_∞ are interpreted as bifree solutions of recursive domain equations which can be constructed as bilimits of appropriate ω-chains of embedding/projection pairs. Adopting techniques from [Pit96] one can show that the **LBD** model of SPCF_∞ is computational adequate. Next we exhibit each SPCF_∞ type as an SPCF_∞ definable retract of the first order type $\mathbf{N} \rightarrow \mathbf{N}$ (where $\mathbf{N}$ is the type of bilifted natural numbers) from which universality of SPCF_∞ follows immediately since every element of $[\![\mathbf{N} \rightarrow \mathbf{N}]\!]$ is obviously SPCF_∞ definable.

In the last chapter we construct a **LBD** model for a (kind of) infinitary untyped λ-calculus CPS_∞ where every element of the model can be denoted by a closed CPS_∞ term. In [RS98] it has been observed that O_∞, i.e. Scott's D_∞ with $D = \mathsf{O} = \{\bot, \top\}$, can be obtained as bifree solution (cf. [Pit96]) of the type equation $D = [D^\omega \rightarrow \mathsf{O}]$. Since solutions of recursive type equations are available in **LBD** we may consider also the

[1] Notice that due to the presence of infinite sum and product types it will be sufficient to have a single control operator **catch** whereas in SPCF there is associated a control operator **catch** with every type $\sigma_1 \rightarrow \ldots \rightarrow \sigma_n \rightarrow \mathbf{N}$.

bifree solution of the equation for D in **LBD**. Canonically associated with this type equation is the language CPS_∞ whose terms are given by the grammar

$$M ::= x \mid \lambda\vec{x}.M\langle\vec{M}\rangle \mid \lambda\vec{x}.\top$$

where $\vec{x}$ ranges over infinite lists of pairwise distinct variables and $\vec{M}$ over infinite lists of terms. Notice that CPS_∞ is more expressive than untyped λ-calculus with an error element $\top$ since one may apply a term to an infinite list of arguments. Consider e.g. the term $\lambda\vec{x}.x_0\langle\vec{\bot}\rangle$ whose interpretation retracts D to O by sending $\top$ to $\top$ and everything else to $\bot$ which is not expressible in λ-calculus with a constant $\top$. We show that CPS_∞ is universal for its model in D. For this purpose we proceed as follows.

We first observe that the finite elements of D all arise from simply typed λ-calculus over O. Since the latter is universal for its **LBD** model (as shown in [Lai05a]) and all retractions of D to finite types are CPS_∞ definable it follows that all finite elements of D are definable in CPS_∞. Then borrowing an idea from [Lai98] we show that the supremum of any sequence of elements in D increasing w.r.t. $\leq_s$ is CPS_∞ definable provided the elements of the sequence are CPS_∞ definable. Thus, universality of CPS_∞ for its **LBD** model D follows from the fact that every element of D appears as supremum of an ω-chain of finite elements increasing w.r.t. $\leq_s$.

Although the interpretation of CPS_∞ in D is surjective it turns out that it may identify terms with different infinite normal form, i.e. that the interpretation is not faithful. Finally, we discuss a way how this shortcoming can be avoided, namely by extending CPS_∞ with a parallel construct $\|$ and refining the observation type O to $\mathsf{O}' \cong \mathsf{List}(\mathsf{O}')$.[2]

[2] This can be considered as a “qualitative” reformulation of a “quantitative” model considered by F. Maurel in his Thesis [Mau04].

2 Locally Boolean Domains

The notion of locally boolean orders and locally boolean domains was first introduced by Jim Laird in [Lai05b]. The following foundation of locally boolean orders and locally boolean domains is based on an unpublished note [Str04] of Thomas Streicher. We start from scratch: first we give the definition of locally boolean orders and locally boolean domains and deduce a set of basic lemmas that will be useful later on. Moreover we introduce the notion of bistable maps between locally boolean domains.

We assume that the reader is familiar with basic categorical and domain-theoretic notions.

2.1 Locally Boolean Orders

We define locally boolean orders as partially ordered sets with an involution satisfying certain constraints:

Definition 2.1.1. *An* involution *on a partial order* $(P, \sqsubseteq)$ *is a function* $\neg : P \to P$ *with* $\neg\neg x = x$ *and* $\neg y \sqsubseteq \neg x$ *whenever* $x \sqsubseteq y$.

A locally boolean order (lbo) *is a triple* $A = (|A|, \sqsubseteq, \neg)$ *where* $(A, \sqsubseteq)$ *is a partial order and* $\neg : |A| \to |A|$ *is an involution such that*

(1) *for every* $x \in |A|$ *the set* $\{x, \neg x\}$ *has a least upper bound* $x^\top = x \sqcup \neg x$ *(and, therefore, also a greatest lower bound* $x_\perp = \neg(x^\top) = x \sqcap \neg x$*)*

(2) *whenever* $x \sqsubseteq y^\top$ *and* $y \sqsubseteq x^\top$ *(notation* $x \uparrow y$*) then* $\{x, y\}$ *has a supremum* $x \sqcup y$ *and an infimum* $x \sqcap y$.

A is complete *if* $(|A|, \sqsubseteq)$ *is a cpo, i.e. every directed subset* X *has a supremum* $\bigsqcup X$. *A is* pointed *if it has a least element* $\perp$ *(and thus also a greatest element* $\top = \neg\perp$*).*

As usual, we write $x \in A$ *(resp.* $X \subseteq A$*) for* $x \in |A|$ *(resp.* $X \subseteq |A|$*).* ◇

We write $x \downarrow y$ as an abbreviation for $\neg x \uparrow \neg y$, and $x \updownarrow y$ for $x \uparrow y$ and $x \downarrow y$. A set $X \subseteq A$ is called *stably coherent* (notation $\uparrow X$) iff $x \uparrow y$ for all $x, y \in X$. Analogously, X is called *costably coherent* (notation $\downarrow X$) iff $x \downarrow y$ for all $x, y \in X$. We call a set $X \subseteq A$ *bistably coherent* (notation $\updownarrow X$) iff $\uparrow X$ and $\downarrow X$.

If $x \downarrow y$ then $x \sqcup y = \neg(\neg x \sqcap \neg y)$ and $x \sqcap y = \neg(\neg x \sqcup \neg y)$. Accordingly, the *dual of a locally boolean order is a locally boolean order again.*

Notice that for elements x and y we have $x \updownarrow y$ iff $x_\perp = y_\perp$ iff $x^\top = y^\top$.

Proposition 2.1.2. *If a lbo A is complete, then it is also cocomplete, i.e. every* $\sqsubseteq$*-codirected subset* X *has an infimum* $\bigsqcap X$.

Proof. If X is a codirected subset of A then the set $\{\neg x \mid x \in X\}$ is directed and has $\bigsqcup\{\neg x \mid x \in X\}$ as supremum. By duality we get $\neg\bigsqcup\{\neg x \mid x \in X\}$ as the infimum of X. □

Furthermore, on a lbo A using $\neg$ one may define a stable and a costable order as follows.

Definition 2.1.3. *For a lbo A we define the following partial orders on $|A|$. For $x, y \in A$*

stable order:	$x \leq_s y$	*iff*	$x \sqsubseteq y$ *and* $x \uparrow y$	
costable order:	$x \leq_c y$	*iff*	$x \sqsubseteq y$ *and* $x \downarrow y$	(*iff* $\neg y \leq_s \neg x$)
bistable order:	$x \leq_b y$	*iff*	$x \leq_s y$ *and* $x \leq_c y$	(*iff* $x \sqsubseteq y$ *and* $x \updownarrow y$) ◇

The following characterisation of $\leq_s$ (and $\leq_c$) turns out as useful.

Lemma 2.1.4. *Let A be a lbo and $x, y \in A$. Then the following are equivalent*

(1) $x \leq_s y$

(2) $x \sqsubseteq y \sqsubseteq x^\top$

(3) $x \sqsubseteq y$ *and* $x_\perp \sqsubseteq y_\perp$.

Proof. ad (1) $\Rightarrow$ (2) : Suppose $x \leq_s y$. Then $x \sqsubseteq y$ and $y \sqsubseteq x^\top$ (because $x \uparrow y$).

ad (2) $\Rightarrow$ (3) : Suppose $x \sqsubseteq y \sqsubseteq x^\top$. Then $x_\perp \sqsubseteq y$ and $x_\perp \sqsubseteq \neg y$, i.e. $x_\perp \sqsubseteq y \sqcap \neg y = y_\perp$.

ad (3) $\Rightarrow$ (1): Suppose $x \sqsubseteq y$ and $x_\perp \sqsubseteq y_\perp$. Then $x \sqsubseteq y \sqsubseteq y^\top$ and $y \sqsubseteq y^\top \sqsubseteq x^\top$, i.e. $x \uparrow y$. □

Notice that by duality we have $x \leq_c y$ iff $y_\perp \sqsubseteq x \sqsubseteq y$ iff $x \sqsubseteq y$ and $x^\top \sqsubseteq y^\top$. Further we have $x \leq_b y$ iff $x \sqsubseteq y$ and $y_\perp = x_\perp$ iff $x \sqsubseteq y$ and $y^\top = x^\top$.

Using Lemma 2.1.4 it is easy to check that $\leq_s$, $\leq_c$ and $\leq_b$ are actually partial orders.

Lemma 2.1.5. *Let A be a lbo and $x, y \in A$. Then $x \uparrow y$ iff x and y are bounded above (by $x \sqcup y$) in the stable order.*

Proof. Suppose $x \uparrow y$. Then we have $x, y \sqsubseteq x \sqcup y$ and $x \sqcup y \sqsubseteq x^\top, y^\top$. Thus $x, y \leq_s x \sqcup y$. Suppose $x, y \leq_s z$. Then $x \sqsubseteq z \sqsubseteq y^\top$ and $y \sqsubseteq z \sqsubseteq x^\top$ and thus $x \uparrow y$. □

Accordingly we have $x \downarrow y$ iff x and y are bounded below (by $x \sqcap y$) in the costable order.

2.2 Locally Boolean Domains

For the definition of locally boolean domains we introduce the notions of *finite* and *prime* elements of a locally boolean order.

Definition 2.2.1. *Let A be a lbo.*

An element $p \in A$ is called prime *iff whenever $x \uparrow y$ or $x \downarrow y$ and $p \sqsubseteq x \sqcup y$ then $p \sqsubseteq x$ or $p \sqsubseteq y$. We write $\mathsf{P}(A)$ for the set $\{p \in A \mid p \text{ is prime}\}$ and $\mathsf{P}(x)$ for the set $\{p \in \mathsf{P}(A) \mid p \leq_s x\}$.*

An element $e \in A$ is called finite *iff the set $\{x \in A \mid x \leq_s e\}$ is finite. We write $\mathsf{F}(A)$ for the set $\{e \in A \mid e \text{ is finite}\}$ and $\mathsf{F}(x)$ for the set $\{e \in \mathsf{F}(A) \mid e \leq_s x\}$.*

Finally, an element $p \in A$ is called finite prime *iff p is finite and prime. We write $\mathsf{FP}(A)$ for the set $\mathsf{P}(A) \cap \mathsf{F}(A)$ and $\mathsf{FP}(x)$ for the set $\mathsf{P}(x) \cap \mathsf{F}(A)$.* ◇

Lemma 2.2.2. *Let A be a lbo, $p \in \mathsf{P}(A)$ and $x, y \in A$ with $x \uparrow y$ or $x \downarrow y$. If $x \not\sqsubseteq \neg p$ and $y \not\sqsubseteq \neg p$ then $x \sqcap y \not\sqsubseteq \neg p$.*

Proof. Suppose $p \in \mathsf{P}(A)$ and $x, y \in A$ with $x \uparrow y$ or $x \downarrow y$. Thus $\neg x \downarrow \neg y$ or $\neg x \uparrow \neg y$. If $x \sqcap y \sqsubseteq \neg p$ then $p \sqsubseteq \neg(x \sqcap y) = \neg x \sqcup \neg y$ from which it follows that $p \sqsubseteq \neg x$ or $p \sqsubseteq \neg y$, i.e. $x \sqsubseteq \neg p$ or $y \sqsubseteq \neg p$. □

Definition 2.2.3. (locally boolean (pre-)domain)
A locally boolean predomain (lbpd) *is a complete locally boolean order A such that for all $x \in A$*

(1) *$x = \bigsqcup \mathsf{FP}(x)$ and*

(2) *all finite primes in A are compact w.r.t. $\sqsubseteq$, i.e. for all $p \in \mathsf{FP}(A)$ and directed sets X with $p \sqsubseteq \bigsqcup X$ there exists an $x \in X$ with $p \sqsubseteq x$.*

A locally boolean domain (lbd) *is a pointed locally boolean predomain.* ◇

Lemma 2.2.4. *Let x and y be elements of a lbpd A with $x \uparrow y$. Then $x \sqcup y$ and $x \sqcap y$ are the stable supremum and infimum of x and y, respectively.*

Proof. Suppose $x \uparrow y$.

From Lemma 2.1.5 we know that $x, y \leq_s x \sqcup y$. Suppose $x, y \leq_s z$. Then $x \sqcup y \sqsubseteq z$. Suppose $p \in \mathsf{FP}(z)$. Then $p \sqsubseteq x^\top$ and $p \sqsubseteq y^\top$. Thus, as p is prime, we have (1) $p \sqsubseteq x$ or (2) $p \sqsubseteq y$ or (3) $p \sqsubseteq \neg x, \neg y$. In cases (1) and (2) we have $p \sqsubseteq x \sqcup y$ and in case (3) we have $p \sqsubseteq \neg x \sqcap \neg y = \neg(x \sqcup y)$. Thus, in any case we have $p \sqsubseteq (x \sqcup y)^\top$. By condition (1) of Def. 2.2.3 it follows that $z \sqsubseteq (x \sqcup y)^\top$. Thus $x \sqcup y \leq_s z$ as desired.

We have $x \sqcap y \sqsubseteq x, y$. Suppose $p \in \mathsf{FP}(x)$. Then $p \sqsubseteq x \sqsubseteq y^\top$ and thus (1) $p \sqsubseteq y$ or (2) $p \sqsubseteq \neg y$. In case (1) we have $p \sqsubseteq x \sqcap y$ and in case (2) we have $p \sqsubseteq \neg y \sqsubseteq \neg x \sqcup \neg y = \neg(x \sqcap y)$. So in any case we have $p \sqsubseteq (x \sqcap y)^\top$. Thus, by condition (1) of Def. 2.2.3 we have $x \sqsubseteq (x \sqcap y)^\top$ and therefore $x \sqcap y \leq_s x$ as desired. Similarly, one shows that $x \sqcap y \leq_s y$. Suppose $z \leq_s x, y$. Then $z \sqsubseteq x \sqcap y$ and $x \sqcap y \sqsubseteq x, y \sqsubseteq z^\top$. Thus $z \leq_s x \sqcap y$ as desired. □

Lemma 2.2.5. *Let x and y be elements of a lbpd A. Then it follows that:*

(1) *The following statements are equivalent:*

  (i) $x \uparrow y$
  (ii) $x \sqcup y$ *and* $x_\perp \sqcup y_\perp$ *exist and* $(x \sqcup y)_\perp = x_\perp \sqcup y_\perp$
  (iii) $x \sqcup y$ *and* $x^\top \sqcap y^\top$ *exist and* $(x \sqcup y)^\top = x^\top \sqcap y^\top$

(2) *The following statements are equivalent:*

  (i) $x \downarrow y$
  (ii) $x \sqcap y$ *and* $x^\top \sqcap y^\top$ *exist and* $(x \sqcap y)^\top = x^\top \sqcap y^\top$
  (iii) $x \sqcap y$ *and* $x_\perp \sqcup y_\perp$ *exist and* $(x \sqcap y)_\perp = x_\perp \sqcup y_\perp$

Proof. ad (1) (i) $\Rightarrow$ (ii) : Suppose $x \uparrow y$.

Then $x_\perp \sqsubseteq x \sqsubseteq y^\top = (y_\perp)^\top$ and $y_\perp \sqsubseteq y \sqsubseteq x^\top = (x_\perp)^\top$, hence $x_\perp \uparrow y_\perp$ and it follows that the suprema $x \sqcup y$ and $x_\perp \sqcup y_\perp$ exist.

For showing that $x_\perp \sqcup y_\perp \sqsubseteq (x \sqcup y)_\perp$, suppose $p \in \mathsf{FP}(x_\perp \sqcup y_\perp)$. Then, as p is prime we have (1) $p \sqsubseteq x_\perp$ or (2) $p \sqsubseteq y_\perp$. In case (1) we get $p \sqsubseteq x_\perp \sqsubseteq x \sqcup y, \neg x, \neg y$ (where $x_\perp \sqsubseteq \neg y$ follows from $y \sqsubseteq x^\top$). Thus, we get $p \sqsubseteq x \sqcup y, \neg x \sqcap \neg y$, and finally, $p \sqsubseteq (x \sqcup y) \sqcap \neg(x \sqcup y) = (x \sqcup y)_\perp$. In case (2) we proceed analogously.

For showing that $(x \sqcup y)_\perp \sqsubseteq x_\perp \sqcup y_\perp$, suppose $p \in \mathsf{FP}((x \sqcup y)_\perp)$. Then, $p \sqsubseteq x \sqcup y, \neg(x \sqcup y)$, thus, $p \sqsubseteq x \sqcup y, \neg x \sqcap \neg y$, thus, $p \sqsubseteq x \sqcup y, \neg x, \neg y$. As p is prime we have (1) $p \sqsubseteq x, \neg x, \neg y$ or (2) $p \sqsubseteq y, \neg x, \neg y$. In case (1) we get $p \sqsubseteq x_\perp \sqsubseteq x_\perp \sqcup y_\perp$. In case (2) we proceed analogously.

Thus, it follows that $x_\perp \sqcup y_\perp = (x \sqcup y)_\perp$ as desired.

ad (1) (ii) $\Rightarrow$ (iii) : Suppose $x \sqcup y$ and $x_\perp \sqcup y_\perp$ exist and $(x \sqcup y)_\perp = x_\perp \sqcup y_\perp$. Then it follows that

$$\begin{aligned}(x \sqcup y)^\top &= \neg((x \sqcup y)_\perp) \\ &= \neg(x_\perp \sqcup y_\perp) \\ &= \neg x_\perp \sqcap \neg y_\perp \\ &= x^\top \sqcap y^\top\end{aligned}$$

as desired.

ad (1) (iii) $\Rightarrow$ (i) : Suppose $(x \sqcup y)^\top = x^\top \sqcap y^\top$, then it follows that

$$\begin{aligned}x, y &\sqsubseteq x \sqcup y \\ &\sqsubseteq (x \sqcup y)^\top \\ &= x^\top \sqcap y^\top \\ &\sqsubseteq x^\top, y^\top\end{aligned}$$

thus, we have $x \uparrow y$.

ad (2) : This follows from (1) by duality. □

Given a set X we write $\mathcal{P}_{\text{f.n.e.}}(X)$ for the set of finite, nonempty subsets of X.

Lemma 2.2.6. *Let A be a lbpd and $X \in \mathcal{P}_{\text{f.n.e.}}(A)$ with $\uparrow X$. Then it follows that:*

(1) *The set X has an infimum $\bigsqcap X$ w.r.t. $\sqsubseteq$ which is an infimum also w.r.t. $\leq_s$ and a supremum $\bigsqcup X$ w.r.t. $\sqsubseteq$ which is a supremum also w.r.t. $\leq_s$.*

(2) *If $y \in A$ with $\uparrow(X \cup \{y\})$ then $y \uparrow \bigsqcap X$ and $y \uparrow \bigsqcup X$.*

(3) $(\bigsqcup X)_\bot = \bigsqcup\{x_\bot \mid x \in X\}$ *and* $(\bigsqcup X)^\top = \bigsqcap\{x^\top \mid x \in X\}$

Proof. We proceed by induction on the size of X. The claims are obvious if X contains precisely one element. Suppose the claims hold for X and $\uparrow X \cup \{y\}$.

ad (1) and (2) : By Lemma 2.2.4 it suffices to show that $y \uparrow \bigsqcap X$ and $y \uparrow \bigsqcup X$. As $x \sqsubseteq y^\top$ for all $x \in X$ we have $\bigsqcap X \sqsubseteq y^\top$ and $\bigsqcup X \sqsubseteq y^\top$. Suppose $p \in \mathsf{FP}(y)$. Then for all $x \in X$ we have $p \sqsubseteq y \sqsubseteq x^\top$ and thus, as p is prime, that $p \sqsubseteq x$ or $p \sqsubseteq \neg x$. Thus either (i) $p \sqsubseteq x$ for all $x \in X$ or (ii) $p \sqsubseteq \neg x$ for some $x \in X$. In case (i) we have $p \sqsubseteq \bigsqcap X$ and thus also $p \sqsubseteq (\bigsqcap X)^\top$. In case (ii) we have $p \sqsubseteq \bigsqcup_{x\in X} \neg x = \neg \bigsqcap X \sqsubseteq (\bigsqcap X)^\top$. Thus, in any case we have $p \sqsubseteq (\bigsqcap X)^\top$. As $p \sqsubseteq y \sqsubseteq x^\top$ for all $x \in X$ we get $p \sqsubseteq \bigsqcap\{x^\top \mid x \in X\}$ and, using (3) for X, that $p \sqsubseteq (\bigsqcup X)^\top$. Accordingly, we have $y \sqsubseteq (\bigsqcup X)^\top$ and $y \sqsubseteq (\bigsqcap X)^\top$ as desired.

ad (3) : We have

$$
\begin{aligned}
(\bigsqcup(X \cup \{y\}))_\bot &= (\bigsqcup X \sqcup y)_\bot \\
&= (\bigsqcup X)_\bot \sqcup y_\bot && (\dagger) \\
&= \bigsqcup\{x_\bot \mid x \in X\} \sqcup y_\bot && \text{(ih)} \\
&= \bigsqcup(\{x_\bot \mid x \in X\} \cup \{y_\bot\})
\end{aligned}
$$

and

$$
\begin{aligned}
(\bigsqcup(X \cup \{y\}))^\top &= (\bigsqcup X \sqcup y)^\top \\
&= (\bigsqcup X)^\top \sqcap y^\top && (\dagger) \\
&= \bigsqcap\{x^\top \mid x \in X\} \sqcap y^\top && \text{(ih)} \\
&= \bigsqcap(\{x^\top \mid x \in X\} \cup \{y^\top\})
\end{aligned}
$$

where (†) follows from Lemma 2.2.5(1). □

Lemma 2.2.7. *An element x of a lbpd is finite iff $\mathsf{FP}(x)$ is finite.*

Proof. Obviously, if x is finite then $\mathsf{FP}(x)$ is finite.

Suppose $\mathsf{FP}(x)$ is finite. If $y \leq_s x$ then $\mathsf{FP}(y) \subseteq \mathsf{FP}(x)$ and $y = \bigsqcup \mathsf{FP}(y)$. Thus there are at most as many $y \leq_s x$ as there are subsets of $\mathsf{FP}(x)$. As a finite set has only finitely many subsets it follows that $\{y \in A \mid y \leq_s x\}$ is finite, i.e. that x is finite. □

Notice that for a lbd A we always have $\bot \in \mathsf{FP}(A)$.

Lemma 2.2.8. *If A is a lbpd and $x \in A$ then $\mathsf{FP}(x) \neq \emptyset$.*

Proof. Suppose $\mathsf{FP}(x) = \emptyset$. By the definition of lbpds $x = \bigsqcup \emptyset = \bot$. Hence, $\bot$ is the least element of A, and we get $\emptyset = \mathsf{FP}(\bot) = \{\bot\}$. □

Lemma 2.2.9. *An element of a lbpd A is compact w.r.t. $\sqsubseteq$ iff it is finite.*

Proof. Suppose c is a compact element of A. By Lemma 2.2.8 it follows that $\mathsf{FP}(c)$ is nonempty. Let $X := \{\bigsqcup Y \mid Y \in \mathcal{P}_{\text{f.n.e.}}(\mathsf{FP}(c))\}$. Obviously, the set X is directed and has supremum c. As c is compact there exists a finite nonempty subset X_0 of $\mathsf{FP}(c)$ with $c = \bigsqcup X_0$. If $p \in \mathsf{FP}(c)$ then $p \leq_s c = \bigsqcup X_0$ and thus there exists a $q \in X_0$ with $p \sqsubseteq q$. As p and q are stably bounded (by c) it follows that $p \leq_s q$. Accordingly, we have $\mathsf{FP}(c) \subseteq \bigcup_{q \in X_0} \mathsf{FP}(q)$ which is finite since X_0 is finite and the $\mathsf{FP}(q)$ are finite. Thus $\mathsf{FP}(c)$ is finite from which it follows by Lemma 2.2.7 that c is finite in A as desired.

For showing the reverse implication suppose e is a finite element of A. Then $\mathsf{FP}(e)$ is finite. As by Def. 2.2.3(2) all elements of $\mathsf{FP}(e)$ are compact and $e = \bigsqcup \mathsf{FP}(e)$ the element e is a supremum of finitely many compact elements and thus compact as well. □

Lemma 2.2.10. *Let A be a lbpd and $c, d \in \mathsf{F}(A)$ with $c \uparrow d$ then $c \sqcup d \in \mathsf{F}(A)$.*

Proof. It follows from Lemma 2.2.9 that c and d are compact w.r.t. $\sqsubseteq$. Thus, it follows that $c \sqcup d$ is compact and thus also finite by Lemma 2.2.9. □

Lemma 2.2.11. *Let x and y be elements of a lbpd A then t.f.a.e.*

(1) $x \sqsubseteq y$

(2) $\forall p {\in} \mathsf{FP}(x). \exists q {\in} \mathsf{FP}(y).\, p \sqsubseteq q$

(3) $\forall c {\in} \mathsf{F}(x). \exists d {\in} \mathsf{F}(y).\, c \sqsubseteq d$

Proof. Suppose $x, y \in A$.

ad (1) $\Rightarrow$ (2) : Suppose $x \sqsubseteq y$ and $p \in \mathsf{FP}(x)$. Then $p \sqsubseteq y = \bigsqcup \mathsf{F}(y)$. As by condition (2) of Def. 2.2.3 the element p is compact w.r.t. $\sqsubseteq$ there is a finite nonempty set $Y_0 \subseteq \mathsf{FP}(y)$ with $p \sqsubseteq \bigsqcup Y_0$. As p is prime and $\uparrow Y_0$ there exists an element $q \in Y_0 \subseteq \mathsf{FP}(y)$ with $p \sqsubseteq q$.

ad (2) $\Rightarrow$ (3) : Suppose $\forall p {\in} \mathsf{FP}(x). \exists q {\in} \mathsf{FP}(y).\, p \sqsubseteq q$ holds and $c \in \mathsf{F}(x)$. As by condition (2) of Def. 2.2.3 the element c is compact w.r.t. $\sqsubseteq$ there is a finite nonempty set $X_0 \subseteq \mathsf{FP}(x)$ with $c \sqsubseteq \bigsqcup X_0$. For all $p \in X_0$ there exists a $\hat{p} \in \mathsf{FP}(y)$ with $p \sqsubseteq \hat{p}$. Now, let $d := \bigsqcup \{\hat{p} \mid p \in X_0\}$ then $c \sqsubseteq d$ and it follows from Lemma 2.2.10 that d is finite.

ad (3) $\Rightarrow$ (1) : This is obvious. □

Lemma 2.2.12. *Every lbpd is algebraic w.r.t. the extensional order.*

Proof. By Lemma 2.2.9 every finite element is compact w.r.t. $\sqsubseteq$. Every $x \in A$ is the supremum of the compact elements $c \sqsubseteq x$ because $x = \bigsqcup \mathsf{FP}(x)$ and all elements of $\mathsf{FP}(x)$ are finite and thus compact.

It remains to show that the set $\{c \mid c \text{ compact and } c \sqsubseteq x\}$ is directed w.r.t. $\sqsubseteq$. Let $X := \{\bigsqcup Y \mid Y \in \mathcal{P}_{\text{f.n.e.}}(\mathsf{FP}(x))\}$. Since by Lemma 2.2.8 the set $\mathsf{FP}(x)$ is nonempty it follows that X is nonempty. Let c and c' be compact elements with $c, c' \sqsubseteq x$. As X is directed and $x = \bigsqcup X$ there exist finite nonempty subsets X_0 and X_1 of $\mathsf{FP}(x)$ with $c \sqsubseteq \bigsqcup X_0$ and $c' \sqsubseteq \bigsqcup X_1$. Thus $c, c' \sqsubseteq \bigsqcup(X_0 \cup X_1) \sqsubseteq x$ and $\bigsqcup(X_0 \cup X_1)$ is compact since it is a finite supremum of compact elements. □

The next two lemmas will show that suprema w.r.t. $\sqsubseteq$ of stably coherent directed sets are also suprema w.r.t. $\leq_s$ and that suprema of arbitrary nonempty stably coherent subsets exist. These facts will be crucial for showing that $(|A|, \leq_s)$ is a dI-predomain (cf. Thm. 2.2.18).

Lemma 2.2.13. *Let A be a lbpd and X be a stably (i.e. w.r.t. $\leq_s$) directed subset of A then $\bigsqcup X$ is the supremum of X w.r.t. $\leq_s$.*

Proof. As A is complete there exists the supremum $\bigsqcup X$ of X w.r.t. $\sqsubseteq$. As X is stably directed we have $x \sqsubseteq y^\top$ for all $x, y \in X$. Thus $\bigsqcup X \sqsubseteq y^\top$ for all $y \in X$ from which it follows that $\bigsqcup X$ is a stable upper bound of X. Suppose $X \leq_s z$. Then $\bigsqcup X \sqsubseteq z$. It remains to show that $z \sqsubseteq (\bigsqcup X)^\top$. For this purpose suppose $p \in \mathsf{FP}(z)$. As $x \leq_s z$ for all $x \in X$ we have $p \sqsubseteq z \sqsubseteq x^\top$ for all $x \in X$. As p is prime we have $p \sqsubseteq x$ or $p \sqsubseteq \neg x$ for all $x \in X$. Thus, either (1) $p \sqsubseteq x$ for some $x \in X$ or (2) $p \sqsubseteq \neg x$ for all $x \in X$. In case (1) we have $p \sqsubseteq \bigsqcup X \sqsubseteq (\bigsqcup X)^\top$. In case (2) we have $p \sqsubseteq \bigsqcap_{x \in X} \neg x = \neg(\bigsqcup X) \sqsubseteq (\bigsqcup X)^\top$. Thus $p \sqsubseteq (\bigsqcup X)^\top$ for all $p \in \mathsf{FP}(z)$ from which it follows that $z \sqsubseteq (\bigsqcup X)^\top$ as desired. □

Lemma 2.2.14. *Let A be a lbpd and X be a nonempty stably coherent subset of A then the supremum $\bigsqcup X$ exists and is also the supremum of X w.r.t. $\leq_s$.*

Proof. Let X be a nonempty stably coherent subset of A, and let $Z := \{\bigsqcup Y \mid Y \in \mathcal{P}_{\text{f.n.e.}}(X)\}$. Obviously, by Lemma 2.2.6 the set Z is directed w.r.t. $\sqsubseteq$ and also $\leq_s$. Thus it follows from Lemma 2.2.13 that $\bigsqcup Z$ is also a supremum w.r.t. $\leq_s$. Thus $\bigsqcup Z$ is a stable upper bound of X (because every element of X is stably below some element of Z). For showing that $\bigsqcup Z$ is the least upper bound of X w.r.t. $\leq_s$ suppose $X \leq_s z$. Then z is also a stable upper bound of Z from which it follows by Lemma 2.2.13 that $\bigsqcup Z \leq_s z$. □

Lemma 2.2.15. *For a lbpd A all elements of $\mathsf{F}(A)$ are compact w.r.t. $\leq_s$.*

Proof. Suppose $c \in \mathsf{F}(A)$ with $c \leq_s \bigsqcup X$. Then $c \sqsubseteq \bigsqcup X \sqsubseteq c^\top$. From Lemma 2.2.9 it follows that c is compact. Thus, there exists an $x \in X$ with $c \sqsubseteq x \sqsubseteq \bigsqcup X \sqsubseteq c^\top$. Thus, $c \leq_s x$. □

Lemma 2.2.16. *Let A be a lbpd and $x \in A$. If x is compact w.r.t. $\sqsubseteq$ then x is compact w.r.t. $\leq_s$.*

Proof. Suppose x is a compact w.r.t. $\sqsubseteq$. Let $X \subseteq A$ be directed w.r.t. $\leq_s$ and $x \leq_s \bigsqcup X$. Then it follows that X is directed w.r.t. $\sqsubseteq$ and $x \sqsubseteq \bigsqcup X$. As x is compact w.r.t. $\sqsubseteq$ there exists a $e \in X$ with $x \sqsubseteq e$. As $x, e \leq_s \bigsqcup X$ it follows that $x \uparrow e$. Thus we have $x \leq_s e$ and it follows that x is a compact w.r.t. $\leq_s$. □

Next we give the definition of dI-(pre)-domains.

Definition 2.2.17. *Let D be an algebraic dcpo. The properties d and I are defined as follows:*

(I) *Each compact element dominates at most finitely many elements.*

(d) *If $\{x, y, z\}$ are bounded then $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.*

A bounded complete algebraic dcpo satisfying properties d and I is called a dI*-predomain. A* dI*-predomain with least element $\perp$ is called a* dI*-domain.* ◇

Next we show that a lbpd A is a dI-predomain w.r.t. $\leq_s$ which has suprema of *all* nonempty bounded subsets. (Notice that the definition of dI-predomains only postulates the existence of suprema of bounded nonempty finite subsets.)

Theorem 2.2.18. *If A is a lbpd then $(|A|, \leq_s)$ is a* dI*-predomain with stable suprema of all nonempty stably coherent subsets.*

Proof. From Lemma 2.2.13 and Lemma 2.2.14 it follows that $(|A|, \leq_s)$ is a dcpo, and that it has stable suprema of nonempty stably coherent sets from which it follows that $(|A|, \leq_s)$ has suprema of nonempty bounded subsets as required for dI-predomains.

Next we show that $(|A|, \leq_s)$ is algebraic. As we already know that $(|A|, \leq_s)$ has suprema of nonempty bounded sets it suffices to show that every element of A is the stable supremum of some set of compact elements. Let $x \in A$ and $Z := \{\bigsqcup Y \mid Y \in \mathcal{P}_{\text{f.n.e.}}(\mathsf{FP}(x))\}$. Then all elements of Z are compact w.r.t. $\sqsubseteq$ and using Lemma 2.2.16 it follows that all elements of Z are compact w.r.t. $\leq_s$. Further, we get that Z is stably directed (as $\mathsf{FP}(x)$ is nonempty and by Lemma 2.2.6) and it follows that $x = \bigsqcup \mathsf{FP}(x) = \bigsqcup Z$.

For verifying the I-property we have to show that every stably compact element c is finite. W.l.o.g. assume that c is different from $\perp$. Let $Z := \{\bigsqcup Y \mid Y \in \mathcal{P}_{\text{f.n.e.}}(\mathsf{FP}(c))\}$. Obviously Z is stably directed and $c = \bigsqcup Z$. As c is assumed as stably compact there exists a finite nonempty subset X_0 of $\mathsf{FP}(c)$ with $c = \bigsqcup X_0$. As the elements of X_0 are compact w.r.t. $\sqsubseteq$ their supremum $\bigsqcup X_0$ is also compact w.r.t. $\sqsubseteq$. Thus, by Lemma 2.2.9 it follows that $c = \bigsqcup X_0$ is finite as desired.

For verifying the d-property suppose $\uparrow\{x, y, z\}$. We have to show that $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$. For showing the nontrivial inequality suppose $p \in \mathsf{FP}(x \sqcap (y \sqcup z))$. Then $p \sqsubseteq x$ and $p \sqsubseteq y \sqcup z$. As p is prime we have (1) $p \sqsubseteq y$ or (2) $p \sqsubseteq z$. In case (1) we have $p \sqsubseteq x \sqcap y$ and in case (2) we have $p \sqsubseteq x \sqcap z$. Thus in any case we have $p \sqsubseteq (x \sqcap y) \sqcup (x \sqcap z)$. Thus, we have $x \sqcap (y \sqcup z) \sqsubseteq (x \sqcap y) \sqcup (x \sqcap z)$ as desired. □

Lemma 2.2.19. *Let A be a lbpd and X be a nonempty subset of A with $\uparrow X$. If $p \in \mathsf{FP}(A)$ and $p \sqsubseteq \bigsqcup X$ then there exists an element $x \in X$ with $p \sqsubseteq x$ and ($p \leq_s x$ whenever $p \leq_s \bigsqcup X$).*

Proof. For all $F \in \mathcal{P}_{\text{f.n.e.}}X$ it holds that $\uparrow F$ and thus $\bigsqcup F$ exists. Accordingly, the set $\hat{X} := \{\bigsqcup F \mid F \in \mathcal{P}_{\text{f.n.e.}}(X)\}$ is stably coherent and directed. As p is compact w.r.t. $\sqsubseteq$ there exists a finite subset F of X with $p \sqsubseteq \bigsqcup F$, and as p is prime, there exists an element $x \in F$ with $p \sqsubseteq x$. Furthermore, if $p \leq_s \bigsqcup X$ then it follows that $p \sqsubseteq x \sqsubseteq \bigsqcup X \sqsubseteq p^\top$, thus, $p \leq_s x$ as desired. □

In the next two lemmas we show that the infimum (resp. the supremum) of stably coherent nonempty set is given by the supremum of the intersection (resp. union) of the sets of finite prime elements.

Lemma 2.2.20. *Let A be a lbpd and X be a nonempty subset of A with $\uparrow X$. Then $\bigsqcap X$ exists, is also the infimum w.r.t. $\leq_s$ and*

$$\bigsqcap X = \bigsqcup\Big(\bigcap_{x \in X} \mathsf{FP}(x)\Big)\,.$$

Proof. Suppose X is a nonempty subset of A with $\uparrow X$. Then the set $Z := \{\bigsqcap Y \mid Y \in \mathcal{P}_{\text{f.n.e.}}(X)\}$ is codirected. Hence its infimum $\bigsqcap Z$ exists and $\bigsqcap Z = \bigsqcap X$. Let $x \in X$ and $p \in \mathsf{FP}(x)$. Then for all $u \in X$ we have $u \uparrow x$, thus it follows that $p \sqsubseteq x \sqsubseteq u^\top = u \sqcup \neg u$ and as p is prime we get $p \sqsubseteq u$ or $p \sqsubseteq \neg u$. Thus we have either $p \sqsubseteq u$ for all $u \in X$ or there exists a $u \in X$ with $p \sqsubseteq \neg u$. In the first case we get $p \sqsubseteq \bigsqcap X \sqsubseteq (\bigsqcap X)^\top$, and in the second case we get $p \sqsubseteq \neg u \sqsubseteq \bigsqcup\{\neg y \mid y \in X\} = \neg \bigsqcap X \sqsubseteq (\bigsqcap X)^\top$ and it follows that $\bigsqcap X \leq_s x$. Thus $\bigsqcap X$ is also the infimum of X w.r.t. $\leq_s$.

For all $y \in X$ we have $\bigcap_{x \in X} \mathsf{FP}(x) \subseteq \mathsf{FP}(y)$. Hence, $\bigcap_{x \in X} \mathsf{FP}(x)$ is stably coherent and $\bigsqcup(\bigcap_{x \in X} \mathsf{FP}(x)) \leq_s y$ for all $y \in X$. Thus, $\bigsqcup(\bigcap_{x \in X} \mathsf{FP}(x)) \leq_s \bigsqcap X$. For showing the reverse inequality suppose $p \in \mathsf{FP}(\bigsqcap X)$. As $\bigsqcap X \leq_s x$ for all $x \in X$ it follows that $p \in \mathsf{FP}(x)$ for all $x \in X$. Thus $p \in \bigcap_{x \in X} \mathsf{FP}(x)$ and we get $p \leq_s \bigsqcup(\bigcap_{x \in X} \mathsf{FP}(x))$ as desired. □

Lemma 2.2.21. *Let A be a lbpd and X be a nonempty subset of A with $\uparrow X$. Then it follows that*

$$\mathsf{FP}\Big(\bigsqcup X\Big) = \bigcup_{x \in X} \mathsf{FP}(x) \qquad \textit{and} \qquad \mathsf{FP}\Big(\bigsqcap X\Big) = \bigcap_{x \in X} \mathsf{FP}(x)\,.$$

Proof. For showing $\mathsf{FP}(\bigsqcup X) = \bigcup_{x \in X} \mathsf{FP}(x)$ suppose $p \in \mathsf{FP}(\bigsqcup X)$. Then from it follows Lemma 2.2.19 that there exists a $x \in X$ with $p \leq_s x$. Hence $\mathsf{FP}(\bigsqcup X) \subseteq \bigcup_{x \in X} \mathsf{FP}(x)$. For the reverse inclusion suppose $p \in \bigcup_{x \in X} \mathsf{FP}(x)$ then there exists a $x \in X$ with $p \in \mathsf{FP}(x)$. As $\uparrow X$ it follows that $p \leq_s x \leq_s \bigsqcup X$. Thus $p \in \mathsf{FP}(\bigsqcup X)$.

For showing the second equation consider

$$\begin{aligned}
\mathsf{FP}(\bigsqcap X) &= \mathsf{FP}(\bigsqcup(\bigcap_{x\in X}\mathsf{FP}(x))) && (\dagger)\\
&= \bigcup\{\mathsf{FP}(y) \mid y \in \bigcap_{x\in X}\mathsf{FP}(x)\} && (\ddagger)\\
&= \bigcap_{x\in X}\mathsf{FP}(x) && (\S)
\end{aligned}$$

where ($\dagger$) follows from Lemma 2.2.20, ($\ddagger$) follows from the first equation of this lemma. Finally we show that (§) holds. Since for all $p \in \mathsf{FP}(A)$ we have that $p \in \mathsf{FP}(p)$ holds it follows immediately that $\bigcap_{x\in X}\mathsf{FP}(x) \subseteq \bigcup\{\mathsf{FP}(y) \mid y \in \bigcap_{x\in X}\mathsf{FP}(x)\}$ holds. For the reverse inclusion suppose $p \in \bigcup\{\mathsf{FP}(y) \mid y \in \bigcap_{x\in X}\mathsf{FP}(x)\}$ then there exists a $y \in \bigcap_{x\in X}\mathsf{FP}(x)$ with $p \in \mathsf{FP}(y)$, thus $p \leq_s y$. Thus it follows that $y \in \mathsf{FP}(x)$ for all $x \in X$, thus as $p \leq_s y$ it follows that $p \in \mathsf{FP}(x)$ for all $x \in X$. Thus we have $p \in \bigcap_{x\in X}\mathsf{FP}(x)$ as desired. $\square$

Notice that a lbpd A gives rise to a *bistable biorder* $(A, \sqsubseteq, \updownarrow)$ as introduced by J. Laird in [Lai05a]. By Lemma 2.2.6 and its dual statement it follows that the relation $\updownarrow$ is an equivalence relation. Further from Lemma 2.2.5 it follows that equivalence classes w.r.t. $\updownarrow$ are closed under binary suprema and infima w.r.t. $\sqsubseteq$ and satisfy the distributivity law (by Thm. 2.2.18).

Definition 2.2.22. *If A is a lbpd and $x \in A$ then we write $[x]_\updownarrow$ for the set $\{y \in A \mid x \updownarrow y\}$. We call $[x]_\updownarrow$ the* bistably connected component *of x.*

If X is a nonempty subset of A with $\updownarrow X$ then we write $[X]_\updownarrow$ for the connected component $\{y \in A \mid \updownarrow(X \cup \{y\})\}$ and $X_\bot$ (resp. $X^\top$) for the bottom (resp. top) element of $[X]_\updownarrow$. $\diamond$

Lemma 2.2.23. *If A is a lbpd and $x \in A$ then $[x]_\updownarrow$ is a boolean algebra w.r.t. $\leq_b$.*

Proof. Suppose $x \in A$. Then using Lemma 2.2.6 it follows that stable suprema and costable infima coincide with those w.r.t. $\sqsubseteq$ and that $\updownarrow\{x, y, x \sqcap y, x \sqcup y\}$ holds whenever $x \updownarrow y$. Using that and Thm. 2.2.18 it follows $[x]_\updownarrow$ satisfies the distributivity law. Negation on $[x]_\updownarrow$ is given by the restriction of $\neg$ to $[x]_\updownarrow$. The bottom element is given by $x_\bot$ and the top element by $x^\top$. $\square$

Lemma 2.2.24. *In a lbpd from $x \sqsubseteq y = y_\bot$ it follows that $x = x_\bot$ and $x \leq_s y$.*

Proof. We have $x \sqsubseteq y = y_\bot \sqsubseteq \neg y \sqsubseteq \neg x$. Thus $x_\bot = x \sqcap \neg x = x$, and as $x_\bot = x \sqsubseteq y_\bot$ it follows $x \leq_s y$ as desired. $\square$

Lemma 2.2.25. *In a lbpd from $x \leq_s y \leq_s z$ and $x \updownarrow z$ it follows that $\updownarrow\{x, y, z\}$.*

Proof. We have $x_\bot \sqsubseteq y_\bot \sqsubseteq z_\bot$ and $x_\bot = z_\bot$. Thus $x_\bot = y_\bot = z_\bot$ as desired. $\square$

Lemma 2.2.26. *In a lbpd from $x \leq_s y$ it follows that $x_\bot \leq_s y_\bot$.*

Proof. From $x \leq_s y$ it follows that $x_\perp \sqsubseteq y_\perp$. Moreover, we have $(x_\perp)_\perp = x_\perp \sqsubseteq y_\perp = (y_\perp)_\perp$. Thus $x_\perp \leq_s y_\perp$. □

Lemma 2.2.27. *Let A be a lbpd. If $X \subseteq A$ is directed w.r.t. $\leq_s$ then $(\bigsqcup X)_\perp = \bigsqcup\{x_\perp \mid x \in X\}$.*

Proof. Suppose $X \subseteq A$ is directed w.r.t. $\leq_s$. If $x \leq_s y$ then from Lemma 2.2.26 it follows that $x_\perp \leq_s y_\perp$. Thus the set $\{x_\perp \mid x \in X\}$ is directed and $\bigsqcup\{x_\perp \mid x \in X\}$ exists. For all $x \in X$ we have $x \leq_s \bigsqcup X$ and further that $x_\perp \sqsubseteq (\bigsqcup X)_\perp$. Thus it follows that $\bigsqcup\{x_\perp \mid x \in X\} \sqsubseteq (\bigsqcup X)_\perp$. Now, suppose $p \in \mathsf{FP}((\bigsqcup X)_\perp)$ then $p = p_\perp$ by Lemma 2.2.24. As p is compact and $p \leq_s \bigsqcup X$ there exists an element $x \in X$ with $p \leq_s x$ and by Lemma 2.2.26 it follows that $p \leq_s x_\perp$. Thus $p \leq_s \bigsqcup\{x_\perp \mid x \in X\}$ and we have $(\bigsqcup X)_\perp \sqsubseteq \bigsqcup\{x_\perp \mid x \in X\}$ as desired. □

Lemma 2.2.28. *Let A be a lbpd. If $X, Y \subseteq A$ are directed w.r.t. $\leq_s$ and $\{[x]_\updownarrow \mid x \in X\} = \{[y]_\updownarrow \mid y \in Y\}$ (i.e. X and Y touch the same bistably connected components of A), then $\bigsqcup X \updownarrow \bigsqcup Y$.*

Proof. As $\{[x]_\updownarrow \mid x \in X\} = \{[y]_\updownarrow \mid y \in Y\}$ it follows that $\{x_\perp \mid x \in X\} = \{y_\perp \mid y \in Y\}$. Using Lemma 2.2.27 we get $(\bigsqcup X)_\perp = \bigsqcup\{x_\perp \mid x \in X\} = \bigsqcup\{y_\perp \mid y \in Y\} = (\bigsqcup Y)_\perp$ as desired. □

Lemma 2.2.29. *Let x and y be elements of a lbpd A.*

(1) *If $x \uparrow y$ then the following statements hold:*

 (i) $(x \sqcap y)_\perp = x_\perp \sqcap y_\perp$

 (ii) $(x \sqcap y)^\top = x^\top \sqcup y^\top$

(2) *If $x \downarrow y$ then the following statements hold:*

 (i) $(x \sqcup y)^\top = x^\top \sqcup y^\top$

 (ii) $(x \sqcup y)_\perp = x_\perp \sqcap y_\perp$

Proof. *ad* (1)(i) : From $x \uparrow y$ it follows that $x_\perp \sqsubseteq x \sqsubseteq y^\top = (y_\perp)^\top$ and $y_\perp \sqsubseteq y \sqsubseteq x^\top = (x_\perp)^\top$. Thus, $\uparrow\{x, x_\perp, y, y_\perp\}$ holds. From Lemma 2.2.6 it follows that $x_\perp \sqcap y_\perp \leq_s x \sqcap y$. Using Lemma 2.2.26 we get $(x_\perp \sqcap y_\perp)_\perp \leq_s (x \sqcap y)_\perp$. Further, from Lemma 2.2.24 it follows that $(x_\perp \sqcap y_\perp)_\perp = x_\perp \sqcap y_\perp$. Thus, we have $x_\perp \sqcap y_\perp \leq_s (x \sqcap y)_\perp$. For showing $(x \sqcap y)_\perp \leq_s x_\perp \sqcap y_\perp$, notice that $x \sqcap y \leq_s x, y$ holds. From Lemma 2.2.26 we get $(x \sqcap y)_\perp \leq_s x_\perp, y_\perp$. Thus, $(x \sqcap y)_\perp \leq_s x_\perp \sqcap y_\perp$ as desired.

ad (1)(ii) : Using (1)(i) we get $(x \sqcap y)^\top = \neg((x \sqcap y)_\perp) = \neg(x_\perp \sqcap y_\perp) = \neg x_\perp \sqcup \neg y_\perp = x^\top \sqcup y^\top$.

ad (2) : These statements follow from (1) by duality. □

Lemma 2.2.30. *Let A be a lbpd and $x, y \in A$ and $x \sqsubseteq y$ then $x \uparrow \neg y$.*

Proof. Suppose $x \sqsubseteq y$. Then $x \sqsubseteq y \sqsubseteq y^\top = (\neg y)^\top$ and $\neg y \sqsubseteq \neg x \sqsubseteq x^\top$ as desired. □

Lemma 2.2.31. *Let A be a lbpd and $x, y \in A$. Then $x \sqsubseteq y$ iff $x \leq_s z \leq_c y$ for some $z \in A$.*

Proof. The reverse implication is obvious as $\leq_s$ and $\leq_c$ are included in $\sqsubseteq$.

For the forward implication assume that $x \sqsubseteq y$. Then $x \uparrow \neg y$ by Lemma 2.2.30. Thus also $x \uparrow y_\bot$ and $x_\bot \uparrow y_\bot$. Putting $z := x \sqcup y_\bot$ we have $x \leq_s z$ because $x \sqsubseteq z$ and using Lemma 2.2.5 it follows that $x_\bot \sqsubseteq x_\bot \sqcup y_\bot = (x \sqcup y_\bot)_\bot = z_\bot$, and $z \leq_c y$ because $x, y_\bot \sqsubseteq y$ and $y_\bot \sqsubseteq x_\bot \sqcup y_\bot = z_\bot$, thus $z \sqsubseteq y$ and $z^\top \sqsubseteq y^\top$. □

Lemma 2.2.32. *Let A be a lbpd and $x, y \in A$. If there exists a $z \in A$ with $z \sqsubseteq x, y$ or $x, y \sqsubseteq z$ then $\mathsf{FP}(x) \cap \mathsf{FP}(y) \neq \emptyset$.*

Proof. Suppose (1) $x, y \sqsubseteq z$ or (2) $z \sqsubseteq x, y$. If (1) then it follows from Lemma 2.2.30 that $x \uparrow \neg z$ and $y \uparrow \neg z$, thus $x_\bot \uparrow z_\bot$ and $y_\bot \uparrow z_\bot$. If (2) then it follows from Lemma 2.2.30 that $z \uparrow \neg x$ and $z \uparrow \neg y$, thus $z_\bot \uparrow x_\bot$ and $z_\bot \uparrow y_\bot$. Thus in either case we get $x_\bot \uparrow z_\bot$ and $y_\bot \uparrow z_\bot$. Thus $x_\bot \sqcap z_\bot \leq_s z_\bot$ and $y_\bot \sqcap z_\bot \leq_s z_\bot$. Hence $x_\bot \sqcap z_\bot \uparrow y_\bot \sqcap z_\bot$ and it follows that $u := (x_\bot \sqcap z_\bot) \sqcap (y_\bot \sqcap z_\bot) \leq_s x_\bot, y_\bot$. From Lemma 2.2.8 it follows that $\mathsf{FP}(u) \neq \emptyset$, and as $\mathsf{FP}(u) \subseteq \mathsf{FP}(x_\bot) \subseteq \mathsf{FP}(x)$ and $\mathsf{FP}(u) \subseteq \mathsf{FP}(y_\bot) \subseteq \mathsf{FP}(y)$ we get $\mathsf{FP}(x) \cap \mathsf{FP}(y) \neq \emptyset$. □

Next we introduce the notion of atom and investigate the structure of the bistably connected components $[x]_\updownarrow$.

Definition 2.2.33. *Let A be a lbpd and $x \in A$. We write $\mathsf{At}(x)$ for the set of atoms of the boolean algebra $([x]_\updownarrow, \leq_b)$.* ◇

Lemma 2.2.34. *Let A be a lbpd and X be a nonempty subset of A with $\updownarrow X$. Then it follows that $\bigsqcup X, \bigsqcap X \in [X]_\updownarrow$.*

Proof. Suppose X is a nonempty subset of A with $\updownarrow X$. Then due to Lemma 2.2.23 the subset $[X]_\updownarrow$ forms a boolean algebra w.r.t. $\leq_b$. For all finite nonempty subsets F of X we have that $\bigsqcup F \in [X]_\updownarrow$. Thus, the set $\hat{X} := \{\bigsqcup F \mid F \in \mathcal{P}_{\text{f.n.e.}}(X)\}$ is bistably coherent and directed. As $\bigsqcup \hat{X}$ is the least upper bound of X, it follows that $\bigsqcup X$ exists and using Lemma 2.2.13 we get that $x \leq_s \bigsqcup X$ holds for all $x \in X$. For all $x \in X$ we have that $x \leq_s x^\top = X^\top$. Thus as $\bigsqcup X$ is supremum w.r.t. $\leq_s$ it follows from Lemma 2.2.13 that $\bigsqcup X \leq_s X^\top$. From Lemma 2.2.25 it follows that $x \updownarrow \bigsqcup X$ for all $x \in X$, thus we get $\bigsqcup X \in [X]_\updownarrow$.

As $\bigsqcap X = \neg \bigsqcup \{\neg x \mid x \in X\} \updownarrow \bigsqcup \{\neg x \mid x \in X\}$ and $\{\neg x \mid x \in X\} \subseteq [X]_\updownarrow$ it follows that $\bigsqcap X \in [X]_\updownarrow$. □

Theorem 2.2.35. *Let A be a lbpd and $x \in A$ then $[x]_\updownarrow$ is a complete atomic boolean algebra w.r.t. $\leq_b$.*

Proof. Suppose $x \in A$. Then from Lemma 2.2.23 and Lemma 2.2.34 it follows that $[x]_\updownarrow$ is a complete boolean algebra.

Suppose $y \in [x]_\updownarrow$ and $p \in \mathsf{FP}(y)$ with $p \neq p_\bot$. Then we get that $p_\bot \sqsubseteq y_\bot = x_\bot$ holds. As $p, x_\bot \leq_s y$ it follows that $p \sqcup x_\bot$ exists. Further, as $x_\bot \leq_s p \sqcup x_\bot \leq_s y$ it follows from Lemma 2.2.25 that $p \sqcup x_\perp \in [x]_\updownarrow$. For showing that $p \sqcup x_\perp$ is an atom suppose $p \sqcup x_\perp = u \sqcup v$ for $u, v \in [x]_\updownarrow$. As $p \sqsubseteq u \sqcup v$ we have $p \sqsubseteq u$ or $p \sqsubseteq v$. If $p \sqsubseteq u$ then $p \leq_s u$ (as $p_\perp \sqsubseteq x_\perp = u_\perp$) and thus $p \sqcup x_\perp \leq_s u \leq_s p \sqcup x_\perp$, i.e. $p \sqcup x_\perp = u$. Similarly, one shows that $p \sqcup x_\perp = v$ if $p \sqsubseteq v$. Thus $u = p \sqcup x_\perp$ or $v = p \sqcup x_\perp$ as desired.

If $y \in [x]_\updownarrow$ and $y \neq x_\perp$ then for every $p \in \mathsf{FP}(y)$ we have $p_\perp \sqsubseteq y_\perp = x_\perp$. Thus every $y \in [x]_\updownarrow$ is the supremum of all $p \sqcup x_\perp$ with $p \in \mathsf{FP}(y)$ and $p_\perp \neq p$, i.e. a supremum of atoms in $[x]_\updownarrow$. □

Notice that in a complete atomic boolean algebra B we have $B \cong (\mathcal{P}(A), \subseteq)$ where A is the set atoms of B and $\bigsqcup_{x \in X}(x \sqcap y) = (\bigsqcup_{x \in X} x) \sqcap y$ and $\bigsqcap_{x \in X}(x \sqcup y) = (\bigsqcap_{x \in X} x) \sqcup y$ holds for all $X \subseteq B$ and $y \in B$.

Lemma 2.2.36. *If A is a lbpd and $p \in \mathsf{FP}(A)$ then either $p = p_\perp$ or $p \in \mathsf{At}(p)$.*

Proof. If neither $p = p_\perp$ nor $p \in \mathsf{At}(p)$ then $p = u \sqcup v$ for some $u, v \in [p]_\updownarrow$ with $u, v \neq p$ in contradiction to p being prime. □

Lemma 2.2.37. *Let A be a lbpd and $p \in \mathsf{FP}(A)$ with $p \neq p_\perp$. Then $p = q$ whenever $p \leq_s q \in \mathsf{FP}(A)$.*

Proof. Suppose $p \in \mathsf{FP}(A)$ with $p \neq p_\perp$. Then by Lemma 2.2.36 we have $p \in \mathsf{At}(p)$. Suppose $p \leq_s q \in \mathsf{FP}(A)$. Then $q \sqsubseteq p^\top = p \sqcup \neg p$, and as q is prime it follows that (1) $q \sqsubseteq p$ or (2) $q \sqsubseteq \neg p$ holds. In case (1) we immediately get $p = q$. In case (2) we have $p \sqsubseteq \neg q$. Thus as $p \sqsubseteq q$ from Lemma 2.2.30 it follows that $p \uparrow \neg q$. Thus $p \leq_s \neq q$. Thus $p \leq_s q \sqcap \neg q = q_\perp$ which entails $p = p_\perp$ (by Lemma 2.2.24) contradicting the assumption $p \neq p_\perp$. □

Lemma 2.2.38. *Let A be a lbpd and $p \in \mathsf{FP}(A)$ such that p is not $\leq_s$-maximal in $\mathsf{FP}(A)$. Then $p = p_\perp$ holds.*

Proof. This is an immediate consequence of Lemma 2.2.37. □

Lemma 2.2.39. *Let A be a lbpd and $x \in A$. Then $x = x_\perp$ iff $p = p_\perp$ for all $p \in \mathsf{FP}(x)$.*

Proof. For the forward implication suppose $p \in \mathsf{FP}(x)$. Then $p \leq_s x = x_\perp$, thus $p = p_\perp$ by Lemma 2.2.24.

For the reverse implication suppose $p = p_\perp$ for all $p \in \mathsf{FP}(x)$. Then $p = p_\perp \sqsubseteq x_\perp$ for all $p \in \mathsf{FP}(x)$. Thus we have $x = \bigsqcup \mathsf{FP}(x) \sqsubseteq x_\perp$ and it follows that $x = x_\perp$. □

Lemma 2.2.40. *Let A be a lbpd, $x \in A$ and $a \in \mathsf{At}(x)$. Then there exists a unique $p \in \mathsf{FP}(a)$ with $a = p \sqcup a_\perp$. Further, for this unique p it holds that $p \neq p_\perp$.*

Proof. Suppose $a \in \mathsf{At}(x)$. Then $a \neq a_\perp$. If $p \in \mathsf{FP}(a)$ with $p = p_\perp$ then $p = p_\perp \leq_s a_\perp$ and $p \sqcup a_\perp = a_\perp \neq a$. Hence, we need a $p \in \mathsf{FP}(A)$ with $p \neq p_\perp$. As $a \neq a_\perp$ it follows from Lemma 2.2.39 that there exists a $p \in \mathsf{FP}(a)$ with $p \neq p_\perp$. Suppose there exists a $q \in \mathsf{FP}(a)$ with $q \neq q_\perp$ and $p \neq q$. Let $Z := \mathsf{FP}(a) \setminus \{q\}$. From Lemma 2.2.39 it follows that $p, q \notin \mathsf{FP}(a_\perp) \subseteq \mathsf{FP}(a)$ thus $\mathsf{FP}(a_\perp) \subsetneqq Z \subsetneqq \mathsf{FP}(a)$. As Z is stably coherent let $z := \bigsqcup Z$ then $a_\perp = \bigsqcup \mathsf{FP}(a_\perp) <_s z <_s \bigsqcup \mathsf{FP}(a) = a$ (because $p \not\leq_s q$ and $q \not\leq_s p$ by

Lemma 2.2.37). Thus $\updownarrow\{a_\perp, z, a\}$ by Lemma 2.2.25, and $a_\perp <_b z <_b a$ contradicting the assumption that $a \in \mathsf{At}(x)$.

Now, given an atom $a \in \mathsf{At}(x)$ let p be the unique element in $\mathsf{FP}(a)$ with $p \neq p_\perp$. We have shown $\mathsf{FP}(a) = \{r \in \mathsf{FP}(a) \mid r = r_\perp\} \cup \{p\}$. As $\{r \in \mathsf{FP}(a) \mid r = r_\perp\} = \mathsf{FP}(a_\perp)$ we get $a = \bigsqcup \mathsf{FP}(a) = p \sqcup \bigsqcup \mathsf{FP}(a_\perp) = p \sqcup a_\perp$ as desired. □

Given an $x \in A$ with $x = x_\perp$ there arises the question how to characterise those finite prime elements p for which the stable supremum of x and p exists and is an atom in $[x]_\updownarrow$.

Lemma 2.2.41. *Let A be a lbpd, $x \in A$ with $x = x_\perp$ and $p \in \mathsf{FP}(A)$ with $p \neq p_\perp \in \mathsf{FP}(x)$. Then the stable supremum of x and p exists iff $x \sqsubseteq \neg p$. Further, if $x \uparrow p$ then $x \sqcup p \in \mathsf{At}(x)$.*

Proof. Suppose $p \neq p_\perp \leq_s x$. Then $x \sqsubseteq p^\top$. The stable supremum of x and p exists iff $x \uparrow p$, i.e. iff $x \sqsubseteq p^\top$ and $p \sqsubseteq x^\top$. Thus $x \uparrow p$ iff $p \sqsubseteq x^\top$ iff $x_\perp \sqsubseteq \neg p$ iff $x \sqsubseteq \neg p$. Now, if $x \uparrow p$ then $(x \sqcup p)_\perp = x_\perp \sqcup p_\perp = x_\perp$, and it follows that $x <_b x \sqcup p$. Thus, there exists an atom $a \in \mathsf{At}(x)$ with $a \leq_b x \sqcup p$. From Lemma 2.2.40 it follows that there exists a unique $q \in \mathsf{FP}(a)$ with $q \neq q_\perp$ and $a = x \sqcup q$. Thus, $q \leq_s x \sqcup p$. As q is prime and $q \not\sqsubseteq x$ we get that $q \sqsubseteq p$ holds. As $p \uparrow x$ it follows that $p \leq_s x \sqcup p$. Thus we have $q, p \leq_s x \sqcup p$. Thus $q \uparrow p$ and as $q \sqsubseteq p$ it follows that $q \leq_s p$. As $q \neq q_\perp$ it follows from Lemma 2.2.37 that $q = p$. Thus we get $a = x \sqcup p$ as desired. □

Lemma 2.2.42. *Let A be a lbpd, $x \in A$ with $\mathsf{At}(x) \neq \emptyset$ and $p \in \mathsf{FP}(A)$ with $p \sqsubseteq x^\top$. Then there exists an $a \in \mathsf{At}(x)$ with $p \sqsubseteq a$. Further, if $p \neq p_\perp$ then there exists a unique $a \in \mathsf{At}(x)$ with $p \sqsubseteq a$.*

Proof. Suppose $x \in A$ with $\mathsf{At}(x) \neq \emptyset$ and $p \in \mathsf{FP}(A)$ with $p \sqsubseteq x^\top$. As $\mathsf{At}(x)$ is nonempty and $\uparrow\mathsf{At}(x)$ and $x^\top = \bigsqcup \mathsf{At}(x)$ hold it follows from Lemma 2.2.19 that there exists an $a \in \mathsf{At}(x)$ with $p \sqsubseteq a$.

Further, suppose $p \neq p_\perp$. If $p \sqsubseteq a_1, a_2$ for different $a_1, a_2 \in \mathsf{At}(x)$ then $p \sqsubseteq a_1 \sqcap a_2 = x_\perp$. Thus it follows that $p = p_\perp$ in contradiction with $p \neq p_\perp$. □

Theorem 2.2.43. *Let A be a lbpd. Then $\mathsf{FP}(A)$ is a tree and downward closed w.r.t. $\leq_s$, i.e. for all $p \in \mathsf{FP}(A)$ the set $\mathsf{FP}(p)$ is linearly ordered by $\leq_s$ and $p' \leq_s p$ implies $p' \in \mathsf{FP}(A)$.*

Proof. Suppose there exists a prime p such that $\mathsf{FP}(p)$ is not linearly ordered by $\leq_s$. As $(A, \leq_s)$ is a dI-domain there exists a $\leq_s$-minimal prime p such that $\mathsf{FP}(p)$ is not linearly ordered by $\leq_s$. We show that this is impossible from which it follows that for all $p \in \mathsf{FP}(A)$ the set $\mathsf{FP}(p)$ is linearly ordered by $\leq_s$ as desired.

Let $p_1, p_2 \in \mathsf{FP}(p)$ such that neither $p_1 \leq_s p_2$ nor $p_2 \leq_s p_1$. Obviously, then both p_1 and p_2 are strictly below p w.r.t. $\leq_s$. Thus, by minimality of p both $\mathsf{FP}(p_1)$ and $\mathsf{FP}(p_2)$ are linearly ordered by $\leq_s$. As $p_1 \uparrow p_2$ there exists $p_0 = p_1 \sqcap p_2$ which is an infimum w.r.t. $\leq_s$ and $\sqsubseteq$. Obviously, both p_1 and p_2 are different from p_0. Thus we have $p_0 <_s p_i <_s p$ for $i \in \{1, 2\}$. From Lemma 2.2.38 it follows that $p_i = p_{i\perp}$ for $i \in \{0, 1, 2\}$.

As $p_0 <_s p$ we get $p_0 \sqsubset p \sqsubseteq p_0^\top$ and it follows that $p_0 \neq p_0^\top$. Thus we get that $\mathsf{At}(p_0) \neq \emptyset$ and as $p \sqsubseteq p_0^\top$ and p is prime it follows from Lemma 2.2.42 that there exists a unique $a \in \mathsf{At}(p_0)$ with $p \sqsubseteq a$. By Lemma 2.2.40 there exists a unique $q \in \mathsf{FP}(a)$ with $q \neq q_\perp$ and $a = p_0 \sqcup q$. As $p_1 \sqsubseteq p \sqsubseteq a = q \sqcup p_0$ and p_1 is prime it follows that $p_1 \sqsubseteq q$ or $p_1 \sqsubseteq p_0$. The latter cannot happen as otherwise $p_0 = p_1$ and, accordingly, we have $p_1 \sqsubseteq q$. Thus, we have $a = q \sqcup p_0 \sqsubseteq q \sqcup p_1 = q \in \mathsf{FP}(A)$, i.e. that $a = q$ and a is prime. As $p_{0\perp} = p_0$ we have $a \sqsubseteq p_0^\top = \neg p_0 = \neg(p_1 \sqcap p_2) = \neg p_1 \sqcup \neg p_2$. As a is prime it follows that $a \sqsubseteq \neg p_1$ or $a \sqsubseteq \neg p_2$ (because $\neg p_1 \downarrow \neg p_2$). Thus $p_1 \sqsubseteq \neg a$ or $p_2 \sqsubseteq \neg a$. As $p_1, p_2 \sqsubseteq p \sqsubseteq a$ we have

$$\begin{aligned} p_{0\perp} = p_0 \sqsubseteq p_1 \sqsubseteq a \sqcap \neg a = a_\perp = p_{0\perp} \qquad &\text{or} \\ p_{0\perp} = p_0 \sqsubseteq p_2 \sqsubseteq a \sqcap \neg a = a_\perp = p_{0\perp}\,, \end{aligned}$$

i.e. $p_1 = p_{0\perp} = p_0$ or $p_2 = p_{0\perp} = p_0$ which is impossible since $p_1, p_2 \neq p_0$.

Finally, suppose $p \in \mathsf{FP}(A)$ and $p' \leq_s p$. As $\mathsf{FP}(p)$ is finite and linearly ordered w.r.t. $\leq_s$, so is $\mathsf{FP}(p')$. Thus, we have $p' = \bigsqcup \mathsf{FP}(p') = \max_{\leq_s} \mathsf{FP}(p')$ and it follows that p' is prime. □

From Lemma 2.2.24 and Thm. 2.2.43 it follows that for a finite prime element p with $p = p_\perp$ and $x \in A$ we have $x \sqsubseteq p$ iff $x \leq_s p$ and x is finite prime with $x = x_\perp$. In Lemma 2.2.47 we will characterise those cases when a prime is extensionally below a cell, i.e. a finite prime q with $q \in \mathsf{At}(q)$ (cf. section 3.1). For this purpose we will need the following auxiliary lemmas.

Lemma 2.2.44. *Let A be a lbpd and $x, y \in A$ with $x \sqsubseteq y$. Then $x \uparrow y_\perp$.*

Proof. We have $x \sqsubseteq y \sqsubseteq y^\top = (y_\perp)^\top$ and $y_\perp \sqsubseteq \neg y \sqsubseteq \neg x \sqsubseteq x^\top$. □

Lemma 2.2.45. *Let A be a lbpd and $x \in A$ with $\mathsf{FP}(x) = \{x\}$. Then x is minimal w.r.t. $\sqsubseteq$.*

Proof. Suppose $\mathsf{FP}(x) = \{x\}$ and $y \sqsubset x$, then by Lemma 2.2.44 it follows that $y \uparrow x_\perp$. Thus we get $y \sqcap x_\perp <_s x$ and $\mathsf{FP}(y \sqcap x_\perp) \subsetneq \mathsf{FP}(x)$. Hence we have $\mathsf{FP}(y \sqcap x_\perp) = \emptyset$ which is impossible by Lemma 2.2.8. □

Lemma 2.2.46. *Let A be a lbpd and $p, q \in \mathsf{FP}(A)$. If p is minimal w.r.t. $\leq_s$ and $p \sqsubseteq q$ then already $p \leq_s q$.*

Proof. Suppose $p \in \mathsf{FP}(A)$ is $\leq_s$-minimal and $q \in \mathsf{FP}(A)$ with $p \sqsubseteq q$. Then $\mathsf{FP}(p) = \{p\}$ and $p = p_\perp$. Due to Lemma 2.2.44 we have $p \uparrow q_\perp$, and thus $p \sqcap q_\perp \leq_s p$. As p is $\leq_s$-minimal we have $p \sqcap q_\perp = p$. Thus we get $p_\perp = p \sqsubseteq q_\perp$ and hence $p \leq_s q$ as desired. □

Lemma 2.2.47. *Let A be a lbpd and $q \in \mathsf{FP}(A)$ with $q \neq q_\perp$. For $p \in \mathsf{FP}(A)$ we have $p \sqsubseteq q$ iff $p \leq_s q$ or $p \leq_c q$.*

Proof. The implication from right to left is obvious. The reverse implication we prove by induction on $|\mathsf{FP}(p)|$.

If $|\mathsf{FP}(p)| = 1$ and $p \sqsubseteq q$ then p is $\leq_s$-minimal and, therefore, by Lemma 2.2.46 we have $p \leq_s q$.

Suppose $|\mathsf{FP}(p)| > 1$ and $p \sqsubseteq q$. As $\mathsf{FP}(p)$ is finite and linearly ordered w.r.t. $\leq_s$, let p_0 be the greatest (w.r.t. $\leq_s$) element in $\mathsf{FP}(p)$ with $p_0 \neq p$. Obviously, we have $p_0 = {p_0}_\perp$ and $|\mathsf{FP}(p_0)| < |\mathsf{FP}(p)|$. Thus, by induction hypothesis we have $p_0 \leq_s q$ or $p_0 \leq_c q$. We show that in either case $p \leq_s q$ or $p \leq_c q$.

Suppose $p_0 \leq_c q$. Then $q_\perp \sqsubseteq {p_0}_\perp \sqsubseteq p_\perp$. As $p \sqsubseteq q$ by assumption we have $p \leq_c q$ as desired.

Suppose $p_0 \leq_s q$. As $p_0 = q$ implies $p_0 \leq_c q$ we can assume that $p_0 <_s q$ holds. We have that $p_0 = {p_0}_\perp \sqsubseteq q_\perp$ holds. Suppose neither $p \leq_s q$ nor $p \leq_c q$, i.e. $p_\perp \not\sqsubseteq q_\perp$ and $q_\perp \not\sqsubseteq p_\perp$. Then $p_\perp \neq p_0$ as otherwise $p_\perp = p_0 = {p_0}_\perp \sqsubseteq q_\perp$ (because $p_0 \leq_s q$). Thus as $p_0 \neq p_\perp$, $p_\perp \leq_s p$ and p_0 is the greatest (w.r.t. $\leq_s$) element in $\mathsf{FP}(p)$ with $p_0 \neq p$ it follows that $p = p_\perp$. As $p_0 <_s q$ we have $q \sqsubseteq {p_0}^\top$. Thus $p_0 \sqsubset p_0^\top$ and it follows that $\mathsf{At}(p_0) \neq \emptyset$ and as q is prime and $q \sqsubseteq {p_0}^\top$ it follows from Lemma 2.2.42 that there exists an atom $a \in \mathsf{At}(p_0)$ with $q \sqsubseteq a$. As $p \sqsubseteq q$ it follows by Lemma 2.2.44 that $p \uparrow q_\perp$. Thus $p \sqcap q_\perp$ is an infimum w.r.t. $\leq_s$ and $\sqsubseteq$. As $p = p_\perp$ and $q_\perp$ are incomparable w.r.t. $\sqsubseteq$ it follows that $p \sqcap q_\perp \neq p$ and $p \sqcap q_\perp \neq p$. Thus, as $p_0 \leq_s p \sqcap q_\perp$ we have $p_0 = p \sqcap q_\perp$ and, accordingly, also ${p_0}^\top = \neg p \sqcup q^\top$. By Lemma 2.2.40 there exists a prime c with $c \uparrow a_\perp = p_0$ and $a = c \sqcup p_0$. Thus, we have $c \sqsubseteq a \sqsubseteq {p_0}^\top = \neg p \sqcup q^\top$. As $p \uparrow q_\perp$ it follows that $\neg p \downarrow q^\top$. Thus as c is prime it follows that $c \sqsubseteq \neg p$ or $c \sqsubseteq q^\top$. As $p_0 \sqsubseteq p$ and $p_0 \sqsubseteq q \sqsubseteq q^\top$ hold anyway it follows that $a \sqsubseteq \neg p$ or $a \sqsubseteq q^\top$. This, however, is impossible as shown by the following reasoning. If $a \sqsubseteq \neg p$ then $p \sqsubseteq \neg a$ and as $p \sqsubseteq q \sqsubseteq a$ it follows that $p \sqsubseteq a \sqcap \neg a = p_0$ in contradiction to $p_0 \sqsubseteq p$ and $p_0 \neq p$. If $a \sqsubseteq q^\top$ then $q_\perp \sqsubseteq \neg a$ and as $q_\perp \sqsubseteq q \sqsubseteq a$ it follows that $q_\perp \sqsubseteq a \sqcap \neg a = p_0$. As $p_0 \sqsubseteq q_\perp$ we get $p_0 = q_\perp$. Thus we have $q_\perp = p_0 \sqsubseteq p = p_\perp$ in contradiction to the fact that $p_\perp$ and $q_\perp$ are incomparable w.r.t. $\sqsubseteq$.

Thus we have shown that it cannot hold that neither $p \leq_s q$ nor $p \leq_c q$, hence it follows that $p \leq_s q$ or $p \leq_c q$ as desired. □

Theorem 2.2.48. *Let A be a lbpd and $p, q \in \mathsf{FP}(A)$. Then $p \sqsubseteq q$ iff $p \leq_s q$ or $p \leq_c q$.*

Proof. The implication from right to left is immediate.

We prove the reverse implication by case analysis on q. If $q = q_\perp$ and $p \sqsubseteq q$ then from Lemma 2.2.24 it follows that $p \leq_s q$. If $q \neq q_\perp$ then it follows from Lemma 2.2.36 that $q \in \mathsf{At}(q)$. As $p \sqsubseteq q$ we get from Lemma 2.2.47 that $p \leq_s q$ or $p \leq_c q$ as desired. □

Thm. 2.2.48 allows us to give the following slightly improved characterisation of the extensional order.

Theorem 2.2.49. *Let A be a lbpd and $x, y \in A$. Then $x \sqsubseteq y$ iff for all $p \in \mathsf{FP}(x)$ there exists a $q \in \mathsf{FP}(y)$ with $p \leq_c q$.*

Proof. The implication from right to left is obvious (using Lemma 2.2.11).

For the reverse implication suppose $x \sqsubseteq y$ and $p \in \mathsf{FP}(x)$. By Lemma 2.2.11 there exists a $q' \in \mathsf{FP}(y)$ with $p \sqsubseteq q'$. By Thm. 2.2.48 we have $p \leq_s q'$ or $p \leq_c q'$. In the first case putting $q := p$ we have $q = p \in \mathsf{FP}(y)$ and $p \leq_c q$. In the second case putting $q := q'$ we have $q \in \mathsf{FP}(y)$ and $p \leq_c q$. $\square$

Theorem 2.2.50. *Let A be a lbpd and $x, y \in A$. Then $x \sqsubseteq y$ iff for all $c \in \mathsf{F}(x)$ there exists a $d \in \mathsf{F}(y)$ with $c \leq_c d$.*

Proof. The implication from right to left is obvious (using Lemma 2.2.11).

For the reverse implication suppose $x \sqsubseteq y$ and $c \in \mathsf{F}(x)$. Then there exists $p_1, \ldots, p_n \in \mathsf{FP}(x)$ with $\bigsqcup\{p_1, \ldots, p_n\} = c$. Using Thm. 2.2.49 we get $q_1, \ldots, q_n \in \mathsf{FP}(y)$ with $q_\perp \sqsubseteq p_i \sqsubseteq q_i$ for $i \in \{1, \ldots, n\}$. Now, we have $\bigsqcup\{p_1, \ldots, p_n\} \sqsubseteq \bigsqcup\{q_1, \ldots, q_n\}$ and, using Lemma 2.2.5(1), $\bigsqcup\{q_1, \ldots, q_n\}_\perp = \bigsqcup\{q_{1\perp}, \ldots, q_{n\perp}\} \sqsubseteq \bigsqcup\{p_1, \ldots, p_n\}$. Thus, $c = \bigsqcup\{p_1, \ldots, p_n\} \leq_c \bigsqcup\{q_1, \ldots, q_n\} \in \mathsf{F}(y)$ as desired. $\square$

Based on Thm. 2.2.49 we will obtain a characterisation of the costable ordering. For this purpose, however, we need the following lemma.

Lemma 2.2.51. *Let A be a lbpd and $p \in \mathsf{FP}(A)$. Then p is minimal w.r.t. $\leq_c$ iff $p = p_\perp$.*

Proof. Let $p \in \mathsf{FP}(A)$. Suppose $p = p_\perp$. If q is an element with $q \leq_c p$ then $q \sqsubseteq p = p_\perp \sqsubseteq q_\perp \sqsubseteq q$ and thus $p = q$. Thus p is minimal w.r.t. $\leq_c$. If $p \in \mathsf{FP}(A)$ is minimal w.r.t. $\leq_c$ then $p = p_\perp$ since $p_\perp \leq_c p$. $\square$

Theorem 2.2.52. *Let A be a lbpd and $x, y \in A$. Then $x \leq_c y$ iff the following two conditions hold*

(1) *for every $p \in \mathsf{FP}(x)$ there exists a $q \in \mathsf{FP}(y)$ with $p \leq_c q$*

(2) *for every $q \in \mathsf{FP}(y_\perp)$ there exists a $p \in \mathsf{FP}(x)$ with $q \leq_c p$.*

Proof. Let $x, y \in A$. We have $x \leq_c y$ iff $y_\perp \sqsubseteq x \sqsubseteq y$. By Thm. 2.2.49 the second inequality is equivalent to (1) and the first inequality is equivalent to (2). $\square$

Lemma 2.2.53. *Let A be a lbpd $x \in A$ and $p \in \mathsf{FP}(x)$. Then the following statements are equivalent:*

(1) $p \leq_s \neg x$

(2) $p \sqsubseteq \neg x$

(3) $p = p_\perp$

Proof. Suppose $x \in A$ and $p \in \mathsf{FP}(x)$.

ad (1) $\Rightarrow$ (2) : This is obvious.

ad (2) $\Rightarrow$ (3) : Suppose $p \sqsubseteq \neg x$. As $p \in \mathsf{FP}(x)$ it follows that $p \leq_s x$. Thus $p \sqsubseteq x \sqcap \neg x = x_\perp$. Using Lemma 2.2.24 we get $p = p_\perp$.

ad (3) $\Rightarrow$ (1) : Suppose $p = p_\perp$. As $p \in \mathsf{FP}(x)$ we have $p \leq_s x$. Thus by Lemma 2.2.26 it follows that $p = p_bot \leq_s x_\perp$. As $x_\perp \leq_s \neg x$ it follows $p \leq_s \neg x$ as desired. $\square$

Lemma 2.2.54. *Let A be a lbpd and $x, y \in A$ with $x \updownarrow y$. Then it holds that*

(1) $\forall p{\in}\mathsf{FP}(x).\exists q{\in}\mathsf{FP}(y).\, p \updownarrow q$

(2) $\forall c{\in}\mathsf{F}(x).\exists d{\in}\mathsf{F}(y).\, c \updownarrow d$

Proof. Suppose $x, y \in A$ with $x \updownarrow y$.

ad (1) : Suppose $p \in \mathsf{FP}(x)$. From Thm. 2.2.43 it follows that $p_\perp \in \mathsf{FP}(x)$. As $p_\perp \leq_s x$ it follows that $p_\perp \leq_s x_\perp$ by Lemma 2.2.26. Thus, $p_\perp \leq_s x_\perp = y_\perp \leq_s y$ and we have $p_\perp \in \mathsf{FP}(y)$ and $p \updownarrow p_\perp$.

ad (2) : Suppose $c \in \mathsf{F}(x)$ then c is finite and $c \leq_s x$. It follows that $c_\perp$ is finite and $c_\perp \leq_s x$. From Lemma 2.2.26 it follows that $c_\perp \leq_s x_\perp$. As $x_\perp = y_\perp$ we have $c_\perp \leq_s x_\perp \leq_s y_\perp \leq_s y$, thus $c_\perp \in \mathsf{F}(y)$ and $c \updownarrow c_\perp$. □

Lemma 2.2.55. *Let A be a lbpd and $x, y \in A$.*

(1) *If $x \downarrow y$ then there exist elements $x', y' \in [x \sqcup y]_\updownarrow$ with $x' \leq_s x$, $y' \leq_s y$ and $x' \sqcup y' = x \sqcup y$.*

(2) *If $x \uparrow y$ then there exist elements $x', y' \in [x \sqcap y]_\updownarrow$ with $x \leq_c x'$, $y \leq_c y'$ and $x' \sqcap y' = x \sqcap y$.*

Proof. Suppose $x, y \in A$.

ad (1) : Putting $x' := \bigsqcup \mathsf{FP}(x{\sqcup}y){\cap}\mathsf{FP}(x)$ and $y' := \bigsqcup \mathsf{FP}(x{\sqcup}y){\cap}\mathsf{FP}(y)$ it follows that $x' \leq_s x, x \sqcup y$ and $y' \leq_s y, x \sqcup y$. From Lemma 2.2.29(2)(ii) we get $(x \sqcup y)_\perp = x_\perp \sqcap y_\perp$. Thus, $(x \sqcup y)_\perp \leq_s x_\perp$. If $p \in \mathsf{FP}((x \sqcup y)_\perp)$ then $p \in \mathsf{FP}(x \sqcup y)$ and $p \in \mathsf{FP}(x_\perp) \subseteq \mathsf{FP}(x)$. Thus $(x \sqcup y)_\perp \leq_s x'$. Now, as $(x \sqcup y)_\perp \leq_s x' \leq_s x \sqcup y$ using Lemma 2.2.25 we get $x \sqcup y \updownarrow x'$. Analogously, it follows that $x \sqcup y \updownarrow y'$. Thus, $x', y' \in [x \sqcup y]_\updownarrow$. For showing $x' \sqcup y' = x \sqcup y$, consider

$$\begin{aligned}
x' \sqcup y' &= \left(\bigsqcup \mathsf{FP}(x \sqcup y) \cap \mathsf{FP}(x)\right) \sqcup \left(\bigsqcup \mathsf{FP}(x \sqcup y) \cap \mathsf{FP}(y)\right) \\
&= \bigsqcup (\mathsf{FP}(x \sqcup y) \cap \mathsf{FP}(x)) \cup (\mathsf{FP}(x \sqcup y) \cap \mathsf{FP}(y)) \\
&= \bigsqcup \mathsf{FP}(x \sqcup y) \cap (\mathsf{FP}(x) \cup \mathsf{FP}(y)) && (\dagger) \\
&= \bigsqcup \mathsf{FP}(x \sqcup y) && (\dagger) \\
&= x \sqcup y
\end{aligned}$$

where (†) follows from Lemma 2.2.21 and (‡) holds as $p \in \mathsf{FP}(x \sqcup y)$ implies $p \in \mathsf{FP}(x)$ or $p \in \mathsf{FP}(y)$ (since p is prime).

ad (2) : This follows from (1) by duality. □

As final result of this section we show that one can reconstruct a lbpd from the underlying bistable biorder (cf. [Lai05a]). Thus being a lbpd is property of a bistable biorder rather than an additional structure.

Theorem 2.2.56. *Let A be a lbpd. Then the involution $\neg : |A| \to |A|$ is uniquely determined by the extensional order $\sqsubseteq$ and the stable order $\leq_s$.*

Proof. Suppose A is a lbpd. Given the stable order $\leq_s$ we can reconstruct the stable coherence relation $\uparrow$ by

$$x \uparrow y \quad \text{iff} \quad \exists z{\in}A.\, x, y \leq_s z$$

for all $x, y \in A$.

Since $\forall x, y{\in}A.(x{\uparrow}y \to y{\sqsubseteq}x^\top)$ it follows that $x^\top = \max_\sqsubseteq\{y \in A \mid x \uparrow y\}$. Thus as $x \updownarrow y$ iff $x^\top = y^\top$ we obtain the bistable coherence relation of A. Finally since for all $x \in A$ the set $[x]_\updownarrow$ is a boolean algebra with $\neg|_{[x]_\updownarrow}$ as boolean negation we get $\neg x$ as the least element of all $y \in [x]_\updownarrow$ with $x \sqcup y = x^\top$.

Thus we have *determined* the involution $\neg$ in terms of $\sqsubseteq$ and $\leq_s$. ☐

2.3 Bistable maps

In this section we introduce *bistable maps*. The notion of bistable map is an extension of Berry's notion of stable maps. A stable map preserves infima of stably coherent pairs while in Lemma 2.3.3 we show that bistable maps can be characterised as those stable maps that are also costable i.e. preserve suprema of costably coherent pairs.

As usual we call a function $f : A \to B$ between lbpds (*Scott*) *continuous* iff f is monotone, i.e. for all $x, y \in A$, $x \sqsubseteq y$ implies $f(x) \sqsubseteq f(y)$ and preserves directed least upper bounds w.r.t. $\sqsubseteq$, i.e.

$$f(\bigsqcup D) = \bigsqcup f(D)$$

for all $\sqsubseteq$-directed subsets $D \subseteq A$.

Definition 2.3.1. *Let A and B be lbpds. A function $f : A \to B$*

- preserves stable (*resp.* costable, *resp.* bistable) coherence *iff for all $x, y \in A$, $x \uparrow y$ (resp. $x \downarrow y$, resp. $x \updownarrow y$) implies*

$$f(x) \uparrow f(y) \ (\textit{resp. } f(x) \downarrow f(y),\ \textit{resp. } f(x) \updownarrow f(y))$$

- preserves stably coherent infima (*resp.* costably coherent suprema, *resp.* bistably coherent infima and suprema) *iff for all $x, y \in A$, $x \uparrow y$ (resp. $x \downarrow y$, resp. $x \updownarrow y$) implies*

$$\begin{gathered} f(x \sqcap y) = f(x) \sqcap f(y) \\ (\textit{resp. } f(x \sqcup y) = f(x) \sqcup f(y), \\ \textit{resp. } f(x \sqcap y) = f(x) \sqcap f(y) \textit{ and } f(x \sqcup y) = f(x) \sqcup f(y)) \end{gathered}$$

A function $f : A \to B$ is called stable (*resp.* costable, *resp.* bistable) *iff*

(1) *f is (Scott) continuous,*

(2) *f preserves stable (resp. costable, resp. bistable) coherence*

(3) *f preserves stably coherent infima (resp. costably coherent suprema, resp. bistably coherent infima and suprema)*

If A and B are pointed then a bistable map f is called strict *if $f(\bot_A) = \bot_B$, and f is called* bistrict *if $f(\bot_A) = \bot_B$ and $f(\top_A) = \top_B$* $\diamond$

Obviously the identity map on a lbpd is bistable and it is an easy exercise to verify that the composition of bistable maps is also bistable. The ensuing category of locally boolean (pre)domains and sequential maps will be denoted by **LBD** (resp. **LBPD**).

Lemma 2.3.2. *Let A and B be lbpds and $f : A \to B$ a monotone function then t.f.a.e.*

(1) *f preserves the bistable order*

(2) *f preserves bistable coherence*

(3) *$f(x)_\bot = f(x_\bot)_\bot$ for all $x \in A$*

(4) *$f(x)^\top = f(x^\top)^\top$ for all $x \in A$*

(5) *f preserves stable and costable coherence*

(6) *f preserves the stable and the costable order*

Proof. Suppose $f : A \to B$ is a monotone function.

ad (1) $\Rightarrow$ (2) : Obvious.

ad (2) $\Rightarrow$ (3) : Suppose f preserves bistable coherence. As $x \updownarrow x_\bot$ it follows that $f(x) \updownarrow f(x_\bot)$. Thus, $f(x)_\bot = f(x_\bot)_\bot$.

ad (3) $\Rightarrow$ (4) : Suppose $f(x)_\bot = f(x_\bot)_\bot$ for all $x \in A$. Using negation we get $f(x)^\top = f(x_\bot)^\top$ for all $x \in A$. In particular, we have $f(x^\top)^\top = f((x^\top)_\bot)^\top = f(x_\bot)^\top$ for all $x \in A$. Thus $f(x)^\top = f(x^\top)^\top$ for all $x \in A$.

ad (4) $\Rightarrow$ (5) : Suppose that $f(x)^\top = f(x^\top)^\top$ holds for all $x \in A$. Suppose $x \uparrow y$, i.e. $x \sqsubseteq y^\top$ and $y \sqsubseteq x^\top$. Then $f(x) \sqsubseteq f(y^\top) \sqsubseteq f(y^\top)^\top = f(y)^\top$. Analogously, we get $f(y) \sqsubseteq f(x)^\top$, thus $f(x) \uparrow f(y)$.
For the preservation of the costable coherence notice that from $\forall x{\in}A.\, f(x)^\top = f(x^\top)^\top$ it follows that $\forall x{\in}A.\, f(x)_\bot = f(x^\top)_\bot$. Thus for all $x \in A$ it follows that $f(x_\bot)_\bot = f((x_\bot)^\top)_\bot = f(x^\top)_\bot = f(x)_\bot$. Suppose $x \downarrow y$, i.e. $x_\bot \sqsubseteq y$ and $y_\bot \sqsubseteq x$. Then $f(x)_\bot = f(x_\bot)_\bot \sqsubseteq f(x_\bot) \sqsubseteq f(y)$, and analogously, $f(y)_\bot \sqsubseteq f(x)$, thus, $f(x) \downarrow f(y)$.

ad (5) $\Rightarrow$ (6) : Suppose f preserves stable and costable coherence. Suppose $x \leq_s y$, then $f(x) \sqsubseteq f(y)$ as f is monotone, and $f(x) \uparrow f(y)$ as f preserves stable coherence. Analogously, it follows that f preserves the costable order.

ad (6) $\Rightarrow$ (1) : Suppose preserves the stable and the costable order and $x \updownarrow y$. Then $x \uparrow y$ and $x \downarrow y$, thus, $f(x) \uparrow f(y)$ and $f(x) \downarrow f(y)$, thus, $f(x) \updownarrow f(y)$. $\square$

Lemma 2.3.3. *Let A and B be lbpds and $f : A \to B$ be monotone. Then f preserves bistable coherence and bistably coherent infima and suprema iff f preserves stable and costable coherence, stably coherent infima and costably coherent suprema.*

Proof. The reverse implication is obvious.

For the forward implication suppose that f preserves bistable coherence and bistably coherent infima and suprema and let $x, y \in A$.

Suppose $x \uparrow y$. As f preserves bistable coherence it follows from Lemma 2.3.2 that f preserves stable coherence. Thus we get $f(x) \uparrow f(y)$. As f is monotone it follows that $f(x \sqcap y) \sqsubseteq f(x) \sqcap f(y)$. Because of Lemma 2.2.55(2) there exist elements $x', y' \in A$ with $x' \updownarrow y'$, $x \leq_c x'$, $y \leq_c y'$ and $x \sqcap y = x' \sqcap y'$. Thus we have $f(x) \sqcap f(y) \sqsubseteq f(x') \sqcap f(y') = f(x' \sqcap y') = f(x \sqcap y)$. Thus we get $f(x) \sqcap f(y) = f(x \sqcap y)$ as desired. The preservation of costably coherent suprema follows by duality and using Lemma 2.2.55(1). □

As an immediate consequence of Lemma 2.3.3 we get the following characterisation of bistable maps which will be used implicitly from now on.

Corollary 2.3.4. *Let A and B be lbpds and $f : A \to B$ a function. Then f is bistable iff f is stable and costable.*

3 Locally boolean domains and Curien-Lamarche games

3.1 Curien-Lamarche games as locally boolean domains

One of the simplest notion of games is the notion of *Curien-Lamarche games* or *Sequential Data Structures* as given in [CCF94, AC98]. The morphism between those games are given by observably sequential functions. In [CCF94] R. Cartwright, P.L. Curien and M. Felleisen have shown that Curien-Lamarche games and observably sequential functions provide a fully abstract model for the language SPCF i.e. an extension of PCF by error elements and control operators **catch**. Since we want to interpret an infinitary extension of SPCF in locally boolean domains we first show that the categories of Curien-Lamarche games and observably sequential algorithms and locally boolean domains and bistable maps (cf. section 2.3) are equivalent. We will establish a translation from locally boolean domains to Curien-Lamarche games and vice versa.

We will only give the basic definition of Curien-Lamarche games. For a detailed introduction we refer to [Lam92, CCF94, AC98].

Definition 3.1.1. *A* Curien-Lamarche game *(or simply a* CL-game *from now on) is a triple $A = (C, V, P)$ where C is a set of* cells*, V is a set of* values *and P is a prefix-closed set of (finite) sequences whose entries at odd positions are cells and whose entries at even positions are values. We write $(C, V)^*$ for the set of all such alternating sequences, Que_A for the set of all s in P whose length is odd and Rsp_A for the set of all s in P whose length is even. We write $\mathsf{Rsp}_A^\top$ for the set $\mathsf{Rsp}_A \cup (\mathsf{Que}_A \times \{\top\})$.*

A strategy *of A is a subset x of $\mathsf{Rsp}_A^\top$ such that*

(1) x is closed under even length prefixes and

(2) $q{\cdot}v_1, q{\cdot}v_2 \in x$ implies $v_1 = v_2$ for all $q \in \mathsf{Que}_A$ and $v_1, v_2 \in V \cup \{\top\}$.

Notice, that we assume w.l.o.g. that $\top \notin V$. We write $\mathsf{Strat}(A)$ for the set of all strategies of A. ◇

Thus a strategy x may be understood as (the graph of) a partial function $\sigma : \mathsf{Que}_A \rightharpoonup V \cup \{\top\}$ whose domain of definition is closed under odd length prefixes and satisfies the condition $q{\cdot}\sigma(q) \in \mathsf{Rsp}_A^\top$ for all $q \in \mathsf{dom}(\sigma)$.

Given a CL-game A the set $\mathsf{Strat}(A)$ ordered by set inclusion is denoted by $\mathbb{D}(A)$. A partial order that is isomorphic to $\mathbb{D}(A)$ is called the *observably sequential domain* generated by A. We write **OSA** for the category of Curien-Lamarche games and observably sequential algorithms (cf. [CCF94, AC98] and section 3.3).

Next we present the object part of an equivalence between the category **LBD** of locally boolean domains and bistable maps and the category **OSA**. For this purpose we first define an extensional order on the strategies of a CL-game.

Definition 3.1.2. *Let $A = (C, V, P)$ be a CL-game and x a strategy of A and $q \in \mathsf{Que}_A$*

- *If $q{\cdot}v \in x$ for some $v \in V \cup \{\top\}$ we write $q \in \mathsf{Fill}(x)$ (q is* filled *in x).*
- *If $r \in x$ and $q = r{\cdot}c$ for some $c \in C$, we say that q is* enabled *in x.*
- *If q is enabled in x but $q \notin \mathsf{Fill}(x)$, we write $q \in \mathsf{Acc}(x)$ (q is* accessible *from x).*

$\diamond$

Definition 3.1.3. *Let A be a CL-game. For elements $r, s \in \mathsf{Rsp}_A^\top$ we write*

$$r \sqsubseteq s \quad \textit{iff} \quad r \textit{ is a prefix of } s \textit{ or } (s = q{\cdot}\top \textit{ and } q \textit{ is a prefix of } r).$$

For strategies $x, y \in \mathsf{Strat}(A)$ we write

$$x \sqsubseteq y \quad \textit{iff} \quad \forall r{\in}x.\exists s{\in}y.\, r \sqsubseteq s\,. \qquad \diamond$$

Lemma 3.1.4. *Let A be a CL-game. Then $\sqsubseteq$ is a partial order on $\mathsf{Rsp}_A^\top$.*

Proof. Obviously, $\sqsubseteq$ is reflexive.

Suppose there are $r, s \in \mathsf{Rsp}_A^\top$ with $r \sqsubseteq s$ and $s \sqsubseteq r$. Then (r is a prefix of s or ($s = q{\cdot}\top$ and q is a prefix of r)) and (s is a prefix of r or ($r = q'{\cdot}\top$ and q is a prefix of s)). Assuming that r is a proper prefix of s it follows that $r = q'{\cdot}\top$ and q is a prefix of s which is impossible. Assuming that $r \neq s$, $s = q{\cdot}\top$ and q is a prefix of r it follows that s is a proper prefix of r which is also impossible. Thus it follows that $r = s$ and we have shown that $\sqsubseteq$ is antisymmetric.

Suppose there are $r, s, t \in \mathsf{Rsp}_A^\top$ with $r \sqsubseteq s$ and $s \sqsubseteq t$. Then (r is a prefix of s or ($s = q{\cdot}\top$ and q is a prefix of r)) and (s is a prefix of t or ($t = q'{\cdot}\top$ and q is a prefix of s)).

Suppose that r is a prefix of s. If s is a prefix of t then obviously r is a prefix of t. If $t = q'{\cdot}\top$ and q is a prefix of s then r is a prefix of q' or q' is a prefix of r. Thus in both cases we get $r \sqsubseteq t$.

Suppose that $s = q{\cdot}\top$ and q is a prefix of r. If s is a prefix of t then it follows that $s = t$. If $t = q'{\cdot}\top$ and q is a prefix of s then q' is a prefix of r. Thus in both cases we get $r \sqsubseteq t$.

Thus we have shown that $\sqsubseteq$ is transitive. $\square$

Lemma 3.1.5. *Let A be a CL-game. Then $\sqsubseteq$ is a partial order on $\mathsf{Strat}(A)$.*

Proof. Reflexivity and transitivity of $\sqsubseteq$ follow immediately from the definition of $\sqsubseteq$ and Lemma 3.1.4.

For showing that $\sqsubseteq$ is antisymmetric suppose there are $x, y \in \mathsf{Strat}(A)$ with $x \sqsubseteq y$ and $y \sqsubseteq x$. Let $r \in x$. Then there exists a $s \in y$ with $r \sqsubseteq s$. If r is a prefix of s then as

y is closed under even length prefixes (by Def. 3.1.1(1)) it follows that $r \in y$. Otherwise we have that $s = q{\cdot}\top$ and q is a prefix of r. If $s = r$ then $r \in y$. So, we can assume that $s = q{\cdot}\top$, q is a prefix of r and $s \neq r$.

As $y \sqsubseteq x$ there exists $r' \in x$ with $s \sqsubseteq r'$. If s is a prefix of r' then as $s = q{\cdot}\top$ it follows that $r' = s$. Otherwise we have that $r' = q'{\cdot}\top$ and q' is a prefix of s. Thus in both cases it follows that $r' = q'{\cdot}\top$ for some q' and that q' is a prefix of s. Thus it follows that q' is a prefix of r. As $r \neq s$ it follows that there exists a $v' \neq \top$ with $q' \cdot v'$ is a prefix of r. Hence it follows from Def. 3.1.1(1) that $q' \cdot v' \in x$. Thus we have $q' \cdot v', q' \cdot \top \in x$ in contradiction with Def. 3.1.1(2)

Thus it follows that $r \in y$. Hence we get $x \subseteq y$. Analogously, it follows that $y \subseteq x$. Thus we get $x = y$ and it follows that $\sqsubseteq$ is antisymmetric. $\square$

Definition 3.1.6. *Given a* CL*-game* $A = (C, V, P)$ *we define* $\mathcal{D}(A) := (\mathsf{Strat}(A), \sqsubseteq, \neg)$ *where negation* $\neg : \mathsf{Strat}(A) \to \mathsf{Strat}(A)$ *is defined by*

$$\neg x := (x \cap \mathsf{Rsp}_A) \cup \{q{\cdot}\top \mid q \in \mathsf{Acc}(x)\}$$

for all $x \in \mathsf{Strat}(A)$. $\diamond$

Lemma 3.1.7. *Let* A *be a* CL*-game and* $x \in \mathcal{D}(A)$. *Then*

(1) $x \sqcup \neg x = x \cup \neg x = x \cup \{q{\cdot}\top \mid q \in \mathsf{Acc}(x)\}$ *and*

(2) $x \sqcap \neg x = x \cap \neg x = x \cap \mathsf{Rsp}_A$

hold.

Proof. Suppose $x \in \mathcal{D}(A)$. The second equality of (1) (resp. (2)) is an immediate consequence of the definition of the negation.

ad (1) : Let $y \in \mathcal{D}(A)$ with $x, \neg x \sqsubseteq y$ and $r \in x \cup \neg x$. Thus $r \in x$ or $r \in \neg x$ and it follows that there exists a $s \in y$ with $r \sqsubseteq s$ as desired.

ad (2) : Obviously, we have $x \cap \neg x \sqsubseteq x, \neg x$. We show that $y \sqsubseteq x \cap \neg x$ whenever $y \sqsubseteq x, \neg x$.

Let $y \in \mathcal{D}(A)$ with $y \sqsubseteq x, \neg x$ and $r \in y$. Thus there exist $s \in x$ and $s' \in \neg x$ with $r \sqsubseteq s, s'$. If $s \in \mathsf{Rsp}_A$ then we get $s \in x \cap \mathsf{Rsp}_A$. Analogously, if $s' \in \mathsf{Rsp}_A$ we get $s' \in x \cap \mathsf{Rsp}_A$. In case that $s, s' \notin \mathsf{Rsp}_A$ we get $s = t{\cdot}c{\cdot}\top$ and $s' = t'{\cdot}c'{\cdot}\top$. We proceed by case analysis on r. If r is a prefix of t or t' we are finished. Otherwise $t{\cdot}c$ and $t'{\cdot}c'$ are both prefixes of r (because $r \sqsubseteq s, s'$). Thus $t{\cdot}c$ is a prefix of $t'{\cdot}c'$ or vice versa. W.l.o.g. suppose that $t{\cdot}c$ is a prefix of $t'{\cdot}c'$. As $t{\cdot}c{\cdot}\top = s \in x$ and $t'{\cdot}c'{\cdot}\top = s' \in \neg x$ it follows from (1) that $t{\cdot}c{\cdot}\top, t'{\cdot}c'{\cdot}\top \in x \sqcup \neg x$. As $t{\cdot}c$ is a prefix of $t'{\cdot}c'$ it follows from Def. 3.1.1(2) that $t{\cdot}c = t'{\cdot}c'$. Thus we have $t{\cdot}c{\cdot}\top \in x, \neg x$ which contradicts Def. 3.1.6.

Thus for each $r \in y$ there exists an $s \in x \cap \neg x$ with $r \sqsubseteq s$ and it follows that $x \cap \neg x = x \sqcap \neg x$. $\square$

Notice that the previous lemma allows for the definition of stable coherence in $\mathcal{D}(A)$. Next we show that infima (resp. suprema) of stably coherent pairs exists and is given by their intersection (resp. union).

Lemma 3.1.8. *Let A be a* CL*-game and $x, y \in \mathcal{D}(A)$ with $x \uparrow y$. If $q{\cdot}\top \in x$ and $r \in y$ and q is a prefix of r then $q{\cdot}\top = r$.*

Proof. Suppose $x, y \in \mathcal{D}(A)$ with $x \uparrow y$ and $q{\cdot}\top \in x$. It suffices to show that $q{\cdot}v \in y$ implies $v = \top$. Thus suppose $q{\cdot}v \in y$. As $x \sqsubseteq y^\top$ there exists a $s \in y^\top$ with $q{\cdot}\top \sqsubseteq s$. Thus we have either case $q{\cdot}\top = s$ and get $v = \top$ by Def. 3.1.1(2), or $s = q'{\cdot}\top$ and q' is a proper prefix of q but this contradicts the assumption that $q{\cdot}\top \in y$. ☐

Lemma 3.1.9. *Let A be a* CL*-game and $x, y \in \mathcal{D}(A)$ with $x \uparrow y$. Then*

(1) $x \sqcup y = x \cup y$ *and*

(2) $x \sqcap y = x \cap y$

hold.

Proof. Suppose $x, y \in \mathcal{D}(A)$. Then obviously $x \cap y \in \mathcal{D}(A)$. For showing that $x \cup y \in \mathcal{D}(A)$ suppose $q{\cdot}v_1 \in x$ and $q{\cdot}v_2 \in y$. As $x \sqsubseteq y^\top$ we get $v_2 = v_1$ or $v_2 = \top$. In case $v_2 = \top$ it follows from $y \sqsubseteq x^\top$ that $v_1 = v_2$ or $v_1 = \top$, hence $v_1 = \top = v_2$.

ad (1) : Let $z \in \mathcal{D}(A)$ with $x, y \sqsubseteq z$ and $r \in x \cup y$. Thus $r \in x$ or $r \in y$ and it follows that there exists a $s \in z$ with $r \sqsubseteq s$ as desired.

ad (2) : Let $z \in \mathcal{D}(A)$ with $z \sqsubseteq x, y$ and $r \in z$. Thus there exist $s \in x$ and $s' \in y$ with $r \sqsubseteq s, s'$. We proceed by case analysis on r. If r is a prefix of s and s' then $r \in x \cap y$. In case that r is not a prefix of s we have $s = q{\cdot}\top$ and q is a prefix of r. If r is a prefix of s' then q is a prefix of s' and it follows from Lemma 3.1.8 that $s = s'$. If r is not a prefix of s' then $s' = q'{\cdot}\top$ and q is a prefix of q' or vice versa. W.l.o.g. suppose that q is a prefix of q'. As $q{\cdot}\top = s \in x$, $q'{\cdot}\top = s' \in y$ and q is a prefix of q' it follows from Lemma 3.1.8 that $s = s'$ holds.

Thus for each $r \in z$ there exists an $s \in x \cap y$ with $r \sqsubseteq s$ and it follows that $x \cap y = x \sqcap y$. ☐

Thus we have shown that $(\mathcal{D}(A), \sqsubseteq, \neg)$ is a locally boolean order. Next we show that $(\mathcal{D}(A), \sqsubseteq, \neg)$ is directed complete.

First we have the following characterisation of the stable order of $\mathcal{D}(A)$.

Lemma 3.1.10. *Let A be a* CL*-game and $x, y \in \mathcal{D}(A)$. Then $x \leq_s y$ iff $x \subseteq y$.*

Proof. The forward implication is an immediate consequence of Lemma 3.1.9. For the reverse implication suppose $x \subseteq y$. Thus, we have $x \sqsubseteq y$.

For showing that $y \sqsubseteq x^\top$ suppose $r \in y$. Then $r \in x$ or $r \notin x$. Notice that by Lemma 3.1.7 we have $x^\top = x \cup \{q{\cdot}\top \mid q \in \mathsf{Acc}(x)\}$. Hence, if $r \in x$ then $r \in x^\top$. Hence we assume that $r \notin x$.

Suppose there exists a $s \in x$ with $r \sqsubseteq s$. If r is a prefix of s then it follows from Def. 3.1.1(1) that $r \in x$ in contradiction with $r \notin x$. If $s = q{\cdot}\top$ and q is a prefix of r then as $x \subseteq y$ we get $s \in y$, thus it follows from Def. 3.1.1(2) that $s = r$ and hence $r \in x$ in contradiction with $r \notin x$.

Thus we have shown that $\neg\exists s{\in}x.\, r \sqsubseteq s$ holds. Hence there exists a maximal prefix t of r with $t \in x$ (t might be ε). As $t \in \mathsf{Rsp}_A$ and t is a proper prefix of r there exists a cell c such that $t{\cdot}c$ is a prefix of r and $t{\cdot}c \in \mathsf{Acc}(x)$. Thus, $t{\cdot}c{\cdot}\top \in x^\top$. As $r \sqsubseteq t{\cdot}c{\cdot}\top$ we get $y \sqsubseteq x^\top$ as desired. □

Next we show that $(\mathcal{D}(A), \sqsubseteq, \neg)$ is directed complete.

Lemma 3.1.11. *Let A be a* CL*-game and $X \subseteq \mathcal{D}(A)$ a directed subset then $\bigsqcup X$ exists and is given by*

$$S_X := \{r \in \bigcup X \mid \neg(\exists y{\in}X, q{\in}\mathsf{Que}_A.\, q{\cdot}\top{\in}y \;\wedge\; r \sqsubset q{\cdot}\top \;\wedge\; q \textit{ prefix of } r)\}$$

Proof. Suppose $X \subseteq \mathcal{D}(A)$ is directed. First we show that S_X is an element of $\mathcal{D}(A)$.

Suppose $r \in S_X$ and r' is an even length prefix of r. If there was a $q{\cdot}\top \in y \in X$ with $r' \sqsubset q{\cdot}\top$ and q' a prefix of r' then we get $r \sqsubset q{\cdot}\top$ and q a prefix of r in contradiction with $r \in S_X$.

Suppose $q{\cdot}v_1, q{\cdot}v_2 \in S_X$ and $v_1 \neq v_2$. As X is directed there exists a $q'{\cdot}\top \in y \in X$ with $q{\cdot}v_1, q{\cdot}v_2 \sqsubseteq q'{\cdot}\top$ and q' is a prefix of q. Thus as $v_1 \neq \top$ or $v_2 \neq \top$ it follows that $q{\cdot}v_1 \notin S_X$ or $q{\cdot}v_2 \notin S_X$.

Next we show that S_X is the supremum of X.

First notice the following fact. Let $x \in X$ and $r \in x$. Then $r \in S_X$ or there exists a $y \in X$ with $q{\cdot}\top \in y$ and q is a prefix of r. Iterating this argument it follows that $r \in S_X$ or there exists a $y \in X$ and $q{\cdot}\top \in y$ with $q{\cdot}\top \in S_X$ and q is a prefix of r. (Since r has only finitely many prefixes we eventually get such a $y \in X$ and a $q{\cdot}\top \in y$.)

Hence it follows that for all $x \in X$ and $r \in x$ there exists a $s \in S_X$ with $r \sqsubseteq s$. Thus S_X is an upper bound of X.

Finally let $y \in \mathcal{D}(A)$ with $x \sqsubseteq y$ for all $x \in X$. If $r \in S_X$ then $r \in x$ for some $x \in X$, thus there exists a $s \in y$ with $r \sqsubseteq s$. Hence it follows that $S_X \sqsubseteq y$. □

Given an element $r \in \mathsf{Rsp}_A^\top$ then we write $\widehat{r}$ for the set of even length prefixes of r. Obviously, it follows that $\widehat{r} \in \mathsf{Strat}(A)$.

Lemma 3.1.12. *Let A be a* CL*-game. Then $p \in \mathsf{FP}(\mathcal{D}(A))$ iff $p = \widehat{r}$ for a uniquely determined element $r \in \mathsf{Rsp}_A^\top$.*

Proof. Suppose $x \in \mathcal{D}(A)$. If $|x|$ is not finite then as $\widehat{r} \subseteq x$ for all $r \in x$ it follows from Lemma 3.1.10 that $x \notin \mathsf{F}(\mathcal{D}(A))$. On the other hand, if $|x|$ is finite then x has at most finitely many subsets hence it follows from Lemma 3.1.10 that $x \in \mathsf{F}(\mathcal{D}(A))$.

So, suppose $|x|$ is finite and suppose that there does not exist an element $r \in \mathsf{Rsp}_A^\top$ with $\widehat{r} = x$. As x is finite there exists a maximal sequence s in x. Thus it follows that $x \setminus \{s\}$ and $\widehat{s}$ are stably coherent elements of $\mathcal{D}(A)$ with $(x \setminus \{s\}) \sqcup \widehat{s} = x$.

Assuming that $x \sqsubseteq x \setminus \{s\}$ holds. Then as $x \setminus \{s\} \subseteq x$ it follows that $x \setminus \{s\} \leq_s x$, hence $x \setminus \{s\} \sqsubseteq x$, thus we get $x = x \setminus \{s\}$ as contradiction.

Assuming that $x \sqsubseteq \widehat{s}$ holds. Let $r \in x$ and suppose $q{\cdot}\top \in \widehat{s}$ such that q is a prefix of r. Then it follows that $q{\cdot}\top = s$ and from Lemma 3.1.8 it follows that $r = s$. Thus is $x \sqsubseteq \widehat{s}$ then for all $r \in x$ it follows that $r = s$ or r is a prefix of s. Hence it follows that $x = \widehat{s'}$ for some prefix s' of s in contradiction with the assumption that x is not of the form $\widehat{r}$ for some $r \in \mathsf{Rsp}_A^\top$.

Thus it follows that x is not prime.

Now, suppose $p = \widehat{r}$ for some $r \in \mathsf{Rsp}_A^\top$. Let $x, y \in \mathcal{D}(A)$ with $x \uparrow y$ or $x \downarrow y$ and $p \sqsubseteq x \sqcup y$.

In case of $x \uparrow y$ there exists an $s \in x \sqcup y = x \cup y$ with $r \sqsubseteq s$. Hence, $p \sqsubseteq x$ or $p \sqsubseteq y$.

In case of $x \downarrow y$ then $x \sqcup y = \neg(\neg x \sqcap \neg y) = \neg(\neg x \cap \neg y)$. Thus there exists an $s \in \neg(\neg x \cap \neg y)$ with $r \sqsubseteq s$. If $s = q{\cdot}v$ with $v \neq \top$ then $s \in \neg x \cap \neg y$, thus $s \in \neg x, \neg y$, thus $s \in x, y$ and it follows that $p \sqsubseteq x$ or $p \sqsubseteq y$.

If $s = r{\cdot}c{\cdot}\top$ then $r \in \neg x \cap \neg y$ and $r{\cdot}c{\cdot}\top \notin \neg x \cap \neg y$, thus w.l.o.g. $r{\cdot}c{\cdot}\top \notin \neg x$ and $r \in \neg x$, thus $r{\cdot}c{\cdot}\top \in \neg\neg x = x$ and it follows that $p \sqsubseteq x$. $\square$

Hence we can identify the finite prime elements of $\mathcal{D}(A)$ with the set $\mathsf{Rsp}_A^\top$.

Lemma 3.1.13. *Let A be a CL-game and $x \in \mathsf{FP}(\mathcal{D}(A))$. Then $x = \bigsqcup \mathsf{FP}(x)$.*

Proof. Let $x \in \mathcal{D}(A)$. It is easy to check that $\mathsf{FP}(x) = \{\widehat{r} \mid r \in x\}$. Thus $\bigsqcup \mathsf{FP}(x) = x$. $\square$

Lemma 3.1.14. *Let A be a CL-game and $p \in \mathsf{FP}(\mathcal{D}(A))$. Then p is compact (w.r.t. $\sqsubseteq$).*

Proof. Suppose $p \in \mathsf{FP}(\mathcal{D}(A))$. Then by Lemma 3.1.12 there exists a $r \in \mathsf{Rsp}_A^\top$ with $\widehat{r} = p$. Let $X \subseteq \mathcal{D}(A)$ be directed with $p \sqsubseteq \bigsqcup X$. Thus there exists a $s \in \bigsqcup X$ with $r \sqsubseteq s$. Using Lemma 3.1.11 it follows that there exists a $x \in X$ with $s \in x$ and hence $p \sqsubseteq x$. $\square$

Theorem 3.1.15. *Let A be a CL-game. Then $\mathcal{D}(A)$ is a locally boolean domain.*

Proof. From Lemma 3.1.7, Lemma 3.1.9 and Lemma 3.1.11 it follows that $\mathcal{D}(A)$ is a complete lbo. From Lemma 3.1.13 and Lemma 3.1.14 ensure that $\mathcal{D}(A)$ fulfils the requirements (1) and (2) of Def. 2.2.3. $\square$

3.2 Locally boolean domains as Curien-Lamarche games

In this section we show how to construct a CL-game from a locally boolean domain. For this purpose we divide the set of finite prime elements of a lbd into a set of *cells* and a set of *values*. The set of positions of the CL-game will be derived from the tree structure of the finite prime elements.

Notice that we will write $\prec_s$ to denote the neighbourhood relation induced by the stable order $\leq_s$, i.e. $x \prec_s y$ iff $x \leq_s y$, $x \neq y$ and there does not exist an element z with $x \leq_s z \leq_s y$ and $x \neq z \neq y$.

Definition 3.2.1. *Let A be a lbpd. A* cell *in A is an element $c \in \mathsf{FP}(A)$ with $c \neq c_\bot$. We write $\mathsf{Cell}(A)$ for the set of cells in A.* ◇

Lemma 3.2.2. *Let A be a lbpd, $p \in \mathsf{FP}(A)$ and $a \in A$ with $a \in \mathsf{At}(a)$. If $a_\bot \prec_s p \sqsubseteq a$ then $a \in \mathsf{Cell}(A)$.*

Proof. Let $p \in \mathsf{FP}(A)$, $a \in A$ with $a \in \mathsf{At}(a)$ and $a_\bot \prec_s p \sqsubseteq a$. From Lemma 2.2.40 it follows that there exists a unique $q \in \mathsf{FP}(a)$ with $q \sqcup a_\bot = a$. As $p \sqsubseteq a = q \sqcup a_\bot$ and p is prime we get $p \sqsubseteq q$ since $p \not\sqsubseteq a_\bot$. Thus $a_\bot \sqsubset q$ and as $q \sqcup a_\bot = a$ we get $q = a \neq a_\bot$. □

Lemma 3.2.3. *Let A be a lbpd and $p \in \mathsf{FP}(A)$. If p is not $\leq_s$-minimal then there exists a unique cell $c \in \mathsf{Cell}(A)$ with $c_\bot \prec_s p \sqsubseteq c$. We also write $\mathsf{C}(p)$ for this unique cell c.*

Proof. Let $p \in \mathsf{FP}(A)$ such that p is not $\leq_s$-minimal. Thus, it follows from Thm. 2.2.43 that there exists a unique $d \in \mathsf{FP}(A)$ with $d \prec_s p$. Further, by Lemma 2.2.38 it follows that $d = d_\bot$. As $d_\bot = d \prec_s p$ it follows that $d_\bot \sqsubset p \sqsubseteq d^\top$. Thus $d_\bot \sqsubset d^\top$. Thus $\mathsf{At}(d) \neq \emptyset$. Hence it follows from Lemma 2.2.42 that there exist an atom $a \in \mathsf{At}(d)$ with $p \sqsubseteq a$ and using Lemma 3.2.2 it follows that $a \in \mathsf{Cell}(A)$.

For showing uniqueness of a suppose there exists a cell $a' \in \mathsf{Cell}(A)$ with $a \neq a'$ and $a'_\bot \prec_s p \sqsubseteq a'$. Then it follows from Thm. 2.2.43 that $a'_\bot = a_\bot$. Thus we have $a'_\bot \updownarrow a_\bot$ and as $p \sqsubseteq a, a'$, $a \neq a'$ and $a, a' \in \mathsf{At}(a')$ it follows that $p \sqsubseteq a \sqcap a' = a'_\bot$ in contradiction with $a'_\bot \prec_s p$. □

Definition 3.2.4. *Let A be a lbpd, $c \in \mathsf{Cell}(A)$ and $x \in A$. We say that x* fills *c* with value *v iff $v \in \mathsf{FP}(x)$ and $c_\bot \prec_s v \sqsubseteq c$. We say that x* fills *c iff there exists a $v \in \mathsf{FP}(x)$ with $c_\bot \prec_s v \sqsubseteq c$. We define*

$$\mathsf{Fill}(x) := \{c \in \mathsf{Cell}(A) \mid x \textit{ fills } c\}.$$

We call a cell c accessible from *x iff $c_\bot \leq_s x$ and x does not fill c. We define*

$$\mathsf{Acc}(x) := \{c \in \mathsf{Cell}(A) \mid c \textit{ is accessible from } x\}.$$

◇

Next we collect a few properties of the notion of filling.

Lemma 3.2.5. *Let A be a lbpd, $c \in \mathsf{Cell}(A)$ and $x \in A$ with $c_\bot \in \mathsf{FP}(x)$. Then x does not fill c iff $x \sqsubseteq \neg c$.*

Proof. Due to the assumption $c_\bot \in \mathsf{FP}(x)$ we have $x \sqsubseteq c^\top$. Thus, for every $p \in \mathsf{FP}(x)$ we have $p \sqsubseteq c^\top = c \sqcup \neg c$ and as p is prime that $p \sqsubseteq c$ or $p \sqsubseteq \neg c$.

The statement $x \sqsubseteq \neg c$ is equivalent to $\forall p{\in}\mathsf{FP}(x).\exists q{\in}\mathsf{FP}(\neg c).\, p \leq_c q$ which in turn is equivalent to the negation of $\exists p{\in}\mathsf{FP}(x).\forall q{\in}\mathsf{FP}(\neg c).\, p \not\leq_c q$. We are finished if we can show that for all $p \in \mathsf{FP}(x)$ it holds that

$$p \text{ fills } c \quad \text{iff} \quad \forall q{\in}\mathsf{FP}(\neg c).\, p \not\leq_c q$$

as then $x \sqsubseteq \neg c$ iff $\neg\exists p \in \mathsf{FP}(x).\,(p \text{ fills } c)$, i.e. iff x does not fill c.

Suppose x fills c, i.e. there exists a $p \in \mathsf{FP}(x)$ with $c_\perp \prec_s p \sqsubseteq c$. Suppose $q \in \mathsf{FP}(\neg c)$ with $p \leq_c q$. Then $p \sqsubseteq c$ and $p \sqsubseteq q \sqsubseteq \neg c$ and thus $c_\perp \sqsubseteq p \sqsubseteq c \sqcap \neg c = c_\perp$, i.e. $p = c_\perp$ contradicting the assumption $c_\perp \prec_s p$.

Suppose that

$$\forall q \in \mathsf{FP}(\neg c).\, p \not\leq_c q \tag{†}$$

holds. Then it cannot hold that $p \leq_s c_\perp$ as then $p \in \mathsf{FP}(\neg c)$ which implies $p \not\leq_c p$ by (†). As $p \sqsubseteq x \sqsubseteq c^\top$ it follows that $p \sqsubseteq c$ or $p \sqsubseteq \neg c$.

Next we show that $p \sqsubseteq \neg c$ cannot hold. Suppose $p \sqsubseteq \neg c$. Then as p is finite prime there is a $q \in \mathsf{FP}(\neg c)$ with $p \sqsubseteq q$. By Thm. 2.2.48 we have $p \leq_s q$ or $p \leq_c q$. Thus we have to consider the cases $p \leq_c q$ and $p <_s q$ (since $p = q$ implies $p \leq_c q$).

If $p \leq_c q$ holds then using (†) we get a contradiction since $q \in \mathsf{FP}(\neg c)$.

If $p <_s q$ holds then $p = p_\perp \sqsubseteq q_\perp \sqsubseteq c_\perp$ and, therefore, by Lemma 2.2.24 it follows that $p \leq_s c_\perp$ in contradiction with $c_\perp \prec_s p$.

Thus we have shown $p \not\sqsubseteq \neg c$ and since $p \sqsubseteq c$ or $p \sqsubseteq \neg c$ it follows that $p \sqsubseteq c$, i.e. $p <_s c$ or $p \leq_c c$. If $p <_s c$ then $p_\perp = p \sqsubseteq c_\perp$ and thus $p \leq_s c_\perp$ which we have already seen to be impossible. Thus we have $p \leq_c c$, i.e. $p \sqsubseteq c$ and $c_\perp \sqsubseteq p_\perp$. Thus $c_\perp \leq_s p_\perp \leq_s p$. It follows that $c_\perp \neq p$ as otherwise $p \leq_s c_\perp$ which we have already refuted. Thus we have shown that $c_\perp <_s p \sqsubseteq c$ holds and using Thm. 2.2.43 it follows that there exists a $v \in \mathsf{FP}(p)$ (namely p itself) with $c_\perp \prec_s v \sqsubseteq c$, i.e. p fills c as desired. ☐

Lemma 3.2.6. *Let A be a lbpd, $p \in \mathsf{FP}(A)$ and $c \in \mathsf{Cell}(A)$. Then p fills c iff $c_\perp <_s p \sqsubseteq c$.*

Proof. For the forward implication suppose p fills c, i.e. there exists an element $v \in \mathsf{FP}(p)$ with $c_\perp \prec_s v \sqsubseteq c$. Thus we get that $c_\perp \in \mathsf{FP}(p)$ and it follows from Lemma 3.2.5 that $p \not\sqsubseteq \neg c$. As $c_\perp <_s p$ it follows that $p \sqsubseteq c^\top = c \sqcup \neg c$ and as p is prime we get $p \sqsubseteq c$ or $p \sqsubseteq \neg c$. Thus $p \sqsubseteq c$ since $p \sqsubseteq \neg c$ is impossible.

For the reverse implication suppose $c_\perp <_s p \sqsubseteq c$. As $\mathsf{FP}(p)$ is finite, $c_\perp \in \mathsf{FP}(p)$ and $c_\perp <_s p$ there exists an element $v \in \mathsf{FP}(p)$ with $c_\perp \prec_s v \leq_s p \sqsubseteq c$. Thus p fills c. ☐

Lemma 3.2.7. *Let A be a lbpd and $p, q \in \mathsf{FP}(A)$. If p fills cell c with value v and $p \leq_s q$ then q also fills c with v.*

Proof. Suppose p fills cell c with value v. Then $v \in \mathsf{FP}(p)$. If $p \leq_s q$ then $v \in \mathsf{FP}(p) \subseteq \mathsf{FP}(q)$, thus q fills c with v. ☐

Lemma 3.2.8. *Let A be a lbpd and c a cell in A. If $p_1, p_2 \in \mathsf{FP}(A)$ with $p_1 \uparrow p_2$ that both fill c then $p_1 \sqcap p_2$ also fills c.*

Proof. As $p_1 \uparrow p_2$ their infimum $p_1 \sqcap p_2$ exists and is an infimum also w.r.t. $\leq_s$. As p_1 and p_2 both fill the cell c we have $c_\perp \leq_s p_1 \sqcap p_2 \sqsubseteq c$. It remains to show that $c_\perp \neq p_1 \sqcap p_2$. Suppose $c_\perp = p_1 \sqcap p_2$. Then $c \sqsubseteq c^\top = \neg p_1 \sqcup \neg p_2$. As c is prime we have $c \sqsubseteq \neg p_1$ or $c \sqsubseteq \neg p_2$, i.e. $p_1 \sqsubseteq \neg c$ or $p_2 \sqsubseteq \neg c$. As $p_1, p_2 \sqsubseteq c$ it follows that $p_1 \sqsubseteq c_\perp$ or $p_1 \sqsubseteq c_\perp$. As $c_\perp \sqsubseteq p_1, p_2$ we have $p_1 = c_\perp$ or $p_2 = c_\perp$ in contradiction with the assumption that both p_1 and p_2 fill c. ☐

Lemma 3.2.9. *Let A be a lbpd and c a cell in A. Let $p_1, p_2 \in \mathsf{FP}(A)$ that both fill c and are $\leq_s$-minimal with this this property. Then $p_1 \uparrow p_2$ implies $p_1 = p_2$.*

Proof. Suppose p_1 and p_2 are finite primes that both fill c and are $\leq_s$-minimal with this property. Assume $p_1 \uparrow p_2$. Then by Lemma 3.2.8 the element $p_1 \sqcap p_2$ fills c as well. Thus, as p_1 and p_2 are $\leq_s$-minimal primes filling c it follows that $p_1 = p_1 \sqcap p_2 = p_2$. □

As an immediate consequence of the above lemmas we get:

Corollary 3.2.10. *Let A be a lbpd, $c \in \mathsf{Cell}(A)$ and $x \in A$. Then x fills c with at most one value.*

Corollary 3.2.11. *Let A be a lbpd, $c \in \mathsf{Cell}(A)$ and $x, y \in A$ with $x \uparrow y$. If x fills c with value v and y fills c with value v' then $v = v'$.*

Further for elements x and y that belong to the same connected component w.r.t. $\leq_s$, i.e. $\mathsf{FP}(x) \cap \mathsf{FP}(y) \neq \emptyset$, we get the following characterisation of stable coherence.

Lemma 3.2.12. *Let A be a lbpd and $x, y \in A$ with $\mathsf{FP}(x) \cap \mathsf{FP}(y) \neq \emptyset$. Then $x \uparrow y$ iff each cell c filled by x and y is filled by both with the same value.*

Proof. Suppose $x, y \in A$ with $\mathsf{FP}(x) \cap \mathsf{FP}(y) \neq \emptyset$.

The forward implication is given in Cor. 3.2.11.

For the reverse implication suppose that each cell c filled by x and y is filled by both with the same value. Let $p \in \mathsf{FP}(x)$. We show that $p \sqsubseteq q^\top$.

If p is $\leq_s$-minimal then as $\mathsf{FP}(x) \cap \mathsf{FP}(y) \neq \emptyset$ it follows that $p \in \mathsf{FP}(y)$ and thus $p \in \mathsf{FP}(y^\top)$. Hence we can assume that p is not $\leq_s$-minimal. Thus by Lemma 3.2.2 there exists a unique cell c with $c_\perp \prec_s p \sqsubseteq c$, thus x fills c with p. We proceed by case analysis:

Suppose y fills c. Then y fills c with p and it follows that $p \in \mathsf{FP}(y)$, thus $p \in \mathsf{FP}(y^\top)$.

Suppose y does not fill c. As p is finite it follows using Thm. 2.2.43 that there exists a $q \in \mathsf{FP}(y)$ that is $\leq_s$-maximal with $q \leq_s p$. As $q <_s p$ there is a $v \in \mathsf{FP}(x)$ with $q \prec_s v$, i.e. there is a cell c' filled by x with v and $c'_\perp = q$. As $v \notin \mathsf{FP}(y)$ it follows that c' is not filled in y. Using Lemma 3.2.5 we get $y \sqsubseteq \neg c'$. Thus, $c' \sqsubseteq \neg y \sqsubseteq y^\top$. As p fills c' it follows that $p \sqsubseteq c'$, thus $p \sqsubseteq y^\top$.

Thus we have shown that $\forall p \in \mathsf{FP}(x).\, p \sqsubseteq y^\top$. Hence we get $x \sqsubseteq y^\top$.

Analogously, one can show that $y \sqsubseteq x^\top$ holds, thus we get $x \uparrow y$ as desired. □

Lemma 3.2.13. *Let A be a lbpd, $x \in A$, $c \in \mathsf{Cell}(A)$ and x fills c with value v. Then either $v = v_\perp$ or $v = c$.*

Proof. Suppose x fills c with v. Then $c_\perp \prec_s v \sqsubseteq c$. Assuming $v \neq c$ it follows that $v \notin [c]_\updownarrow$ since c is an atom. Thus $c_\perp \neq v_\perp$ and as $c_\perp <_s v$ it follows that $c_\perp <_s v_\perp \leq_s v$ and hence $v_\perp = v$ since $c_\perp \prec_s v$. □

Lemma 3.2.14. *Let A be a lbpd, $c \in \mathsf{Cell}(A)$ and $x, y \in A$ with $x \sqsubseteq y$ and $c_\perp \in \mathsf{FP}(x) \cap \mathsf{FP}(y)$. If x fills c with value v then y fills c with v or c.*

Proof. Suppose $c \in \mathsf{Cell}(A)$, $x, y \in A$ with $x \sqsubseteq y$, $c_\bot \in \mathsf{FP}(x) \cap \mathsf{FP}(y)$ and x fills c with value v. Then by Lemma 3.2.5 it follows that $x \not\sqsubseteq \neg c$, thus $y \not\sqsubseteq \neg c$ and using Lemma 3.2.5 again we get that y fills c. Thus, there exists a $v' \in \mathsf{FP}(y)$ with $c_\bot \prec_s v' \sqsubseteq c$. If $v' = c$ we are done. Hence we assume that $v' \neq c$. Thus, we have $v' = v'_\bot$ by Lemma 3.2.13 and it follows that $v' \in \mathsf{FP}(y_\bot) \subseteq \mathsf{FP}(\neg y)$. As $x \uparrow \neg y$ it follows that $v \uparrow v'$ and by Lemma 3.2.9 we get $v = v'$ as desired. $\square$

Using the above lemma we get the following characterisation of the extensional order of lbds:

Lemma 3.2.15. *Let A be a lbd and $x, y \in A$. Then $x \sqsubseteq y$ iff*

$$\forall c{\in}\mathsf{Cell}(A).\forall v{\in}\mathsf{FP}(A).\,(x \textit{ fills } c \textit{ with } v \ \rightarrow \\ (y \textit{ fills } c \textit{ with } v \ \ \vee \ \ \exists c'{\in}\mathsf{Cell}(A).\,(v \textit{ fills } c' \ \ \wedge \ \ y \textit{ fills } c' \textit{ with } c')))\,.$$

Proof. Suppose $x, y \in A$.

For the forward implication suppose that $x \sqsubseteq y$, c is a cell and x fills c with v. Thus $v \in \mathsf{FP}(x)$ and it follows from Thm. 2.2.49 that there exists a $q \in \mathsf{FP}(y)$ with $v \leq_c q$. If $v \leq_s q$ then $v \in \mathsf{FP}(y)$ and it follows that y fills c with v. Hence we can assume that $v \not\leq_s q$. Thus $v <_c q$. Thus $q_\bot \sqsubseteq v_\bot$ and $v \sqsubset q$. As $q_\bot \sqsubseteq v_\bot$ it follows that $q_\bot \leq_s v_\bot \prec_s v$. Thus we get $q_\bot <_s v \sqsubseteq q$, i.e. q is a cell filled by v and y fills q with q.

For the reverse implication we show that $\forall p{\in}\mathsf{FP}(x).\exists q{\in}\mathsf{FP}(y).\, p{\sqsubseteq}q$ holds.

Suppose $p \in \mathsf{FP}(x)$ and w.l.o.g. $p \neq \bot$. Then there exists a cell c filled by x with value p and we have to consider two cases:

(1) y fills c with p : Then $p \in \mathsf{FP}(y)$ and we are finished.

(2) $\exists c'{\in}\mathsf{Cell}(A).\,(p \text{ fills } c' \wedge y \text{ fills } c' \text{ with } c')$: Then by Lemma 3.2.6 we get that $p \sqsubseteq c'$. As y fills c' with c' we have $c' \in \mathsf{FP}(y)$. Thus $p \sqsubseteq y$ as desired. $\square$

Next we show how to construct a CL-game from a lbd. We will interpret the elements of a lbd as strategies of the associated CL-game. Notice that for cells c we have $c_\bot \prec_s c \sqsubseteq c$ and we interpret $c \in \mathsf{FP}(x)$ as the strategy x fills the cell c with $\top$. The following notion will be useful:

Definition 3.2.16. *Let A be a lbd and $p \in \mathsf{FP}(A)$ with $p \neq \bot$. We write $\overleftarrow{p}$ for the sequence*

- $c_1 v_1 \dots c_n v_n$ *iff* $p = p_\bot$ *and* $v_n = p$
- $c_1 v_1 \dots c_{n-1} v_{n-1} c_n$ *iff* $p \neq p_\bot$ *and* $c_n = p$

with $c_{1\bot} = \bot$, $c_{i\bot} \prec_s v_i \sqsubseteq c_i$ for all $i \in \{1, \dots, n\}$ and $v_i = c_{i+1\bot}$ for all $i \in \{1, \dots, n{-}1\}$. We call $\overleftarrow{p}$ the position generated by p.
Notice that Lemma 3.2.3 and Thm. 2.2.43 ensure that $\overleftarrow{p}$ is well defined. $\diamond$

Theorem 3.2.17. *Let A be a lbd. Then $\mathcal{G}(A) := (C_A, V_A, P_A)$ with*

- $C_A := \mathsf{Cell}(A)$
- $V_A := \mathsf{FP}(A) \setminus (\{\bot\} \cup \mathsf{Cell}(A))$
- $P_A := \{\overleftarrow{p} \mid p \in \mathsf{FP}(A) \setminus \{\bot\}\}$

is a CL*-game.*

Proof. The set of positions P_A is obviously closed under non-empty prefixes. Thus (C_A, V_A, P_A) is a CL-game. □

Lemma 3.2.18. *Let A be a lbd and $p, q \in \mathsf{FP}(A)$. If $\overleftarrow{p} \wedge \overleftarrow{q} \in \mathsf{Rsp}_{\mathcal{G}(A)}$ then $p \uparrow q$.*

Proof. Suppose $p, q \in \mathsf{FP}(A)$ with $\overleftarrow{p} \wedge \overleftarrow{q} \in \mathsf{Rsp}_{\mathcal{G}(A)}$. W.l.o.g. we assume that $p \neq q$. Thus there exists a response $r \in \mathsf{Rsp}_{\mathcal{G}(A)}$ and $c, d \in \mathsf{Cell}(A)$ with $c \neq d$ such that $r{\cdot}c$ is a prefix of $\overleftarrow{p}$ and $r{\cdot}d$ is a prefix of $\overleftarrow{q}$. It follows from Def. 3.2.16 that $c_\bot = d_\bot$. Now using Lemma 3.2.3 and Thm. 2.2.43 it follows that p does not fill the cell d, and as $d_\bot = c_\bot \in \mathsf{FP}(p)$ it follows from Lemma 3.2.5 that $p \sqsubseteq \neg d$. Thus $d \sqsubseteq \neg p \sqsubseteq p^\top$, and since q fills d using Lemma 3.2.6 we get $q \sqsubseteq d \sqsubseteq \neg p \sqsubseteq p^\top$. Analogously, we can show that $p \sqsubseteq q^\top$ holds. Thus $p \uparrow q$ as desired. □

Lemma 3.2.19. *Let A be a lbd and $x \in A$. Then*

$$S_x := \left\{\overleftarrow{p} \mid p \in \mathsf{FP}(x) \cap V_A\right\} \cup \left\{\overleftarrow{c}{\cdot}\top \mid c \in \mathsf{FP}(x) \cap C_A\right\}$$

is an element of $\mathcal{G}(A)$. Further, $|\mathcal{G}(A)| = \{S_x \mid x \in A\}$.

Proof. Suppose $x \in A$. Then it follows from Thm. 2.2.43 that S_x is closed under response prefixes. Let $r_1, r_2 \in S_x$ with $r_1 \wedge r_2 \neq \varepsilon$. Suppose $r_1 \wedge r_2$ is not a response, i.e. $r_1 \wedge r_2$ is a query $q = r'{\cdot}c$. Thus there exist $v_1, v_2 \in V$ such that $q{\cdot}v_1$ is a prefix of r_1, $q{\cdot}v_2$ is a prefix of r_2 and $v_1 \neq v_2$. As $v_1, v_2 \in \mathsf{FP}(x)$ it follows that $v_1 \uparrow v_2$, and since v_1 and v_2 both fill c and are minimal with this property it follows from Lemma 3.2.9 that $v_1 = v_2$ in contradiction with $v_1 \neq v_2$.

For showing that each strategy of $\mathcal{G}(A)$ is of the form S_x for some $x \in X$ suppose $s \in \mathcal{G}(A)$. Let

$$\mathsf{FP}_s := \{v \in V_A \mid \overleftarrow{v} \in s\} \cup \{c \in C_A \mid \overleftarrow{c}{\cdot}\top \in s\}\,.$$

we will prove that $\uparrow\mathsf{FP}_s$ and $S_{\bigsqcup \mathsf{FP}_s} = s$. Suppose $p, q \in \mathsf{FP}_s$ and w.l.o.g. we assume $p \neq p'$. As s is a strategy it follows that $\overleftarrow{p} \wedge \overleftarrow{q} \in \mathsf{Rsp}_{\mathcal{G}(A)}$. Hence, it follows from Lemma 3.2.18 that $p \uparrow q$. For showing that $S_{\bigsqcup \mathsf{FP}_s} = s$ suppose $r \in s$. Thus there exists a $p \in \mathsf{FP}(A)$ with $r = \overleftarrow{p}$ or $r = \overleftarrow{p}{\cdot}\top$. In either case it follows that $p \in \mathsf{FP}_s$. Thus $p \in \mathsf{FP}(\bigsqcup \mathsf{FP}_s)$, thus $r \in S_{\bigsqcup \mathsf{FP}_s}$. For the reverse inclusion suppose $r \in S_{\bigsqcup \mathsf{FP}_s}$. Hence, there exists a $p \in \mathsf{FP}(\bigsqcup \mathsf{FP}_s)$ with $r = \overleftarrow{p}$ or $r = \overleftarrow{p}{\cdot}\top$. Thus, there exists a $p' \in \mathsf{FP}_s$ with $p \leq_s p'$ and as s is closed under response prefixes it follows that $p \in \mathsf{FP}_s$. Hence, we have $r \in s$ as desired. □

3.3 Observable sequentiality vs. bistability

In this section we show that the notion of bistable maps between lbpds coincides with the notion of observably sequential maps between CL-games as introduced in [CCF94, Cur05]. For this purpose we show that bistable maps are exactly those maps that are sequential in the sense of Milner-Vuillemin (cf. [Mil77, Vui74, KP93]) and error propagating.

Definition 3.3.1. *Let A and B be lbpds and $f : A \to B$ a function that is continuous w.r.t. $\leq_s$. If c' is a cell accessible from $f(x)$ and there exists a $y \in A$ with $x \leq_s y$ and $f(y)$ fills c' then a* sequentiality index for f at (x, c') *is a cell c accessible from x such that for every y with $x \leq_s y$ if $f(y)$ fills c' then y fills c.* $\diamond$

Definition 3.3.2. *Let A and B be lbpds. A function $f : A \to B$ is called* sequential in the sense of Milner-Vuillemin *(or simply* MV-sequential*), iff f is continuous w.r.t. $\leq_s$ and whenever $x \leq_s y$, $c' \in \mathsf{Acc}(f(x))$ and $f(y)$ fills c' then there exists a unique sequentiality index for f at (x, c').*

We say that a MV*-sequential function f is* error propagating *iff for any sequentiality index q at (x, q'), if q' is filled by $f(y)$ for some y with $x <_s y$ then $f(x \sqcup q)$ fills q' with q'.*

We further say that f is observably sequential *if f is* MV*-sequential and error propagating.* $\diamond$

In the following sequence of lemmas we show that a bistable map between locally boolean predomains is also observably sequential.

Lemma 3.3.3. *Let A and B be lbpds and $f : A \to B$ a bistable map. Suppose $x, y \in A$ with $x \leq_s y$ and $c' \in \mathsf{Acc}(f(x))$ such that $f(y)$ fills c'. Then there exists a unique $a \in \mathsf{At}(x)$ with $f(a) \not\sqsubseteq \neg c'$ and for this unique a it holds that $f(\neg a) \sqsubseteq \neg c'$.*

Proof. Suppose $x, y \in A$ with $x \leq_s y$ and $c' \in \mathsf{Acc}(f(x))$ such that $f(y)$ fills c. As $c'_\perp \leq_s f(x) \leq_s f(y)$ and $f(y)$ fills c' it follows from Lemma 3.2.5 that $f(y) \not\sqsubseteq \neg c'$. As $x \leq_s y$ we get $y \sqsubseteq x^\top$, thus $f(x^\top) \not\sqsubseteq \neg c'$ (as otherwise $f(y) \sqsubseteq \neg c'$). As $f(x^\top) = f(\bigsqcup \mathsf{At}(x)) = \bigsqcup_{a \in \mathsf{At}(x)} f(a)$ (because $\updownarrow\mathsf{At}(x)$ and f is bistable) there exists an atom $a \in \mathsf{At}(x)$ with $f(a) \not\sqsubseteq \neg c'$.

Suppose there exists an atom $a' \in \mathsf{At}(x)$ with $a' \neq a$ and $f(a') \not\sqsubseteq \neg c'$. As c' is prime, $f(a) \updownarrow f(a)$ and $f(a), f(a') \not\sqsubseteq \neg c'$ it follows from Lemma 2.2.2 that $f(a) \sqcap f(a') \not\sqsubseteq \neg c'$. Thus $f(x_\perp) = f(a \sqcap a') = f(a) \sqcap f(a') \not\sqsubseteq \neg c'$ and it follows that $f(x) \not\sqsubseteq \neg c'$. By Lemma 3.2.5 it follows that $f(x)$ fills c' in contradiction with $c' \in \mathsf{Acc}(f(x))$.

Since $c' \in \mathsf{Acc}(f(x))$ it follows from Lemma 3.2.5 that $f(x) \sqsubseteq \neg c'$. Thus we have $f(a) \sqcap f(\neg a) = f(a \sqcap \neg a) = f(x_\perp) \sqsubseteq f(x) \sqsubseteq \neg c'$ and as c' is prime it follows that $f(a) \sqsubseteq \neg c'$ or $f(\neg a) \sqsubseteq \neg c'$. As $f(a) \sqsubseteq \neg c'$ is impossible it follows that $f(\neg a) \sqsubseteq \neg c'$ as desired. $\square$

Lemma 3.3.4. *Let A and B be lbpds. If $f : A \to B$ is bistable then f is observably sequential.*

Proof. Suppose $f : A \to B$ is bistable.

First we show that f is continuous w.r.t. $\leq_s$. Let $X \subseteq A$ be directed w.r.t. $\leq_s$. Then $X \subseteq A$ be directed w.r.t. $\sqsubseteq$ and as f is continuous it follows that $f(\bigsqcup X) = \bigsqcup f[X]$. As f is monotone w.r.t. $\leq_s$ it follows that $f[X]$ is directed w.r.t. $\leq_s$. Thus it follows from Lemma 2.2.13 that $\bigsqcup f[X]$ is also the supremum of $f[X]$ w.r.t. $\leq_s$.

Next we show that f is MV-sequential. Suppose $x, y \in A$ with $x \leq_s y$ and $c' \in \mathsf{Acc}(f(x))$ such that $f(y)$ fills c'. Then by Lemma 3.3.3 there exists a unique atom $a \in \mathsf{At}(x)$ with

$$f(a) \not\sqsubseteq \neg c' \tag{$\dagger$}$$

and for this unique a it holds that

$$f(\neg a) \sqsubseteq \neg c' \tag{$\ddagger$}$$

From Lemma 2.2.40 it follows that there exists a unique $c \in \mathsf{FP}(a)$ with $a = x_\perp \sqcup c$ and for this c it holds that $c \neq c_\perp$. Thus c is a cell. We show that c is a sequentiality index for f at (x, c').

First notice that for all $z \in A$ with $x_\perp \leq_s z$ it holds that $z \sqsubseteq x^\top$, thus

$$\begin{aligned} z \sqsubseteq \neg a \quad &\Leftrightarrow \quad z \sqsubseteq \neg(x_\perp \sqcup c) \\ &\Leftrightarrow \quad z \sqsubseteq x^\top \sqcap \neg c \\ &\Leftrightarrow \quad z \sqsubseteq x^\top \quad \text{and} \quad z \sqsubseteq \neg c \\ &\Leftrightarrow \quad z \sqsubseteq \neg c \end{aligned} \tag{§}$$

Suppose $x \not\sqsubseteq \neg c$. Then by (§) we get $x \not\sqsubseteq \neg a$. Thus as $a \in \mathsf{At}(x)$ it follows that $a \leq_b x$. As $f(a) \not\sqsubseteq \neg c'$ it follows that $f(x) \not\sqsubseteq \neg c'$. Thus by Lemma 3.2.5 it follows that $f(x)$ fills c' in contradiction with $c' \in \mathsf{Acc}(f(x))$. Hence we have $x \sqsubseteq \neg c$, and as $c \leq_s a$ we have $c_\perp \leq_s a_\perp = x_\perp \leq_s x$. Thus from Lemma 3.2.5 it follows that x does not fill c. Hence as $c_\perp \leq_s x$ we get $c \in \mathsf{Acc}(x)$.

Let $z \in A$ with $x \leq_s z$ and suppose $f(z)$ fills c'. Then from Lemma 3.2.5 we get $f(z) \not\sqsubseteq \neg c'$. As $f(\neg a) \sqsubseteq \neg c'$ it follows that $z \not\sqsubseteq \neg a$. Thus by (§) it follows that $z \not\sqsubseteq \neg c$, and as $c_\perp \leq_s x \leq_s y$ it follows from Lemma 3.2.5 that z fills c. Thus we have shown that c is a sequentiality index for f at (x, c').

For showing uniqueness of this sequentiality index suppose there exists a sequentiality index d for f at (x, c') with $d \neq c$. As $c'_\perp \leq_s f(x)$ it follows that $c'_\perp \leq_s f(x)_\perp = f(a)_\perp \leq_s f(a)$ and as $f(a) \not\sqsubseteq \neg c'$ it follows from Lemma 3.2.5 that $f(a)$ fills c'. Thus a fills d with some value v.

Suppose $v = v_\perp$. Then as $v \leq_s a$ (since a fills d with v) it follows that $v_\perp \leq_s a_\perp$. Thus $v = v_\perp \leq_s a_\perp = x_\perp \leq_s x$. Thus x fills d in contradiction to $d \in \mathsf{Acc}(x)$. Thus we have $v \neq v_\perp$ and it follows from Lemma 3.2.13 that $v = d$. Thus $d \in \mathsf{FP}(a)$. As $x_\perp \uparrow d$ it follows from Lemma 2.2.41 that $x_\perp \sqcup d \in \mathsf{At}(x)$. Thus as $x_\perp, d \leq_s a$ we get $x_\perp \sqcup d = a$. From Lemma 2.2.40 it follows that d is the unique element in $\mathsf{FP}(a)$ with $d \neq d_\perp$ and $a = d \sqcup x_\perp$ in contradiction to $c \in \mathsf{FP}(a)$, $c \neq c_\perp$, $a = c \sqcup x_\perp$ and $c \neq d$.

Finally, for showing that f is error propagating notice that we have already shown that $f(a)$ fills c'. Thus there exists a $v' \in \mathsf{FP}(f(a))$ with $c'_\perp \prec_s v' \sqsubseteq c'$. As $a \updownarrow x$ and f is bistable it follows that $f(a) \updownarrow f(x)$ thus $f(a)_\perp = f(x)_\perp$. Suppose $v' = v'_\perp$ then $v' = v'_\perp \leq_s f(a)_\perp = f(x)_\perp \leq_s f(x)$, thus $v' \in \mathsf{FP}(f(x))$ in contradiction to $c' \in \mathsf{Acc}(f(x))$. Thus $v' \neq v'_\perp$ and it follows that $v' = c'$. As $c' \in \mathsf{FP}(f(a))$ and $a = x_\perp \sqcup c \leq_s x \sqcup c$ it follows that $c' \in \mathsf{FP}(f(x \sqcup c))$. Thus $f(x \sqcup c)$ fills c' with c'. $\square$

In the next lemma we show that an observably sequential map between locally boolean predomains is also bistable.

Lemma 3.3.5. *Let A and B be lbpds and $f : A \to B$. If f is observably sequential then f is bistable.*

Proof. Suppose f is observably sequential.

f is monotonic w.r.t. $\sqsubseteq$: Suppose $x \sqsubseteq y$ and $p \in \mathsf{FP}(f(x))$. We show that $p \sqsubseteq f(y)$ holds.

As f is continuous w.r.t. $\leq_s$ there exists an $e \in \mathsf{F}(x)$ with $p \leq_s f(e)$. Since $x \sqsubseteq y$ by Thm. 2.2.50 it follows that there exists a $d \in \mathsf{F}(y)$ with $e \leq_c d$. Let $s := \bigsqcup(\mathsf{FP}(e) \cap \mathsf{FP}(d))$. (Since $e \sqsubseteq d$ it follows from Lemma 2.2.32 that $\mathsf{FP}(e) \cap \mathsf{FP}(d) \neq \emptyset$, thus s exists.)

If $p \sqsubseteq f(s)$ then $p \sqsubseteq f(s) \leq_s f(d) \leq_s f(y)$.

Hence we can assume that $p \not\sqsubseteq f(s)$. Then $p \notin \mathsf{FP}(f(s))$. Let q be the greatest element w.r.t. $\leq_s$ of $\{q \in \mathsf{FP}(p) \mid q \leq_s f(s)\}$ (q exists because $\mathsf{FP}(p)$ is finite and by Thm. 2.2.43 linearly ordered w.r.t. $\leq_s$.) As $q <_s p$ we have $q_\perp = q$. Then there exists a unique cell $c' \in \mathsf{Cell}(B)$ such that $c'_\perp = q$ and c' is filled by p and $f(e)$ with some value w. As $c'_\perp = q \leq_s f(s) \leq_s f(e)$ it follows from Cor. 3.2.11 that if $f(s)$ fills c' then $f(s)$ fills c' with w which is impossible since otherwise $q = c'_\perp <_s w \leq_s p, f(s)$ in contradiction to maximality of q. Thus $c' \in \mathsf{Acc}(f(s))$. Since f is MV-sequential there exists a unique sequentiality index c for f at (s, c'). As $f(e)$ fills c' and $s \leq_s e$ it follows that e fills c with some value v. As $c_\perp \leq_s s \leq_s d$ it follows from Lemma 3.2.14 that d fills c with v or c. If d fills c with v then since both e and d fill c with v and $s = \bigsqcup(\mathsf{FP}(e) \cap \mathsf{FP}(d))$ it follows that s fills c with v in contradiction to $c \in \mathsf{Acc}(s)$. Thus d fills c with c from which it follows that $c' \leq_s f(c) \leq_s f(d)$ as f is error propagating. As p fills c' it follows from Lemma 3.2.6 that $p \sqsubseteq c'$. Thus $p \sqsubseteq c' \leq_s f(c) \leq_s f(d) \leq_s f(y)$.

f is continuous w.r.t. $\sqsubseteq$: Let $X \subseteq A$ be directed w.r.t. $\sqsubseteq$. As we already know that f is monotone w.r.t. $\sqsubseteq$ it follows that $f(X)$ is directed, $\bigsqcup f(X)$ exists and $\bigsqcup f(X) \sqsubseteq f(\bigsqcup X)$. For showing the reverse inequality let $p \in \mathsf{FP}(f(\bigsqcup X))$. As f is continuous w.r.t. $\leq_s$ there exists an element $e \in \mathsf{F}(\bigsqcup X)$ with $p \leq_s f(e)$. As e is compact there exists a $x \in X$ with $e \sqsubseteq x$, and as f is monotone w.r.t. $\sqsubseteq$ we get $f(e) \sqsubseteq f(x)$. Thus, we have that $p \leq_s f(e) \sqsubseteq f(x) \sqsubseteq \bigsqcup f(X)$.

f preserves bistable coherence: By Lemma 2.3.2 it suffices to show that $f(x_\perp)_\perp = f(x)_\perp$ for all $x \in A$. Suppose $x \in A$. As f is monotone w.r.t. $\leq_s$ it follows that $f(x_\perp) \leq_s f(x)$, thus $f(x_\perp)_\perp \leq_s f(x)_\perp$. For showing the reverse inequality suppose $p \in \mathsf{FP}(f(x)_\perp)$. Then we have $p = p_\perp$. If p is $\leq_s$-minimal we get $p \leq_s f(x_\perp)_\perp$ since

$p \uparrow f(x_\perp)_\perp$. Hence we can assume that p is not $\leq_s$-minimal. Thus, there exists a unique cell c with $c_\perp \prec_s p \sqsubseteq c$, i.e. p fills c with value p.

We show that $f(x_\perp)_\perp$ fills c'. Suppose $f(x_\perp)_\perp$ does not fill c. Hence $p \notin \mathsf{FP}(f(x_\perp)_\perp)$. Let q be the greatest element w.r.t. $\leq_s$ of $\{q \in \mathsf{FP}(p) \mid q \leq_s f(x_\perp)_\perp\}$ (q exists because $p \uparrow f(x_\perp)_\perp$ and $\mathsf{FP}(p)$ is finite and by Thm. 2.2.43 linearly ordered w.r.t. $\leq_s$.) As $q \leq_s f(x_\perp)_\perp$ we get $q_\perp <_s q$ and as $q <_s p$ it follows that there exists a cell c' with $c'_\perp = q$ that is filled by p value v'. By maximality of q it follows that $v' \notin \mathsf{FP}(f(x_\perp)_\perp)$. Thus as $p \uparrow f(x_\perp)_\perp$ it follows from Cor. 3.2.11 that $f(x_\perp)_\perp$ does not fill c'. As $c'_\perp = q \in \mathsf{FP}(f(x_\perp)_\perp)$ we get $c' \in \mathsf{Acc}(f(x_\perp)_\perp)$.

Suppose that $f(x_\perp)$ fills c' with value w'. If $w' = w'_\perp$ then from $w' = w'_\perp \leq_s f(x_\perp)$ we get $w' \leq_s f(x_\perp)_\perp$ which is impossible since $f(x_\perp)_\perp$ does not fill c'. Assuming $w' \neq w'_\perp$ then $w' = c'$ and as $w' = c' \leq_s f(x_\perp)$ and f is monotone w.r.t. $\leq_s$ it follows that $c' \in \mathsf{FP}(f(x))$. As $p_\perp = p \in \mathsf{FP}(f(x))$ fills c' with value $v' = v'_\perp$ it follows that $f(x)$ fills c' with $v' \neq c'$ and we have a contradiction.

Thus as $c'_\perp \leq_s f(x_\perp)_\perp \leq_s f(x_\perp)$ and $f(x_\perp)$ does not fill c' we have $c' \in \mathsf{Acc}(f(x_\perp))$. As f is observably sequential and p fills c' and $p \leq_s f(x)_\perp$ it follows that there exists a unique sequentiality index r for f at $(x_\perp, c')$.

As $f(x)_\perp$ and therefore also $f(x)$ fills c' it follows that x fills r with some value u. Assuming $u = u_\perp$ it follows from $u = u_\perp \leq_s x$ that $u \leq_s x_\perp$. Hence $x_\perp$ fills r in contradiction with $r \in \mathsf{Acc}(x_\perp)$. Thus $u_\perp \neq u = r$ and it follows that $r \in \mathsf{FP}(x)$. As f is error propagating and $x_\perp \leq_s x$ and $f(x)$ fills c' it follows that $f(x_\perp \sqcup r)$ fills c' with c'. Thus $c' \in \mathsf{FP}(f(x_\perp \sqcup r))$ and as $x_\perp, r \leq_s x$, thus $x_\perp \sqcup r \leq_s x$, and f is monotone w.r.t. $\leq_s$ it follows that $c' \in \mathsf{FP}(f(x))$. As $p = p_\perp \in \mathsf{FP}(f(x))$ fills c' with some value $v' = v'_\perp$ we get a contradiction and it follows that $f(x_\perp)_\perp$ fills c'.

Thus $f(x_\perp)_\perp$ fills c with some value v. As $f(x_\perp)_\perp \uparrow f(x)_\perp$ by Cor. 3.2.11 we get $v = p$. Thus we have $p \in \mathsf{FP}(f(x_\perp)_\perp)$.

f preserves bistably coherent infima and suprema: Suppose $x, y \in A$ with $x \updownarrow y$. As we already know that f preserves bistable coherence it follows that $f(x) \updownarrow f(y)$. Hence $f(x) \sqcap f(y)$ and $f(x) \sqcup f(y)$ exist, and as f is monotonic w.r.t. $\leq_s$ we have $f(x \sqcap y) \leq_s f(x) \sqcap f(y)$ and $f(x) \sqcup f(y) \leq_s f(x \sqcup y)$.

Suppose $f(x \sqcap y) <_s f(x) \sqcap f(y)$ then there exists a cell c' that is accessible from $f(x \sqcap y)$ and filled by $f(x) \sqcap f(y)$. Let c be the sequentiality index at $(x \sqcap y, c')$. Then $c \in \mathsf{Acc}(x \sqcap y)$ and there exist v and w such that c is filled by x with v and by y with w. As $x \uparrow y$ it follows from Cor. 3.2.11 that $v = w$. Thus using Lemma 2.2.21 we get $v \in \mathsf{FP}(x \sqcap y)$ in contradiction with $c \in \mathsf{Acc}(x \sqcap y)$.

For showing that $f(x \sqcup y) \sqsubseteq f(x) \sqcup f(y)$ suppose $p \in \mathsf{FP}(f(x \sqcup y))$. We have either (1) $p = p_\perp$ or (2) $p \neq p_\perp$. In case (1) we get $p = p_\perp \leq_s f(x \sqcup y)_\perp = f(x)_\perp \leq_s f(x) \leq_s f(x) \sqcup f(y)$ since f is monotone w.r.t. $\leq_s$ and preserves bistable coherence.

In case (2) first notice that:

(‡) If c is a cell filled by x but not filled by $x_\perp$ then x fills c with value c (since for all finite prime elements p with $p = p_\perp$ we have $p \in \mathsf{FP}(x)$ iff $p \in \mathsf{FP}(x_\perp)$) and $c \in \mathsf{Acc}(x_\perp)$ (since $c_\perp \leq_s x$ and hence $c_\perp \leq_s x_\perp$).

We have $p \neq p_\perp$, hence p is a cell.

Let us assume that $f(x \sqcup y)_\perp$ fills p. Then as $f(x \sqcup y)_\perp \uparrow f(x \sqcup y)$ and as $f(x \sqcup y)$ fills p with p it follows from Cor. 3.2.11 that $f(x \sqcup y)_\perp$ fills p with p. Hence $p \in \mathsf{FP}(f(x \sqcup y)_\perp)$ in contradiction with $p \neq p_\perp$. Thus $f(x \sqcup y)_\perp$ does not fill p.

If $f(x)$ fills p then as $f(x)_\perp = f(x \sqcup y)_\perp$ (since f preserves bistable coherence) it follows from (‡) that $f(x)$ fills p with p, and hence $p \leq_s f(x) \sqsubseteq f(x) \sqcup f(y)$.

Thus, suppose $f(x)$ does not fill p. Thus $f(x_\perp)$ does not fill p (since $f(x_\perp) \leq_s f(x)$). As $p_\perp \leq_s f(x \sqcup y)_\perp = f((x \sqcup y)_\perp)_\perp = f(x_\perp)_\perp = f(x)_\perp \leq_s f(x_\perp)$ we have $p \in \mathsf{Acc}(f(x_\perp))$. As $x_\perp \leq_s x \sqcup y$ and $p \in \mathsf{FP}(f(x \sqcup y))$, i.e. $f(x \sqcup y)$ fills p (with p), there exists a sequentiality index c for f at $(x_\perp, p)$. As $(x \sqcup y)_\perp = x_\perp$ and $x_\perp$ does not fill p it follows by (‡) that $x \sqcup y$ fills c with c, i.e. $c \leq_s x \sqcup y$. As c is prime it follows that $c \sqsubseteq x$ or $c \sqsubseteq y$. As $c \in \mathsf{Acc}(x_\perp)$ we have $c_\perp \sqsubseteq x_\perp = y_\perp$ and thus $c \leq_s x$ or $c \leq_s y$.

As f is error propagating $f(x_\perp \sqcup c)$ fills p with p. If $c \leq_s x$ then $x_\perp \sqcup c \leq_s x$ and thus $f(x_\perp \sqcup c) \leq_s f(x)$ from which it follows that $f(x)$ fills p with p contradicting the assumption that $f(x)$ does not fill p. Thus we have $c \leq_s y$. As $x_\perp \sqcup c \leq_s y$ and thus $f(x_\perp \sqcup c) \leq_s f(y)$ it follows that $f(y)$ fills p with p. Thus $p \leq_s f(y) \leq_s f(x) \sqcup f(y)$. □

Using Lemma 3.3.4 and Lemma 3.3.5 we get the following characterisation of bistable maps between locally boolean domains.

Theorem 3.3.6. *Let A and B be lbpds and $f : A \to B$. Then f is bistable iff f is observably sequential.*

3.4 Equivalence of the categories **LBD** and **OSA**

Using the results of the previous sections of this chapter we show that the categories **LBD** and **OSA** are equivalent.

Theorem 3.4.1. *Let A be a lbd. Then $A \cong \mathcal{D}(\mathcal{G}(A))$.*

Proof. We have $\mathcal{D}(\mathcal{G}(A)) = (|\mathcal{G}(A)|, \neg, \sqsubseteq) = (\{S_x \mid x \in A\}, \neg, \sqsubseteq)$ by Lemma 3.2.19. Obviously, the mapping $x \mapsto S_x$ is a bijection from $|A|$ to $\{S_x \mid x \in A\}$.

Suppose $x, y \in A$ with $x \sqsubseteq y$, i.e. $\forall p{\in}\mathsf{FP}(x).\exists q{\in}\mathsf{FP}(y).\, p \sqsubseteq q$, or equivalently (†) $\forall p{\in}\mathsf{FP}(x).\exists q{\in}\mathsf{FP}(y).\, p \leq_s q \vee p \leq_c q$ (by Thm. 2.2.48). Since (†) is equivalent to the fact that $\forall r{\in}S_x.\exists s{\in}S_y.\, r \sqsubseteq s$ we get $x \sqsubseteq y$ iff $S_x \sqsubseteq S_y$.

Suppose $x \in A$. Then

$$S_{\neg x} = \left\{\overleftarrow{p} \mid p \in \mathsf{FP}(\neg x) \cap V_A\right\} \cup \left\{\overleftarrow{c}\cdot\top \mid c \in \mathsf{FP}(\neg x) \cap C_A\right\}$$

and

$$\neg S_x = (S_x \cap \mathsf{Rsp}_{\mathcal{G}(A)}) \cup \{q\cdot\top \mid q \in \mathsf{Acc}(S_x)\}\,.$$

Suppose $r \in \mathsf{Rsp}_{\mathcal{G}(A)}$. Then

$$
\begin{aligned}
r \in \neg S_x \quad &\Leftrightarrow \quad r \in S_x \\
&\Leftrightarrow \quad \exists p{\in}\mathsf{FP}(x){\cap}V_A.\, r = \overleftarrow{p} \\
&\Leftrightarrow \quad \exists p{\in}\mathsf{FP}(x).\, p = p_\perp \wedge r = \overleftarrow{p} \\
&\Leftrightarrow \quad \exists p{\in}\mathsf{FP}(\neg x).\, p = p_\perp \wedge r = \overleftarrow{p} \\
&\Leftrightarrow \quad \exists p{\in}\mathsf{FP}(\neg x){\cap}V_A.\, r = \overleftarrow{p} \\
&\Leftrightarrow \quad r \in S_{\neg x}\,.
\end{aligned}
$$

Suppose $r \in \mathsf{Rsp}_{\mathcal{G}(A)}$, $c \in C_A$ and $q = r{\cdot}c \in \mathsf{Que}_{\mathcal{G}(A)}$. First notice that:

(†) If q is enabled in S_x then either $q = c$ or there exist $q' \in \mathsf{Que}_{\mathcal{G}(A)}$ and $v \in V_A$ such that $q = q'{\cdot}v{\cdot}c$. In the first case we have $c_\perp = \perp$ and in the second case we have $c_\perp = v$, thus in both cases we have $c_\perp \in \mathsf{FP}(x)$. On the other hand, if $c_\perp \in \mathsf{FP}(x)$ then obviously $\overleftarrow{c}$ is enabled in S_x.

Thus, we get

$$
\begin{aligned}
q{\cdot}\top \in \neg S_x \quad &\Leftrightarrow \quad q \in \mathsf{Acc}(S_x) \\
&\Leftrightarrow \quad q \text{ is enabled but not filled in } S_x \\
&\Leftrightarrow \quad q \text{ is enabled in } S_x \text{ and } x \sqsubseteq \neg c && (\ddagger) \\
&\Leftrightarrow \quad q \text{ is enabled in } S_x \text{ and } c \sqsubseteq \neg x \\
&\Leftrightarrow \quad c \in \mathsf{FP}(\neg x) && (\S) \\
&\Leftrightarrow \quad q{\cdot}\top \in S_{\neg x}
\end{aligned}
$$

where (‡) follows by (†) and Lemma 3.2.5, and (§) holds as $\overleftarrow{c} = q$ is enabled in S_x iff $c_\perp \in \mathsf{FP}(x)$ iff $c_\perp \in \mathsf{FP}(\neg x)$, and $c_\perp \leq_s x_\perp$ and $c \sqsubseteq \neg x$ iff $c \leq_s \neg x$.

Thus, we have shown that $S_{\neg x} = \neg S_x$ and it follows that $A \cong \mathcal{D}(\mathcal{G}(A))$. □

Notice that in the category of CL-games and observably sequential maps two objects A and B are isomorphic iff $(\mathsf{Strat}(A), \subseteq) \cong (\mathsf{Strat}(B), \subseteq)$.

Theorem 3.4.2. *Let $A = (C, V, P)$ be a CL-game. Then $A \cong \mathcal{G}(\mathcal{D}(A))$.*

Proof. Suppose $s \in A$. Then $s \in \mathcal{D}(A)$ and it follows from Lemma 3.1.12 that $\mathsf{FP}(s) = s \cap \mathsf{Rsp}_A^\top$. Hence, we get by Lemma 3.2.19 that

$$
S_s = \left\{ \overleftarrow{p} \mid p \in s \cap \mathsf{Rsp}_A \right\} \cup \left\{ \overleftarrow{c}{\cdot}\top \mid c \in s \cap (\mathsf{Que}_A{\times}\{\top\}) \right\}
$$

is a strategy of $\mathcal{G}(\mathcal{D}(A))$. The map $s \mapsto S_s$ is obviously injective and preserves and reflects the subset relation. As the strategies of A are the elements of $\mathcal{D}(A)$ it follows from Lemma 3.2.19 that $s \mapsto S_s$ is surjective. □

Theorem 3.4.3. *The category* **LBD** *of locally boolean domains and bistable maps is equivalent to the category* **OSA** *of Curien-Lamarche games and observably sequential algorithms.*

Proof. We define the functors $\mathcal{D} : \mathbf{OSA} \to \mathbf{LBD}$ and $\mathcal{G} : \mathbf{LBD} \to \mathbf{OSA}$ on objects as given above. If $f \in \mathbf{OSA}(A,B)$ then we put $\mathcal{D}(f)(x) = f(x)$ for all $x \in \mathcal{D}(A)$. If $f \in \mathbf{LBD}(A,B)$ then we put $\mathcal{G}(f)(x) = S_{f(\bigsqcup \mathsf{FP}_x)}$ for all $x \in \mathcal{G}(A)$.

In Thm. 3.4.1 and Thm. 3.4.2 we have show that the object parts of the functors $\mathcal{D}$ and $\mathcal{G}$ are essentially surjective. Further it follows from Thm. 3.3.6 that for all observably sequential domains A and B the induced map $\mathbf{OSA}(A,B) \to \mathbf{LBD}(\mathcal{D}(A), \mathcal{D}(B))$ is a bijection. □

In [McC96] G. McCusker constructed a category of games with sums. In an analogous way we can construct a category **POSA** which is a free coproduct completion of the category **OSA**. Further, J. Laird has shown in [Lai05b] that every lbd is the limit of an ω-chain of prenex normal forms constructed using only products and sums. Hence it follows that the category **POSA** is equivalent to the category **LBPD**.

In [CCF94] it is shown that **OSA** is cartesian closed. Thus we get cartesian closedness of **POSA**, and hence also of **LBD** and **LBPD**, for free.

Theorem 3.4.4. *The categories* **LBD** *and* **LBPD** *are cartesian closed.*

3.5 Exponentials in the categories LBD and OSA

In the final section of this chapter we give a characterisation of the extensional order and the involution of exponentials in the category **LBD**. The extensional order will happen to coincide with the pointwise extensional order for morphisms in **LBD**. Finally we show that the category **LBD** is cpo-enriched w.r.t. the extensional order $\sqsubseteq$ and w.r.t. the stable order $\leq_s$.

Next we present the definition of the exponential of CL-games as introduced in [CCF94] (see also for further details). Notice that given a sequence $s = p_0 \cdot \ldots \cdot p_n$ then we write $s@i$ for p_i.

Definition 3.5.1. *Let $A = (C,V,P)$ be a* CL*-game. A* path sequence *s over A is a sequence over the alphabet $(\mathsf{Que}_A \cup \mathsf{Rsp}_A)$ such that:*

1. *$s \in (\mathsf{Que}_A, \mathsf{Rsp}_A)^*$, and s is non-repetitive in Que_A (which implies that s is also non-repetitive in Rsp_A);*
2. *for all $i \geq 1$ such that $2i+1 \leq |s|$ there exists a $v \in V$ such that $s@(2i+1) = s@(2i) \cdot d$;*
3. *$s@0 \in C$; and*
4. *for all $i \geq 1$ such that $2i \leq |s|$ there exists a j such that $2j+1 < 2i$ and $s@(2i) = s@(2j+1) \cdot c$ for some $c \in C$.*

◇

A path sequence is constructed from tokens that are paths. Thus we can consider a path sequences as a linearisation of a state. In [CCF94] it is shown that the exponential of CL-games is given by the following construction.

Given a path sequence s over A we write $\|s\|$ for the set $\{s@(2i+1) \mid 2i+1 \leq |s|\}$. In [CCF94, Lemma 6.8] it is shown that for any path sequence s over A the set $\|s\|$ is a state of A.

Theorem 3.5.2. *Let $G_1 = (C_1, V_1, P_1)$ and $G_2 = (C_2, V_2, P_2)$ be CL-games. Let $\mathsf{Rsp}_1 = \mathsf{Rsp}_{G_1}$, $\mathsf{Que}_1 = \mathsf{Que}_{G_1}$ and S_1 be the set of path sequences over G_1. Then the exponential $[G_1 \rightarrow G_2]$ is given by $G = (C, V, P)$ where*

$$\begin{aligned}
C &= \mathsf{Rsp}_{G_1} + C_2 \\
V &= \mathsf{Que}_{G_1} + V_2 \\
P &= \{p \in (C, V)^* \mid \pi_1^{\Rightarrow}(p) \in S_1, \pi_2^{\Rightarrow}(p) \in P_2 \\
&\qquad\qquad p@0 \in C_2, \\
&\qquad\qquad \textit{if } p@(i{+}1) \in C_2 \textit{ then } p@i \in V_2, \\
&\qquad\qquad \textit{if } p@(i{+}1) \in \mathsf{Rsp}_1 \textit{ then } p@i \in \mathsf{Que}_1\}
\end{aligned}$$

The corresponding functions $\pi_1^{\Rightarrow}, \pi_2^{\Rightarrow} : P \to P$ are defined as follows:

$$\begin{aligned}
\pi_i^{\Rightarrow}(\varepsilon) &= \varepsilon \\
\pi_i^{\Rightarrow}(p \cdot \top) &= \pi_i^{\Rightarrow}(p) \cdot \top \\
\pi_i^{\Rightarrow}(p \cdot \langle x, i\rangle) &= \pi_i^{\Rightarrow}(p) \cdot x \\
\pi_i^{\Rightarrow}(p \cdot \langle x, j\rangle) &= \pi_i^{\Rightarrow}(p) && \textit{if } i \neq j
\end{aligned}$$

$\pi_i^{\Rightarrow}(\varepsilon) = \varepsilon$, $\pi_i^{\Rightarrow}(p{\cdot}\langle x, i\rangle) = \pi_i^{\Rightarrow}(p){\cdot}x$, $\pi_i^{\Rightarrow}(p{\cdot}\langle x, j\rangle) = \pi_i^{\Rightarrow}(p)$ *if* $i \neq j$, $\pi_i^{\Rightarrow}(p{\cdot}\top) = \pi_i^{\Rightarrow}(p){\cdot}\top$ *for* $x \in C_2 \cup V_2 \cup \mathsf{Rsp}_1 \cup \mathsf{Que}_1$ *and* $i, j \in \{1, 2\}$.

The application of some $f \in \mathbb{D}([G_1 \rightarrow G_2])$ to some $x \in \mathbb{D}(G_1)$ is given by

$$\begin{aligned}
f(x) = &\{\pi_2^{\Rightarrow}(p) {\in} \mathsf{Rsp}_{G_2} \mid \|\pi_1^{\Rightarrow}(p)\| \subseteq x, p {\in} f\} \quad \cup \\
&\{\pi_2^{\Rightarrow}(p){\cdot}\top {\in} \mathsf{Rsp}_{G_2}^{\top} \mid \\
&\qquad \exists q {\in} \mathsf{Que}_{G_1}.(\|\pi_1^{\Rightarrow}(p)\| {\cup} \{q{\cdot}\top\} {\subseteq} x \wedge p{\cdot}\langle q, 1\rangle {\in} f)\}\,.
\end{aligned}$$

Lemma 3.5.3. *Let $f \in \mathbb{D}([G_1 \rightarrow G_2])$ and $x \in \mathbb{D}(G_1)$ then $f(x) = \bigcup\{\widehat{p}(x) \mid p \in f\}$.*

Proof. This follows immediately from the definition of $f(x)$ in Thm. 3.5.2 $\square$

Definition 3.5.4. *Let A, B be lbds and $f, g : A \to B$ be bistable maps. Then we write $f \sqsubseteq_{\mathsf{pw}} g$ iff $\forall x \in A.\, f(x) \sqsubseteq g(x)$. If $f \sqsubseteq_{\mathsf{pw}} g$ then we say that f is* pointwise extensionally below *g.* $\diamond$

Lemma 3.5.5. *Let $A = (C_A, V_A, P_A)$ and $B = (C_B, V_B, P_B)$ be CL-games and $p, q \in \mathsf{FP}([\mathcal{D}(A) \rightarrow \mathcal{D}(B)])$. Then $p \sqsubseteq_{\mathsf{pw}} q$ implies $p \sqsubseteq q$.*

Proof. Suppose $p, q \in \mathsf{FP}([\mathcal{D}(A) \to \mathcal{D}(B)])$. As $[\mathcal{D}(A) \to \mathcal{D}(B)]$ is isomorphic to $\mathcal{D}(A \to B)$ it follows from Thm. 3.2.17 that $\overleftarrow{p}$ and $\overleftarrow{q}$ are positions of the CL-game $\mathcal{G}(\mathcal{D}(A \to B))$ which is isomorphic to the CL-game $A \to B$. Thus we have $\overleftarrow{p} = c_1 v_1 \cdots c_n v_n$ and $\overleftarrow{q} = c'_1 v'_1 \cdots c'_m v'_m$.

Consider the following proposition

$$P(i) := \forall j \leq i.(c_j = c'_j \wedge v_j = v'_j) \vee \exists j \leq i.(\overleftarrow{q} = c_1 v_1 \cdots c_j \top)\,.$$

With induction we show that $\forall i \leq n.\, P(i)$ holds.

Suppose $i < n$ and $P(i)$ holds. We show that $P(i+1)$ holds.

Since $P(i)$ holds we have that $\forall j \leq i.(c_j = c'_j \wedge v_j = v'_j)$ or $\exists j \leq i.(q = c_1 v_1 \cdots c_j \top)$ holds. If the second proposition holds then we immediately get $P(i+1)$.

So, suppose that $\forall j \leq i.(c_j = c'_j \wedge v_j = v'_j)$ holds. We have that $c_{i+1} \in C_B$ or $c_{i+1} \in \mathsf{Rsp}_A$. If $c_{i+1} \in C_B$ (resp. $c_{i+1} \in \mathsf{Rsp}_A$) then it follows that $v_i \in V_B$ (resp. $v_i \in \mathsf{Que}_A$). As $v_i = v'_i$ we get $c'_{i+1} \in C_B$ (resp. $c'_{i+1} \in \mathsf{Rsp}_A$) or c'_{i+1} does not exist, i.e. $m < i+1$.

Suppose $c_{i+1} \neq c'_{i+1}$ or c'_{i+1} does not exist. Let

$$x := \begin{cases} \|\pi_1^{\Rightarrow}(c_1 v_1 \cdots c_{i+1} v_{i+1})\| & \text{if } v_{i+1} \in V_B \cup \{\top\}, \\ \|\pi_1^{\Rightarrow}(c_1 v_1 \cdots c_{i+1} v_{i+1}) \cdot \top\| & \text{if } v_{i+1} \in \mathsf{Que}_A. \end{cases}$$

and let k be the maximal index $k \leq i+1$ with $c_k \in C_B$. Then it follows that $p(x)$ fills c_k with v_{i+1} (resp. with $\top$). If c'_{i+1} does not exist then $q(x)$ obviously does not fill c_k. If $c'_{i+1} \in C_B \setminus \{c_{i+1}\}$ then $k = i+1$ and $q(x)$ does not fill c_k. Finally if $c'_{i+1} \in \mathsf{Rsp}_A \setminus \{c_{i+1}\}$ then there exist $w, w' \in V_A$ with $c_{i+1} = v_i \cdot w$ and $c'_{i+1} = v_i \cdot w'$, hence as $v_i \cdot w' \notin x$ it follows that $q(x)$ does not fill c_k. So, $q(x)$ does not fill c_k and it follows that $p(x) \not\sqsubseteq q(x)$ in contradiction with $p \sqsubseteq_{\mathsf{pw}} q$. Thus we have that

$$c_1 v_1 \cdots c_i v_i c_{i+1} = c'_1 v'_1 \cdots c'_i v'_i c'_{i+1}\,. \tag{†}$$

Suppose $v_{i+1} \in V_B \cup \{\top\}$. Then $p(x)$ fills c_k with value v_{i+1}. If $v'_{i+1} \in (V_B \setminus \{v_{i+1}\}) \cup \mathsf{Que}_A$ then it follows from (†) that $q(x)$ enables c_k but does not fill it. Thus from (†) and $p(x) \sqsubseteq q(x)$ it follows that $v'_{i+1} = v_{i+1}$ or $v'_{i+1} = \top$ as desired.

Finally suppose $v_{i+1} \in \mathsf{Que}_A$. Then $p(x)$ fills c_k with $\top$. If $v'_{i+1} \in \mathsf{Que}_A \setminus \{v_{i+1}\}$ then it follows from (†) that $q(x)$ enables c_k but does not fill it. If $v'_{i+1} \in V_B$ then $q(x)$ fills ck with value v'_{i+1}. As in both cases it follows that $p(x) \not\sqsubseteq q(x)$ we get $v'_{i+1} = v_{i+1}$ or $v'_{i+1} = \top$ as desired.

Now using induction it follows that $P(n)$ holds. Thus we have

$$\begin{aligned} q &= c_1 v_1 \cdots c_n v_n c'_{n+1} v'_{n+1} \cdots c'_m v'_m && \text{with } m \geq n \text{ or} \\ q &= c_1 v_1 \cdots c_j \top && \text{with } j \leq n. \end{aligned}$$

and it follows that $p \sqsubseteq q$. □

Lemma 3.5.6. *Let $A = (C_A, V_A, P_A)$ and $B = (C_B, V_B, P_B)$ be CL-games and $p \in \mathsf{FP}([\mathcal{D}(A) \to \mathcal{D}(B)])$ and $g \in [\mathcal{D}(A) \to \mathcal{D}(B)]$. If $p \sqsubseteq_{\mathsf{pw}} g$ then there exists a $q \in \mathsf{FP}(g)$ with $p \sqsubseteq_{\mathsf{pw}} q$.*

Proof. Suppose $p \in \mathsf{FP}([\mathcal{D}(A) \to \mathcal{D}(B)])$ and $g \in [\mathcal{D}(A) \to \mathcal{D}(B)]$ with $p \sqsubseteq_{\mathsf{pw}} g$. As $[\mathcal{D}(A) \to \mathcal{D}(B)]$ is isomorphic to $\mathcal{D}(A \to B)$ it follows from Thm. 3.2.17 that $\overleftarrow{p}$ is a position of the CL-game $\mathcal{G}(\mathcal{D}(A \to B))$ which is isomorphic to the CL-game $A \to B$. Thus we have

$$\overleftarrow{p} = c_1 v_1 \cdots c_n v_n \,.$$

Let

$$x := \begin{cases} \|\pi_1^{\Rightarrow}(c_1 v_1 \cdots c_{i+1} v_{i+1})\| & \text{if } v_{i+1} \in V_B \cup \{\top\}, \\ \|\pi_1^{\Rightarrow}(c_1 v_1 \cdots c_{i+1} v_{i+1}) \cdot \top\| & \text{if } v_{i+1} \in \mathsf{Que}_A. \end{cases}$$

Then as $p(x)$ is prime it follows from Lemma 3.5.3 that there exists a $q \in \mathsf{FP}(g)$ with $p(x) \sqsubseteq q(x)$. We show that $p \sqsubseteq_{\mathsf{pw}} q$ holds.

Suppose there exists a $y \in A$ such that $p(y) \not\sqsubseteq q(y)$. Let

$$y' := (x \cap y \cap \mathsf{Rsp}_A) \cup \{q \cdot \top \in y \mid \exists v.\, q \cdot v \in x\} \,.$$

Then it follows that $p(y) = p(y')$ (since y' contains all the responses (possibly ending with $\top$) of y that may be inspected by p) and hence $p(y') \not\sqsubseteq q(y')$. As $p(y') \sqsubseteq g(y')$ there exists a $q' \in \mathsf{FP}(g)$ with $p(y') \sqsubseteq q'(y')$.

As $q, q' \in \mathsf{FP}(g)$ it follows that both are stably coherent. Thus as $q \neq q'$ we get that

$$\begin{array}{lll} & c_{i+1} r_{i+1} \cdots c_n r_n & = q \\ c_1 r_1 \cdots c_i r_i & & \\ & c'_{i+1} r'_{i+1} \cdots c'_m r'_m & = q' \end{array}$$

where $c_{i+1} \neq c'_{i+1}$. From the definition of the positions given in Thm. 3.5.2 it follows that either $c_{i+1}, c'_{i+1} \in C_B$ or $c_{i+1}, c'_{i+1} \in \mathsf{Rsp}_A$.

Suppose that $c_{i+1}, c'_{i+1} \in C_B$. If $q'(y')$ does not fill c'_{i+1} then it follows that $q'(y') \sqsubseteq q(y')$ and thus $p(y') \sqsubseteq q(y')$ in contradiction with $p(y') \not\sqsubseteq q(y')$. Otherwise suppose $q'(y')$ fills c'_{i+1}. Then c'_{i+1} is also filled by $p(y')$ (as otherwise it follows that $p(y') \sqsubseteq q(y')$) but then in order to get extensionally above $p(x)$, $q(x)$ fills some cell c_j with $j \leq i$ (and $c_j \in C_B$) with $\top$. Since $q'(y')$ fills c'_{i+1} it follows that $q'(y')$ fills c_j and thus $q(y')$ fills c_j. From the definition of y' it follows that $q(y')$ fills c_j with $\top$. Thus $q'(y')$ fills c_j with $\top$ in contradiction with the fact that $q'(y')$ fills c'_{i+1}.

Now suppose $c_{i+1}, c'_{i+1} \in \mathsf{Rsp}_A$. Thus as $c_{i+1} \neq c'_{i+1}$ there exists values $v, v' \in V_A$ with $v \neq v'$ and a $q \in \mathsf{Que}_A$ such that $c_{i+1} = q \cdot v$ and $c'_{i+1} = q \cdot v'$. If $q'(y')$ does not fill c'_{i+1} then it follows that $q'(y') \sqsubseteq q(y')$ and thus $p(y') \sqsubseteq q(y')$ in contradiction with $p(y') \not\sqsubseteq q(y')$. Otherwise suppose $q'(y')$ fills c'_{i+1}. Then $q \cdot v' \in y'$ and by definition of y' it follows that $q \cdot v' \in x$. Thus $q \cdot v \notin x$. Hence in order to get extensionally above $p(x)$, $q(x)$ fills some cell c_j with $j \leq i$ (and $c_j \in C_B$) with $\top$. Since $q'(y')$ fills c'_{i+1} it follows that $q'(y')$ fills c_j and thus $q(y')$ fills c_j. From the definition of y' it follows that $q(y')$ fills c_j with $\top$. Thus $q'(y')$ fills c_j with $\top$ in contradiction with the fact that $q'(y')$ fills c'_{i+1}. $\square$

Theorem 3.5.7. *Let A, B be lbds and $f, g : A \to B$ be bistable maps. Then $f \sqsubseteq g$ iff $f \sqsubseteq_{\mathsf{pw}} g$ holds.*

Proof. Suppose $f, g : A \to B$ are bistable. For the forward implication suppose $f \sqsubseteq g$. Let $x \in A$. Since the category **LBD** is cartesian closed it follows that

$$f(x) = \mathsf{eval}_{A,B}(f, x) \sqsubseteq \mathsf{eval}_{A,B}(g, x) = g(x)$$

as desired.

For the reverse implication suppose $f \sqsubseteq_{\mathsf{pw}} g$. Let $p \in \mathsf{FP}(f)$. Then $p \sqsubseteq f$ holds and using the already shown forward implication of this lemma we get $p \sqsubseteq_{\mathsf{pw}} f$. Thus it follows that $p \sqsubseteq_{\mathsf{pw}} g$ holds. As $[A{\to}B] \cong [\mathcal{D}(\mathcal{G}(A)){\to}\mathcal{D}(\mathcal{G}(B))]$ it follows from Lemma 3.5.6 that there exists a $q \in \mathsf{FP}(g)$ with $p \sqsubseteq_{\mathsf{pw}} q$. Finally it follows from Lemma 3.5.5 that $p \sqsubseteq q$ as desired. □

Notice, that a bistable map f between lbpds A and B is continuous w.r.t. the extensional order and continuous w.r.t. the stable order. However, a bistable map will in general not be cocontinuous w.r.t. the costable order as shown by the following example.

Example 3.5.8. *Let O be the lbd $(\{\bot, \top\}, \sqsubseteq_{\mathsf{O}}, \neg_{\mathsf{O}})$ with $\bot \sqsubseteq_{\mathsf{O}} \top$ and $\neg_{\mathsf{O}}(\bot) = \top$, and N be the lbd $(\mathbb{N} \cup \{\bot, \top\}, \sqsubseteq_{\mathsf{N}}, \neg_{\mathsf{N}})$ with $\bot \sqsubseteq_{\mathsf{N}} n \sqsubseteq_{\mathsf{N}} \top$ and $\neg_{\mathsf{O}}(\bot) = \top$ and $\neg_{\mathsf{O}}(n) = n$ for all $n \in \mathbb{N}$.*

Let $F : [\mathsf{N}{\to}\mathsf{O}^{\mathsf{O}}] \to \mathsf{O}$ be defined recursively as $F(f) = f(0)(F(\lambda n.f(n{+}1)))$. Let $f = \lambda n.\mathrm{id}_{\mathsf{O}}$ and $f_n(k) = \mathrm{id}_{\mathsf{O}}$ for $k < n$ and $f_n(k) = \lambda n.\top$ for $k \geq n$. Obviously, the set $X := \{f_n \mid n \in \mathbb{N}\}$ is costably coherent, codirected w.r.t. $\leq_c$ and $f = \bigsqcap X$. As F is least with the properties of its defining equation we have $F(f) = \bot$ and $F(f_n) = \top$ for all n. Thus, we have $F(\bigsqcap X) = \bot$ whereas $\bigsqcap F[X] = \top$, i.e. F fails to be cocontinuous w.r.t. $\leq_c$.

Whereas in the finite case there is a perfect symmetry between $\bot$ and $\top$ this symmetry is broken in the infinitary case where $\bot$ can manifest itself as *nontermination which cannot be detected in finite time.*

In the following two lemmas we analyse for bistable maps $f : A \to B$ the strategy corresponding to the map $\neg f$. These observations lead to the characterisation of the involution $\neg$ for exponentials in the category **LBD** given in Thm. 3.5.11.

Lemma 3.5.9. *Let $A = (C_A, V_A, P_A)$ and $B = (C_B, V_B, P_B)$ be CL-games, $x \in \mathcal{D}(A)$, $f \in \mathbf{LBD}(\mathcal{D}(A), \mathcal{D}(B))$ and $c \in \mathsf{Cell}(\mathcal{D}(B))$. If $f(x)$ fills c with $\top$ then there exists a unique minimal (w.r.t. the prefix order) path $s := c_1 v_1 \cdots c_i v_i \in f$ such that $\widehat{s}(x)$ fills c with $\top$.*

Proof. Suppose $f \in \mathbf{LBD}(\mathcal{D}(A), \mathcal{D}(B))$, $x \in \mathcal{D}(A)$ and $c \in \mathsf{Cell}(\mathcal{D}(B))$ such that $f(x)$ fills c with $\top$. Then from Lemma 3.5.3 it follows that there exists a path $s := c_1 v_1 \cdots c_i v_i \in f$ such that $\widehat{s}(x)$ fills c with $\top$. W.l.o.g. we can assume that s is minimal.

Suppose there exists another minimal path $s' := c'_1 v'_1 \cdots c'_j v'_j \in f$ such that $s \neq s'$ and $\widehat{s'}(x)$ fills c with $\top$. As $\widehat{s} \uparrow \widehat{s'}$ and $\widehat{s} \sqcap \widehat{s'} = \widehat{s} \cap \widehat{s'}$ it follows that $\widehat{s} \sqcap \widehat{s'} \subsetneqq \widehat{s}$. As $(\widehat{s} \sqcap \widehat{s'})(x) = \widehat{s}(x) \sqcap \widehat{s'}(x)$ fills c with $\top$ we get a contradiction to the minimality of s. □

Lemma 3.5.10. *Let $A = (C_A, V_A, P_A)$ and $B = (C_B, V_B, P_B)$ be CL-games, $f \in$ $\mathbf{LBD}(\mathcal{D}(A), \mathcal{D}(B))$ and $e \in \mathsf{F}(\mathcal{D}(A))$. Then $(\neg f)(e) = \neg f(\neg e)$ holds.*

Proof. Suppose $f \in \mathbf{LBD}(\mathcal{D}(A), \mathcal{D}(B))$ and $e \in \mathsf{F}(\mathcal{D}(A))$. Let $c \in \mathsf{Cell}(\mathcal{D}(B))$.

Suppose that $(\neg f)(e)$ fills c with value v and $v \neq \top$. Thus there exists a path $p \in \neg f$ with $p = c_1 v_1 \cdots c_i v_i$ and $v_i = v$ and $c_j = c$ for some $j \leq i$ and $\forall k > j.\, c_k \notin C_B$. As $v \neq \top$ it follows from Def. 3.1.6 that $p \in f$. Thus we get that $f(e)$ fills c with v. As $s := \|\pi_1^{\Rightarrow}(c_1 v_1 \cdots c_i v_i)\| \subseteq e$ and s does not fill any cell with $\top$ it follows that $s \in \neg e$. Thus $f(\neg e)$ fill c with v. And as $v \neq \top$ it follows that $\neg f(\neg e)$ fills c with v.

Suppose that $(\neg f)(e)$ fills c with $\top$. Then by Lemma 3.5.9 there exists a unique minimal path $s := c_1 v_1 \cdots c_i v_i \in \neg f$ such that $\widehat{s}(e)$ fills c with $\top$. We consider the case $v_i = \top$ and $v_i \neq \top$.

Suppose $v_i = \top$. Then it follows that s is a maximal path in $\neg f$. Thus s is the only path in $\neg f$ such that $\widehat{s}(e)$ fills c with $\top$. Further from minimality of s and since $v_i = \top$ it follows that $\widehat{s}(e_\perp)$ fills c with $\top$. Thus $\widehat{s}(\neg e)$ fills c with $\top$. Thus $(\neg f)(\neg e)$ fills c with $\top$. As $s \notin f$ but all $s' \in f$ for all prefixes of s it follows that c is accessible from $f(\neg e)$. Thus $\neg f(\neg e)$ fills c with $\top$.

Suppose $v_i \neq \top$. Then $s \in f$ and it follows that $f(e)$ fills c with $\top$. As s is the least path w.r.t. the prefix order such that $\widehat{s}(e)$ fills c with $\top$ it follows that $\|\pi_1^{\Rightarrow}(c_1 v_1 \cdots c_i v_i) \cdot \top\| \subseteq e$. As $\|\pi_1^{\Rightarrow}(c_1 v_1 \cdots c_i v_i) \cdot \top\| \nsubseteq \neg e$ but $\|\pi_1^{\Rightarrow}(c_1 v_1 \cdots c_i)\| \subseteq \neg e$ it follows that c is accessible from $f(\neg e)$. Thus $\neg f(\neg e)$ fills c with $\top$.

If cell c is enabled but not filled by $(\neg f)(e)$ then as e is finite this cannot result from an infinite sequence of queries and responses in A. Thus with a similar argument as above one can show that $c \in \mathsf{Acc}((\neg f)(e))$ implies $c \in \mathsf{Acc}(\neg f(\neg e))$. Finally if c is not enabled in $(\neg f)(e)$ then it follows from Def. 3.1.6 that c is not enabled in $\neg f(\neg e)$.

Thus we have shown that $(\neg f)(e)$ and $\neg f(\neg e)$ fill the cells of B identically and it follows that $(\neg f)(e) = \neg f(\neg e)$. ☐

Theorem 3.5.11. *Let A and B be lbds, $f : A \to B$ be bistable and $x \in A$. Then*

$$(\neg f)(x) = \bigsqcup \{\neg f(\neg e) \mid e \in \mathsf{F}(x)\}$$

holds.

Proof. Suppose $f : A \to B$ is bistable. Then we get

$$\begin{aligned} (\neg f)(x) &= (\neg f)(\bigsqcup \mathsf{F}(x)) \\ &= \bigsqcup \{(\neg f)(e) \mid e \in \mathsf{F}(x)\} && (\dagger) \\ &= \bigsqcup \{\neg f(\neg e) \mid e \in \mathsf{F}(x)\} && (\ddagger) \end{aligned}$$

where $(\dagger)$ follows as $\mathsf{F}(x)$ is directed and $\neg f$ is continuous and $(\ddagger)$ from Lemma 3.5.10 and as $\mathcal{D}(\mathcal{G}(A)) \cong A$ and $\mathcal{D}(\mathcal{G}(B)) \cong B$. ☐

Corollary 3.5.12. *Let A and B be lbds and $f : A \to B$ be bistable. Then*

(1) $(\neg f)(e) = \neg f(\neg e)$ *for all* $e \in \mathsf{F}(A)$ *and*

(2) $(\neg f)(x) \leq_b \neg f(\neg x)$ *for all* $x \in A$

holds.

Proof. Suppose A and B are lbds and $f : A \to B$ is a bistable map.

ad (1) : This follows immediately from Thm. 3.5.11.

ad (2) : Suppose $x \in A$. As $f \updownarrow \neg f$ it follows from Lemma 3.6.2 that $f(x) \updownarrow (\neg f)(x)$. Thus we get $\updownarrow\{(\neg f)(x), f(x), \neg f(x), \neg f(\neg x)\}$. As $\neg f(\neg c) \leq_s \neg f(\neg x)$ for all $c \in \mathsf{F}(x)$, we get that $(\neg f)(x) \leq_s \neg f(\neg x)$ holds. Thus, it follows that $(\neg f)(x) \leq_b \neg f(\neg x)$ as desired. □

3.6 Exponentials as function spaces

In this section we show that the exponential $[A{\to}B]$ of lbds A and B can also be considered as function space of bistable maps. The main results of this section are the fact that infima and suprema of stably or costably coherent bistable maps are computed pointwise and that the category **LBD** is cpo-enriched w.r.t. the extensional order $\sqsubseteq$ and w.r.t. the stable order $\leq_s$.

Notice, that throughout this section we use the fact that the binary products of lbds A and B is given by $(|A| \times |B|, \sqsubseteq, \neg)$ where $\sqsubseteq$ and $\neg$, and thus also the (co-/bi-)stable relations and orders are defined componentwise (cf. section 4.1).

Lemma 3.6.1. *Let A and B be lbpds, $f : A \to B$ a bistable map and $x \in A$. Then*

(1) $f_\bot(x) = f(x) \sqcap (\neg f)(x)$ *and*

(2) $f^\top(x) = f(x) \sqcup (\neg f)(x)$

holds.

Proof. Suppose $f : A \to B$ is bistable and let $x \in A$. Then $f \updownarrow \neg f$ and hence $(f, x) \updownarrow (\neg f, x)$. As $\mathsf{eval}_{A,B}$ is a bistable map it follows that

$$\begin{aligned} f_\bot(x) &= (f \sqcap \neg f)(x) \\ &= \mathsf{eval}_{A,B}(f \sqcap \neg f, x) \\ &= \mathsf{eval}_{A,B}((f, x) \sqcap (\neg f, x)) \\ &= \mathsf{eval}_{A,B}(f, x) \sqcap \mathsf{eval}_{A,B}(\neg f, x) \\ &= f(x) \sqcap (\neg f)(x) \end{aligned}$$

holds. Analogously, one can show that $f^\top(x) = f(x) \sqcup (\neg f)(x)$ holds. □

Lemma 3.6.2. *Let A and B be lbpds and $f, g : A \to B$ bistable maps. Then*

(1) $f \uparrow g$ *implies* $\forall x{\in}A.\, f(x) \uparrow g(x)$ *and*

(2) $f \downarrow g$ *implies* $\forall x{\in}A.\, f(x) \downarrow g(x)$ *.*

Proof. *ad* (1) : Suppose $f, g : A \to B$ be bistable maps and let $x \in A$. If $f \uparrow g$ then $(f, x) \uparrow (g, x)$ and as $\mathsf{eval}_{A,B}$ preserves stable coherence it follows that $f(x) \uparrow g(x)$.

ad (2) : This follows analogously. □

Lemma 3.6.3. *Let A and B be lbpds and $f, g : A \to B$ bistable maps with $f \uparrow g$. Then*

(1) $(f \sqcap g)(x) = f(x) \sqcap g(x)$ *and*

(2) $(f \sqcup g)(x) = f(x) \sqcup g(x)$

holds.

Proof. Suppose $f, g : A \to B$ are bistable maps with $f \uparrow g$ and let $x \in A$. Then $(f, x) \uparrow (g, x)$ holds and as $\mathsf{eval}_{A,B}$ is a bistable map it follows that

$$\begin{aligned}(f \sqcap g)(x) &= (f \sqcap g)(x)\\ &= \mathsf{eval}_{A,B}(f \sqcap g, x)\\ &= \mathsf{eval}_{A,B}((f, x) \sqcap (g, x))\\ &= \mathsf{eval}_{A,B}(f, x) \sqcap \mathsf{eval}_{A,B}(g, x)\\ &= f(x) \sqcap g(x)\end{aligned}$$

holds.

For showing that $(f \sqcup g)(x) = f(x) \sqcup g(x)$ notice that from Lemma 3.1.9 it follows that the supremum of the strategies S_f and S_g is given by their union. Thus it follows from the definition of the application in Thm. 3.5.2 that $(S_f \sqcup S_g)(x) = S_f(x) \cup S_g(x) = S_f(x) \sqcup S_g(x)$ (where the last equation follows as $S_f(x) \uparrow S_g(x)$ and by Lemma 3.6.2). As $[A{\to}B] \cong \mathcal{D}(\mathcal{G}([A{\to}B])) \cong \mathcal{D}(\mathcal{G}(A){\to}\mathcal{G}(B))$ we get $(f \sqcup g)(x) = f(x) \sqcup g(x)$. □

Lemma 3.6.4. *Let A and B be lbpds and $f, g : A \to B$ bistable maps with $f \downarrow g$. Then*

(1) $(f \sqcap g)(x) = f(x) \sqcap g(x)$ *and*

(2) $(f \sqcup g)(x) = f(x) \sqcup g(x)$

holds.

Proof. Suppose $f, g : A \to B$ are bistable maps with $f \uparrow g$ and let $x \in A$. Then $(f, x) \downarrow (g, x)$ holds and as $\mathsf{eval}_{A,B}$ is a bistable map it follows that

$$\begin{aligned}(f \sqcup g)(x) &= (f \sqcup g)(x)\\ &= \mathsf{eval}_{A,B}(f \sqcup g, x)\\ &= \mathsf{eval}_{A,B}((f, x) \sqcup (g, x))\\ &= \mathsf{eval}_{A,B}(f, x) \sqcup \mathsf{eval}_{A,B}(g, x)\\ &= f(x) \sqcup g(x)\end{aligned}$$

holds.

Next we show that $(f \sqcap g)(x) = f(x) \sqcap g(x)$ holds. As $f \downarrow g$ it follows that $\neg f \uparrow \neg g$. Thus we get

$$\begin{aligned}
(f \sqcap g)(x) &= (\neg(\neg f \sqcup \neg g))(x) \\
&= \bigsqcup \{\neg(\neg f \sqcup \neg g)(\neg e) \mid e \in \mathsf{F}(x)\} \\
&= \bigsqcup \{\neg((\neg f)(\neg e) \sqcup (\neg g)(\neg e)) \mid e \in \mathsf{F}(x)\} && (\dagger) \\
&= \bigsqcup \{\neg(\neg f)(\neg e) \sqcap \neg(\neg g)(\neg e) \mid e \in \mathsf{F}(x)\} \\
&= \bigsqcup \{(\neg\neg f)(e) \sqcap (\neg\neg g)(e) \mid e \in \mathsf{F}(x)\} && (\ddagger) \\
&= \bigsqcup \{f(e) \sqcap g(e) \mid e \in \mathsf{F}(x)\} \\
&= f(x) \sqcap g(x) && (\S)
\end{aligned}$$

where $(\dagger)$ follows as $\neg f \uparrow \neg g$ and from Lemma 3.6.3 and $(\ddagger)$ follows from Cor. 3.5.12(1).

Finally, $(\S)$ holds for the following reason: obviously we have $\bigsqcup\{f(e) \sqcap g(e) \mid e \in \mathsf{F}(x)\} \sqsubseteq f(x) \sqcap g(x)$. For showing the reverse inequality suppose $p \in \mathsf{FP}(f(x) \sqcap g(x))$ holds. Thus, as $p \sqsubseteq f(x), g(x)$ and p is compact there exist $e_1, e_2 \in \mathsf{F}(x)$ with $p \sqsubseteq f(e_1)$ and $p \sqsubseteq g(e_2)$. As $\mathsf{F}(x)$ is directed there exists a $e_3 \in \mathsf{F}(x)$ with $e_1, e_2 \sqsubseteq e_3$. Hence, we have $p \sqsubseteq f(e_3), g(e_3)$ and it follows that $p \sqsubseteq f(e_3) \sqcap g(e_3) \sqsubseteq \bigsqcup\{f(e) \sqcap g(e) \mid e \in \mathsf{F}(x)\}$ as desired. □

Thus, we have shown that infima and suprema of stably or costably coherent bistable maps are computed pointwise. Next we show that the category **LBD** is cpo-enriched w.r.t. $\sqsubseteq$ and w.r.t. $\leq_s$.

Lemma 3.6.5. *Let A and B be lbpds. If F is a directed subset of $[A \to B]$ then*

$$(\bigsqcup F)(x) = \bigsqcup\{f(x) \mid f \in F\}$$

for all $x \in A$.

Proof. Suppose F is a directed subset of $[A \to B]$ and $x \in A$. Then $F \times \{x\}$ is a directed subset of $[A \to B] \times A$ with $\bigsqcup(F \times \{x\}) = (\bigsqcup F, x)$. As $\mathsf{eval}_{A,B}$ is continuous it follows that

$$\begin{aligned}
(\bigsqcup F)(x) &= \mathsf{eval}_{A,B}(\bigsqcup F, x) \\
&= \mathsf{eval}_{A,B}(\bigsqcup(F \times \{x\})) \\
&= \bigsqcup\{\mathsf{eval}_{A,B}(f, x) \mid f \in F\} \\
&= \bigsqcup\{f(x) \mid f \in F\}
\end{aligned}$$

as desired. □

Theorem 3.6.6. *The categories* **LBD** *and* **LBPD** *are cpo-enriched w.r.t.* $\sqsubseteq$ *and w.r.t.* $\leq_s$.

Proof. Suppose A, B and C are lbpds and $f \in [A \to B]$ and $g \in [B \to C]$. Let F be a directed (resp. stably directed) subset of $[A \to B]$ with $\bigsqcup F = f$ and $x \in A$. Then it follows that

$$\begin{aligned} g((\bigsqcup F)(x)) &= g(\bigsqcup \{f'(x) \mid f' \in F\}) && (\dagger) \\ &= \bigsqcup \{g(f'(x)) \mid f' \in F\} && (\ddagger) \\ &= \bigsqcup \{(g \circ f')(x) \mid f' \in F\} \\ &= (\bigsqcup \{g \circ f' \mid f' \in F\})(x) && (\dagger) \end{aligned}$$

holds, where $(\dagger)$ follows from Lemma 3.6.5 and $(\ddagger)$ holds as $\bigsqcup \{f'(x) \mid f' \in F\}$ is directed. Thus it follows that $g \circ (\bigsqcup F) = \bigsqcup_{f' \in F} \{g \circ f'\}$ holds.

Analogously one can show that $(\bigsqcup G) \circ f = \bigsqcup_{g' \in G} \{g' \circ f\}$ holds for all directed subsets G of $[B \to C]$. Thus it follows that composition of morphisms is continuous w.r.t. $\sqsubseteq$ and w.r.t. $\leq_s$. □

4 Properties of the category $\mathbf{LBD}$

In this chapter we show that **LBD** and **LBPD** are closed under basic categorical constructions. In the first section we show how to construct products, coproducts, biliftings and sums in **LBPD** (resp. **LBD**). In the next sections we introduce the notion of embedding/projection pair and show that inverse limits of ω-chains of embedding/projection pairs (w.r.t. $\leq_s$) exist in **LBD** and are constructed as usual. Finally, adapting a result of J. Longley in [Lon02] we show that every countably based locally boolean domain appears as retract of $\mathsf{U} = [\mathsf{N} \to \mathsf{N}]$ where N are the bilifted natural numbers, i.e. that U is a universal object for countably based locally boolean domains.

4.1 Products, biliftings and sums

In this section we show how to construct products, coproducts, biliftings and sums in **LBPD** (resp. **LBD**). We only present the results and leave the proofs as an easy exercise to the reader. Further notice that for notational convenience we assume set unions to be disjoint when building coproducts, biliftings and sums.

Definition 4.1.1. *Let $(A_i)_{i \in I}$ be a family of lbpds. If $j \in I$ then we define $\varepsilon_j : |A_j| \to \prod_{i \in I} |A_i|$ by*

$$(\varepsilon_j(x))_i := \begin{cases} x & \text{if } i = j \text{ and} \\ \bot & \text{otherwise} \end{cases}$$

for all $x \in |A_j|$ and $i \in I$. ◇

Theorem 4.1.2. *Let $(A_i)_{i \in I}$ be a family of lbpds. Further let*

$$\prod_{i \in I} A_i := (\prod_{i \in I} |A_i|, \sqsubseteq, \neg)$$

with $x \sqsubseteq y$ iff $x_i \sqsubseteq y_i$ for all $i \in I$ and $\neg x = (\neg x_i)_{i \in I}$ for all $x, y \in \prod_{i \in I} |A_i|$. Then

$$\prod_{i \in I} A_i \quad \text{with} \quad \pi_i : \prod_{i \in I} A_i \to A_i : (x_i)_{i \in I} \mapsto x_i \quad \text{for all } i \in I$$

is a product of the A_i in **LBPD** *(resp.* **LBD***).*
An element $x \in \prod_{i \in I} A_i$ is finite prime (resp. a cell) iff $x = \varepsilon_i(p)$ for some $i \in I$ and $p \in \mathsf{FP}(A_i)$ (resp. $p \in \mathsf{Cell}(A_i)$).
Further, $\prod_{i \in I} A_i$ is pointed iff all the A_i are.

As an immediate consequence we get:

Corollary 4.1.3. *The object* $\mathbf{1} := (\{*\}, \sqsubseteq, \neg)$ *with* $* \sqsubseteq *$ *and* $\neg(*) = *$ *is terminal in* **LBPD** (*resp.* **LBD**).

Next, we consider coproducts.

Theorem 4.1.4. *Let* $(A_i)_{i \in I}$ *be a family of lbpds. Further let*

$$\coprod_{i \in I} A_i := (\coprod_{i \in I} |A_i|, \sqsubseteq, \neg)$$

where $(i, x) \sqsubseteq (j, y)$ *iff* $i = j$ *and* $x \sqsubseteq y$ *and* $\neg(i, x) = (i, \neg(x))$ *for all* $(i, x), (j, y) \in \coprod_{i \in I} |A_i|$ *Then*

$$\coprod_{i \in I} A_i \quad \textit{with} \quad \iota_i : A_i \to \coprod_{i \in I} A_i : x \mapsto (i, x) \quad \textit{for all } i \in I$$

is a coproduct of the A_i *in* **LBPD**.
An element $(i, x) \in \coprod_{i \in I} A_i$ *is finite prime* (*resp. a cell*) *iff* $x \in \mathsf{FP}(A_i)$ (*resp.* $x \in \mathsf{Cell}(A_i)$).

In contrast to the category **LBPD**, the category **LBD** does not have coproducts, since coproducts in general do not have a least element. Nevertheless we can construct separated sum of lbds by bilifting the coproduct given in **LBPD**.

Theorem 4.1.5. *Let* A *be a lbpd then its* bilifting $A_\uparrow$ *is a lbd with*

> $|A_\uparrow| := |A| \cup \{\bot', \top'\}$ (*we assume* $\bot', \top' \notin |A|$),
>
> *the extensional order is the extensional order on* A *extended by* $\bot' \sqsubseteq x \sqsubseteq \top'$ *for all* $x \in |A_\uparrow|$,
>
> *negation is given by the negation on* A *extended by* $\neg\bot' = \top'$, $\neg\top' = \bot'$.

Further we have $\mathsf{FP}(A_\uparrow) = \mathsf{FP}(A) \cup \{\bot', \top'\}$ *and* $\mathsf{Cell}(A_\uparrow) = \mathsf{Cell}(A) \cup \{\top'\}$.

Theorem 4.1.6. *Let* A *and* B *be lbpds and* $f : A \to B$ *a sequential map. Then* $f_\uparrow : A_\uparrow \to B_\uparrow$, *defined by*

$$f_\uparrow(x) = \begin{cases} \bot' & \textit{if } x = \bot' \\ \top' & \textit{if } x = \top' \\ f(x) & \textit{otherwise} \end{cases}$$

is sequential.
Further $(\_)_\uparrow : \mathbf{LBPD} \to \mathbf{LBD}$ *is a locally continuous functor, and for all lbds* A *we have sequential functions* $\mathsf{up}_A : A \to A_\uparrow$ *and* $\mathsf{down}_A : A_\uparrow \to A$ *with*

$$\mathsf{up}_A(x) = x$$

and

$$\mathsf{down}_A(x) = \begin{cases} \bot & \textit{if } x = \bot' \\ \top & \textit{if } x = \top' \\ x & \textit{otherwise} \end{cases}$$

Theorem 4.1.7. *Let $(A_i)_{i \in I}$ be a family of lbpds. Then*

$$\sum_{i \in I} A_i := (\coprod_{i \in I} A_i)_{\uparrow} .$$

is the separated sum of the A_i, i.e. given a lbd B and bistable maps $f_i : A_i \to B$ for all $i \in I$, there exists a unique bistrict bistable map $f : \sum_{i \in I} A_i \to B$ with $f_i = f \circ \mathsf{up}_{\coprod_{i \in I} A_i} \circ \iota_i$.

Finally we get the following result which is crucial for the interpretation of product and sum types in **LBD**.

Theorem 4.1.8. *For all sets I we have the following local continuous functors:*

(1) $\prod_{i \in I} : \mathbf{LBD}^I \to \mathbf{LBD}$ *with* $\prod_{i \in I} f_i = (f_i \circ \pi_i)_{i \in I}$

(2) $\coprod_{i \in I} : \mathbf{LBPD}^I \to \mathbf{LBPD}$ *with* $\coprod_{i \in I} f_i = [\iota_i \circ f_i]_{i \in I}$

(3) $\sum_{i \in I} : \mathbf{LBD}^I \to \mathbf{LBD}$ *with* $\sum_{i \in I} f_i = (\coprod_{i \in I} f_i)_{\uparrow}$

4.2 Embedding/Projection Pairs in **LBD**

For constructing recursive types in **LBD** we have to introduce an appropriate notion of embedding/projection pairs in **LBD** and, further, a notion of inverse limit for ω-chains of embedding/projection pairs.

Definition 4.2.1. *An* embedding/projection pair (ep-pair) *from A to B in* **LBD** *(notation $(\iota, \pi) : A \to B$) is a pair of* **LBD** *morphisms $\iota : A \to B$ and $\pi : B \to A$ with $\pi\iota = \mathrm{id}_A$ and $\iota\pi \leq_s \mathrm{id}_B$.*

If $(\iota, \pi) : A \to B$ and $(\iota', \pi') : B \to C$ then their composition is defined as $(\iota', \pi') \circ (\iota, \pi) = (\iota' \circ \iota, \pi \circ \pi')$. We write $\mathbf{LBD}^{\mathrm{ep}}$ *for the ensuing category of embedding/projection pairs in* **LBD**. ◇

Notice that this is the usual definition of ep-pair when viewing **LBD** as order enriched by the stable and not by the extensional order.

Lemma 4.2.2. *Suppose $(\iota, \pi) : A \to B$ then ι is left adjoint to π w.r.t. $\leq_s$, i.e. for all $x \in A$ and $y \in B$ we have $\iota(x) \leq_s y$ iff $x \leq_s \pi(y)$.*

Proof. Suppose $\iota(x) \leq_s y$ then $x = \pi(\iota(x)) \leq_s \pi(y)$. If $x \leq_s \pi(y)$ then $\iota(x) \leq_s \iota(\pi(y)) \leq_s y$ where the last inequality holds since $\iota\pi \leq_s \mathrm{id}_B$ and $\iota\pi(y) = \mathsf{eval}_{B,B}(\iota\pi, y) \leq_s \mathsf{eval}_{B,B}(\mathrm{id}, y) = y$. □

Lemma 4.2.3. *Suppose $(\iota, \pi) : A \to B$. Then*

(1) $\iota(x_\perp) = \iota(x)_\perp$

(2) $\pi(y_\perp) = \pi(y)_\perp$

holds for all $x \in A$ and $y \in B$.

Proof. Suppose $(\iota, \pi) : A \to B$.

ad (1) : Obviously, we have $\iota(x)_\perp \leq_b \iota(x_\perp)$. Further, we get $\pi(\iota(x)_\perp) \leq_b \pi(\iota(x_\perp)) = x_\perp$, thus, $x_\perp = \pi(\iota(x)_\perp)$. It follows that $\iota(x_\perp) = \iota(\pi(\iota(x)_\perp)) \leq_s \iota(x)_\perp$ as desired.

ad (2) : As $\iota(\pi(y_\perp)) \leq_s y_\perp$ we have $\iota(\pi(y_\perp)) = \iota(\pi(y_\perp))_\perp = \iota(\pi(y))_\perp$ by Lemma 2.2.24. Using $\iota(\pi(y))_\perp = \iota(\pi(y)_\perp)$ which holds by (1) we get

$$\pi(y_\perp) = \pi(\iota(\pi(y_\perp))) = \pi(\iota(\pi(y))_\perp) = \pi(\iota(\pi(y)_\perp)) = \pi(y)_\perp$$

as desired. □

Corollary 4.2.4. *Suppose $(\iota, \pi) : A \to B$. Then*

(1) $\iota(\neg x) \leq_b \neg\iota(x)$

(2) $\iota(\neg x) = \neg\iota(x) \sqcap \iota(x^\top)$

(3) $\pi(\neg y) \leq_b \neg\pi(y)$

(4) $\pi(\neg y) = \neg\pi(y) \sqcap \pi(y^\top)$

holds for all $x \in A$ and $y \in B$.

Proof. Suppose $(\iota, \pi) : A \to B$.

ad (1) : Suppose $x \in A$. As $[x]_\updownarrow$ is a boolean algebra and $\iota(x) \sqcap \iota(\neg x) = \iota(x \sqcap \neg x) = \iota(x_\perp) = \iota(x)_\perp$ it follows that $\iota(\neg x) \leq_b \neg\iota(x)$.

ad (2) : Suppose $x \in A$. Then it follows that

$$\begin{aligned}
\neg\iota(x) \sqcap \iota(x^\top) &= \neg\iota(x) \sqcap \iota(x \sqcup \neg x) \\
&= \neg\iota(x) \sqcap (\iota(x) \sqcup \iota(\neg x)) \\
&= (\neg\iota(x) \sqcap \iota(x)) \sqcup (\neg\iota(x) \sqcap \iota(\neg x)) \\
&= \iota(x)_\perp \sqcup (\neg\iota(x) \sqcap \iota(\neg x)) \\
&= \neg\iota(x) \sqcap \iota(\neg x) \\
&= \iota(\neg x) && \text{by (1)}
\end{aligned}$$

ad (3) : Suppose $y \in B$. Then as $[y]_\updownarrow$ is a boolean algebra and $\pi(y) \sqcap \pi(\neg y) = \pi(y \sqcap \neg y) = \pi(y_\perp) = \pi(y)_\perp$ we get $\pi(\neg y) \leq_b \neg\pi(y)$.

ad (4) : Suppose $y \in B$. Then it follows that

$$\begin{aligned}
\neg\pi(y) \sqcap \pi(y^\top) &= \neg\pi(y) \sqcap \pi(y \sqcup \neg y) \\
&= \neg\pi(y) \sqcap (\pi(y) \sqcup \pi(\neg y)) \\
&= (\neg\pi(y) \sqcap \pi(y)) \sqcup (\neg\pi(y) \sqcap \pi(\neg y)) \\
&= \pi(y)_\perp \sqcup (\neg\pi(y) \sqcap \pi(\neg y)) \\
&= \neg\pi(y) \sqcap \pi(\neg y) \\
&= \pi(\neg y) && \text{by (3)}
\end{aligned}$$

□

Lemma 4.2.5. *Let* $f, g \in \mathbf{LBD}(A, B)$ *then*

$$f \leq_s g \quad \Leftrightarrow \quad \forall x, y{\in}A.\, x \leq_s y \to (f(y){\uparrow}g(x) \wedge f(x){=}f(y){\sqcap}g(x))\,.$$

Proof. This is proven in [AC98, Lemma 12.2.7.]. □

Lemma 4.2.6. *Suppose* $(\iota, \pi) : A \to B$ *and* $x, y \in B$. *Then* $x \leq_s y = \iota\pi(y)$ *implies* $x = \iota\pi(x)$.

Proof. Suppose $\iota\pi \leq_s \mathrm{id}_B$ holds. It follows from Lemma 4.2.5 that $\iota\pi(x) = \iota\pi(y) \sqcap x = y \sqcap x = x$. □

As an immediate consequence of Lemma 4.2.6 we get

Corollary 4.2.7. *Suppose* $(\iota, \pi) : A \to B$. *Then* $\iota\pi(A)$ *is downward closed w.r.t.* $\leq_s$.

Lemma 4.2.8. *Suppose* $(\iota, \pi) : A \to B$ *and* $x \in B$. *Then* $\exists y{\in}[x]_{\updownarrow}.\, y = \iota\pi(y)$ *holds iff* $x_\perp = \iota\pi(x_\perp)$.

Proof. Suppose $y \in [x]_{\updownarrow}$ with $y = \iota\pi(y)$. Hence, $x_\perp \leq_s y$ and using Lemma 4.2.6 we get $x_\perp = \iota\pi(x_\perp)$. The reverse implication is trivial as $x_\perp \in [x]_{\updownarrow}$. □

Lemma 4.2.9. *Suppose* $(\iota, \pi) : A \to B$. *Let* $X \subseteq A$ *be* $\sqsubseteq$*-codirected then* $\pi(\bigsqcap X) = \bigsqcap \pi(X)$.

Proof. As π is monotone the set $\pi(X)$ is codirected and $\pi(\bigsqcap X) \sqsubseteq \bigsqcap \pi(X)$. If $x \in X$ then $\iota(\pi(x)) \leq_s x$, thus $\bigsqcap \iota(\pi(X)) \sqsubseteq \bigsqcap X$. As ι is monotone we get $\iota(\bigsqcap \pi(X)) \sqsubseteq \bigsqcap \iota(\pi(X)) \sqsubseteq \bigsqcap X$. Thus, $\bigsqcap \pi(X) = \pi(\iota(\bigsqcap \pi(X))) \sqsubseteq \pi(\bigsqcap X)$ as desired. □

Lemma 4.2.10. *Suppose* $(\iota, \pi) : A \to B$. *Then*

(1) $\iota(x \sqcup y) = \iota(x) \sqcup \iota(y)$ *if* $x, y \in A$ *with* $x \uparrow y$

(2) $\pi(x \sqcup y) = \pi(x) \sqcup \pi(y)$ *if* $x, y \in B$ *with* $x \uparrow y$

hold.

Proof. Suppose $(\iota, \pi) : A \to B$.

ad (1) : As ι is a left adjoint w.r.t. $\leq_s$ it preserves stable suprema.

ad (2) : Let $\varrho : B \to B$ denote the retraction $\iota \circ \pi$, then from $\varrho \leq_s \mathrm{id}_B$ it follows that $\varrho(x) = \varrho(x \sqcup y) \sqcap x$ and $\varrho(y) = \varrho(x \sqcup y) \sqcap y$. Hence, $\varrho(x) \sqcup \varrho(y) = (\varrho(x \sqcup y) \sqcap x) \sqcup (\varrho(x \sqcup y) \sqcap y) = \varrho(x \sqcup y) \sqcap (x \sqcup y) = \varrho(x \sqcup y)$. As ι preserves stable suprema we get $\iota(\pi(x) \sqcup \pi(y)) = \iota(\pi(x)) \sqcup \iota(\pi(y)) = \iota(\pi(x \sqcup y))$. Finally, as ι is an injection it follows that $\pi(x \sqcup y) = \pi(x) \sqcup \pi(y)$. □

Notice that embeddings and projections preserve stable infima and suprema but they do *not* preserve negation.

Example 4.2.11. *Consider the embedding $\iota : \mathbb{1} \to \mathsf{O}$ then we have $\iota(*) = \bot = \iota(\neg *)$ whereas $\neg\bot = \top$. Further, consider the projection $\pi : \mathsf{O}^{\mathsf{O}} \to \mathsf{O} : f \mapsto f(\bot)$ (whose corresponding embedding sends u to $\lambda x.u$). We have $\pi(\mathrm{id}_{\mathsf{O}}) = \bot = \pi(\neg\mathrm{id}_{\mathsf{O}})$ whereas $\neg\bot = \top$.*

Moreover, projections π need not to be constant on bistably connected components not containing any fixpoint of $\iota\pi$. Consider the first projection $\mathsf{O} \times \mathsf{O}^{\mathsf{O}} \to \mathsf{O}$ whose associated embedding sends u to $(u, \bot)$. Notice that π is not constant on the equivalence class $\{(\bot, \mathrm{id}_{\mathsf{O}}), (\top, \mathrm{id}_{\mathsf{O}})\}$ as its image under π is $\{\bot, \top\}$.

However, we can show that embeddings send atoms to atoms and projections send atoms to atoms or $\bot$-elements.

Lemma 4.2.12. *Let $(\iota, \pi) : A \to B$ be an ep-pair in **LBD**. Then for all $x \in A$ and $a \in \mathsf{At}(x)$ it follows that $\iota(a) \in \mathsf{At}(\iota(a))$.*

Proof. Suppose $x \in A$ and $a \in \mathsf{At}(x)$. As $\pi(\iota(a)_\bot) = \pi(\iota(a))_\bot = a_\bot \neq a$ it follows that $\iota(a) \neq \iota(a)_\bot$. As $[\iota(a)]_\updownarrow$ is a complete atomic boolean algebra it suffices to show that $\iota(a)$ bistably dominates at most one atom. Hence, suppose $b, b' \in \mathsf{At}(\iota(a))$ with $b \neq b'$ and $b, b' \leq_b \iota(a)$. Thus, $b, b' <_b \iota(a)$. Then as $b \updownarrow b'$ and $b \sqcup b' \leq_b \iota(a)$ and we get $\pi(b) \sqcup \pi(b') = \pi(b \sqcup b') \leq_b \pi(\iota(a)) = a$. Thus $\pi(b) = a$ or $\pi(b') = a$ since a is an atom and $\updownarrow\{a, \pi(b), \pi(b')\}$. Assuming w.l.o.g. that $\pi(b) = a$ it follows that $\iota(\pi(b)) = \iota(a) >_b b$ in contradiction with $\iota(\pi(b)) \leq_s b$. $\square$

Lemma 4.2.13. *Let $(\iota, \pi) : A \to B$ be an ep-pair in **LBD**. Then for all $y \in B$ and $b \in \mathsf{At}(y)$ either $\pi(b) \in \mathsf{At}(\pi(b))$ or $\pi(b)_\bot = \pi(b)$.*

Proof. W.l.o.g. we assume that ι is an inclusion. Suppose $b \in \mathsf{At}(y)$. As $[\pi(b)]_\updownarrow$ is a complete atomic boolean algebra it suffices to show that from $a_1, a_2 \in \mathsf{At}(\pi(b))$ and $a_1, a_2 \leq_b \pi(b)$ it follows that $a_1 = a_2$. Hence, suppose $a_1, a_2 \in \mathsf{At}(\pi(b))$ with $a_1 \neq a_2$ and $a_1, a_2 \leq_b \pi(b)$. First we show that (†) $a_i \sqcup b_\bot = b$ holds for $i = 1, 2$. Obviously b is an upper bound for a_i and $b_\bot$ w.r.t. $\leq_s$ since $b_\bot \leq_b b$ and $a_i \leq_b \pi(b) \leq_s b$. Thus $a_i \sqcup b_\bot$ exists. Suppose $a_i, b_\bot \leq_s y \leq_s b$. Then we have $b_\bot \sqsubseteq y_\bot \sqsubseteq b_\bot$, i.e. $b_\bot = y_\bot$, from which it follows that $b_\bot \leq_b y \leq_b b$. Accordingly, as b is an atom, we have $b_\bot = y$ or $y = b$. If $b_\bot = y$ then $a_i \leq_s y = b_\bot$ from which it follows by Lemma 2.2.24 that $a_{i\bot} = a_i$ in contradiction with $a_i \in \mathsf{At}(\pi(b))$. Thus $y = b$, which finishes the proof of (†).

As $\uparrow\{a_1, a_2, b_\bot\}$ we have

$$b = b \sqcap b = (a_1 \sqcup b_\bot) \sqcap (a_2 \sqcup b_\bot) = (a_1 \sqcap a_2) \sqcup b_\bot = \pi(b_\bot)_\bot \sqcup b_\bot = b_\bot$$

contradicting the assumption that b is an atom. $\square$

4.3 Inverse Limits of Projections in **LBD**

This section is dedicated to the proof, that inverse limits exist in $\mathbf{LBD}^{\mathrm{ep}}$ and are computed in the usual way. So, let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor. Then, we write

$(\iota_{n+1,n}, \pi_{n,n+1})$ for the ep-pair $A(n, n+1) : A_n \to A_{n+1}$. The *inverse limit of* A (notation A_∞), provided it exists, has the set of all sequences $x \in \prod_{n\in\omega} A_n$ with $x_n = \pi_{n,n+1}(x_{n+1})$ for all $n \in \omega$ as underlying set. The extensional order on A_∞ is defined pointwise, i.e. $x \sqsubseteq y$ iff $x_n \sqsubseteq y_n$ for all $n \in \omega$.

Unfortunately, we can not simply define negation on A_∞ in a pointwise way. The reason is that projections in general do not commute with negation (cf. Example 4.2.11).

Definition 4.3.1. *Let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor. Then we write $(\iota_{n+1,n}, \pi_{n,n+1})$ for the ep-pair $A(n, n+1)$ for all $n \in \omega$ and set*

$$A_\infty := \{x \in \prod_{n\in\omega} A_n \mid x_n = \pi_{n,n+1}(x_{n+1}) \text{ for all } n \in \omega\}\,.$$

For elements $x, y \in A_\infty$ we write $x \sqsubseteq y$ iff $x_n \sqsubseteq y_n$ for all $n \in \omega$. $\diamond$

Further, we will use the following notation: given a functor $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ and $n < k$ then we write $\pi_{n,k}$ for $\pi_{n,n+1} \circ \cdots \circ \pi_{k-1,k}$ and $\iota_{k,n}$ for $\iota_{k,k+1} \circ \cdots \circ \iota_{n-1,n}$. For $n = k$, we put $\pi_{n,n} = \iota_{n,n} = \mathrm{id}_{A_n}$.

Lemma 4.3.2. *Let A and B be lbpds, $f : A \to B$ a bistable map and $x \in A$. Then*

$$f(\neg x) = (\neg f(x) \sqcap f(x^\top)) \sqcup f(x_\perp) = (\neg f(x) \sqcup f(x_\perp)) \sqcap f(x^\top)$$

resp.

$$\neg f(\neg x) = (f(x) \sqcup \neg f(x^\top)) \sqcap \neg f(x_\perp) = (f(x) \sqcap \neg f(x_\perp)) \sqcup \neg f(x^\top)$$

holds.

Proof. As f is bistable and $x \updownarrow \neg x$ it follows that $\updownarrow\{f(x), f(\neg x), \neg f(x)\}$, thus

$$\begin{aligned}
(\neg f(x) \sqcap f(x^\top)) \sqcup f(x_\perp) &= (\neg f(x) \sqcap f(x \sqcup \neg x)) \sqcup f(x \sqcap \neg x)\\
&= (\neg f(x) \sqcap (f(x) \sqcup f(\neg x))) \sqcup (f(x) \sqcap f(\neg x))\\
&= ((\neg f(x) \sqcap f(x)) \sqcup (\neg f(x) \sqcap f(\neg x))) \sqcup (f(x) \sqcap f(\neg x))\\
&= f(x)_\perp \sqcup (\neg f(x) \sqcap f(\neg x)) \sqcup (f(x) \sqcap f(\neg x))\\
&= (\neg f(x) \sqcap f(\neg x)) \sqcup (f(x) \sqcap f(\neg x))\\
&= (\neg f(x) \sqcup f(x)) \sqcap f(\neg x)\\
&= f(x)^\top \sqcap f(\neg x)\\
&= f(\neg x)
\end{aligned}$$

further, we have

$$\begin{aligned}
(\neg f(x) \sqcap f(x^\top)) \sqcup f(x_\perp) &= (\neg f(x) \sqcup f(x_\perp)) \sqcap (f(x^\top) \sqcup f(x_\perp))\\
&= (\neg f(x) \sqcup f(x_\perp)) \sqcap f(x^\top)
\end{aligned}$$

as desired. $\square$

In the next lemma we show that bistable maps restricted to a bistably connected component preserve arbitrary infima and suprema.

Lemma 4.3.3. *Let A and B be lbpds, $f : A \to B$ a bistable map and X be a nonempty subset of A with $\updownarrow X$ then*

(1) $f(\bigsqcup X) = \bigsqcup\{f(x) \mid x \in X\}$ *and*

(2) $f(\bigsqcap X) = \bigsqcap\{f(x) \mid x \in X\}$.

Proof. Suppose $f : A \to B$ is bistable and X is a nonempty subset of A with $\updownarrow X$.

ad (1) : As the set $\{\bigsqcup F \mid F \in \mathcal{P}_{\text{f.n.e.}}(X)\}$ is directed, it follows that

$$\begin{aligned} f(\bigsqcup X) &= f(\bigsqcup\{\bigsqcup F \mid F \in \mathcal{P}_{\text{f.n.e.}}(X)\}) \\ &= \bigsqcup\{f(\bigsqcup F) \mid F \in \mathcal{P}_{\text{f.n.e.}}(X)\} \\ &= \bigsqcup\{\bigsqcup\{f(x) \mid x \in F\} \mid F \in \mathcal{P}_{\text{f.n.e.}}(X)\} \\ &= \bigsqcup\{f(x) \mid x \in X\} \end{aligned}$$

as desired.

ad (2) : As f preserves bistable coherence it follows that for all $x \in X$ the terms $f(x)$, $f(\neg x)$, $f(X^\top)$, $f(X_\bot)$ and their respective negation happen to be bistably coherent, thus we get

$$\begin{aligned} & f(\bigsqcap X) \\ &= f(\neg\bigsqcup\{\neg x \mid x \in X\}) \\ &= (\neg f(\bigsqcup\{\neg x \mid x \in X\}) \sqcap f(X^\top)) \sqcup f(X_\bot) && (\dagger) \\ &= (\neg\bigsqcup\{f(\neg x) \mid x \in X\} \sqcap f(X^\top)) \sqcup f(X_\bot) && (\ddagger) \\ &= (\neg\bigsqcup\{(\neg f(x) \sqcup f(X_\bot)) \sqcap f(X^\top) \mid x \in X\} \sqcap f(X^\top)) \sqcup f(X_\bot) && (\dagger) \\ &= (\neg\bigsqcup\{\neg((f(x) \sqcap \neg f(X_\bot)) \sqcup \neg f(X^\top)) \mid x \in X\} \sqcap f(X^\top)) \sqcup f(X_\bot) \\ &= (\bigsqcap\{(f(x) \sqcap \neg f(X_\bot)) \sqcup \neg f(X^\top) \mid x \in X\} \sqcap f(X^\top)) \sqcup f(X_\bot) \\ &= (((\bigsqcap\{f(x) \mid x \in X\} \sqcap \neg f(X_\bot)) \sqcup \neg f(X^\top)) \sqcap f(X^\top)) \sqcup f(X_\bot) && (\S) \\ &= (((\bigsqcap\{f(x) \mid x \in X\} \sqcap \neg f(X_\bot)) \sqcap f(X^\top)) \sqcup (\neg f(X^\top) \sqcap f(X^\top))) \sqcup f(X_\bot) \\ &= (\bigsqcap\{f(x) \mid x \in X\} \sqcap \neg f(X_\bot)) \sqcup f(X^\top)_\bot \sqcup f(X_\bot) && (\P) \\ &= (\bigsqcap\{f(x) \mid x \in X\} \sqcap \neg f(X_\bot)) \sqcup f(X_\bot) \\ &= (\bigsqcap\{f(x) \mid x \in X\} \sqcup f(X_\bot)) \sqcap (\neg f(X_\bot) \sqcup f(X_\bot)) \\ &= \bigsqcap\{f(x) \mid x \in X\} \sqcap f(X_\bot)^\top && (\|) \\ &= \bigsqcap\{f(x) \mid x \in X\} && (\dagger\dagger) \end{aligned}$$

where (†) follows from Lemma 4.3.2, (‡) follows from (1), (§) follows from Thm. 2.2.35, (¶) follows as $\bigsqcap\{f(x) \mid x \in X\} \sqcap \neg f(X_\perp) \leq_b f(X^\top)$ (since $f(x) \leq_b f(X^\top)$ for all $x \in X$), (||) follows as $f(X_\perp) \leq_b f(x)$ for all $x \in X$ and (††) follows as $f(x) \leq_b f(X_\perp)^\top$ for all $x \in X$. □

Lemma 4.3.4. *Let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor and $x \in A_\infty$. Then for all $n, k \in \omega$ with $n \leq k$ we have*

(1) $\pi_{n,k}(\neg x_k) \leq_b \pi_{n,k'}(\neg x_{k'})$ *for all $k' \in \omega$ with $n \leq k' \leq k$*

(2) $\bigsqcap_{k \geq n} \pi_{n,k}(\neg x_k) \in A_n$

(3) $\bigsqcap_{k \geq n} \pi_{n,k}(\neg x_k) = \pi_{n,n+1}(\bigsqcap_{k \geq n+1} \pi_{n+1,k}(\neg x_k))$

Proof. Suppose $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ is a functor and $x \in A_\infty$.

ad (1) : Suppose $k' \in \omega$ with $n \leq k' \leq k$. Then we have $\pi_{n,k}(\neg x_k) = \pi_{n,k'}(\pi_{k',k}(\neg x_k))$. As $\pi_{k,k'}(\neg x_{k'}) \leq_b \neg \pi_{k,k'}(x_{k'})$ by Cor. 4.2.4(3), it follows that

$$\pi_{n,k}(\neg x_k) \leq_b \pi_{n,k'}(\neg \pi_{k,k'}(x_k)) = \pi_{n,k}(\neg x_{k'})$$

as desired.

ad (2) : From (1) it follows that $\updownarrow\{\pi_{n,k}(\neg x_k) \mid k \geq n\}$. As bistably connected components are complete boolean algebras it follows that $\bigsqcap_{k \geq n} \pi_{n,k}(\neg x_k)$ exists and is in A_n.

ad (3) : From Lemma 4.3.3(2) it follows that

$$\begin{aligned}
\pi_{n,n+1}(\bigsqcap_{k \geq n+1} \pi_{n+1,k}(\neg x_k)) &= \bigsqcap_{k \geq n+1} \pi_{n,n+1}(\pi_{n+1,k}(\neg x_k)) \\
&= \bigsqcap_{k \geq n+1} \pi_{n,k}(\neg x_k) \\
&= \bigsqcap_{k \geq n} \pi_{n,k}(\neg x_k)
\end{aligned}$$

where the last equation follows from (1). □

Using Lemma 4.3.4 we can define negation A_∞. Notice that the sequence in the definition of $(\neg x)_n$ is decreasing w.r.t. the bistable order. Taking the infimum of this sequence eliminates all those occurrences of $\top$ in $\neg x_n$ whose corresponding occurrences of $\perp$ in x_n "develop" to something different from $\perp$ in subsequent x_k.

Definition 4.3.5. *Let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor and $x \in A_\infty$. Then we define $\neg x : A_\infty \to A_\infty$ by*

$$(\neg x)_n := \bigsqcap_{k \geq n} \pi_{n,k}(\neg x_k)\,.$$

◇

Lemma 4.3.6. *Let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor and $x \in A_\infty$. Then for all $n \in \omega$ we have $(\neg x)_n \updownarrow x_n$ and $(\neg x)_n \leq_b \neg x_n$.*

Proof. Suppose $x \in A_\infty$ and $n \in \omega$. Then we have $x_n = \pi_{n,k}(x_k) \updownarrow \pi_{n,k}(\neg x_k)$ for all $k \in \omega$ with $n \leq k$. Thus $\updownarrow\{\pi_{n,k}(\neg x_k) \mid k \geq n\}$. From Lemma 2.2.34 it follows that $\updownarrow\{\pi_{n,k}(\neg x_k) \mid k \geq n\} \cup \{(\neg x)_n\}$. Thus, $(\neg x)_n \updownarrow x_n$ and $(\neg x)_n = \bigsqcap_{k \geq n} \pi_{n,k}(\neg x_k) \leq_b \pi_{n,n}(\neg x_n) = \neg x_n$ as desired. □

Lemma 4.3.7. *Let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor and $(\iota_{n+1,n}, \pi_{n,n+1}) = A(n, n+1) : A_n \to A_{n+1}$ for all $n \in \omega$. Then*

$$\bigsqcap_{j \geq i} \bigsqcup_{k \geq j} \pi_{i,j}(\neg \pi_{j,k}(x_k^\top)) = x_{i\perp}$$

holds for all $i \in \omega$ and $x \in A_\infty$.

Proof. Suppose $i \in \omega$ and $x \in A_\infty$. Then

$$\begin{aligned}
\bigsqcap_{j \geq i} \bigsqcup_{k \geq j} \pi_{i,j}(\neg \pi_{j,k}(x_k^\top)) &= \bigsqcap_{j \geq i} \bigsqcup_{k \geq j} \left(\pi_{i,j}(\pi_{j,k}(x_k^\top)^\top) \sqcap \neg \pi_{i,j}(\pi_{j,k}(x_k^\top))\right) && (\dagger)\\
&= \bigsqcap_{j \geq i} \bigsqcup_{k \geq j} \left(\pi_{i,j}(x_j^\top) \sqcap \neg \pi_{i,k}(x_k^\top)\right)\\
&= \bigsqcap_{j \geq i} \left(\pi_{i,j}(x_j^\top) \sqcap \bigsqcup_{k \geq j} \neg \pi_{i,k}(x_k^\top)\right)\\
&= \bigsqcap_{j \geq i} \left(\pi_{i,j}(x_j^\top) \sqcap \neg \bigsqcap_{k \geq j} \pi_{i,k}(x_k^\top)\right)\\
&= \bigsqcap_{j \geq i} \left(\pi_{i,j}(x_j^\top) \sqcap \neg \bigsqcap_{k \geq i} \pi_{i,k}(x_k^\top)\right) && (\ddagger)\\
&= \left(\bigsqcap_{j \geq i} \pi_{i,j}(x_j^\top)\right) \sqcap \left(\neg \bigsqcap_{k \geq i} \pi_{i,k}(x_k^\top)\right)\\
&= x_i \sqcap \neg x_i\\
&= x_{i\perp}
\end{aligned}$$

where $(\dagger)$ follows from Cor. 4.2.4(4) and $(\ddagger)$ holds since $\pi_{k',k}(x_k^\top) \leq_b x_{k'}^\top$ (since $x_{k'} = \pi_{k',k}(x_k)$) and therefore $\pi_{i,k}(x_k^\top) \leq_b \pi_{i,k'}(x_{k'}^\top)$ for all $k \geq k' \geq i$. □

Lemma 4.3.8. *Let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor. Then for all $x \in A_\infty$ and $n \in \omega$ it holds that $(\neg\neg x)_n = x_n$.*

Proof. Suppose $x \in A_\infty$ and $n \in \omega$. Then we have

$$\begin{aligned}
(\neg\neg x)_n &= \bigsqcap_{k \geq n} \pi_{n,k}(\neg(\neg x)_k) \\
&= \bigsqcap_{k \geq n} \pi_{n,k}(\neg \bigsqcap_{l \geq k} \pi_{k,l}(\neg x_l)) \\
&= \bigsqcap_{k \geq n} \pi_{n,k}(\bigsqcup_{l \geq k} \neg \pi_{k,l}(\neg x_l)) \\
&= \bigsqcap_{k \geq n} \bigsqcup_{l \geq k} \pi_{n,k}(\neg \pi_{k,l}(\neg x_l)) && (\dagger) \\
&= \bigsqcap_{k \geq n} \bigsqcup_{l \geq k} \pi_{n,k}(\pi_{k,l}(x_l) \sqcup \neg \pi_{k,l}(x_l^\top)) && (\ddagger) \\
&= \bigsqcap_{k \geq n} \bigsqcup_{l \geq k} \pi_{n,k}(\pi_{k,l}(x_l)) \sqcup \pi_{n,k}(\neg \pi_{k,l}(x_l^\top)) \\
&= \bigsqcap_{k \geq n} \bigsqcup_{l \geq k} x_n \sqcup \pi_{n,k}(\neg \pi_{k,l}(x_l^\top)) \\
&= x_n \sqcup \bigsqcap_{k \geq n} \bigsqcup_{l \geq k} \pi_{n,k}(\neg \pi_{k,l}(x_l^\top)) && (\S) \\
&= x_n \sqcup x_{n\perp} && (\P) \\
&= x_n
\end{aligned}$$

where $(\dagger)$ follows as $\neg \pi_{k,l}(\neg x_l)$ is ascending w.r.t. $\leq_b$, $(\ddagger)$ follows by Cor. 4.2.4(4), $(\S)$ by Lemma 2.2.34 and $(\P)$ by Lemma 4.3.7. □

Using the above lemma we now can show that $\neg : A_\infty \to A_\infty$ is an involution.

Lemma 4.3.9. *Let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor and $(\iota_{n+1,n}, \pi_{n,n+1})$ ep-pairs $A(n, n+1) : A_n \to A_{n+1}$ for all $n \in \omega$. Then for all $x, y \in A_\infty$ we have that $x \sqsubseteq y$ implies $\neg y \sqsubseteq \neg x$.*

Proof. Suppose $x, y \in A_\infty$ with $x \sqsubseteq y$. As $x_n \sqsubseteq y_n$ for all $n \in \omega$ it follows that $\neg y_n \sqsubseteq \neg x_n$ for all $n \in \omega$. Thus $\pi_{n,k}(\neg y_k) \sqsubseteq \pi_{n,k}(\neg x_k)$ for all $n, k \in \omega$ with $n \leq k$. Hence, we have

$$(\neg y)_n = \bigsqcap_{k \geq n} \pi_{n,k}(\neg y_k) \sqsubseteq \bigsqcap_{k \geq n} \pi_{n,k}(\neg x_k) = (\neg x)_n$$

for all $n \in \omega$ as desired. □

Next we show that for all $x \in A_\infty$ the infimum $x \sqcap \neg x$ (resp. the supremum $x \sqcup \neg x$) exist and is computed pointwise.

Lemma 4.3.10. *Let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor and $x \in A_\infty$ then $x \sqcap \neg x = (x_n \sqcap (\neg x)_n)_{n \in \omega}$ and $x \sqcup \neg x = (x_n \sqcup (\neg x)_n)_{n \in \omega}$.*

Proof. Suppose $x \in A_\infty$. From Lemma 4.3.6 it follows that $x_n \updownarrow (\neg x)_n$ for all $n \in \omega$. Thus,

$$\pi_{n,n+1}(x_{n+1} \sqcap (\neg x)_{n+1}) = \pi_{n,n+1}(x_{n+1}) \sqcap \pi_{n,n+1}((\neg x)_{n+1}) = x_n \sqcap (\neg x)_n$$

and

$$\pi_{n,n+1}(x_{n+1} \sqcup (\neg x)_{n+1}) = \pi_{n,n+1}(x_{n+1}) \sqcup \pi_{n,n+1}((\neg x)_{n+1}) = x_n \sqcup (\neg x)_n \,.$$

Thus, we have $(x_n \sqcap (\neg x)_n)_{n\in\omega}, (x_n \sqcup (\neg x)_n)_{n\in\omega} \in A_\infty$. Since $(x_n \sqcap (\neg x)_n)_{n\in\omega}$ (resp. $(x_n \sqcup (\neg x)_n)_{n\in\omega}$) is the infimum (resp. supremum) of x and $\neg x$ in $\prod_{n\in\omega} A_n$ it follows that it is also the infimum (resp. supremum) in A_∞. □

Lemma 4.3.11. *Let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor and $x \in A_\infty$. Then $x_\perp = ((x_n)_\perp)_{n\in\omega}$.*

Proof. Suppose $x \in A_\infty$ and $n \in \omega$. From Lemma 4.3.10 it follows that $(x \sqcap \neg x)_n = x_n \sqcap (\neg x)_n$ and from Lemma 4.3.6 it follows that $x_n \updownarrow (\neg x)_n$. As $(\neg x)_n \leq_b \neg x_n$ by Lemma 4.3.6 we get $x_n \sqcap (\neg x)_n \leq_b x_n \sqcap \neg x_n = (x_n)_\perp$. Thus, $x_n \sqcap (\neg x)_n = (x_n)_\perp$ as desired. □

Lemma 4.3.12. *Let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor. If $x \in A_\infty$ and $n \in \omega$ then*

$$(x^\top)_n = \bigsqcap_{k\geq n} \pi_{n,k}((x_k)^\top) \leq_b (x_n)^\top \,.$$

Proof. The inequality $(x^\top)_n \leq_b (x_n)^\top$ follows as $x_n \updownarrow (\neg x)_n$ for all $n \in \omega$ and by Lemma 4.3.10 the supremum $x \sqcup \neg x$ is computed pointwise. As $x_\perp = ((x_n)_\perp)_{n\in\omega}$ by Lemma 4.3.11, we get

$$\begin{aligned}
(x^\top)_n &= (\neg(x_\perp))_n \\
&= \bigsqcap_{k\geq n} \pi_{n,k}(\neg(x_\perp)_k) \\
&= \bigsqcap_{k\geq n} \pi_{n,k}(\neg(x_k)_\perp) \\
&= \bigsqcap_{k\geq n} \pi_{n,k}((x_k)^\top)
\end{aligned}$$

as desired. □

Lemma 4.3.13. *Let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor and $x, y \in A_\infty$. Then $x \sqsubseteq y^\top$ holds iff $x_n \sqsubseteq y_n^\top$ holds for all $n \in \omega$.*

Proof. Suppose $x, y \in A_\infty$. The forward implication is trivial as $(y^\top)_n \sqsubseteq y_n^\top$ by Lemma 4.3.12 for all $n \in \omega$. For the reverse implication suppose $x_n \sqsubseteq y_n^\top$ holds for all $n \in \omega$. Thus $x_n = \pi_{n,k}(x_k) \sqsubseteq \pi_{n,k}(y_k^\top)$ for all $n \leq k$ and it follows, also by Lemma 4.3.12, that $x_n \sqsubseteq \bigsqcap_{k\geq n} \pi_{n,k}({y_k}^\top) = (y^\top)_n$ for all $n \in \omega$ as desired. □

Lemma 4.3.14. *Let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor and $x, y \in A_\infty$. Then we have*

$$
\begin{array}{llcll}
(1) & x \uparrow y & \Leftrightarrow & x_n \uparrow y_n & \text{for all } n \in \omega \\
(2) & x \downarrow y & \Leftrightarrow & x_n \downarrow y_n & \text{for all } n \in \omega \\
(3) & x \updownarrow y & \Leftrightarrow & x_n \updownarrow y_n & \text{for all } n \in \omega \\
(4) & x \leq_s y & \Leftrightarrow & x_n \leq_s y_n & \text{for all } n \in \omega \\
(5) & x \leq_c y & \Leftrightarrow & x_n \leq_c y_n & \text{for all } n \in \omega \\
(6) & x \leq_b y & \Leftrightarrow & x_n \leq_b y_n & \text{for all } n \in \omega
\end{array}
$$

Proof. Suppose $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ is a functor and $x, y \in A_\infty$.

ad (1) : This follows immediately from Lemma 4.3.13.

ad (2) : Using Lemma 4.3.11 we get $(z_\perp)_n = (z_n)_\perp$ for all $z \in A_\infty$. Thus, we have $x \downarrow y$ iff $x_\perp \sqsubseteq y$ and $y_\perp \sqsubseteq x$ iff $x_{n\perp} \sqsubseteq y_n$ and $y_{n\perp} \sqsubseteq x_n$ for all $n \in \omega$ iff $x_n \downarrow y_n$ for all $n \in \omega$.

ad (3) : This follows immediately from (1) and (2).

ad (4), (5) and (6) : These are immediate consequences of (1), (2) and (3) and the fact that $\sqsubseteq$ is defined pointwise on A_∞. □

Lemma 4.3.15. *Let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor and $x, y \in A_\infty$.*

(1) *If $x \uparrow y$ then $x \sqcap y = (x_n \sqcap y_n)_{n\in\omega}$ and $x \sqcup y = (x_n \sqcup y_n)_{n\in\omega}$.*

(2) *If $x \downarrow y$ then $x \sqcup y = (x_n \sqcup y_n)_{n\in\omega}$.*

Proof. Suppose $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ is a functor and $x, y \in A_\infty$.

ad (1) : Suppose $x, y \in A_\infty$ with $x \uparrow y$. Then by Lemma 4.3.14 we have $x_n \uparrow y_n$ for all $n \in \omega$. Thus, $x_n \sqcap y_n$ and $x_n \sqcup y_n$ exist for all $n \in \omega$. From Cor. 2.3.4 it follows that

$$\pi_{n,n+1}(x_{n+1} \sqcap y_{n+1}) = \pi_{n,n+1}(x_{n+1}) \sqcap \pi_{n,n+1}(y_{n+1}) = x_n \sqcap y_n$$

and from Lemma 4.2.10 it follows that

$$\pi_{n,n+1}(x_{n+1} \sqcup y_{n+1}) = \pi_{n,n+1}(x_{n+1}) \sqcup \pi_{n,n+1}(y_{n+1}) = x_n \sqcup y_n\,.$$

Further $(x_n \sqcap y_n)_{n\in\omega}$ (resp. $(x_n \sqcup y_n)_{n\in\omega}$) is the pointwise infimum (resp. supremum) of x and y, thus the infimum (resp. supremum) in $\prod_{n\in\omega} A_n$ and hence also in A_∞.

ad (2) : Suppose $x, y \in A_\infty$ with $x \downarrow y$ then by Lemma 4.3.14 we have $x_n \downarrow y_n$ for all $n \in \omega$. Thus, $x_n \sqcup y_n$ exists for all $n \in \omega$. From Cor. 2.3.4 it follows that

$$\pi_{n,n+1}(x_{n+1} \sqcup y_{n+1}) = \pi_{n,n+1}(x_{n+1}) \sqcup \pi_{n,n+1}(y_{n+1}) = x_n \sqcup y_n\,.$$

As $(x_n \sqcup y_n)_{n\in\omega}$ is the pointwise supremum of x and y it the supremum in $\prod_{n\in\omega} A_n$ and hence also in A_∞. □

Lemma 4.3.16. *Let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor. Then A_∞ is a lbo.*

Proof. From Lemma 4.3.9 and Lemma 4.3.8 it follows that $\neg : |A_\infty| \to |A_\infty|$ is an involution. Lemma 4.3.10 ensures the existence of $x^\top$ and $x_\perp$ for all $x \in |A_\infty|$, and Lemma 4.3.15 provides binary infima and suprema of stably coherent elements. $\square$

Lemma 4.3.17. *Let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor then $(A_\infty, \sqsubseteq)$ is a cpo. If X is a directed subset of A_∞ then $\bigsqcup X = (\bigsqcup\{x_n \mid x \in X\})_{n\in\omega}$.*

Proof. Suppose $X \subseteq A_\infty$ is $\sqsubseteq$-directed. As the extensional order $\sqsubseteq$ is given pointwise on A_∞ it follows that for all $n \in \omega$ the set $\{x_n \mid x \in X\}$ is directed w.r.t. $\sqsubseteq$. As $\pi_{n,n+1}(\bigsqcup\{x_{n+1} \mid x \in X\}) = \bigsqcup\{\pi_{n,n+1}(x_{n+1}) \mid x \in X\} = \bigsqcup\{x_n \mid x \in X\}$ and $(\bigsqcup\{x_n \mid x \in X\})_{n\in\omega}$ is the supremum of X in $\prod_{n\in\omega} A_n$ it follows that it is also the supremum in A_∞. $\square$

Next we study finite prime elements of A_∞.

Definition 4.3.18. *If $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ is a functor. Then we define $\pi_n : A_\infty \to A_n$ by*

$$\pi_n(x) := x_n$$

and $\iota_n : A_n \to A_\infty$ by

$$(\iota_n(x))_k := \begin{cases} \pi_{k,n}(x) & \text{if } k \leq n, \\ \iota_{k,n}(x) & \text{otherwise,} \end{cases}$$

for all $k \in \omega$. Further, we put $r_n := \iota_n \circ \pi_n$ for all $n \in \omega$. $\diamond$

Lemma 4.3.19. *For all $n \in \omega$ the functions $\pi_n : A_\infty \to A_n$, $\iota_n : A_n \to A_\infty$ and $r_n : A_\infty \to A_\infty$ from Def. 4.3.18 are bistable.*

Proof. This follows as $\pi_{k,n}$ and $\iota_{k,n}$ are bistable for all $k \leq n$ and using Lemma 4.3.14. $\square$

Lemma 4.3.20. *Let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor, $x \in A_\infty$ and $y \in A_n$ for some $n \in \omega$. Then*

(1) *$y \sqsubseteq x_n$ implies $\iota_n(y) \sqsubseteq x$ and*

(2) *$y \leq_s x_n$ implies $\iota_n(y) \leq_s x$.*

Proof. Suppose $x \in A_\infty$ and $y \in A_n$.

ad (1) : If $y \sqsubseteq x_n$ then $(\iota_n(y))_n = y \sqsubseteq x_n$. Let $k \in \omega$, if $k \leq n$ then $(\iota_n(x))_k = \pi_{k,n}(y) \sqsubseteq \pi_{k,n}(x_n) = x_k$. If $k > n$ then $(\iota_n(x))_k = \iota_{k,n}(y) \sqsubseteq \iota_{k,n}(x_n) \sqsubseteq x_k$. Thus, $\iota_n(y) \sqsubseteq x$ as desired.

ad (2) : If $y \leq_s x_n$ then $(\iota_n(y))_n = y \leq_s x_n$. Let $k \in \omega$, if $k \leq n$ then $(\iota_n(x))_k = \pi_{k,n}(y) \leq_s \pi_{k,n}(x_n) = x_k$. If $k > n$ then $(\iota_n(x))_k = \iota_{k,n}(y) \leq_s \iota_{k,n}(x_n) = \iota_{k,n}(\pi_{n,k}(x_k)) \leq_s x_k$. Thus, $\iota_n(y) \leq_s x$ as desired. $\square$

Lemma 4.3.21. *Let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor then*

$$\mathsf{FP}(A_\infty) = \{\iota_n(p) \mid n \in \omega \text{ and } p \in \mathsf{FP}(A_n)\}.$$

Proof. Suppose $p \in \mathsf{FP}(A_\infty)$ then the set $\{q \in A_\infty \mid q \leq_s p\}$ is finite. Let $n := |\{q \in A_\infty \mid q \leq_s p\}|$. By Lemma 4.3.20 n is an upper bound for $|\{q \in A_i \mid q \leq_s p_i\}|$ for all $i \in \omega$. Further, there exists a $k \in \omega$ such that $|\{q \in A_k \mid q \leq_s p_k\}| = n$. As p_{i+1} stably dominates at least as many elements as p_i it follows that $|\{q \in A_l \mid q \leq_s p_l\}| = n$ for all $l \geq k$. As all the maps $\iota_{l,k}$ are injective and preserve the stable order it follows that $|\{r \in A_l \mid r \leq_s \iota_{l,k}(p_k)\}| \geq n$. Thus, as $\iota_{l,k}(p_k) \leq_s p_l$ we get $\iota_{l,k}(p_k) = p_l$ for all $l \geq k$. Hence, $p = \iota_k(p_k)$ and p_k is finite.

For showing that p_k is prime suppose $x, y \in A_k$ with $x \uparrow y$ or $x \downarrow y$. If $p_k \sqsubseteq x \sqcup y$ then from Lemma 4.3.20 it follows that $p \sqsubseteq \iota_k(x \sqcup y)$. Using Cor. 2.3.4 (in case $x \downarrow y$) and Lemma 4.2.10 (in case $x \uparrow y$) we get $\pi_{l,k}(x \sqcup y) = \pi_{l,k}(x) \sqcup \pi_{l,k}(y)$ for all $l \leq k$ and $\iota_{l,k}(x \sqcup y) = \iota_{l,k}(x) \sqcup \iota_{l,k}(y)$ for all $l > k$. Thus, it follows that $\iota_k(x \sqcup y) = \iota_k(x) \sqcup \iota_k(y)$. As $\iota_k(x) \uparrow \iota_k(y)$ or $\iota_k(x) \downarrow \iota_k(y)$ (since ι_k is bistable) and $p \sqsubseteq \iota_k(x \sqcup y) = \iota_k(x) \sqcup \iota_k(y)$ we get $p \sqsubseteq \iota_k(x)$ or $p \sqsubseteq \iota_k(y)$. Thus, $p_k \sqsubseteq x$ or $p_k \sqsubseteq y$ as desired.

For the reverse inclusion suppose $p \in \mathsf{FP}(A_k)$ for some $k \in \omega$ and $y, z \in A_\infty$ with $y \uparrow z$ or $y \downarrow z$. If $\iota_k(p) \sqsubseteq y \sqcup z$ then it follows from Lemma 4.3.15 that $(y \sqcup z)_k = y_k \sqcup z_k$, and from Lemma 4.3.14 that $y_k \uparrow z_k$ or $y_k \downarrow z_k$. As $(\iota_k(p))_k \sqsubseteq y_k \sqcup z_k$ and $(\iota_k(p))_k = p \in \mathsf{FP}(A_k)$ we get $p \sqsubseteq y_k$ or $p \sqsubseteq z_k$. W.l.o.g. assume $p \sqsubseteq y_k$. Then by Lemma 4.3.20 it follows that $\iota_k(p) \sqsubseteq y$. Thus, $\iota_k(p) \in \mathsf{P}(A_\infty)$. Suppose $l > k$. From Lemma 4.2.6 it follows that $\iota_{l,k}(\pi_{k,l}(r)) = r$ for all $r \leq_s \iota_{l,k}(p)$. Thus, if $x \in A_\infty$ with $x \leq_s \iota_k(p)$ then $x_l \leq_s \iota_k(p)_l = \iota_{l,k}(p)$ and it follows that $x = \iota_k(x_k) = \iota_k(q)$ for some $q \leq_s p$. As p is finite there exist only finitely many q with $q \leq_s p$. Thus, $\iota_k(p)$ is also finite. $\square$

Lemma 4.3.22. *Let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor and $x \in A_\infty$ then*

$$\mathsf{FP}(x) = \{\iota_n(p) \mid n \in \omega \text{ and } p \in \mathsf{FP}(x_n)\}.$$

Proof. Suppose $x \in A_\infty$. Then we have $\mathsf{FP}(x) = \{y \in \mathsf{FP}(A_\infty) \mid y \leq_s x\}$. Thus, from Lemma 4.3.21 it follows that

$$\mathsf{FP}(x) = \{\iota_n(q) \mid n \in \omega, q \in \mathsf{FP}(A_n) \text{ and } \iota_n(q) \leq_s x\}.$$

Using Lemma 4.3.14 we get

$$\begin{aligned} \mathsf{FP}(x) &= \{\iota_n(q) \mid n \in \omega, q \in \mathsf{FP}(A_n) \text{ and } q \leq_s x_n\} \\ &= \{\iota_n(p) \mid n \in \omega \text{ and } p \in \mathsf{FP}(x_n)\} \end{aligned}$$

as desired. $\square$

Theorem 4.3.23. *Let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor. Then A_∞ is a lbd.*

Proof. We already know from Lemma 4.3.16 and Lemma 4.3.17 that A_∞ is a complete lbo. Further, as every A_n has a least element $\bot$ it follows that A_∞ has also a least element $\bot$. Thus, A_∞ is pointed. It remains to show that A_∞ fulfils the requirements (1) and (2) of Def. 2.2.3:

ad (1) : Suppose $x \in A_\infty$. Then it follows from Lemma 4.3.22 that $\mathsf{FP}(x) = \{\iota_n(p) \mid n \in \omega \text{ and } p \in \mathsf{FP}(x_n)\}$. For all $n \in \omega$ and $p \in \mathsf{FP}(x_n)$ we have $(\iota_n(p))_n = p$ and it follows that $x_n \leq_s (\bigsqcup\{\iota_n(p) \mid p \in \mathsf{FP}(x_n)\})_n$. As $\iota_n(p) \leq_s x$ for all $n \in \omega$ and $p \in \mathsf{FP}(x_n)$ by Lemma 4.3.20, it follows that $x_n = (\bigsqcup\{\iota_n(p) \mid p \in \mathsf{FP}(x_n)\})_n$ Thus, $x = \bigsqcup \mathsf{FP}(x)$ as desired.

ad (2) : Suppose $p \in \mathsf{FP}(A_\infty)$, X a directed subset of A_∞ and $p \sqsubseteq \bigsqcup X$. From Lemma 4.3.21 it follows that there exists an $i \in \omega$ and a $q \in A_i$ with $p = \iota_i(q)$. Further, $\bigsqcup X = (\bigsqcup\{x_n \mid x \in X\})_{n\in\omega}$ by Lemma 4.3.17. Thus, there exists an $x \in X$ with $q \sqsubseteq x_i$. From Lemma 4.3.20 it follows that $\iota_i(q) \sqsubseteq x$. Thus, $p \sqsubseteq x$ as desired. □

Theorem 4.3.24. *Let $A : \omega \to \mathbf{LBD}^{\mathrm{ep}}$ be a functor then the following holds:*

(1) *The locally boolean domain A_∞ together with the morphisms $\pi_n : A_\infty \to A_n$ for all $n \in \omega$ is a limit over the diagram $\underleftarrow{D} := ((A_n)_{n\in\omega}, (\pi_{n,n+1})_{n\in\omega})$ in* **LBD**.

(2) *The locally boolean domain A_∞ together with the morphisms $\iota_n : A_n \to A_\infty$ for all $n \in \omega$ is a colimit over the diagram $\underrightarrow{D} := ((A_n)_{n\in\omega}, (\iota_{n+1,n})_{n\in\omega})$ in* **LBD**.

(3) *(ι_n, π_n) is an ep-pair for every $n \in \omega$.*

(4) $\bigsqcup_{n\in\omega} \iota_n \circ \pi_n = \mathrm{id}_{A_\infty}$

Proof. ad (1) : The construction of the limiting object A_∞ and the projections π_n is the usual one for categories of domains. Thus, having another cone $(B, (f_n)_{n\in\omega})$ over the diagram $\underleftarrow{D}$ it follows that the function $f : B \to A_\infty$ with $f(x) := (f_n(x))_{n\in\omega}$ is Scott continuous. Thus, it remains to show that f is bistable. Suppose $x, y \in B$ with $x \updownarrow y$. Then it follows that $f_n(x \sqcap y) = f_n(x) \sqcap f_n(y)$ and $f_n(x \sqcup y) = f_n(x) \sqcup f_n(y)$ for all $n \in \omega$ and we get

$$\begin{aligned} f(x \sqcap y) &= (f_n(x \sqcap y))_{n\in\omega} \\ &= (f_n(x) \sqcap f_n(y))_{n\in\omega} \\ &= (f_n(x))_{n\in\omega} \sqcap (f_n(y))_{n\in\omega} && (\dagger) \\ &= f(x) \sqcap f(y) \end{aligned}$$

and

$$\begin{aligned} f(x \sqcup y) &= (f_n(x \sqcup y))_{n\in\omega} \\ &= (f_n(x) \sqcup f_n(y))_{n\in\omega} \\ &= (f_n(x))_{n\in\omega} \sqcup (f_n(y))_{n\in\omega} && (\dagger) \\ &= f(x) \sqcup f(y) \end{aligned}$$

where (†) holds as $f_n(x) \updownarrow f_n(y)$ for all $n \in \omega$ and the infimum (resp. supremum) is computed pointwise (by Lemma 4.3.15).

ad (2) : Like in the proof of (1) it remains to show that if $(B, (f_n)_{n\in\omega})$ is a co-cone over the diagram $\underrightarrow{D}$ then the function $f : A_\infty \to B$ with $f(x) := \bigsqcup_{n\in\omega} f_n(x_n)$ is bistable. First, notice that for $x \in A_\infty$ we have $f_n(x_n) = f_{n+1}(\iota_{n+1,n}(x_n)) = f_{n+1}(\iota_{n+1,n}(\pi_{n,n+1}(x_{n+1}))) \leq_s f_{n+1}(x_{n+1})$, thus we have

$$f_n(x_n) \leq_s f_{n+1}(x_{n+1}) \tag{†}$$

for all $n \in \omega$. Suppose $x, y \in A_\infty$. If $x \updownarrow y$ then it follows that $x_n \updownarrow y_n$ and $f_n(x_n) \updownarrow f_n(y_n)$ for all $n \in \omega$. Thus, using Lemma 2.2.28 it follows that $f(x) \updownarrow f(y)$. Further, we have $f_n(x_n \sqcup y_n) = f_n(x_n) \sqcup f_n(y_n)$ for all $n \in \omega$ and it follows that

$$\begin{aligned}
f(x \sqcup y) &= \bigsqcup_{n\in\omega} f_n((x \sqcup y)_n) \\
&= \bigsqcup_{n\in\omega} f_n(x_n \sqcup y_n) \\
&= \bigsqcup_{n\in\omega} f_n(x_n) \sqcup f_n(y_n) \\
&= \bigsqcup_{n\in\omega} f_n(x_n) \sqcup \bigsqcup_{n\in\omega} f_n(y_n) \\
&= f(x) \sqcup f(y)
\end{aligned}$$

From monotonicity of f we get $f(x \sqcap y) \sqsubseteq f(x) \sqcap f(y)$. Suppose $p \in \mathsf{FP}(\bigsqcup_{n\in\omega} f_n(x_n) \sqcap \bigsqcup_{n\in\omega} f_n(y_n))$ then $p \in \mathsf{FP}(\bigsqcup_{n\in\omega} f_n(x_n))$ and $p \in \mathsf{FP}(\bigsqcup_{n\in\omega} f_n(y_n))$. Thus, there exists an $i \in \omega$ such that $p \sqsubseteq f_i(x_i) \sqcap f_i(y_i) \sqsubseteq f(x) \sqcap f(y)$ as desired.

ad (3) : Suppose $n \in \omega$. Then we have $\pi_n \circ \iota_n(x) = \pi_n(\iota_n(x)) = \iota_n(x)_n = x$, thus $\pi_n \circ \iota_n = \mathrm{id}_{A_n}$. If $x, y \in A_\infty$ and $x \leq_s y$ then for all $n, k \in \omega$ it follows that

$$((\iota_n \circ \pi_n)(x))_k = (\iota_n(x_n))_k = \begin{cases} \pi_{k,n}(x_n) & \text{if } k \leq n, \\ \iota_{k,n}(x_n) & \text{if } k > n, \end{cases}$$

$$((\iota_n \circ \pi_n)(y) \sqcap x)_k = \iota_n(y_n)_k \sqcap x_k = \begin{cases} \pi_{k,n}(x_n) & \text{if } k \leq n, \\ \iota_{k,n}(y_n) \sqcap x_k & \text{if } k > n. \end{cases}$$

As $\iota_{k,n} \circ \pi_{n,k} \leq_s \mathrm{id}_{A_k}$ for all $k > n$ and $x_k \leq_s y_k$ it follows that $\iota_{k,n} \circ \pi_{n,k}(x_k) = \iota_{k,n} \circ \pi_{n,k}(y_k) \sqcap x_k$. Thus, we have $\iota_{k,n}(x_n) = \iota_{k,n}(y_n) \sqcap x_k$ as desired.

ad (4) : We have $\iota_n \circ \pi_n \leq_s \mathrm{id}_{A_\infty}$ for all $n \in \omega$ thus $\bigsqcup_{n\in\omega} \iota_n \circ \pi_n \leq_s \mathrm{id}_{A_\infty}$. On the other hand, we have $(\mathrm{id}_{A_\infty}(x))_m = (\iota_m \circ \pi_m(x))_m \leq_s \left(\bigsqcup_{n\in\omega} \iota_n \circ \pi_n(x)\right)_m$. Thus, $\bigsqcup_{n\in\omega} \iota_n \circ \pi_n = \mathrm{id}_{A_\infty}$. □

4.4 Countably based Locally Boolean Domains

We will now restrict to those locally boolean (pre)domains where the set of finite prime elements is countable. Notice that countably based lbds will be sufficient for the interpretation of the language SPCF_∞ (cf. chapter 5).

Further, adapting a result of J. Longley in [Lon02] we show that every countably based locally boolean domain appears as retract of $\mathsf{U} = [\mathsf{N}{\to}\mathsf{N}]$ where N are the bilifted natural numbers, i.e. that U is a universal object for countably based locally boolean domains.

Definition 4.4.1. *A lbpd (resp. lbd) A is* countably based *(a* cblbpd *(resp.* cblbd*)) iff the set $\mathsf{FP}(A)$ is countable.*
*We write ω**LBPD** (resp. ω**LBD**) for the category of countably based locally boolean (pre)domains and sequential maps.* ◇

Obviously, for a lbpd A the set $\mathsf{FP}(A)$ is countable iff $\mathsf{F}(A)$ is countable.

Lemma 4.4.2. *The category ω**LBD** has countable bilifted sums and products, is cartesian closed, and is closed under inverse limits of ω-chains of embedding projection pairs.*

Proof. It is well known that the category **OSA** restricted to Curien-Lamarche games with countable sets of cells and values forms a cartesian closed category and this category is obviously equivalent to ω**LBD**.

The other statements are left as an exercise to the reader. □

We now define abbreviations for some frequently used cblbds.

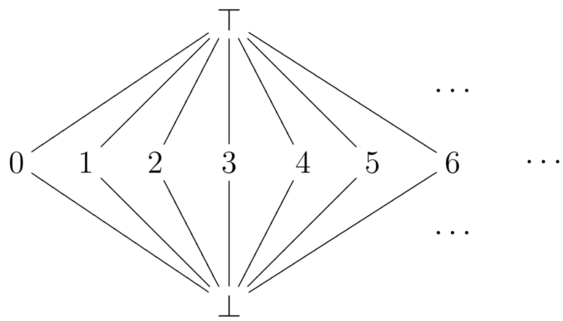
Definition 4.4.3. *We define the following cblbds:*

$$
\begin{aligned}
\mathbb{1} &:= \prod_{i\in\emptyset} && \text{(The empty product.)}\\
\mathsf{O} &:= \sum_{i\in\emptyset} && \text{(The empty sum.)}\\
\mathsf{N} &:= \sum_{i\in\omega} \mathbb{1} && \text{(The bilifted natural numbers.)}\\
\mathsf{U} &:= [\mathsf{N}{\to}\mathsf{N}] && \text{(The bistable endomaps on } \mathsf{N}.)
\end{aligned}
$$

Further, given a lbpd A we introduce the abbreviation A^ω for $\prod_{i\in\omega} A$. ◇

We call the type O the *type of observations*. More explicitly O can be described as the lbd $(\{\bot,\top\},\sqsubseteq,\neg)$ with $\bot \sqsubseteq \top$ and $\neg\bot = \top$. Notice that $[A{\to}\mathsf{O}]$ separates points in A for any lbd A.

The data type N will serve as the type of bilifted natural numbers. More explicitly N can be described as the lbd $(\mathbb{N} \cup \{\bot,\top\},\sqsubseteq,\neg)$ with $x \sqsubseteq y$ iff $x = \bot$ or $y = \top$ or $x = y$, and negation is given by $\neg\bot = \top$ and $\neg n = n$ for all $n \in \mathbb{N}$. The extensional order of N can by visualised by the following diagram



where the bistably connected components are $[\bot]_{\updownarrow} = [\top]_{\updownarrow} = \{\bot, \top\}$ and $[n]_{\updownarrow} = \{n\}$ for all $n \in \omega$. Thus, N is isomorphic to the bilifted set of natural numbers, and we will refer to the elements of N as given in the diagram above. In terms of CL-games the lbd N has exactly one cell $c = \top$ and $\mathsf{FP}(\mathsf{N}) = \{c\} \cup \{v_n \mid n \in \omega\}$.

Further using Thm. 4.1.2 it follows that the locally boolean domain N^ω has as cells the set $\{c_n \mid n \in \omega\}$

In [Lon02] J. Longley has shown that in **SA** (i.e. the category of sequential data structures (without error elements) and sequential algorithms/function) the sequential data structure of partial functions on the natural numbers is universal. We will modify this proof and show that the lbds U and N^ω both are universal in the category ω**LBD**.

Definition 4.4.4. *Let A and B be lbpds and $e : A \to B$ and $p : B \to A$ be sequential maps. We call the pair $[e, p]$ a* retraction pair *and write $[e, p] : A \to B$ iff $p \circ e = \mathrm{id}_A$.* $\diamond$

If $[e, p]$ is a retraction pair, then e is called *embedding*, p is called *projection*. Notice that we do not impose any condition on $e \circ p$. Thus, p is not necessarily a projection in the sense of Def. 4.2.1.

In the following we will show that the locally boolean domain N^ω is universal in the category ω**LBD**, i.e. if A is a cblbd then there exists a retraction pair $[e, p] : A \to \mathsf{N}^\omega$.

As A is a countably based we can pick some identification $\nu : \mathsf{FP}(A) \hookrightarrow \omega$ of $\mathsf{FP}(A)$ with a subset of the natural numbers. Further, we write ϵ for the (partial) inverse function of ν, and write $\epsilon(n)\downarrow$ iff there is a $p \in \mathsf{FP}(A)$ with $\nu(p) = n$ and $\epsilon(n)\uparrow$ otherwise.

Next we define functions e_A and $e_A^\flat$ from a cblbpd A to N^ω. The intuition behind the definition of $e_A^\flat$ is that $e_A^\flat(x)$ fills cell i with j (resp. $\top$) iff $\epsilon(i)$ is a cell that is filled by x with value $\epsilon(j) \neq \epsilon(i)$ (resp. $\epsilon(i)$). The definition of e_A is like the definition of $e_A^\flat$ but $e_A(x)$ additionally fills a cell c_i with $\top$ if $\epsilon(i)$ fills some cell $c \in \mathsf{FP}(x)$ with c. Notice that $e_A(x)$ and $e_A^\flat(x)$ agree on all cells that are filled by $e_A^\flat(x)$, and if a cell c is filled by $e_A(x)$ with value $v \neq \top$ then c is also filled by $e_A^\flat(x)$ with value v.

Definition 4.4.5. *If A is a cblbd we define the functions $e_A, e^\flat_A : A \to \mathsf{N}^\omega$ by*

$$(e^\flat_A(x))_i := \begin{cases} \top & \text{if } \epsilon(i)\downarrow,\ \epsilon(i) \in \mathsf{Cell}(A) \text{ and } x \text{ fills } \epsilon(i) \text{ with } \epsilon(i) \\ j & \text{if } \epsilon(i)\downarrow,\ \epsilon(i) \in \mathsf{Cell}(A),\ \epsilon(j)\downarrow,\ \epsilon(j) = (\epsilon(j))_\perp \\ & \qquad \text{and } x \text{ fills } \epsilon(i) \text{ with } \epsilon(j) \\ \perp & \text{otherwise} \end{cases}$$

$$(e_A(x))_i := \begin{cases} \top & \text{if } \epsilon(i)\downarrow,\ \epsilon(i) \in \mathsf{Cell}(A) \text{ and} \\ & \qquad \exists c{\in}\mathsf{FP}(x){\cap}\mathsf{Cell}(A).\, c \text{ filled by } \epsilon(i) \\ j & \text{if } \epsilon(i)\downarrow,\ \epsilon(i) \in \mathsf{Cell}(A),\ \epsilon(j)\downarrow,\ \epsilon(j) = (\epsilon(j))_\perp \\ & \qquad \text{and } x \text{ fills } \epsilon(i) \text{ with } \epsilon(j) \\ \perp & \text{otherwise} \end{cases}$$

for all $x \in A$ and $i \in \omega$. ◇

Notice that in the above definitions the cases for $\top$ and $j \in \omega$ are mutually exclusive. Thus both functions are well-defined. Further from the preceeding remarks we get that:

Lemma 4.4.6. *Let A be a cblbd. Then $e^\flat_A(x) \leq_s e_A(x)$ for all $x \in A$.*

Lemma 4.4.7. *Let A be a cblbd. Then the functions $e^\flat_A$ and e_A preserve bistable coherence and bistably coherent infima and suprema.*

Proof. Suppose $x \updownarrow y$.

For showing that $e_A(x) \updownarrow e_A(y)$ holds, we have to check that $(e_A(x))_i = j$ for some $j \in \omega$ implies $(e_A(y))_i = j$ and vice versa. If $(e_A(x))_i = j$ then $\epsilon(j) = (\epsilon(j))_\perp$ and x fills $\epsilon(i)$ with $\epsilon(j)$. Thus, $\epsilon(j) \in \mathsf{FP}(x)$ and as $\epsilon(j) = (\epsilon(j))_\perp$ it follows that $\epsilon(j) \in \mathsf{FP}(x_\perp) = \mathsf{FP}(y_\perp) \subseteq \mathsf{FP}(y)$. Thus, y fills $\epsilon(i)$ with $\epsilon(j)$ and we have that $(e_A(y))_i = j$.

For showing that $e_A(x \sqcap y) = e_A(x) \sqcap e_A(y)$ and $e_A(x \sqcup y) = e_A(x) \sqcup e_A(y)$ hold it suffices to check the cases where $(e_A(x))_i, (e_A(y))_i \in \{\perp, \top\}$. Using Lemma 2.2.21 it follows that $(e_A(x \sqcap y))_i = \top$ iff $(e_A(x))_i = \top$ and $(e_A(y))_i = \top$, and $(e_A(x \sqcup y))_i = \top$ iff $(e_A(x))_i = \top$ or $(e_A(y))_i = \top$.

Analogously, one shows that $e^\flat_A$ has the required properties. □

Lemma 4.4.8. *Let A be a cblbd. Then functions e_A and $e^\flat_A$ preserve $\perp$-elements, i.e. $e_A(x_\perp) = e_A(x)_\perp$ (resp. $e^\flat_A(x_\perp) = e^\flat_A(x)_\perp$) for all $x \in A$.*

Proof. Immediately from the definition of e_A and $e^\flat_A$. □

Lemma 4.4.9. *Let A be a cblbd. Then the functions e_A and $e^\flat_A$ reflect stable coherence, i.e. if $x, y \in A$ then $e_A(x) \uparrow e_A(y)$ (resp. $e^\flat_A(x) \uparrow e^\flat_A(y)$) implies $x \uparrow y$.*

Proof. Suppose $x, y \in A$ with $x \not\uparrow y$. Then as A has a least element it follows that $\mathsf{FP}(x) \cap \mathsf{FP}(y) \neq \emptyset$. Thus from Lemma 3.2.12 it follows that there is a cell c that is filled by x with v_x and by y with v_y and $v_x \neq v_y$. Thus, by Def. 4.4.5 we get $(e_A(x))_{\nu(c)} = \nu(v_x) \neq \nu(v_y) = (e_A(y))_{\nu(c)}$ and it follows that $e_A(x) \not\uparrow e_A(y)$. Analogously, one can show that $x \not\uparrow y$ implies $e^\flat_A(x) \not\uparrow e^\flat_A(y)$. □

Lemma 4.4.10. *Let A be a cblbd. Then the function e_A is bistable.*

Proof. Suppose $x, y \in A$ with $x \sqsubseteq y$. If $(e_A(x))_i = \top$ then $\epsilon(i) \in \mathsf{Cell}(A)$ and $\exists c{\in}\mathsf{FP}(x){\cap}\mathsf{Cell}(A).\, c$ filled by $\epsilon(i)$ and it follows from Lemma 3.2.15 that $(e_A(y))_i = \top$. Now suppose $(e_A(x))_i = j$. Then $\epsilon(j) = (\epsilon(j))_\perp$ and x fills $\epsilon(i)$ with $\epsilon(j)$. By Lemma 3.2.15 there exist two cases:
(1) y fills $\epsilon(i)$ with $\epsilon(j)$: Thus $(e_A(y))_i = j$ and we are finished.
(2) There exists a cell c' filled by $\epsilon(j)$ and y fills c' with c' : As $\epsilon(j)$ and $\epsilon(i)$ fill the same cells it follows that c' is filled by $\epsilon(i)$, and we get $(e_A(y))_i = \top$.

Continuity follows immediately from monotonicity, as the definition of e_A refers only to compact elements of A.
Applying Lemma 4.4.7 finishes the proof. □

Lemma 4.4.11. *Let A be a cblbpd. Then for all $x \in \mathsf{N}^\omega$ the set $\{y \in A \mid e^\flat_A(y) \leq_s x\}$ has a greatest element w.r.t. $\leq_s$.*

Proof. If $x \in \mathsf{N}^\omega$ then it follows from Lemma 4.4.9 that the set $\{y \in A \mid e^\flat_A(y) \leq_s x\}$ is stably coherent. Thus, $m_x := \bigsqcup\{y \in A \mid e^\flat_A(y) \leq_s x\}$ exists. From Lemma 2.2.21 it follows that $\mathsf{FP}(m_x) = \bigcup\{\mathsf{FP}(y) \mid y \in A, e^\flat_A(y) \leq_s x\}$. Assuming $p \in \mathsf{FP}(m_x)$ and w.l.o.g. $p \neq \perp$ it follows that there exists a unique cell c with $c_\perp \prec_s p \sqsubseteq c$, further $p \in \mathsf{FP}(y)$ for some $y \in A$ with $e^\flat_A(y) \leq_s x$. Hence, we get

$$(x)_{\nu(c)} = \begin{cases} \top & \text{if } c = p, \text{ and} \\ \nu(p) & \text{otherwise.} \end{cases}$$

Thus, $e^\flat_A(m_x) \leq_s x$. □

Using the just proven lemma we can define a projection from N^ω to a cblbd A.

Definition 4.4.12. *If A is a cblbd then we define the function $p_A : \mathsf{N}^\omega \to A$ by*

$$p_A(x) := \bigsqcup\{y \in A \mid e^\flat_A(y) \leq_s x\}$$

for all $x \in \mathsf{N}^\omega$. ◇

Notice that Lemma 4.4.11 ensures that p_A is well-defined.

Lemma 4.4.13. *Let A be a cblbd. Then the function p_A is observably sequential.*

Proof. For showing that p_A is continuous w.r.t. $\leq_s$ suppose $x, y \in \mathsf{N}^\omega$ with $x \leq_s y$. Let $z \leq_s p_A(x)$ then $e^\flat_A(z) \leq_s x \leq_s y$, thus $z \leq_s p_A(y)$. Further let $X \subseteq \mathsf{N}^\omega$ be directed w.r.t. $\leq_s$. As p_A is monotone w.r.t. $\leq_s$ it follows that $\bigsqcup p_A(X) \leq_s p_A(\bigsqcup X)$. For the showing reverse inequality suppose $q \in \mathsf{FP}(p_A(\bigsqcup X))$, thus $e^\flat_A(q) \leq_s \bigsqcup X$. As q is finite it follows that q fills at most finitely many cells. Thus $(e^\flat_A(q))_i \neq \perp$ for at most finitely many $i \in \omega$. It follows from Thm. 4.1.2 that $e^\flat_A(q)$ is finite. Hence as X is directed w.r.t. $\leq_s$ there exists a $x \in X$ with $e^\flat_A(q) \leq_s x$, thus $q \leq_s p_A(x) \leq_s \bigsqcup p_A(X)$.

For showing that p_A is observably sequential suppose $x, y \in \mathsf{N}^\omega$ with $x \leq_s y$, $c' \in \mathsf{Acc}(p_A(x))$ and $p_A(y)$ fills c'. Then $e_A^\flat(c'_\perp) <_s y$ and it follows that y fills the cell $c_{\nu(c')}$, thus $c_{\nu(c')}$ is the unique sequentiality index for p_A at (x, c'). Uniqueness follows as $p_A(x \sqcup c_{\nu(c')})$ fills c'. Further by definition of $e_A^\flat$ it follows that $p_A(x \sqcup c_{\nu(c')})$ fills c' with value c', thus p_A is error propagating. □

Thus we can show that each cblbd is a retract of the lbd N^ω.

Theorem 4.4.14. *The cblbd N^ω is universal in the category ω**LBD**, i.e. if A is a cblbd then there exists a retraction pair $[e, p] : A \to \mathsf{N}^\omega$.*

Proof. We show that $[e_A, p_A] : A \to \mathsf{N}^\omega$ is a retraction pair. By Thm. 3.3.6, Lemma 4.4.10 and Lemma 4.4.13 and it follows that e_A and p_A are bistable maps.

For showing that $p_A \circ e_A = \mathrm{id}_A$ suppose $x \in A$. As $e_A^\flat(x) \leq_s e_A(x)$ by Lemma 4.4.6 it follows that $x \leq_s p_A(e_A(x))$.

For showing the reverse inequality suppose $p \in \mathsf{FP}(p_A(e_A(x)))$ and w.l.o.g. $p \neq \perp$. Then $e_A^\flat(p) \leq_s e_A(x)$ and p fills a unique cell $c \in \mathsf{Cell}(A)$ with value p. We proceed by case analysis on p. In case that $p = p_\perp$ then $(e_A^\flat(p))_{\nu(c)} = \nu(p)$. As $e_A^\flat(p) \leq_s e_A(x)$ it follows that $(e_A(x))_{\nu(c)} = \nu(p)$, thus $p \in \mathsf{FP}(x)$. In case that $p \neq p_\perp$ then $p = c$ and $(e_A^\flat(p))_{\nu(c)} = \top$. Thus $(e_A(x))_{\nu(c)} = \top$ and it follows that there exists a cell c' that is filled by c and x fills c' with value c'. As $p, x \leq_s p_A(e_A(x))$ we have $p \uparrow x$ and it follows from Cor. 3.2.11 that the filling of p and x agrees on those cells filled by both. Thus, as all cells except c that are filled by p are filled with values $v = v_\perp$ it follows that $c = c'$. Hence we get $p = c \in \mathsf{FP}(x)$ as desired. □

As an easy consequence it follows that type of bistable endomaps on the bilifted natural numbers is also universal.

Corollary 4.4.15. *The cblbd U is universal in the category ω**LBD**.*

Proof. It is an easy exercise to verify that the maps $e : \mathsf{N}^\omega \to \mathsf{U}$ and $p : \mathsf{U} \to \mathsf{N}^\omega$ given by

$$e(x) := \begin{cases} \top & \mapsto \top \\ i & \mapsto x_i \quad \text{for all } i \in \omega \\ \perp & \mapsto \perp \end{cases}$$

and

$$p(f)_i := f(i) \quad \text{for all } i \in \omega$$

are both bistable and form a retraction pair $[e, p] : \mathsf{N}^\omega \to \mathsf{U}$. Using Thm. 4.4.14 it follows that U is universal. □

5 A universal model for the language SPCF_∞ in $\mathbf{LBD}$

In the first part of this chapter we introduce the language SPCF_∞ and its operational semantics. The language SPCF_∞ is an infinitary version of SPCF as introduced in [CCF94]. More explicitly, it is obtained from simply typed λ-calculus by adding (countably) infinite sums and products, error elements, a control operator **catch** and recursive types. We give a call-by-name operational semantics for SPCF_∞ where we use evaluation contexts in order to formalise the behaviour of the control operator **catch**.

In the second part of this chapter we present a computationally adequate model for SPCF_∞ in the category **LBD**. Further we exhibit each SPCF_∞ type as an SPCF_∞ definable retract of the type $\mathbf{N}\to\mathbf{N}$ from which we deduce universality of SPCF_∞ for its **LBD** model.

5.1 Definition of SPCF_∞

First we define the types of the language SPCF_∞. Since SPCF_∞ has recursive types we also have to consider types with free type variables.

We assume a given set of type variables (denoted by α, α' and so on) and generate the types of SPCF_∞ as follows:

$$\sigma ::= \alpha \mid \sigma\to\sigma \mid \mu\alpha.\sigma \mid \Sigma_{i\in n}\sigma \mid \Pi_{i\in n}\sigma$$

where $n \in \omega + 1$.

A type σ is called *closed* iff it does not contain a free type variable α, i.e. each occurrence of a type variable α is bound under by some $\mu\alpha$. We introduce the following abbreviations for types:

$$\begin{aligned}
\mathbf{0} &:= \Sigma_{i\in\emptyset} && \text{(type of observations)}\\
\mathbb{1} &:= \Pi_{i\in\emptyset} && \text{(empty product)}\\
\mathbf{N} &:= \Sigma_{i\in\omega}\mathbb{1} && \text{(natural numbers)}\\
\mathbf{n} &:= \Sigma_{i\in n}\mathbb{1} && \text{(for all } n\in\omega{+}1)\\
\sigma_\uparrow &:= \Sigma_{i\in 1}\sigma && \text{(bilifting)}\\
\sigma_0+\cdots+\sigma_{n-1} &:= \Sigma_{i\in n}\sigma_i && (n\text{-ary sum})\\
\sigma_0\times\cdots\times\sigma_{n-1} &:= \Pi_{i\in n}\sigma_i && (n\text{-ary product})
\end{aligned}$$

Additionally, for a given SPCF_∞ type σ and $n \in \omega + 1$ we define the abbreviation

$$\sigma^n := \Pi_{i\in n}\sigma$$

The terms of SPCF_∞ are derived using the following grammar:

$$\begin{aligned} t ::= {} & x \mid (\lambda x : \sigma.t) \mid (tt) \mid \\ & \langle t_i \rangle_{i\in n} \mid \mathbf{pr}_i^{\Pi_{i\in n}\sigma_i}(t) \\ & \mathbf{in}_i^{\Sigma_{i\in n}\sigma_i}(t) \mid \mathbf{case}^{\Sigma_{i\in n}\sigma_i,\tau}\, t\,\mathbf{of}\,(\mathbf{in}_i\, x \Rightarrow t_i)_{i\in n} \\ & \mathbf{fold}^{\mu\alpha.\sigma}(t) \mid \mathbf{unfold}^{\mu\alpha.\sigma}(t) \mid \\ & \top^{\Sigma_{i\in n}\sigma_i} \mid \mathbf{catch}(t) \end{aligned}$$

for any variable x, $n \in \omega + 1$ and all types σ, σ_i, τ and type variables α.

By $\langle t_i \rangle_{i\in n}$ we denote

$$\langle t_0, \ldots, t_{n-1} \rangle \qquad \text{if } n \in \omega$$

and

$$\langle t_0, t_1, t_2, \ldots \rangle \qquad \text{if } n \in \omega\,.$$

Accordingly, by $\mathbf{case}^{\Sigma_{i\in n}\sigma_i,\tau}\, t\,\mathbf{of}\,(\mathbf{in}_i\, x \Rightarrow t)_{i\in n}$ we denote

$$\mathbf{case}^{\Sigma_{i\in n}\sigma_i,\tau}\, t\,\mathbf{of}\,(\mathbf{in}_0 \Rightarrow t_0, \ldots, \mathbf{in}_{n-1} \Rightarrow t_{n-1}) \qquad \text{if } n \in \omega$$

and

$$\mathbf{case}^{\Sigma_{i\in n}\sigma_i,\tau}\, t\,\mathbf{of}\,(\mathbf{in}_0 \Rightarrow t_0, \mathbf{in}_1 \Rightarrow t_1, \mathbf{in}_2 \Rightarrow t_2, \ldots) \qquad \text{if } n \in \omega\,.$$

Further we define the *values* of SPCF_∞ by the grammar

$$\begin{aligned} v ::= {} & (\lambda x : \sigma.t) \mid \\ & \langle t_i \rangle_{i\in n} \\ & \mathbf{in}_i^{\Sigma_{i\in n}\sigma_i}(t) \\ & \mathbf{fold}^{\mu\alpha.\sigma}(t) \\ & \top^{\Sigma_{i\in n}\sigma_i} \end{aligned}$$

for all terms t. The values $\top^{\Sigma_{i\in n}\sigma_i}$ are called *error values* the other values are called *proper values*.

Type annotations are merely used for type inference, we will omit them when they are clear from the context. Terms not containing any free variables are called *closed terms* or *programs* the other terms are called *open terms*.

For the typing rules (as given in table Table 5.1), we look at terms-in-context of the form $\Gamma \vdash M : \tau$, where M is a term, τ a closed type and $\Gamma \equiv x_1 : \sigma_1, \ldots, x_n : \sigma_n$ is a *context* assigning closed types $\sigma_1, \ldots, \sigma_n$ to a finite set of variables $x_1, \ldots, x_n$. If Γ is the empty context, then we also write $M : \tau$ for $\Gamma \vdash M : \tau$.

Notice that even in the presence of infinite constructions (countable product and sum) we do not consider contexts with infinitely many free variables as we are interested only in those terms-in-context that can be transformed into closed terms.

$$\frac{}{x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash x_i : \sigma_i}\ (\mathsf{Ax\ var}) \qquad \frac{}{\Gamma \vdash \top^{\Sigma_{i\in n}\sigma_i} : \Sigma_{i\in n}\sigma_i}\ (\mathsf{Ax}\top)$$

$$\frac{\Gamma, x : \sigma \vdash t : \tau}{\Gamma \vdash (\lambda x : \sigma.t) : \sigma \to \tau}\ (\mathsf{I}\to) \qquad \frac{\Gamma \vdash t : \sigma \to \tau \quad \Gamma \vdash s : \sigma}{\Gamma \vdash (ts) : \tau}\ (\mathsf{E}\to)$$

$$\frac{\Gamma \vdash t_i : \sigma_i \quad \text{for all } i \in n}{\Gamma \vdash \langle t_i \rangle_{i\in n} : \Pi_{i\in n}\sigma_i}\ (\mathsf{I}\Pi) \qquad \frac{\Gamma \vdash t : \Pi_{i\in n}\sigma_i}{\Gamma \vdash \mathbf{pr}_i^{\Pi_{i\in n}\sigma_i}(t) : \sigma_i}\ (\mathsf{E}\Pi)$$

$$\frac{\Gamma \vdash t : \sigma_i}{\Gamma \vdash \mathbf{in}_i^{\Sigma_{i\in n}\sigma_i}(t) : \Sigma_{i\in n}\sigma_i}\ (\mathsf{I}\Sigma) \qquad \frac{\Gamma \vdash t : \Sigma_{i\in n}\sigma_i \quad \Gamma, x : \sigma_i \vdash s_i : \tau \quad \text{for all } i \in n}{\Gamma \vdash \mathbf{case}^{\Sigma_{i\in n}\sigma_i, \tau}\, t\, \mathbf{of}\ (\mathbf{in}_i\, x \Rightarrow s_i)_{i\in n} : \tau}\ (\mathsf{E}\Sigma)$$

$$\frac{\Gamma \vdash t : \sigma[\mu\alpha.\sigma/\alpha]}{\Gamma \vdash \mathbf{fold}^{\mu\alpha.\sigma}(t) : \mu\alpha.\sigma}\ (\mathsf{I}\mu) \qquad \frac{\Gamma \vdash t : \mu\alpha.\sigma}{\Gamma \vdash \mathbf{unfold}^{\mu\alpha.\sigma}(t) : \sigma[\mu\alpha.\sigma/\alpha]}\ (\mathsf{E}\mu)$$

$$\frac{\Gamma \vdash t : \mathbf{0}^\omega \to \mathbf{0}}{\Gamma \vdash \mathbf{catch}(t) : \mathbf{N}}\ (\mathsf{catch})$$

Table 5.1: Typing rules for SPCF$_\infty$

For sake of convenience we introduce the following abbreviations:

$$
\begin{aligned}
\mathbf{Y}_\sigma :\equiv & k(\mathbf{fold}^\tau(k)) \\
& \quad \text{with } \tau :\equiv \mu\alpha.(\alpha\to(\sigma\to\sigma)\to\sigma) \\
& \quad \text{and } k :\equiv \lambda x : \tau.\lambda f : \sigma\to\sigma. f(\mathbf{unfold}^\tau(x)xf) \\
\mathbf{id}_\sigma :\equiv & \lambda x : \sigma.x \\
\bot_\sigma :\equiv & \mathbf{Y}_\sigma\, \mathbf{id}_\sigma \\
* :\equiv & \langle\rangle^{\mathbb{1}} \\
\mathbf{zero} :\equiv & \mathbf{in}_0^{\mathbf{N}}(*) \\
\mathbf{succ} :\equiv & \lambda n : \mathbf{N}.\, \mathbf{case}^{\mathbf{N},\mathbf{N}}\, n\, \mathbf{of}\, \left(\mathbf{in}_i^{\mathbf{N}}\, x \Rightarrow \mathbf{in}_{i+1}^{\mathbf{N}}\, x\right)_{i\in\omega} \\
\mathbf{pred} :\equiv & \lambda n : \mathbf{N}.\, \mathbf{case}^{\mathbf{N},\mathbf{N}}\, n\, \mathbf{of}\, \begin{pmatrix} \mathbf{in}_i^{\mathbf{N}}\, x \Rightarrow \bot_{\mathbf{N}} & \text{if } i = 0, \\ \mathbf{in}_i^{\mathbf{N}}\, x \Rightarrow \mathbf{in}_{i-1}^{\mathbf{N}}\, x & \text{otherwise} \end{pmatrix}_{i\in\omega} \\
\mathbf{ifz} :\equiv & \lambda n : \mathbf{N}.\lambda k : \mathbf{N}.\lambda l : \mathbf{N}.\, \mathbf{case}^{\mathbf{N},\mathbf{N}}\, n\, \mathbf{of}\, \begin{pmatrix} \mathbf{in}_i^{\mathbf{N}}\, x \Rightarrow k & \text{if } i = 0, \\ \mathbf{in}_i^{\mathbf{N}}\, x \Rightarrow l & \text{otherwise} \end{pmatrix}_{i\in\omega} \\
\mathbf{catch}^{\sigma_0\to\ldots\to\sigma_{n-1}\to\mathbf{N}} :\equiv & \lambda f.\, \mathbf{catch}(\lambda x : \mathbf{0}^\omega.\, \mathbf{case}\, f(e_0(\mathbf{pr}_0(x)), \ldots, e_{n-1}(\mathbf{pr}_{n-1}(x))) \\
& \qquad \mathbf{of}\, (\mathbf{in}_i\, y \Rightarrow \mathbf{pr}_{i+n}(x))_{i\in\omega}) \\
& \quad \text{with } e_i :\equiv \lambda x : \mathbf{0}.\, \mathbf{case}^{\mathbf{0},\sigma_i}\, x\, \mathbf{of}\, () \quad \text{for all } i \in \{0, \ldots, n-1\}
\end{aligned}
$$

If $n \in \omega{+}1$ and $m < n$ then we also write $\underline{m}$ for the term $\mathbf{in}_m^{\mathbf{n}}(*)$ of type $\mathbf{n}$.

Hence we get SPCF_∞ as an extension of ordinary SPCF defined by R. Cartwright, P.L. Curien and M. Felleisen in [CCF94] and of $\mathsf{SPCF}+$ defined by Jim Laird in [Lai03a]. (It is tedious but straightforward to check that the operational semantics of SPCF_∞ (as given in section 5.2) is sound w.r.t. to the other operational semantics.)

We also remark that due to the countably infinite case construct any bistable function from $[\![\mathbf{N}]\!]$ to $[\![\mathbf{N}]\!]$ is implementable in SPCF_∞ (and not only functions that are computable in the classical sense).

5.2 Operational semantics

In order to formalise the behaviour of the control operator **catch** we introduce the notion of evaluation contexts.

Definition 5.2.1. *A* (call-by-name) evaluation context E *is defined by the grammar*

$$
\begin{aligned}
E ::= & [\,] \mid \\
& Et \mid \\
& \mathbf{pr}_i(E) \mid \\
& \mathbf{case}\, E\, \mathbf{of}\, (\mathbf{in}_i\, s_i \Rightarrow t)_{i\in n} \mid \\
& \mathbf{unfold}(E) \mid \\
& \mathbf{catch}(\lambda x : \mathbf{0}^\omega.E)
\end{aligned}
$$

for any $i \in \omega$ and where t and the s_i range over SPCF_∞ terms. $\diamond$

The notion $E[t]$ stands for E with the $[\,]$ hole filled by t. Further, if we write $E[x]$ we assume that the occurrence of the variable x in the hole of E is a free occurrence, and, analogously, if t is an open term. We allow only those evaluation contexts E that are typeable, i.e. there exists a context Γ and closed types σ and τ with $\Gamma \vdash E[x : \sigma] : \tau$, then we say that E is of type τ.

Definition 5.2.2 (SPCF_∞-redexes)**.** *The SPCF_∞-redexes are given by the following production rules:*

$$\begin{aligned}\Delta ::=&(\lambda x : \sigma.t)s \mid \\ &\mathbf{pr}_i(\langle t_i\rangle_{i\in n}) \mid \\ &\mathbf{case}\,\mathbf{in}_i\, s\,\mathbf{of}\,(\mathbf{in}_i\, x \Rightarrow t_i)_{i\in n} \mid \\ &\mathbf{unfold}(\mathbf{fold}(t)) \mid \\ &\mathbf{catch}(\lambda x : \mathbf{0}^\omega.E'[x]) \mid \\ &E[\top]\end{aligned}$$

for all $n \in \omega+1$, terms t, t_i and s and evaluation contexts E and E', with the constraint that $E \neq [\,]$. A redex of the form $E[\top]$ is called an error redex*, the other redexes are called* proper redexes. $\diamond$

Lemma 5.2.3. *Let t be a SPCF_∞ term. Then either t is a proper value or there exists a unique decomposition of t into an evaluation context E and a term R where R is either a free variable, $\top$ or a proper redex with $t \equiv E[R]$.*

Proof. The proof is a standard induction on the structure of the term t and similar to the proof of Lemma 8.5 in [CCF94]. $\square$

The operational semantics of the language SPCF_∞ is described by means of the following deterministic evaluation relation $\to_{\mathsf{op}}$.

Definition 5.2.4. *For all terms t, t_i and s and evaluation contexts E we define the following redex reductions:*

$$\begin{aligned}(\lambda x : \sigma.t)s &\to_{\mathsf{red}} t[s/x] && (\mathsf{beta})\\ \mathbf{pr}_i(\langle t_i\rangle_{i\in n}) &\to_{\mathsf{red}} t_i && (\mathsf{prod})\\ \mathbf{case}\,\mathbf{in}_i\, s\,\mathbf{of}\,(\mathbf{in}_i\, x \Rightarrow t_i) &\to_{\mathsf{red}} t_i[s/x] && (\mathsf{case})\\ \mathbf{unfold}(\mathbf{fold}(t)) &\to_{\mathsf{red}} t && (\mathsf{fold})\end{aligned}$$

For all evaluation contexts E of type $\mathbf{0}$ the evaluation relation $\to_{\mathsf{op}}$ is given by

$$\begin{aligned}E[t] &\to_{\mathsf{op}} E[t'] && \textit{if } t \to_{\mathsf{red}} t' && (\mathsf{red})\\ E[\top] &\to_{\mathsf{op}} \top && \textit{if } E \neq [\,] && (\top)\\ E[\mathbf{catch}\, t] &\to_{\mathsf{op}} t\langle E[\underline{n}]\rangle_{n\in\omega} && && (\mathsf{catch})\end{aligned}$$ $\diamond$

It follows from Lemma 5.2.3 that $\to_{\mathsf{op}}$ is deterministic.

5.3 Interpretation of types

As the language SPCF_∞ includes recursive types the interpretation of SPCF_∞-types has to be defined inductively on the type structure. Hence we cannot restrict ourselves to closed types but have to define the interpretation of types relative to a context.

The canonical way of interpreting a type $\alpha_1, \ldots, \alpha_n \vdash \sigma$ in domain theory is as a locally continuous functor $F_{\alpha_1,\ldots,\alpha_n \vdash \sigma} : \mathcal{C}^n \to \mathcal{C}$ over some suitable category $\mathcal{C}$.

For example the type $\alpha_1, \alpha_2 \vdash \alpha_1 {\to} \alpha_2$ corresponds to the functor

$$\begin{aligned} F((X_1,Y_1),(X_2,Y_2)) =& Y_2^{X_1} \quad \text{for objects } X_1, Y_1, X_2, Y_2 \in \mathcal{C} \\ F((f_1,g_1),(f_2,g_2)) =& g_2^{f_1} \quad \text{for morphisms } f_i : X_i' \to X_i,\ g_i : Y_i \to Y_i' \\ & \text{with } i \in \{1,2\} \text{ where} \\ & g_2^{f_1} : Y_2^{X_1} \to {Y'}_2^{X_1'},\ g_2^{f_1}(h) = g_2 \circ h \circ f_1 \end{aligned}$$

It is straightforward to define such functors for function, sum and product types. Taking recursive types into account, things get more complicated. In the theory of domains recursive types are usually interpreted as the solution of recursive domain equations, which can be constructed as bilimits over some suitable diagram of embeddings and projections. For this purpose we have to restrict our category **LBD** to the category $\mathbf{LBD}_\mathrm{s}$ of locally boolean domains and *strict* bistable maps. (This is in fact no restriction as all embeddings and projections are strict.) Our approach is related to Freyd's results on initial algebras and final co-algebras, see [Fre91] and [Fre92]. Following the notational convention introduced in [Pit96] we will decorate variable names with superscripts $+$ and $-$. to distinguish between 'co- and contravariant arguments'.

As W. K. Ho pointed out in [Ho06] we have to carry out the constructions in the category $\widetilde{\mathbf{LBD}_\mathrm{s}}$ which is the diagonal category of $\mathbf{LBD}_\mathrm{s}^{\mathrm{op}} \times \mathbf{LBD}_\mathrm{s}$, i.e. the full subcategory of $\mathbf{LBD}_\mathrm{s}^{\mathrm{op}} \times \mathbf{LBD}_\mathrm{s}$ whose objects are those of $\mathbf{LBD}_\mathrm{s}$ and morphisms are pairs of $\mathbf{LBD}_\mathrm{s}$ morphisms of the form

$$\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \end{tikzcd}$$

Definition 5.3.1. *Let* $\Theta \equiv \alpha_1, \ldots, \alpha_n$ *by a type context. For any type-in-context* $\Theta \vdash \sigma$ *we define a corresponding functor* $F_{\Theta\vdash\sigma} : \widetilde{\mathbf{LBD}_\mathrm{s}}^n \to \widetilde{\mathbf{LBD}_\mathrm{s}}$ *by induction on the structure of the type* σ*.*

For any collection $\vec{x}^{\mp} := (x_1^-, x_1^+, \ldots, x_n^-, x_n^+)$ *of either objects or morphisms in the* *category* $\widetilde{\mathbf{LBD}_{\mathrm{s}}}$*, respectively, we define*

$$\begin{aligned}
F_{\Theta\vdash\alpha_i}(\vec{x}^{\mp}) &:= x_i^{+} \\
F_{\Theta\vdash\sigma\to\tau}(\vec{x}^{\mp}) &:= [F_{\Theta\vdash\sigma}(\vec{x}^{\pm})\to F_{\Theta\vdash\tau}(\vec{x}^{\mp})] \\
F_{\Theta\vdash\Pi_{i\in I}\sigma_i}(\vec{x}^{\mp}) &:= \prod_{i\in I} F_{\Theta\vdash\sigma_i}(\vec{x}^{\mp}) \\
F_{\Theta\vdash\Sigma_{i\in I}\sigma_i}(\vec{x}^{\mp}) &:= \sum_{i\in I} F_{\Theta\vdash\sigma_i}(\vec{x}^{\mp}) \\
F_{\Theta\vdash\mu\alpha.\sigma}(\vec{x}^{\mp}) &:= \mathsf{rec}_1(F_{\alpha,\Theta\vdash\sigma})(\vec{x}^{\mp}) \quad \textit{for } \alpha\notin\Theta
\end{aligned}$$

where $\mathsf{rec}_1(F_{\alpha,\Theta\vdash\sigma})$ *is the functor* $H : \widetilde{\mathbf{LBD}_{\mathrm{s}}}^{n-1} \to \widetilde{\mathbf{LBD}_{\mathrm{s}}}$ *such that for all* $\vec{x}' := (x_2, x_2, \ldots, x_n, x_n)$ *we have*

$$F_{\Theta\vdash\mu\alpha.\sigma}(H(\vec{x}'), H(\vec{x}'), \vec{x}') \simeq H(\vec{x}')$$

According to the results of Freyd, $H(\vec{x}')$ *is the free algebra for the functor* $F_{\Theta\vdash\mu\alpha.\sigma}$*.* $\diamond$

Lemma 5.3.2. *For any type-in-context* $\Theta\vdash\sigma$ *the functor* $F_{\Theta\vdash\sigma}$ *is locally continuous.*

Proof. The proof is done by induction on the structure of the type-in-context $\Theta\vdash\sigma$

ad $\Theta\vdash\alpha_i$: We have $F_{\Theta\vdash\alpha_i}(\vec{x}^{\mp}) = x_i^{+}$, which is obviously locally continuous.

ad $\Theta\vdash\sigma\to\tau$: We have $F_{\Theta\vdash\sigma\to\tau}(\vec{x}^{\mp}) = [F_{\Theta\vdash\sigma}(\vec{x}^{\pm})\to F_{\Theta\vdash\tau}(\vec{x}^{\mp})]$. If $\vec{x}^{\mp}$ is a collection of morphisms and X a directed set with $\bigsqcup X = \vec{x}^{\mp}$ then

$$\begin{aligned}
F_{\Theta\vdash\sigma\to\tau}(\vec{x}^{\mp}) &= [F_{\Theta\vdash\sigma}(\vec{x}^{\pm})\to F_{\Theta\vdash\tau}(\vec{x}^{\mp})] \\
&= F_{\Theta\vdash\tau}(\vec{x}^{\mp}) \circ _ \circ F_{\Theta\vdash\sigma}(\vec{x}^{\pm}) \\
&= \bigsqcup_{y^{\mp}\in X} \{F_{\Theta\vdash\tau}(\vec{y}^{\mp})\} \circ _ \circ \bigsqcup_{y^{\mp}\in X} \{F_{\Theta\vdash\sigma}(\vec{y}^{\pm})\} \\
&= \bigsqcup_{y^{\mp}\in X} \{F_{\Theta\vdash\tau}(\vec{y}^{\mp}) \circ _ \circ F_{\Theta\vdash\sigma}(\vec{y}^{\pm})\} & (\dagger) \\
&= \bigsqcup_{y^{\mp}\in X} \{[F_{\Theta\vdash\sigma}(\vec{y}^{\pm})\to F_{\Theta\vdash\tau}(\vec{y}^{\mp})]\} \\
&= \bigsqcup_{y^{\mp}\in X} \{F_{\Theta\vdash\sigma\to\tau}(\vec{y}^{\mp})\}
\end{aligned}$$

where $(\dagger)$ holds as $\mathbf{LBD}_{\mathrm{s}}$ is cpo-enriched.

ad $\Theta\vdash\Pi_{i\in I}\sigma_i$ and $\Theta\vdash\Sigma_{i\in I}\sigma_i$: For all $i\in I$ the functor $F_{\Theta\vdash\sigma_i}$ is locally continuous by induction hypothesis. Further, $\prod_{i\in I}$ and $\sum_{i\in I}$ are locally continuous by Thm. 4.1.8.

ad $\Theta\vdash\mu\alpha.\sigma$: According to the results of W. K. Ho in [Ho06] it follows that the functor H is locally continuous. $\square$

Definition 5.3.3. *Let σ be a closed* SPCF$_\infty$*-type. Then we define its interpretation by*

$$[\![\sigma]\!] := F_{\vdash\sigma}$$

and identify the constant functor $F_{\vdash\sigma}$ with the corresponding object $[\![\sigma]\!]$ in the category **LBD**$_s$*. Further, if $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$ is an* SPCF$_\infty$ *contexts then we put $[\![\Gamma]\!] := [\![\sigma_1]\!] \times \ldots \times [\![\sigma_n]\!]$.* ◇

5.4 Denotational semantics of SPCF$_\infty$

In the previous section we have given an interpretation of closed SPCF$_\infty$-types. In order to give a denotational semantics we introduce the following sequential maps between locally boolean domains.

Lemma 5.4.1. *For all i, n with $i \in n \in \omega + 1$ and lbds X_i and Y the function*

$$\mathsf{case} : (\sum_{i \in n} X_i \times \prod_{i \in n} [X_i \to Y]) \to Y$$

$$\mathsf{case}(x, (f_i)_{i \in n}) = \begin{cases} f_j(w) & \textit{iff } x = \iota_j(w) \textit{ for some } j \in n \textit{ and } w \in X_i \\ \top_Y & \textit{iff } x = \top_{\sum_{i \in n} X_i} \\ \bot_Y & \textit{iff } x = \bot_{\sum_{i \in n} X_i} \end{cases}$$

is sequential.

Proof. It is an easy exercise to check that for all i, n, X_i and Y the function case is continuous and bistable. □

Next, we take a closer look at the locally boolean domain $[\mathsf{O}^\omega \to \mathsf{O}]$. We show that for all $f \in [\mathsf{O}^\omega \to \mathsf{O}]$ the function f is either constant or a projection. As $\updownarrow \mathsf{O}^\omega$ it follows from Lemma 4.3.3 that f preserves arbitrary infima and suprema. If $a, b \in \mathsf{At}(\mathsf{O}^\omega)$ with $a \neq b$ and $f(a) = f(b) = \top$ then it follows that $f(\bot) = f(a \sqcap b) = f(a) \sqcap f(b) = \top$, thus f is constant $\top$. If $f(a) = \bot$ for all $a \in \mathsf{At}(\mathsf{O}^\omega)$ then $f(\top) = f(\bigsqcup \mathsf{At}(\mathsf{O}^\omega)) = \bigsqcup f[\mathsf{At}(\mathsf{O}^\omega)] = \bot$, thus f is constant $\bot$.

Thus, it follows that f is either constant or $f = \pi_i$, i.e. the i-th projection, for some $i \in \omega$.

Further, one easily verifies that $\neg\bot = \top$ and $\neg\pi_i = \pi_i$ for all $i \in \omega$. Thus, it follows that $[\mathsf{O}^\omega \to \mathsf{O}] \simeq \mathsf{N}$ and we have isomorphisms

$$[\mathsf{O}^\omega \to \mathsf{O}] \underset{\widehat{\mathsf{case}}}{\overset{\mathsf{catch}}{\rightleftarrows}} \mathsf{N}$$

where $\widehat{\mathsf{case}}$ is the transpose of $\mathsf{case} : (\mathsf{N} \times \mathsf{O}^\omega) \to \mathsf{O}$.[1]

Definition 5.4.2. *The inductive definition of the interpretation of* SPCF$_\infty$ *terms-in-context is given in table Table 5.2.* ◇

[1]We used the fact $[\mathbb{1} \to X] \simeq X$ implicitly.

$$
\begin{aligned}
[\![x_1:\sigma_1,\dots,x_n:\sigma_n \vdash x_i:\sigma_i]\!] &:= \pi_i\\
[\![\Gamma \vdash \top : \Sigma_{i\in n}\sigma_i]\!] &:= x^{[\![\Gamma]\!]} \mapsto \top_{[\![\Sigma_{i\in n}\sigma_i]\!]}\\
[\![\Gamma \vdash (\lambda x:\sigma.t):\sigma{\to}\tau]\!] &:= \mathsf{curry}_{[\![\Gamma]\!],[\![\sigma]\!]}([\![\Gamma, x:\sigma \vdash t:\tau]\!])\\
[\![\Gamma \vdash ts:\tau]\!] &:= \mathsf{eval}\circ\langle [\![\Gamma\vdash t:\sigma{\to}\tau]\!], [\![\Gamma\vdash s:\sigma]\!]\rangle\\
[\![\Gamma \vdash \langle t_i\rangle_{i\in n}^{\Pi_{i\in n}\sigma_i} : \Pi_{i\in n}\sigma_i]\!] &:= \langle [\![\Gamma\vdash t_i:\sigma_i]\!]\rangle_{i\in n}\\
[\![\Gamma \vdash \mathbf{pr}_i(t):\sigma]\!] &:= \pi_i\circ[\![\Gamma\vdash t:\sigma]\!]\\
[\![\Gamma \vdash \mathbf{case}^{\Sigma_{i\in n}\tau_i,\sigma}\, t\,\mathbf{of}\,(\mathbf{in}_i\, x\Rightarrow t_i):\sigma]\!] &:= \mathsf{case}\circ\langle[\![\Gamma\vdash t]\!],\\
&\qquad \langle[\![\Gamma\vdash(\lambda x:\tau_i.t_i):\tau_i{\to}\sigma]\!]\rangle_{i\in n}\rangle\\
[\![\Gamma \vdash \mathbf{in}_i(t):\Sigma_{i\in n}\sigma_i]\!] &:= \iota_i\circ[\![\Gamma\vdash t:\sigma_i]\!]\\
[\![\Gamma \vdash \mathbf{catch}(t):\mathbf{N}]\!] &:= \mathsf{catch}\circ[\![\Gamma\vdash t:\mathbf{0}^\omega{\to}\mathbf{0}]\!]\\
[\![\Gamma \vdash \mathbf{fold}^{\mu\alpha.\sigma}(t):\mu\alpha.\sigma]\!] &:= \mathsf{fold}\circ[\![\Gamma\vdash t:\sigma[\mu\alpha.\sigma/\alpha]]\!]\\
[\![\Gamma \vdash \mathbf{unfold}^{\mu\alpha.\sigma}(t):\sigma[\mu\alpha.\sigma/\alpha]]\!] &:= \mathsf{unfold}\circ[\![\Gamma\vdash t:\mu\alpha.\sigma]\!]
\end{aligned}
$$

where fold and unfold are the respective isomorphisms between $[\![\mu\alpha.\sigma]\!]$ and $[\![\sigma[\mu\alpha.\sigma/\alpha]]\!]$ from the construction of the minimal invariant

Table 5.2: Interpretation of SPCF$_\infty$-terms-in-context

For showing that the **LBD** model of SPCF$_\infty$is computationally adequate we have to show that the model is correct w.r.t. the operational semantics, i.e. evaluation of terms does not change their denotational values. This can be done as usual by induction on the reduction rules and hence is omitted. Further, we have to ensure that the operational semantics is complete w.r.t. to the model, i.e. $[\![t]\!] \neq \bot$ implies $t \to^*_{\mathsf{op}} \top$ for any closed term $t:\mathbf{0}$. For this purpose one can adopt the method in [Plo85] and use results from [Pit96] to establish a type-indexed family of formal-approximations to deduce completeness. The proof for this is similar to the one for a call-by-name variant of the language FPC given in [Roh02] and hence also omitted.

5.5 Universality of SPCF$_\infty$

In this section we show that the first order type $\mathbf{U} = \mathbf{N}{\to}\mathbf{N}$ is universal for the language SPCF$_\infty$ by proving that every type is a SPCF$_\infty$ definable retract of $\mathbf{U}$. Since all elements of the lbd $[\![\mathbf{U}]\!]$ can be defined syntactically we get universality of SPCF$_\infty$ for its model in **LBD**.

Definition 5.5.1. *A closed* SPCF$_\infty$*-type* σ *is called a* SPCF$_\infty$-definable retract *of a* SPCF$_\infty$*-type* τ *(denoted* $\sigma \lhd \tau$*) iff there exist closed terms* $e:\sigma\to\tau$ *and* $p:\tau\to\sigma$ *such that*

$$[\![p]\!]\circ[\![e]\!] = \mathrm{id}_{[\![\sigma]\!]}\,. \qquad \diamond$$

Definition 5.5.2. *An* SPCF_∞*-type* σ *is called* universal *iff every closed* SPCF_∞*-type* τ *is a* SPCF_∞*-definable retract of* σ. ◇

For the next lemma we will not give a prove here as it is a standard induction on the structure of type in contexts and can be found for a call-by-name variant of the language FPC in [Roh02].

Lemma 5.5.3. *An* SPCF_∞*-type* U *is universal iff for all* $n \in \omega + 1$ *the types*

$$U \to U, \qquad \Pi_{i\in n} U, \qquad \Sigma_{i \in n} U$$

are definable retracts of U.

And as an immediate consequence of Lemma 5.5.3 we get:

Lemma 5.5.4. *Suppose the* SPCF_∞*-type* U *is universal. If for the types* $\sigma \in \{U \to U, \Pi_{i\in n} U, \Sigma_{i\in n} U\}$ *there exists terms* e_σ, p_σ *such that*

$$[\![p_\sigma]\!] \circ [\![e_\sigma]\!] = \mathrm{id}_{[\![\sigma]\!]} \tag{†}$$

holds, then for all SPCF_∞*-types* σ *there exist terms* e_σ, p_σ *such that* (†) *holds.*

Lemma 5.5.5. *For any closed* SPCF_∞*-type* σ *and any* $n \in \omega + 1$ *we have*

$$\Pi_{i\in n}\sigma \lhd \mathbf{n} \to \sigma \quad \textit{and} \quad \Sigma_{i\in n}\sigma \lhd \mathbf{n} \times \sigma\,.$$

Proof. The retractions are given by the following terms

$$\begin{aligned} e_{\Pi_{i\in n}\sigma, \mathbf{n}\to\sigma} &:= \lambda p : \Pi_{i\in n}\sigma. \lambda k : \mathbf{n}.\, \mathbf{case}\, k\, \mathbf{of}\, (\underline{i} \Rightarrow \mathbf{pr}_i\, p)_{i \in n} \\ p_{\Pi_{i\in n}\sigma, \mathbf{n}\to\sigma} &:= \lambda f : \mathbf{n}\to\sigma. \langle f\, \underline{i}\rangle_{i\in n} \end{aligned}$$

and

$$\begin{aligned} e_{\Sigma_{i\in n}\sigma, \mathbf{n}\times\sigma} &:= \lambda s : \Sigma_{i\in n}\sigma.\, \mathbf{case}\, s\, \mathbf{of}\, (\mathbf{in}_i\, x \Rightarrow \langle i, x\rangle)_{i\in n} \\ p_{\Sigma_{i\in n}\sigma, \mathbf{n}\times\sigma} &:= \lambda p : \mathbf{n}\times\sigma.\, \mathbf{case}\, \mathbf{pr}_0\, p\, \mathbf{of}\, (\underline{i} \Rightarrow \mathbf{in}_i(\mathbf{pr}_1\, p))_{i\in n} \end{aligned}$$

□

Lemma 5.5.6. *The* SPCF_∞*-type* $\mathbf{U} := \mathbf{N} \to \mathbf{N}$ *is universal.*

Proof. If we can show that the types

$$\mathbf{U} \to \mathbf{U}, \qquad \Pi_{i\in n}\mathbf{U}, \qquad \Sigma_{i\in n}\mathbf{U}$$

are retracts of $\mathbf{U}$ we get the proposition using Lemma 5.5.3. We will only give the term for the respective embedding and projection and leave it to the reader, to check that $[\![p]\!] \circ [\![e]\!]$ is equal to the identity of the appropriate type.

ad $\Pi_{i\in n}\mathbf{U} \lhd \mathbf{U}$: The idea for the definition of this retraction is to encode an n-tupel of functions of type $\mathbf{U}$ as one single function by encoding the type $\mathbf{n}\times\mathbf{N}$ in the type $\mathbf{N}$. Unfortunately, the type $\mathbf{n}\times\mathbf{N}$ is in general not a retract of the type $\mathbf{N}$. Nevertheless, we can define closed SPCF_∞-terms $\iota_\mathbf{n} : (\mathbf{n}\times\mathbf{N})\to\mathbf{N}$ and $\pi_\mathbf{n} : \mathbf{N}\to(\mathbf{n}\times\mathbf{N})$ such that $[\![\pi_\mathbf{n}(\iota_\mathbf{n}\langle\underline{i},\underline{m}\rangle)]\!] = [\![\langle\underline{i},\underline{m}\rangle]\!]$ for all $i \in n$ and $m \in \omega$.[2] Hence, we have to take special care of those functions, that return a value without evaluating their argument. This can be done using the control operator **catch**. We take the terms

$$e_{\Pi_{i\in n}\mathbf{U}} := \lambda f : \Pi_{i\in n}\mathbf{U}.\lambda n : \mathbf{N}.\,\mathbf{case}\,\mathbf{pr}_0(\pi_\mathbf{2}(n))\,\mathbf{of}\begin{pmatrix}\mathbf{in}_0\,x \Rightarrow \mathbf{catch}^\mathbf{U}(\mathbf{pr}_i(f))\\ \mathbf{in}_1\,x \Rightarrow T\end{pmatrix}$$

with

$$T := \mathbf{case}\,\mathbf{pr}_0(\pi_\mathbf{n}(\mathbf{pr}_1(\pi_\mathbf{2}(n))))\,\mathbf{of}\,(\underline{j} \Rightarrow \mathbf{pr}_j(f)(\mathbf{pr}_1(\pi_\mathbf{n}(\mathbf{pr}_1(\pi_\mathbf{2}(n))))))_{j\in n}$$

and

$$p_{\Pi_{i\in n}\mathbf{U}} := \lambda f : \mathbf{U}.\langle\mathbf{case}\,f(\iota_\mathbf{2}\langle\underline{0},\underline{i}\rangle)\,\mathbf{of}\begin{pmatrix}\mathbf{in}_0\,x \Rightarrow \lambda n : \mathbf{N}.f(\iota_\mathbf{2}\langle\underline{1},\iota_\mathbf{n}\langle\underline{i},n\rangle\rangle)\\ \mathbf{in}_{j+1}\,x \Rightarrow \lambda n : \mathbf{N}.\underline{j}\end{pmatrix}_{j\in\omega}\rangle_{i\in n}$$

which form a retraction pair.

ad $\Sigma_{i\in n}\mathbf{U}\lhd\mathbf{U}$: We have already shown that $\mathbf{U}\times\mathbf{U}\lhd\mathbf{U}$ holds. Further, by Lemma 5.5.5 it follows that

$$\Sigma_{i\in n}\mathbf{U} \;\lhd\; \mathbf{n}\times\mathbf{U} \;\lhd\; \mathbf{U}\times\mathbf{U} \;\lhd\; \mathbf{U}$$

holds.

ad $\mathbf{U}\to\mathbf{U} \lhd \mathbf{U}$: By currying we have $\mathbf{U}\to\mathbf{U} \cong (\mathbf{U}\times\mathbf{N})\to\mathbf{N}$. As $\mathbf{U}\times\mathbf{N} \lhd \mathbf{U}\times\mathbf{U} \lhd \mathbf{U}$ it suffices to construct a retraction $\mathbf{U}\to\mathbf{N} \lhd \mathbf{U}$ for showing that $\mathbf{U}\to\mathbf{U} \lhd \mathbf{U}$ holds. For this purpose we adapt an analogous result given by J. Longley in [Lon02] for ordinary sequential algorithms without error elements. The function p interprets elements of $\mathbf{U}$ as sequential algorithms for functionals of type $\mathbf{U}\to\mathbf{N}$ as described in [Lon02]. For a given $F : \mathsf{U}\to\mathsf{N}$ the element $[\![e]\!](F) : \mathsf{N}\to\mathsf{N}$ is a strategy / sequential algorithm for computing F. This is achieved by computing sequentiality indices iteratively using **catch**.

Suppose, we have given a functional $F : \mathbf{U}\to\mathbf{N}$ and a function $f : \mathbf{U}$ such that $F(f)$ evaluates to some value. Then f has been evaluated at only finitely many terms. As the set of all finite subgraphs of f is countable, this gives us the possibility of coding F as term of type $\mathbf{N}\to\mathbf{N}$.

For this purpose, we assume that we have given functions[3] $\alpha : (\mathbb{1}+\mathbf{N}+\mathbf{N})\to\mathbf{N}$ and $\alpha^* : \mathbf{N}\to(\mathbb{1}+\mathbf{N}+\mathbf{N})$ satisfying $[\![\alpha^*(\alpha(\mathbf{in}_0\langle\rangle))]\!] = [\![\mathbf{in}_0\langle\rangle]\!]$ and $[\![\alpha^*(\alpha(\mathbf{in}_i\,n))]\!] = [\![\mathbf{in}_i\,n]\!]$ for

[2]Those terms can already be given as primitive recursive functions, i.e. the terms $\iota_\mathbf{n}$ and $\pi_\mathbf{n}$ can be coded without the use of coding functions on the integers as an infinite case-construction. Additionally, for all those $\iota_\mathbf{n},\pi_\mathbf{n}$ we have $[\![\pi_\mathbf{n}(\iota_\mathbf{n}\langle\bot,\underline{m}\rangle)]\!] = [\![\pi_\mathbf{n}(\iota_\mathbf{n}\langle\underline{i},\bot\rangle)]\!] = \bot$ (resp. $[\![\pi_\mathbf{n}(\iota_\mathbf{n}\langle\top,\underline{m}\rangle)]\!] = [\![\pi_\mathbf{n}(\iota_\mathbf{n}\langle\underline{i},\top\rangle)]\!] = \top$).

[3]All those codings can be given in terms of primitive recursion.

$i = 1, 2$ and $n \in \omega$, and the following auxiliary list-handling functions in Haskell-style where γ encodes lists of pairs of natural numbers as natural numbers: nil represents the encoded empty list. The function $\mathsf{cons} : (\mathbf{N}\times(\mathbf{N}\times\mathbf{N}))\to\mathbf{N}$ decodes a given (encoded) list, appends a pair of natural numbers and encodes the result. Finally, the function $\mathsf{find} : (\mathbf{N}\times\mathbf{N})\to(\mathbf{N}+\mathbb{1})$ applied to a pair (g, x) returns $\mathbf{in}_0\, y$ if the encoded list g contains the pair (x, y) and otherwise it returns $\mathbf{in}_1\langle\rangle$. (Notice that find will be applied only to such (g, x) where $\gamma^{-1}(g)$ is a finite subset of the graph of a function $f : \mathbf{N}\to\mathbf{N}$.)

$$\begin{aligned}
\mathsf{nil} &:= \gamma(\texttt{[]})\\
\mathsf{cons}(g, (x, y)) &:= \gamma((x, y) : \gamma^{-1}(g))\\
\mathsf{find}(g, x) &:= \texttt{case}\ \gamma^{-1}(g)\ \texttt{of}\\
&\qquad \begin{array}{lll}
\texttt{[]} & \texttt{->} & \mathbf{in}_1\langle\rangle\\
((x, y) : r) & \texttt{->} & \mathbf{in}_0\, y\\
(_ : r) & \texttt{->} & \mathsf{find}(\gamma(r), x)
\end{array}
\end{aligned}$$

The embedding $e : (\mathbf{U}\to\mathbf{N})\to(\mathbf{N}\to\mathbf{N})$ is given by the following term:

$$e := \lambda F : \mathbf{U}\to\mathbf{N}.\lambda n : \mathbf{N}.\, \mathbf{case}\, \alpha^*(n)\, \mathbf{of}$$
$$\left(\begin{array}{l}
\mathbf{in}_0\, t \Rightarrow \mathbf{case}\,\mathbf{catch}^{\mathbf{U}\to\mathbf{N}}(F)\,\mathbf{of}\, \begin{pmatrix} \mathbf{in}_0\, x \Rightarrow \alpha(\mathbf{in}_0\langle\rangle)\\ \mathbf{in}_{i+1}\, x \Rightarrow \alpha(\mathbf{in}_1\, \underline{i})\end{pmatrix}_{i\in\omega}\\
\mathbf{in}_1\, t \Rightarrow \alpha(\mathbf{in}_1(F(\lambda x : \mathbf{N}.t)))\\
\mathbf{in}_2\, t \Rightarrow \mathbf{case}\, R\, \mathbf{of}\, \begin{pmatrix} \mathbf{in}_{2i}\, x \Rightarrow \alpha(\mathbf{in}_1\, \underline{i})\\ \mathbf{in}_{2i+1}\, x \Rightarrow \alpha(\mathbf{in}_2\, \underline{i})\end{pmatrix}_{i\in\omega}
\end{array}\right)$$

with

$$R := \mathbf{catch}(\lambda x : \mathbf{0}^\omega.\, \mathbf{case}\, F(\lambda n : \mathbf{N}.\, \mathbf{case}\, \mathsf{find}(t, n)\, \mathbf{of}$$
$$\begin{pmatrix} \mathbf{in}_0\, s \Rightarrow s\\ \mathbf{in}_1\, s \Rightarrow \mathbf{case}^{\mathbf{0},\mathbf{N}}(\mathbf{case}\, n\, \mathbf{of}\, (\mathbf{in}_j\, u \Rightarrow \mathbf{pr}_{2j+1}\, x)_{j\in\omega})\, \mathbf{of}\, ()\end{pmatrix})\, \mathbf{of}\, (\mathbf{in}_i\, s \Rightarrow \mathbf{pr}_{2i}\, x)_{i\in\omega})$$

The projection $p : (\mathbf{N}\to\mathbf{N})\to(\mathbf{U}\to\mathbf{N})$ is given by the following term:

$$p := \lambda r : \mathbf{N}\to\mathbf{N}.\lambda f : \mathbf{N}\to\mathbf{N}.\, \mathbf{case}\, \alpha^*(r(\alpha(\mathbf{in}_0\langle\rangle)))\, \mathbf{of}\, \begin{pmatrix} \mathbf{in}_0\, t \Rightarrow S\\ \mathbf{in}_1\, t \Rightarrow t\\ \mathbf{in}_2\, t \Rightarrow \bot\end{pmatrix}$$

with

$$S := \mathbf{case}\,\mathbf{catch}^{\mathbf{N}\to\mathbf{N}}(f)\,\mathbf{of}\, \left(\begin{array}{l} \mathbf{in}_0\, t \Rightarrow T(\mathsf{nil})\\ \mathbf{in}_{i+1}\, t \Rightarrow \mathbf{case}\, \alpha^*(r(\alpha(\mathbf{in}_1\, \underline{i})))\, \mathbf{of}\, \begin{pmatrix} \mathbf{in}_0\, t \Rightarrow \bot\\ \mathbf{in}_1\, t \Rightarrow t\\ \mathbf{in}_2\, t \Rightarrow \bot\end{pmatrix}\end{array}\right)_{i\in\omega}$$

and

$$T := \mathbf{Y}_{\mathbf{N}\to\mathbf{N}}(\lambda h : \mathbf{N}\to\mathbf{N}.\lambda g : \mathbf{N}.\,\mathbf{case}\,\alpha^*(r(\alpha(\mathbf{in}_2\, g)))\,\mathbf{of} \begin{pmatrix} \mathbf{in}_0\, t \Rightarrow \bot \\ \mathbf{in}_1\, t \Rightarrow t \\ \mathbf{in}_2\, t \Rightarrow h(\mathsf{cons}(g, (t, f(t)))) \end{pmatrix})$$

□

Lemma 5.5.7. *All elements in the locally boolean domain* $[\![\mathbf{U}]\!]$ *are* SPCF_∞*-definable, i.e. if* $f \in [\![\mathbf{U}]\!]$ *then there exists a closed* SPCF_∞*-term* $t : \mathbf{U}$ *with* $[\![t : \mathbf{U}]\!] = f$.

Proof. Suppose $f : [\![\mathbf{N}]\!] \to [\![\mathbf{N}]\!]$ is a sequential map. If $f(\bot) = m$ (resp. $f(\bot) = \top$) then we take the term $t := \lambda n : \mathbf{N}.\underline{m}$ (resp. $t := \lambda n : \mathbf{N}.\top$) and it follows that $[\![t]\!] = f$. We proceed analogously if $f(\top) = m$ (resp. $f(\top) = \bot$). In all other cases we have $f(\bot) = \bot$, $f(\top) = \top$ and $f(n) = m_n$. Hence, we can take the term $t := \lambda n : \mathbf{N}.\,\mathbf{case}\, n\,\mathbf{of}\,(\mathbf{in}_i\, x \Rightarrow \underline{i_m})_{i\in\omega}$ and get $[\![t]\!] = f$. □

Thus it follows that SPCF_∞ is universal for its **LBD** model.

Theorem 5.5.8. *The language* SPCF_∞ *is universal for its* **LBD** *model, i.e. for all closed* SPCF_∞*-types* σ *and elements* $d \in [\![\sigma]\!]$ *there exists a closed* SPCF_∞*-term* $t : \sigma$ *with* $[\![t]\!] = d$.

Proof. Suppose σ is a closed SPCF_∞-type and $d \in [\![\sigma]\!]$, then from Lemma 5.5.7 it follows that there exists a term $t : \mathbf{U}$ with $[\![e_\sigma]\!](d) = [\![t]\!]$. Thus, we get

$$[\![p_\sigma(t)]\!] = [\![p_\sigma]\!]([\![t]\!]) = [\![p_\sigma]\!]([\![e_\sigma]\!](d)) = d$$

as desired. □

6 CPS_∞: An infinitary CPS target language

The interpretation of the SPCF_∞ type $\delta := \mu\alpha.(\alpha^\omega{\to}\mathbf{0})$ is the minimal solution of the domain equation $D \cong [D^\omega{\to}\mathsf{O}]$. Obviously, we have $D \cong [D{\to}D]$. Moreover, it has been shown in [RS98] that D is isomorphic to O_∞, i.e. what one obtains by performing D. Scott's D_∞ construction in **LBD** when instantiating D by O.

We now describe an untyped infinitary language CPS_∞ canonically associated with the domain equation $D \cong [D^\omega{\to}\mathsf{O}]$.

6.1 The untyped language CPS_∞

The language $\mathsf{CPS}_\infty$is untyped call-by-name λ-calculus with abstraction (resp. application) extended to countably-infinite lists of variables (resp. terms). In addition CPS_∞ contains an non-recuperable error-element $\top$.

The terms of the language CPS_∞ are given by the following grammar:

$$\begin{aligned} M &::= x \mid \lambda\vec{x}.t & \qquad\qquad \vec{x} &\equiv (x_i)_{i\in\omega} \\ t &::= \top \mid M\langle\vec{M}\rangle & \qquad\qquad \vec{M} &\equiv (M_i)_{i\in\omega} \end{aligned}$$

The operational semantics of CPS_∞ is given by the following big step reduction rules:

$$\frac{}{\top \Downarrow \top} \qquad \frac{t[M_i/x_i]_{i\in\omega} \Downarrow \top}{(\lambda\vec{x}.t)\langle\vec{M}\rangle \Downarrow \top}$$

The language CPS_∞ is an extension of pure untyped λ-calculus since applications MN can be expressed by $\lambda\vec{y}.M\langle N,\vec{y}\rangle$ and abstraction $\lambda x.M$ by $\lambda x\vec{y}.M\langle\vec{y}\rangle$ where $\vec{y}$ are fresh variables. Thus, CPS_∞ allows for recursion and we can define recursion combinators in the usual way.

To allow a more compact representation of CPS_∞-terms, we will write

$$\lambda y_1 \ldots y_n\vec{x} \qquad \text{for} \qquad \lambda(z_i)_{i\in\omega} \quad \text{with } z_i \equiv \begin{cases} y_{i+1} & \text{if } i < n, \\ x_{i-n} & \text{otherwise} \end{cases}$$

and

$$\langle N_1,\ldots,N_n,\vec{M}\rangle \qquad \text{for} \qquad \langle Z_i\rangle_{i\in\omega} \quad \text{with } Z_i \equiv \begin{cases} N_{i+1} & \text{if } i < n, \\ M_{i-n} & \text{otherwise} \end{cases}$$

Notice that we will use the above abbreviations mostly in the form $\lambda\vec{x}$ and $\langle\vec{M}\rangle$, i.e. with $n = 0$. Additionally, we define the term

$$\bot :\equiv \lambda y\vec{x}.W\langle W,\vec{x}\rangle \quad \text{with} \quad W :\equiv \lambda y\vec{x}.y\langle y,\vec{x}\rangle$$

which has an infinite reduction-tree and denotes $\bot$.

Finally, we introduce the following abbreviation

$$R_0 :\equiv \lambda\vec{x}.x_0\langle\vec{\bot}\rangle$$

6.2 Universality of CPS_∞

In this section we show that the language CPS_∞ is universal w.r.t. the lbd D. Universality of CPS_∞ will be shown in two steps. First we argue why all finite elements of D are CPS_∞ definable. Then adapting a trick from [Lai98] we show that suprema of chains increasing w.r.t. $\leq_s$ are CPS_∞definable, too.

Lemma 6.2.1. *The lbd* O *is a* CPS_∞ *definable retract of the lbd* D*.*

Proof. The CPS_∞ term R_0 retracts D to O as it sends $\top_D$ to $\top_D$ and all other elements of D to $\bot_D$. □

Notice that the language CPS_∞ is more expressive than pure untyped λ-calculus as it does not contain a term semantically equivalent to R_0.[1]

Lemma 6.2.2. *The lbds* N *and* U *are both* CPS_∞ *definable retract of the lbd* D*.*

Proof. Since we can retract the lbd D to the lbd O (by Lemma 6.2.1) and $[\mathsf{O}^\omega\to\mathsf{O}] \cong \mathsf{N}$ it follows that N is a CPS_∞ definable retract of D. As $D \cong [D\to D]$ is a CPS_∞ definable retract of D it follows that $\mathsf{U} = [\mathsf{N}\to\mathsf{N}]$ is a CPS_∞ definable retract of D. □

Thus, we can do arithmetic within CPS_∞. Natural numbers are encoded by $\underline{n} \equiv \lambda\vec{x}.x_n\langle\vec{\bot}\rangle$ and a function $f{:}\mathbb{N}\to\mathbb{N}$ by its graph, i.e. $\underline{f} \equiv \lambda x\vec{y}.x\langle\lambda\vec{z}.\underline{f(i)}\langle\vec{y}\rangle\rangle_{i\in\omega}$. Notice that CPS_∞ allows for the implementation of an infinite case construct.

Lemma 6.2.3. *All finite elements of the lbd* D *are* CPS_∞ *definable.*

Proof. In [Lai05a] Jim Laird has shown that the language $\Lambda_\bot^\top$, i.e. simply typed λ-calculus over the base type $\{\bot,\top\}$ is universal for its model in **LBD**. Thus, since all retractions of D to its finitary approximations D_n are CPS_∞ definable and all compact elements use only finitely many arguments it follows that all finite elements of D are CPS_∞ definable. □

[1] Since $[\![\lambda\vec{x}.x_0\langle\vec{\bot}\rangle]\!]$ is certainly "computable" pure λ-calculus with constant $\top$ cannot denote all "computable" elements.

Definition 6.2.4. *Let $f : A \to \mathsf{O}$ be a* **LBD** *morphism then we define the map $\widetilde{f} : A \to [\mathsf{O}{\to}\mathsf{O}]$ with*

$$\widetilde{f}(a)(u) := \begin{cases} u & \text{if } f(a^\top) = \bot_\mathsf{O} \text{ and} \\ f(a) & \text{otherwise.} \end{cases} \qquad \diamond$$

Informally, the map $\widetilde{f}$ can be described as the function where

"in the strategy of f all occurrences of $\bot$ are replaced by u".

Next we show that for all $f : A \to \mathsf{O}$ in **LBD** the map $\widetilde{f} : A \to [\mathsf{O}{\to}\mathsf{O}]$ is an **LBD** morphism as well.

Lemma 6.2.5. *If $f : A \to \mathsf{O}$ is a sequential map between lbds then the function $\widetilde{f} : A \to [\mathsf{O}{\to}\mathsf{O}]$ given by Def. 6.2.4 is sequential.*

Proof. For showing monotonicity suppose $a_1, a_2 \in A$ with $a_1 \sqsubseteq a_2$ and $u \in \mathsf{O}$. We proceed by case analysis on $f(a_1^\top)$.

Suppose $f(a_1^\top) = \bot_\mathsf{O}$. Thus, $\widetilde{f}(a_1)(u) = u$. If $f(a_2^\top) = \bot_\mathsf{O}$ then $\widetilde{f}(a_2)(u) = u$, and we get $\widetilde{f}(a_1)(u) = u = \widetilde{f}(a_2)(u)$. If $f(a_2^\top) = \top_\mathsf{O}$ then $\widetilde{f}(a_2)(u) = f(a_2)$. As $f(a_1^\top) = \bot_\mathsf{O}$ it follows that $f(\neg a_1) = \bot_\mathsf{O}$ and $f(\neg a_2) = \bot_\mathsf{O}$ (because $\neg a_2 \sqsubseteq \neg a_1$). As $\top_\mathsf{O} = f(a_2^\top) = f(a_2) \sqcup f(\neg a_2)$ it follows that $f(a_2) = \top_\mathsf{O}$ as desired.

If $f(a_1^\top) = \top_\mathsf{O}$ then $\widetilde{f}(a_1)(u) = f(a_1)$. W.l.o.g. assume $f(a_1) = \top_\mathsf{O}$. Then $\top_\mathsf{O} = f(a_1) \sqsubseteq f(a_2) \sqsubseteq f(a_2^\top)$. Hence, $f(a_2) = \top_\mathsf{O} = f(a_2^\top)$ and we get $\widetilde{f}(a_2)(u) = f(a_2) = \top_\mathsf{O}$.

Next we show that $\widetilde{f}$ is bistable. Let $a_1 \updownarrow a_2$, thus $(\dagger)$ $a_1^\top = a_2^\top = (a_1 \sqcap a_2)^\top$.

If $f(a_1^\top) = f(a_2^\top) = \bot_\mathsf{O}$ then $\widetilde{f}(a_1) = \mathrm{id}_\mathsf{O} = \widetilde{f}(a_2)$. If $f(a_1^\top) = f(a_2^\top) = \top_\mathsf{O}$ then $\widetilde{f}(a_i) = \lambda x{:}\mathsf{O}.\, f(a_i)$ for $i \in \{1,2\}$. Since $\lambda x{:}\mathsf{O}.\, \bot_\mathsf{O} \updownarrow \lambda x{:}\mathsf{O}.\, \top_\mathsf{O}$ it follows that $\widetilde{f}$ preserves bistable coherence.

Finally we show that $\widetilde{f}$ preserves bistably coherent suprema and infima. If $f((a_1 \sqcap a_2)^\top) = \bot_\mathsf{O}$ then $\widetilde{f}(a_1 \sqcap a_2)(u) = u = \widetilde{f}(a_1)(u) \sqcap \widetilde{f}(a_2)(u)$ (since $f(a_1^\top) = f(a_2^\top) = \bot_\mathsf{O}$ by $(\dagger)$). Otherwise, if $f((a_1 \sqcap a_2)^\top) = \top_\mathsf{O}$ then $\widetilde{f}(a_1 \sqcap a_2)(u) = f(a_1 \sqcap a_2) = f(a_1) \sqcap f(a_2) = \widetilde{f}(a_1)(u) \sqcap \widetilde{f}(a_2)(u)$ (since f is bistable and $f(a_1^\top) = f(a_2^\top) = \top_\mathsf{O}$ by $(\dagger)$).

Analogously, one shows that $\widetilde{f}$ preserves bistably coherent suprema. $\square$

The following observation is useful when computing with functions of the form $\widetilde{f}$.

Lemma 6.2.6. *If $f : A \to \mathsf{O}$ is a* **LBD** *morphism then $\widetilde{f}(a)(\bot_\mathsf{O}) = f(a)$.*

Proof. If $f(a) = \bot_\mathsf{O}$ then $\widetilde{f}(a)(\bot_\mathsf{O}) = \bot_\mathsf{O} = f(a)$ since $\bot$ and $f(a)$ are the only possible values of $\widetilde{f}(a)(\bot_\mathsf{O})$. If $f(a) = \top_\mathsf{O}$ then $f(a^\top) = \top_\mathsf{O}$ and thus $\widetilde{f}(a)(\bot_\mathsf{O}) = f(a)$ as desired. $\square$

Lemma 6.2.7. *For $f, g : A \to \mathsf{O}$ with $f \leq_s g$ it holds that $\widetilde{g}(x) = \widetilde{f}(x) \circ \widetilde{g}(x)$ for all $x \in A$.*

Proof. Suppose $f \leq_s g$. Let $x \in A$ and $u \in \mathsf{O}$. We have to show that $\widetilde{g}(x)(u) = \widetilde{f}(x)(\widetilde{g}(x)(u))$.

If $g(x^\top) = \bot_\mathsf{O}$ then $f(x^\top) = \bot_\mathsf{O}$ (since $f \leq_s g$) and thus $\widetilde{g}(x)(u) = u = \widetilde{f}(x)(\widetilde{g}(x)(u))$. Thus, w.l.o.g. suppose $g(x^\top) = \top_\mathsf{O}$. Then $\widetilde{g}(x)(u) = g(x)$.

If $f(x) = \top_\mathsf{O}$ then $f(x^\top) = \top_\mathsf{O} = g(x)$ and, therefore, we have $\widetilde{f}(x)(\widetilde{g}(x)(u)) = f(x) = \top_\mathsf{O} = g(x) = \widetilde{g}(x)(u)$.

Suppose $f(x) = \bot_\mathsf{O}$.

If $g(x) = \bot_\mathsf{O}$ then we have $\widetilde{f}(x)(\widetilde{g}(x)(u)) = \widetilde{f}(x)(g(x)) = \widetilde{f}(x)(\bot_\mathsf{O}) = \bot_\mathsf{O}$ where the last equality holds by Lemma 6.2.6.

Now suppose $g(x) = \top_\mathsf{O}$. We proceed by case analysis on the value of $f(x^\top)$.

If $f(x^\top) = \bot_\mathsf{O}$ then $\widetilde{f}(x)(\widetilde{g}(x)(u)) = \widetilde{g}(x)(u)$ and we are finished.

We show that the case $f(x^\top) = \top_\mathsf{O}$ cannot happen. Suppose $f(x^\top) = \top_\mathsf{O}$. Then by bistability we have $\top_\mathsf{O} = f(x^\top) = f(x) \sqcup f(\neg x) = \bot_\mathsf{O} \sqcup f(\neg x) = f(\neg x)$ and thus also $\neg f(\neg x) = \bot_\mathsf{O}$. Since $f \leq_s g$ we have $g \sqsubseteq f^\top$. Moreover, by Cor. 3.5.12(2) we have $(\neg f)(x) \sqsubseteq \neg f(\neg x)$. Thus, we have $\top_\mathsf{O} = g(x) \sqsubseteq f^\top(x) = f(x) \sqcup (\neg f)(x) = (\neg f)(x) \sqsubseteq \neg f(\neg x) = \bot_\mathsf{O}$ which clearly is impossible. □

In the following we denote by $i : \mathsf{O} \to D$ and $p : D \to \mathsf{O}$ the embedding of O into D (resp. projection from D to O) given by

$$i(x) := \begin{cases} \top_D & \text{if } x = \top_\mathsf{O}, \\ \bot_D & \text{otherwise} \end{cases} \qquad\qquad p(x) := \begin{cases} \top_\mathsf{O} & \text{if } x = \top_D, \\ \bot_\mathsf{O} & \text{otherwise.} \end{cases}$$

Definition 6.2.8. *Let $f \in D \cong [D^\omega \to \mathsf{O}]$. Then we write $\widehat{f}$ for that element of D with*

$$\widehat{f}(d_0, \vec{d}) := \widetilde{f}_n(\vec{d})(p(d_0))$$ ◇

Lemma 6.2.9. *For every finite f in D the element $\widehat{f}$ is also finite and thus CPS_∞ definable.*

Proof. If A is a finite lbd then for every $f : A \to \mathsf{O}$ the **LBD** map $\widetilde{f} : A \to [\mathsf{O}\to\mathsf{O}]$ is also finite. This holds in particular for f in the finite type hierarchy over O.

Since embeddings of lbds preserves finiteness of elements we conclude that for every finite f in D the element $\widehat{f}$ is finite as well. Thus, by Lemma 6.2.3 the element $\widehat{f}$ is CPS_∞ definable. □

Now we are ready to prove our universality result for CPS_∞.

Theorem 6.2.10. *All elements of the lbd D are CPS_∞ definable.*

Proof. Suppose $f \in D$. Then $f = \bigsqcup f_n$ for some increasing (w.r.t. $\leq_s$) chain $(f_n)_{n\in\omega}$ of finite elements. Since by Lemma 6.2.9 all $\widehat{f}_n$ are CPS_∞ definable there exists a CPS_∞ term F with $[\![F\underline{n}]\!] = \widehat{f}_n$ for all $n \in \omega$.

Since recursion is available in CPS_∞ one can exhibit a CPS_∞ term Ψ such that

$$\Psi g = \lambda x.\, g(\underline{0})(\Psi(\lambda n.\, g(n{+}1))x) = \bigsqcup_{n\in\omega}(g(\underline{0})\circ\cdots\circ g(\underline{n}))(\bot)$$

holds. (Using computational adequacy of the model one can show that Ψg denotes the least fixpoint of the sequence $((g(\underline{0})\circ\cdots\circ g(\underline{n}))(\bot))_{n\in\omega}$.)

Thus, the term $M_f \equiv \lambda\vec{x}.\,\Psi(\lambda y.\lambda z.F\langle y, i(z), \vec{x}\rangle)$ denotes f since

$$\begin{aligned}
M_f(\vec{d}) &= \Psi(\lambda y.\lambda z.F(y, i(z), \vec{d}))\\
&= \bigsqcup_{n\in\omega}(\lambda z.F\underline{0}(i(z),\vec{d}))\circ\cdots\circ(\lambda z.F\underline{n}(i(z),\vec{d}))(\bot)\\
&= \bigsqcup_{n\in\omega}(\lambda z.\widehat{f}_0(i(z),\vec{d}))\circ\cdots\circ(\lambda z.\widehat{f}_n(i(z),\vec{d}))(\bot)\\
&= \bigsqcup_{n\in\omega}((\lambda z.\widetilde{f}_0(\vec{d})(p(i(z))))\circ\cdots\circ(\lambda z.\widetilde{f}_n(\vec{d})(p(i(z)))))(\bot)\\
&= \bigsqcup_{n\in\omega}((\lambda z.\widetilde{f}_0(\vec{d})(z))\circ\cdots\circ(\lambda z.\widetilde{f}_n(\vec{d})(z)))(\bot)\\
&= \bigsqcup_{n\in\omega}(\widetilde{f}_0(\vec{d})\circ\cdots\circ\widetilde{f}_n(\vec{d}))(\bot)\\
&= \bigsqcup_{n\in\omega}(\widetilde{f}_n(\vec{d}))(\bot) && \text{(by Lemma 6.2.7)}\\
&= \bigsqcup_{n\in\omega} f_n(\vec{d}) && \text{(by Lemma 6.2.6)}\\
&= f(\vec{d})
\end{aligned}$$

for all $\vec{d}\in D^\omega$. □

6.3 Lack of faithfulness of the interpretation

In the previous section we have shown that the interpretation of closed CPS_∞ terms in the lbd D is surjective. Recall that infinite normal forms for CPS_∞ are given by the grammar

$$N ::= x \mid \lambda\vec{x}.\top \mid \lambda\vec{x}.x\langle\vec{N}\rangle$$

understood in a coinductive sense.

Definition 6.3.1. *We call a model* faithful *iff for all normal forms N_1, N_2 if $[\![N_1]\!] = [\![N_2]\!]$ then $N_1 = N_2$.* ◇

We will show that the **LBD** model of CPS_∞ is not faithful.[2] For a closed CPS_∞ term M consider

$$M^* \equiv \lambda\vec{x}.x_0\langle\ \lambda\vec{y}.x_0\langle\bot, M, \vec{\bot}\rangle, \vec{\bot}\ \rangle$$

[2]For an affine version of CPS_∞ on can show that the **LBD** model is faithful.

Lemma 6.3.2. *For closed* CPS$_\infty$ *terms* M_1, M_2 *it holds that* $[\![M_1^*]\!] = [\![M_2^*]\!]$.

Proof. We will show that for all terms M the term M^* is semantically equivalent to the term R_0, i.e. for all $\vec{d} \in D^\omega$ we have $[\![M^*]\!](\vec{d}) = \top$ iff $d_0 = \top$.

If $d_0 = \bot$ or $d_0 = \top$ then we are finished.

Otherwise there is an n such that d_0 evaluates the n-th argument first. If $n = 0$ then $d_0\langle\bot, M, \vec{\bot}\rangle = \bot$, thus

$$d_0\langle\lambda\vec{y}.d_0\langle\bot, M, \vec{\bot}\rangle, \vec{\bot}\rangle = d_0\langle\vec{\bot}\rangle = \bot\,.$$

which is also the case if $n \neq 0$. $\square$

Suppose N_1 and N_2 are different infinite normal forms. Then N_1^* and N_2^* have different infinite normal forms and we get $[\![N_1^*]\!] = [\![N_2^*]\!]$ by Lemma 6.3.2. Thus, the **LBD** model of CPS$_\infty$ is not faithful.

Lemma 6.3.3. *There exist infinite normal forms* N_1, N_2 *in* CPS$_\infty$ *that can not be separated by* CPS$_\infty$ *terms.*

Proof. We have different normal forms N_1, N_2 with $[\![N_1]\!] = [\![N_2]\!]$. Since all CPS$_\infty$ terms preserve model equality the terms N_1 and N_2 cannot be separated by CPS$_\infty$ terms. $\square$

Notice that in pure untyped λ-calculus different normal forms can always be separated (cf. [Bar84]).

7 Conclusion and possible extensions

We think that the defect that interpretation of CPS_∞ in the locally boolean domain D is not faithful (cf. section 6.3) can be overcome by extending the language by a parallel construct and refining the observation type O to $\mathsf{O}' \cong \mathsf{List}(\mathsf{O}')$. The language $\mathsf{CPS}^{\|}_\infty$ associated with the domain equation $D \simeq D^\omega \to \mathsf{O}'$ is given by

$$\begin{aligned} M &::= x \mid \lambda\vec{x}.t \\ t &::= \top \mid M\langle\vec{M}\rangle \mid (\!| t \parallel \ldots \parallel t |\!) \end{aligned}$$

The syntactic values are given by the grammar $V ::= \top \mid (\!| V \| \ldots \| V |\!)$. The operational semantics of $\mathsf{CPS}^{\|}_\infty$ is the operational semantics of CPS_∞ extended by the rule

$$\frac{(\lambda\vec{x}.t_i)\langle\vec{M}\rangle \Downarrow V_i \quad \text{for all } i \in \{1,\ldots,n\}}{(\lambda\vec{x}.(\!| t_1 \| \ldots \| t_n |\!))\langle\vec{M}\rangle \Downarrow (\!| V_1 \| \ldots \| V_n |\!)}$$

and the normal forms of $\mathsf{CPS}^{\|}_\infty$ are given by the grammar

$$\begin{aligned} N &::= x \mid \lambda\vec{x}.t \\ t &::= \top \mid x\langle\vec{N}\rangle \mid (\!| t \| \ldots \| t |\!) \end{aligned}$$

understood in a coinductive sense.

Separability of normal forms can be shown for an affine version of CPS_∞ by substituting the respective projections for head variables. Using the parallel construct $(\!| \ldots \| \ldots |\!)$ of $\mathsf{CPS}^{\|}_\infty$ we can substitute for a head variable quasi simultaneously *both* the respective projection *and* the head variable itself. Since the interpretation of $\mathsf{CPS}^{\|}_\infty$ is faithful w.r.t. the parallel construct $(\!| \ldots \| \ldots |\!)$ we get separation for $\mathsf{CPS}^{\|}_\infty$ normal forms as in the affine case. This kind of argument can be seen as a “qualitative” reformulation of a related “quantitative” method introduced by F. Maurel in his Thesis [Mau04] albeit in the somewhat more complex context of J.-Y. Girard’s *Ludics* [Gir01].

In a sense this is not surprising since our parallel construct introduced above allows one to make the same observations as with **parallel-or**. The only difference is that our parallel construct keeps track of all possibilities simultaneously whereas the traditional semantics of **parallel-or** takes their supremum thus leading out of the realm of sequentiality. This is avoided by our parallel construct at the price of a more complicated domain of observations. For an approach in a similar spirit see [HM99].

Another prospect is the development of a theory of computability for locally boolean domains. In [Asp90] A. Asperti has successfully developed a notion of computability for the stable model of PCF. We are convinced that this approach can be extended to locally boolean domains. Curien-Lamarche games A arising as interpretation of a type expressions are effective (i.e. $P_A \subseteq \mathsf{Rsp}^\top(A)$ is decidable). An element $s \in \mathsf{Strat}(A)$ is computable iff s is an r.e. subset of $\mathsf{Rsp}^\top(A)$. Obviously, an element $f \in \mathsf{U} = [\mathsf{N}{\to}\mathsf{N}]$ is computable in this sense iff it can be denoted by an r.e. term. Since all $e_\sigma : \sigma \lhd \mathbf{U} : p_\sigma$ can be denoted by r.e. terms it follows that all computable elements of type $[\![\sigma]\!]$ can be denoted by r.e. terms. Obviously, denotations of r.e. terms of type σ denote computable elements of $[\![\sigma]\!]$. Thus elements of $[\![\sigma]\!]$ are computable iff they can be denoted by r.e. terms.

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