SIAM J. DISCRETE MATH.
Vol. 34, No. 2, pp. 1409–1443

TIGHT ANALYSIS OF THE SMARTSTART ALGORITHM FOR ONLINE DIAL-A-RIDE ON THE LINE*

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Abstract. The online DIAL-A-RIDE problem is a fundamental online problem in a metric space, where transportation requests appear over time and may be served in any order by a single server with unit speed. Restricted to the real line, online DIAL-A-RIDE captures natural problems like controlling a personal elevator. Tight results in terms of competitive ratios are known for the general setting and for online TSP on the line (where the source and target of each request coincide). In contrast, online DIAL-A-RIDE on the line has resisted tight analysis so far, even though it is a very natural online problem. We conduct a tight competitive analysis of the SMARTSTART algorithm that gave the best known results for the general, metric case. In particular, our analysis yields a new upper bound of 2.94 for open, nonpreemptive online DIAL-A-RIDE on the line, which improves the previous bound of 3.41 [S. O. Krumke, "Online Optimization Competitive Analysis and Beyond," Habilitation thesis, Technische Universität Berlin, 2001]. The best known lower bound remains 2.04 [A. Bjelde et al., in Proceedings of the 28th Annual Symposium on Discrete Algorithms (SODA), SIAM, 2017, pp. 994–1005]. We also show that the known upper bound of 2 [N. Ascheuer, S. O. Krumke, and J. Rambau, in Proceedings of the 17th Annual Symposium on Theoretical Aspects of Computer Science (STACS), Springer, 2000, pp. 639–650] regarding SMARTSTART's competitive ratio for closed, nonpreemptive online DIAL-A-RIDE is tight on the line.

Key words. DIAL-A-RIDE on the line, elevator problem, online algorithms, competitive analysis, SMARTSTART, competitive ratio

AMS subject classifications. 68W27, 90C27

DOI. 10.1137/19M1268513

1. Introduction.

Online optimization deals with settings where algorithmic decisions have to be made over time without knowledge of the future. A typical introductory example is the problem of controlling an elevator/conveyor system, where requests to transport passengers/goods arrive over time and the elevator needs to decide online how to adapt its trajectory along the real line. In terms of competitive analysis, the central question in this context is how much longer our trajectory will be in the worst case, relative to an optimum offline solution that knows all requests ahead of time, i.e., we ask for solutions with good *competitive ratio*.

While the elevator problem is a natural online problem, even simplified versions of it have long resisted tight analysis. *Online TSP on the line* is such a simplification, where a single server on the real line needs to serve requests that appear over time at arbitrary positions by visiting their location, i.e., requests do not need to be transported. We distinguish the *closed* and *open* variants of this problem, depending on whether the server needs to eventually return to the origin or not. Determining the exact competitive ratios for either variant had been an open problem for more than two decades [3, 6, 13, 14, 16, 17], when Bjelde et al. [5] were finally able to conduct a tight analysis that established competitive ratios of roughly 1.64 for the closed case and 2.04 for the open case.

*Received by the editors June 14, 2019; accepted for publication (in revised form) March 23, 2020; published electronically June 22, 2020. A preliminary version of this paper was published as [4].
https://doi.org/10.1137/19M1268513
Funding: This work was supported by the "Excellence Initiative" of the German Federal and State Governments and the Graduate School CE at TU Darmstadt.
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The next step towards formally capturing the intuitive elevator problem is to allow transportation requests that appear over time, and to fix a capacity $c \in \mathbb{N} \cup \{\infty\}$ of the server that limits the number of transportation requests that can be served simultaneously. The resulting *online DIAL-A-RIDE problem on the line* has received considerable attention in the past [1, 5, 9, 14, 15, 17], but still resists tight analysis. The best known (nonpreemptive) bounds put the competitive ratio in the range $[1.75, 2]$ for the closed variant (see [5, 1]). For the open variant the best known (nonpreemptive) bounds put the competitive ratio in the range $[2.04, 3]$ for $c = 1$ and in the range $[2.04, 3.41]$ for $c > 1$ (see [5, 14]). In this paper, we show an improved upper bound of (roughly) 2.94 for open online DIAL-A-RIDE on the line for arbitrary capacity $c \in \mathbb{N} \cup \{\infty\}$.

A straightforward algorithm for online DIAL-A-RIDE on the line is the algorithm IGNORE [1]: Whenever the server is idle and unserved requests R_t are present at the current time t, compute an optimum schedule to serve these requests from the current location, and follow this schedule while *ignoring* newly incoming requests. IGNORE has a competitive ratio of exactly 4 (see Appendix A). This competitive ratio can be improved by potentially waiting before starting the optimum schedule, in order to protect against requests that come in right after we decide to start. Ascheuer, Krumke, and Rambau [1] proposed the algorithm SMARTSTART (see Algorithm 2.1) that delays starting the optimum schedule until a certain time t relative to the length $L(t, p, R_t)$ of this schedule (formal definitions below).

SMARTSTART is parameterized by a factor $\Theta > 1$ that scales this waiting period. In this paper, we conduct a tight analysis of the best competitive ratio of SMARTSTART for open/closed online DIAL-A-RIDE on the line, over all parameter values $\Theta > 1$.

Results and techniques. The SMARTSTART algorithm is of particular importance for online DIAL-A-RIDE, since, on arbitrary metric spaces, it achieves the best possible competitive ratio of 2 for the closed variant [1, 3], and the best known competitive ratio of $2 + \sqrt{2} \approx 3.41$ for the open variant [14]. We provide a conclusive treatment of this algorithm for online DIAL-A-RIDE on the line in terms of competitive analysis, both for the open and the closed variant.

Regarding the open case, we show that SMARTSTART attains a competitive ratio of $\rho^* \approx 2.94$ for parameter value $\Theta^* \approx 2.05$ (section 3). To show this, we derive two separate upper bounds depending on Θ (cf. Figure 1): an upper bound $f_1(\Theta)$ in the case that SMARTSTART has a waiting period before starting its last schedule (Proposition 3.3), and an upper bound $f_2(\Theta)$ in the case that SMARTSTART begins its final schedule immediately (Proposition 3.7). The resulting general upper bound of $\max\{f_1(\Theta), f_2(\Theta)\}$ has its minimum precisely at the intersection point (Θ^*, ρ^*) of f_1 and f_2.

On the other hand, we show that for $\Theta \in (2, 3)$ there are instances where SMARTSTART waits before starting its final schedule and has a competitive ratio of at least $f_1(\Theta)$ (Proposition 4.2). Similarly, we show that for $\Theta \in [2, 2.303]$ there are instances where SMARTSTART does not wait before starting its final schedule and has competitive ratio at least $f_2(\Theta)$ (Proposition 4.3). Together, this implies that the general upper bound of $\max\{f_1(\Theta), f_2(\Theta)\}$ is tight for $\Theta \in (2, 2.303]$, and thus for $\Theta = \Theta^*$ (cf. Figure 1).

To complete our analysis of SMARTSTART, we give lower bound constructions for different domains of Θ (g_1 through g_4 in Figure 1) that establish that Θ^* is indeed the best parameter choice for SMARTSTART in the worst case (Lemma 4.8). The key ingredient to all our lower bounds is a way to *lure* SMARTSTART away from the origin (Lemma 4.1).

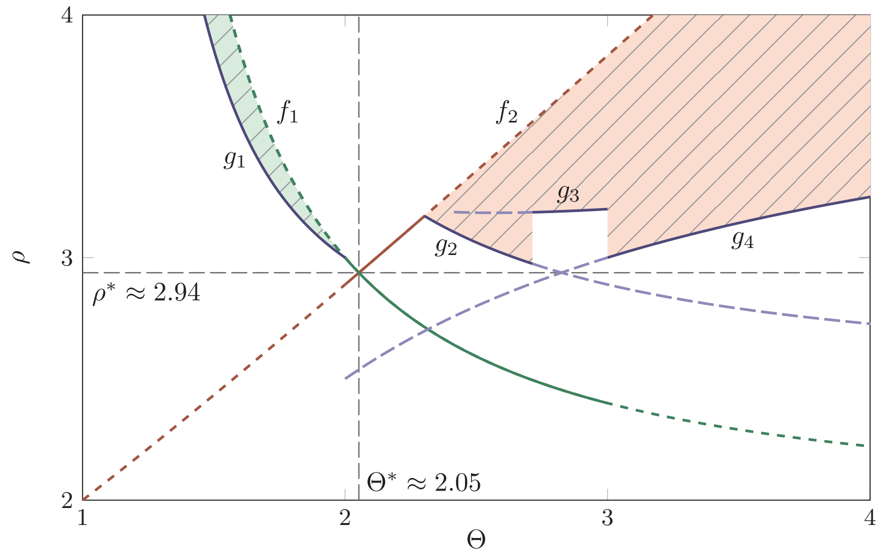

FIG. 1. *Overview of our bounds for* SMARTSTART. *The functions f_1 (green) / f_2 (red) are upper bounds for the cases where* SMARTSTART *waits / does not wait before starting the final schedule, respectively. The upper bounds are drawn solid in the domains where they are tight for their corresponding case. The functions g_1 through g_4 (blue) are general lower bounds; dashed continuations indicate how far these bounds could be extended.*

Finally, for the closed variant of the problem, we provide a lower bound of 2 on the best-possible competitive ratio of SMARTSTART over all possible choices of the parameter $\Theta > 1$ (section 5). This tightly matches the known upper bound for general metric spaces [1].

Significance. The main contribution of this paper is a conclusive treatment of the algorithm SMARTSTART for online DIAL-A-RIDE on the line in terms of competitive analysis. Additionally, our analysis yields an improved upper bound of (roughly) 2.94 for nonpreemptive, open online DIAL-A-RIDE on the line. This is the first bound below 3 and narrows the gap for the competitive ratio to $[2.04, 2.94]$. Our work is likely to serve as a starting point towards devising better algorithms (preemptive or nonpreemptive) that narrow the gaps for both the open and closed settings by avoiding critical "mistakes" of SMARTSTART, as evidenced by our lower bound constructions.

Further related work. In this paper, we focus on the nonpreemptive variant of online DIAL-A-RIDE on the line, where requests cannot be unloaded on the way in reaction to the arrival of new requests. For the case where preemption is allowed, the best known bounds for the closed version are $[1.64, 2]$ (see [3, 1]), which is slightly worse than the gap of $[1.75, 2]$ in the nonpreemptive case. On the other hand, the best bounds for the open, preemptive variant are $[2.04, 2.41]$ (see [5]), which is better than the gap of $[2.04, 2.94]$ in the nonpreemptive case. In particular, the preemptive and nonpreemptive cases can currently not be separated in terms of competitive ratios.

A variant of the online DIAL-A-RIDE problem where the objective is to minimize the maximal flow time, instead of the makespan, has been studied by Krumke et al. [15, 16]. They established that in many metric spaces no online algorithm can be competitive with respect to this objective. Hauptmeier, Krumke, and Rambau [12] showed that a competitive algorithm is possible if we restrict ourselves to instances with "reasonable" load, which roughly means that requests that appear over a sufficiently large time period T can always be served in time at most T.

Lipmann et al. [18] studied a natural variant of closed, online DIAL-A-RIDE where the destinations of requests are only revealed upon collection by the server. For general metric spaces and server capacity c, they showed a tight competitive ratio of 3 in the preemptive setting, and lower/upper bounds of $\max\{3.12, c\}$ and $2c+2$, respectively, in the nonpreemptive setting.

Yi and Tian [19] considered the online DIAL-A-RIDE problem with deadlines, with the objective of serving the maximum number of requests. They provided bounds on the competitive ratio depending on the diameter of the metric space. In [20] they further studied this setting when the destination of requests are only revealed upon collection by the server.

The offline version of DIAL-A-RIDE on the line has been studied in various settings; for an overview, see [8]. For the closed, nonpreemptive case without release times, Gilmore and Gomory [10] and Atallah and Kosaraju [2] gave a polynomial time algorithm for a server with unit capacity $c = 1$, and Guan [11] showed that the problem is hard for $c = 2$. Bjelde et al. [5] extended this result to any finite $c \geq 2$ and both the open and closed cases. They further showed that with release times the problem is already hard for finite $c \geq 1$. On the other hand, the complexity of the case $c = \infty$ has not yet been established. The closed, preemptive case without release times was shown to be polynomial time solvable for $c = 1$ by Atallah and Kosaraju [2], and for $c \geq 2$ by Guan [11].

For the closed, nonpreemptive case with finite capacity, Krumke [14] provided a 3-approximation algorithm. Finally, Charikar and Raghavachari [7] gave approximation algorithms for the closed case without release times, both preemptive and nonpreemptive, on general metric spaces. They also claimed to have a 2-approximation for the line, but this result appears to be incorrect (personal communication).

2. Preliminaries. Formally, an instance of DIAL-A-RIDE on the line is given by a set of requests denoted by $\sigma = \{(a_1, b_1; r_1), (a_2, b_2; r_2), \dots, (a_n, b_n; r_n)\}$ that need to be served by a single server with capacity $c \in \mathbb{N} \cup \{\infty\}$, traveling with unit speed and starting at the origin on the real line. Request σ_i appears at time $r_i > 0$ at position $a_i \in \mathbb{R}$ of the real line and needs to be transported to position $b_i \in \mathbb{R}$. The objective of the DIAL-A-RIDE problem on the line is to find a shortest schedule for the server to transport all requests without carrying more than c requests at once, where the length of a schedule is the length of the resulting trajectory. In the *closed* version of the problem, the server eventually needs to return to the origin, in the *open* version it does not. In the *online* DIAL-A-RIDE problem on the line, each request σ_i is revealed only at time r_i, and n is only revealed implicitly by the fact that no more requests appear. In contrast, in the *offline* problem, all requests are known ahead of time (but release times still need to be respected).

We define $L(t, p, R)$ to be the length of a shortest schedule that starts at position p at time t and serves all requests in $R \subseteq \sigma$ after they appeared (i.e., the schedule must respect release times). Observe that, for all $0 \leq t \leq t'$, $p, p' \in \mathbb{R}$, and $R \subseteq \sigma$, we have

$$L(t, p, R) \geq L(t', p, R), \tag{2.1}$$

$$L(t, p, R) \leq |p - p'| + L(t, p', R). \tag{2.2}$$

By $x_- := \min\{0, \min_{i=1,\dots,n} a_i, \min_{i=1,\dots,n} b_i\}$ we denote the leftmost and by $x_+ := \max\{0, \max_{i=1,\dots,n} a_i, \max_{i=1,\dots,n} b_i\}$ the rightmost position that needs to be visited by the server. Here and throughout, we orient the real line from left to right. Obviously, there is an optimum trajectory that only visits points in $[x_-, x_+]$, and we let $\textsc{Opt}$ be such a trajectory and $\textsc{Opt}(\sigma) := L(0, 0, \sigma)$ be its length.

For the description of online algorithms, we denote by t the current time and by R_t the set of requests that have appeared until time t but have not been served yet. The algorithm $\textsc{Smartstart}$ is given in Algorithm 2.1. Essentially, at time t, $\textsc{Smartstart}$ waits before starting an optimal schedule to serve all available requests at time

$$\min_{t' \geq t} \left\{ t' \geq \frac{L(t', p, R_{t'})}{\Theta - 1} \right\}, \tag{2.3}$$

where p is the current position of the server and $\Theta > 1$ is a parameter of the algorithm that scales the waiting time. Importantly, $\textsc{Smartstart}$ ignores incoming requests while executing a schedule. Whenever we need to distinguish the behavior of $\textsc{Smartstart}$ for different values of $\Theta > 1$, we write $\textsc{Smartstart}_\Theta$ to make the choice of Θ explicit. The length of $\textsc{Smartstart}$'s trajectory is denoted by $\textsc{Smartstart}(\sigma)$. Note that the schedules used by $\textsc{Smartstart}$ are NP-hard to compute for $1 < c < \infty$; see [5].

Algorithm 2.1. $\textsc{Smartstart}$.

```
p_1 ← 0
for j = 1, 2, ... do
    while t < L(t, p_j, R_t)/(Θ − 1) do
        wait
    t_j ← t
    S_j ← optimal offline schedule serving R_t starting from p_j
    execute S_j
    p_{j+1} ← current position
```

We let $N \in \mathbb{N}$ be the number of schedules needed by $\textsc{Smartstart}$ to serve σ. The jth schedule is denoted by S_j, its starting time by t_j, its starting point by p_j, its ending point by p_{j+1} (cf. Algorithm 2.1), and the set of requests served in S_j by σ_{S_j}. For convenience, we set $t_0 = p_0 = 0$. Finally, we denote by $y_-^{S_j}$ the leftmost and by $y_+^{S_j}$ the rightmost position that occurs in the requests σ_{S_j}. Note that $y_-^{S_j}$ and $y_+^{S_j}$ need not lie on different sides of the origin, in contrast to $x_{-/+}$.

3. Upper bound for the open version.

In this section, we give an upper bound on the completion time

$$\textsc{Smartstart}(\sigma) = t_N + L(t_N, p_N, \sigma_{S_N}) \tag{3.1}$$

of $\textsc{Smartstart}$, relative to $\textsc{Opt}(\sigma)$. To do this, we consider two cases, depending on whether or not $\textsc{Smartstart}$ postpones the execution of the final schedule S_N. If $\textsc{Smartstart}$ postpones the execution of S_N (i.e., it waits even though there are unserved requests), the starting time of schedule S_N is given by

$$t_N = \frac{1}{\Theta - 1} L(t_N, p_N, \sigma_{S_N}). \tag{3.2}$$

Otherwise, we have

$$t_N = t_{N-1} + L(t_{N-1}, p_{N-1}, \sigma_{S_{N-1}}) \tag{3.3}$$

if the final schedule S_N is executed directly after the second-to-final schedule and

$$t_N = r_n \tag{3.4}$$

if there are no unserved requests at the point of time the execution of S_{N-1} is finished and the last requests are released at time r_n. We start by giving a lower bound on the starting time of a schedule.

LEMMA 3.1. *Algorithm* SMARTSTART *does not start schedule S_j earlier than time $\frac{|p_{j+1}|}{\Theta}$, i.e., we have $t_j \geq \frac{|p_{j+1}|}{\Theta}$.*

Proof. Since SMARTSTART at least has to move from p_j to p_{j+1}, we have

$$L(t_j, p_j, \sigma_{S_j}) \geq |p_j - p_{j+1}|.$$

Note, however, that SMARTSTART needs at least time $|p_j|$ to reach p_j. Therefore, we have

$$\begin{aligned} t_j &\overset{(2.3)}{\geq} \min\left\{t \geq |p_j| : t \geq \frac{|p_j - p_{j+1}|}{\Theta - 1}\right\} \\ &= \max\left\{|p_j|, \frac{|p_j - p_{j+1}|}{\Theta - 1}\right\}. \end{aligned} \tag{3.5}$$

It remains to show

$$\max\left\{|p_j|, \frac{|p_j - p_{j+1}|}{\Theta - 1}\right\} \geq \frac{|p_{j+1}|}{\Theta}.$$

For $|p_j| \geq \frac{|p_{j+1}|}{\Theta}$ we trivially have

$$\max\left\{|p_j|, \frac{|p_j - p_{j+1}|}{\Theta - 1}\right\} \geq |p_j| \geq \frac{|p_{j+1}|}{\Theta}. \tag{3.6}$$

For $|p_j| < \frac{|p_{j+1}|}{\Theta}$ we have

$$|p_j - p_{j+1}| \overset{|p_j| < \frac{|p_{j+1}|}{\Theta} < |p_{j+1}|}{>} \left|\frac{p_{j+1}}{\Theta} - p_{j+1}\right|. \tag{3.7}$$

This leads to

$$\max\left\{|p_j|, \frac{|p_j - p_{j+1}|}{\Theta - 1}\right\} \geq \frac{|p_j - p_{j+1}|}{\Theta - 1} \overset{(3.7)}{>} \frac{|\frac{p_{j+1}}{\Theta} - p_{j+1}|}{\Theta - 1} = \frac{|p_{j+1}|}{\Theta}. \tag{3.8}$$

To sum it up, we have

$$t_j \overset{(3.5)}{\geq} \max\left\{|p_j|, \frac{|p_j - p_{j+1}|}{\Theta - 1}\right\} \overset{(3.8),(3.6)}{\geq} \frac{|p_{j+1}|}{\Theta},$$

as claimed. ∎

The following bound on the length of SMARTSTART's schedules is an essential ingredient in our upper bounds.

LEMMA 3.2. *For every schedule S_j of* SMARTSTART, *we have*

$$L(t_j, p_j, \sigma_{S_j}) \leq \left(1 + \frac{\Theta}{\Theta + 2}\right) \mathrm{OPT}(\sigma).$$

Proof. First, we notice that by the triangle inequality we have

$$L(t_j, p_j, \sigma_{S_j}) \le |p_j| + L(t_j, 0, \sigma_{S_j}) \le \textsc{Opt}(\sigma) + |p_j|. \tag{3.9}$$

Now, let $\sigma_{S_j}^{\textsc{Opt}}$ be the first request of σ_{S_j} that is picked up by Opt, let $a_j^{\textsc{Opt}}$ be its starting point, and let $r_j^{\textsc{Opt}}$ be its release time. We have

$$L(t_j, p_j, \sigma_{S_j}) \le |a_j^{\textsc{Opt}} - p_j| + L(t_j, a_j^{\textsc{Opt}}, \sigma_{S_j}), \tag{3.10}$$

again by the triangle inequality. Since Opt serves all requests of σ_{S_j} starting at position $a_j^{\textsc{Opt}}$ no earlier than time $r_j^{\textsc{Opt}}$, we have

$$L(t_j, a_j^{\textsc{Opt}}, \sigma_{S_j}) \overset{r_j^{\textsc{Opt}} \le t_j}{\le} L(r_j^{\textsc{Opt}}, a_j^{\textsc{Opt}}, \sigma_{S_j}) \le \textsc{Opt}(\sigma) - r_j^{\textsc{Opt}}, \tag{3.11}$$

which yields

$$\begin{aligned} L(t_j, p_j, \sigma_{S_j}) &\overset{(3.10)}{\le} |a_j^{\textsc{Opt}} - p_j| + L(t_j, a_j^{\textsc{Opt}}, \sigma_{S_j}) \\ &\overset{(3.11)}{\le} \textsc{Opt}(\sigma) + |a_j^{\textsc{Opt}} - p_j| - r_j^{\textsc{Opt}} \\ &\overset{t_{j-1} < r_j^{\textsc{Opt}}}{<} \textsc{Opt}(\sigma) + |a_j^{\textsc{Opt}} - p_j| - t_{j-1}. \end{aligned} \tag{3.12}$$

Since p_j is the destination of a request, Opt needs to visit it. In the case that Opt visits p_j before collecting $\sigma_{S_j}^{\textsc{Opt}}$, Opt still has to collect and serve every request of σ_{S_j} after it has visited position p_j the first time, which directly implies

$$\left(1 + \frac{\Theta}{\Theta + 2}\right) \textsc{Opt}(\sigma) > \textsc{Opt}(\sigma) \ge L(|p_j|, p_j, \sigma_{S_j}) \overset{|p_j| \le t_j}{\ge} L(t_j, p_j, \sigma_{S_j}).$$

On the other hand, if Opt collects $\sigma_{S_j}^{\textsc{Opt}}$ before visiting the position p_j, we have

$$t_{j-1} + |a_j^{\textsc{Opt}} - p_j| \overset{t_{j-1} < r_j^{\textsc{Opt}}}{<} r_j^{\textsc{Opt}} + |a_j^{\textsc{Opt}} - p_j| \le \textsc{Opt}(\sigma), \tag{3.13}$$

since Opt cannot collect $\sigma_{S_j}^{\textsc{Opt}}$ before time $r_j^{\textsc{Opt}}$ and then still has to visit position p_j. Thus, we have

$$\begin{aligned} L(t_j, p_j, \sigma_{S_j}) &\overset{(3.12)}{<} \textsc{Opt}(\sigma) + |a_j^{\textsc{Opt}} - p_j| - t_{j-1} \\ &\overset{(3.13)}{\le} 2\textsc{Opt}(\sigma) - 2t_{j-1} \\ &\overset{\text{Lem. 3.1}}{\le} 2\textsc{Opt}(\sigma) - 2\frac{|p_j|}{\Theta}. \end{aligned} \tag{3.14}$$

This implies

$$\begin{aligned} L(t_j, p_j, \sigma_{S_j}) &\overset{(3.9),(3.14)}{\le} \min\left\{\textsc{Opt}(\sigma) + |p_j|, 2\textsc{Opt}(\sigma) - \frac{2}{\Theta}|p_j|\right\} \\ &\le \left(1 + \frac{\Theta}{\Theta + 2}\right) \textsc{Opt}(\sigma), \end{aligned}$$

since the minimum above is largest if the two terms are equal, which is the case for $|p_j| = \frac{\Theta}{\Theta+2}\textsc{Opt}(\sigma)$. ∎

The following proposition uses Lemma 3.2 to provide an upper bound for the competitive ratio of SMARTSTART in the case where SMARTSTART does have a waiting period before starting the final schedule.

PROPOSITION 3.3. *In the case that* SMARTSTART *postpones executing S_N, we have*

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)} \leq f_1(\Theta) := \frac{2\Theta^2 + 2\Theta}{\Theta^2 + \Theta - 2}.$$

Proof. Assume SMARTSTART waits before starting the final schedule. Then we have

$$\textsc{Smartstart}(\sigma) \overset{(3.1)}{=} t_N + L(t_N, p_N, \sigma_{S_N}) \overset{(3.2)}{=} \frac{\Theta}{\Theta - 1} L(t_N, p_N, \sigma_{S_N}).$$

Lemma 3.2 thus yields the claimed bound:

$$\begin{aligned} \textsc{Smartstart}(\sigma) \quad &= \quad \frac{\Theta}{\Theta - 1} L(t_N, p_N, \sigma_{S_N}) \\ &\overset{\text{Lem. 3.2}}{\leq} \quad \frac{\Theta}{\Theta - 1}\left(1 + \frac{\Theta}{\Theta + 2}\right) \textsc{Opt}(\sigma) \\ &= \quad \frac{2\Theta^2 + 2\Theta}{\Theta^2 + \Theta - 2} \textsc{Opt}(\sigma). \end{aligned}$$ ∎

It remains to examine the case where the algorithm SMARTSTART has no waiting period before starting the final schedule. We start with two lemmas that give us an upper bound for the length of a schedule depending on its extreme positions.

LEMMA 3.4. *Let S_j with $j \in \{1, \ldots, N\}$ be a schedule of* SMARTSTART. *Moreover, let* $\textsc{Opt}(\sigma) = |x_-| + x_+ + y$ *for some $y \geq 0$. Then we have*

$$L(t_j, 0, \sigma_{S_j}) \leq |\min\{0, y_-^{S_j}\}| + \max\{0, y_+^{S_j}\} + y.$$

Proof. We need to analyze the amount of time the server needs to serve σ_{S_j} starting from position 0 at time t_j. First of all, note that the server does not wait at any point, since all requests of σ_{S_j} already have appeared at time t_j. Because of that, the server cannot go to the left of $\min\{0, y_-^{S_j}\}$ or to the right of $\max\{0, y_+^{S_j}\}$ while staying on an optimal route. Furthermore, we notice that the route OPT takes to serve σ is a valid route to serve σ_{S_j}, since $\sigma_{S_j} \subseteq \sigma$. However, we can skip every part of the route OPT takes to collect σ that lies left of $\min\{0, y_-^{S_j}\}$ or right of $\max\{0, y_+^{S_j}\}$, since no requests of σ_{S_j} have a starting or ending point that lies in those regions. Since all requests have already appeared at time t_j, this does not produce additional waiting time; i.e., we can just delete the parts of the route that lie left of $\min\{0, y_-^{S_j}\}$ and right of $\max\{0, y_+^{S_j}\}$ and still have a valid route for serving σ_{S_j} when starting at time t_j. This shortens the length of the route by at least

$$|x_-| - |\min\{0, y_-^{S_j}\}| + x_+ - \max\{0, y_+^{S_j}\},$$

which gives us

$$\begin{aligned} L(t_j, 0, \sigma_{S_j}) &\leq \textsc{Opt}(\sigma) - (|x_-| - |\min\{0, y_-^{S_j}\}| + x_+ - \max\{0, y_+^{S_j}\}) \\ &= |\min\{0, y_-^{S_j}\}| + \max\{0, y_+^{S_j}\} + y, \end{aligned}$$

as desired. ∎

Lemma 3.5. *Let S_j with $j \in \{1, \dots, N\}$ be a schedule of* Smartstart. *Moreover, let* $\textsc{Opt}(\sigma) = |x_-| + x_+ + y$ *for some* $y \geq 0$. *Then we have*

$$L(t_j, \max\{0, y_-^{S_j}\} + \min\{0, y_+^{S_j}\}, \sigma_{S_j}) \leq y_+^{S_j} - y_-^{S_j} + y.$$

Proof. First note that the case $\max\{0, y_-^{S_j}\} = \min\{0, y_+^{S_j}\} = 0$ follows from Lemma 3.4. Assume we have $\max\{0, y_-^{S_j}\} = y_-^{S_j}$. Then all requests of σ_{S_j} have starting and ending points on the right side of the origin, and we have

$$0 \leq y_-^{S_j} \leq y_+^{S_j},$$

i.e., $\min\{0, y_+^{S_j}\} = 0$. Therefore, we need to examine $L(t_j, y_-^{S_j}, \sigma_{S_j})$, i.e., the length of the optimal offline schedule serving the set of requests σ_{S_j} and starting from position $y_-^{S_j}$ at time t_j. We note that the server does not wait at any point, since all requests of σ_{S_j} already have appeared at time t_j. Because of that, the server cannot go to the left of $y_-^{S_j}$ or to the right of $y_+^{S_j}$ while staying on an optimal route. Furthermore, we notice that Opt cannot collect any requests of σ_{S_j} before passing $y_-^{S_j}$ for the first time, since Opt starts at the origin. Therefore, removing the parts of the path that Opt takes until it first crosses $y_-^{S_j}$ gives us a valid route to serve σ_{S_j} since $\sigma_{S_j} \subseteq \sigma$. Additionally, we can skip every part of the route Opt takes to collect requests that lie left of 0 or right of $y_+^{S_j}$ since no requests of σ_{S_j} have a starting or ending point that lies in those regions. Again, this does not produce additional waiting time. This shortens the length of the route by at least

$$|x_-| + y_-^{S_j} + x_+ - y_+^{S_j},$$

which gives us

$$L(t_j, y_-^{S_j}, \sigma_{S_j}) \leq \textsc{Opt}(\sigma) - (|x_-| + y_-^{S_j} + x_+ - y_+^{S_j}) = y_+^{S_j} - y_-^{S_j} + y,$$

as desired.

It remains to examine the case $\min\{0, y_+^{S_j}\} = y_+^{S_j}$. In this case all requests of σ_{S_j} have starting and ending points on the left side of the origin, and we have

$$y_-^{S_j} \leq y_+^{S_j} \leq 0,$$

i.e., $\max\{0, y_-^{S_j}\} = 0$. Therefore, we have to examine $L(t_j, y_+^{S_j}, \sigma_{S_j})$. From this point the proof works analogously to the former case with the roles of $y_+^{S_j}$ and $y_-^{S_j}$ switched; in particular, we have

$$L(t_j, y_+^{S_j}, \sigma_{S_j}) \leq \textsc{Opt}(\sigma) - (|x_-| + y_-^{S_j} + x_+ - y_+^{S_j}) = y_+^{S_j} - y_-^{S_j} + y,$$

as desired. ∎

Next, we give an upper bound for the rightmost position that can be reached during a schedule.

Lemma 3.6. *Let S_j with $j \in \{1, \dots, N\}$ be a schedule of* Smartstart. *Moreover, let $|x_-| \leq x_+$ and* $\textsc{Opt}(\sigma) = |x_-| + x_+ + y$ *for some $y \geq 0$. Then, for every point p that is visited by S_j we have*

$$p \leq |p_j| + |p_j - p_{j+1}| + y - |\min\{0, y_-^{S_j}\}|.$$

Proof. First of all, we notice that S_j does not wait at any point since all requests of σ_{S_j} already have appeared at time t_j. Because of that, S_j cannot go to the left of $\min\{p_j, y_-^{S_j}\}$ or to the right of $\max\{p_j, y_+^{S_j}\}$ while staying on a optimal route. It suffices to show

$$\max\{p_j, y_+^{S_j}\} \leq |p_j| + |p_j - p_{j+1}| + y - |\min\{0, y_-^{S_j}\}|. \tag{3.15}$$

Since $|x_-| \leq x_+$ implies $\textsc{Opt}(\sigma) \geq 2|x_-| + x_+$, we have $y \geq |x_-| \geq |\min\{0, y_-^{S_j}\}|$. Note that $|x_-| \geq |\min\{0, y_-^{S_j}\}|$ holds since we always have $|x_-| \geq 0$ and $x_- \leq y_-^{S_j}$, i.e., $|x_-| \geq |y_-^{S_j}|$ if $x_- < 0$ and $y_-^{S_j} < 0$ holds. This implies that if we have $\max\{p_j, y_+^{S_j}\} = p_j$, inequality (3.15) holds. Thus, we may assume $\max\{p_j, y_+^{S_j}\} = y_+^{S_j}$ in the following. Similarly as before, if we have $y_+^{S_j} < 0$, inequality (3.15) again holds, since the right-hand side is always nonnegative. We may thus assume $y_+^{S_j} \geq 0$, i.e.,

$$\max\{0, y_+^{S_j}\} = y_+^{S_j}, \tag{3.16}$$

in the following. According to the triangle inequality and Lemma 3.4, we have

$$\begin{aligned}
L(t_j, p_j, \sigma_{S_j}) &\leq |p_j| + L(t_j, 0, \sigma_{S_j}) \\
&\overset{\text{Lem. 3.4}}{\leq} |p_j| + |\min\{0, y_-^{S_j}\}| + \max\{0, y_+^{S_j}\} + y \\
&\overset{(3.16)}{=} |p_j| + |\min\{0, y_-^{S_j}\}| + y_+^{S_j} + y. \qquad (3.17)
\end{aligned}$$

For the sake of contradiction, we assume

$$y_+^{S_j} > |p_j| + |p_j - p_{j-1}| + y - |\min\{0, y_-^{S_j}\}|. \tag{3.18}$$

Since S_j has to visit both extreme points $\max\{p_j, y_+^{S_j}\} = y_+^{S_j}$ and $\min\{p_j, y_-^{S_j}\}$, we have two possible scenarios: S_j either visits $\min\{p_j, y_-^{S_j}\}$ before $y_+^{S_j}$ or S_j visits $\min\{p_j, y_-^{S_j}\}$ after $y_+^{S_j}$. In both cases S_j ends in p_{j+1}. In the first case, we have

$$\begin{aligned}
L(t_j, p_j, \sigma_{S_j}) &\geq |p_j - \min\{p_j, y_-^{S_j}\}| + |\min\{p_j, y_-^{S_j}\} - y_+^{S_j}| + |y_+^{S_j} - p_{j+1}| \\
&= p_j - \min\{p_j, y_-^{S_j}\} + y_+^{S_j} - \min\{p_j, y_-^{S_j}\} + y_+^{S_j} - p_{j+1} \\
&= p_j - 2\min\{p_j, y_-^{S_j}\} + 2y_+^{S_j} - p_{j+1} \\
&\overset{(3.18)}{>} p_j - 2\min\{p_j, y_-^{S_j}\} + y_+^{S_j} + |p_j| + |p_j - p_{j+1}| \\
&\quad + y - |\min\{0, y_-^{S_j}\}| - p_{j+1} \\
&\geq y_+^{S_j} + |p_j| + y - |\min\{0, y_-^{S_j}\}| - 2\min\{p_j, y_-^{S_j}\}. \qquad (3.19)
\end{aligned}$$

In the second case, we obtain the same result:

$$\begin{aligned}
L(t_j, p_j, \sigma_{S_j}) &\geq |p_j - y_+^{S_j}| + |y_+^{S_j} - \min\{p_j, y_-^{S_j}\}| + |\min\{p_j, y_-^{S_j}\} - p_{j+1}| \\
&= y_+^{S_j} - p_j + y_+^{S_j} - \min\{p_j, y_-^{S_j}\} + p_{j+1} - \min\{p_j, y_-^{S_j}\} \\
&= p_{j+1} + 2y_+^{S_j} - 2\min\{p_j, y_-^{S_j}\} - p_j \\
&\overset{(3.18)}{>} p_{j+1} + y_+^{S_j} + |p_j| + |p_j - p_{j+1}| + y \\
&\quad - |\min\{0, y_-^{S_j}\}| - 2\min\{p_j, y_-^{S_j}\} - p_j \\
&\geq y_+^{S_j} + |p_j| + y - |\min\{0, y_-^{S_j}\}| - 2\min\{p_j, y_-^{S_j}\}. \qquad (3.20)
\end{aligned}$$

Now we again consider two cases.

Case 1: $\min\{p_j, y_-^{S_j}\} \leq 0$. In this case, we claim that

$$-\min\{p_j, y_-^{S_j}\} \geq |\min\{0, y_-^{S_j}\}| \tag{3.21}$$

holds. This is clear for $\min\{0, y_-^{S_j}\} = 0$ and for $\min\{p_j, y_-^{S_j}\} = y_-^{S_j}$. In the remaining case, we have $\min\{0, y_-^{S_j}\} = y_-^{S_j}$ and $\min\{p_j, y_-^{S_j}\} = p_j$, i.e., $p_j \leq y_-^{S_j} \leq 0$, which implies $-p_j \geq -y_-^{S_j} = |y_-^{S_j}|$, as desired. This gives us

$$\begin{aligned} L(t_j, p_j, \sigma_{S_j}) &\overset{(3.19),(3.20)}{>} y_+^{S_j} + |p_j| + y - |\min\{0, y_-^{S_j}\}| - 2\min\{p_j, y_-^{S_j}\} \\ &\overset{(3.21)}{\geq} y_+^{S_j} + |p_j| + y + |\min\{0, y_-^{S_j}\}|, \end{aligned}$$

which is a contradiction to inequality (3.17).

Case 2: $\min\{p_j, y_-^{S_j}\} > 0$. Inequality $y_+^{S_j} \geq y_-^{S_j} > 0$ implies

$$\max\{0, y_-^{S_j}\} + \min\{0, y_+^{S_j}\} = y_-^{S_j}. \tag{3.22}$$

Therefore, we can apply Lemma 3.5 and the triangle inequality to obtain

$$\begin{aligned} L(t_j, p_j, \sigma_{S_j}) &\leq |p_j - y_-^{S_j}| + L(t_j, y_-^{S_j}, \sigma_{S_j}) \\ &\overset{(3.22),\text{ Lem. } 3.5}{\leq} |p_j - y_-^{S_j}| + y_+^{S_j} - y_-^{S_j} + y \\ &= \max\{p_j, y_-^{S_j}\} - \min\{p_j, y_-^{S_j}\} + y_+^{S_j} - y_-^{S_j} + y. \end{aligned} \tag{3.23}$$

We have

$$\max\{p_j, y_-^{S_j}\} - \min\{p_j, y_-^{S_j}\} - y_-^{S_j} = p_j - 2\min\{p_j, y_-^{S_j}\}. \tag{3.24}$$

This gives us

$$\begin{aligned} L(t_j, p_j, \sigma_{S_j}) &\overset{(3.23)}{\leq} \max\{p_j, y_-^{S_j}\} - \min\{p_j, y_-^{S_j}\} + y_+^{S_j} - y_-^{S_j} + y \\ &\overset{(3.24)}{=} p_j - 2\min\{p_j, y_-^{S_j}\} + y_+^{S_j} + y. \end{aligned} \tag{3.25}$$

Finally, we have

$$\begin{aligned} L(t_j, p_j, \sigma_{S_j}) &\overset{(3.20)}{>} y_+^{S_j} + |p_j| + y - |\min\{0, y_-^{S_j}\}| - 2\min\{p_j, y_-^{S_j}\} \\ &= y_+^{S_j} + p_j + y - 2\min\{p_j, y_-^{S_j}\}, \end{aligned}$$

which is a contradiction to inequality (3.25). We conclude that (3.18) does not hold, which in turn proves (3.15) in the case that $\max\{p_j, y_+^{S_j}\} = y_+^{S_j}$ holds. ∎

Now we can give an upper bound for the competitive ratio of SMARTSTART if the server is not waiting before starting the final schedule.

PROPOSITION 3.7. *If* SMARTSTART *does not postpone executing S_N, we have*

$$\frac{\text{SMARTSTART}(\sigma)}{\text{OPT}(\sigma)} \leq f_2(\Theta) := \left(\Theta + 1 - \frac{\Theta - 1}{3\Theta + 3}\right).$$

Proof. Assume algorithm SMARTSTART does not postpone the last schedule, i.e., SMARTSTART starts the final schedule S_N either immediately after finishing S_{N-1} or immediately after the last requests are released.

Let the latter be the case; then the final schedule is started at the release time r_n of the last request. Since OPT also has to serve the last request, we have $\text{OPT}(\sigma) \geq r_n$, and since the execution of the final schedule is not postponed, we have $r_n > \frac{1}{\Theta-1} L(t_N, p_N, \sigma_{S_N})$, i.e.,

$$L(t_N, p_N, \sigma_{S_N}) < (\Theta - 1)\text{OPT}(\sigma). \tag{3.26}$$

In total we have

$$\begin{aligned}
\text{SMARTSTART}(\sigma) &\overset{(3.1)}{=} t_N + L(t_N, p_N, \sigma_{S_N}) \\
&\overset{(3.4)}{=} r_n + L(t_N, p_N, \sigma_{S_N}) \\
&\overset{(3.26)}{<} \Theta\text{OPT}(\sigma) \\
&< \left(\Theta + 1 - \frac{\Theta - 1}{3\Theta + 3}\right) \text{OPT}(\sigma).
\end{aligned}$$

Now let the last schedule be started immediately after the second-to-last one. Without loss of generality, we assume $|x_-| \leq x_+$ throughout the rest of the proof by symmetry. Let $\sigma_{S_N}^{\text{OPT}}$ be the first request of σ_{S_N} that is served by OPT, let a_N^{OPT} be its starting point, and let r_N^{OPT} be its release time. We have

$$\begin{aligned}
\text{SMARTSTART}(\sigma) &\overset{(3.1)}{=} t_N + L(t_N, p_N, \sigma_{S_N}) \\
&\overset{(3.3)}{=} t_{N-1} + L(t_{N-1}, p_{N-1}, \sigma_{S_{N-1}}) + L(t_N, p_N, \sigma_{S_N}) \\
&\overset{t_N \geq r_N^{\text{OPT}}}{\leq} t_{N-1} + L(t_{N-1}, p_{N-1}, \sigma_{S_{N-1}}) + L(r_N^{\text{OPT}}, p_N, \sigma_{S_N}).
\end{aligned} \tag{3.27}$$

Since OPT serves all requests of σ_{S_N} after time r_N^{OPT}, starting with a request with starting point a_N^{OPT}, we also have

$$\text{OPT}(\sigma) \geq r_N^{\text{OPT}} + L(r_N^{\text{OPT}}, a_N^{\text{OPT}}, \sigma_{S_N}). \tag{3.28}$$

Furthermore, we have

$$r_N^{\text{OPT}} > t_{N-1} \tag{3.29}$$

since otherwise $\sigma_{S_N}^{\text{OPT}} \in \sigma_{S_{N-1}}$ would hold. This gives us

$$\begin{aligned}
\text{SMARTSTART}(\sigma) &\overset{(3.27)}{\leq} t_{N-1} + L(t_{N-1}, p_{N-1}, \sigma_{S_{N-1}}) + L(r_N^{\text{OPT}}, p_N, \sigma_{S_N}) \\
&\overset{(2.2)}{\leq} t_{N-1} + L(t_{N-1}, p_{N-1}, \sigma_{S_{N-1}}) \\
&\quad + |a_N^{\text{OPT}} - p_N| + L(r_N^{\text{OPT}}, a_N^{\text{OPT}}, \sigma_{S_N}) \\
&\overset{(3.28)}{\leq} t_{N-1} + L(t_{N-1}, p_{N-1}, \sigma_{S_{N-1}}) + |a_N^{\text{OPT}} - p_N| + \text{OPT}(\sigma) - r_N^{\text{OPT}} \\
&\overset{(3.29)}{<} L(t_{N-1}, p_{N-1}, \sigma_{S_{N-1}}) + |a_N^{\text{OPT}} - p_N| + \text{OPT}(\sigma).
\end{aligned} \tag{3.30}$$

We denote by $\sigma_{S_{N-1}}^{\text{SMART}}$ the last request that is delivered during schedule S_{N-1} by SMARTSTART. Note that the destination of $\sigma_{S_{N-1}}^{\text{SMART}}$ is p_N. We consider two cases.

Case 1: $\textsc{Opt}$ *collects* $\sigma^{\textsc{Opt}}_{S_N}$ *before delivering* $\sigma^{\textsc{Smart}}_{S_{N-1}}$. Obviously $\textsc{Opt}$ cannot collect the request $\sigma^{\textsc{Opt}}_{S_N}$ before its release time $r^{\textsc{Opt}}_N$. Furthermore, since $\textsc{Opt}$ still has to go to position p_N for delivering request $\sigma^{\textsc{Smart}}_{S_{N-1}}$ after collecting $\sigma^{\textsc{Opt}}_{S_N}$, we have

$$\textsc{Opt}(\sigma) \geq r^{\textsc{Opt}}_N + |a^{\textsc{Opt}}_N - p_N| \overset{(3.29)}{>} t_{N-1} + |a^{\textsc{Opt}}_N - p_N|. \tag{3.31}$$

The inequality above gives us

$$\begin{aligned}\textsc{Smartstart}(\sigma) &\overset{(3.30)}{<} L(t_{N-1}, p_{N-1}, \sigma_{S_{N-1}}) + |a^{\textsc{Opt}}_N - p_N| + \textsc{Opt}(\sigma)\\ &\overset{(3.31)}{<} L(t_{N-1}, p_{N-1}, \sigma_{S_{N-1}}) + 2\textsc{Opt}(\sigma) - t_{N-1}. \end{aligned}\tag{3.32}$$

By definition of $\textsc{Smartstart}$, we have

$$t_{N-1} \geq \frac{1}{\Theta - 1} L(t_{N-1}, p_{N-1}, \sigma_{S_{N-1}}). \tag{3.33}$$

This leads to

$$\begin{aligned}\textsc{Smartstart}(\sigma) &\overset{(3.32)}{<} L(t_{N-1}, p_{N-1}, \sigma_{S_{N-1}}) + 2\textsc{Opt}(\sigma) - t_{N-1}\\ &\overset{(3.33)}{\leq} L(t_{N-1}, p_{N-1}, \sigma_{S_{N-1}}) + 2\textsc{Opt}(\sigma)\\ &\qquad - \frac{1}{\Theta - 1} L(t_{N-1}, p_{N-1}, \sigma_{S_{N-1}})\\ &= \frac{\Theta - 2}{\Theta - 1} L(t_{N-1}, p_{N-1}, \sigma_{S_{N-1}}) + 2\textsc{Opt}(\sigma)\\ &\overset{\text{Lem. 3.2}}{\leq} \frac{\Theta - 2}{\Theta - 1}\left(1 + \frac{\Theta}{\Theta + 2}\right)\textsc{Opt}(\sigma) + 2\textsc{Opt}(\sigma)\\ &= \frac{4\Theta^2 - 8}{\Theta^2 + \Theta - 2}\textsc{Opt}(\sigma).\end{aligned}$$

It can be shown that

$$\frac{4\Theta^2 - 8}{\Theta^2 + \Theta - 2} < \Theta + 1 - \frac{\Theta - 1}{3\Theta + 3}$$

holds for all $\Theta > 1$, which concludes this case.

Case 2: $\textsc{Opt}$ *delivers* $\sigma^{\textsc{Smart}}_{S_{N-1}}$ *before collecting the request* $\sigma^{\textsc{Opt}}_{S_N}$. In this case we notice that we have

$$\begin{aligned}\textsc{Smartstart}(\sigma) &\overset{(3.27)}{\leq} t_{N-1} + L(t_{N-1}, p_{N-1}, \sigma_{S_{N-1}}) + L(r^{\textsc{Opt}}_N, p_N, \sigma_{S_N})\\ &\overset{(3.33)}{\leq} \Theta t_{N-1} + L(r^{\textsc{Opt}}_N, p_N, \sigma_{S_N})\\ &\overset{(2.1)}{\leq} \Theta t_{N-1} + |p_N - a^{\textsc{Opt}}_N| + L(r^{\textsc{Opt}}_N, a^{\textsc{Opt}}_N, \sigma_{S_N})\\ &\overset{(3.28)}{\leq} \Theta t_{N-1} + |p_N - a^{\textsc{Opt}}_N| + \textsc{Opt}(\sigma) - r^{\textsc{Opt}}_N\\ &\overset{(3.29)}{<} (\Theta - 1) r^{\textsc{Opt}}_N + |p_N - a^{\textsc{Opt}}_N| + \textsc{Opt}(\sigma)\\ &\overset{r^{\textsc{Opt}}_N \leq \textsc{Opt}(\sigma)}{\leq} \Theta\textsc{Opt}(\sigma) + |p_N - a^{\textsc{Opt}}_N|.\end{aligned}$$

This means the claim is shown if we have

$$|p_N - a_N^{\textsc{Opt}}| < \textsc{Opt}(\sigma) - \frac{\Theta - 1}{3\Theta + 3}\textsc{Opt}(\sigma).$$

Therefore, we may assume in the following that

$$|p_N - a_N^{\textsc{Opt}}| \geq \textsc{Opt}(\sigma) - \frac{\Theta - 1}{3\Theta + 3}\textsc{Opt}(\sigma). \tag{3.34}$$

Let $\textsc{Opt}(\sigma) = |x_-| + x_+ + y$ for some $y \geq 0$. By definition of x_- and x_+ we have

$$|p_N - a_N^{\textsc{Opt}}| + y \leq \textsc{Opt}(\sigma). \tag{3.35}$$

Since by assumption Opt delivers $\sigma_{S_{N-1}}^{\textsc{Smart}}$ to position p_N before collecting $\sigma_{S_N}^{\textsc{Opt}}$ at position $a_N^{\textsc{Opt}}$, we have

$$|p_N - a_N^{\textsc{Opt}}| + |p_N| \leq \textsc{Opt}(\sigma), \tag{3.36}$$

and since $\sigma_{S_{N-1}}^{\textsc{Smart}}$ appears after time t_{N-2}, we also have

$$|p_N - a_N^{\textsc{Opt}}| + t_{N-2} < \textsc{Opt}(\sigma). \tag{3.37}$$

To sum it up, we may assume that

$$\max\{y, |p_N|, t_{N-2}\} \overset{(3.34),(3.35),(3.36),(3.37)}{\leq} \frac{\Theta - 1}{3\Theta + 3}\textsc{Opt}(\sigma) \tag{3.38}$$

holds. In the following, denote by $y_-^{S_{N-1}}$ the leftmost starting or ending point and by $y_+^{S_{N-1}}$ the rightmost starting or ending point of the requests in $\sigma_{S_{N-1}}$. We compute

$$\begin{aligned}
\textsc{Smartstart}(\sigma) &\overset{(3.30)}{<} L(t_{N-1}, p_{N-1}, \sigma_{S_{N-1}}) + |p_N - a_N^{\textsc{Opt}}| + \textsc{Opt}(\sigma) \\
&\overset{(3.37)}{<} L(t_{N-1}, p_{N-1}, \sigma_{S_{N-1}}) + 2\textsc{Opt}(\sigma) - t_{N-2} \\
&\overset{(2.1)}{\leq} |p_{N-1}| + L(t_{N-1}, 0, \sigma_{S_{N-1}}) + 2\textsc{Opt}(\sigma) - t_{N-2} \\
&\overset{\text{Lem. 3.1}}{\leq} (\Theta - 1)t_{N-2} + L(t_{N-1}, 0, \sigma_{S_{N-1}}) + 2\textsc{Opt}(\sigma) \\
&\overset{\text{Lem. 3.4}}{\leq} (\Theta - 1)t_{N-2} + \max\{0, |y_-^{S_{N-1}}|\} + \max\{0, y_+^{S_{N-1}}\} \\
&\quad + y + 2\textsc{Opt}(\sigma).
\end{aligned} \tag{3.39}$$

Obviously, position $y_+^{S_{N-1}}$ is visited by Smartstart in schedule S_{N-1}. Therefore, $y_+^{S_{N-1}}$ is smaller than or equal to the rightmost point that is visited by Smartstart during schedule S_{N-1}, which gives us

$$y_+^{S_{N-1}} \overset{\text{Lem. 3.6}}{\leq} |p_{N-1}| + |p_{N-1} - p_N| + y - \max\{0, |y_-^{S_{N-1}}|\}. \tag{3.40}$$

On the other hand, because of $|x_-| \leq x_+$, we have $\textsc{Opt}(\sigma) \geq 2|x_-| + x_+$, which implies $y \geq |x_-|$. By definition of x_- and $y_-^{S_{N-1}}$, we have $|x_-| \geq \max\{0, |y_-^{S_{N-1}}|\}$. This gives us $y \geq \max\{0, |y_-^{S_{N-1}}|\}$ and

$$0 \leq |p_{N-1}| + |p_{N-1} - p_N| + y - \max\{0, |y_-^{S_{N-1}}|\}. \tag{3.41}$$

To sum it up, we have

$$\max\{0, y_+^{S_{N-1}}\} \overset{(3.40),(3.41)}{\leq} |p_{N-1}| + |p_{N-1} - p_N| + y - \max\{0, |y_-^{S_{N-1}}|\}. \tag{3.42}$$

The inequality above gives us

$$\begin{aligned}
\textsc{Smartstart}(\sigma) &\overset{(3.39)}{<} (\Theta - 1)t_{N-2} + \max\{0, |y_-^{S_{N-1}}|\} + \max\{0, y_+^{S_{N-1}}\} \\
&\qquad + y + 2\textsc{Opt}(\sigma) \\
&\overset{(3.42)}{\leq} (\Theta - 1)t_{N-2} + |p_{N-1}| + |p_{N-1} - p_N| + 2y + 2\textsc{Opt}(\sigma) \\
&\leq (\Theta - 1)t_{N-2} + |p_{N-1}| + |p_{N-1}| + |p_N| + 2y + 2\textsc{Opt}(\sigma) \\
&\overset{\text{Lem. 3.1}}{\leq} (\Theta - 1)t_{N-2} + 2\Theta t_{N-2} + |p_N| + 2y + 2\textsc{Opt}(\sigma) \\
&\overset{(3.38)}{\leq} (3\Theta + 2)\frac{\Theta - 1}{3\Theta + 3}\textsc{Opt}(\sigma) + 2\textsc{Opt}(\sigma) \\
&= \left(\Theta + 1 - \frac{\Theta - 1}{3\Theta + 3}\right)\textsc{Opt}(\sigma). \qquad \square
\end{aligned}$$

We combine the results of Propositions 3.3 and 3.7 to obtain the main result of this section.

THEOREM 3.8. *Let* Θ^* *be the only positive, real solution of* $f_1(\Theta) = f_2(\Theta)$, *i.e.,*

$$\Theta^* + 1 - \frac{\Theta^* - 1}{3\Theta^* + 3} = \frac{2\Theta^{*2} + 2\Theta^*}{\Theta^{*2} + \Theta^* - 2}.$$

Then $\textsc{Smartstart}_{\Theta^*}$ *is* ρ^**-competitive with* $\rho^* := f_1(\Theta^*) = f_2(\Theta^*) \approx 2.93768$.

Proof. For the case where SMARTSTART does wait before starting the final schedule, we have established the upper bound

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)} \leq \frac{2\Theta^2 + 2\Theta}{\Theta^2 + \Theta - 2} = f_1(\Theta)$$

in Proposition 3.3, and for the case where SMARTSTART starts the final schedule immediately after the second-to-last one, we have established the upper bound

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)} \leq \Theta + 1 - \frac{\Theta - 1}{3\Theta + 3} = f_2(\Theta)$$

in Proposition 3.7. Therefore the parameter for SMARTSTART with the smallest upper bound is

$$\Theta^* = \operatorname*{argmin}_{\Theta > 1} \left\{\max\{f_1(\Theta), f_2(\Theta)\}\right\}.$$

We note that f_1 is strictly decreasing for $\Theta > 1$ and that f_2 is strictly increasing for $\Theta > 1$. Therefore the minimum above lies at the intersection point of f_1 and f_2 that is larger than 1, i.e., Θ^* is the only positive, real solution of

$$\Theta + 1 - \frac{\Theta - 1}{3\Theta + 3} = \frac{2\Theta^2 + 2\Theta}{\Theta^2 + \Theta - 2}.$$

The resulting upper bound for the competitive ratio is

$$\rho^* = f_1(\Theta^*) = f_2(\Theta^*) \approx 2.93768. \qquad \square$$

4. Lower bound for the open version. In this section, we explicitly construct instances that demonstrate that the upper bounds given in the previous section are tight for certain ranges of $\Theta > 1$, in particular for $\Theta = \Theta^*$ (as in Theorem 3.8). Further, we show that choices of $\Theta > 1$ different from Θ^* yield competitive ratios worse than $\rho^* \approx 2.94$. Together, this implies that ρ^* is exactly the best possible competitive ratio for SMARTSTART.

All our lower bounds rely on the following lemma, which gives a way to lure SMARTSTART away from the origin, with almost no time overhead. More specifically, the lemma provides a way to make SMARTSTART move to any position $p > 0$ within time $p + \mu$, where $\mu > 0$ is arbitrarily small.

LEMMA 4.1. *Let the capacity $c \in \mathbb{N} \cup \{\infty\}$ of the server be arbitrary but fixed, let $p > 0$ be any position on the real line, and let $\mu > 0$ be any positive number. Furthermore, let $\delta > 0$ be such that $\frac{p}{\delta\Theta} = n \in \mathbb{N}$ and $\delta < (\Theta - 1)\mu$. Algorithm* SMARTSTART *finishes serving the set of requests $\sigma = \{\sigma_1, \dots, \sigma_{n+1}\}$ with*

$$\begin{aligned}
\sigma_1 &= (\delta, \delta; 0),\\
\sigma_i &= \left(i\delta, i\delta; \frac{1}{\Theta - 1}\delta + (i-1)\delta\right) \text{ for } i \in \{2, \dots, n\},\\
\sigma_{n+1} &= (p, p; n\delta + \mu) = \left(p, p; \frac{p}{\Theta} + \mu\right)
\end{aligned}$$

and reaches the position p at time $p + \mu$, provided that no additional requests appear until time $\frac{p}{\Theta} + \mu$. The final schedule serving σ_{n+1} is started at time $\frac{p}{\Theta} + \mu$.

Proof. We show via induction that every request σ_i with $i \in \{1, \dots, n\}$ is served in a separate schedule S_i with starting position $p_i = (i-1)\delta$ and starting time

$$t_i = \frac{1}{\Theta - 1}\delta + (i-1)\delta.$$

This is clear for $i = 1$: By definition, SMARTSTART starts from $p_1 = 0$. The schedule S_1 to serve σ_1 is started at time

$$t_1 = \min\left\{t \geq 0 \;\middle|\; \frac{L(t, 0, \{\sigma_1\})}{\Theta - 1} \leq t\right\} = \frac{1}{\Theta - 1}\delta$$

and reaches position δ at time $\frac{1}{\Theta-1}\delta + \delta = \frac{\Theta}{\Theta-1}\delta$. Note that the release time of every request σ_i is larger than t_1, ensuring that S_1 indeed only serves σ_1.

We assume the claim is true for some $k \in \{1, \dots, n-1\}$. Consider $i = k + 1$. By reduction, the server finishes schedule S_k at position $p_{k+1} = k\delta$ at time $\frac{1}{\Theta-1}\delta + k\delta$. Therefore, we have

$$t_{k+1} \geq \frac{1}{\Theta - 1}\delta + k\delta.$$

On the other hand, we have

$$\frac{L\left(\frac{\delta}{\Theta-1} + k\delta, k\delta, \{\sigma_{k+1}\}\right)}{\Theta - 1} = \frac{\delta}{\Theta - 1} < \frac{1}{\Theta - 1}\delta + k\delta.$$

Since there are no other unserved requests at time $\frac{1}{\Theta-1}\delta + k\delta$, the schedule S_{k+1} is started at time $t_{k+1} = \frac{1}{\Theta-1}\delta + k\delta$ and only serves σ_{k+1}, as claimed. It remains to examine the final request σ_{n+1}. The above shows that in the schedule S_n is finished at time

$$t_n + L(t_n, p_n, \{\sigma_n\}) = \frac{1}{\Theta-1}\delta + (n-1)\delta + \delta = \frac{1}{\Theta-1}\delta + n\delta < \mu + n\delta$$

at position $n\delta = \frac{p}{\Theta}$, i.e., before the request σ_{n+1} is released at time $\mu + n\delta$. On the other hand, we have

$$\frac{L\left(\mu + n\delta, \frac{p}{\Theta}, \{\sigma_{n+1}\}\right)}{\Theta-1} = \frac{\frac{\Theta-1}{\Theta}p}{\Theta-1} = \frac{p}{\Theta} = n\delta < \mu + n\delta.$$

Therefore the final schedule S_{n+1} is started at time $t_{n+1} = \mu + n\delta = \mu + \frac{p}{\Theta}$, and we get

$$\begin{aligned}\textsc{Smartstart}((\sigma_i)_{i\in\{1,\dots,n+1\}}) &= t_{n+1} + L(t_{n+1}, p_{n+1}, \{\sigma_{n+1}\})\\ &= \mu + \frac{p}{\Theta} + \frac{\Theta-1}{\Theta}p\\ &= \mu + p.\end{aligned}$$

Note that for every request the starting point is identical to the ending point. Thus, our construction remains valid for every capacity $c \in \mathbb{N} \cup \{\infty\}$. Furthermore, there is no interference with requests that are released after time $t_{n+1} = \mu + \frac{p}{\Theta}$. ∎

Equipped with this strategy to lure Smartstart away from the origin, we now move on to establish lower bounds matching Propositions 3.3 and 3.7.

Proposition 4.2. *Let the capacity $c \in \mathbb{N} \cup \{\infty\}$ of the server be arbitrary but fixed, and let $2 < \Theta < 3$. For every sufficiently small $\varepsilon > 0$, there is a set of requests σ such that* Smartstart *waits before starting the final schedule and such that the inequality*

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)} \geq \frac{2\Theta^2 + 2\Theta}{\Theta^2 + \Theta - 2} - \varepsilon$$

holds, i.e., the upper bound established in Proposition 3.3 *is tight for* $\Theta \in (2, 3)$.

Proof. Let $\varepsilon > 0$ with $\varepsilon < \frac{2}{9}(\frac{2\Theta^2}{\Theta^2+\Theta-2})$ and $\varepsilon' = \frac{\Theta^2+\Theta-2}{2\Theta^2}\varepsilon$. We apply Lemma 4.1 with $p = 1$ and $\mu = \frac{\varepsilon'}{2}$. For convenience, we start the enumeration of schedules with the first schedule after the application of Lemma 4.1. Smartstart reaches position $p_1 = 1$ at time $1 + \frac{\varepsilon'}{2}$. Now let the requests

$$\begin{aligned}\sigma_1^{(1)} &= \left(-\frac{1}{\Theta} + \varepsilon', 0; \frac{1}{\Theta} + \varepsilon'\right),\\ \sigma_1^{(2)} &= \left(\frac{1}{\Theta}, 1; \frac{1}{\Theta} + \varepsilon'\right)\end{aligned}$$

appear. Note that both requests appear after time $\frac{1}{\Theta} + \frac{\varepsilon'}{2}$ and therefore do not interfere with the application of Lemma 4.1. If Smartstart delivers $\sigma_1^{(2)}$ before collecting $\sigma_1^{(1)}$, the time it needs is at least

$$\begin{aligned}\left|1 - \frac{1}{\Theta}\right| + \left|\frac{1}{\Theta} - 1\right| + \left|1 - \left(-\frac{1}{\Theta} + \varepsilon'\right)\right| + \left|\left(-\frac{1}{\Theta} + \varepsilon'\right) - 0\right| &= \frac{2\Theta-2}{\Theta} + \frac{\Theta+2}{\Theta} - 2\varepsilon'\\ &= 3 - 2\varepsilon'.\end{aligned}$$

The best schedule that delivers $\sigma_1^{(2)}$ after collecting $\sigma_1^{(1)}$ delivers $\sigma_1^{(1)}$ before visiting the starting point $-\frac{1}{\Theta}+\varepsilon'$ of $\sigma_1^{(2)}$ and needs time

$$\left|1-\left(-\frac{1}{\Theta}+\varepsilon'\right)\right|+\left|\left(-\frac{1}{\Theta}+\varepsilon'\right)-0\right|+\left|0-\frac{1}{\Theta}\right|+\left|\frac{1}{\Theta}-1\right|=2+\frac{2}{\Theta}-2\varepsilon'.$$

By assumption, we have $\Theta > 2$, which implies $2+\frac{2}{\Theta}-2\varepsilon' < 3-2\varepsilon'$. Therefore, SMARTSTART delivers $\sigma_1^{(2)}$ after collecting $\sigma_1^{(1)}$ and, for all $t \geq 1+\frac{\varepsilon'}{2}$, we have

$$L(t,p_1,\{\sigma_1^{(1)},\sigma_1^{(2)}\}) = L(t,1,\{\sigma_1^{(1)},\sigma_1^{(2)}\}) = 2+\frac{2}{\Theta}-2\varepsilon'.$$

Again, by assumption, we have $\Theta < 3$ and $\varepsilon < \frac{2}{9}(\frac{2\Theta^2}{\Theta^2+\Theta-2})$, i.e., $\varepsilon' < \frac{2}{9}$, which implies that for the time $1+\frac{\varepsilon'}{2}$, when SMARTSTART reaches position $p_1 = 1$, the inequality

$$\frac{L(1+\frac{\varepsilon'}{2},p_1,\{\sigma_1^{(1)},\sigma_1^{(2)}\})}{\Theta-1} = \frac{2+\frac{2}{\Theta}-2\varepsilon'}{\Theta-1} \overset{\Theta<3}{>} 1+\frac{1}{3}-\varepsilon' \overset{\varepsilon'<\frac{2}{9}}{>} 1+\frac{\varepsilon'}{2} \tag{4.1}$$

holds. (Note that inequality (4.1) also holds for slightly larger Θ if we let $\varepsilon \to 0$.) Because of inequality (4.1), SMARTSTART has a waiting period and starts the schedule S_1 at time

$$\begin{aligned} t_1 &= \min\left\{ t \geq 1+\frac{\varepsilon'}{2} \;\middle|\; \frac{L(t,p_1,\{\sigma_1^{(1)},\sigma_1^{(2)}\})}{\Theta-1} \leq t \right\} \\ &= \min\left\{ t \geq 1+\frac{\varepsilon'}{2} \;\middle|\; \frac{2+\frac{2}{\Theta}-2\varepsilon'}{\Theta-1} \leq t \right\} \\ &= \frac{2+\frac{2}{\Theta}-2\varepsilon'}{\Theta-1} \\ &= \frac{2\Theta+2-2\varepsilon'\Theta}{\Theta(\Theta-1)}. \end{aligned}$$

To sum it up, we have

$$\begin{aligned} \text{SMARTSTART}(\sigma) &= t_1 + L(t_1,p_1,\{\sigma_1^{(1)},\sigma_1^{(2)}\}) \\ &= \frac{2\Theta+2-2\varepsilon'\Theta}{\Theta(\Theta-1)} + 2 + \frac{2}{\Theta} - 2\varepsilon' \\ &= \frac{2\Theta+2-2\varepsilon'\Theta}{\Theta-1}. \end{aligned}$$

On the other hand, OPT goes from the origin to $-\frac{1}{\Theta}+\varepsilon'$ to collect $\sigma_1^{(1)}$ at time $\frac{1}{\Theta}+\varepsilon'$ (i.e., it has to wait for $2\varepsilon'$ units of time after it reaches position $-\frac{1}{\Theta}+\varepsilon'$) and delivers $\sigma_1^{(1)}$ to the origin at time $\frac{2}{\Theta}$. Let $q > 0$ be the position of a request arising from the application of Lemma 4.1 at the beginning of this proof. Then this request is released earlier than time $q+\frac{\varepsilon'}{2}$. On the other hand, OPT reaches position q not earlier than time $\frac{2}{\Theta}+q$. Since we have $\Theta < 3$ and $\varepsilon < \frac{2}{9}(\frac{2\Theta^2}{\Theta^2+\Theta-2})$, i.e., $\varepsilon' < \frac{2}{9}$, we have $\frac{2}{\Theta}+q > q+\frac{\varepsilon'}{2}$ and OPT can go straight from the origin to position 1, collecting and delivering all requests that occur by the application of Lemma 4.1 as well as $\sigma_1^{(2)}$ on the way. Therefore, we have

$$\text{OPT}(\sigma) = \left|0-\left(-\frac{1}{\Theta}+\varepsilon'\right)\right| + 2\varepsilon' + \left|\left(-\frac{1}{\Theta}+\varepsilon'\right)-1\right| = \frac{1}{\Theta}+\frac{\Theta+1}{\Theta} = \frac{\Theta+2}{\Theta}.$$

Note that OPT can do this even if the capacity is $c = 1$, since no additional requests need to be carried over $[0, \frac{1}{\Theta}] \cup \{1\}$, where the requests of the application of Lemma 4.1 appear, and because the carrying paths of $\sigma_1^{(1)}$ and $\sigma_1^{(2)}$ are disjoint. Since we have $\varepsilon' = \frac{\Theta^2+\Theta-2}{2\Theta^2}\varepsilon$, we obtain

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)} = \frac{2\Theta^2 + 2\Theta - 2\varepsilon'\Theta^2}{\Theta^2 + \Theta - 2} = \frac{2\Theta^2 + 2\Theta}{\Theta^2 + \Theta - 2} - \varepsilon,$$

as claimed. ∎

PROPOSITION 4.3. *Let the capacity $c \in \mathbb{N} \cup \{\infty\}$ of the server be arbitrary but fixed, and let $2 \leq \Theta \leq \frac{1}{2}(1 + \sqrt{13})$. For every sufficiently small $\varepsilon > 0$ there is a set of requests σ such that* SMARTSTART *immediately starts S_N after S_{N-1} and such that*

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)} \geq \Theta + 1 - \frac{\Theta - 1}{3\Theta + 3} - \varepsilon,$$

i.e., the upper bound established in Proposition 3.7 *is tight for $\Theta \in [2, \frac{1}{2}(1 + \sqrt{13})] \approx [2, 2.303]$.*

Proof. Let $\varepsilon > 0$ with $\varepsilon < \frac{1}{5\Theta}\frac{3\Theta^2-\Theta}{3\Theta+3}$ and $\varepsilon' = \frac{3\Theta+3}{3\Theta^2-\Theta}\varepsilon$. We apply Lemma 4.1 with $p = 1$ and $\mu = \frac{\varepsilon'}{2}$. For convenience, we start the enumeration of the schedules with the first schedule after the application of Lemma 4.1. Algorithm SMARTSTART reaches position $p_1 = 1$ at time $1 + \frac{\varepsilon'}{2}$. Now let the requests

$$\begin{aligned} \sigma_1^{(1)} &= \left(2 + \frac{1}{\Theta} - \varepsilon', 2 + \frac{1}{\Theta} - \varepsilon'; \frac{1}{\Theta} + \varepsilon'\right), \\ \sigma_1^{(2)} &= \left(-\frac{1}{\Theta}, -\frac{1}{\Theta}; \frac{1}{\Theta} + \varepsilon'\right) \end{aligned}$$

appear. Note that both requests are released after time $\frac{1}{\Theta} + \frac{\varepsilon'}{2}$ and therefore do not interfere with the application of Lemma 4.1. If SMARTSTART serves $\sigma_1^{(2)}$ before serving $\sigma_1^{(1)}$, the time it needs is at least

$$\left|1 - \left(-\frac{1}{\Theta}\right)\right| + \left|\left(-\frac{1}{\Theta}\right) - \left(2 + \frac{1}{\Theta} - \varepsilon'\right)\right| = 1 + \frac{1}{\Theta} + 2 + \frac{2}{\Theta} - \varepsilon' = 3 + \frac{3}{\Theta} - \varepsilon'.$$

The best schedule that serves $\sigma_1^{(2)}$ after serving $\sigma_1^{(1)}$ needs time

$$\begin{aligned} \left|1 - \left(2 + \frac{1}{\Theta} - \varepsilon'\right)\right| + \left|\left(2 + \frac{1}{\Theta} - \varepsilon'\right) - \left(-\frac{1}{\Theta}\right)\right| &= 1 + \frac{1}{\Theta} - \varepsilon' + 2 + \frac{2}{\Theta} - \varepsilon' \\ &= 3 + \frac{3}{\Theta} - 2\varepsilon'. \end{aligned}$$

Thus, SMARTSTART serves $\sigma_1^{(2)}$ after serving $\sigma_1^{(1)}$, and, for all $t \geq 1 + \frac{\varepsilon'}{2}$, we obtain

$$L\left(t, p_1, \{\sigma_1^{(1)}, \sigma_1^{(2)}\}\right) = L\left(t, 1, \{\sigma_1^{(1)}, \sigma_1^{(2)}\}\right) = 3 + \frac{3}{\Theta} - 2\varepsilon'.$$

By assumption, we have $\Theta \leq \frac{1}{2}(1 + \sqrt{13})$ and $\varepsilon < \frac{1}{5\Theta}\frac{3\Theta^2-\Theta}{3\Theta+3}$, i.e., $\varepsilon' < \frac{1}{5\Theta} < 1$, which implies that for the time $1 + \frac{\varepsilon'}{2}$, when SMARTSTART reaches position $p_1 = 1$, the inequality

$$\begin{aligned}
\frac{L\left(1+\frac{\varepsilon'}{2}, p_1, \{\sigma_1^{(1)}, \sigma_1^{(2)}\}\right)}{\Theta-1} &= \frac{3+\frac{3}{\Theta}-2\varepsilon'}{\Theta-1} \\
&= \frac{3-2\varepsilon'}{\Theta-1} + \frac{3}{\Theta(\Theta-1)} \\
&\overset{1<\Theta\le\frac{1}{2}(1+\sqrt{13})}{\geq} \frac{3-2\varepsilon'}{\frac{1}{2}(\sqrt{13}-1)} + \frac{3}{\frac{1}{4}(\sqrt{13}-1)(1+\sqrt{13})} \\
&= \frac{3-2\varepsilon'}{\frac{1}{2}(\sqrt{13}-1)} + 1 \\
&\overset{\frac{1}{2}(\sqrt{13}-1)<2}{>} 1+\frac{\varepsilon'}{2}
\end{aligned}$$

holds. Thus, SMARTSTART has a waiting period and starts schedule S_1 at time

$$\begin{aligned}
t_1 &= \min\left\{ t \geq 1+\frac{\varepsilon'}{2} \;\middle|\; \frac{L(t, p_1, \{\sigma_1^{(1)}, \sigma_1^{(2)}\})}{\Theta-1} \leq t \right\} \\
&= \min\left\{ t \geq 1+\frac{\varepsilon'}{2} \;\middle|\; \frac{3+\frac{3}{\Theta}-2\varepsilon'}{\Theta-1} \leq t \right\} \\
&= \frac{3+\frac{3}{\Theta}-2\varepsilon'}{\Theta-1} \\
&= \frac{3\Theta+3}{\Theta(\Theta-1)} - \frac{2\varepsilon'}{\Theta-1}.
\end{aligned}$$

Next, we let the final request

$$\sigma_2 = \left(\frac{3\Theta+3}{\Theta(\Theta-1)} - \frac{2}{\Theta} - \varepsilon', \frac{3\Theta+3}{\Theta(\Theta-1)} - \frac{2}{\Theta} - \varepsilon'; \frac{3\Theta+3}{\Theta(\Theta-1)} \right)$$

appear. SMARTSTART finishes schedule S_1 at time

$$t_1 + L(t_1, p_1, \{\sigma_1^{(1)}, \sigma_1^{(2)}\}) = \frac{3\Theta+3}{\Theta(\Theta-1)} - \frac{2\varepsilon'}{\Theta-1} + 3 + \frac{3}{\Theta} - 2\varepsilon' = \frac{3\Theta+3}{\Theta-1} - \frac{2\Theta\varepsilon'}{\Theta-1}$$

at position $p_2 = -\frac{1}{\Theta}$. For all $t \geq \frac{3\Theta+3}{\Theta-1} - \frac{2\Theta}{\Theta-1}\varepsilon'$, we obtain

$$L\left(t, -\frac{1}{\Theta}, \{\sigma_2\}\right) = \frac{3\Theta+3}{\Theta(\Theta-1)} - \frac{1}{\Theta} - \varepsilon'.$$

By assumption, we have $2 \leq \Theta \leq \frac{1}{2}(1+\sqrt{13}) < 3$ and $\varepsilon < \frac{1}{5\Theta}\frac{3\Theta^2-\Theta}{3\Theta+3}$, i.e., $\varepsilon' < \frac{1}{5\Theta}$, which implies that, for the finishing time $\frac{3\Theta+3}{\Theta-1} - \frac{2\Theta\varepsilon'}{\Theta-1}$ of schedule S_1, the inequality

$$\begin{aligned}
\frac{L\left(\frac{3\Theta+3}{\Theta-1} - \frac{2\Theta\varepsilon'}{\Theta-1}, -\frac{1}{\Theta}, \{\sigma_2\}\right)}{\Theta-1} &= \frac{3\Theta+3}{\Theta(\Theta-1)^2} - \frac{1+\Theta\varepsilon'}{\Theta(\Theta-1)} \\
&\overset{\Theta\geq 2}{<} \frac{3\Theta+3}{\Theta-1} - \frac{1+\Theta\varepsilon'}{\Theta(\Theta-1)} \\
&\overset{5\Theta\varepsilon'<1}{<} \frac{3\Theta+3}{\Theta-1} - \frac{6\varepsilon'}{\Theta-1} \\
&\overset{\Theta<3}{<} \frac{3\Theta+3}{\Theta-1} - \frac{2\Theta\varepsilon'}{\Theta-1} \qquad (4.2)
\end{aligned}$$

holds. (Note that inequality (4.2) still holds for slightly smaller Θ if we let $\varepsilon \to 0$.) Because of inequality (4.2), the final schedule S_2 is started at time

$$t_2 = \frac{3\Theta + 3}{\Theta - 1} - \frac{2\Theta\varepsilon'}{\Theta - 1}$$

without waiting. To sum it up, we have

$$\begin{aligned}\textsc{Smartstart}(\sigma) &= t_2 + L(t_2, p_2, \{\sigma_2\}) \\ &= \frac{3\Theta + 3}{\Theta - 1} - \frac{2\Theta\varepsilon'}{\Theta - 1} + \frac{3\Theta + 3}{\Theta(\Theta - 1)} - \frac{1}{\Theta} - \varepsilon' \\ &= \frac{3\Theta + 3}{\Theta - 1} + \frac{3\Theta + 3}{\Theta(\Theta - 1)} - \frac{1}{\Theta} - \frac{3\Theta - 1}{\Theta - 1}\varepsilon'.\end{aligned}$$

On the other hand, $\textsc{Opt}$ goes from the origin straight to position $-\frac{1}{\Theta}$ serving request $\sigma_1^{(2)}$ at time $\frac{1}{\Theta} + \varepsilon'$ (i.e., it has to wait for ε' units of time after it reaches position $-\frac{1}{\Theta}$) and returns to the origin at time $\frac{2}{\Theta} + \varepsilon'$. Let $q > 0$ be the position of a request that has occurred by the application of Lemma 4.1 at the beginning of this proof. Then this request is released earlier than time $q + \frac{\varepsilon'}{2}$. Since $\textsc{Opt}$ reaches position q not earlier than time $\frac{2}{\Theta} + \varepsilon' + q > q + \frac{\varepsilon'}{2}$, $\textsc{Opt}$ can go straight from the origin to the right and can serve all remaining requests without waiting. Note that the position $\frac{3\Theta+3}{\Theta(\Theta-1)} - \frac{2}{\Theta} - \varepsilon'$ of σ_2 is equal to or to the right of the position $2 + \frac{1}{\Theta} - \varepsilon'$ of $\sigma_1^{(2)}$ because of $\Theta \le \frac{1}{2}(1 + \sqrt{13})$. Thus, $\textsc{Opt}$ finishes at position $\frac{3\Theta+3}{\Theta(\Theta-1)} - \frac{2}{\Theta} - \varepsilon'$ and we have

$$\begin{aligned}\textsc{Opt}(\sigma) &= \left|0 - \left(-\frac{1}{\Theta}\right)\right| + \varepsilon' + \left|-\frac{1}{\Theta} - \left(\frac{3\Theta + 3}{\Theta(\Theta - 1)} - \frac{2}{\Theta} - \varepsilon'\right)\right| \\ &= \frac{1}{\Theta} + \varepsilon' + \frac{1}{\Theta} + \frac{3\Theta + 3}{\Theta(\Theta - 1)} - \frac{2}{\Theta} - \varepsilon' \\ &= \frac{3\Theta + 3}{\Theta(\Theta - 1)}.\end{aligned}$$

Note that $\textsc{Opt}$ can do this even if $c = 1$ since for all requests the starting point is equal to the destination. Since we have $\varepsilon' = \frac{3\Theta+3}{3\Theta^2-\Theta}\varepsilon$, we finally obtain

$$\begin{aligned}\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)} &= \frac{\frac{3\Theta+3}{\Theta-1} + \frac{3\Theta+3}{\Theta(\Theta-1)} - \frac{1}{\Theta} - \frac{3\Theta-1}{\Theta-1}\varepsilon'}{\frac{3\Theta+3}{\Theta(\Theta-1)}} \\ &= \Theta + 1 - \frac{\Theta - 1}{3\Theta + 3} - \frac{3\Theta^2 - \Theta}{3\Theta + 3}\varepsilon' \\ &= \Theta + 1 - \frac{\Theta - 1}{3\Theta + 3} - \varepsilon,\end{aligned}$$

as claimed. ◻

Recall that the optimal parameter Θ^* established in Theorem 3.8 is the only positive, real solution of the equation

$$\Theta + 1 - \frac{\Theta - 1}{3\Theta + 3} = \frac{2\Theta^2 + 2\Theta}{\Theta^2 + \Theta - 2},$$

which is $\Theta^* \approx 2.0526$. Therefore, according to Propositions 4.2 and 4.3 the parameter Θ^* lies in the ranges where the upper bounds of Propositions 3.3 and 3.7 are both tight.

It remains to make sure that for all Θ that lie outside of this range the competitive ratio of $\textsc{Smartstart}_\Theta$ is larger than $\rho^* \approx 2.93768$.

Lemma 4.4. *Let the capacity $c \in \mathbb{N} \cup \{\infty\}$ of the server be arbitrary but fixed, and let $1 < \Theta \leq 2$. There is a set of requests σ such that*

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)} > \rho^* \approx 2.93768.$$

Proof. Let $\varepsilon > 0$ with $\varepsilon < \frac{1}{100}$. We apply Lemma 4.1 with $p = 1$ and $\mu = \varepsilon$. For convenience, we start the enumeration of the schedules with the first schedule after the application of Lemma 4.1. Smartstart reaches position $p_1 = 1$ at time $1 + \varepsilon$. Now let the requests

$$\sigma_1^{(1)} = \left(\frac{1}{\Theta}, 1 + \frac{\varepsilon}{2}; \frac{1}{\Theta} + 2\varepsilon\right),$$
$$\sigma_1^{(2)} = \left(-\frac{1}{\Theta}, -\varepsilon; \frac{1}{\Theta} + 2\varepsilon\right)$$

appear. Note that both requests appear after time $\frac{1}{\Theta} + \varepsilon$ and therefore do not interfere with the application of Lemma 4.1. If Smartstart collects $\sigma_1^{(2)}$ before delivering $\sigma_1^{(1)}$, the time it needs is at least

$$\left|1 - \left(-\frac{1}{\Theta}\right)\right| + \left|\left(-\frac{1}{\Theta}\right) - \left(1 + \frac{\varepsilon}{2}\right)\right| = 2 + \frac{2}{\Theta} + \frac{\varepsilon}{2}.$$

The best schedule that collects $\sigma_1^{(2)}$ after delivering $\sigma_1^{(1)}$ needs time

$$\left|1 - \frac{1}{\Theta}\right| + \left|\frac{1}{\Theta} - \left(1 + \frac{\varepsilon}{2}\right)\right| + \left|\left(1 + \frac{\varepsilon}{2}\right) - \left(-\frac{1}{\Theta}\right)\right| + \left|\left(-\frac{1}{\Theta}\right) - (-\varepsilon)\right| = 3.$$

By assumption, we have $\Theta \leq 2$, which implies $3 < 2 + \frac{2}{\Theta} + \frac{\varepsilon}{2}$. Therefore, Smartstart delivers $\sigma_1^{(2)}$ after collecting $\sigma_1^{(1)}$, and for all $t \geq 1 + \varepsilon$ we have

$$L(t, p_1, \{\sigma_1^{(1)}, \sigma_1^{(2)}\}) = L(t, 1, \{\sigma_1^{(1)}, \sigma_1^{(2)}\}) = 3.$$

Again, by assumption, we have $\Theta \leq 2$ and $\varepsilon < \frac{1}{100}$, which implies that for the time $1 + \varepsilon$, when Smartstart reaches position $p_1 = 1$, the inequality

$$\frac{L\left(1 + \varepsilon, p_1, \{\sigma_1^{(1)}, \sigma_1^{(2)}\}\right)}{\Theta - 1} = \frac{3}{\Theta - 1} > 1 + \varepsilon$$

holds. Thus, Smartstart has a waiting period and starts schedule S_1 at time

$$\begin{aligned} t_1 &= \min\left\{t \geq 1 + \varepsilon \;\middle|\; \frac{L(t, p_1, \{\sigma_1^{(1)}, \sigma_1^{(2)}\})}{\Theta - 1} \leq t\right\} \\ &= \min\left\{t \geq 1 + \varepsilon \;\middle|\; \frac{3}{\Theta - 1} \leq t\right\} \\ &= \frac{3}{\Theta - 1}. \end{aligned}$$

To sum it up, we have

$$\textsc{Smartstart}(\sigma) = t_1 + L(t_1, p_1, \{\sigma_1^{(1)}, \sigma_1^{(2)}\}) = \frac{3}{\Theta - 1} + 3 = \frac{3\Theta}{\Theta - 1}.$$

On the other hand, $\textsc{Opt}$ goes from the origin to $-\frac{1}{\Theta}$ to collect $\sigma_1^{(1)}$ at time $\frac{1}{\Theta} + 2\varepsilon$ (i.e., it has to wait for 2ε units of time after it reaches position $-\frac{1}{\Theta}$) and returns to the origin at time $\frac{2}{\Theta} + 2\varepsilon$. Let q be the position of a request that has occurred by the application of Lemma 4.1 at the beginning of this proof. Then this request is released earlier than time $q + \varepsilon$. Since $\textsc{Opt}$ reaches position q not earlier than time $\frac{2}{\Theta} + 2\varepsilon + q > q + \varepsilon$, $\textsc{Opt}$ can go straight from position $-\frac{1}{\Theta}$ to position $1 + \frac{\varepsilon}{2}$ collecting and delivering all requests that occur by the application of Lemma 4.1 as well as $\sigma_1^{(2)}$ on the way. Therefore, we have

$$\textsc{Opt}(\sigma) = \left|0 - \left(-\frac{1}{\Theta}\right)\right| + 2\varepsilon + \left|-\frac{1}{\Theta} - \left(1 + \frac{\varepsilon}{2}\right)\right| = \frac{2}{\Theta} + 1 + \frac{5\varepsilon}{2} = \frac{\Theta + 2 + \frac{5}{2}\varepsilon\Theta}{\Theta}.$$

Note that $\textsc{Opt}$ can do this even if the capacity is $c = 1$ since no additional requests need to be carried over $[0, \frac{1}{\Theta}] \cup \{1\}$, where the requests of the application of Lemma 4.1 appear, and since the carrying paths of $\sigma_1^{(1)}$ and $\sigma_1^{(2)}$ are disjoint. Finally, we have

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)} = \frac{3\Theta^2}{\Theta^2 + \Theta - 2 + \frac{5}{2}\varepsilon(\Theta^2 - \Theta)}. \tag{4.3}$$

Note that the fraction in equality (4.3) becomes larger with decreasing ε. By assumption, we have $\varepsilon < \frac{1}{100}$, which implies

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)} > \frac{3\Theta^2}{\frac{41}{40}\Theta^2 + \frac{39}{40}\Theta - 2} =: g_1(\Theta).$$

The function g_1 is monotonically decreasing on $(1, 2]$. Therefore, we have

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)} > g_1(2) = \frac{80}{27} > 2.95 > \rho^*$$

for all $\Theta \in (1, 2]$. ∎

Lemma 4.5. *Let the capacity $c \in \mathbb{N} \cup \{\infty\}$ of the server be arbitrary but fixed, and let $\frac{1}{2}(1 + \sqrt{13}) < \Theta \leq 1 + \sqrt{2}$. There is a set of requests σ such that*

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)} > \rho^* \approx 2.93768.$$

Proof. Let $\varepsilon > 0$ with $\varepsilon < \frac{1}{25}$. We apply Lemma 4.1 with $p = 1$ and $\mu = \varepsilon$. For convenience, we start the enumeration of the schedules with the first schedule after the application of Lemma 4.1. $\textsc{Smartstart}$ reaches position $p_1 = 1$ at time $1 + \varepsilon$. Now let the requests

$$\sigma_1^{(1)} = \left(2 + \frac{1}{\Theta} - \frac{\varepsilon}{2}, 2 + \frac{1}{\Theta} - \frac{\varepsilon}{2}; \frac{1}{\Theta} + 2\varepsilon\right),$$
$$\sigma_1^{(2)} = \left(-\frac{1}{\Theta} - \varepsilon, -\frac{1}{\Theta} - \varepsilon; \frac{1}{\Theta} + 2\varepsilon\right)$$

appear. Note that both requests appear after time $\frac{1}{\Theta}+\varepsilon$ and therefore do not interfere with the application of Lemma 4.1. If SMARTSTART serves $\sigma_1^{(2)}$ before serving $\sigma_1^{(1)}$, the time it needs is at least

$$\begin{aligned}\left|1-\left(-\frac{1}{\Theta}-\varepsilon\right)\right|+\left|\left(-\frac{1}{\Theta}-\varepsilon\right)-\left(2+\frac{1}{\Theta}-\frac{\varepsilon}{2}\right)\right| &= \frac{\Theta+1}{\Theta}+\varepsilon+\frac{2\Theta+2}{\Theta}+\frac{\varepsilon}{2}\\ &= 3+\frac{3}{\Theta}+\frac{3}{2}\varepsilon.\end{aligned}$$

The best schedule that serves $\sigma_1^{(2)}$ after serving $\sigma_1^{(1)}$ needs time

$$\begin{aligned}&\left|1-\left(2+\frac{1}{\Theta}-\frac{\varepsilon}{2}\right)\right|+\left|\left(2+\frac{1}{\Theta}-\frac{\varepsilon}{2}\right)-\left(-\frac{1}{\Theta}-\varepsilon\right)\right|\\ &=\frac{\Theta+1}{\Theta}-\frac{\varepsilon}{2}+\frac{2\Theta+2}{\Theta}+\frac{\varepsilon}{2}\\ &=3+\frac{3}{\Theta}.\end{aligned}$$

Therefore SMARTSTART serves $\sigma_1^{(2)}$ after serving $\sigma_1^{(1)}$ and, for all $t\geq 1+\varepsilon$, we have

$$L(t,p_1,\{\sigma_1^{(1)},\sigma_1^{(2)}\})=L(t,1,\{\sigma_1^{(1)},\sigma_1^{(2)}\})=3+\frac{3}{\Theta}.$$

By assumption, we have $\Theta\leq 1+\sqrt{2}$ and $\varepsilon<\frac{1}{25}$, which implies that for the time $1+\varepsilon$, when SMARTSTART reaches position $p_1=1$, the inequality

$$\frac{L\left(1+\varepsilon,p_1,\{\sigma_1^{(1)},\sigma_1^{(2)}\}\right)}{\Theta-1}=\frac{3+\frac{3}{\Theta}}{\Theta-1}\overset{\Theta\leq 1+\sqrt{2}}{\geq}\frac{3}{\sqrt{2}}+\frac{3}{2+\sqrt{2}}\overset{\varepsilon<\frac{1}{10}}{>}1+\frac{\varepsilon}{2}$$

holds. Thus, SMARTSTART has a waiting period and starts schedule S_1 at time

$$\begin{aligned}t_1&=\min\left\{t\geq 1+\varepsilon\;\middle|\;\frac{L(t,p_1,\{\sigma_1^{(1)},\sigma_1^{(2)}\})}{\Theta-1}\leq t\right\}\\ &=\min\left\{t\geq 1+\varepsilon\;\middle|\;\frac{3+\frac{3}{\Theta}}{\Theta-1}\leq t\right\}\\ &=\frac{3+\frac{3}{\Theta}}{\Theta-1}\\ &=\frac{3\Theta+3}{\Theta(\Theta-1)}.\end{aligned}$$

Next, we let the final request

$$\sigma_2=\left(2+\frac{1}{\Theta}-\varepsilon,2+\frac{1}{\Theta}-\varepsilon;\frac{3\Theta+3}{\Theta(\Theta-1)}+\varepsilon\right)$$

appear. SMARTSTART finishes schedule S_1 at time

$$t_1+L\left(t_1,p_1,\{\sigma_1^{(1)},\sigma_1^{(2)}\}\right)=\frac{3\Theta+3}{\Theta(\Theta-1)}+3+\frac{3}{\Theta}=\frac{3\Theta+3}{\Theta-1}$$

at position $p_2 = -\frac{1}{\Theta} - \varepsilon$. For all $t \geq \frac{3\Theta+3}{\Theta-1}$, we have

$$L\left(t, -\frac{1}{\Theta} - \varepsilon, \{\sigma_2\}\right) = 2 + \frac{2}{\Theta}.$$

By assumption, we have $\Theta > \frac{1}{2}(1+\sqrt{13})$, which implies that for the finishing time $\frac{3\Theta+3}{\Theta-1}$ of schedule S_1 the inequality

$$\frac{L\left(\frac{3\Theta+3}{\Theta-1}, p_2, \{\sigma_2\}\right)}{\Theta - 1} = \frac{2+\frac{2}{\Theta}}{\Theta-1} < \frac{3\Theta+3}{\Theta-1}$$

holds. Therefore the final schedule S_2 is started at time $t_2 = \frac{3\Theta+3}{\Theta-1}$. To sum it up, we have

$$\text{SMARTSTART}(\sigma) = t_2 + L(t_2, p_2, \{\sigma_2\}) = \frac{3\Theta+3}{\Theta-1} + 2 + \frac{2}{\Theta}.$$

On the other hand, OPT goes from the origin straight to position $-\frac{1}{\Theta} - \varepsilon$ to serve request $\sigma_1^{(2)}$ at time $\frac{1}{\Theta} + 2\varepsilon$ (i.e., it has to wait for ε units of time after it reaches position $-\frac{1}{\Theta} - \varepsilon$) and returns to the origin at time $\frac{1}{\Theta} + 3\varepsilon$. Let $q > 0$ be the position of a request that has occurred by the application of Lemma 4.1 at the beginning of this proof. Then this request is released earlier than time $q + \varepsilon$. Since OPT reaches position q not earlier than time $\frac{2}{\Theta} + 3\varepsilon + q > q + \varepsilon$, OPT can go straight from position $-\frac{1}{\Theta} - \varepsilon$ to position $2 + \frac{1}{\Theta} - \frac{\varepsilon}{2}$ serving the requests that occur by applying Lemma 4.1 as well as $\sigma_1^{(1)}$ and σ_2 on the way. Therefore, we have

$$\begin{aligned}
\text{OPT}(\sigma) &= \left|0 - \left(-\frac{1}{\Theta} - \varepsilon\right)\right| + \varepsilon + \left|-\frac{1}{\Theta} - \varepsilon - \left(2 + \frac{1}{\Theta} - \varepsilon\right)\right| \\
&= \frac{1}{\Theta} + 3\varepsilon + 2 + \frac{2}{\Theta} - \frac{\varepsilon}{2} \\
&= \frac{2\Theta+3}{\Theta} + \frac{5\varepsilon}{2}.
\end{aligned}$$

Note that OPT reaches position $2 + \frac{1}{\Theta} - \frac{\varepsilon}{2}$ at time $\frac{2\Theta+3}{\Theta} + \frac{5\varepsilon}{2}$ and can immediately serve $\sigma_1^{(1)}$ since the assumption $\Theta > \frac{1}{2}(1+\sqrt{13})$ implies

$$\frac{2\Theta+3}{\Theta} + \frac{5\varepsilon}{2} > \frac{2\Theta+3}{\Theta} + \varepsilon > \frac{3\Theta+3}{\Theta(\Theta-1)} + \varepsilon.$$

The latter inequality holds, because of the monotonicity of the curves $2\Theta + 3$ and $3 + \frac{3}{\Theta}$ and their intersection at $\Theta = \frac{1}{2}(1+\sqrt{13})$. Note, furthermore, that OPT can serve all requests on the way even if capacity $c = 1$ holds since for all requests the starting point is equal to the ending point. To sum it up, we have

$$\frac{\text{SMARTSTART}(\sigma)}{\text{OPT}(\sigma)} = \frac{\frac{3\Theta+3}{\Theta-1} + 2 + \frac{2}{\Theta}}{\frac{2\Theta+3}{\Theta} + \frac{5\varepsilon}{2}}. \tag{4.4}$$

Note that the fraction in (4.4) becomes larger with decreasing ε. By assumption, we have $\varepsilon < \frac{1}{25}$, which implies

$$\frac{\text{SMARTSTART}(\sigma)}{\text{OPT}(\sigma)} > \frac{\frac{3\Theta+3}{\Theta-1} + 2 + \frac{2}{\Theta}}{\frac{2\Theta+3}{\Theta} + \frac{1}{10}} =: g_2(\Theta).$$

The function g_2 is monotonically decreasing on $(\frac{1}{2}(1+\sqrt{13}), 1+\sqrt{2}]$. Therefore, we have

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)} \geq g_2(1+\sqrt{2}) = \frac{10}{573}(109+45\sqrt{2}) > 3 > \rho^*$$

for all $\Theta \in (\frac{1}{2}(1+\sqrt{13}), 1+\sqrt{2}]$. ∎

LEMMA 4.6. *Let the capacity $c \in \mathbb{N} \cup \{\infty\}$ of the server be arbitrary but fixed, and let $1+\sqrt{2} < \Theta < 3$. There is a set of requests σ such that*

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)} > \rho^* \approx 2.93768.$$

Proof. Let $\varepsilon > 0$ with $\varepsilon < \frac{1}{20}$. We apply Lemma 4.1 with $p = 1$ and $\mu = \varepsilon$. For convenience, we start the enumeration of the schedules with the first schedule after the application of Lemma 4.1. SMARTSTART reaches position $p_1 = 1$ at time $1+\varepsilon$. Now let the requests

$$\sigma_1^{(1)} = \left(\frac{1}{\Theta}, 1; \frac{1}{\Theta} + 2\varepsilon\right),$$
$$\sigma_1^{(2)} = \left(-\frac{1}{\Theta}, -\frac{1}{\Theta}; \frac{1}{\Theta} + 2\varepsilon\right)$$

appear. Note that both requests appear after time $\frac{1}{\Theta}+\varepsilon$ and therefore do not interfere with the application of Lemma 4.1. If SMARTSTART serves $\sigma_1^{(2)}$ before delivering $\sigma_1^{(1)}$, the time it needs is at least

$$\left|1 - \left(-\frac{1}{\Theta}\right)\right| + \left|\left(-\frac{1}{\Theta}\right) - 1\right| = 2 + \frac{2}{\Theta}.$$

The best schedule that serves $\sigma_1^{(2)}$ after delivering $\sigma_1^{(1)}$ needs time

$$\left|1 - \left(\frac{1}{\Theta}\right)\right| + \left|\left(\frac{1}{\Theta}\right) - 1\right| + \left|1 - \left(-\frac{1}{\Theta}\right)\right| = \frac{2\Theta-2}{\Theta} + \frac{\Theta+1}{\Theta} = 3 - \frac{1}{\Theta}.$$

By assumption, we have $\Theta < 3$, which implies $3-\frac{1}{\Theta} < 2+\frac{2}{\Theta}$. Therefore SMARTSTART serves $\sigma_1^{(2)}$ after delivering $\sigma_1^{(1)}$, and for all $t \geq 1+\varepsilon$ we have

$$L(t, p_1, \{\sigma_1^{(1)}, \sigma_1^{(2)}\}) = L(t, 1, \{\sigma_1^{(1)}, \sigma_1^{(2)}\}) = 3 - \frac{1}{\Theta}.$$

Again, by assumption, we have $\Theta < 3$ and $\varepsilon < \frac{1}{20}$, which implies that for the time $1+\varepsilon$, when SMARTSTART reaches position $p_1 = 1$ the inequality

$$\frac{L\left(1+\varepsilon, p_1, \{\sigma_1^{(1)}, \sigma_1^{(2)}\}\right)}{\Theta - 1} = \frac{3-\frac{1}{\Theta}}{\Theta-1} > 1+\varepsilon$$

holds. Thus, SMARTSTART has a waiting period and starts schedule S_1 at time

$$\begin{aligned} t_1 &= \min\left\{ t \geq 1+\varepsilon \;\middle|\; \frac{L(t, p_1, \{\sigma_1^{(1)}, \sigma_1^{(2)}\})}{\Theta - 1} \leq t \right\} \\ &= \min\left\{ t \geq 1+\varepsilon \;\middle|\; \frac{3-\frac{1}{\Theta}}{\Theta-1} \leq t \right\} \\ &= \frac{3-\frac{1}{\Theta}}{\Theta-1} \\ &= \frac{3\Theta-1}{\Theta(\Theta-1)}. \end{aligned}$$

Next, we let the final request

$$\sigma_2 = \left(1, 1; \frac{3\Theta-1}{\Theta(\Theta-1)} + \varepsilon\right)$$

appear. SMARTSTART finishes schedule S_1 at time

$$t_1 + L(t_1, p_1, \{\sigma_1^{(1)}, \sigma_1^{(2)}\}) = \frac{3-\frac{1}{\Theta}}{\Theta-1} + 3 - \frac{1}{\Theta} = \frac{3\Theta-1}{\Theta-1}$$

at position $p_2 = -\frac{1}{\Theta}$. For all $t \geq \frac{3\Theta-1}{\Theta-1}$, we have

$$L\left(t + L\left(t_1, -\frac{1}{\Theta}, \{\sigma_2\}\right)\right) = 1 + \frac{1}{\Theta}.$$

By assumption, we have $\Theta > 1+\sqrt{2}$, which implies that for the finishing time $\frac{3\Theta-1}{\Theta-1}$ of schedule S_1 the inequality

$$\frac{L\left(\frac{3\Theta-1}{\Theta-1}, p_2, \{\sigma_2\}\right)}{\Theta-1} = \frac{1+\frac{1}{\Theta}}{\Theta-1} < \frac{3\Theta-1}{\Theta-1}$$

holds. Therefore the final schedule S_2 is started at time $t_2 = \frac{3\Theta-1}{\Theta-1}$. To sum it up, we have

$$\text{SMARTSTART}(\sigma) = t_2 + L(t_2, p_2, \{\sigma_2\}) = \frac{3\Theta-1}{\Theta-1} + 1 + \frac{1}{\Theta}.$$

On the other hand, OPT goes from the origin to $-\frac{1}{\Theta}$ to collect $\sigma_1^{(1)}$ at time $\frac{1}{\Theta} + 2\varepsilon$ (i.e., it has to wait for 2ε units of time after reaching position $-\frac{1}{\Theta}$) and returns to the origin at time $\frac{2}{\Theta} + 2\varepsilon$. Let $q > 0$ be the position of a request that has occurred by the application of Lemma 4.1 at the beginning of this proof. Then this request is released earlier than time $q + \varepsilon$. Since OPT reaches position q not earlier than time $\frac{2}{\Theta} + 2\varepsilon + q > q + \varepsilon$, OPT can go straight from position $-\frac{1}{\Theta}$ to position 1 collecting and delivering all requests that occur by the application of Lemma 4.1 as well as $\sigma_1^{(2)}$. Note that OPT can also collect σ_2 at arrival at position 1 at time $1 + \frac{2}{\Theta} + 2\varepsilon$ since the assumption $\Theta > 1+\sqrt{2}$ implies

$$1 + \frac{2}{\Theta} + 2\varepsilon = \frac{\Theta^2 + \Theta - 2}{\Theta(\Theta-1)} + 2\varepsilon \geq \frac{3\Theta-1}{\Theta(\Theta-1)} + \varepsilon.$$

The latter inequality holds, because of the monotonicity of the curves $\Theta^2+\Theta-2$ and $3\Theta-1$ and intersection at $\Theta=1+\sqrt{2}$. Therefore, we have

$$\textsc{Opt}(\sigma)=\left|0-\left(-\frac{1}{\Theta}\right)\right|+2\varepsilon+\left|-\frac{1}{\Theta}-1\right|=1+\frac{2}{\Theta}+2\varepsilon.$$

Note that Opt can do this even if capacity $c=1$ holds since no additional requests need to be carried over $[0,\frac{1}{\Theta}]\cup\{1\}$, where the requests of the application of Lemma 4.1 appear. Finally, we have

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)}=\frac{\frac{3\Theta-1}{\Theta-1}+1+\frac{1}{\Theta}}{1+\frac{2}{\Theta}+2\varepsilon}. \tag{4.5}$$

Note that the fraction in equality (4.5) becomes larger with decreasing ε. By assumption, we have $\varepsilon<\frac{1}{20}$, which implies

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)}>\frac{\frac{3\Theta-1}{\Theta-1}+1+\frac{1}{\Theta}}{1.1+\frac{2}{\Theta}}=:g_3(\Theta).$$

The function g_3 has exactly one local minimum in the range $(1+\sqrt{2},3)$ at

$$\hat{\Theta}=\frac{349}{247}+\frac{\sqrt{84998}}{247}.$$

Therefore, we have

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)}>g_3(\hat{\Theta})\approx 3.01454>3>\rho^*. \qquad \square$$

Lemma 4.7. *Let the capacity $c\in\mathbb{N}\cup\{\infty\}$ of the server be arbitrary but fixed, and let $\Theta\geq 3$. There is a set of requests σ such that*

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)}>\rho^*\approx 2.93768.$$

Proof. Let $\varepsilon>0$ with $\varepsilon<\frac{1}{75}$. We apply Lemma 4.1 with $p=1$ and $\mu=\varepsilon$. For convenience, we start the enumeration of the schedules with the first schedule after the application of Lemma 4.1. Algorithm Smartstart reaches position $p_1=1$ at time $1+\varepsilon$. Now let the requests

$$\sigma_1^{(1)}=\left(\frac{\Theta+1}{2\Theta}+\frac{\varepsilon}{2},1;\frac{1}{\Theta}+2\varepsilon\right),$$
$$\sigma_1^{(2)}=\left(\frac{1}{\Theta},\frac{1}{\Theta};\frac{1}{\Theta}+2\varepsilon\right)$$

appear. Note that both requests appear after time $\frac{1}{\Theta}+\varepsilon$ and therefore do not interfere with the application of Lemma 4.1 and that the carrying path of $\sigma_1^{(1)}$ does not cross the position $\frac{1}{\Theta}$ of $\sigma_1^{(2)}$ since the assumption $\Theta\geq 3$ implies $\frac{1}{\Theta}<\frac{\Theta+1}{2\Theta}$. Thus, if Smartstart serves $\sigma_1^{(2)}$ before delivering $\sigma_1^{(1)}$ the time it needs is at least

$$\left|1-\frac{1}{\Theta}\right|+\left|\frac{1}{\Theta}-1\right|=2-\frac{2}{\Theta}.$$

The best schedule that serves $\sigma_1^{(2)}$ after delivering $\sigma_1^{(1)}$ needs time

$$\left|1-\left(\frac{\Theta+1}{2\Theta}+\frac{\varepsilon}{2}\right)\right|+\left|\left(\frac{\Theta+1}{2\Theta}+\frac{\varepsilon}{2}\right)-1\right|+\left|1-\frac{1}{\Theta}\right| = 2\left(\frac{\Theta-1}{2\Theta}-\frac{\varepsilon}{2}\right)+\frac{\Theta-1}{\Theta}$$
$$= 2-\frac{2}{\Theta}-\varepsilon.$$

Therefore SMARTSTART serves $\sigma_1^{(2)}$ after delivering $\sigma_1^{(1)}$, and for all $t \geq 1+\varepsilon$ we have

$$L\left(t, p_1, \{\sigma_1^{(1)}, \sigma_1^{(2)}\}\right) = L\left(t, 1, \{\sigma_1^{(1)}, \sigma_1^{(2)}\}\right) = 2-\frac{2}{\Theta}-\varepsilon.$$

By assumption, we have $\Theta \geq 3$, which implies that for the finishing time $1+\varepsilon$ of schedule S_1 the inequality

$$\frac{L\left(1+\varepsilon, p_1, \{\sigma_1^{(1)}, \sigma_1^{(2)}\}\right)}{\Theta-1} = \frac{2-\frac{2}{\Theta}-\varepsilon}{\Theta-1} \leq 1+\varepsilon$$

holds. Thus, the schedule S_1 is started immediately after the application of Lemma 4.1 at time $t_1 = 1+\varepsilon$. Next, we let the final request

$$\sigma_2 = (1, 1; 1+2\varepsilon)$$

appear. SMARTSTART finishes schedule S_1 at time

$$t_1 + L(t_1, p_1, \{\sigma_1^{(1)}, \sigma_1^{(2)}\}) = 1+\varepsilon+2-\frac{2}{\Theta}-\varepsilon = 3-\frac{2}{\Theta}$$

at position $p_2 = \frac{1}{\Theta}$. For all $t \geq 3-\frac{2}{\Theta}$, we have

$$L(t, p_2, \{\sigma_2\}) = 1-\frac{1}{\Theta}.$$

By assumption, we have $\Theta \geq 3$, which implies that for the finishing time $3-\frac{2}{\Theta}$ of schedule S_1 the inequality

$$\frac{L\left(3-\frac{2}{\Theta}, \frac{1}{\Theta}, \{\sigma_2\}\right)}{\Theta-1} = \frac{1-\frac{1}{\Theta}}{\Theta-1} \leq 3-\frac{2}{\Theta}$$

holds. Therefore the final schedule S_2 is started at time $t_2 = 3-\frac{2}{\Theta}$. To sum it up, we have

$$\text{SMARTSTART}(\sigma) = t_2 + L(t_2, p_2, \{\sigma_2\}) = 3-\frac{2}{\Theta}+1-\frac{1}{\Theta} = 4-\frac{3}{\Theta}.$$

On the other hand, OPT waits at the origin until time 2ε. Let q be the position of a request that has occurred by the application of Lemma 4.1 at the beginning of this proof. Then this request is released earlier than time $q+\varepsilon$. Since OPT reaches position q not earlier than time $q+2\varepsilon > q+\varepsilon$, OPT can go straight from the origin to position 1 collecting and delivering all requests that occur by the application of Lemma 4.1, as well as $\sigma_1^{(1)}$, $\sigma_1^{(2)}$, and σ_2. Therefore, we have

$$\text{OPT}(\sigma) = 1+2\varepsilon.$$

Note that OPT can do this even if capacity $c = 1$ holds since no additional requests need to be carried over $[0, \frac{1}{\Theta}] \cup \{1\}$, where the requests of the application of Lemma 4.1 appear. To sum it up, we have

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)} = \frac{4 - \frac{3}{\Theta}}{1 + 2\varepsilon}. \tag{4.6}$$

Note that the fraction in equality (4.6) becomes larger with decreasing ε. By assumption, we have $\varepsilon < \frac{1}{75}$, which implies

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)} > \frac{4 - \frac{3}{\Theta}}{1 + \frac{1}{75}} =: g_4(\Theta).$$

The function g_4 is strictly monotonically increasing on $[3, \infty)$. Therefore, we have

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)} > g_4(3) = \frac{225}{76} > 2.95 > \rho^*. \qquad \square$$

We summarize Lemmas 4.4, 4.5, 4.6, and 4.7 into one lemma.

LEMMA 4.8. *Let*

$$I_1 = (1, 2], \quad I_2 = (\tfrac{1}{2}(1 + \sqrt{13}), 1 + \sqrt{2}], \quad I_3 = (1 + \sqrt{2}, 3), \quad I_4 = [3, \infty)$$

be intervals. For every $i \in \{1, 2, 3, 4\}$ there is a set of requests σ such that, for all $\Theta \in I_i$,

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)} > \rho^* \approx 2.93768.$$

Proof. This is an immediate consequence of Lemmas 4.4, 4.5, 4.6, and 4.7. $\square$

Our main theorem now follows from Theorem 3.8 combined with Propositions 4.2 and 4.3, as well as Lemma 4.8.

THEOREM 4.9. *The competitive ratio of* $\textsc{Smartstart}_{\Theta^*}$ *is exactly*

$$\rho^* = f_1(\Theta^*) = f_2(\Theta^*) \approx 2.93768.$$

For every other $\Theta > 1$ with $\Theta \neq \Theta^*$ the competitive ratio of $\textsc{Smartstart}_{\Theta}$ *is larger than ρ^*.*

Proof. We have shown in Proposition 4.2 that the upper bound

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)} \leq f_1(\Theta) = \frac{2\Theta^2 + 2\Theta}{\Theta^2 + \Theta - 2},$$

established in Proposition 3.3 for the case where SMARTSTART waits before starting the final schedule, is tight for all $\Theta \in (2, 3)$. Furthermore, we have shown in Proposition 4.3 that the upper bound

$$\frac{\textsc{Smartstart}(\sigma)}{\textsc{Opt}(\sigma)} \leq f_2(\Theta) = \left(\Theta + 1 - \frac{\Theta - 1}{3\Theta + 3}\right),$$

established in Proposition 3.7 for the case where SMARTSTART does not wait before starting the final schedule, is tight for all $\Theta \in (2, \frac{1}{2}(1 + \sqrt{13})]$. Since $\Theta^* \approx 2.0526$ lies in those ranges, the competitive ratio of $\textsc{Smartstart}_{\Theta^*}$ is indeed exactly ρ^*.

It remains to show that for every $\Theta > 1$ with $\Theta \neq \Theta^*$ the competitive ratio is larger. First, according to Lemma 4.8, the competitive ratio of SMARTSTART with parameter $\Theta \in (1, 2]$ or $\Theta \in (\frac{1}{2}(1+\sqrt{13}), \infty)$ is larger than ρ^*. By monotonicity of f_1, every function value in $(2, \Theta^*)$ is larger than $f_1(\Theta^*) = \rho^*$. Thus, the competitive ratio of SMARTSTART with parameter $\Theta \in (2, \Theta^*)$ is larger than ρ^*, since f_1 is tight on $(2, \Theta^*)$ by Proposition 4.2. Similarly, by monotonicity of f_2, every function value in $(\Theta^*, \frac{1}{2}(1+\sqrt{13})]$ is larger than $f_2(\Theta^*) = \rho^*$. Thus, the competitive ratio of SMARTSTART with parameter $\Theta \in (\Theta^*, \frac{1}{2}(1+\sqrt{13})]$ is larger than ρ^*, since f_1 is tight on $(\Theta^*, \frac{1}{2}(1+\sqrt{13})]$ by Proposition 4.3. □

5. Lower bound for the closed version.

We provide a lower bound for SMARTSTART for closed online DIAL-A-RIDE on the line that matches the upper bound given in [1] for arbitrary metric spaces. Note that in this setting, by definition, every schedule of SMARTSTART is a closed walk that returns to the origin.

THEOREM 5.1. *The competitive ratio of* SMARTSTART *for closed online* DIAL-A-RIDE *on the line with* $\Theta = 2$ *is exactly* 2. *For every other* $\Theta > 1$ *with* $\Theta \neq 2$ *the competitive ratio of* SMARTSTART_Θ *is larger than* 2.

Proof. We show that the competitive ratio of SMARTSTART_2 is at least 2 and that the competitive ratio of SMARTSTART_Θ is larger than 2 for all $\Theta \neq 2$. From the fact that SMARTSTART is 2-competitive even for general metric spaces [1, Thm. 6], it follows that SMARTSTART_2 has a competitive ratio of exactly 2 on the line.

Let $\Theta \leq 2$ and consider the set of requests $\{\sigma_1\}$ with $\sigma_1 = (0.5, 0.5; 0)$. Obviously, OPT can serve this request and return to the origin in time $\text{OPT}(\{\sigma_1\}) = 1$. Thus, for all $t \geq 0$, we have $L(t, 0, \{\sigma_1\}) = 1$. On the other hand, SMARTSTART waits until time

$$t_1 = \frac{L(t_1, 0, \{\sigma_1\})}{\Theta - 1} = \frac{1}{\Theta - 1}$$

to start its only schedule and finishes at time $\frac{\Theta}{\Theta-1}$. To sum it up, we have

$$\frac{\text{SMARTSTART}(\{\sigma_1\})}{\text{OPT}(\{\sigma_1\})} = \frac{\Theta}{\Theta - 1}$$

with $\frac{\Theta}{\Theta-1} > 2$ for all $\Theta < 2$ and $\frac{\Theta}{\Theta-1} = 2$ for $\Theta = 2$. Now let $2 < \Theta \leq 3$ and $\varepsilon \in (0, \min\{1 - \frac{1}{\Theta-1}, \frac{\Theta-2}{2(\Theta-1)}\})$, and consider the set of requests $\{\sigma_1, \sigma_2\}$ with

$$\begin{aligned} \sigma_1 &= (0.5, 0.5; 0) \quad \text{and} \\ \sigma_2 &= \left(1 - \frac{1}{\Theta - 1} - \varepsilon, 1 - \frac{1}{\Theta - 1} - \varepsilon; \frac{1}{\Theta - 1} + \varepsilon\right). \end{aligned}$$

By assumption, we have $\Theta > 2$ and $\varepsilon < 1 - \frac{1}{\Theta-1}$, which implies

$$0 \overset{\varepsilon < 1 - \frac{1}{\Theta-1}}{<} 1 - \frac{1}{\Theta - 1} - \varepsilon \overset{\Theta \leq 3}{<} 0.5,$$

i.e., the position of request σ_2 lies between 0 and 0.5. If OPT moves to position 0.5 and then returns to the origin, it is at position

$$a_2 = 0.5 - \left| \underbrace{\left(\frac{1}{\Theta - 1} + \varepsilon\right)}_{>0.5} - 0.5 \right| = 1 - \frac{1}{\Theta - 1} - \varepsilon$$

at time $r_2 = \frac{1}{\Theta-1} + \varepsilon$. Thus, OPT can serve σ_2 on the way, and we have

$$\textsc{Opt}(\{\sigma_1, \sigma_2\}) = 1.$$

For all $t \geq 0$, we have $L(t, 0, \{\sigma_1\}) = 1$. Therefore, SMARTSTART waits until time

$$t_1 = \frac{L(t_1, 0, \{\sigma_1\})}{\Theta - 1} = \frac{1}{\Theta - 1}$$

before starting its first schedule. Since we have $\frac{1}{\Theta-1} < \frac{1}{\Theta-1} + \varepsilon$, SMARTSTART starts to serve σ_1 at time t_1 and returns to the origin at time $\frac{\Theta}{\Theta-1}$. For all $t \geq 0$, we have

$$L(t, 0, \{\sigma_2\}) = 2 - \frac{2}{\Theta - 1} - 2\varepsilon;$$

thus SMARTSTART does not start the second and final schedule before time $\frac{2-\frac{2}{\Theta-1}-2\varepsilon}{\Theta-1}$. By assumption, we have $\Theta > 2$, which implies $\frac{\Theta}{\Theta-1} > \frac{2-\frac{2}{\Theta-1}-2\varepsilon}{\Theta-1}$. Thus, the second schedule is started at time $t_2 = \frac{\Theta}{\Theta-1}$ and finished at time

$$\textsc{Smartstart}(\{\sigma_1, \sigma_2\}) = \frac{\Theta}{\Theta - 1} + 2 - \frac{2}{\Theta - 1} - 2\varepsilon.$$

To sum it up, we have

$$\begin{aligned} \frac{\textsc{Smartstart}(\{\sigma_1, \sigma_2\})}{\textsc{Opt}(\{\sigma_1, \sigma_2\})} &= \frac{\Theta}{\Theta - 1} + 2 - \frac{2}{\Theta - 1} - 2\varepsilon \\ &\overset{\varepsilon < \frac{\Theta-2}{2(\Theta-1)}}{>} \frac{3\Theta - 4}{\Theta - 1} - 2\frac{\Theta - 2}{2(\Theta - 1)} \\ &= 2. \end{aligned}$$

Now let $\Theta > 3$ and $\varepsilon \in (0, 0.5 - \frac{1}{\Theta-1})$, and consider the set of requests $\{\sigma_1, \sigma_2\}$ with

$$\sigma_1 = (0.5, 0.5; 0) \quad \text{and}$$
$$\sigma_2 = \left(0.5, 0.5; \frac{1}{\Theta - 1} + \varepsilon\right).$$

By assumption, we have $\varepsilon < 0.5 - \frac{1}{\Theta-1}$, which implies

$$\frac{1}{\Theta - 1} + \varepsilon < 0.5,$$

i.e., σ_2 is released before position 0.5 is reachable. If OPT moves to position 0.5 and then returns to the origin, it can serve both requests without additional waiting time and we have $\textsc{Opt}(\{\sigma_1, \sigma_2\}) = 1$. For all $t \geq 0$, we have $L(t, 0, \{\sigma_1\}) = 1$. Therefore, SMARTSTART waits until time

$$t_1 = \frac{L(t_1, 0, \{\sigma_1\})}{\Theta - 1} = \frac{1}{\Theta - 1}$$

before starting its first schedule. Since we have $\frac{1}{\Theta-1} < \frac{1}{\Theta-1} + \varepsilon$, SMARTSTART starts to serve σ_1 at time t_1 and returns to the origin at time $\frac{\Theta}{\Theta-1}$. For all $t \geq 0$, we have

$$L(t, 0, \{\sigma_2\}) = 1;$$

thus SMARTSTART does not start the second and final schedule before time $\frac{1}{\Theta-1}$. By assumption, we have $\Theta > 3$, which implies $\frac{\Theta}{\Theta-1} > \frac{1}{\Theta-1}$. Thus, the second schedule is started at time $t_2 = \frac{\Theta}{\Theta-1}$ and finished at time

$$\textsc{Smartstart}(\{\sigma_1, \sigma_2\}) = \frac{\Theta}{\Theta - 1} + 1.$$

To sum it up, we have

$$\frac{\textsc{Smartstart}(\{\sigma_1, \sigma_2\})}{\textsc{Opt}(\{\sigma_1, \sigma_2\})} = \frac{\Theta}{\Theta - 1} + 1 > 2. \qquad \square$$

Appendix A. Algorithm IGNORE. The algorithm IGNORE was described in [1] (though the authors do not claim originality for the algorithm) for the closed case of the online DIAL-A-RIDE problem in arbitrary metric spaces. We describe the algorithm for the open case as introduced in [14] (see Algorithm A.1): The server remains idle until the point in time t when the first request appears. It then serves the requests released at time t immediately by following a shortest schedule S. All requests that appear during the time when the algorithm follows S are temporarily ignored. After S has been completed the server is at the destination of the last served request p, computes a shortest schedule for the unserved requests starting in position p, and follows this schedule. Again all new requests appearing during the time that the server is following the schedule are temporarily ignored. The algorithm keeps on following schedules and temporarily ignoring requests this way.

Algorithm A.1. IGNORE.

```
repeat
    if R_t ≠ ∅ then
        Start optimal offline schedule serving R_t starting from the current position
    else
        wait
```

It was shown in [14] that 4 is an upper bound for the competitive ratio of IGNORE. We show that this is tight on the line.

PROPOSITION A.1. *The competitive ratio of* IGNORE *is* $\rho_{\textsc{Ignore}} = 4$.

Proof. It was shown in [14, Thm. 2.29] that 4 is a upper bound for the competitive ratio of IGNORE for arbitrary metric spaces and therefore in particular for the real line. It remains to show that for every $\varepsilon > 0$ there is a set of requests σ such that

$$\frac{\textsc{Ignore}(\sigma)}{\textsc{Opt}(\sigma)} \geq 4 - \varepsilon.$$

Let $\varepsilon > 0$. We consider the set of requests σ consisting of

$$\sigma_1 = \left(1 - \frac{1}{5}\varepsilon, 1 - \frac{1}{5}\varepsilon; 0\right),$$
$$\sigma_2^{(1)} = \left(\frac{1}{2}, 1 - \frac{1}{5}\varepsilon; \frac{1}{5}\varepsilon\right),$$

$$\sigma_2^{(2)} = \left(0, 0; \frac{1}{5}\varepsilon\right),$$
$$\sigma_3 = \left(1 - \frac{1}{5}\varepsilon, 1 - \frac{1}{5}\varepsilon; 1\right).$$

IGNORE first serves request σ_1 in time $1-\frac{1}{5}\varepsilon$. Then it serves the requests $\sigma_2^{(1)}$ and $\sigma_2^{(2)}$. Note that serving $\sigma_2^{(1)}$ before collecting $\sigma_2^{(2)}$ takes time $2-\frac{3}{5}\varepsilon$, while collecting $\sigma_2^{(2)}$ first takes time $2-\frac{2}{5}\varepsilon$. Therefore IGNORE serves $\sigma_2^{(1)}$ first and the second schedule ends in the origin. The third and final schedule again needs time $1-\frac{1}{5}$ to serve σ_3. To sum it up, we have

$$\text{IGNORE}(\sigma) = 4 - \varepsilon.$$

OPT, on the other hand, waits until time $\frac{1}{5}\varepsilon$ at the origin for the request $\sigma_2^{(2)}$ and then just collects and delivers the remaining requests on its way to $p = 1 - \frac{1}{5}\varepsilon$, resulting in

$$\text{OPT}(\sigma) = 1.$$

Therefore, we have

$$\rho_{\text{IGNORE}} = \frac{\text{IGNORE}(\sigma)}{\text{OPT}(\sigma)} = 4 - \varepsilon,$$

as desired. ∎

REFERENCES

[1] N. Ascheuer, S. O. Krumke, and J. Rambau, *Online dial-a-ride problems: Minimizing the completion time*, in Proceedings of the 17th Annual Symposium on Theoretical Aspects of Computer Science (STACS), Springer, 2000, pp. 639–650.

[2] M. J. Atallah and S. R. Kosaraju, *Efficient solutions to some transportation problems with applications to minimizing robot arm travel*, SIAM J. Comput., 17 (1988), pp. 849–869, https://doi.org/10.1137/0217053.

[3] G. Ausiello, E. Feuerstein, S. Leonardi, L. Stougie, and M. Talamo, *Algorithms for the on-line travelling salesman*, Algorithmica, 29 (2001), pp. 560–581.

[4] A. Birx and Y. Disser, *Tight analysis of the Smartstart algorithm for online dial-a-ride on the line*, in Proceedings of the 36th International Symposium on Theoretical Aspects of Computer Science (STACS), Schloss Dagstuhl, 2019.

[5] A. Bjelde, Y. Disser, J. Hackfeld, C. Hansknecht, M. Lipmann, J. Meissner, K. Schewior, M. Schlöter, and L. Stougie, *Tight bounds for online TSP on the line*, in Proceedings of the 28th Annual Symposium on Discrete Algorithms (SODA), SIAM, 2017, pp. 994–1005, https://doi.org/10.1137/1.9781611974782.63.

[6] M. Blom, S. O. Krumke, W. E. de Paepe, and L. Stougie, *The online TSP against fair adversaries*, INFORMS J. Comput., 13 (2001), pp. 138–148.

[7] M. Charikar and B. Raghavachari, *The finite capacity dial-a-ride problem*, in Proceedings of the 39th Annual Symposium on Foundations of Computer Science (FOCS), IEEE Computer Society Press, 1998, pp. 458–467.

[8] W. E. de Paepe, J. K. Lenstra, J. Sgall, R. A. Sitters, and L. Stougie, *Computer-aided complexity classification of dial-a-ride problems*, INFORMS J. Comput., 16 (2004), pp. 120–132.

[9] E. Feuerstein and L. Stougie, *On-line single-server dial-a-ride problems*, Theoret. Comput. Sci., 268 (2001), pp. 91–105.

[10] P. C. Gilmore and R. E. Gomory, *Sequencing a one state-variable machine: A solvable case of the traveling salesman problem*, Oper. Res., 12 (1964), pp. 655–679.

[11] D. J. Guan, *Routing a vehicle of capacity greater than one*, Discrete Appl. Math., 81 (1998), pp. 41–57.

[12] D. Hauptmeier, S. O. Krumke, and J. Rambau, *The online dial-a-ride problem under reasonable load*, in Proceedings of the 4th Italian Conference on Algorithms and Complexity (CIAC), Springer, 2000, pp. 125–136.

[13] P. Jaillet and M. R. Wagner, *Generalized online routing: New competitive ratios, resource augmentation, and asymptotic analyses*, Oper. Res., 56 (2008), pp. 745–757.

[14] S. O. Krumke, *Online Optimization Competitive Analysis and Beyond*, Habilitation thesis, Technische Universität Berlin, 2001.

[15] S. O. Krumke, W. E. de Paepe, D. Poensgen, M. Lipmann, A. Marchetti-Spaccamela, and L. Stougie, *On minimizing the maximum flow time in the online dial-a-ride problem*, in Proceedings of the Third International Conference on Approximation and Online Algorithms (WAOA), Springer, 2006, pp. 258–269.

[16] S. O. Krumke, L. Laura, M. Lipmann, A. Marchetti-Spaccamela, W. de Paepe, D. Poensgen, and L. Stougie, *Non-abusiveness helps: An $O(1)$-competitive algorithm for minimizing the maximum flow time in the online traveling salesman problem*, in Proceedings of the 5th International Workshop on Approximation Algorithms for Combinatorial Optimization (APPROX), Springer, 2002, pp. 200–214.

[17] M. Lipmann, *On-Line Routing*, Ph.D. thesis, Technical University Eindhoven, 2003.

[18] M. Lipmann, X. Lu, W. E. de Paepe, R. A. Sitters, and L. Stougie, *On-line dial-a-ride problems under a restricted information model*, Algorithmica, 40 (2004), pp. 319–329.

[19] F. Yi and L. Tian, *On the online dial-a-ride problem with time-windows*, in Proceedings of the 1st International Conference on Algorithmic Applications in Management (AAIM), Springer, 2005, pp. 85–94.

[20] F. Yi, Y. Xu, and C. Xin, *Online dial-a-ride problem with time-windows under a restricted information model*, in Proceedings of the 2nd International Conference on Algorithmic Aspects in Information and Management (AAIM), Springer, 2006, pp. 22–31.