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An improved lower bound for competitive graph exploration ☆

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ABSTRACT

We give an improved lower bound of 10/3 on the competitive ratio for the exploration of an undirected, edge-weighted graph with a single agent that needs to return to the starting location after visiting all vertices. We assume that the agent has full knowledge of all edges incident to visited vertices, and, in particular, vertices have unique identifiers. Our bound improves a lower bound of 2.5 by Dobrev et al. [10] and also holds for planar graphs, where it complements an upper bound of 16 by Megow et al. [20]. The question whether a constant competitive ratio can be achieved in general remains open.

1. Introduction

We consider the problem of exploring an initially unknown, undirected and edge-weighted graph with an agent starting at some vertex called the *origin*. In each step, the agent may move along any edge incident to its current location, which incurs a cost equal to the weight of the edge. The agent has full knowledge of all edges incident to visited vertices, including their weights and the identities of any unvisited vertices they may lead to. In this setting, the objective of the agent is to visit all vertices and return to the origin while minimizing the total cost of edge traversals. We measure the performance of an exploration algorithm by its competitive ratio, i.e., the worst case ratio between its cost and the cost of an optimum offline solution that knows the graph beforehand.

This problem was first introduced by Kalyanasundaram and Pruhs [19].[1] Megow et al. [20] gave an upper bound of $16(1+2g)$ for graphs of genus g. In particular, the underlying algorithm is 16-competitive on planar graphs. The best known lower bound for the weighted case is 2.5 and was given by Dobrev et al. [10]; it holds even for planar graphs. The question whether a constant competitive ratio can be achieved in general remains open.

Our results We improve the lower bound of Dobrev et al. [10] from 2.5 to 10/3. To do this, we first tweak their construction within each recursive layer to obtain an improved bound of 3 and then modify the resulting graph macroscopically to further improve this bound to 10/3. Note that our bound still holds for planar graphs, where the best known upper bound is 16, due

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[1] Note that early works referred to the exploration problem as "online TSP", a name now reserved for the problem where the graph is known but the vertices that need to be visited are revealed over time (see [3]).

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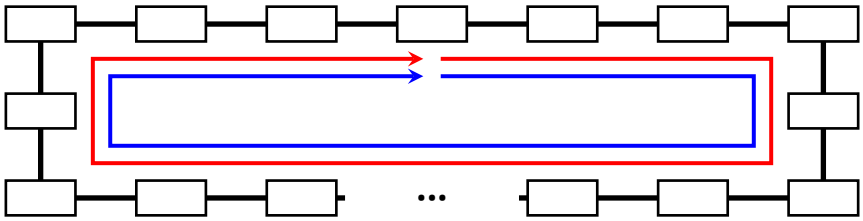

Fig. 1. The basic design of the graph of instance G. The blocks can only be explored inefficiently by Alg. Note that, while the routes of Alg (red) and Opt (blue) look the same in this high level view, their actions within blocks differ. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

to Megow et al. [20]. We hope that our construction gives some helpful insights towards the question whether a constant competitive ratio is attainable.

Related work A natural exploration algorithm is the nearest-neighbor algorithm that always explores the unexplored vertex that is cheapest to reach for the agent. This algorithm is a well-known heuristic for TSP and is $\Theta(\log n)$-competitive [24]. Hurkens and Woeginger [18] showed that the lower bound for the competitive ratio of this heuristic already holds for unweighted, planar graphs. Note that, for unweighted graphs, DFS is obviously 2-competitive, and Miyazaki et al. [21] showed that this is best-possible. Improved bounds are known for special classes of (weighted) graphs [2,5,21]. Most prominently, Miyazaki et al. [21] gave a $(1+\sqrt{3})/2$-competitive algorithm for cycles and showed that this is best-possible.

For exploring all vertices of a directed graph, Foerster and Wattenhofer [13] gave upper and lower bounds on the best-possible competitive ratio that are linear in the number of vertices. The problem changes significantly if we require all edges to be explored [1,6,12]. Deng and Papadimitriou [6] showed that the best competitive ratio for this setting depends on the deficiency d of the graph, i.e., the number of edges we need to add to make the graph Eulerian. More specifically, they showed that no deterministic algorithm can explore every graph (V, E) in less than $\frac{d^2}{4}|E|$ steps. The best known algorithm is due to Fleischer and Trippen [12] and has competitive ratio $\mathcal{O}(d^8)$.

For collaborative exploration with teams of agents, several bounds have been shown for undirected, unweighted graphs [7,9,11,14,17,22]. A constant competitive ratio can be achieved for teams of constant or exponential size by using DFS or BFS, respectively. Dereniowski et al. [7] showed that a constant competitive ratio is possible already (roughly) for a quadratic number of agents. Tight results are not yet known for smaller, non-constant teams (see [9] for a survey).

Exploration of undirected, unweighted graphs has also been studied from an information perspective, usually with the assumption that vertices do not have unique identifiers. For example, the tradeoff between the size of *advice* available to the algorithm and its performance has been studied [4,10,16]. Also, it is known that $\Theta(\log n)$ memory bits are needed for exploration [15,23], and this number can be lowered if the agent has $\Theta(\log\log n)$ pebbles available to it [8].

Outline We first describe the basic construction with a single layer in Section 2. This construction proceeds along the same lines as the construction by Dobrev et al. [10] but introduces an adaptation that allows us to obtain an improved factor of 3 when applying this construction recursively in Section 3. In Section 4 we introduce an additional modification of the macroscopic structure that improves our result to a lower bound of 10/3. Finally, in Section 5, we discuss some properties of our construction regarding planarity and the number of different weights involved.

2. Lower bound of 2

Let Alg be a fixed deterministic algorithm for solving the graph exploration problem. We write $\textsc{Alg}(G)$ for the costs Alg accumulates when traversing an edge-weighted graph G. Similarly, we write $\textsc{Opt}(G)$ for the costs of the optimum traversal of G. The basic idea is to construct a graph G_{simple} with origin v_o consisting of a cycle of *block gadgets* (see Fig. 1). These blocks are constructed in a way that Opt can traverse them cheaply, while Alg traverses them inefficiently the first time it visits them. Alg adds up roughly three times the cost of Opt for traversing a block the first time. However, both incur the same cost for any traversal of an explored block and for the transition between the blocks. We will prove the following result in this section.

Theorem 2.1. *For every $\varepsilon > 0$, there is a graph G_{simple} such that $\textsc{Alg}(G_{\text{simple}}) \geq (2-\varepsilon)\cdot \textsc{Opt}(G_{\text{simple}})$.*

2.1. Basic block strategy

We start by describing block gadgets. In the following, $x \in \mathbb{N}$ is a sufficiently large number. We construct the blocks in a way that exploring one block and entering another block afterwards can be done for an optimum cost of only $2x$, while Alg incurs a cost of at least $4x-1$.

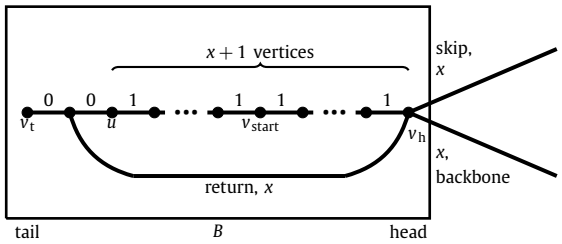

Fig. 2. A normal block. The position of v_{start} depends on the actions of ALG.

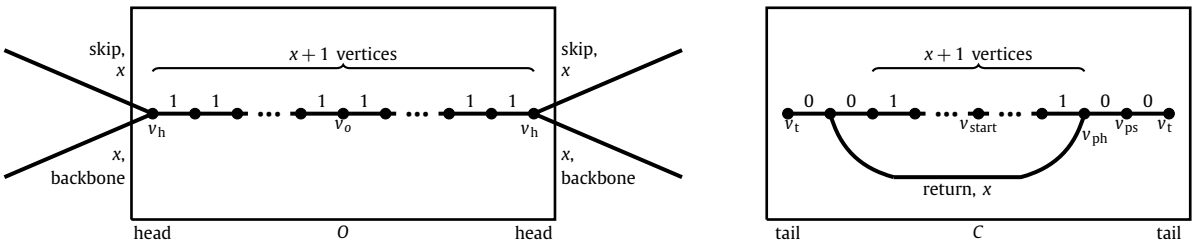

Fig. 3. An origin block (left) and a closing block (right).

See Fig. 2 for an illustration of a normal block gadget (defined below). Note that this gadget is similar to the one in [10]. However, importantly, in our gadget, the position of the start vertex v_{start}, which is the first vertex inside of the gadget that is visited by ALG, is variable and not fixed.

Normal block gadget. Assume ALG starts at some vertex, which we call start vertex v_{start}, and has two incident edges of weight 1 to choose from. Every time ALG chooses to traverse an unexplored edge of weight 1, it arrives at a new vertex that is incident to another edge of weight 1. This way, ALG explores the graph and every new vertex it visits is incident to exactly one unexplored edge of weight 1. We stop this procedure once ALG has explored x vertices and call the current position of ALG *head vertex* v_h. The head vertex is incident to three edges of weight x. ALG now has the choice between backtracking to the first unvisited vertex u to the other side of the path or taking one of the three edges of weight x. If ALG chooses to traverse an edge of weight x, we call it a *return edge* (the other two edges we call *skip edge* and *backbone edge*, depending on ALG's behavior later on). The backbone edge and skip edge are incident to vertices of other gadgets. The return edge is incident to a new vertex that is incident to two edges of weight 0. One of those two edges of weight 0 is incident to u. The other edge of weight 0 is incident to a new vertex, which we call *tail vertex* v_t.

2.2. Special blocks and graph G_{simple}

The idea now is to connect multiple normal blocks to each other and form a cycle of connected gadgets. We will construct our graph G_{simple} in a way, such that normal blocks are always entered from a block connected to their tail vertex. The gadget containing the origin, called *origin block*, is different in the sense that, initially, it is surrounded by gadgets that are not yet visited by ALG. Therefore, we have to construct it in a way that it makes no difference through which side ALG chooses to leave it. Likewise, the last block that is explored by ALG is different in the sense that it is surrounded by blocks that are already visited. Therefore, it has some special structure to be able to act as link connecting both ends of the path of blocks to a cycle. We call this special gadget *closing block* since it closes the cycle of blocks.

Origin block gadget. An *origin block* O (see Fig. 3) is a subgraph of G_{simple} that is a path of $x+1$ vertices. It contains the origin v_o, which has two incident edges of weight 1. The basic construction is the same as a normal block, however, there are no edges of weight 0 and we call both end vertices of the path *head vertices*. Furthermore, there is no return edge. Instead, both head vertices have an incident skip edge and a backbone edge.

Closing block gadget. A *closing block* C (see Fig. 3) is a subgraph of G_{simple} that is a path of $x+5$ vertices. The basic construction is the same as a normal block, however, there are two edges of weight 0 on both sides of the path and we call both end vertices of the path *tail vertices*. Furthermore, there is no outgoing skip edge. We call the unique vertex of C that is incident to one edge of weight 0, one edge of weight 1 and the return edge the *pseudo head vertex* v_{ph}. Furthermore, we call the unique vertex of C that is connected to the pseudo head vertex via an edge of weight 0 the *pseudo start vertex* v_{ps}.

Observation 2.2. We can always connect the head side of a block to the tail side of another block by connecting a backbone edge to a tail vertex and a skip edge to a start vertex v_{start}. This can be done independently of the types of the two blocks.

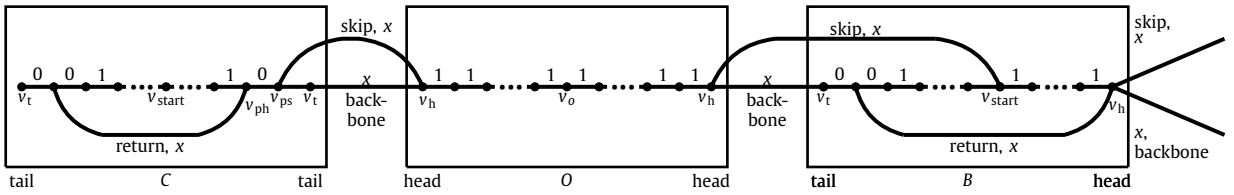

Fig. 4. A closing block connected to an origin block connected to a normal block. Note that the scenario drawn in this figure occurs if the agent never backtracks and only explores blocks to the right of the origin block. In this case the closing block is first entered from the left via the x-th explored normal block.

See Fig. 4 for an example of a closing block connected to an origin block connected to a normal block. Note that in case of a closing block, we connect the pseudo start vertex and its adjacent tail vertex to the same head side (in case of Fig. 4 one head side of an origin block). Both special blocks as well as normal blocks can be explored optimally with cost x by omitting skip and return edges and only taking backbone edges between blocks. Finally, we present the graph G_{simple}.

Graph G_{simple}. ALG starts at the origin v_o inside an origin block. We let ALG explore the block until it reaches a head vertex leading outside of the origin block. In order to continue the exploration, at some point, ALG takes one of the two edges of weight x at a head vertex of the origin block. Since ALG cannot distinguish between the endpoints of the edges, we let the edge chosen by ALG be the skip edge leading to a vertex v_{start} of a normal block. If ALG chooses to backtrack and leave the origin block through the other side of it, it enters a normal block via a skip edge in the same way. Note it is possible to connect normal blocks to both sides of an origin block by Observation 2.2. If ALG leaves a normal block B via the skip edge it already used to enter B before it has explored the head vertex of B and reenters B at the tail side via the backbone edge, we connect one (arbitrarily chosen) end of the explored path in B to the vertex inside of B that is adjacent to the tail vertex. The first time ALG takes an edge of weight x at the head vertex of a block, it takes the return edge; the second time it takes a yet unused edge of weight x in the same head vertex, it takes a skip edge, which leads to the next unvisited block. By construction, ALG then arrives at the (pseudo-)start vertex of an unexamined block. Every new block examined by ALG is a normal block until ALG has examined $x+2$ blocks (including the origin block). We let the $(x+3)$-rd block examined by ALG be a closing block connecting both ends of the path of blocks (see Fig. 1). Note that both sides of a closing block can be connected to a normal block or an origin block by Observation 2.2.

Note that G_{simple} is a planar graph: If we exclude the skip and return edges, G_{simple} is just a cycle. If we embed this cycle on a plane, we can add the skip edges on the outer side of the cycle and the return edges on the inner side of the cycle. The planarity of G_{simple} then follows from the fact that skip edges never intersect with each other and return edges never intersect with each other.

2.3. Analysis

We start our analysis of G_{simple} by formally defining the notions of the *exploration* and the *examination* of a block.

Definition 2.3 *(Exploration).* The *exploration* of a block refers to the process of visiting every vertex inside of the block.

Definition 2.4 *(Examination).* A normal block is *examined* once its head vertex has been visited. A closing block is *examined* once its pseudo-head vertex has been visited. An origin block is *examined* once ALG has left it for the first time.

We now analyze the optimum cost of exploring a block.

Observation 2.5. Let B be any block. Assume B has backbone edges incident to all tail and head vertices (i.e., by construction, the block contains exactly two vertices with incident backbone edges; one on each side) and assume OPT enters B via a backbone edge. Exploring the whole block and reaching the vertex incident to the backbone edge on the other side incurs costs of x.

It remains to analyze the cost of ALG for exploring G_{simple}. For a normal block B that is at least once left via its outgoing skip edge we introduce the following notation (cf. Definition 2.3):

$\tilde{u}_B \coloneqq$ ALG's cost incurred inside B for exploring B
plus an optional cost incurred by using the backbone edge incident to the tail vertex of B
plus an optional cost incurred by leaving B via the incoming skip edge
plus the cost incurred by leaving B via the outgoing skip edge for the first time.

We prove that $\tilde{u}_B$ yields a valid accounting scheme in the sense that the cost of one action of ALG is charged to at most one normal block B.

Lemma 2.6. *Let X be the set of normal blocks contained in the graph G_{simple} that are at least once left via their outgoing skip edges. We have $|X| = x$ and $\text{ALG}(G_{\text{simple}}) \geq \sum_{B \in X} \tilde{u}_B$.*

Proof. Let B be a normal block that is at least once left via its outgoing skip edge. First, we show that no action of ALG charged to B according to $\tilde{u}_B$ is also charged for a different normal block.

Since every vertex is part of exactly one block, the cost incurred by walking an edge between to vertices of B is only charged for the block B. In particular, the action of leaving B via an incoming/outgoing skip edge is only charged to B. By Observation 2.2, the backbone edge incident to the tail vertex of B is incident to the head vertex of another block, i.e., in particular it is not incident to the tail vertex of another normal block. Thus, the cost of using the backbone edge is only charged for B.

This proves that ALG incurs a cost of $\tilde{u}_B$ for every normal block B that is at least once left via its outgoing skip edge, i.e. $\text{ALG}(G_{\text{simple}}) \geq \sum_{B \in X} \tilde{u}_B$.

It remains to show that at G_{simple} contains x normal blocks that are left at least once via their outgoing skip edges. By definition of G_{simple}, the first $x+1$ blocks examined after leaving the initial origin block are normal blocks that are entered via incoming skip edges. We consider two cases depending on whether or not the origin block is connected to a closing block. In case the origin block is connected to the closing block, one normal block is entered via incoming skip edges coming from the origin block. Therefore, x normal blocks are entered via incoming skip edges coming from normal blocks, i.e., x normal blocks are left via outgoing skip edges, which shows the claim for this case.

In case the origin block is not connected to the closing block, two normal blocks are entered via incoming skip edges coming from the origin block. Therefore, $x-1$ normal blocks are entered via incoming skip edges coming from normal blocks, i.e., $x-1$ normal blocks are left via outgoing skip edges. Since the closing block is not connected to the origin block, it also is entered by an incoming skip edge, i.e., by an outgoing skip edge of a normal block. Thus, in total x normal blocks are left via outgoing skip edges, which shows the claim. □

Lemma 2.7. *For every normal block B that is at least once left via its outgoing skip edge, we have $\tilde{u}_B \geq 4x - 1$.*

Proof. Let B be a normal block that is at least once left via its outgoing skip edge. By definition of G_{simple}, the block B is entered via its incoming skip edge. Assume ALG leaves B before exploring it and instead goes back through the skip edge it first arrived by. This incurs a cost of x. For exploring B, ALG has to reenter B either by using the incoming skip edge or the backbone edge. In case that ALG reenters B via the incoming skip edge, it is in the same situation as in the beginning of the proof. In case that ALG reenters B via the backbone, it incurs a cost of x. Visiting every vertex of B then again incurs a cost of at least x. Finally, ALG has to take the outgoing skip edge of weight x incident to the head vertex of B to enter the unexplored block adjacent to the head side of B. Thus, in this case, we have $\tilde{u}_B \geq 4x$.

Therefore, it is now sufficient to consider the case that ALG stays inside B and explores B. ALG incurs a cost of at least $x-1$ until it reaches the head vertex. At this point only the three vertices on the tail side are left to explore. ALG can either backtrack to the first unvisited vertex on the other side of the path or take one of the edges of weight x. Both cases add a cost of x. To enter the unexplored block adjacent to the head side of B, ALG needs to backtrack to the head vertex of B for costs of x. Then, ALG has to take the outgoing skip edge of weight x incident to the head vertex of B to enter the unexplored block adjacent to the head side of B. This results in $\tilde{u}_B \geq 4x - 1$. □

Proof of Theorem 2.1. Let $\varepsilon > 0$. By Lemmas 2.6 and 2.7 we have

$$\text{ALG}(G_{\text{simple}}) \geq \sum_{B \in X} \tilde{u}_B \geq 4x^2 - x.$$

OPT on the other hand can just traverse every block in one direction and can transit via backbone edges, moving a distance of x per block (Lemma 2.7) and x per backbone edge. Thus

$$\text{OPT}(G_{\text{simple}}) = (x+3)x + (x+3)x = 2x^2 + 6x.$$

Therefore, this construction gives a lower bound for the competitive ratio of

$$\frac{\text{ALG}(G_{\text{simple}})}{\text{OPT}(G_{\text{simple}})} \geq \frac{4x^2 - x}{2x^2 + 6x} \xrightarrow{x \to \infty} 2 > 2 - \varepsilon. \quad \square$$

3. Recursive construction: lower bound of 3

We refine the approach of Section 2: The macroscopic cycle structure of G_{simple} remains the same as before, but we introduce a recursive construction inside the blocks, i.e., the blocks now consist of blocks instead of vertices. Whenever

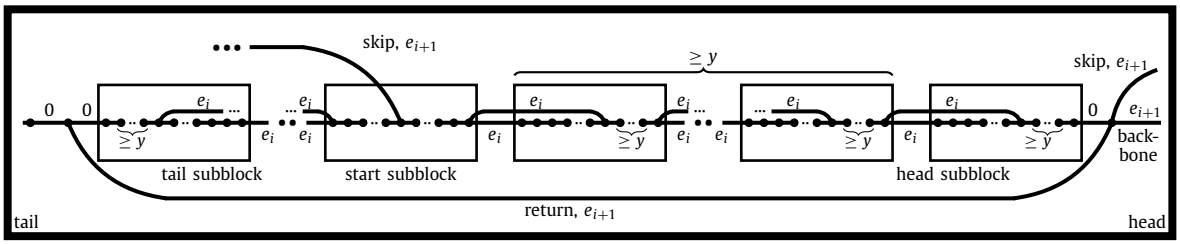

Fig. 5. Example of a block of level $i+1$. Instead of vertices it consists of blocks of level i. The basic design of the blocks from Section 2 remains the same, i.e., there is a skip edge leading to some block in the middle of the path of blocks, called origin subblock. As before, there is a return edge going from the *head vertex* to a vertex at the tail side. Additionally, there are at least y subblocks between the origin subblock and the subblock connected to the head vertex. The dots inside the blocks of level i represent blocks of level $i-1$. The edge weights will be defined later.

we talk about a block A that is inside a block B, we say that A is a subblock of B. Furthermore, we introduce a constant $y \in \{0, \dots, \lfloor x/2 \rfloor\}$ and construct every normal block (except the ones on the highest level) to have at least y subblocks between the origin subblock and last subblock on the head side (see Fig. 5). This constant y ensures that there are at least y subblocks to traverse between two skip edges, forcing the agent to accumulate additional costs in case of backtracking. We will see that the optimum choice of y in the graph constructed in this section is $y = 0$. However, we will include y in our construction since the graph constructed in Section 4 reuses parts of the recursive structure introduced in this section and has $y > 0$.

We call this refined graph G_{rec} since it contains a recursive structure. In this graph, ALG accumulates at least $3-2/(N+2)$ times the cost the optimum accumulates, where N is the number of recursive levels. This improves our lower bound to 3. In particular, we will show the following.

Theorem 3.1. *For every $\varepsilon > 0$ there is a graph G_{rec} such that $\text{ALG}(G_{\text{rec}}) \geq (3-\varepsilon) \cdot \text{OPT}(G_{\text{rec}})$.*

We now formalize the idea above. In general, for $i > 0$, the block of level i contains subblocks of level $i-1$. The edges inside of the block of level i are the backbone/skip edges of subblocks of level $i-1$. Blocks of level 0 contain vertices instead of subblocks and are similarly structured as the blocks of Section 2.

Fig. 5 shows the interaction of three levels. It shows a block of level $i+1$ consisting of subblocks of level i. These subblocks replace the vertices of the blocks from Section 2. Note that the block that is marked as *start subblock* as well as the rightmost and the leftmost block, which are marked as *head subblock* and *tail subblock* in Fig. 5, differ from the other blocks and thus have a special construction. The block that is marked as *start subblock* shares similarity with the origin block of Section 2, i.e., it is the first block of level i in Fig. 5 that is entered by ALG and is surrounded by two unvisited blocks. Fittingly, we will call this type of blocks *origin blocks of level i*. The blocks of level i, which are marked as *head subblock* and *tail subblock* in Fig. 5 are special in that they have no skip edges. We will call this type of blocks *final blocks of level i*. All remaining blocks are called *normal blocks of level i*. We give a detailed description of all blocks in the next subsections. By e_i we denote the weight of a backbone edge of level i and for now postpone the exact definition. The skip edges and return edges of a block of level i also have weight e_i.

3.1. Recursive construction

We introduce a recursive construction of levels in which blocks contain blocks, i.e., blocks of level i contain blocks of level $i-1$. We let level -1 be lowest level that only contains vertices, i.e., a block of level -1 is a vertex.

Definition 3.2 *(Adjacency and incidence).* A vertex is *adjacent* to a block if it is adjacent to a vertex inside of the block. Two blocks are adjacent if a vertex contained in one block is adjacent to a vertex in the other. A block is *incident* to an edge if it contains a vertex that is incident to the edge.

Definition 3.3 *(Traversal). Traversing* a block B means moving from the block on one side of B to the block on the other side of B, while using at least one vertex of B.

We introduce blocks of level $i \in \{0, \dots, N-1\}$ inductively, where N is the highest level. For convenience, we omit the skip edges in figures and label blocks with their type (see Fig. 6). Unlabeled blocks are normal blocks. We will sometimes omit the indices of blocks that denote their level if it is clear from the context. Note that we already omitted the skip edges of the blocks of level $i-1$ in Fig. 5. In the following we introduce normal blocks, origin blocks and final blocks of levels $i \in \{0, \dots, N\}$. Level -1 serves as induction basis in which all three kinds of blocks are just single vertices.

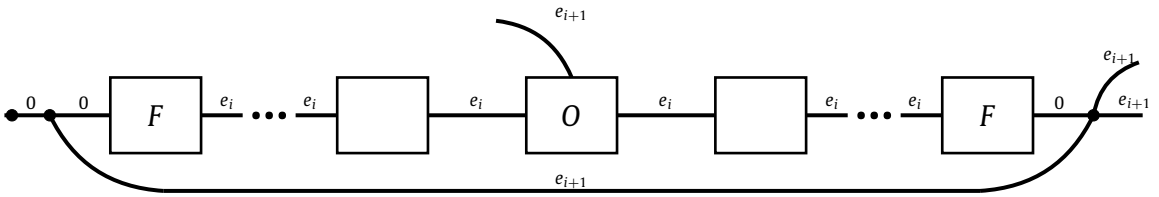

Fig. 6. A series of blocks of level i that form a block of level $i+1$ in simplified notation.

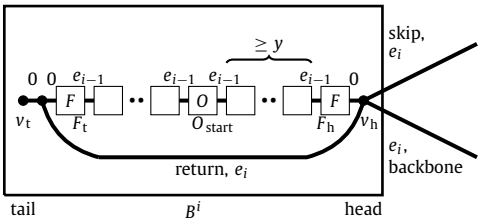

Fig. 7. An illustration of a normal block of type B^i using the simplified notation of Fig. 6.

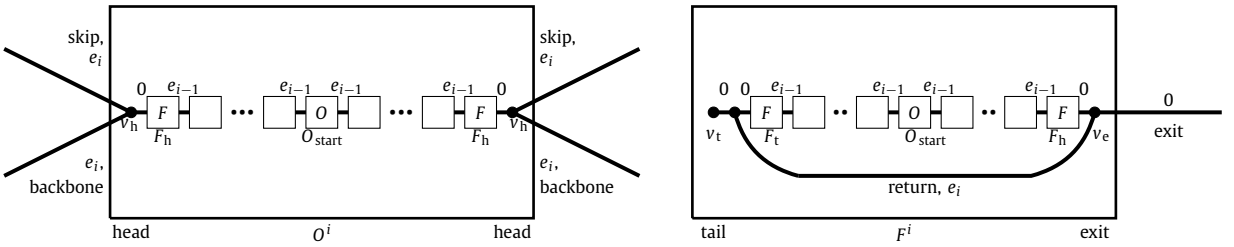

Fig. 8. An illustration of an origin block of type O^i (left) and a final block of type F^i (right).

Normal block gadget of level $0 \leq i < N$. A normal block of level i (see Fig. 7), also called *block of type* B^i, is defined as follows. Assume ALG starts in some subblock O_{start} that is an origin block of type O^{i-1} (defined below). Once ALG leaves O_{start} on one side, we let it enter a subblock that is a normal block of type B^{i-1} via a skip edge.

If ALG chooses to backtrack and to continue on the other side of O_{start}, we add normal blocks of type B^{i-1} to the other side of O_{start} in the same way. The first time ALG takes an edge of weight e_{i-1} at the head vertex of a subblock of type B^{i-1}, it takes the return edge; the second time it takes a yet unused edge of weight e_{i-1} in the same head vertex, it takes a skip edge, which leads to the next unvisited subblock of type B^{i-1}. Every new subblock examined by ALG is chosen to be a normal block of type B^{i-1} until ALG has examined $x+1$ subblocks (including O_{start}). Then, the unexplored subblock adjacent to the last examined subblock is chosen to be a final block of type F^{i-1} (defined below). Likewise, the first unexplored subblock on the other end of the path of subblocks is also chosen to be a final block of type F^{i-1}. In the case that y or more subblocks lie on the path from the last examined subblock to O_{start} (excluding O_{start}), the final block adjacent to the last examined subblock is called *head subblock* F_{h} and the final block on the other end of the path of subblocks is called *tail subblock* F_{t}. Otherwise, the unexplored subblock adjacent to the last examined block is called tail subblock, while the first unexplored subblock on the other end of the path of subblocks is called head subblock. Note that in both cases there are at least y normal blocks of type B^{i-1} on the path from the F_{h} to O_{start} (excluding F_{h} and O_{start}). The head subblock is connected to a vertex via its backbone edge of weight 0. This vertex is called *head vertex* v_h and it is incident to three edges of weight e_i. If ALG is at the head vertex and chooses to take an edge of weight e_i (which we then call *return edge*), it reaches a vertex surrounded by two edges of weight 0. The other two edges, we call *skip edge* and *backbone edge*, depending on the behavior of ALG later on. One of the two edges of weight zero is the backbone edge of the tail subblock F_{t}. The vertex incident to the other edge of weight 0 is called *tail vertex* v_{t}.

Origin block gadget of level $0 \leq i < N$. An *origin block of level* i (see Fig. 8), also called *block of type* O^i, is a path of $x+3$ subblocks of level $i-1$ and two vertices. The two subblocks on the ends of the path of blocks are final blocks of type F^{i-1} and are called *head subblocks* F_{h}. Both head subblocks are connected to a vertex via their backbone edge of weight 0. Those two vertices are called *head vertices* v_{h}. All edges connecting blocks with each other have weight e_{i-1}. All other subblocks contained in the path are of type B^{i-1} except one origin block of type O^{i-1}, which we call *start subblock* O_{start}. The position of O_{start} in the path of blocks of type B^{i-1} can be chosen arbitrarily.

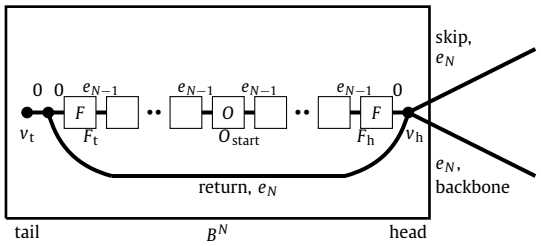

Fig. 9. An illustration of a normal block of type B^N using the simplified notation of Fig. 6.

Final block gadget of level $0 \leq i < N$. A *final block of level* i (see Fig. 8), also called *block of type* F^i, is a block very similar to a block of type B^i, i.e., it contains of $x+3$ blocks. However, in contrast to a block of type B^i, we call the head vertex *exit vertex* v_e and the head side *exit side*. Additionally, the *exit vertex* has no skip edge and the backbone edge incident to the exit vertex has weight 0 and is called *exit edge*.

Lemma 3.4. *We can always connect the head side of a block of level* $0 \leq i < N$ *to the tail side of another block of level* $0 \leq i < N$ *by connecting a backbone edge to a tail vertex and a skip edge to a start subblock. This can be done independently of the types of the two blocks.*

Proof. We notice that every block of level $0 \leq i < N$ has the same number of head sides as backbone and skip edges. Furthermore, every block has the same number of tail sides as start subblocks and tail vertices (except origin blocks, which have no tail side, but a start subblock). We can connect a head side to a tail side by connecting the backbone edge to a tail vertex and the skip edge to a start subblock. Note that every start subblock is an origin block and thus either has a start vertex or a start subblock to which we can connect the skip edge. □

3.2. The highest recursive level

It remains to examine the blocks of the highest level. The highest level is different from all other levels, because the blocks form a cycle as in Section 2. For the highest level, we do not require having at least y blocks between the origin subblock and the head subblock inside a normal block of level N.

Normal block gadget of level N. A *normal block of level* N (see Fig. 9), also called *block of type* B^N is defined as follows. Assume ALG starts in some origin block O_{start} of type O^{N-1}. Once ALG leaves O_{start} on one side, we let it enter a subblock of type B^{N-1} via a skip edge. If ALG chooses to backtrack and to continue on the other side of O_{start}, we add normal blocks to the other side of O_{start} in the same way. The first time ALG takes an edge of weight e_{N-1} at the head vertex of a subblock of type B^{N-1}, it takes the return edge; the second time it takes a yet unused edge of weight e_{N-1} in the same head vertex, it takes a skip edge, which leads to the next unvisited subblock of type B^{N-1}. Every new block examined by ALG is chosen to be a normal block of type B^{N-1} until ALG has examined $x+1$ blocks (including the origin block). The $(x+2)$-nd explored block is chosen to be a final block of type F^{N-1} and is called *head subblock* F_{h} and the first unexplored block on the other end of the path of blocks is chosen to be a final subblock of type F^{N-1} as well and is called the *tail subblock* F_{t}. The head subblock is connected to a vertex via its backbone edge of weight 0. This vertex is called the *head vertex* v_h and it is incident to three edges of weight e_N. If ALG chooses to take an edge of weight e_N (which we then call *return edge*), it reaches a vertex surrounded by two edges of weight 0. The other two edges, we call *skip edge* and *backbone edge*, depending on the behavior of ALG later on. One of the two edges of weight zero is the backbone edge of the tail subblock F_{t}. The vertex incident to the other edge of weight 0 is called *tail vertex* v_{t}.

Origin block gadget of level N. An *origin block of the level* N, also called *block of type* O^N is defined as block of type O^i with $i = N$.

Final block gadget of level N. A *final block of the level* N, also called *block of type* F^N is defined as block of type F^i with $i = N$.

Closing block gadget of level N. A *closing block of level* N (see Fig. 10), also called *block of type* C^N, is a path of $x+3$ subblocks of level $N-1$ and four vertices. The two subblocks on the ends of the path of blocks are final blocks of type F^{N-1} and one is called *tail subblock* F_{t} and the other one is called *pseudo head subblock* F_{ph}. Both final blocks are connected to a vertex via their backbone edge of weight 0. The vertex connected to F_{ph} is called *pseudo start vertex* v_{ps} and it is connected to the vertex adjacent to F_{t} via a *return edge* of weight e_N. Furthermore, v_{ps} is connected to another vertex via an edge of weight 0 and the vertex adjacent to F_{t} is connected to yet another vertex via an edge of weight 0. Those two vertices are both called *tail vertex* v_{t}. All edges connecting blocks with each other have weight e_{N-1}. All subblocks besides F_{t} and F_{ph} contained in

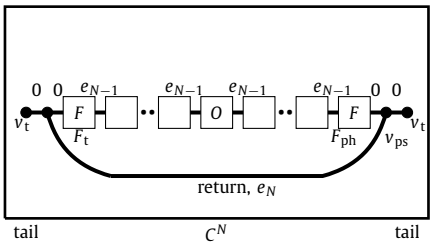

Fig. 10. An illustration of a closing block gadget of type C^N.

the path are of type B^{N-1} except one origin block of type O^{N-1}, which we call *start subblock* O_{start}. The position of O_{start} in the path of blocks of type B^{i-1} is dependent on the behavior of ALG.

Lemma 3.5. *We can always connect the head side of a block of level N to the tail side of another block of level N by connecting the backbone edge to the tail vertex and the skip edge to the (pseudo) start subblock/vertex. This can be done independently of the types of the two blocks.*

Proof. Every block has the same number of head sides as backbone and skip edges and the same number of tail sides as start subblocks/(pseudo) start vertices and tail vertices. To complete the proof, we need to check if the claimed connections work: Since start subblocks are always origin blocks we can connect the skip edges to their contained start subblocks/start vertices. □

We describe the macroscopic design of the graph G_{rec}, i.e., how the blocks in the highest level N are connected to each other.

Graph G_{rec}. We construct G_{rec} similarly to G_{simple}: ALG starts at the origin inside an origin block of type O^0, which is (through multiple levels) contained in an origin block of type O^N. We let ALG explore the block until it reaches a head vertex leading outside of the origin block. If ALG takes one of the two edges of weight e_N at a head vertex of the origin block, we fix it to be the skip edge. Independently of the head side chosen by ALG, we let ALG enter a normal block of type B^N. If, on the other hand, ALG chooses to backtrack and continue on the other side of the origin, we add normal blocks to the other side of the origin block the same way. Note that we can connect blocks of type B^N to both sides of the origin block, by Lemma 3.5. If ALG leaves a normal block B of type B^N via the skip edge it used to enter B before it has visited the head vertex and reenters B via the backbone edge, we connect one (arbitrarily chosen) end of the explored path in B to a final subblock of type F^{N-1} that is adjacent to the vertex, which is adjacent to the tail vertex of B. Every time ALG is at a head vertex of a normal block and takes one of the two remaining (it already took the return edge; cf. construction of normal of type B^N) edges of weight e_N, we let it be the skip edge. Every new block examined by ALG is chosen to be a normal block of type B^N until ALG has examined $x+2$ blocks (including the origin block). The $(x+3)$-nd block examined by ALG is chosen to be a closing block of type C^N connecting both ends of our path of blocks.

Note that G_{rec} is a planar graph: As with G_{simple}, if we exclude the skip and return edges, G_{rec} is just a cycle. If we embed this cycle in the plane, we can add the skip edges on the outer side of the cycle and the return edges on the inner side of the cycle. As before, the planarity of G_{rec} then follows from the fact that skip edges and return edges can be drawn without crossing.

3.3. Analysis of the recursive costs

Recall the variable $\tilde{u}_B$ of Section 2 for accounting the cost of ALG for a normal block B that is at least once left via its outgoing skip edge. We use the accounting scheme $\tilde{u}_B$ again for graph G_{rec} and show that it is valid in the sense that the cost of one action of ALG is charged to at most one normal block B of the same level.

Lemma 3.6. *Let $i \in \{0, \dots, N\}$ and let X^i be the set of normal blocks of type B^i contained in the graph G_{rec} that are at least once left via their outgoing skip edges. Every action of* ALG *contributes to at most one variable $\tilde{u}_B$ with $B \in X^i$.*

Proof. Let $i \in \{0, \dots, N\}$ and B be a block of type B^i that is at least once left via its outgoing skip edge. We need to show that no action of ALG charged to B according to $\tilde{u}_B$ is also charged to a different block of type B^i.

Since every vertex is part of exactly one block of level i, the cost incurred by walking an edge between to vertices of B is only charged for the block B. In particular, the action of leaving B via an incoming/outgoing skip edge is only charged to B. By Lemmas 3.4 and 3.5, the backbone edge incident to the tail vertex of B is incident to the head vertex of another

block, i.e., in particular it is not incident to the tail vertex of another normal block of level i. Thus, the cost of using the backbone edge is only charged to B. □

Lemma 3.7. *Let X^N be the set of normal blocks of type B^N contained in the graph G_{rec} that are at least once left via their outgoing skip edges. We have $|X^N| = x$ and $\text{ALG}(G_{\text{simple}}) \geq \sum_{B \in X^N} \tilde{u}_B$.*

Proof. The inequality $\text{ALG}(G_{\text{simple}}) \geq \sum_{B \in X^N} \tilde{u}_B$ is a consequence of Lemma 3.6. It remains to show that at G_{rec} contains at least x normal blocks of type B^N that are left at least once via their outgoing skip edges.

By definition of G_{rec}, the first $x+1$ blocks examined after leaving the initial origin block of type O^N are normal blocks of type B^N that are entered via incoming skip edges. We consider two cases depending on whether or not the origin block of type O^N is connected to a closing block of type C^N. In case the origin block is connected to the closing block, one normal block of type B^N is entered via incoming skip edges coming from the origin block. Therefore, x normal blocks of type B^N are entered via incoming skip edges coming from normal blocks of type B^N, i.e., x normal blocks of type B^N are left via outgoing skip edges, which shows the claim for this case.

In case the origin block is not connected to the closing block, two normal blocks of type B^N are entered via incoming skip edges coming from the origin block. Therefore, $x-1$ normal blocks of type B^N are entered via incoming skip edges coming from normal blocks of type B^N, i.e., $x-1$ normal blocks of type B^N are left via outgoing skip edges. Since the closing block is not connected to the origin block, it also is entered by an incoming skip edge, i.e., by an outgoing skip edge of a normal block of type B^N. Thus, in total x normal blocks of type B^N are left via outgoing skip edges, which shows the claim. □

We define

$$t_i \coloneqq \text{ cost of traversing an explored block of type } B^i \text{ optimally}$$

and set the weight of edges of level -1 to be $e_{-1} \coloneqq 1$. Consequently, we have and $t_{-1} = 0$ since blocks of level -1 are vertices that incur no cost for exploring/traversing. Furthermore, we assume that ALG can explore/traverse blocks that are not of type B^i for cost 0.

Note that the cheapest way to traverse an explored block of type B^i is to enter and leave via skip edges (recall that one skip edge is connected to the start subblock and one to the head vertex). This way the subblocks between the tail vertex and the start subblock cam be skipped entirely. Thus, only the subblocks between the origin subblock and the head vertex need to be traversed for a traversal. By construction, the number of normal subblocks between the origin subblock and the head vertex is at least y. Therefore, the cost of traversing an explored block of type B^i is bounded by $y(t_{i-1} + e_{i-1})$, i.e.,

$$t_i = y(t_{i-1} + e_{i-1}). \tag{1}$$

Furthermore, we define the edge weights

$$e_i \coloneqq x(t_{i-1} + e_{i-1}). \tag{2}$$

Note that the weight e_i is smaller than or equal to the cost of moving from the head side to the tail side of an already explored block B^i. For the optimum cost, we introduce the following notation (cf. Definition 2.3):

$$o_i \coloneqq \text{ OPT's cost of exploring an unvisited block of level } i \text{ and entering a new unvisited block}$$

For convenience we set $o_{-1} \coloneqq 1$ since blocks of level -1 are vertices that incur no cost for exploring, but incur a cost of $e_{-1} = 1$ for moving to the next unvisited vertex.

Note that o_i is well-defined since the different kinds of blocks can all be explored optimally by omitting all skip and return edges and just moving from one side to another.

Lemma 3.8. *Let B be any block of level $0 \leq i < N$ that* OPT *enters B via a backbone edge or exit edge. Exploring the whole block and reaching the vertex adjacent to the backbone/exit edge on the other side incurs costs of*

$$o_i = (x+3)o_{i-1} - e_{i-1} + e_i.$$

Proof. OPT has to explore $(x+3)$ subblocks and walk $(x+2)$ edges of weight e_{i-1} inside B. This incurs costs of $(x+3)o_{i-1} - e_{i-1}$. Since the agent still has to enter a new block via backbone edge, an additional cost of e_i is added. □

Lemma 3.9. *We have $\text{OPT}(G_{\text{rec}}) = (x+3)o_N$.*

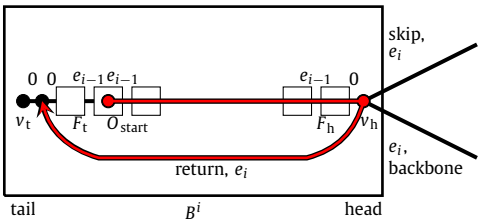

Fig. 11. ALG moves from the origin block directly to the head subblock (i.e., ALG does not backtrack) and then takes the return edge. The last vertex of the block that is explored either is one of the last two vertices on the tail side or a vertex within the tail subblock.

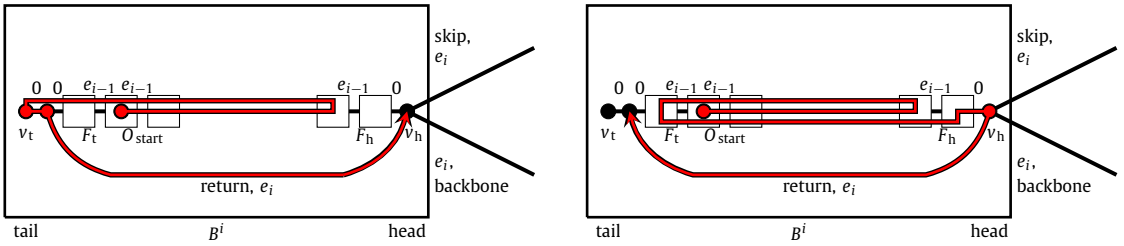

Fig. 12. ALG moves from the origin block $x+1$ blocks in one direction and then backtracks, i.e., the $(x+2)$-th subblock that is being explored is the tail subblock. Left: ALG explores the tail of the block before the head. Right: ALG backtracks again and explores the head before the tail.

Proof. Let OPT choose an arbitrary direction at the start vertex inside the origin block and from there follow its chosen direction always taking the backbone edges between blocks until it again reaches the start vertex. This way, every block is traversed from one side to the other except the origin block, for which one half is traversed at the beginning of the tour and one half at the end of the tour. Thus, we can apply Lemma 3.8 for all $x+3$ blocks in the graph G_{rec}, which shows the claim. □

It remains to bound the costs for ALG in a normal block of type B^i and eventually in G_{rec}. For this, we need to analyze the recursive formula for the costs of ALG. Recall that for simplicity we assume that ALG is able to explore every block except normal blocks of type B^i for costs of 0. Thus, we only examine the exploration of normal blocks in detail.

Lemma 3.10. *Let $0 \leq i < N$ and B be a normal block of type B^i that* ALG *entered via a skip edge, i.e., the vertex of B that is first visited by* ALG *is inside of the start subblock. Furthermore, let X_B^{i-1} be the set of normal subblocks of type B^{i-1} inside B that are at least once left via their outgoing skip edge. Assume* ALG *does not leave B before it is fully explored.*

1. *If the $(x+2)$-nd examined subblock in B is the head subblock, then*
   - ALG*'s cost inside B until B is fully explored is at least $\sum_{b \in X_B^{i-1}} \tilde{u}_b + e_i$ with $|X_B^{i-1}| = x$ and*
   - ALG*'s last explored vertex is either one of the last two vertices on the tail side or inside of the tail subblock.*
2. *If the $(x+2)$-nd examined subblock in B is the tail subblock, then either*
   - ALG*'s cost inside B until B is fully explored is at least $\sum_{b \in X_B^{i-1}} \tilde{u}_b + (2 - y/x)e_i$ with $|X_B^{i-1}| = x$ and*
   - ALG*'s last explored vertex of B is the head vertex or inside of the head subblock*

   or
   - ALG*'s cost inside B until B is fully explored is at least $\sum_{b \in X_B^{i-1}} \tilde{u}_b + (3 - y/x)e_i$ with $|X_B^{i-1}| = x$ and*
   - ALG*'s last explored vertex is either one of the last two vertices on the tail side or inside of the tail subblock.*

Proof. First we notice that, independently of which subblock the $(x+2)$-nd examined subblock is, ALG has to leave at least x normal subblocks are left via their outgoing skip edge, i.e., $|X_B^{i-1}| = x$.

Assume the $(x+2)$-nd subblock examined by ALG is the head subblock F_h. See Fig. 11 for an illustration of this case. Then there are at least y subblocks between F_h and O_{start}. The cheapest way for ALG to be in this scenario is by moving straight in one direction and never backtrack. This incurs costs of at least $\sum_{b \in X_B^{i-1}} \tilde{u}_b$. It remains to explore the last two vertices on the tail side, the tail subblock F_t, the head vertex and possibly vertices in F_h. The head subblock is adjacent to the head vertex which has three incident edges of weight e_i. If ALG explores the head vertex, it can either backtrack to the tail subblock or take one of the edges of weight e_i, which by construction will be the return edge. Both cases add a cost of at least e_i (note that backtracking includes at least the traversal of x explored normal blocks of type B^{i-1} and x edges of weight e_{i-1}, i.e., has at least cost e_i (cf. Definition 2))). This raises the total cost to at least $\sum_{b \in X_B^{i-1}} \tilde{u}_b + e_i$.

Now assume the $(x+2)$-nd subblock examined by ALG is the tail subblock F_t. See Fig. 12 for an illustration of this case. Then there are at most $y-1$ normal subblocks between F_t and O_{start}. In other words, ALG has explored at least $x-y+1$ normal subblocks on the other side of O_{start}. Thus, ALG has incurred costs of at least $\sum_{b\in X_B^{i-1}} \tilde{u}_b + (x-y+1)(t_{i-1}+e_{i-1})$ until it has examined $x+2$ subblocks. It remains to explore the last two vertices on the tail side, the head subblock and the head vertex.

If ALG explored the last two vertices on the tail side first, it incurs costs of 0 since they are connected via 0-weighted edges. ALG can either backtrack to the head subblock or take the return edge of weight e_i. Both cases add a cost of at least e_i. This raises the total cost to at least

$$\sum_{b\in X_B^{i-1}} \tilde{u}_b + (x-y+1)(t_{i-1}+e_{i-1}) + e_i > \sum_{b\in X_B^{i-1}} \tilde{u}_b + (x-y)(t_{i-1}+e_{i-1}) + e_i$$
$$= \sum_{b\in X_B^{i-1}} \tilde{u}_b + \left(2-\frac{y}{x}\right) e_i$$

and the last explored vertex is the head vertex or inside of the head subblock.

If ALG explores the head vertex and head subblock before exploring both vertices on the tail side and the tail subblock, it has to either backtrack to the head side of the block or take the return edge of weight e_i to explore the head vertex and head subblock. As before, both cases add a cost of at least e_i. Afterwards it has to backtrack yet again to the tail side to explore the remaining vertices there. This adds again a cost of at least e_i, raising the total cost to at least

$$\sum_{b\in X_B^{i-1}} \tilde{u}_b + (x-y+1)(t_{i-1}+e_{i-1}) + e_i > \sum_{b\in X_B^{i-1}} \tilde{u}_b + (x-y)(t_{i-1}+e_{i-1}) + 2e_i$$
$$= \sum_{b\in X_B^{i-1}} \tilde{u}_b + \left(3-\frac{y}{x}\right) e_i. \quad \square$$

Next, we examine ALG's cost for exploring a normal block of level $0 \le i < N$.

Lemma 3.11. *Let $i \in \{0,\dots,N-1\}$. For all normal blocks B that are at least once left via their outgoing skip edge, we have*

$$\tilde{u}_B \ge \sum_{b\in X_B^{i-1}} \tilde{u}_b + \left(3-\frac{y}{x}\right) e_i$$

with X_B^{i-1} with $|X_B^{i-1}| = x$ being the set of normal subblocks of type B^{i-1} inside B that are at least once left via their outgoing skip edge.

Proof. Let B be a normal block of type B^i that is at least once left via its outgoing skip edge. By definition of G_{rec}, the block B is entered via skip edge

Assume ALG leaves B before exploring it and instead goes back through the skip edge it first arrived by. This incurs a cost of e_i. For exploring B, ALG has to reenter B either by using the incoming skip edge or the backbone edge. If ALG reenters B via its incoming skip edge, it is in the same situation as in the beginning of the proof. If ALG reenters B via the backbone edge, it incurs a cost of e_i. Exploring every subblock of B then incurs an additional cost of at least $\sum_{b\in X_B^{i-1}} \tilde{u}_b$. Finally, ALG has to leave B to enter a new unexplored block. This again incurs a cost of at least e_i. Thus, we have $\tilde{u}_B \ge \sum_{b\in X_B^{i-1}} \tilde{u}_b + 3e_i$.

Therefore, it is sufficient to consider the case that ALG stays inside B and explores B. Now let $0 \le i < N$. According to Lemma 3.10 we have to consider two cases. If ALG's last explored vertex is one of the two vertices on the tail side of B or inside of the tail subblock of B, it has costs of at least $\sum_{b\in X_B^{i-1}} \tilde{u}_b + e_i$ for exploring B. In the case that ALG now leaves B via the backbone edge adjacent to the tail vertex of B, it incurs a cost of e_i. Since the block adjacent to the tail side of B is already examined, ALG has to do at least one more transition to another block incurring again a cost of e_i and resulting in $\tilde{u}_B \ge \sum_{b\in X_B^{i-1}} \tilde{u}_b + 3e_i$. Otherwise ALG backtracks to the head side of B for costs of at least e_i and enters the unexamined block adjacent to the head side of B by taking either the skip or the backbone edge adjacent to the head vertex of B for a cost of e_i, also resulting in $\tilde{u}_B \ge \sum_{b\in X_B^{i-1}} \tilde{u}_b + 3e_i$.

If ALG's last explored vertex is the head vertex on the head side of B, it has costs of at least $\sum_{b\in X_B^{i-1}} \tilde{u}_b + (2-y/x)e_i$ for exploring B according to Lemma 3.10. It remains to enter a new unexplored block by taking either the skip or the backbone edge adjacent to the head vertex of B for a cost of e_i, resulting in $\tilde{u}_B \ge \sum_{b\in X_B^{i-1}} \tilde{u}_b + (3-y/x)e_i$. $\square$

Lemma 3.12. *Let B be a normal block of type B^N that ALG entered via a skip edge, i.e., the vertex of B that is first visited by ALG is inside of the start subblock. Furthermore, let X_B^{N-1} be the set of normal subblocks of type B^{N-1} inside B that are at least once left*

via their outgoing skip edge. Assume ALG *does not leave B before it is fully explored. Then, the total cost of exploring B is at least $\sum_{b\in X_B^{N-1}} \tilde{u}_b + (x - y_N - 1)(t_{N-1} + e_{N-1}) + e_N$, where $y_N \in \{0, \dots, x\}$ is the number of normal blocks between the origin subblock and the head subblock of B and $|X_B^{N-1}|$. The last explored vertex is either one of the last two vertices on the tail side or inside of the tail subblock.*

Proof. First we notice that ALG has to leave at least x normal subblocks are left via their outgoing skip edge, i.e., $|X_B^{i-1}| = x$ until the $(x+2)$-nd subblock is examined.

By definition of a normal block of type B^N the $(x+2)$-nd examined subblock is always the head subblock. Thus, the proof is the same as for Lemma 3.10 Case 1 except one difference: If there are $y_N \in \{0, \dots, x\}$ normal subblocks between the origin subblock and the head subblock of B, the agent has examined $x - y_N$ normal subblocks on one side of the origin subblock, then has backtracked to the origin subblock and has examined y_N normal subblocks on the other side before visiting the head subblock. This backtracking to the origin subblock adds additional costs of $(x - y_N)(t_{N-1} + e_{N-1})$. □

Next, we examine ALG's cost for exploring a normal block of level N.

Lemma 3.13. *Let B be a normal block of type B^N that is at least once left via its outgoing skip edge and let X_B^{N-1} be the set of normal subblocks of type B^{N-1} inside B that are at least once left via their outgoing skip edge. We have*

$$\tilde{u}_B \geq \sum_{b\in X_B^{N-1}} \tilde{u}_b + (x - y_N^B)(t_{N-1} + e_{N-1}) + 3e_N,$$

where $y_N^B \in \{0, \dots, x\}$ is the number of normal blocks between the origin subblock and the head subblock of B and with $|X_B^{N-1}| = x$.

Proof. By definition of G_{rec}, the block B is entered via its incoming skip edge.

Assume ALG leaves B before exploring it and instead goes back through the skip edge it first arrived by. This incurs a cost of e_N. For exploring B, ALG has to reenter B either by using the skip edge or the backbone edge. If ALG reenters B via its incoming skip edge, it is in the same situation as in the beginning of the proof. If ALG reenters B via the backbone edge, it incurs a cost of e_N. Exploring every subblock of B then incurs an additional cost of at least $\sum_{b\in X_B^{N-1}} \tilde{u}_b$. Finally, ALG has to leave B to enter a new unexplored block. This again incurs a cost of at least e_N. Thus, in total, ALG has a cost of at least $\sum_{b\in X_B^{N-1}} \tilde{u}_b + 3e_N$.

Therefore, it is sufficient to consider the case that ALG stays inside B and explores B. According to Lemma 3.12, ALG has costs of at least $\sum_{b\in X_B^{N-1}} \tilde{u}_b + (x - y_N)(t_{N-1} + e_{N-1}) + e_N$ for exploring B and its last explored vertex is one of the two vertices on the tail side of B or inside of the tail subblock of B. Since the block adjacent to the tail side of B is already examined ALG has to do at least one more transition to another block incurring again a cost of e_N and resulting in a total cost of at least $\sum_{b\in X_B^{N-1}} \tilde{u}_b + (x - y_N)(t_{N-1} + e_{N-1}) + 3e_N$. Otherwise ALG backtracks to the head side of B for costs of at least e_N and enters the unexamined block adjacent to the head side of B by taking either the skip or the backbone edge adjacent to the head vertex of B for a cost of e_N, also resulting in a total cost of at least $\sum_{b\in X_B^{N-1}} \tilde{u}_b + (x - y_N)(t_{N-1} + e_{N-1}) + 3e_N$. □

We define

$$u_i^{\text{rec}} := x u_{i-1}^{\text{rec}} + \left(3 - \frac{y}{x}\right) e_i \tag{3}$$

for $i \geq 0$ with $u_{-1}^{\text{rec}} := 1$. Let $0 \leq i \leq N$. For every normal block B of type B^i that is at least once left via its outgoing skip edge, we have

$$\tilde{u}_B \geq u_i^{\text{rec}}, \tag{4}$$

i.e., u_i^{rec} is a lower bound for $\tilde{u}_B$ with B being a normal block of type B^i. We will use u_i^{rec} instead of $\tilde{u}_B$ in the next section since it allows an easier analysis of G_{rec}.

3.4. Analysis of the competitive ratio

It remains to compute the ratio between ALG's costs exploring G_{rec} and the optimum costs. First we examine the degree of the polynomials u_i^{rec}, o_i, t_i and e_i with respect to x.

Lemma 3.14. *Let $i \geq 0$. We have*

$$u_i^{\text{rec}} \in \Theta(x^{i+1}), \qquad o_i \in \Theta(x^{i+1}), \qquad e_i \in \Theta(x^{i+1}), \qquad t_i \in O(x^{i+1}).$$

Proof. We prove the statement inductively. According to equation 3 we have $u_0^{\text{rec}} = 4x - y$ since $e_0 = x$ (see equation 2). Furthermore, we have $o_0 = 2x + 2$ (Lemma 3.8) and $t_0 = y$ (equation 1). Since we have $y \in \{0, \dots, \lfloor x/2 \rfloor\}$, i.e., $y \in O(x)$, the claim is true for $i = 0$. Now assume, the claim is true for $i - 1$. We have

$$e_i \overset{(2)}{=} x(e_{i-1} + t_{i-1}) \in x \cdot \Theta(x^i) + x \cdot O(x^i) = \Theta(x^{i+1})$$

and

$$t_i \overset{(1)}{=} y(e_{i-1} + t_{i-1}) \in y \cdot \Theta(x^i) + y \cdot O(x^i) \subseteq O(x^{i+1}).$$

Thus, the claim is true for e_i and t_i. Furthermore, we have

$$o_i \overset{\text{Lem. 3.8}}{=} (x+3)o_{i-1} - e_{i-1} + e_i \in (x+3)\Theta(x^i) - \Theta(x^i) + \Theta(x^{i+1}) = \Theta(x^{i+1}),$$

and

$$u_i^{\text{rec}} \overset{(3)}{=} x u_{i-1}^{\text{rec}} + \left(3 - \tfrac{y}{x}\right) e_i \in x \cdot \Theta(x^i) + \left(3 - \tfrac{y}{x}\right) \Theta(x^{i+1}) = \Theta(x^{i+1}),$$

i.e., the claim is also true for o_i and u_i^{rec}. □

Lemma 3.15. *Let $i \geq 0$. We have $e_i = (x + y)e_{i-1}$.*

Proof. We have

$$e_i = x(e_{i-1} + t_{i-1}) \qquad \text{and} \qquad t_{i-1} = y(e_{i-2} + t_{i-2}) = \frac{y}{x} e_{i-1}$$

and thus

$$e_i = x\left(e_{i-1} + \frac{y}{x} e_{i-1}\right) = (x + y)e_{i-1}. \quad \square$$

Lemma 3.16. *Let $i \geq 0$ and B be a normal block of type B^i. We have*

$$u_i^{\text{rec}} = x^{i+1} u_{-1}^{\text{rec}} + \sum_{j=0}^{i} x^j \left(3 - \tfrac{y}{x}\right) e_{i-j}.$$

Proof. We show the claim by showing inductively that for every $0 \leq k \leq i$ we have

$$u_i^{\text{rec}} = x^{k+1} u_{i-k-1}^{\text{rec}} + \sum_{j=0}^{k} x^j \left(3 - \tfrac{y}{x}\right) e_{i-j}. \tag{5}$$

For $k = 0$ we have

$$u_i^{\text{rec}} \overset{(3)}{=} x u_{i-1}^{\text{rec}} + \left(3 - \tfrac{y}{x}\right) e_i.$$

Now assume the claim is true for $k - 1$. Then we have

$$\begin{aligned}
u_i^{\text{rec}} &= x^k u_{i-(k-1)-1}^{\text{rec}} + \sum_{j=0}^{k-1} x^j \left(3 - \tfrac{y}{x}\right) e_{i-j} \\
&\overset{(3)}{=} x^k \left(x u_{i-k-1}^{\text{rec}} + \left(3 - \tfrac{y}{x}\right) e_{i-k}\right) + \sum_{j=0}^{k-1} x^j \left(3 - \tfrac{y}{x}\right) e_{i-j} \\
&= x^{k+1} u_{i-k-1}^{\text{rec}} + x^k \left(3 - \tfrac{y}{x}\right) e_{i-k} + \sum_{j=0}^{k-1} x^j \left(3 - \tfrac{y}{x}\right) e_{i-j} \\
&= x^{k+1} u_{i-k-1}^{\text{rec}} + \sum_{j=0}^{k} x^j \left(3 - \tfrac{y}{x}\right) e_{i-j},
\end{aligned}$$

i.e., the equality (5) holds for k. The claim now follows since (5) also holds for $k = i$. □

If we choose $y = 0$ the ratio of ALG's costs of exploring G_{rec} and the optimum cost is largest. This is the case since ALG is only forced to do a moderate amount of backtracking in G_{rec}. In Section 4 we will present a graph, where the best choice is $y > 0$. For now we assume $y = 0$.

Lemma 3.17. *Let $i \geq 0$ and $y = 0$. We have*

$$u_i^{\text{rec}} = \left(3 - \frac{2}{i+2}\right) o_i - O(x^i).$$

Proof. We have

$$u_0^{\text{rec}} \overset{(3)}{=} 4x = 2(2x+2) - 4 = 2o_0 - O(1).$$

Thus, the claim holds for $i = 0$. Now let $i \geq 1$. We show the claim by showing the equation

$$u_i^{\text{rec}} = x^{i-k}\left(3 - \frac{2}{i+2}\right) o_k + \sum_{j=0}^{i-k-1} x^j \left(3 - \frac{2}{i+2}\right) e_{i-j} - O(x^i) \tag{6}$$

inductively. Let $k = 0$. We have

$$\begin{aligned}
u_i^{\text{rec}} &\overset{\text{Lem. 3.16}}{=} x^{i+1} u_{-1}^{\text{rec}} + \sum_{j=0}^{i} 3x^j e_{i-j} \\
&\overset{u_{-1}^{\text{rec}} = o_{-1}}{=} x^{i+1} o_{-1} + 3x^i e_0 + \sum_{j=0}^{i-1} 3x^j e_{i-j} \\
&\overset{o_0 = xo_{-1} + e_0 + O(1)}{=} x^i o_0 + 2x^i e_0 + \sum_{j=0}^{i-1} 3x^j e_{i-j} - O(x^i) \\
&\overset{e_j = x^j e_0}{=} x^i o_0 + \left(2 + \frac{2i}{i+2}\right) x^i e_0 + \sum_{j=0}^{i-1} \left(3 - \frac{2}{i+2}\right) x^j e_{i-j} - O(x^i) \\
&= x^i o_0 + \left(4 - \frac{4}{i+2}\right) x^i e_0 + \sum_{j=0}^{i-1} \left(3 - \frac{2}{i+2}\right) x^j e_{i-j} - O(x^i) \\
&\overset{o_0 = 2e_0 + O(1)}{=} x^i \left(3 - \frac{2}{i+2}\right) o_0 + \sum_{j=0}^{i-1} \left(3 - \frac{2}{i+2}\right) x^j e_{i-j} - O(x^i)
\end{aligned}$$

as claimed. Now assume equation (6) is satisfied for $k-1$. We have

$$\begin{aligned}
u_i^{\text{rec}} &\overset{(6)}{=} x^{i-k+1}\left(3 - \frac{2}{i+2}\right) o_{k-1} + \sum_{j=0}^{i-k} \left(3 - \frac{2}{i+2}\right) x^j e_{i-j} - O(x^i) \\
&\overset{o_k = xo_{k-1} + e_k + O(x^k)}{=} x^{i-k}\left(3 - \frac{2}{i+2}\right) o_k + \sum_{j=0}^{i-k-1} \left(3 - \frac{2}{i+2}\right) x^j e_{i-j} - O(x^i),
\end{aligned}$$

i.e., equality (6) is satisfied for k. The claim now follows from the fact that equation (6) is satisfied for $k = i$. □

Proof of Theorem 3.1. By Lemma 3.7, we have

$$\text{ALG}(G_{\text{rec}}) \geq x u_N^{\text{rec}} \tag{7}$$

and by Lemma 3.9, we have

$$\text{OPT}(G_{\text{rec}}) = (x+3) o_N = x o_N + O(x^{N+1}). \tag{8}$$

The competitive ratio is

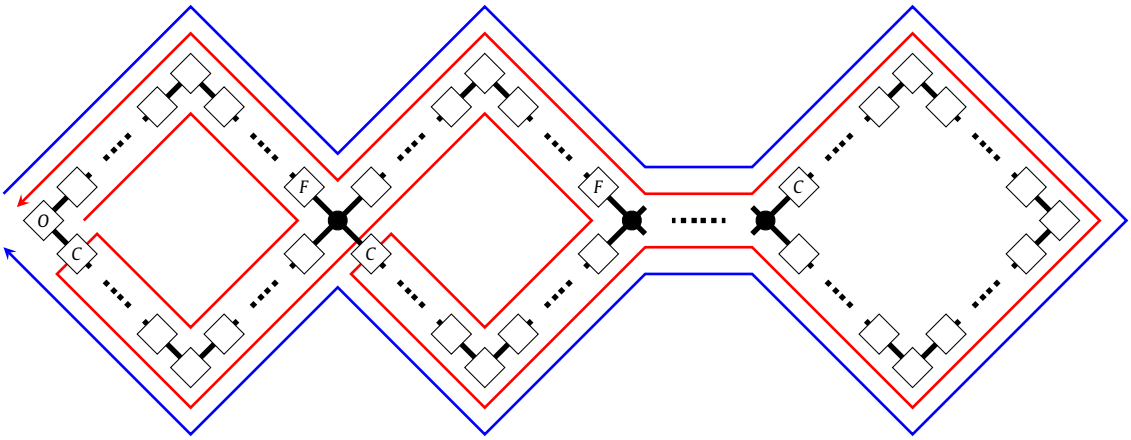

Fig. 13. The structure of the highest level of G_{chain}. Instead of one cycle, we have a series of cycles. In blue the route Opt takes, in red a possible route of Alg.

$$f(x) \coloneqq \frac{\textsc{Alg}(G_{\text{rec}})}{\textsc{Opt}(G_{\text{rec}})} \overset{(7),(8)}{\geq} \frac{xu_N^{\text{rec}}}{xo_N + O(x^{N+1})} \overset{\text{Lem. 3.17}}{\geq} \frac{x\left(3 - \frac{2}{N+2}\right)o_N - O(x^{N+1})}{xo_N + O(x^{N+1})}.$$

Note that we have

$$x\left(3 - \frac{2}{N+2}\right)o_N \in \Theta(x^{N+2}) \quad \text{and} \quad xo_N \in O(x^{N+2}),$$

by Lemma 3.14. Thus letting the number of blocks per block x as well as the number of levels N go to infinity, we get

$$f(x) \geq 3 - \frac{2}{N+2} \overset{N\to\infty}{\longrightarrow} 3 > 3 - \varepsilon. \quad \square$$

4. Further improvements to the lower bound

Even though the construction presented in Section 3 has a quite complex recursive structure, from a macroscopic point of view, the resulting graph is simply a cycle. In this section, we replace the cycle structure by a series of cycles (see Fig. 13). We call the resulting graph G_{chain}. In this graph, Alg is forced to traverse certain blocks twice, while the optimum is still able to traverse the entire graph and return to the origin traversing every block only once. The recursive construction of the blocks itself remains the same. This approach further increases the lower bound on the competitive ratio to 10/3. In particular, we will show the following.

Theorem 4.1. *For every $\varepsilon > 0$ we can construct a graph G_{chain} such that*

$$\textsc{Alg}(G_{\text{chain}}) \geq \left(\frac{10}{3} - \varepsilon\right) \cdot \textsc{Opt}(G_{\text{chain}}).$$

Let $x, N \in \mathbb{N}$ be large and x be an even number. We use the same level-based recursive structure as introduced in Section 3. However, the highest level is not constructed like a cycle, but like a series of connected cycles. The basic idea is to construct G_{chain} in such a way that Alg takes the wrong path at the vertices connecting the cycles. More precisely, Alg first completely traverses a cycle before moving on to the next one, effectively traversing every cycle twice: One time by taking the wrong path, half a time to get to the next cycle and another half a time on the way back to the origin. Every cycle is similarly structured as the macroscopic cycle of Section 3. We describe the graph G_{chain}.

Graph G_{chain}. Let N be the highest level. Alg starts at the origin in an origin block of type O^0, which is (through multiple levels) contained in an origin block of type O^N. Once Alg leaves the origin block, we let Alg enter a normal block of type B^N connected to the origin block via a skip edge as before. Similarly, every time Alg enters a new block it is a normal block. The $(x/2+2)$-nd block Alg examines on one side of the origin block (excluding the origin block) is a final block of type F^N that is connected to a vertex via its exit edge. We call the path of blocks between the origin block and the final block "upper half" and the path of blocks on the other side of the origin block "lower half". The vertex that is adjacent to the final block is called *connection vertex* v_{con} and it is incident to six edges of weight e_N and one edge of weight 0 (three skip edges and three backbone edges leading to three unexplored blocks and the exit edge of the final block of weight 0).

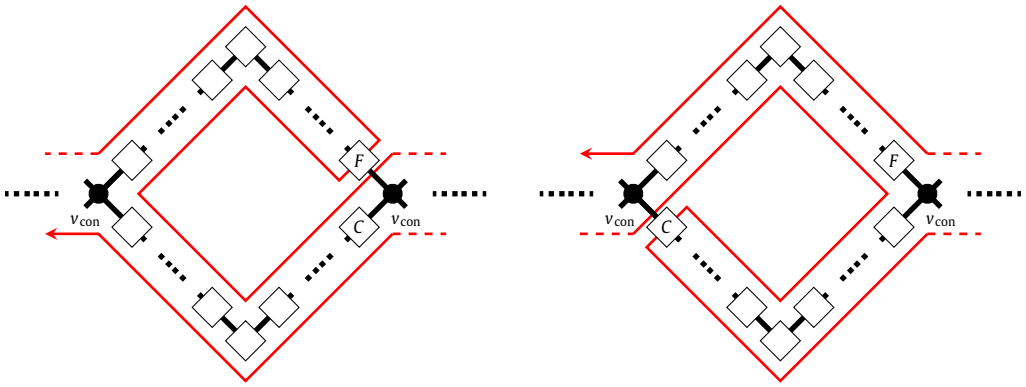

Fig. 14. The path ALG takes if it always decides to backtrack after having traversed a connection block for the first time (left) and the path ALG takes if it does not backtrack, i.e., $z = 0$ (right).

Now ALG has the choice to either take one of the six edges of weight e_N at v_{con} or to backtrack to the origin and explore the blocks on the "lower half" further.

If ALG backtracks to the origin block and explores the lower half, it continues to explore normal blocks until we connect the $(x/2+1)$-st normal block of type B^N on the lower half to a closing block of type C^N. This closing block is connected to v_{con} by connecting one of the six edges of weight e_N incident to v_{con} to the tail vertex of the closing block and another one to the pseudo start vertex of the closing block. See an illustration of this scenario for a cycle that is not the first one on the left side of Fig. 14.

Now, has examined z normal blocks on the lower half before it visits v_{con} the first time from the upper half. If ALG chooses to take one of the six edges of weight e_N at the connection vertex, we let it enter a normal block of type B^N and we continue adding normal blocks of type B^N next to every traversed normal block of type B^N. Note that v_{con} has six incident edges of weight e_N, which can be used as three pairs of skip and backbone edges. Therefore we can add up to three paths of normal blocks of type B^N to v_{con}, if ALG chooses to take another edge of weight e_N. Once ALG has examined $x/2 - z + 1$ blocks of type B^N in one of the three paths, we add a closing block of type C^N. This block will be connected to the last examined block on the lower half or the origin block (in the case $z = 0$). See an illustration of this scenario with $z = 0$ for a cycle that is not the first one on the right side of Fig. 14. Note that ALG now needs to backtrack to v_{con} to get to the next cycle.

This procedure can be repeated for every cycle. Note that apart from the cycle containing the origin, every other cycle is visited by ALG through a connection vertex. In the last cycle, after ALG has explored a total of x normal blocks on the upper and lower half combined (excluding the connection vertex), we connect both halves with a closing block.

As G_{simple} and G_{rec} before, G_{chain} is also a planar graph: The argument is the same as before with G_{rec}, with the slight change that, if we exclude skip and return edges, the resulting graph is a series of cycles instead of just one cycle. We bound the cost of ALG for traversing one cycle which allows us to prove Theorem 4.1.

The structure of the levels lower than N is the same as in Section 3. Consequently, we connect blocks in the same way (cf. Lemmas 3.4 and 3.5) and use the same accounting schemes t_i, e_i, o_i, $\tilde{u}_B$ and u_i^{rec}. In particular, $\tilde{u}_B$ is still a valid accounting scheme for normal blocks of type B^i inside G_{chain} that are at least once left via their outgoing skip edge.

Observation 4.2. Let $i \in \{0, \dots, N\}$ and let X^i be the set of normal blocks of type B^i contained in the graph G_{chain} that are at least once left via their outgoing skip edge. Every action of ALG increases at most one accounting variable $\tilde{u}_B$ with $B \in X^i$.

Lemma 4.3. *Let R be a cycle in G_{chain} and let X_R^N be the set of normal blocks in R of type B^N that are left at least once via their outgoing skip edge. Furthermore, let R not be the last cycle.* ALG *incurs costs of at least*

$$\sum_{b \in X_R^N} \tilde{u}_b + x(t_N + e_N)$$

inside of R during its exploration of G_{chain} and we have $|X_R^N| \geq x$.

Proof. The cycle R consists of $x+4$ blocks ($x+5$ blocks in case of the first cycle since the first contains an additional origin block). If we exclude the closing block of type C^N, the final block of type F^N as well as origin blocks of type O^N in case of the first cycle, exactly $x+2$ normal blocks remain in any cycle. Without loss of generality R is not the first cycle, i.e., it contains two connection vertices instead of one connection vertex and an origin block. We denote by v_1 the

connection vertex that is first reached by ALG and the other by v_2. Note that the proof works similarly for the first cycle by just replacing v_1 with an origin block. It is clear that every normal block of type B^N in R needs to be explored and that at least $x/2$ normal blocks per half are least at least once via their outgoing skip edge. This proves $|X_R^N| \geq x$. Thus, taking both halves of R into account, ALG accumulates costs of at least $\sum_{b \in X_R^N} \tilde{u}_b$. Consider the situation where ALG reaches for v_2 the first time. We call the path of blocks on the side of v_1 that ALG took to move from v_1 to v_2 upper half. Consequently, the blocks of R on the other side of v_1 are called lower half. We distinguish between two cases:

Case 1: ALG *chooses to backtrack from* v_2 *to* v_1*:* In this case, ALG accumulates costs of at least $x/2(t_N + e_N)$ by backtracking from v_2 to v_1 and thus traversing at least $x/2$ already explored normal blocks. Moving back towards v_2 via the upper half only accumulates additional costs. Therefore, it is now sufficient to consider the case that ALG explores the blocks on the lower half and reaches v_2. At this point ALG has traversed the complete cycle and starts to explore new cycles. However, at some point it needs to backtrack to the origin to complete its tour. On its way back, ALG again needs to move from v_2 to v_1 and thus traverse at least $x/2$ already examined normal blocks again, accumulating a cost of $x/2(t_N + e_N)$. So, in total, ALG accumulates costs of at least

$$\sum_{b \in X_R^N} \tilde{u}_b + x(t_N + e_N)$$

as claimed.

Case 2: ALG *takes one of the edges of weight* e_N *at* v_2*:* Note that if ALG takes multiple skip edges at v_2 it eventually leaves R. Let $z < x/2$ be the number of normal blocks to the lower half of v_1 that have been explored by ALG. Since by assumption ALG has reached v_2 via the upper half, ALG has incurred additional costs of $z(t_N + e_N)$ on the lower half for backtracking to v_1. Let B be the last block on the lower half that has been examined by ALG. By construction, after ALG has left v_2, we add normal blocks of type B^N until ALG has explored $x/2 - z + 1$ of them consecutively after leaving v_2 and then add a closing block of type C^N that is connected to B. Observe that ALG has accumulated costs of at least

$$\sum_{b \in X_R^N} \tilde{u}_b + z(t_N + e_N)$$

until it enters the closing block. At this point, ALG needs to move back to v_2 to find unexplored cycles. This costs again at least $(\frac{1}{2}x - z)(t_N + e_N)$ since at least $\frac{1}{2}x - z$ already explored normal blocks need to be traversed. Finally, ALG has to backtrack to the origin at some point, which means it needs to move from v_2 to v_1 again and thus traverse at least $x/2$ already examined normal blocks again. This incurs costs of at least $\frac{1}{2}x(t_N + e_N)$. So, in total, we have

$$\sum_{b \in X_R^N} \tilde{u}_b + z(t_N + e_N) + (\tfrac{1}{2}x - z)(t_N + e_N) + \tfrac{1}{2}x(t_N + e_N) = \sum_{b \in X_R^N} \tilde{u}_b + x(t_N + e_N),$$

as claimed. □

In the following, we set

$$u_N^{\text{chain}} := x u_{N-1}^{\text{rec}} + 3e_N. \tag{9}$$

For every normal block B of type B^N that is at least once left via its outgoing skip edge we have

$$\tilde{u}_B \geq u_N^{\text{chain}} \tag{10}$$

by Lemma 3.13, i.e., u_N^{chain} is a lower bound for $\tilde{u}_B$.

Lemma 4.4. *Let* $N \geq 0$ *and* B *be a normal block of type* B^N *that is at least once left via its outgoing skip edge. We have*

$$\tilde{u}_B + t_N + e_N \geq u_N^{\text{chain}} + 2e_N.$$

Proof. Let X_B^{N-1} be the set of normal subblocks of type B^{N-1} inside B that are at least once left via their outgoing skip edge. We have $|X_B^{N-1}| = x$ and

$$\begin{aligned}
\tilde{u}_B + t_N + e_N &\overset{\text{Lem. 3.13}}{\geq} \sum_{b \in X_B^{N-1}} \tilde{u}_b + (x - y_N)(t_{N-1} + e_{N-1}) + 4e_N + t_N \\
&\overset{(1)}{\geq} x u_{N-1}^{\text{rec}} + x(t_{N-1} + e_{N-1}) + 4e_N \\
&\overset{(2)}{=} x u_{N-1}^{\text{rec}} + 5e_N \\
&\overset{(9)}{=} u_N^{\text{chain}} + 2e_N. \quad \square
\end{aligned}$$

The best choice for y in the graph G_{chain} is $y = x/2$. This is the case since ALG is forced to do more backtracking as in G_{rec}.

Lemma 4.5. *Let $N \geq 0$ and $y = x/2$. We have*

$$u_N^{\text{chain}} + 2e_N = x^{N+1} u_{-1}^{\text{rec}} + \sum_{j=1}^{N} \tfrac{5}{2} x^j e_{N-j} + 5e_N.$$

Proof. We show the claim by showing inductively that for every $0 \leq k \leq N$ we have

$$u_N^{\text{chain}} + 2e_N = x^{k+1} u_{N-k-1}^{\text{rec}} + \sum_{j=1}^{k} \tfrac{5}{2} x^j e_{N-j} + 5e_N. \tag{11}$$

For $k = 0$ we have

$$u_N^{\text{chain}} + 2e_N \overset{(9)}{=} x u_{N-1}^{\text{rec}} + 5e_N,$$

as claimed. Now assume the claim is true for $k-1$. Then we have

$$\begin{aligned}
u_N^{\text{chain}} + 2e_N &\overset{(11)}{=} x^k u_{N-(k-1)-1}^{\text{rec}} + \sum_{j=1}^{k-1} \tfrac{5}{2} x^j e_{N-j} + 5e_N \\
&\overset{(3)}{=} x^k \left(x u_{N-k-1}^{\text{rec}} + \tfrac{5}{2} e_{N-k} \right) + \sum_{j=1}^{k-1} \tfrac{5}{2} x^j e_{N-j} + 5e_N \\
&= x^{k+1} u_{N-k-1}^{\text{rec}} + \sum_{j=1}^{k} \tfrac{5}{2} x^j e_{N-j} + 5e_N,
\end{aligned}$$

i.e., the equality (11) holds for k. The claim now follows since (11) also holds for $k = N$. $\square$

Lemma 4.6. *Let $N \geq 0$ and $y = x/2$. We have*

$$u_N^{\text{chain}} + 2e_N = \left(\frac{10}{3} - \frac{2}{3N+6} \right) o_N - O(x^N),$$

Proof. We have

$$u_0^{\text{chain}} + 2e_0 \overset{(9)}{=} 6x = 3(2x+2) - 6 = 3o_0 - O(1).$$

Thus, the claim holds for $N = 0$. Now let $N \geq 1$. We show the claim by showing the equality

$$u_N^{\text{chain}} + 2e_N = x^{N-k} \left(\frac{10}{3} - \frac{2}{3N+6} \right) o_k + \sum_{j=0}^{N-k-1} x^j \left(\frac{10}{3} - \frac{2}{3N+6} \right) e_{N-j} - O(x^N) \tag{12}$$

inductively. Let $k = 0$. We have

$$
\begin{aligned}
u_N^{\text{chain}} + 2e_N &\overset{\text{Lem. 4.5}}{=} x^{N+1}u_{-1}^{\text{rec}} + \sum_{j=1}^{N}\frac{5}{2}x^j e_{N-j} + 5e_N \\
&\overset{u_{-1}^{\text{rec}}=o_{-1}}{=} x^{N+1}o_{-1} + \sum_{j=1}^{N}\frac{5}{2}x^j e_{N-j} + 5e_N \\
&\overset{5xe_{j-1}=\frac{5}{2}xe_{j-1}+\frac{5}{3}e_j}{=} x^{N+1}o_{-1} + 5x^N e_0 + \sum_{j=0}^{N-1}\frac{10}{3}x^j e_{N-j} \\
&\overset{o_0=xo_{-1}+e_0+O(1)}{=} x^N o_0 + 4x^N e_0 + \sum_{j=0}^{N-1}\frac{10}{3}x^j e_{N-j} - O(x^N) \\
&= x^N o_0 + x^N\left(4+\frac{2N}{3N+6}\right)e_0 + \sum_{j=0}^{N-1}x^j\left(\frac{10}{3}-\frac{2}{3N+6}\right)e_{N-j} - O(x^N) \\
&= x^N o_0 + x^N\left(\frac{14}{3}-\frac{4}{3N+6}\right)e_0 + \sum_{j=0}^{N-1}x^j\left(\frac{10}{3}-\frac{2}{3N+6}\right)e_{N-j} - O(x^N) \\
&\overset{o_0=2e_0+O(1)}{=} x^N\left(\frac{10}{3}-\frac{2}{3N+6}\right)o_0 + \sum_{j=0}^{N-1}x^j\left(\frac{10}{3}-\frac{2}{3N+6}\right)e_{N-j} - O(x^N)
\end{aligned}
$$

as claimed. Now assume equation (12) is satisfied for $k-1$. We have

$$
\begin{aligned}
u_N^{\text{chain}} + 2e_N &\overset{(12)}{=} x^{N-k+1}\left(\frac{10}{3}-\frac{2}{3N+6}\right)o_{k-1} + \sum_{j=0}^{N-k}x^j\left(\frac{10}{3}-\frac{2}{3N+6}\right)e_{N-j} - O(x^N) \\
&\overset{o_k=xo_{k-1}+e_k+O(x^k)}{=} x^{N-k}\left(\frac{10}{3}-\frac{2}{3N+6}\right)o_k + \sum_{j=0}^{N-k-1}x^j\left(\frac{10}{3}-\frac{2}{3N+6}\right)e_{N-j} - O(x^N)
\end{aligned}
$$

i.e., equation (12) is satisfied for k. The claim now follows from the fact that equation (12) is also satisfied $k = N$. □

Proof of Theorem 4.1. For convenience, we only calculate ALG's costs for the first x cycles of G_{chain}. Note that this suffices since the true costs of cost of ALG can only be higher. According to Lemma 4.3, ALG incurs a cost of

$$\sum_{b\in X_R^N}\tilde{u}_b + x(t_N + e_N) \overset{Lem.4.4}{\geq} x(u_N^{\text{chain}} + 2e_N)$$

per cycle R. Thus, in total we have

$$\text{ALG}(G_{\text{chain}}) \geq x^2(u_N^{\text{chain}} + 2e_N). \tag{13}$$

Graph G_{chain} contains $x+1$ cycles consisting of $x+2$ blocks of level N each (except the first cycle, which consists of $x+3$ blocks). Thus, G_{chain} contains a total of $(x+1)(x+2)+1$ blocks of level N. Thus, OPT explores $(x+1)(x+2)+1$ blocks. Furthermore, there are $(x-1)x+2(x+2)$ backbone edges of weight e_N to traverse ($x+2$ in the first and the last cycle, x in the remaining $x-1$ cycles). Thus, there are $2x-1$ more backbone edges to traverse than blocks to explore. In total, we have

$$\text{OPT}(G_{\text{chain}}) = ((x+1)(x+2)+1)o_N + (2x-1)e_N = x^2 o_N + O(x^{N+2}). \tag{14}$$

The competitive ratio is

$$
\begin{aligned}
g(x) &:= \frac{\text{ALG}(G_{\text{chain}})}{\text{OPT}(G_{\text{chain}})} \\
&\overset{(13),(14)}{\geq} \frac{x^2(u_N^{\text{chain}} + 2e_N)}{x^2 o_N + O(x^{N+2})} \\
&\overset{\text{Lem. 4.6}}{=} \frac{x^2\left(\frac{10}{3}-\frac{2}{3N+6}\right)o_N - O(x^{N+2})}{x^2 o_N + O(x^{N+2})}.
\end{aligned}
$$

Note that

$$x^2\left(\frac{10}{3}-\frac{2}{3N+6}\right)o_N \in O(x^{N+3}) \quad \text{and} \quad x^2 o_N \in O(x^{N+3}).$$

Thus, we get

$$g(x) \geq \frac{10}{3}-\frac{2}{3N+6} \overset{N\to\infty}{\longrightarrow} \frac{10}{3} > \frac{10}{3}-\varepsilon,$$

which completes the proof. □

5. Conclusion

In this final section we summarize the improvements that have been achieved in the sections above. Every block constructed in the prior sections is planar, making the complete graph G planar. Since the competitive ratio for planar graphs is at most 16 [20], the gap of the competitive ratio of online graph exploration for planar graphs has been narrowed.

Corollary 5.1. *There is no algorithm for online graph exploration with competitive ratio smaller than* $10/3$*, even for planar graphs.*

Megow et al. [20] present an online algorithm for online graph exploration that achieves $2k$-competitiveness in the case that the graph G only has k distinct weights. By limiting the number N of recursive levels, our construction yields the following lower bound.

Corollary 5.2. *There is no algorithm for online graph exploration with competitive ratio smaller than* $10/3-2/(3k)$ *for planar graphs with* $k>1$ *distinct weights.*

Proof. If we have a fixed number $N \geq 0$ of levels, the graph contains $N+3$ distinct weights. However, it is possible to replace the edges of weight 0 in G_{chain} with edges of weight 1, reducing the amount of distinct edge weights to $k := N+2$. Let this graph be called G^1_{chain} This modification increases the costs for Opt for traversing a block of level 0 by not more than 4. In general, for every block, not more than 0-weighted edges are replaced with 1-weighted edges. Since every block of level i has at most $x+3$ subblocks of level $i-1$ and we have less than $(x+3)^2$ blocks of level N, we can bound the number of edges of weight 0 in graph G_{chain} by

$$4(x+3)^{N+2} \in O(x^{N+2}).$$

We have

$$h(x) := \frac{\text{Alg}\left(G^1_{\text{chain}}\right)}{\text{Opt}\left(G^1_{\text{chain}}\right)} \geq \frac{\text{Alg}(G_{\text{chain}})}{\text{Opt}(G_{\text{chain}})+O(x^{N+2})}.$$

Asymptotically, we get

$$h(x) \overset{x\to\infty}{\longrightarrow} \frac{10}{3}-\frac{2}{3N+6} = \frac{10}{3}-\frac{2}{3k},$$

which shows the claim. □

Finally, we remark that our construction easily carries over to the case where degrees are bounded.

Corollary 5.3. *There is no algorithm for online graph exploration with competitive ratio smaller than* $10/3-2/(3k)$ *for planar graphs with maximum degree* 3*.*

Proof. We can split the vertices with degree higher than 3 into multiple vertices of degree 3 each, connected by edges of weight 0. Since all new edges can be explored for free by Opt, and since Alg does not gain any additional information when visiting a split vertex, the construction works as before. Furthermore, the construction remains planar. □

Note that we can avoid edges of weight 0 in the construction of Corollary 5.3 by replacing them with ε-weighted edges with sufficiently small ε.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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