Generators for complemented modular lattices and the von-Neumann-Jónsson Coordinatization Theorems

Christian Herrmann

Abstract. Extending work of von Neumann, Jónsson has shown that each complemented modular lattice, L, admitting a large partial n-frame, with $n \geq 4$ or $n \geq 3$ and L Arguesian, can be coordinatized as the lattice of all principal right ideals of some regular ring. His proof built on the embedding of L into the subgroup lattice of an abelian group which follows from Frink's embedding of L into to a direct product of subspace lattices of irreducible projective spaces and coordinatization of the latter. We offer a proof which, in addition to these results, employs only some elementary Linear Algebra. Luca Giudici's thesis [6] is an important source for this approach.

1. Introduction

A ring R (associative with unit) is (von Neumann) *regular*, if for any a there is x such that $axa = a$. Equivalently, each principal right (resp. left) ideal is generated by an idempotent. See [24, 27, 7]. For a right module, M_S, the lattice $\mathsf{L}(M_S)$ of all submodules is modular and even Arguesian [16]. Observe that interval sublattices $[U, V]$ of $\mathsf{L}(M_S)$ correspond to quotient modules V/U. A ring R is regular if and only if the principal right ideals form a complemented sublattice $\mathbb{L}(R)$ of $\mathsf{L}(R_R)$. A lattice L is *coordinatizable* if it is isomorphic to some $\mathbb{L}(R)$.

Von Neumann has shown that any complemented modular lattice admitting an n-frame (i.e. homogeneous basis of order n) with $n \geq 4$ is coordinatizable. His proof as well as those given by Amemiya, Fryer, Halperin, and Maeda (cf [19, 29]) mimick the classical or the group theoretic approach to the coordinatization of irreducible projective spaces. Jónsson [17, 19] generalized the result to large partial n-frames as well as to $n = 3$ under the hypothesis of the Arguesian identity, giving so the still strongest result. The common feature is that the isomorphism is construced from L to $\mathbb{L}(R)$. That no substantial extension is possible might be indicated by the result resp. the methods of Wehrung [30] showing that the class of coordinatizable lattices is not first-order axiomatizable.

The Arguesian identity is valid in all lattices $\mathsf{L}(A_{\mathbb{Z}})$ of subgroups of abelian groups [16] (but not a basis for the equational theory of these - even no finite basis exists [13]). However, any sectionally complemented Arguesian lattice L can be embedded into some $\mathsf{L}(A_{\mathbb{Z}})$ and any sectionally complemented modular lattice admitting a large partial n-frame with $n \geq 4$ is Arguesian [18]. The basis of these

2000 *Mathematics Subject Classification*: Primary: 06C20; Secondary: 16E50, 16W10.
Key words and phrases: complemented modular lattice, regular ring, frame, coordinatization.

results is Frink's [5] embedding of L into the subspace lattice of a projective space and the Coordinatization Theorem of Projective Geometry.

Jónsson used this embedding and methods introduced to Geometric Algebra by Baer and E. Artin, to construct an extension of L with L in analogy to the hyperplane at infinity in a projective space and then mimicked the coordinatization of this hyperplane. This required rather sophisticated considerations but provided also the representation of sectionally complemented Arguesian lattices with large partial n-frames ($n \geq 3$) by means of locally projective modules over regular rings [18].

[14] followed Jónsson considering L a sublattice of some $\mathsf{L}(A_{\mathbb{Z}})$, but dealt with the case of n-frames, only. The coordinatization was in terms of the lattice $\mathbb{L}(S^n_S)$ of finitely generated submodules of S^n_S where S is a regular ring (recall that $\mathbb{L}(S^n_S) \cong \mathbb{L}(R)$ where R is the ring of $n \times n$-matrices over S). The ring S was obtained as a subring of $\mathsf{End}(A_{\mathbb{Z}})$ consisting of the endomorphisms with graph a member of L - a point of view dating back to von Staudt and Remak [26, 25] cf. von Neumann [24, Ch.VI App.] and Hutchinson [15]. Then the isomorphism $\mathbb{L}(S^n_S) \to L$ was defined via cyclic submodules - surjectivity derived from the fact that these form a generating set. A similar approach has been outlined in [13] for primary Arguesian lattices.

The calculations would become rather cumbersome in the case of partial frames. Fortunately, it turns out that, given a skew n-frame, the coordinatizing ring R can be obtained, naturally, as a subring of $\mathsf{End}(A_{\mathbb{Z}})$. This was first observed by Luca Giudici in his thesis [6, Thm.4.2.1] - including the uniqueness results. The basic idea is to replace the abstract lattice computations of von Neumann [24] and Halperin [8, 9, 10] by their Linear Algebra counterparts. Here, the isomorphism is from $\mathbb{L}(R)$ to L associating $\mathsf{im}\,\phi \in \mathsf{L}(A_{\mathbb{Z}})$ with ϕR for $\phi \in R$.

In the present note we give a somehow simplified version of this approach: We introduce R (in sect.7) directly without first defining the 'auxiliary ring', instead we refer to the ringoid of morphisms between the subgroups constituting base elements of the skew frame. We also use (in sect.8) a smaller set of generators for complemented modular lattices, namely the skew frame and the 'coordinate domain' corresponding to the auxiliary ring. The embedding and coordinatization results are presented in sect.9-14 including the case of lattices with involution cf. von Neumann [24, Part II, Thm.4.3].

An important fact also observed by Giudici [6, Thm.4.2.1] is that R is determined by the sublattice L of $\mathsf{L}(A_{\mathbb{Z}})$ without reference to the skew frame. This allows a simple proof of Jónsson's [18] coordinatization of sectionally complemented Arguesian lattices admitting a large partial 3-frame in terms of locally projective modules over regular rings. An outline has been given on p.178 [6] based on [3]. In sect.12 we give a short proof referring to the above coordinatization, directly.

The fact that a complemented modular lattice with an n-frame is generated by the frame and the coordinate domain has been already established by von Neumann [24, Part II, Lemma 4.2]. Also, as the appendices in [24, Part II] show, von Neumann was well aware of the fact that his lattice calculations had their Linear Algebra counterparts. So one may speculate whether, Frink's Theorem being available at the time Continuous Geometry was developped, von Neumann would have bothered with lattice calculations. Of course, to avoid these we have to invoke the Axiom of Choice for the representation in some $\mathsf{L}(A_{\mathbb{Z}})$. According to Halperin [10] a proof in von Neumann's style is also possible for skew frames. We refer to [6] as the most comprehensive source on the history and related areas of coordinatization.

The author is much indebted to Luca Giudici, Fred Wehrung, and the referee for numerous helpful hints and remarks. Many ideas are taken from the work of Florence Micol [22] and Niklas Niemann [23].

2. Orthogonal systems of idempotents

Rings will always be associative and, with the sole exception of Thm.14.1, have a unit 1. Most of the results in sections 2-6 are well known, cf. the quoted books and [29]. Idempotents $e_i\,(0 \leq i < m)$ of a ring R form an *orthogonal system* if $e_ie_j = 0$ for $i \neq j$ and *spanning* if $\sum_{i=0}^{m-1} e_i = 1$. Then each $r \in R$ has unique representation $r = \sum_{i,j<m} r_{ji}$ with $r_{ji} \in e_jRe_i$, namely $r_{ji} = e_jre_i$. Indeed, $r = 1r1 = (\sum_{j<m} e_j)r(\sum_{i<m} e_i)$.

Given $S_{ji} \subseteq e_jRe_i$ for $0 \leq i,j < m$ such that $0,\ e_i \in S_{ii}$ for all i and

$$-r \in S_{ji},\ \ r+s \in S_{ji} \quad \text{for all } r,s \in S_{ji} \text{ where } i,j < m$$

$$sr \in S_{ki} \quad \text{for all } r \in S_{ji},\ s \in S_{kj} \text{ where } i,j,k < m$$

we say that the $S_{ij}\ (i,j < m)$ from a *subringoid* of R *compatible* with the spanning orthogonal system of idempotents. Given a ring R', an spanning orthogonal system $e'_i\ (0 \leq i < m)$ of idempotents, and a compatible subringoid S'_{ji} of R', a *homomorphism* from one subringoid into the other is a homomorphism of multisorted structures, i.e. a family of additive homomorphisms $\iota_{ji}\colon S_{ji} \to S'_{ji}\,(i,j < m)$ such that $\iota_{ki}(sr) = \iota_{kj}(s)\cdot\iota_{ji}(r)$ for all $i,j,k < m$ and all $(r,s) \in S_{ji} \times S_{kj}$.

Proposition 2.1. *Given a subringoid $S_{ji}\,(i,j < m)$ of R compatible with the spanning orthogonal system $e_i\,(0 \leq i < m)$ of idempotents then*

$$S = \{r \in R \mid e_jre_i \in S_{ji} \text{ for all } i,j < m\}$$

is a unital subring of R. Moreover, $e_i \in S$ and any homomorphism $\iota_{ji}\,(i,j < m)$ of subringoids extends to a homomorphism $\iota\colon S \to S'$ of the associated rings

$$\iota r = \sum_{i,j<m} \iota_{ji}(e_jre_i)$$

which is injective resp. surjective if so is $\iota_{ji}\,(i,j < m)$.

Proof. With the above unique representation, one has $r \in S$ if and only if $r_{ji} \in S_{ji}$. Moreover addition, additive inversion, and multiplication of elements in R are carried out in analogy with matrix operations

$$\sum_{ji} r_{ji} + \sum_{ji} s_{ji} = \sum_{ji}(r_{ji}+s_{ji}), \quad -\sum_{ji} r_{ji} = \sum_{ji} -r_{ji}$$

$$(\sum_{lk} s_{lk})(\sum_{ji} r_{ji}) = \sum_{lji} s_{lj}r_{ji}$$

since $s_{lk}r_{ji} = s_{lk}e_ke_jr_{ji} = 0$ for $j \neq k$. Thus, the hypotheses on the S_{ji} guarantee that S is a subring. Now, the same applies to R' and S'. Since $\iota(\sum_{ij} r_{ij}) = \sum_{ij} \iota_{ji}(r_{ji})$ it follows that ι is a homomorphism. □

3. Systems of generalized matrix units

A family e_{ji} $(0 \leq i, j < m)$ of elements in a ring is an *n-system of generalized matrix units* where $n \leq m$ if $e_i = e_{ii}$ $(i < m)$ is a spanning orthogonal system of idempotents, $e_{ji} \in e_jRe_i$, and

$$e_{kj}e_{ki} = e_{ji} \text{ for all } i, j < m \text{ and } k < n.$$

Given the e_{ji} with $i = j$ or $0 \in \{i, j\}$, only, satisfying the pertinent relations, one obtains an n-system of generalized matrix units defining the additional elements as $e_{ji} = e_{j0}e_{0i}$.

Lemma 3.1. *In any ring with an n-system $(n \geq 2)$ of generalized matrix units, the $e_i - r$ with $r \in e_jRe_i$ and $i \neq j$ are idempotent and form together with the e_i, e_{0i}, and e_{i0} a generating set.*

Proof. The subring S generated by these elements contains all e_jRe_i where $i \neq j$. Since $e_iRe_i = e_{i0}e_{0i}Re_i$ it follows $S = R$. For $r \in e_jRe_i$ with $i \neq j$ one computes $(e_i - r)^2 = e_i^2 - e_ie_jr - re_i + re_ie_jr = e_i - r$. □

Lemma 3.2. *If the e_{ij} $(i, j < m)$ form a generalized system of matrix units in a ring R and if the ring e_0Re_0 (with addition and multiplication inherited from R) is regular then R is regular.*

Proof. According to [7, Lemma 1.6] it suffices to show that for each $x \in e_iRe_j$ there is y in e_jRe_i with $xyx = x$. Now, consider $a = e_{0i}xe_{j0}$ and observe that

$$e_{i0}ae_{0j} = e_{i0}e_{0i}xe_{j0}e_{0j} = e_ixe_j = x$$

$$ae_{0j}e_{j0} = e_{0i}xe_{j0}e_{0j}e_{j0} = e_{0i}xe_je_{j0} = e_{0i}xe_{j0} = a$$

and, similarly, $e_{0i}e_{i0}a = a$. Since e_0Re_0 is regular and $a \in e_0Re_0$ there is $s \in e_0Re_0$ such that $asa = a$. With $y = e_{j0}se_{0i} \in e_jRe_i$ it follows

$$xyx = e_{i0}ae_{0j}e_{j0}se_{0i}e_{i0}ae_{0j} = e_{i0}asae_{0j} = e_{i0}ae_{0j} = x.$$

□

4. Graphs of morphisms

In abstract lattices, join is denoted by $a + b$, meet by ab. All lattices will have smallest element 0 and greatest element 1, considered as constants. We write $a + b = a \oplus b$ if $ab = 0$. Elements $a_0, \ldots, a_{m-1}$ in a lattice L with zero are *independent* if

$$(\sum_{i \in I} a_i)(\sum_{j \in J} a_j) = \sum_{k \in I \cap J} a_k \quad \text{for all } I, J \subseteq \{1, \ldots, m\}$$

- in case L is modular, it suffices to require $a_{k+1}\sum_{i<k} a_i = 0$ for all $k < m$. Submodules $A_0, \dots, A_{m-1}$ in $\mathsf{L}(M_S)$ are independent if and only if the sum $\sum_{i<m} A_i$ is direct.

For S-modules A, B we write $\phi\colon A \to B$ if ϕ is a homomorphism of A to B (especially, we do not assume ϕ to be one-to-one) and we consider this also as a homomorphism $\phi\colon A \to C$ for any $C \supseteq B$. We write $\phi\colon\colon A \to B$ if the domain $\mathsf{dom}\,\phi$ of definition of ϕ is a submodule of A and $\phi\colon \mathsf{dom}\,\phi \to B$ is a homomorphism. The following three results require just some elementary Linear Algebra.

Proposition 4.1. *Suppose $A \cap B = 0$ in $\mathsf{L}(M_S)$. Then there is a bijective correspondence $\phi \leftrightarrow \Gamma(\phi)$ between the $\phi\colon\colon A \to B$ and the $C \in \mathsf{L}(M_S)$ such that $C \subseteq A + B$ and $C \cap B = 0$. It is given by*

$$C = \{x - \phi x \mid x \in \mathsf{dom}\,\phi\}, \quad \mathsf{dom}\,\phi = (C + B) \cap A,\ \phi x = y \Leftrightarrow x - y \in C$$

The case $\phi\colon A \to B$ is characterized by $\Gamma(\phi) + B = A + B$.

$\Gamma(\phi)$ is called the *graph* of ϕ (anti-graph in [6]). The fact that abelian groups may be equationally described in terms of subtraction, but not in terms of additon, may be seen as an explanation why not to adhere to the usual definition of graph.

Proposition 4.2. *Suppose $A \cap B = 0$ in $\mathsf{L}(M_S)$ and $\phi\colon\colon A \to B$. Then ϕ has image $\mathsf{im}\,\phi = B \cap (A + \Gamma(\phi))$ and kernel $\mathsf{ker}\,\phi = A \cap \Gamma(\phi)$. Injectivity of ϕ means $A \cap \Gamma(\phi) = 0$ and then $\Gamma(\phi) = \Gamma(\phi^{-1})$ for the inverse $\phi^{-1}\colon \mathsf{im}\,\phi \to A$. For the restriction to $D \subseteq A$ one has $\Gamma(\phi|D) = (D+B) \cap \Gamma(\phi)$. If $A = Z \oplus \mathsf{dom}\,\phi$ then there is unique ψ such that $\psi|\mathsf{dom}\,\phi = \phi$ and $\psi|Z = 0$ and it follows $\Gamma(\psi) = Z + \Gamma(\phi)$.*

Lemma 4.3. *If A_0, A_1, A_2 are independent in $\mathsf{L}(M_S)$ then*

$$\Gamma(\psi \circ \phi) = (A_0 + A_2) \cap (\Gamma(\psi) + \Gamma(\phi)) \quad \textit{for } \phi\colon\colon A_0 \to A_1,\ \psi\colon\colon A_1 \to A_2.$$

5. Direct decompositions

A *basis* of a modular lattice L with 0 and 1 consists of independent $a_0, \dots, a_{m-1}$ such that $1 = \sum_{i=0}^{m-1} a_i$. For $L = \mathsf{L}(M_S)$ and $A_i = a_i$ this means $M_S = \bigoplus_{i=0}^{m-1} A_i$ and a 1-1-correspondence with spanning orthogonal systems π_i of idempotents in the endomorphism ring $\mathsf{End}(M_S)$, namely $A_i = \mathsf{im}\,\pi_i$ and $\sum_{j \neq i} A_j = \mathsf{ker}\,\pi_i$. Let $\varepsilon_i\colon A_i \to M$ denote the canonical embedding.

Each $\psi\colon A_i \to M$ lifts uniquely to $\hat{\psi}\colon M \to M$ such that $\hat{\psi}|A_j = 0$ for $j \neq i$; namely, $\hat{\psi} = \psi \circ \varepsilon_i$. Also, $\psi\colon A_i \to A_j$ if and only if $\psi = \pi_j \phi \varepsilon_i$ for some $\phi \in \mathsf{End}(M_S)$. Here, ϕ may be choosen as $\hat{\psi}$. Observe that $\pi_i = \hat{\varepsilon}_i$ and that for $\phi_{ji}\colon A_i \to A_j$ and $\phi\colon M \to M$

$$\phi = \sum_{ji} \widehat{\phi_{ji}} \ \Leftrightarrow\ \phi_{ji} = \pi_j \phi \varepsilon_i \ \text{ for all } j, i < m.$$

By a *subringoid compatible* R_{ji} with the direct decomposition $M_S = \bigoplus_{i<m} A_i$ we understand one of $\mathsf{End}(M_S)$ w.r.t. the canonical projections π_i. We prefer to

think of R_{ji} as a subset of $\mathsf{Hom}_S(A_i, A_j)$ replacing $\pi_j\phi\pi_i$ by $\pi_j\phi\varepsilon_i$. In particular, $\mathsf{id}_{A_i} \in R_{ii}$ and $0_{ji} \in R_{ji}$. Then the associated ring R is

$$\{\phi \in \mathsf{End}(M_S) \mid \pi_j\phi\varepsilon_i \in R_{ji} \text{ for all } i, j < m\} = \{\sum_{i,j<m} \widehat{\phi_{ji}} \mid \phi_{ji} \in R_{ji}\},$$

$\varepsilon_i \in R_{ii}$, and a homomorphism of subringoids extends as follows

$$\iota(\sum_{i,j<m} \widehat{\phi_{ji}}) = \sum_{i,j<m} \widehat{\iota_{ji}(\phi_{ji})}.$$

6. Frames

In a modular lattice, b is *perspective to* a (with *axis* c), in signs $b \sim a$ ($b \sim_c a$) if $b \oplus c = a \oplus c$. If $b \oplus c \leq a \oplus c$ then b is *subperspective* to a with *axis* c, in signs $b \lesssim_c a$. In this case, by modularity, $c(a+b)$ is an axis, too, $b \sim_c a(b+c)$, and $d \lesssim_c a$ for any $d \leq b$.

A *skew* n*-frame* Φ of a lattice L is given by a basis $a_i = a_{ii}$ $(0 \leq i < m)$ where $m \geq n$ and axes $a_{0i} \leq a_0 + a_i$ of subperspectivities from a_i to a_0 for $0 < i < m$. Moreover, for $0 < i < n$ these are required to be perspectivities, i.e. $a_i \oplus a_{0i} = a_0 \oplus a_i$ for $0 < i < n$. For $0 < l < n$ and $l \neq i$ it follows from modularity that a_i is subperspective to a_l with axis $a_{li} = (a_i + a_l)(a_{0i} + a_{0l})$. We say that Φ is an n*-frame* if $m = n$.

For a sublattice L of $\mathsf{L}(M_S)$ and $A_i = a_i$, by Prop.4.1 the $A_{0i} = a_{0i}$ correspond to injective $\eta_{0i}\colon A_i \to A_0$ with inverse $\eta_{i0}\colon \mathsf{im}\,\eta_{0i} \to A_i$, namely $\Gamma(\eta_{0i}) = \Gamma(\eta_{i0}) = A_{0i}$. Moreover, $\eta_{0i}\colon A_i \to A_0$ is an isomorphism for $i < n$. Define for $i, j > 0$

$$\eta_{ji} = \eta_{j0}\eta_{0i} :: A_i \to A_j, \ \ \mathsf{dom}\,\eta_{ji} = \eta_{0i}^{-1}(\mathsf{im}\,\eta_{0j}).$$

It follows that

$$\eta_{ii} = \mathsf{id}_{A_i} \ \text{ and } \ \eta_{jk}\eta_{ki} = \eta_{ji} \text{ for } k < n.$$

If L is complemented, one may lift η_{ji} to $\gamma_{ji}\colon A_i \to A_j$ such that $\Gamma(\gamma_{ji}) \in L$ (use Prop.4.2). In particular, $\gamma_{ij}\gamma_{ji} = \gamma_{ii}$ for $j < n$.

Corollary 6.1. *Any n-system π_{ji} of generalized matrix units for $\mathsf{End}(M_S)$ defines a skew n-frame of $\mathsf{L}(M_S)$ of L, namely $a_i = \mathsf{im}\,\pi_i, ; a_{0i} = \mathsf{im}(\pi_i - \pi_{0i})$. Any skew n-frame of a complemented sublattice L of $\mathsf{L}(M_S)$ may be obtained in this way.*

Proof. The first claim is again elementary Linear Algebra. Lift γ_{ji} to $\pi_{ji} = \hat{\gamma}_{ji}$ in the second. □

Jónsson [17] defined a *large partial n-frame* in a complemented modular lattice L as an n-frame in an interval $[0, u]$ together with a join v of elements subperspective to a_0 such that $u \oplus v = 1$. In view of [17, Lemma 1.5], the condition on v means that L is the neutral ideal generated by $[0, u]$. Surprisingly, Wehrung [30, Prop.A.1] characterized coordinatizable lattices admitting a large partial 3-frame by a first-order sentence.

Proposition 6.2. *A complemented modular lattice admits a large partial n-frame if and only if it admits a skew n-frame.*

Proof. Of course, any skew n-frame gives rise to a large partial n-frame. To show the converse, inductively, one has to consider a skew n-frame in some $[0, w]$ and $b \lesssim_c a_0$. To obtain a skew n-frame of $[0, w+b]$ add d as a new basis element where $b = wb \oplus d$ and add $c(a_0 + d)$ as axis of subperspectivity. $\square$

A modular lattice L is *sectionally complemented*, if it has 0 and each interval $[0, u]$ is complemented. A *large partial n-frame* in L is an n-frame in an interval $[0, u]$ such that each $x \in L$ is a join of elements perspective to parts of a_0. By the above observation on neutral ideals this coincides with the orginal concept in the complemented case. Any simple L of height $\geq n$ admits such frame (cf. [18, Cor.8.4]).

7. Endomorphism ring of a frame

Consider a skew n-frame $\Phi\colon A_i, A_{0i}$ in $\mathsf{L}(M_S)$ with associated η_{ji} and $\phi :: A_i \to A_j$. If $\mathsf{dom}\,\phi = 0$, then put $\Gamma_\Phi(\phi) = 0$; otherwise, define $\Gamma_\Phi(\phi) = \Gamma(\chi)$ where $\chi = \phi$ if $i \neq j$, $\chi = \eta_{0i}\phi$ if $i = j > 0$ and $\chi = \phi\eta_{01}$ if $i = j = 0$. Define for $i, j < m$

$$\mathsf{End}_{ji}(M_S; \Phi, L) = \{\phi\colon A_i \to A_j \mid \Gamma_\Phi(\phi) \in L\}.$$

Theorem 7.1. *Given a module M_S and a 0-1-sublattice L of $\mathsf{L}(M_S)$ with skew n-frame $\Phi\colon A_i$, $A_{0i}\,(i < m)$, $m \geq n \geq 3$. Then the $\mathsf{End}_{ji}(M_S; \Phi, L)$, $(i, j < m)$ form a subringoid of $\mathsf{End}(M_S)$ which admits a first-order logic interpretation in the 0-1-lattice with constants from Φ. In particular,*

$$R = \{\phi \in \mathsf{End}(M_S) \mid \pi_j\phi\varepsilon_i \in \mathsf{End}_{ji}(M_S; \Phi, L) \text{ for all } i, j < m\}$$

is a subring of $\mathsf{End}(M_S)$. If L is complemented then R is regular and the generalized matrix units π_{ji} in Cor.6.1 may be chosen in R.

The ring R will be denoted as $\mathsf{End}(M_S; \Phi, L)$. In steps (5) and (7) in the proof of [6, Thm.4.2.1], first the 'auxiliary ring' $\mathsf{C}(\Phi, L)$ (see below) is defined and then R as a ring of matrices over this ring.

Proof. Let P denote the set of all $\phi :: A_i \to A_j$ with $i, j < m$ and $\Gamma_\Phi(\phi) \in L$. With Prop.4.2 and Lemma 4.3 it follows $\eta_{ki} \in P$ for all i, k. Consequently, for any $\phi :: A_i \to A_j$

$$(*) \quad \phi \in P \;\Leftrightarrow\; \eta_{lj}\phi \in P \;\Leftrightarrow\; \phi\eta_{il} \in P \quad \text{where } l < n.$$

Clearly, $\eta_{ii} = \mathsf{id}_{A_i} \in \mathsf{End}_{ii}(M_S; \Phi, L)$ and $0_{ji} \in \mathsf{End}_{ji}(M_S; \Phi, L)$. To verify the properties of a subringoid amounts to the following

$$\begin{array}{llll}
(1) & \phi, \psi \in P \Rightarrow \psi\phi \in P & & \text{for } \phi :: A_i \to A_j,\ \psi :: A_j \to A_k \\
(2) & \phi \in P \Rightarrow -\phi \in P & & \text{for } \phi :: A_i \to A_j \\
(3) & \phi, \psi \in P \Rightarrow \phi + \psi \in P & & \text{for } \phi, \psi\colon D \to A_j,\ D \subseteq A_i,\ D \in L.
\end{array}$$

This is done in the usual way dating back to von Staudt: In view of $(*)$, (1) reduces to the case where i, j, k are pairwise distinct and then Lemma 4.3 applies. (2) and (3) reduce to the case where $i = 0$, $j = 1$. Now

$$\Gamma(-\eta_{10}) = ((A_1 + A_{02}) \cap (A_0 + \Gamma(\eta_{21})) + A_2) \cap (A_0 + A_1).$$

and (2) follows with (1). Finally, in the context of (3),

$$\Gamma(\phi+\psi) = [(\Gamma(\phi)+A_2)\cap(A_{02}+A_1)+\Gamma(\psi\eta_{02})]\cap(D+A_1).$$

In particular, an isomorphic ringoid can be first-order defined within the 0-1-lattice L^{m^2} using constants from Φ. By Prop.2.1, R is a subring of $\mathsf{End}(M_S)$.

Now, assume L to be complemented. The η_{ji} lift to γ_{ji} and then to an n-system of generalized matrix units according to Cor.6.1. In view of Lemma 3.2 we have to show that $\pi_0 R\pi_0$ is regular. Given $\phi\colon A_0\to A_1$ with $\Gamma(\phi)\in L$, choose $D,C\in L$ such that $D\oplus\ker\phi = A_0$ and $C\oplus\mathsf{im}\,\phi = A_1$. Then $\mathsf{im}\,\phi = \mathsf{im}(\phi|D)$ and $\phi|D$ is injective. Lifting $(\phi|D)^{-1}$ to $\psi\colon A_1\to A_0$ according to Prop.4.2 it follows $\Gamma(\psi)\in L$ and $\phi=\phi\psi\phi$. For $\alpha\in\pi_0R\pi_0$ apply this to $\phi=\pi_1\gamma_{10}\alpha\varepsilon_0$ and choose $\xi=\hat{\psi}\gamma_{10}\in\pi_0R\pi_0$ to obtain $\alpha=\alpha\xi\alpha$. □

Corollary 7.2. *Under the hypotheses of Thm.7.1, if $n\le k\le m$ and if Ψ is the skew n-frame $A_i, A_{0i}\,(i<k)$ of the interval $[0,B]$ of L then $\mathsf{End}(B_S;\Psi,[0,B]) = \pi_B\,\mathsf{End}(M_S;\Phi,L)\varepsilon_B$ where $B=\sum_{i<k}A_i$ and $M=B\oplus\sum_{k\le i<m}A_i$ with associated π_B and ε_B.*

Proof. This is immediate observing that $\mathsf{End}_{ji}(M_S;\Phi,L)=\mathsf{End}_{ji}(B_S;\Psi,L_B)$, $\pi_i^\Psi = \pi_B\pi_i\varepsilon_B$, and $\varepsilon_i^\Psi=\pi_B\varepsilon_i\varepsilon_B$ for $i,j<k$ where π_i^Ψ and ε_i^Ψ are associated with the direct decomposition of B given by Ψ. □

8. Generators

Lemma 8.1. *In a complemented modular lattice, let $c\le a\oplus b$. Then there are $a_i\le a$, $b_i\le b$ for $i\in\{1,2\}$, and $c_1\le c$ such that a_1,a_2,b_1,b_2 are independent, $a_1\sim_{c_1} b_1$, and $c=c_1+a_2+b_2$.*

Proof. Choose $a_2=ac$, $b_2=bc$, $a(b+c)=a_1\oplus a_2$, $b(a+c)=b_1\oplus b_2$ and $c_1=c(a_1+b_1)$. □

Given a basis $\Phi\colon a_0,\dots,a_{m-1}$ of L and $i\neq j$ define the *coordinate domains*

$$\mathsf{C}_{ji}(\Phi,L)=\{x\in L\mid a_j\oplus x=a_i+a_j\},\quad \mathsf{C}(\Phi,L)=\mathsf{C}_{10}(\Phi,L).$$

Proposition 8.2. *For any basis $\Phi\colon A_0,\dots,A_{m-1}$ of the sublattice L of $\mathsf{L}(M_S)$, the coordinate domain $\mathsf{C}_{ji}(\Phi,L)$ consists of the graphs $\Gamma(\psi)=\mathsf{im}(\pi_i-\hat{\psi})$ where $\psi\colon A_i\to A_j$ and $\Gamma(\psi)\in L$.*

Proof. In view of Prop.5 it suffices to observe that one has $\Gamma(\psi)=\{\pi_ix-\psi\pi_ix\mid x\in M\}=\mathsf{im}(\pi_i-\hat{\psi})$. □

Theorem 8.3. *For any module M_S and complemented sublattice L of $\mathsf{L}(M_S)$, L is generated by $\Phi\cup\mathsf{C}(\Phi,L)$ if Φ is a skew n-frame of L with $n\ge 2$.*

For n-frames this is [24, Part II, Lemma 4.3]. In the proof of [6, Thm.4.2.1] a larger set of generators has been established. Observe that step (6) has to be modified in order to allow additional summands $\le a_j$ which also applies to the corresponding statements in [8, 9, 10].

Proof. Let Φ be given by the A_i and A_{ji} $(j < n)$ and L' the sublattice of L generated by $\Phi \cup \mathsf{C}(\Phi, L)$. The proof proceeds as follows.

(1) $X \in L'$ for $A_i \supseteq X \in L$,
(2) $\Gamma(\phi) \in L'$ for $\phi\colon\colon A_i \to A_j$, $i \neq j$, $\Gamma(\phi) \in L$,
(3) $X \in L'$ for $X \leq \sum_{j=0}^{i} A_j$, $X \in L$ by induction on i.

Ad (1). First, consider $X \in L$ with $X \subseteq A_1$ and observe that $\Gamma(\gamma_{01}) = \Gamma(\gamma_{10}) = A_{01} \in L'$. By Prop.4.2 it follows $Y = \mathsf{im}(\gamma_{01}|X) \in L$ and, lifting $\gamma_{10}|Y$ as in Prop.4.2 to $\psi\colon A_0 \to A_1$, also $\Gamma(\psi) \in L$. Thus, by Prop.4.1 and definition, $\Gamma(\psi) \in L'$ whence by Prop.4.2 $X = \mathsf{im}\,\psi \in L'$.

Now, if $X \in L$ and $X \subseteq A_i$ with $i \neq 1$ then $Y = A_1(X + A_{1i}) \in L'$ whence $X = A_i(Y + A_{1i}) \in L'$.

Ad (2). Consider $i = 0, j = 1$ first. Lifting ϕ to $\psi\colon A_0 \to A_1$ according to Prop.4.2 one obtains $\Gamma(\psi) \in L'$ whence $\Gamma(\phi) \in L'$. In the general case, with $\psi = \gamma_{1j}\phi\gamma_{i0}$, by Lemma 4.3, $\Gamma(\psi) \in L$ and so $\Gamma(\psi) \in L'$ by the case just dealt with. Again with Lemma 4.3 and $\phi = \gamma_{j1}\psi\gamma_{0i}$, it follows $\Gamma(\phi) \in L'$.

Ad (3). The case $i = 0$ is done by (1). So consider $i > 0$ and $B = \sum_{j=0}^{i-1} A_j$. By Lemma 8.1 and Prop.4.1 there are $Y \subseteq A_i$, $Z \subseteq B$, and $\phi\colon\colon A_i \to B$ such that $Y, Z, \Gamma(\phi) \in L$ and $X = Y + Z + \Gamma(\phi)$. By (1) and inductive hypothesis it follows $Y, Z \in L'$. Then

$$\Gamma(\pi_j\phi) = (A_j + A_i) \cap \left(\Gamma(\phi) + \sum_{k \neq i,j} A_k\right) \in L'.$$

Finally, $\phi = \sum_{j=0}^{i-1} \pi_j\phi$ implies

$$\Gamma(\phi) = \bigcap_{j=0}^{i-1} \left(\Gamma(\pi_j\phi) + \sum_{k \neq i,j} A_k\right) \in L'.$$

□

9. The coordinatizing ring and module

Proposition 9.1. *Let M be a right module over a ring S and let R be a regular subring of $\mathsf{End}(M_S)$. Then $\mathbb{L}(R)$ embeds into $\mathsf{L}(M_S)$ via $\iota(\phi R) = \mathsf{im}\,\phi$.*

This is (1) in the proof of [6, Thm.4.2.1].

Proof. For simplicity of notation, we consider M an R-S-bimodule. ι is well defined and order-preserving since $s \in rR$ implies $sM \subseteq rM$. Also, if $t = rx + sy$ then $tv \in rM + sM$ for all $v \in M$ so ι is join-preserving. Now recall (cf [27, Ch.II§4 (II)]) that for idempotent e, f one has $eR \cap fR = (f - fg)R$ where g is an idempotent such that $Rg = R(f - ef)$. Let $g = r(f - ef)$ and consider $v \in eM \cap fM$. Then $v = ev = fv$ and $v = fv - fr(v - v) = fv - fr(fv - efv) = fv - fr(f - ef)v = (f - fg)v$. Thus, ι is also meet-preserving. Clearly, $rM = 0$ if and only if $r = 0$. Since $\mathbb{L}(R)$ is complemented, it follows that ι is an embedding. □

Corollary 9.2. *Let L be a complemented modular 0-1-sublattice of $\mathsf{L}(M_S)$ admitting a skew n-frame Φ with $n \geq 3$ and $R = \mathsf{End}(M_S; \Phi, L)$. Then $\iota\colon \mathbb{L}(R) \to L$ is an isomorphism where $\iota(\phi R) = \mathsf{im}\,\phi$.*

The proof corresponds to step (8) of [6, Thm.4.2.1]. The simplification is due to the smaller set of generators used here.

Proof. $R = \mathsf{End}(M_S; \Phi, L)$ is regular with n-system of generalized matrix units $\pi_{ji} \in R$ and $\iota\colon \mathbb{L}(R) \to \mathsf{L}(M_S)$ is an embedding, see Thm.7.1 and Prop.9.1. Let Ψ denote the skew n-frame of $\mathbb{L}(R)$ defined by the π_{ji} according to Cor.6.1. Then $\iota(\Psi) = \Phi$. Moreover, by Prop.8.2, $\mathsf{C}(\Phi, L)$ consists of the $\mathsf{im}(\pi_0 - \phi)$ where $\phi = \widehat{\phi_{10}} \in \pi_1 R \pi_0$ and $\mathsf{C}(\Psi, \mathbb{L}(R))$ of the $(\pi_0 - \phi)R$ since the endomorphisms of R_R are just left multiplications. It follows that ι maps $\mathsf{C}(\Psi, \mathbb{L}(R))$ onto $\mathsf{C}(\Phi, L)$. Since both L and $\mathbb{L}(R)$ are complemented, they are generated by $\Phi \cup \mathsf{C}(\Phi, L)$ and $\Psi \cup \mathsf{C}(\Psi, \mathbb{L}(R))$ respectively. Thus, ι maps $\mathbb{L}(R)$ onto L, isomorphically. $\square$

Corollary 9.3. *The ring $R = \mathsf{End}(M_S; \Phi, L)$ in Cor.9.2 is generated by its idempotents and an idempotent $\phi \in \mathsf{End}(M_S)$ belongs to R if and only if $\mathsf{im}\,\phi \in L$ and $\mathsf{ker}\,\phi \in L$. In particular, as a subring of $\mathsf{End}(M_S)$, R is uniquely determined by the sublattice L of $\mathsf{L}(M_S)$.*

Thus, we may define the *coordinatizing ring* in Cor.9.2 as $\mathsf{End}(M_S; L)$ without mentioning a particular frame. This result is due to Luca Giudici, it is part of [6, Thm.4.2.1].

Proof. The first claim follows from Lemma 3.1. Now, consider the isomorphism of Cor.9.2. Consider idempotent $\phi \in \mathsf{End}(M_S)$ with $X = \mathsf{im}\,\phi$ and $Y = \mathsf{ker}\,\phi = \mathsf{im}(\mathsf{id} - \phi)$. If $\phi \in R$ then also $\mathsf{id} - \phi \in R$ whence $X, Y \in \iota\,\mathbb{L}(R) = L$. Conversely, assume $X, Y \in L$. Then $R = \iota^{-1}(X) \oplus \iota^{-1}(Y)$ whence $\iota^{-1}(X) = \varepsilon R$ and $\iota^{-1}(Y) = (\mathsf{id} - \varepsilon)R$ for some idempotent $\varepsilon \in R$. But this implies $\mathsf{im}\,\varepsilon = X$ and $\mathsf{ker}\,\varepsilon = Y$ and so $\phi = \varepsilon \in R$. $\square$

The following coordinatization in terms of a module builds on the well known correspondence between submodules of S^n_S and right ideals of the matrix ring S_n. Let $\mathbb{L}(M_S)$ denote the poset of finitely generated submodules of M_S.

Proposition 9.4. *Given a complemented 0-1-sublattice L of $\mathsf{L}(A_{\mathbb{Z}})$ with skew n-frame A_i, A_{0i}, $n \geq 3$, and coordinatizing ring $R = \mathsf{End}(A_{\mathbb{Z}}; L)$, then*

$$R_0 = \{\phi \in \mathsf{End}(A_0) \mid \phi = \psi|A_0 \text{ for some } \psi \in R\}$$

is a regular subring of $\mathsf{End}(A_0)$ and

$$M = \{\phi \in \mathsf{Hom}(A_0, A) \mid \phi = \psi|A_0 \text{ for some } \psi \in R\}$$

is a submodule of the right R_0-module $\mathsf{Hom}(A_0, A)$ and a direct sum of finitely many cyclic R_0-modules. Moreover, one has an isomorphism

$$\theta\colon \mathbb{L}(M_{R_0}) \to L \text{ given by } \theta(N) = \sum_{\phi \in N} \mathsf{im}\,\phi.$$

Proof. Let $\pi_i, \varepsilon_i, \eta_i$ denote the maps associated with the given frame. $\pi_0 R \pi_0$ is a regular ring in view of Thm.7.1. Also, this ring is embedded into $\mathsf{End}(A_0)$ via $\psi \mapsto \psi|A_0$. Thus, the image R_0 under this embedding is a regular ring, too. Clearly, M is an R_0-submodule and

$$\psi\varepsilon_0 = (\sum_i \varepsilon_i \pi_i)\psi\varepsilon_0 = \sum_i \varepsilon_i \eta_{i0}\rho_i \text{ with } \rho_i = \eta_{0i}\pi_i\psi\varepsilon_0 \in R_0$$

for any $\psi \in R$. It follows that $M = \bigoplus_i \varepsilon_i \eta_{i0} R_0$. Define

$$\delta(N) = \{\sum_i \phi_i \eta_{0i}\pi_i \mid \phi_i \in N\} \text{ for } N \in \mathsf{L}(M_{R_0})$$

$$\gamma(I) = \{\psi|A_0 \mid \psi \in I\} \text{ for } I \in \mathsf{L}(R_R).$$

$\gamma(I)$ is a submodule, obviously. If N is a submodule, and if $\phi_i \in N$ and $\psi = \sum_{jk} \psi_{jk} \in R$ with $\psi_{jk} \in \pi_j R \pi_k$, then

$$(\sum_i \phi_i \eta_{0i}\pi_i)\psi = \sum_{ijk} \phi_i \eta_{0i}\pi_i\psi_{jk} = \sum_{ik} \phi_i \eta_{0i}\psi_{ik} = \sum_k \chi_k \eta_{0k}\pi_k \in \delta(N)$$

with

$$\rho_i = \eta_{0i}\psi_{ik}\eta_{k0} \in R_0 \text{ and } \chi_k = \sum_i \phi_i \rho_i \in N.$$

Thus, $\delta(N)$ is a right ideal. Obviously, both maps preserve inclusion. Moreover, for $\phi_i \in N$ one has $(\sum_i \phi_i \eta_{0i}\pi_i)\varepsilon_0 = \phi_0 \in N$ whence $\gamma\delta(N) = N$. On the other hand, for $\psi_i, \psi \in I$ one has

$$\psi_i\varepsilon_0 \in \gamma(I),\ \sum_i \psi_i \varepsilon_0 \eta_{0i}\pi_i \in \delta\gamma(I)$$

$$\psi = \sum_i \psi\varepsilon_i\pi_i = \sum_i \phi_i\eta_{0i}\pi_i \in \delta\gamma(I) \text{ with } \phi_i = \psi\varepsilon_i\eta_{i0} \in \gamma(I)$$

whence $\delta\gamma(I) = I$. This shows that one has mutually inverse lattice isomorphisms between $\mathsf{L}(M_{R_0})$ and $\mathsf{L}(R_R)$.

Obvioulsy, finitely generated right ideals are matched with finitely generated submodules. Thus, with the isomorphism ι of Cor.9.2, one has an isomorphism $\theta = \iota \circ \delta \colon \mathbb{L}(M_{R_0}) \to L$ denoting the restriction of δ to $\mathbb{L}(M_{R_0})$ also by δ. From the definition of δ and ι it follows immediately that $\theta(N) = \sum_{\phi \in N} \mathsf{im}\,\phi$. □

10. Embedding

Theorem 10.1. (Jónsson). *Any sectionally complemented modular lattice L which is Arguesian or admits a large partial n-frame with $n \geq 4$ can be embedded into some $\mathsf{L}(A_{\mathbb{Z}})$.*

Proof. (Outline) L embeds into an atomic modular lattice M, namely the lattice of all filters of L ordered by dual inclusion. The atoms of M are the points of a projective space P with collinearity given by $p + q = q + r = q + r$. According to Frink's Theorem, L embeds into the subspace lattice of $\mathsf{L}(P)$ - given $p \leq a \oplus b$ modularity readily supplies $q \leq a$ and $r \leq b$ with p, q, r, collinear. Decomposing

P into its irreducible components P_i one has $\mathsf{L}(P)$ isomomorphic to the direct product of the $\mathsf{L}(P_i)$. If L is Arguesian, then so are the $\mathsf{L}(P_i)$ since lattice identities are inherited by the filter and the ideal lattice and since $\mathsf{L}(P)$ may be seen as a sublattice of the ideal lattice of M cf. [4]. If $n \geq 4$ then the $\mathsf{L}(P_i)$ have height ≥ 4 since in an n-frame a_i, a_{0i}, if one a_i is identified with 0 then so are all - due the perspectivities.

In any case, for $\mathsf{L}(P_i)$ of height ≥ 3, by the Coordinatization Theorem of Projective Geometry, we have $\mathsf{L}(P_i) \cong \mathsf{L}((V_i)_{D_i})$ for some vector space V_i. For $\mathsf{L}(P_i)$ of height ≤ 2 we have an embedding into some $\mathsf{L}(V_i)$, trivially. This results into an embedding of $\prod_i \mathsf{L}(P_i)$ into $\mathsf{L}(A_{\mathbb{Z}})$ where $A = \prod_i V_i$. $\square$

11. Coordinatization

Theorem 11.1. (Jónsson.) *Let L be an complemented modular lattice admitting a skew n-frame Φ with $n \geq 4$ or with $n \geq 3$ and L Arguesian. Then there is a regular ring R such that $L \cong \mathbb{L}(R)$.*

This is [17, Thm.8.3].

Proof. By Thm.10.1 we may assume L a 0-1-sublattice of $\mathsf{L}(A_{\mathbb{Z}})$ for some abelian group A. Now, apply Cor.9.2. $\square$

A coordinatizable lattice is *uniquely coordinatizable* if, for all regular rings R, R', $\mathbb{L}(R) \cong L \cong \mathbb{L}(R')$ implies that for any isomorphism $\omega\colon \mathbb{L}(R) \to \mathbb{L}(R')$ there is an isomorphism $\alpha\colon R \to R'$ such that $\omega(aR) = \alpha(a)R'$ for all $a \in R$.

A coordinatizable L is *rigidly coordinatizable* if for any R such that $L \cong \mathbb{L}(R)$, if α is an automorphism of R with $\alpha(a)R = aR$ for all $a \in R$ then $\alpha = \mathsf{id}_R$. The following is [18, Thm.10.4] resp. its proof. For n-frames this is [24, Ch.IV] including rigidity in case $n \geq 2$. The latter has been extended in [22, Cor.4.11] to skew frames.

Theorem 11.2. (Jónsson.) *Let L be a complemented modular lattice admitting a skew n-frame Φ with $n \geq 4$ or with $n \geq 3$ and L Arguesian. Then L is uniquely and rigidly coordinatizable.*

Proof. By Thm.11.1 one may assume $L = \mathbb{L}(R)$. Observe that $R \cong \mathsf{End}(R_R)$ mapping r to the left multiplication $r \mapsto rx$. Moreover, for every spanning orthogonal system e_i of idempotents in R, say the preimages of the π_i, all $\phi\colon e_iR \to e_jR$ with $i \neq j$ have a graph of form $\Gamma(\phi) = (e_i - e_jr)R \in L$. It follows that $\mathsf{End}(R_R) = \mathsf{End}(R_R; L)$.

Now, for the first property, it suffices to consider an isomorphism $\omega\colon L = \mathbb{L}(R) \to L' = \mathbb{L}(R')$. Then the image Φ' of Φ under ω is a skew n-frame of L' and by Thm.7.1 ω induces an isomorphism of the subringoids associated with L and L'. With Prop.2.1 we get an isomorphism $\alpha\colon R \to R'$ composed as follows

$$R \cong \mathsf{End}(R_R) = \mathsf{End}(R_R; L) \cong \mathsf{End}(R'_{R'}; \Phi', L') = \mathsf{End}(R'_{R'}) \cong R'.$$

Moreover, for either $a = e_i$ or for $a \in e_i R e_j$ with $i \neq j$ we have $\alpha(a)R' = \omega(aR)$. Since these form a generating set of the lattice, ω is the lattice isomorphism induced by α.

For the second property, consider an automorphism α of R such that the induced lattice isomorphism ω is the identity. In particular, the frame elements and graphs are fixed, which implies that α is the identity. □

12. Sectionally complemented lattices

Here, for a lattice L no largest element 1 is required. L is *sectionally complemented* if each interval $[0, u]$ is complemented. A *basis* for L is a family a_i $(i \in I)$ such that every finite subfamily is independent and such that for any $x \in L$ there is a finite $J \subseteq I$ with $x \leq a_J$ where $a_J = \sum_{i \in J} a_i$. A *generalized skew n-frame* of L consists of a basis a_i $(i \in I)$, where $\{0, \dots, n-1\} \subseteq I$, together with axes a_{0i} of subperspectivities from a_i to a_0 which are perspectivities for $i = 0, \dots, n-1$. Actually, the hypothesis in [18, Thm.9.4] was formulated according to the following.

Proposition 12.1. *A sectionally complemented modular lattice admits a generalized skew n-frame if and only if it admits a basis and a large partial n-frame.*

Proof. Given an n-frame of $[0, u]$ and a basis $b_i, (i \in I)$ choose J with $b_J \geq u$ and apply Prop.6.2 to obtain a skew n-frame of $[0, u]$. Then, for all $i \in I \setminus J$, simultaneously, as in the proof of Prop.6.2 extend this skew n-frame to one of $[0, b_J + b_i]$. □

The following is [18, Thm.9.4]. Here, rings are no more required to have a unit and regularity amounts to $\mathbb{L}(R)$ being a sectionally complemented sublattice of $\mathsf{L}(R_R)$.

Theorem 12.2. (*Jónsson*) *Given a sectionally complemented modular lattice, L, with a generalized skew n-frame where either $n \geq 4$ or $n \geq 3$ and L Arguesian, there is a regular ring R such that $L \cong \mathbb{L}(R)$,*

Proof. (Outline) As observed by Jónsson [18], Thm.10.1 extends to the sectional case and, given an infinitary skew frame, one may assume $A = \bigoplus_{i \in I} A_i$. In the definition of systems of generalized matrix units. instead of $\sum_i e_i = 1$, one has to require that for each $a \in R$ there is a finite sum e of the e_i such that $a = ea = ae$. Also, in the definition of $R(\Phi, L, M_S)$ in $\sum_{ij} \widehat{\phi_{ij}}$, all but finitely many ϕ_{ij} have to be 0. Then the proof carries over with no major changes. □

Here, unique coordinatization fails e.g. if L is the lattice of all finite-dimensional subspaces of a vector space of infinite dimension [18, Sect.10]. On the other hand, coordinatization is always possible if L is the union of a countable increasing sequence of uniquely coordinatizable principal ideals - see [18, Thm.10.3] for the ultimate proof. Recently, a thorough analysis of coordinatizability by regular rings without unit has been given by Wehrung [31, 32].

A module M over a regular ring S with unit is *locally projective* is each of its finitely generated submodules is projective. In particular, a direct sum of finitely many cyclic modules is locally projective. The finitely generated submodules of a locally projective module form a sectionally complemented modular lattice $\mathbb{L}(M_S)$ cf. [18, Cor.7.13; Cor.7.14]. The following is the main result of Jónsson [18, Thm.8.2].

Theorem 12.3. *Any Arguesian sectionally complemented lattice admitting a large partial* 3*-frame is isomorphic to* $\mathbb{L}(M_S)$ *for some locally projective module* M_S *over a regular ring* S*.*

Proof. According to Thm.10.1 we may assume L a 0-sublattice of some $\mathsf{L}(A_{\mathbb{Z}})$ with a large partial 3-frame $\Phi\colon A_i, A_{0i}$. Put $U = \sum_{i<3} A_i$ and apply Prop.9.4 to $U_{\mathbb{Z}}$ and the interval $[0, U]$ of L to define the regular ring

$$R_0 = \{\phi \in \mathsf{End}(A_0) \mid \phi = \psi|A_0 \text{ for some } \psi \in \mathsf{End}(U; [0, U])\}.$$

Now, consider $X \supseteq U$ in L. By Prop.6.2 there is a skew 3-frame $\Psi\colon B_i, B_{0i}$ in $[0, X]$ with $B_i = A_i$ and $B_{0i} = A_{0i}$ for $i < 3$. In view of Cor.9.3 there is a unique subring R_X of $\mathsf{End}(X_{\mathbb{Z}})$ such that $R_X = \mathsf{End}(X; [0, X])$ not depending on Ψ. By Cor.7.2 one has $R_U = \pi_U R_X \varepsilon_U$ where π_U, ε_U are associated with the decomposition $X = U \oplus \sum_{i \geq 3} B_i$. It follows that

$$R_0 = \{\rho \in \mathsf{End}(A_0) \mid \rho = \sigma|A_0 \text{ for some } \sigma \in R_X\}.$$

Applying Prop.9.4 to $X_{\mathbb{Z}}$ and the interval $[0, X]$ of L one obtains the locally projective R_0-submodule

$$M_X = \{\phi \in \mathsf{Hom}(A_0, X) \mid \phi = \psi|A_0 \text{ for some } \psi \in R_X\}$$

of $\mathsf{Hom}(A_0, A)$ and the isomorphism

$$\theta_X\colon \mathbb{L}(M_{X R_0}) \to [0, X] \ \text{ with } \theta_X(N) = \sum_{\phi \in N} \mathsf{im}\, \phi.$$

Now, consider $U \subseteq X \subseteq Y$ in L. Again, by the proof of Prop.6.2 the skew frame in $[0, X]$ can be extended to one of $[0, Y]$ and, in view of Cor.7.2 and Cor.9.3, we have $R_X = \pi_X R_Y \varepsilon_X$ where π_X, ε_X are associated with some decomposition $Y = X \oplus Z$. It follows that M_X is an R_0-submodule of M_Y. Moreover, $\theta_Y(N) = \theta_X(N)$ for $N \in \mathbb{L}(M_{X R_0})$.

Thus, $M = \bigcup_{U \subseteq X \in L} M_X$ is a directed sum of the M_X and a locally projective R_0-module and $\bigcup_{U \subseteq X \in L} \theta_X$ is a lattice isomorphism of $\mathbb{L}(M_{R_0})$ onto L. □

13. Rings and lattices with involution

A $*$-*ring* is a ring endowed with an involutory anti-automorphism $r \mapsto r^*$. A $*$-ring is $*$-*regular* if it is regular and $x^* x = 0$ only for $x = 0$.

Proposition 13.1. *Given* $*$*-rings* R, R' *and an* n*-system of generalized matrix units* e_{ji} *of* R $(n \geq 2)$ *where the* $e_i = e_{ii}$ *are projections, a ring homomorphism* $\eta\colon R \to R'$ *is a* $*$*-ring homomorphism if* $\eta e_i = (\eta e_i)^*$ *for all* i *and* $\eta(r^*) = \eta(r)^*$ *for all* $i \neq j$ *and* $r \in e_j R e_i$*.*

Proof. For $r \in e_i R e_i$ and $i \neq k < n$ one has $e_{ki} r \in e_k R e_i$ whence $\eta((e_{ki}r)^*) = (\eta(e_{ki}r))^* = (\eta e_{ki} \eta r)^* = (\eta r)^* (\eta e_{ki})^*$. It follows now that $(\eta r)^* = (\eta r)^* (\eta e_{ii})^* = (\eta r)^* (\eta e_{ki})^* (\eta e_{ik})^* = \eta((e_{ki}r)^*)(\eta e_{ik}^*) = \eta((e_{ki}r)^* e_{ik}^*) = \eta((e_{ik}e_{ki}r)^*) = \eta(r^*)$. □

Given a ring R and $X \subseteq R$ we define *left* and *right annihilators*

$$X^{\mathrm{l}} = \{r \in R \mid \forall x \in X.\, rx = 0\},\quad Y^{\mathrm{r}} = \{r \in R \mid \forall y \in Y.\, yr = 0\},$$

the closed sets in the Galois correspondence on R induced by the binary relation given by $xy = 0$. The maps $X \mapsto X^{\mathrm{lr}}$ and $X \mapsto X^{\mathrm{rl}}$ are closure operators on $\mathsf{L}(R_R)$ resp. $\mathsf{L}({}_R R)$ (the lattice of left ideals) and the maps $X \mapsto X^{\mathrm{l}}$, $Y \mapsto Y^{\mathrm{r}}$ form a pair of mutually inverse anti-isomorphisms between the lattices $L_{\mathrm{c}}(R_R)$ and $L_{\mathrm{c}}({}_R R)$ of closed sets.

Observe that for an idempotent e we have $eR \in L_{\mathrm{c}}(R_R)$ and $Re \in L_{\mathrm{c}}({}_R R)$. In particular, for a regular ring R the above maps constitute a pair of mutually inverse anti-isomorphisms between $\mathbb{L}(R)$ and $\mathbb{L}(R^{\mathrm{op}})$, the lattice of principal left ideals.

An *involution* on a poset is an involutory dual automorphism $x \mapsto x^{\perp}$. For any $*$-ring there is a canonical involution on $L_{\mathrm{c}}(R_R)$ given by $X^{\perp} = \{x^* \mid x \in X\}^{\mathrm{r}}$. In particular, this restricts to the involution $(eR)^{\perp} = (1 - e^*)R$ on the poset $L_{\mathrm{e}}(R)$ of all eR, e idempotent. Thus, for a regular $*$-ring R one has a canonical involution induced on $\mathbb{L}(R)$. An *orthocomplementation* on a 0-1-lattice is an involution such that $a \oplus a^{\perp} = 1$ for all a. The following is in essence von Neumann [24, II, Thm.4.3] cf. [21, XII, Satz 2.2].

Theorem 13.2. *For any uniquely and rigidly coordinatizable lattice, L, endowed with an involution, there is an regular $*$-ring R, unique up to isomorphism, such that L is isomorphic to $\mathbb{L}(R)$ as a lattice with involution. If the involution on L is an orthocomplementation, then R is $*$-regular.*

Proof. Here, one has an isomorphism

$$\omega\colon\ \mathbb{L}(R) \to \mathbb{L}(R^{\mathrm{op}}),\ \ \omega(aR) = ((aR)^{\perp})^{\mathrm{l}}$$

By unique coordinatizability, there is

$$\alpha\colon R \to R^{\mathrm{op}},\ \ \omega(aR) = \alpha(a)R^{\mathrm{op}} = R\alpha(a).$$

Write $a^* = \alpha(a)$ and put $\delta(aR) = (Ra^*)^{\mathrm{r}}$. So δ is an automorphism of $\mathbb{L}(R)$ and

$$\delta^2(aR) = ((((aR)^{\perp})^{\mathrm{lr}})^{\perp})^{\mathrm{lr}} = ((aR)^{\perp})^{\perp} = aR.$$

On the other hand, observe that $\delta(X) = \{x^* \mid x \in X\}^{\mathrm{r}}$ for all $X \in \mathbb{L}(R)$ whence $\delta^2(aR) = \delta((Ra^*)^{\mathrm{r}}) = \delta\{x \mid a^*x = 0\} = \{x^* \mid a^*x = 0\}^{\mathrm{r}} = \{x^* \mid x^*a^{**} = 0\}^{\mathrm{r}} = \{y \mid ya^{**} = 0\}^{\mathrm{r}} = (a^{**}R)^{\mathrm{lr}} = a^{**}R = \alpha^2(a)R$. From $\delta^2 = \mathsf{id}$ and rigidity it follows $\alpha^2 = \mathsf{id}$.

Given a second involution β with $(aR)^{\perp} = \beta(a)R$ we get $(\beta\alpha)(a)R = aR$ whence $\beta\alpha = \mathsf{id}$ and so $\beta = \alpha$.

Finally, $a^*a = 0$ implies $aR \subseteq (aR)^{\perp}$ and, in the case of an orthocomplementation, $a = 0$. □

14. Modular lattices of annihilators

A *projection* in a $*$-ring R is an idempotent e such that $e^* = e$. If $R_R = \bigoplus_{i=0}^m A_i$ where $A_i = e_i R$ with idempotents e_i then $A_i \leq A_j^\perp$ for $i \neq j$ if and only if the e_i are projections. Indeed, from $A_i \leq A_j^\perp$ it follows $A_i = (\sum_{j \neq i} A_j)^\perp$ and $e_i R = A_i = ((1-e_i)R)^\perp = e_i^* R$ whence $e_i = e_i e_i^* = (e_i e_i^*)^* = e_i^*$.

A skew n-frame $\Phi\colon a_i, a_{ji}$ in a lattice with involution $x \mapsto x^\perp$ is *orthogonal*, if $a_i \leq a_j^\perp$ for all $i \neq j$. Any Arguesian ortholattice L admitting a skew n-frame with $n \geq 3$ admits an orthogonal one (indeed, by Thm.11.1 we may assume $L = \mathbb{L}(R_R)$ with skew n-frame A_i, A_{0i} and find B_i, B_{0i}, with $\sum_{i<k} B_i = U_k\colon = \sum_{i<k} A_i$ for all k; namely $B_k := U_{k+1} \cap U_k^\perp \sim A_k \sim X \leq A_0 = B_0$, so $B_k \cong X$ and $B_k \sim X$ via the graph of an isomorphism cf. Prop.4.2 and [7, Prop.4.22]). The following is due, in essence, to Handelman [11, sect.5].

Theorem 14.1. *Let R be a $*$-ring such that $L_{\mathrm{e}}(R)$ is a modular sublattice of $L_{\mathrm{c}}(R_R)$ closed under involution and admits an orthogonal skew n-frame Φ for some $n \geq 3$. Then $L_{\mathrm{e}}(R)$ is complemented and R embeds into the coordinatizing $*$-ring R' of the lattice $L_{\mathrm{e}}(R)$ with involution, canonically. The analoguous result holds for rings and lattices without involution.*

Proof. Since $L_{\mathrm{e}}(R)$ is supposed to be a sublattice of $L_{\mathrm{c}}(R)$, meets in $L_{\mathrm{e}}(R)$ are intersections $X \cap Y$, i.e. meets in $\mathsf{L}(R_R)$. For joins in $L_{\mathrm{e}}(R)$ we have $X \vee Y \in \mathsf{L}(R_R)$ whence $X \vee Y \supseteq X + Y$. It follows by induction, that evaluating a lattice term $p(X_1, \ldots, X_n)$ in $L_{\mathrm{e}}(R)$ one gets a value V with $V \supseteq U$ where U is obtained evaluating the term in $\mathsf{L}(R_R)$.

$L_{\mathrm{e}}(R)$ is complemented since $R = eR \oplus (1-e)R$ for any idempotent e. Let Φ be given by $A_i = e_i R$ and $A_{ji} = e_{ji} R$ with idempotents e_i, e_{ji}. In particular, since $e_i + e_j$ is an idempotent we have $A_i \vee A_j = A_i + A_j$.

$\mathsf{End}(R_R)$ is the ring associated with its subringoid $R_{ji} = \mathsf{Hom}_R(A_i, A_j)$ according to Prop.2.1. For $\phi \in R_{ji}$ and $i \neq j$ we have $\Gamma_\Phi(\phi) = (e_i - r)R$ for suitable $r \in R$ whence $\Gamma_\Phi(\phi) \in L_{\mathrm{e}}(R)$. This extends to the case $i = j$ by definition of Γ_Φ.

On the other hand, the isomorphism $\omega\colon L_{\mathrm{e}}(R) \to \mathbb{L}(R')$ defines a frame Φ' in $\mathbb{L}(R')$ and the subringoid $R'_{ji} = \mathsf{Hom}_{R'}(A'_i, A'_j)$ of $\mathsf{End}(R'_{R'})$. In particular, we have well-defined injective maps

$$\iota_{ji}\colon R_{ji} \to R'_{ji}, \;\; \Gamma_\Phi(\iota(\phi)) = \omega(\Gamma_\Phi(\phi)).$$

Now, consider the lattice polynomials $p(x, y, \mathsf{z})$ used in the proof of Thm.7.1. Substituting $x = \Gamma(\phi)$ and $y = \Gamma(\psi)$ with ϕ, ψ in the subringoid of R and $\mathsf{z} = \Phi$ we may evaluate in $\mathsf{L}(R_R)$ and in $L_{\mathrm{e}}(R)$ to obtain U resp. V. The latter is transferred to $\mathbb{L}(R')$ via ω. Now, as observed above, $U \subseteq V$ but both U and V are complements of A_j in some interval $[0, A_i + A_j]$ in $\mathsf{L}(R_R)$ resp. $L_{\mathrm{e}}(R)$. In particular, $U + A_j = A_i + A_j$ and $V \cap A_j = 0$ in the modular lattice $\mathsf{L}(R_R)$. This implies $U = V$.

This shows that the ι_{ji} constitute an injective homomorphism between subringoids whence by Prop.2.1 an injective homomorphism $\iota\colon \mathsf{End}(R_R) \to \mathsf{End}(R'_{R'})$ and an embedding of $\eta\colon R \to R'$.

Coming to the case of $*$-rings, observe that for $i \neq j$, $r \in e_j R e_i$, and $s \in e_i R e_j$ one has

$$s = r^* \text{ if and only if } (e_j + s)R \subseteq ((e_i - r)R)^\perp.$$

Indeed, the second condition is equivalent to $0 = (e_i - r)^*(e_j + s)$ and the latter term equals $(e_i - r^*)(e_j + s) = s - r^*$. The second condition carries over via the isomorphism ω of lattices with involution whence $\eta(r^*) = \eta(r)^*$ for $r \in e_j R e_i$ and $i \neq j$. By Prop.13.1 it follows that η is a $*$-ring embedding. □

A $*$-ring R is a *Rickart $*$-ring* if for any $x \in R$ there is a projection $e \in R$ such that $\{x\}^{\mathrm{r}} = eR$ cf. Berberian [1, 2]. Then for each idempotent f there is a projection e such that $fR = (1-f)^{\mathrm{r}} = eR$, i.e. $L_{\mathrm{e}}(R)$ can be identified with the lattice of projections of R (ordered by $e \leq f \Leftrightarrow fe = e$) and is a sublattice of $L_{\mathrm{c}}(R)$ cf. [2, sect.1]. A $*$-ring R is *finite* if $xx^* = 1$ implies $x^*x = 1$ and *directly finite* if $xy = 1$ implies $yx = 1$. A $*$-ring is a Baer $*$-ring if any right annihilator is of the form eR, e a projection.

Modularity of the lattice of projections is then granted e.g. by any of the following conditions

(1) If $e \leq f$ have a common complement then $e = f$ [2, Prop.20.14].
(2) R is a finite C^*-algebra [12, Cor.1.1].
(3) Every non-zero right ideal contains a non-zero projection and $\{x\}^{\mathrm{r}} = 0$ implies $\{x\}^{\mathrm{l}} = 0$ [2, Prop.21.16].
(4) R is a directly finite Baer $*$-ring and $eR \cong fR$ if e, f are projections such that $eR = \{x\}^{\mathrm{lr}}$ and $Rf = \{x\}^{\mathrm{rl}}$ for some $x \in R$ [11, Prop.2.9].
(5) R is a Baer $*$-ring such that the involution extends to its maximal ring of right quotients [2, Prop.21.2].
(6) The matrix rings $M_n(R)$ are Baer $*$-rings cf. [28].

In (iv) and (v) the projection lattice is an orthocomplemented complete lattice [2, Prop.1.24], whence a direct product of lattices admitting an orthogonal n-frame for some $n \in \{3,4,5\}$ and a 2-distributive one cf. [20, sect.9,10]. In (vi) $L_{\mathrm{e}}(M_n(R))$ has an orthogonal n-frame, obviously. According to [23, Cor.1.4.15] every simple orthocomplemented modular lattice of height $\geq n$ admits an orthogonal skew n-frame.

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(C. Herrmann) TUD FB4, Schlossgartenstr. 7, 64289 Darmstadt, Germany
E-mail address: `herrmann@mathematik.tu-darmstadt.de`