ON THE EQUATIONAL THEORY OF THE PROJECTION LATTICES OF FINiTE VON NEUMANN ALGERBA FACTORS

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This work is dedicated to the memory of Herbert Gross,
a true friend and teacher of much more than mathematics.

Abstract

For a finite von Neumann-algebra factor $\mathbf{M}$, the projections form a modular ortholattice $L(\mathbf{M})$. We show that the equational theory of $L(\mathbf{M})$ coincides with that of some resp. all $L(\mathbb{C}^{n\times n})$ and is decidable. In contrast, the uniform word problem for the variety geneated by all $L(\mathbb{C}^{n\times n})$ is shown to be undecidable.

Keywords: Von Neumann algebra, projection lattice, continuous geometry, equational theory

1 Introduction

Projection lattices $L(\mathbf{M})$ of finite von Neumann algebra factors $\mathbf{M}$ are continuous orthocomplemented modular lattices and have been considered as logics resp. geometries of quantum meachnics cf. [28]. In the finite dimensional case, the correspondence between irreducible lattices and algebras, to wit the matrix rings $\mathbb{C}^{n\times n}$, has been completely clarified by Birkhoff and von Neumann [5]. Combining this with Tarski's decidability result for real closed fields and elementary geometry, decidability of the first order theory of $L(\mathbf{M})$ for a finite dimensional factor $\mathbf{M}$ has been observed by Dunn, Hagge, Moss, and Wang [8].
For modern quantum logicians the next natural step would be the characterization of infinite dimensional Hilbert spaces by means of their ortholattices of closed subspaces. This line of research was the subject of the last of H. Gross's papers [39], and finally resulted in Soler's beutiful characterization of (real, complex or quaternionic) Hilbert spaces as orthomodular inner product spaces with a infinite orthonormal sequence of vectors [49]. There is a general agreement about the relevance of this result for quantum logics (see [42, 37, 45, 47, 48, 31, 50] and their references). The only different opinion seems to be the one in chapter 2 of [9] (with important corrections in [10]), which gives a comparation of the above line of research with von Neumann's point of view [26]. A point of view which stressed modularity ("finiteness") and so was quite

different, despite the fact that the decisive homogeneity axiom has already a important role in von Neumann and the fact that a (physically meaninigful) Hilbert space embedding can be easily obtained for AC orthoalgebras whose finite dimensional intervals satisfy von Neumann's axioms *except* completeness. Von Neumann himself used metric completeness only after having reached a completable structure by means of the other axioms. No physical motivation of lattice completeness *together* with orthomodularity is known (i.e. no sufficiently general completion process for orthomodular structures is known to preserve orthomodularity), except the one that can be obtained by von Neumann's methods.

Von Neumann continued with the study of the infinite dimensional case, in the landmark series of papers on 'Rings of Operators' (jointly with Murray) [23], his lectures on 'Continuous Geometry' [25], and in the treatment of traces resp. transition probabilities in a ring resp. lattice-theoretic framework [27, 26].

The key to an algebraic treatment is the coordinatization of $L(\mathbf{M})$ by a $*$-regular ring $U(\mathbf{M})$ derived from $\mathbf{M}$ and having the same projections: $L(\mathbf{M})$ is isomorphic to the lattice of principal right ideals of $U(\mathbf{M})$ (cf. [9] for a thorough discussion of coordinatization theory). For finite factors this has been achieved in [23], more generally for finite AW*-algebras and certain Baer-*-rings by Berberian in [2, 3]. Further extensions by Pyle, Hafner, Handelman (see references in [35, 51]) also identified $U(\mathbf{M})$ with the maximal ring of quotients of $\mathbf{M}$. Handelman, Ara and Menal, and coauthors (see [32, 33, 34, 38] and their references) studied finite Rickart C^*-algebras $\mathbf{M}$: again, $U(\mathbf{M})$ can be defined, coordinatizes the $\aleph_0$-continuous modular ortholattice $L(\mathbf{M})$ and is the classical ring of quotients of $\mathbf{M}$.

In the present note we show that the equational theory of $L(\mathbf{M})$ coincides with that of $L(\mathbb{C}^{n\times n})$ if $L(\mathbf{M})$ is $n+1$- but not n-distributive for some n; and with that of all $L(\mathbb{C}^{n\times n})$, $n<\infty$, otherwise - which applies to the type II$_1$ factors. In the latter case, the equational theory is decidable, but the theory of quasi-identities is not.

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2 Modular Ortholattices: Equations and representations

An algebraic structure $(L,\cdot,+,',0,1)$ is an *ortholattice* if there is a partial order $\leq$ on L such that, for all $a,b\in L$, $0\leq a\leq 1$, $a\cdot b=ab=\inf\{a,b\}$, $a+b=\sup\{a,b\}$, $a''=a$, and $a\leq b$ iff $b'\leq a'$. It is a *modular ortholattice* (shortly: MOL) if, in addition, $a\geq b$ implies $a(b+c)=b+ac$. One can define this class by a finite set of equations, easily ([4, 5]).

If V is a unitary space then the subspaces of finite dimensions together with their orthogonal complements form an MOL $L_f(V)$ - a sublattice of the lattice $L(V)$ of all subspaces. For V of finite dimension n, we have $L(V)=L_f(V)\cong L(\mathbb{C}^n)$.

A lattice is *n-distributive* if and only if it satsifies

$$x\sum_{i=0}^{n} y_i=\sum_{i=0}^{n} x\sum_{j\neq i} y_j.$$

Lemma 1 *$L(\mathbb{C}^k)$ n-distributive if and only if $k\leq n$.*

Proof. Huhn [20, p. 304] cf. [14]. □
For a class $\mathcal{C}$ of algebraic structures, e.g. ortholattices, let $\mathsf{V}\mathcal{C}$ denote the smallest equationally definable class (variety) containing $\mathcal{C}$ cf. [6]. By Tarski's version of Birkhoff's Theorem, $\mathsf{V}\mathcal{C} = \mathsf{HSP}\mathcal{C}$ where $\mathsf{H}\mathcal{C}$, $\mathsf{S}\mathcal{C}$, and $\mathsf{P}\mathcal{C}$ denote the classes of all homomomorphic images, subalgebras, and direct products, resp., of members of $\mathcal{C}$. Define

$$\mathcal{N} = \mathsf{V}\{L(\mathbb{C}^k) \mid k < \infty\}.$$

Clearly, $L(\mathbb{C}^k) \in \mathsf{SH}L(\mathbb{C}^n)$ for $k \leq n$. Within the variety of MOLs, each ortholattice identity is equivalent to an identity $t = 0$ (namely, $a = b$ if and only if $a(ab)' + b(ab)' = 0$). If L is an MOL and $u \in L$ then the section $[0, u]$ is naturally an MOL with orthocomplement $x \mapsto x^u = x'u$.

Lemma 2 *An ortholattice identity $t = 0$ with m occurences of variables holds in a given atomic MOL L if any only if it holds in all sections $[0, u]$ of L with $\dim u \leq m$.*

Proof. We show by induction on complexity: if $g(z_1, \dots, z_k)$ is a lattice term with each variable occuring exactly once and if p is an atom of L and b_i in L with

$$p \leq g(b_1, \dots, b_k) \text{ in } L$$

then there are $p_1, \dots, p_k$ in L which are atoms or 0 such that

$$(*) \;\; p \leq g(p_1, \dots, p_k) \text{ and } p_i \leq b_i \text{ for } i = 1, \dots, k.$$

Indeed, if $g = z_1$ let $p_1 = p$. If

$$g(z_1, \dots, z_k) = g_1(z_1, \dots, z_l) \cdot g(z_{l+1}, \dots, z_k)$$

then

$$p \leq g_1(b_1, \dots b_l), \quad p \leq g_2(b_{l+1}, \dots, b_k)$$

and we may choose the p_i for g_1 and g_2 by inductive hypothesis. Now, let

$$g(z_1, \dots, z_k) = g_1(z_1, \dots, z_l) + g(z_{l+1}, \dots, z_k)$$

If $g_2(b_{l+1}, \dots, b_k) = 0$ then $p \leq g_1(b_1, \dots b_l)$ and we may choose the $p_i \leq b_i$, $i \leq l$ by induction and the $p_i = 0$ for $i > l$. Similarly, if $g_1(b_1, \dots, b_l) = 0$. Otherwise, there are atoms q_1, q_2 such that (cf. [1])

$$q_1 \leq g_1(b_1, \dots b_l), \;\; q_2 \leq g_2(b_{l+1}, \dots, b_k), \;\; p \leq q_1 + q_2$$

and, applying the inductive hypothesis, we may choose $p_i \leq b_i$, atoms or 0, such that

$$q_1 \leq g_1(p_1, \dots p_l), \;\; q_2 \leq g_2(p_{l+1}, \dots, p_k)$$

whence

$$p \leq g(p_1, \dots, p_k).$$

Now, consider an identity $t(x_1, \ldots, x_n) = 0$. By de Morgan's laws, we may assume that t is in so called negation normal form, i.e. a lattice term

$$f(x_1, \ldots, x_n, x_1', \ldots, x_n')$$

Let $g(z_1, \ldots, z_m)$ the lattice term obtained from f by replacing each positive (i.e. x_i) and each negative (i.e. x_i') occurence of a variable by a new variable. Now, assume $t(a_1, \ldots, a_n) > 0$ in L. Since L is atomic, there is an atom p with

$$p \leq f(a_1, \ldots, a_n, a_1', \ldots, a_n').$$

Let $b_j = a_i$ if z_j replaces a positive occurence of x_i and $b_j = a_i'$ if z_j replaces a negative occurrence of x_i. Choose $p_j \leq b_j$ according to $(*)$. Let c_i the join of all p_j where z_j replaces a positive occurence of x_i and d_i the join of all p_j where z_j replaces a negative occurence of x_i. Then $c_i \leq a_i$ and $d_i \leq a_i' \leq c_i'$. Let

$$u = \sum_{i=1}^{n} c_i + d_i.$$

Then

$$c_i \leq u, \;\; d_i \leq c_i^u \leq u$$

and for the MOL $[0, u]$ it follows by monotonicity

$$0 < p \leq f(c_1, , c_n, d_1, \ldots, d_n) \leq f(c_1, , c_n, c_1', \ldots, c_n') \leq t(c_1, \ldots, c_n)$$

□

A unitary representation of an MOL L is a 0-lattice embedding $\varepsilon : L \to L(V)$ into the lattice of all subspaces of a unitary space such that

$$\varepsilon(a') = \varepsilon(a)^{\perp} \quad \text{for all } a \in L.$$

This means that ε is an embedding of the ortholattice L into the orthostable lattice associated with the unitary space V in the sense of Herbert Gross [12].

Corollary 3 *$L \in \mathcal{N}$ for any MOL admitting a unitary representation.*

Proof. By [16, Prop.3.12] (cf. [17]) L embeds into an atomic MOL $\hat{L}$ such that the sections $[0, u]$, $\dim u < \infty$ are subspace ortholattices of finite dimensional unitary spaces (namely, if L is represented in V then $\hat{L}$ consists of all closed subspces X such that $\dim[X \cap \varepsilon a, \, X + \varepsilon a] < \infty$ for some $a \in L$). By Lemma 2, $\hat{L}$ whence also L belong to the variety$\mathcal{N}$ generated by these. □

Corollary 4 *$\mathcal{N} = \mathsf{V} L_f(V)$ for any unitary space of infinite dimension.*

3 Regular rings with positive involution

An associative ring (with or without unit) R is (von Neumann) *regular* if for any $a \in R$ there is a *quasi-inverse* $x \in R$ such that $axa = a$ cf. [25, 21, 11]. A $*$-ring, i.e. a ring with an involution * as additional operation:

$$(x+y)^* = x^* + y^*, \ \ (xy)^* = y^*x^*, \ \ x^{**} = x$$

A $*$-ring is $*$-*regular* if it is regular and, moreover, *positive*: $xx^* = 0$ only for $x = 0$. Equivalently, for any $a \in R$ there is a (unique) *projection* e (i.e. $e = e^* = e^2$) such that $aR = eR$. Particular examples are the rings $\mathbb{C}^{n\times n}$ of all complex n-by$n$-matrices with r^* the adjoint matrix, i.e. the transpose of the conjugate.
The right ideals aR of a $*$-regular ring with unit form a modular ortholattice $\mathbb{L}(R)$, a sublattice of the lattice of all right ideals. $\mathbb{L}(R)$ is isomorphic (via $eR \mapsto e$) to the ortholattice $L(R)$ of projections of R where $e \leq f \Leftrightarrow e = ef$ and $e' = 1 - e$. Observe that

$$L(\mathbb{C}^n) \cong \mathbb{L}(\mathbb{C}^{n\times n}) \cong L(\mathbb{C}^{n\times n})$$

canonically, where a subspace X corresponds to the set of all matrices with columns in X cf. Prop.5, below.

Proposition 5 (Giudici). *Let M be a right module over a ring S and let R be a regular subring of the endomorphism ring $\mathsf{End}(M_S)$. Then $\mathbb{L}(R)$ embeds into the lattice $L(M_S)$ of submodules via $\varepsilon(\phi R) = \mathsf{Im}\phi$.*

Proof. This is (1) in the proof of [9, Thm.4.2.1] in the thesis of Luca Giudici, cf. [18, Prop.9.1]. For simplicity of notation, we consider M an R-S-bimodule. ε is well defined and order-preserving since $s \in rR$ implies $sM \subseteq rM$. Also, if $t = rx + sy$ then $tv \in rM + sM$ for all $v \in M$ so ε is join-preserving. Now recall (cf [30, Ch.II§4 (II)]) that for idempotent e, f one has $eR \cap fR = (f - fg)R$ where g is an idempotent such that $Rg = R(f - ef)$. Let $g = r(f - ef)$ and consider $v \in eM \cap fM$. Then $v = ev = fv$ and $v = fv - fr(v - v) = fv - fr(fv - efv) = fv - fr(f - ef)v = (f - fg)v$. Thus, ε is also meet-preserving. Clearly, $rM = 0$ if and only if $r = 0$. Since $\mathbb{L}(R)$ is complemented, it follows that ε is an embedding. $\square$

Corollary 6 *If R and S are $*$-regular rings, R a $*$-subring of S then $L(R)$ is a sub-ortholattice of $L(S)$.*

Proof. R embeds into $\mathsf{End}S_S$ via $r \mapsto \hat{r}$ where $\hat{r}(x) = rx$ for $x \in S$. By Prop.5 qq this yields an embedding of $\mathbb{L}(R)$ into $\mathbb{L}(S)$ with $eR \mapsto \mathsf{Im}\hat{e} = eS$ for $e \in L(R)$. Since $e' = 1 - e$ in both OLs, we have $L(R)$ a sub-ortholattice of $L(S)$. $\square$

Corollary 7 *For any $*$-regular ring S,*

$$\mathsf{V}L(S) = \mathsf{V}\{L(R) \mid R \textit{ at most countable, regular } *\textit{-subring of } S\}$$

Proof. '$\supseteq$' follows from Cor.6. Conversely, $L(S)$ belongs to the variety generated by its finitely generated sub-ortholattices L. Endow S with a unary operation q such that $a\mathsf{q}(a)a$ for all a in S. Now, for any such L there is an at most countable $*$-subring R of S containing L and also closed under the operation q. Observe that for $e, f \in L(R)$ one has $e \leq f$ if and only if $ef = e$, i.e. $e \leq f$ in $L(S)$. Thus L is also a sublattice of $L(R)$: assume we have join $e \vee f = g$ in L and $h \in L(R)$ with $h \geq e$, f in $L(R)$. Then $h \geq g$ in $L(S)$ whence $h \geq g$ which means $e \vee f = g$ also in $L(R)$. Similarly for meets. Also, since L is closed under the orthocomplement $e \mapsto 1 - e$ in $L(S)$, the same is true in $L(R)$. It follows, that L is a sub-ortholattice of $L(R)$. □

Let V be a unitary space. Denote by ϕ^* the adjoint of ϕ - if it exists. A *unitary representation* of a $*$-ring R is a ring embedding $\iota : R \to \mathsf{End}(V)$ such that $\iota(r^*) = \iota(r)^*$ for any $r \in R$.

Corollary 8 *If $\iota : R \to \mathsf{End}(V)$ is a unitary representation of the $*$-regular ring R, then*

$$\varepsilon(eR) = \mathsf{Im}\iota(e) \quad e \in L(R)$$

is a unitary representation of the MOL $\mathbb{L}(R)$ in V.

Proof. The lattice embedding follows from Prop.5. Now, observe that

$$\varepsilon(eR)^{\perp} = \mathsf{Im}(\mathsf{id} - \iota(e)) = \varepsilon((1 - e)R) = \varepsilon(eR)')$$

since e and $\iota(e)$ are selfadjoint idempotents. □

4 Finite von Neumann-algebras

A *von Neumann algebra* (cf. [19]) $\mathbf{M}$ is an unital involutive $\mathbb{C}$-subalgebra of the algebra $\mathcal{B}(H)$ of all bounded operators of a separable Hilbert space H with $\mathbf{M} = \mathbf{M}''$ where $\mathbf{A}' = \{\phi \in \mathcal{B}(H) \mid \phi\psi = \psi\phi \;\; \forall \phi \in \mathbf{A}\}$ is the *commutant* of $\mathbf{A}$. $\mathbf{M}$ is *finite* if $rr^* = 1$ implies $r^*r = 1$. For such, the *projections* of $\mathbf{M}$, i.e. the $e = e^2 = e^*$, form a (continuous) modular ortholattice $L(\mathbf{M})$. Here, the order is given by $e \leq f \Leftrightarrow e = ef$ and one has $e' = 1 - e$. A finite von Neumann-algebra is a *factor* if its center is $\mathbb{C} \cdot 1$. Particular examples of a finite factors are the algebras $\mathbb{C}^{n\times n}$ of all complex n-by-n-matrices.

Theorem 9 *Any finite von Neumann-algebra factor is either isomorphic to $\mathbb{C}^{n\times n}$ for some $n < \infty$* (type I_n) *or contain for any $n < \infty$ a subalgebra isomorphic to $\mathbb{C}^{n\times n}$* (type II_1).

Proof. [23, 14.1] and [24, Thm. XIII]. □

For any operator ϕ defined on some linear subspace of H, write $\phi\eta\mathbf{M}$ if $\psi\phi\psi^{-1} = \phi$ for all unitary $\psi \in \mathbf{M}'$ (cf [23, Def.4.2.1]). Let $U(\mathbf{M})$ consist of all closed linear operators $\phi\eta\mathbf{M}$ having dense linear domain with operations given as as the closures of the algebraic operations, e.g.

$$(\phi, \psi) \mapsto [\phi + \psi]$$

where $[\chi]$ denotes the closure of the linear operator χ.

Theorem 10 *For every finite factor $\mathbf{M}$, $U(\mathbf{M})$ is a $*$-regular ring having $\mathbf{M}$ as $*$-subring and such that ϕ^* is adjoint to ϕ. Moreover, $\mathbf{M}$ and $U(\mathbf{M})$ have the same projections.*

Proof. This is trivial for type I_n. For II_1 factors this is [23, Thm. XV] together with [25, Part II, Ch.II, App 2.(VI)] and [26, p.191] for $*$-regularity. Now, consider $\pi : D \to H$ in $U(\mathbf{M})$ such that $\pi = \pi^* = \pi^2$. Then $U = \mathsf{Im}\pi \subseteq D$ so π is a projection of D, i.e. $D = U \oplus^\perp V$. By density of D it follows $U^{\perp\perp} \oplus^\perp V^{\perp\perp} = H$ and π extends to a projection $\hat{\pi}$ of H onto $U^{\perp\perp}$. From $\pi\eta\mathbf{M}$ it follows $\hat{\pi}\eta\mathbf{M}$, whence $\hat{\pi} \in U(\mathbf{M})$ and $\pi = \hat{\pi} \in \mathbf{M}$ by [23] Lemmas 16.4.2 and 4.2.1. □

An important concept in the Murray-von-Neumann construction is that of an *essentially dense* linear subspace X of H (w.r.t. $\mathbf{M}$). Here, we need only the following properties:

1. Essentially dense X is dense in H [23, Lemma 16.2.1].
2. The domains of members of $U(\mathbf{M})$ are essentially dense [23, Lemma 16.4.3].
3. For any $\phi \in U(\mathbf{M})$ and essentially dense X, the preimage $\phi^{-1}(X)$ is essentially dense [23, Lemma 16.2.3].
4. Any finite or countable intersection of essentially dense X_n is essentially dense [23, Lemma 16.2.2].

Theorem 11 *Any countable $*$-subring of $U(\mathbf{M})$ is representable.*

Proof. Consider any countable $*$-subring R of $U(\mathbf{M})$. A representation of R is constructed from the given Hilbert space H. Let H_0 be the intersection of all domains of operators $\phi \in R$. By (2), H_0 is essentially dense. Define, recursively, H_{n+1} as the intersection of H_n and all preimages $\phi^{-1}(H_n)$ where $\phi \in R$. By (3) and (4), H_{n+1} is essentially dense. By (4), the intersection $H_\omega = \bigcap_{n<\omega} H_n$ is essentially dense and, by (1), dense in H. By construction, H_ω is invariant under R.

Now, for $\phi \in R$ define $\varepsilon(\phi) = \phi|H_\omega$. Then $\varepsilon : R \to \mathsf{End}_{\mathbb{C}}(H_\omega)$ is a $*$-ring homomorphism. Indeed, e.g. $[\phi + \psi]|H_\omega$ is an extension of $\phi|H_\omega + \psi|H_\omega$ and equality holds since both are maps with the same domain. Also $\varepsilon(\phi^*)$ is the restriction of the adjoint ϕ^* in H, whence the adjoint in H_ω. If $\varepsilon(\phi) = 0$, then H_ω is contained in the closed subspace $\ker\phi$ and it follows $\phi = 0$ by density. Thus, ε is a representation. □

5 Equational theory of projection lattices

Theorem 12 *For any class $\mathcal{M}$ of finite von Neumann algebra factors, let $\mathcal{V} = \mathsf{V}\{L(\mathbf{M}) \mid \mathbf{M} \in \mathcal{M}\}$. $\mathcal{V} = \mathsf{V}L(\mathbb{C}^n)$ if and only if $\mathcal{V}$ satisfies the $n+1$-distributive law but not the n-distributive law. $\mathcal{V} = \mathcal{N}$ if and only if $\mathcal{V}$ satisfies no n-distributive law. In any case, the equational theory of $\mathcal{V}$ is decidable.*

Proof. Let $\mathbf{M}$ be a finite von Neumann-algebra factor. In view of Thm.10 and Cor.7, we have to consider countable regular $*$-subrings R of $U(\mathbf{M})$. By Thm.11, each such R is representable. By Cor.8 and Cor.3 we have $L(R) \in \mathcal{N}$ and it follows $L(\mathbf{M}) \in \mathcal{N}$.

By Lemma 1, Cor.6, and Thm.9, $\mathcal{M}$ contains factors of arbitrarily large finite dimensions or a type II_1 factor if and only if $\mathcal{V}$ is n-distributive for no n. In this case, $\mathcal{V} = \mathcal{N}$. Otherwise, there is a maximal n such that $\mathcal{V}$ is n-distributive, in particular all members of $\mathcal{M}$ are of the form $\mathbb{C}^{k\times k}$ with $k \leq n$ and $k = n$ occurs, so $\mathcal{V} = \mathsf{V}L(\mathbb{C}^{n\times n})$.

Recall that $\mathbb{C}$ whence $\mathbb{C}^{n\times n}$ and $L(\mathbb{C}^{n\times n})$ have a decidable first order theory - this is due to Tarski, actually, the method of quantifier elimination applies cf. [22]. This settles the case of $\mathcal{V} = \mathsf{V}L(\mathbb{C}^{n\times n})$. To decide whether an identity $t = 0$ holds in $\mathcal{N}$, by Lemma 2 it suffices to decide validity in $L(\mathbb{C}^{m\times m})$, m the number of occurences of variables in t. $\square$

6 Von Neumann frames

Let $n \geq 3$ fixed. An *n-frame*, in the sense of von Neumann [25], in a lattice L is a list $\overline{a}$: $a_i, a_{ij}, 1 \leq i,j \leq n, i \neq j$ of elements of L such that for any 3 distinct j,k,l

$$a_j \sum_{i\neq j} a_i = 0 = a_j a_{jk}, \quad \sum_i a_i = 1$$

$$a_j + a_{jk} = a_j + a_k, \quad a_{jl} = a_{lj} = (a_j + a_l)(a_{jk} + a_{kl}).$$

If L can be embedded into the subgroup lattice of some abelian group A or if $n \geq 4$ then

$$R(L,\overline{a}) = \{r \in L \,|\, ra_2 = 0, \ \ r + a_2 = a_1 + a_2\}$$

can be turned into a ring (a subring of $\mathsf{End}(a_0)$), the *coordinate ring* - see [25, 21, 9] for a systematic study, [18] for a short exposition. In particular, for each ring term $t(\overline{x})$ there is a lattice polynomial $\hat{t}(\overline{a},\overline{x})$ such that $\hat{t}(\overline{a},\overline{r}) = t(\overline{r})$ for all substitutions $\overline{r}$ in $R(L,\overline{a})$. This ring is regular if L is complemented. If $R^{n\times n}$ is the $n\times n$-matrix ring of some regular ring R with unit and $L = \mathbb{L}(R^{n\times n})$ with the canonical n-frame $\overline{a}$ then $R(L,\overline{a}) \cong R$ - here $\overline{a}$ consists of the $E_{ii}R^{n\times n}$ and $(E_{ii} - E_{ij})R^{n\times n}$ where the E_{ij} form the canonical basis of the R-module $R^{n\times n}$.

7 Quasivarieties and word problems

A *quasi-identity* is an universally quantified implication the premise of which is a conjunction of identities and the conclusion an identity. A *quasivariety* is an axiomatic class closed under substructures and direct products. A solution of the *uniform word problem* for a quasivariety consists in a decision procedure for quasi-identities (i.e. a solution for all finite presentations). For a class $\mathcal{C}$ of algebraic structures let $\mathsf{Th}_q\mathcal{C}$ denote the set of all quasi-identities valid in $\mathcal{C}$.
Let $\mathcal{S}$ ($\mathcal{S}_{fin}$) denote the class of all (finite) semigroups, and $\mathcal{S}_p$ the set of semigroups $F_p^{n\times n}$ ($n \geq 1$) where F_p is the prime field of characteristic p, prime or 0. Let $\mathcal{M}$ denote the class of all modular lattices, $\mathcal{M}_p$ the set of lattices $\mathbb{L}(F_p^n)$ ($n \geq 1$) . For a class $\mathcal{C}$ of structures denote by $\mathsf{R}_S\mathcal{C}$ and $\mathsf{R}_L\mathcal{C}$ the class of all semigroup resp. lattice reducts of structures in $\mathcal{C}$.

Theorem 13 *A quasivariety $\mathcal{Q}$ has unsolvable uniform word problem if $\mathcal{S}_p \subseteq \mathsf{SR}_S\mathcal{Q} \subseteq \mathcal{S}$ or $\mathcal{M}_p \subseteq \mathsf{SR}_L\mathcal{Q} \subseteq \mathcal{M}$ for some p.*

Proof. Given a finite semigroup S, one may consider the semigroup ring $F_p[S]$ as a F_p-vector space V and thus embed S into $\mathsf{End}_{F_p}(V) \cong F_p^{n\times n}$ where $n = |S|$. It follows $\mathsf{Th}_q\mathcal{S}_p \subseteq \mathsf{Th}_q\mathcal{S}_{fin}$ for all p and equality for $p > 0$. Since $\mathbb{Q}^{n\times n} \in \mathsf{SP}_u\{F_p^{n\times n} \mid p \text{ prime}\}$, one has

$$\mathsf{Th}_q\mathcal{S}_p = \mathsf{Th}_q\mathcal{S}_{fin} \quad \text{for all } p.$$

The claim in the semigroup case follows from the result of Gurevich and Lewis [13] that there is no recursive Γ such that $\mathsf{Th}_q\mathcal{S} \subseteq \Gamma \subseteq \mathsf{Th}_q\mathcal{S}_{fin}$.
One may associate with each quasi-identity ϕ as above in the semigroup language a quasi-identity $\hat{\phi}$ in the lattice language

$$\forall\overline{a}\forall\overline{x}\ \alpha(\overline{a}) \wedge \bigwedge_i (x_i a_a = 0 \wedge x_i + a_2 = a_1 + a_2)$$

$$\wedge \bigwedge_j \hat{s}_j(\overline{a},\overline{x}) = \hat{t}_j(\overline{a},\overline{x}) \Rightarrow \hat{s}_(\overline{a},\overline{x}) = \hat{t}(\overline{a},\overline{x})$$

where $\alpha(\overline{a})$ states that $\overline{a}$ is a 4-frame. In particular, if $\phi \in \mathsf{Th}_q\mathcal{S}$ then $\hat{\phi} \in \mathsf{Th}_q\mathcal{M}$ and if $\hat{\phi}$ holds in $\mathbb{L}(R^4)$, substituting the canoncial 4-frame for $\overline{a}$, then ϕ holds in R. Considering $R = F_p^{n\times n}$ it follows $\phi \in \mathsf{Th}_q\mathcal{S}_p$ if $\hat{\phi} \in \mathcal{M}_p$.
Now, given $\mathsf{Th}_q\mathcal{M} \subseteq \Delta \subseteq \mathsf{Th}_q\mathcal{M}_p$ define Γ as the set of those quasi-identities ϕ in semigroup language with $\hat{\phi} \in \Delta$. Then

$$\mathsf{Th}_q\mathcal{S} \subseteq \Gamma \subseteq \mathsf{Th}_q\mathcal{S}_p$$

and if Δ is recursive then so is Γ. $\square$

Corollary 14 *$\mathcal{N}$ has an undecidable uniform word problem. The quasivariety $\mathcal{Q}$ generated by all ortholattices $\mathbb{L}(\mathbb{C}^{n\times n})$ ($n < \omega$) has an undecidable uniform word problem and is not a variety.*

Proof. The undecidability claim is immediate by Thm.13. By decidability of the $\mathbb{L}(\mathbb{C}^{n\times n})$, the complement of $\mathsf{Th}_q\mathcal{Q}$ within the set of quasi-identities is recursively enumerable. If $\mathcal{Q}$ were a variety, then by Thm.12 it would coincide with $\mathcal{N}$ and be recursively axiomatizable. Thus $\mathsf{Th}_q\mathcal{Q}$ would be recursivley enumerable, too, and this would imply solvability of the uniform word problem. $\square$
For the class of all modular ortholattices a finite presentation with unsolvable word problem has been given by M.S. Roddy [29]

8 Further developments

The equational theory of $L(\mathbb{C}^{n\times n})$ with a fixed frame of reference is bi-interpretable (using [5]) in the equational theory of the *-field $\mathbb{C}$ which in turn, fixing the imaginary unit, is bi-interpretable in the equational theory for $\mathbb{R}$. So Ono's axiomatization [46] can be applied, giving a easy explicit axiomatization for the equational theory of $L(\mathbb{C}^{n\times n})$ with a fixed frame, and also its non-finite axiomatizability (since there are fields whose failure of formally reality requires an arbitrarilly large number of sums of squares, see [43]). But then also the equational theory of $L(\mathbb{C}^{n\times n})$ without a fixed frame cannot be finitely axiomatixable (since the defining axioms for a frame, and for the imaginary unit and the involution in $\mathbb{C}$, are finite in number).
Further extensions are easily obtained (see [10] and references therein for what follows, and more). For varieties generated by sets of projection ortholattices of (non factorial, non separable, real or complex) von Neumann algebras, or even JBW algebras (see [40]), one has essentially

the same theorem.[1] Note that in [26] von Neumann abandoned complex scalars (and elsewhere he even considered Jordan structures) in order to have a equivalence between suitable orto-lattices (the propositional logic) and suitable "rings of operators"; infact the theorem has an equivalent in ring rather than lattice language. In the general classificaton theorem, one has four kinds of varieties: bounded (i.e. n-distributive) versus unbounded; orthoarguesian (i.e. all generators come from special Jordan algebras) versus exceptional. The bounded varieties are all modular and have decidable equational theory; a set S of JBW algebras generates a n-distributive variety iff each element of S has type I_n at most. The bounded varieties are no more a chain (as it happens for complex von Neumann algebras): the new generators are the projective orthogeometries over the real and quaterinon number systems, and the elliptic plane over the octonions (the exceptional factor, corresponding to the Albert algebra). There are exactly four unbounded varieties: the smallest is the modular orthoarguesian one, the largest is the non-modular non-orthoarguesian one; in between on has a non comparable pair (modular non-othoarguesian; non-modular orthoarguesian). The unbounded modular orthoarguesian variety is generated by any infinite set of non-isomorphic finite dimensional factors (except the exceptional one), or more generally by any set of finite algebras provided that either there is at least one nonzero type II component or there is no bound on the n of the nonzero I_n components. Adding the exceptional factor (or any finite algebra with nonzero exceptional component) gives the modular non-orthorguesian variety. All modular varieties have decidable equational theory. The two nonmodular varieties are generated by sets where at least one generator has a non-zero infinite component; one has the orthoarguesian variety iff all generators have zero exceptional component. Concerning decidability, one surely has decidability for the lattice equations (since the free lattice on three generators is a sublattice of the lattice of closed subspaces of any infinite dimensional Hilbert space)[2] But for ortholattice equations, their set is still quite misterious: see [44] and references therein. However, the lattice result is sufficient for an interesting corollary: the projection lattice $L(\mathbf{M})$ of any orthoseparable (hence simple) type III factor $\mathbf{M}$ generates the non-modular orthoarguesian variety (since it has the type I_∞ factor as subfactor), hence it is a fractal lattice (even fractal as ortholattice since all corners $e\mathbf{M}e$ are $*$-isomorphic to $\mathbf{M}$ by dimension theory) which generates the variety of all lattices. This answers a question of Czédli [36].

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[1] The theorem also extends to real or complex Rickart C^*-algebras, but a new proof for the decisive case of finite AW* factors is needed.

[2] G. Kalmbach (who previously proved the weaker theorem that "general" orthomodular lattices do not satisfy any special lattice equation) cites this result in In Discrete Math. 53, 125-135. She wrote (p. 129): "M. M. Denneau observed that the equational theory of the lattice $C(H)$ of closed subspaces of an infinite dimensional Hilbert space H is decidable and is precisely the set of identities valid in the class of all lattices. Moreover the free lattice on countably many generators can be embedded in $C(H)$." This result is possibly unpublished, but it should be contained in M. M. Denneau's Ph. D. thesis. Citing from `http://www.math.uiuc.edu/GraduateProgram/phd_defense.html` "Denneau, Monty M. (Thesis Area: Logic) Thesis Title: On the Decidability of the Identities Valid in the Lattice of Closed Subspaces of an Infinite Dimensional Hilbert Space. Thesis Director: Takeuti, Gaisi. Final exam: 7/31/1978; Ph. D. 10/15/1978".

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