Sparsity of Integer Formulations for Binary Programs

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June 2019

Abstract

This paper considers integer formulations of binary sets X of minimum sparsity, i.e., the maximal number of non-zeros for each row of the corresponding constraint matrix is minimized. Providing a constructive mechanism for computing the minimum sparsity, we derive sparsest integer formulations of several combinatorial problems, including the traveling salesman problem. We also show that sparsest formulations are $\mathcal{NP}$-hard to separate, while (under mild assumptions) there exists a dense formulation of X separable in polynomial time.

1 Introduction

Given a non-empty set $X \subseteq \{0,1\}^n$, we call a linear system $Ax \leq b$, with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, an *integer formulation* of X if $X = \{x \in \mathbb{Z}^n : Ax \leq b\}$. In the following, we investigate the sparsity of such formulations. Sparsity is a desirable property of integer formulations, since it often allows optimization algorithms to perform faster in comparison with dense formulations, see, e.g., Suhl and Suhl [11], Yen et al. [12] or McCormick [9].

An inequality $a^\top x \leq \beta$, where $a \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$, is called *s-sparse* if at most s entries of a are non-zero. Thus, the sparser an inequality the smaller s. Using the notation $[n] \coloneqq \{1,\dots,n\}$, we define the *sparsity function* σ for a vector $a \in \mathbb{R}^n$ by $\sigma(a) \coloneqq |\{a_i \neq 0 : i \in [n]\}|$. By overloading notation, we define for a matrix A the sparsity $\sigma(A) \coloneqq \max\{\sigma(a) : a \text{ is a row of } A\}$, and finally, the *minimum sparsity* of a set X is

$$\sigma(X) \coloneqq \min_{(A,b)}\{\sigma(A) : Ax \leq b \text{ integer formulation of } X\}.$$

In particular, the minimum sparsity of X is s, if X admits an integer formulation by s-sparse inequalities but not by $(s-1)$-sparse inequalities.

The main result of this article is an exact combinatorial characterization of the minimum sparsity of integer formulations of X.

In the literature, sparsity was discussed, e.g., by Dey et al. [3], who studied the approximation of polyhedra by sparse cutting planes, and the same authors also investigated the impact of sparse cutting planes for sparse integer programs, see [4]. Among others, the approximation of polyhedra using a fixed number of dense inequalities and arbitrarily many sparse ones is investigated by Dey et al. [2] and Iroume [6].

Assumptions Throughout this article, $X \subsetneq \{0,1\}^n$ denotes a non-empty set. Moreover, we use the notation $\overline{X} \coloneqq \{0,1\}^n \setminus X$ to denote the complement of X within $\{0,1\}^n$.

2 Lower Bounds on Minimum Sparsity

To derive lower bounds on the number of non-zero coefficients needed in an integer formulation of a given set X, let $\hat{x} \in \{0,1\}^n$ and let $N(\hat{x})$ be the neighbors of $\hat{x}$ in the 0/1-cube, i.e.,

$$N(\hat{x}) \coloneqq \{x \in \{0,1\}^n : \|x - \hat{x}\|_1 = 1\},$$

where $\|\cdot\|_1$ denotes the 1-norm. That is, the neighbors of $\hat{x}$ are those 0/1-points that differ from $\hat{x}$ in exactly one coordinate.

Lemma 1. *Let $\bar{x} \in \overline{X}$. Then $\sigma(X) \geq |N(\bar{x}) \cap X|$.*

Proof. Define $s = |N(\bar{x}) \cap X|$ and assume that $s > 1$, since otherwise the statement is trivial. For the sake of contradiction assume $X = \{x \in \mathbb{Z}^n : Ax \leq b\}$, where each row of A has at most $s-1$ non-zeros. Since $\bar{x} \notin X$, there exists an inequality $a^\top x \leq \beta$ in $Ax \leq b$ with $a^\top \bar{x} > \beta$ and $a^\top \hat{x} \leq \beta$ for all $\hat{x} \in N(\bar{x}) \cap X$. Assume w.l.o.g. that $a_1 = a_2 = \cdots = a_{n-s+1} = 0$. Thus, the remaining $s-1$ entries $a_{n-s+2}, \ldots, a_n$ might have non-zero coefficients. Since $|N(\bar{x}) \cap X| = s$, there exists $\hat{x} \in N(\bar{x}) \cap X$ such that $\bar{x} = \hat{x}$ except $\bar{x}_i \neq \hat{x}_i$ for some $i \in [n-s+1]$. But then $\beta < a^\top \bar{x} = a^\top \hat{x} \leq \beta$, a contradiction. □

We demonstrate some applications of Lemma 1.

Example 2. Let X_e and X_o be the sets of binary points that have an even and odd number of 1-entries, respectively. Each point in X_e has its n neighbors in X_o and vice versa. Hence, Lemma 1 implies that both X_e and X_o need completely dense inequalities in any integer formulation.

The next example will also show that the sparsity bound derived in Lemma 1 can be tight for every sparsity level.

Example 3. Let $X_s \coloneqq \{x \in \{0,1\}^n : \mathbb{1}^\top x \leq s\}$, where $s \in [n-1]$. The set X_s is the feasible region of a 0/1-knapsack problem, and we claim that $\sigma(X_s) = s + 1$.

Let $\bar{x} \in \{0,1\}^n$ such that $\mathbb{1}^\top \bar{x} = s + 1$. Then $\bar{x} \notin X_s$ and there are $s+1$ neighbors of $\bar{x}$ in X_s. Hence, Lemma 1 implies that X_s cannot be represented by an integer formulation that consists of s-sparse inequalities only. But X_s admits an $(s+1)$-sparse integer formulation via box constraints and

$$x(I) \leq s, \qquad I \subseteq [n],\ |I| = s + 1,$$

where $x(I)$ abbreviates $\sum_{i \in I} x_i$.

3 Characterization of Minimum Sparsity

Lemma 1 allows to derive bounds on the sparsity of any integer formulation of X by a simple neighborhood argument. But this bound is not always tight. To be able to compute the minimum sparsity of an integer formulation, we introduce the concept of infeasible face coverings.

Definition 4. A face F of $[0,1]^n$ is called *infeasible w.r.t.* $X \subseteq \{0,1\}^n$ if no integer point in F is contained in X and is called *maximally infeasible* if it is infeasible and there does not exist an infeasible face F' of $[0,1]^n$ with $F \subsetneq F'$.

A collection $\mathcal{F}$ of infeasible faces of $[0,1]^n$ w.r.t. X is called an *infeasible face collection* of X. If $\mathcal{F}$ is an infeasible face collection and for each $x \in \overline{X}$ there exists $F \in \mathcal{F}$ such that $x \in F$, then $\mathcal{F}$ is called an *infeasible face covering* of X. An infeasible face covering $\mathcal{F}$ of X is called *maximal* if for each face $F \in \mathcal{F}$ there is no infeasible face F' of $[0,1]^n$ with $F \subsetneq F'$ and $F' \notin \mathcal{F}$. An infeasible face covering $\mathcal{F}$ is called *irredundant* if for each $F \in \mathcal{F}$ there is no $F' \in \mathcal{F}$ with $F \subsetneq F'$.

Lemma 5. *Every* $X \subseteq \{0,1\}^n$ *has a unique maximal irredundant infeasible face covering. It consists of all maximally infeasible faces of* $[0,1]^n$ *w.r.t.* X.

Proof. Let $\mathcal{F}$ be the collection of all infeasible faces of $[0,1]^n$ w.r.t. X. The collection $\mathcal{F}$ turns into a poset if we order the faces in $\mathcal{F}$ w.r.t. inclusion. Obviously, the $\subseteq$-maximal elements in $\mathcal{F}$ form a maximal irredundant infeasible face covering $\hat{\mathcal{F}}$ of X.

To show uniqueness, assume there exists a maximal irredundant infeasible face covering $\mathcal{F}' \neq \hat{\mathcal{F}}$. Since $\mathcal{F}'$ is maximal and covers $\overline{X}$, $\mathcal{F}'$ contains every maximally infeasible face of $[0,1]^n$. Thus, $\hat{\mathcal{F}} \subseteq \mathcal{F}'$. If $\hat{\mathcal{F}}$ was a proper subset of $\mathcal{F}'$, there would exist $F' \in \mathcal{F}' \setminus \hat{\mathcal{F}}$. Since $F' \notin \hat{\mathcal{F}}$, F' cannot be a maximally infeasible face. Hence, there exists a maximally infeasible face $\hat{F}$ of $[0,1]^n$ with $F' \subsetneq \hat{F}$. By definition of $\hat{\mathcal{F}}$, we find $\hat{F} \in \hat{\mathcal{F}}$. Consequently, $F' \subsetneq \hat{F} \in \hat{\mathcal{F}} \subseteq \mathcal{F}'$ showing that $\mathcal{F}'$ cannot be irredundant, a contradiction. □

We call the smallest dimension of a face in the unique maximal irredundant infeasible face covering of X the *maximal irredundant covering bound* of X, abbreviated as $\operatorname{micb}(X)$. By Lemma 5, $\operatorname{micb}(X)$ is the smallest dimension of a maximally infeasible face of $[0,1]^n$. In the following, we show that $\operatorname{micb}(X)$ completely characterizes the sparsity of any integer formulation of X.

If F and F' are faces of $[0,1]^n$, we denote by $\operatorname{sf}(F, F')$ the smallest face of $[0,1]^n$ that contains both F and F'. If F' consists of a single vertex x only, we write $\operatorname{sf}(F, x)$. Note that $\operatorname{sf}(F, F')$ can also be written as $F \vee F'$, the *join* of these two faces in the face lattice of $[0,1]^n$.

To show the main result characterizing the minimum sparsity in Theorem 8, we first need some technical lemmata.

Lemma 6. *Let $\bar{x} \in \{0,1\}^n$ and let F be a face of $[0,1]^n$. Moreover, let $a^\top x \leq \beta$ be an inequality valid for $[0,1]^n$ with $a^\top \bar{x} = \beta$. If all binary points in $\{\bar{x}\} \cup F$ violate the inequality $a^\top x \leq \beta'$, $\beta' \in \mathbb{R}$, then all binary points in $\operatorname{sf}(F, \bar{x})$ violate $a^\top x \leq \beta'$.*

Proof. Since flipping/permuting coordinates has no impact on the sparsity of an inequality, we may assume w.l.o.g. that $\bar{x} = \mathbb{1}$. The proof proceeds via induction on n. If $n = 1$, the statement is trivially fulfilled. In the inductive step, consider all facets $F_i = \{x \in [0,1]^n : x_i = 1\}$, $i \in [n]$, of $[0,1]^n$ that contain $\bar{x}$. If F is a subset of any of these facets, the induction hypothesis can be used to show the statement, since each facet is an $(n-1)$-dimensional cube. Otherwise, for every F_i, the face F contains a vertex of $[0,1]^n$ that is not contained in F_i. For this reason, there exists $S \subseteq [n]$ such that $F = \{x \in [0,1]^n : x_i = 0,\ i \in S\}$, where $S = \varnothing$ is possible. Hence, $0 \in F$.

Since $a^\top \bar{x} = \beta$ by assumption, $a^\top \bar{x} = a^\top \mathbb{1} = \beta$. Thus, $a_i \geq 0$ holds for all $i \in [n]$, because $a^\top x \leq \beta$ is valid for $[0,1]^n$. This means that $a^\top y \leq a^\top x' \leq a^\top \bar{x}$ for all $x' \in \{0,1\}^n$. Since $y \in F$ and F violates $a^\top x \leq \beta'$ by assumption, the statement follows by $\beta' < a^\top y \leq a^\top x'$ for all $x' \in \{0,1\}^n$. □

Now we are able to connect sparsity and faces of $[0,1]^n$.

Lemma 7. *Let F be a face of $[0,1]^n$ and let $a^\top x \leq \beta$ be an inequality with $a^\top \bar{x} > \beta$ for all $\bar{x} \in F$. If every face F' of $[0,1]^n$ with $F \subsetneq F'$ contains a point that satisfies $a^\top x \leq \beta$, then the vector a has at least $n - \dim(F)$ non-zero entries. Furthermore, there is an inequality $\bar{a}^\top x \leq \bar{\beta}$ such that the binary points that violate this inequality are exactly the binary points in F and such that $\bar{a}$ has exactly $n - \dim(F)$ non-zero entries.*

Proof. Let F be an $(n-k)$-dimensional face of $[0,1]^n$ and suppose every face F' of $[0,1]^n$ with $F \subsetneq F'$ contains a point x' that satisfies $a^\top x' \leq \beta$. By convexity of $[0,1]^n$, there also exists $x' \in F' \cap \{0,1\}^n$ with $a^\top x' \leq \beta$. W.l.o.g. we may assume $F = \{x \in [0,1]^n : x_i = v_i,\ i \in [k]\}$ for some $v \in \{0,1\}^k$.

Consider the faces $F_j \supseteq F$ of $[0,1]^n$, $j \in [k]$, that are defined by dropping constraint $x_j = v_j$ from the definition of F. By assumption, each F_j contains a binary point $y^j \notin F$ satisfying $a^\top y^j \leq \beta$. By construction, $y^j_j = 1 - v_j$ holds. Let $z^j \in \{0,1\}^n$ be the point that coincides with y^j except in coordinate j. Then, $z^j \in F$. Since, for all $j \in [k]$, $a^\top y^j \leq \beta$ and $a^\top z^j > \beta$, we conclude that $a_j \neq 0$ for every $j \in [k]$. Hence, $\sigma(a) \geq k = n - \dim(F)$.

To prove the second part of the lemma, let $\bar{a} \in \mathbb{R}^n$ be defined by

$$\bar{a}_i = \begin{cases} 1, & \text{if } i \in [k] \text{ and } v_i = 1, \\ -1, & \text{if } i \in [k] \text{ and } v_i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

for every $i \in [n]$, and define

$$\bar{\beta} = \max\{\bar{a}^\top x : x \in [0,1]^n\} \text{ and}$$
$$\beta' = \max\{\bar{a}^\top x : x \in \{0,1\}^n \setminus F\}.$$

Then, $\operatorname{argmax}\{\bar{a}^\top x : x \in [0,1]^n\} = F$ and thus $\bar{\beta} > \beta'$. Consequently, $\bar{a}^\top x \leq \beta'$ is an inequality which is violated by exactly those binary points that are contained in F. Moreover, $\bar{a}$ has exactly $k = n - \dim(F)$ non-zero entries. □

This allows us to completely characterize the minimum sparsity of an integer formulation of X.

Theorem 8. *For a non-trivial set $X \subseteq \{0,1\}^n$ it holds that $\sigma(X) = n - \mathrm{micb}(X)$.*

Proof. To prove the assertion, we first show that each integer formulation of X contains an inequality with at least $n - \mathrm{micb}(X)$ non-zero entries. Afterwards, we construct an integer formulation all of whose inequalities are $(n - \mathrm{micb}(X))$-sparse to prove that this bound is tight.

To show the first part, let $a^\top x \leq \beta$ be an inequality valid for $\mathrm{conv}(X)$ that cuts off at least one point $\bar{x} \in \overline{X}$ and denote the set of all such a by $\mathcal{A}$. Consider the face

$$F_a \coloneqq \{x \in [0,1]^n \,:\, x_i = 1 \text{ if } a_i > 0 \text{ and } x_i = 0 \text{ if } a_i < 0,\ i \in [n]\}$$

of $[0,1]^n$, which coincides with $\operatorname{argmax}\{a^\top x \,:\, x \in [0,1]^n\}$. Note that $\sigma(a) = n - \dim(F_a)$.

Since $a^\top x \leq \beta$ separates $\bar{x}$ from $\mathrm{conv}(X)$, it separates at least one point in F_a. Hence, all points in F_a are cut off. Moreover, let F be a maximal face of $[0,1]^n$ that is completely cut off by $a^\top x \leq \beta$. On the one hand, if $F_a \subseteq F$, then

$$\dim(F) \geq \dim(F_a). \tag{1}$$

On the other hand, if $F_a \not\subseteq F$, then $a^\top x \leq \beta$ cuts off $\mathrm{sf}(F, F_a)$ by applying Lemma 6 to all points of F_a. Thus, all points in $\mathrm{sf}(F, F_a)$ have to be infeasible, which is a contradiction to the maximality of F. Consequently, all maximal infeasible faces of $[0,1]^n$ that are cut off by $a^\top x \leq \beta$ contain F_a and fulfill (1).

Let $F^\star$ denote a maximal infeasible face of $[0,1]^n$ of minimal dimension. Then,

$$\max_{a \in \mathcal{A}} \sigma(a) = \max_{a \in \mathcal{A}} (n - \dim(F_a)) \overset{(1)}{\geq} \max_{a \in \mathcal{A}} (n - \dim(F^\star)) = n - \mathrm{micb}(X),$$

which proves the proposed lower bound.

To prove the theorem, it suffices to construct an integer formulation all of whose inequalities are $(n - \mathrm{micb}(X))$-sparse. Let $\mathcal{F}$ be the maximal irredundant infeasible face covering of X. To construct the desired integer formulation, we take the box constraints $x \in [0,1]^n$ to guarantee that each feasible point is contained in the hypercube. Note that each such constraint is 1-sparse. Now it suffices to cut off the infeasible faces of $[0,1]^n$ w.r.t. X.

By Lemma 7, each face $F \in \mathcal{F}$ can be cut off by an inequality that has exactly $n - \dim(F)$ non-zero left-hand side coefficients. If we take these inequalities for all faces in $\mathcal{F}$, we have ensured that we cut off each point in $\overline{X}$ since $\mathcal{F}$ is a maximal infeasible face covering. These inequalities are $(n - \operatorname{micb}(X))$-sparse, which proves the assertion. □

Remark 9. *Since the normal cone of any face of $[0,1]^n$ contains a vector in $\{0, \pm 1\}^n$, there always exists a sparsest integer formulation of any non-trivial $X \subseteq \{0,1\}^n$ all of whose left-hand side coefficients are contained in $\{0, \pm 1\}$.*

We now apply Theorem 8 to investigate whether there exist sparse formulations for several combinatorial optimization problems. First, we consider the solution set of the *traveling salesman problem* (TSP), which is to find a weight minimal Hamiltonian cycle in an undirected graph.

Theorem 10. *Let $K_n = (V, E)$ be the complete undirected graph with $n \geq 5$ nodes and let $X \subseteq \{0,1\}^E$ be the set of incidence vectors of Hamiltonian cycles in K_n. It holds that $\sigma(X) = n - 2$.*

Proof. To be able to apply Theorem 8, we have to analyze the binary points that are not contained in X. To this end, we construct for each infeasible binary point x a certificate, i.e., fixings of variables that ensure infeasibility of x, of minimum size. The following situations cover all possible cases of infeasibility: The graph G' induced by edges $e \in E$ with $x_e = 1$

- contains a subgraph isomorphic to $K_{1,3}$,
- contains a connected component that is a path graph,
- is the empty graph,
- contains an induced subgraph which is a cycle of length less than $|V|$.

In the first case, a certificate of infeasibility is given by fixing the three edges of $K_{1,3}$ to 1, because this implies that there is a node in a solution with degree at least 3. In the second and third case, there is a node v of G' whose degree is at most 1. Thus, by fixing $n - 2$ pairwise different edges incident to v to 0, we obtain a certificate of infeasibility of size $n - 2$, since this enforces that v has at most degree 1 in any solution. Finally, in the fourth case, every node has degree 2 and if the solution is infeasible, a subtour contains at most $n - 3$ nodes. Fixing the corresponding $n - 3$ edges to 1 generates a certificate of infeasibility.

Thus, for each infeasible $x \in \{0,1\}^E$, there exists a certificate of infeasibility of size at most $n - 2$. Therefore, each infeasible point is contained in an infeasible face of $[0,1]^n$ of dimension at least $|E| - n + 2$. To be able to apply Theorem 8, we have to show that there is indeed an infeasible binary point for which no certificate of size less than $n-2$ exists. To see this, consider the infeasible point $x = 0$. If at most $n-3$ variables are fixed to 0, a Hamiltonian cycle exists on the remaining edges by Ore's Theorem [10], which guarantees the existence of a Hamiltonian cycle if for every pair of distinct non-adjacent nodes the sum of their degrees is at least the number of nodes. Consequently, a minimum size certificate of infeasibility for the zero vector has size $n-2$, and Theorem 8 implies the assertion. □

Note that the minimum sparsity of the TSP for $n = 4$ is 3, because the certificate for a $K_{1,3}$ subgraph has size 3, which is larger than the bound $n-2$. For $n = 3$, however, Theorem 10 holds, since K_3 does not contain a $K_{1,3}$ subgraph.

The technique used in the proof of Theorem 10 can be used to show that other formulations are as sparse as possible.

Lemma 11. *Let $G = (V, E)$ be an undirected graph and let $\delta(S) \subseteq E$ denote the cut induced by $S \subseteq V$. If the sets*

$$\begin{aligned} X_C &\coloneqq \{x \in \{0,1\}^E : x(\delta(S)) \geq 1, \varnothing \neq S \subsetneq V\}, \\ X_M &\coloneqq \{x \in \{0,1\}^E : x(\delta(\{v\})) = 1, v \in V\}, \\ X_k &\coloneqq \{x \in \{0,1\}^{V \times [k]} : \sum_{i=1}^{k} x_{vi} = 1, v \in V, \\ &\qquad x_{ui} + x_{vi} \leq 1, i \in [k], \{u, v\} \in E\} \end{aligned}$$

of incidence vectors of connected subgraphs, perfect matchings, and k-colorings, respectively, are non-empty, then

$$\sigma(X_C) = \max_{S \subseteq V} |\delta(S)|, \ \sigma(X_M) = \max_{v \in V} |\delta(\{v\})|, \ \sigma(X_k) = k.$$

Proof. Since the above integer formulations are as sparse as stated in the lemma, it suffices to show that no sparser formulations exist. As for the proof of Theorem 10, we will state infeasible points for which no certificate of smaller size exists. Further details will be omitted. For X_C consider the point $x(\delta(S)) = 0$ with some maximal cut $\delta(S)$, for X_M consider $x(\delta(\{v\})) = 0$ with some node $v \in V$ of maximal degree and finally for X_k consider $x_{vi} = 0$ for some node $v \in V$ and all $i \in [k]$. □

Remark 12. *In some cases, sparse integer formulations of X describe the convex hull* conv(X) *completely: Let $G_X = (V_X, E_X)$ be the subgraph of* *the graph of* $[0,1]^n$ *induced by* $V_X = \overline{X}$*. Let* $\mathcal{F}$ *be the maximal irredundant infeasible face covering of* X *and let* $F \in \mathcal{F}$ *with* $k = n - \dim(F)$*. There exists a vector* $v \in \{0,1\}^k$ *and a map* $\sigma\colon [k] \to [n]$ *such that* $F = \{x \in [0,1]^n : x_{\sigma(i)} = v_i,\ i \in [k]\}$*. Let* $\bar{a}_F$ *be the vector defined by*

$$\bar{a}_{F,j} = \begin{cases} 1, & \text{if } j = \sigma(i),\ v_i = 1 \text{ for some } i \in [k], \\ -1, & \text{if } j = \sigma(i),\ v_i = 0 \text{ for some } i \in [k], \\ 0, & \text{otherwise.} \end{cases}$$

for $j \in [n]$*. Provided that each connected component of* G_X *is a cycle of length greater than* 4 *or a path, Cornuéjols and Lee [1] proved that* conv(X) *is completely described by box constraints and inequalities* $\bar{a}_F^\top x \leq \beta_F$*, where* $F \in \mathcal{F}$ *and* β_F *an appropriately chosen scalar.*

Finally, we show that it is hard to compute the minimum sparsity of an integer formulation or to separate inequalities of such formulations.

Theorem 13. *Computing* $\sigma(X)$ *for* $X \subseteq \{0,1\}^n$ *is* $\mathcal{NP}$*-hard, even if* X *corresponds to the independent sets of a graphic matroid.*

Proof. Let $G = (V, E)$ be an undirected simple graph and let $X \subseteq \{0,1\}^E$ correspond to the independent sets of a graphic matroid of G, i.e., X contains the incidence vectors of cycle free edge sets in G. Let $\bar{x} \in \{0,1\}^E \setminus X$, and let F be an infeasible face of $[0,1]^E$. If $\bar{x}$ is the incidence vector of an induced cycle C of G,

$$F_C \coloneqq \operatorname{conv}(\{x \in \{0,1\}^E : x_e = 1,\ e \in E \text{ with } \bar{x}_e = 1\})$$

is a face of $[0,1]^E$ that contains only infeasible binary points including $\bar{x}$. All the faces F_C, where C is an induced cycle of G form the maximal irredundant infeasible face covering of X, since every infeasible point contains an induced cycle. If a point does not contain any induced cycle, it is contained in the incidence vector of a tree and is therefore feasible. Thus, the dimension of each maximal infeasible face of $[0,1]^E$ that contains $\bar{x}$ is at least $|E| - |C|$ for the longest induced cycle C encoded by $\bar{x}$. Hence, computing the minimum sparsity of an integer formulation of X is equivalent to computing the length of a longest induced cycle in G by Theorem 8 and the definition of F_C. Computing the maximum length of an induced cycle is $\mathcal{NP}$-hard, see Garey and Johnson [5, Problem GT23]. □

In general, formulations with minimum sparsity are $\mathcal{NP}$-hard to separate as the following example of a knapsack set shows.

Example 14. Let X be the feasible set of a binary knapsack problem. Similar to the proof of Theorem 13, one can show that the faces

$$F_C \coloneqq \operatorname{conv}(\{x \in \{0,1\}^n \,:\, x_i = 1 \text{ for all } i \in C\}),$$

for every minimal cover C of X, form the maximal irredundant infeasible face covering of X. Since the sparsity of the cover inequality for a minimal cover C equals $n - \dim(F_C)$, an integer formulation with minimum sparsity of X is given by box constraints and all minimal cover inequalities for X. Separating this integer formulation is (weakly) $\mathcal{NP}$-hard, see Klabjan et al. [8].

4 Separation of Dense Formulations

In the previous section, we have seen that sparse formulations are $\mathcal{NP}$-hard to separate in general. By dropping the sparsity requirement, it is possible to find tractable, i.e., polynomial time separable, formulations with $\{0, \pm 1\}$-coefficients on the left-hand side of many 0/1-problems. To see this, we make use of the concept of infeasibility cuts: Given a set $X \subseteq \{0,1\}^n$ and a point $\bar{x} \in \overline{X}$, the *infeasibility* or *no-good cut* w.r.t. $\bar{x}$ is given by

$$\sum_{i\colon \bar{x}_i = 0} x_i + \sum_{i\colon \bar{x}_i = 1} (1 - x_i) \geq 1. \tag{2}$$

Observe that this inequality is completely dense.

The only binary point that violates this inequality is $\bar{x}$. Thus, the box constraints and infeasibility cuts for all infeasible binary points define an integer formulation of X with coefficients in $\{0, \pm 1\}$. Obviously, the tractability of these inequalities depends on the way X is given. For example, if $|\overline{X}|$ is small and can be enumerated explicitly, (2) yields a polynomial size integer formulation. The following theorem gives a sufficient condition on when the formulation by infeasibility cuts is tractable.

Theorem 15. *Consider some 0/1-problem that defines $X^I \subseteq \{0,1\}^{n(I)}$ for each instance I, and let the membership problem for X^I be solvable in polynomial time in n. Then the integer formulation given by (2) for every $\bar{x} \in \{0,1\}^{n(I)} \setminus X^I$ and the bounds $0 \leq x \leq \mathbb{1}$ is tractable in n.*

Proof. Since the box constraints $0 \leq x \leq \mathbb{1}$ can be separated in linear time, it suffices to show that the separation problem for (2) and $x^\star \in [0,1]^n$ can be solved in polynomial time.

Note that the left-hand side of (2) is $\|\bar{x} - x\|_1$. To solve the separation problem of infeasibility cuts for X, we solve the auxiliary problem $\min_{\hat{x} \in \{0,1\}^n} \|\hat{x} - x^\star\|_1$ first. Obviously,

$$\hat{x}_i = \begin{cases} 0, & \text{if } x_i^\star \leq \frac{1}{2}, \\ 1, & \text{otherwise,} \end{cases}$$

is a solution of this problem and it can be computed in linear time.

If $\|\hat{x} - x^\star\|_1 \geq 1$, the point $x^\star$ cannot violate (2) for any $\bar{x} \in \{0,1\}^n$. Hence, $x^\star$ lies inside the integer formulation of X via infeasibility cuts. Otherwise, $\|\hat{x} - x^\star\|_1 < 1$, and we are done if $\hat{x} \in \overline{X}$, because the infeasibility cut for $\hat{x}$ is violated by $x^\star$. Thus, we can assume in the following that $\hat{x} \in X$.

Assume now that $\|\bar{x} - \hat{x}\|_1 \geq 2$. Then plugging $x^\star$ into (2) yields

$$\begin{aligned} &\sum_{i\colon \bar{x}_i = 0} x_i^\star + \sum_{i\colon \bar{x}_i = 1} (1 - x_i^\star) = \|\bar{x} - x^\star\|_1 \\ =&\|(\bar{x} - \hat{x}) + (\hat{x} - x^\star)\|_1 \geq \underbrace{\|\bar{x} - \hat{x}\|_1}_{\geq 2} - \underbrace{\|\hat{x} - x^\star\|_1}_{<1} > 1. \end{aligned}$$

Thus, (2) cannot be violated in this case. It therefore suffices to check at most $n + 1$ points $\bar{x} \in \{0,1\}^n$ with $\|\bar{x} - \hat{x}\|_1 \leq 1$. Thus, we call the membership problem for each of these points and check whether (2) is violated by $x^\star$ if $\bar{x}$ is infeasible. □

Since every $X \subseteq \{0,1\}^n$ admits an integer formulation via infeasibility cuts, Theorem 15 shows that there always exist a completely dense formulation that can be separated efficiently, provided the membership problem is efficiently solvable. Exemplary applications of this result are many combinatorial problems like the traveling salesman, longest path, stable set, or max-cut problem, which are all $\mathcal{NP}$-hard but have a polynomial time solvable membership problem.

5 Sparsification by Additional Variables

Theorem 8 shows that general sets $X \subseteq \{0,1\}^n$ may not admit sparse integer formulations due to the structure of $\overline{X}$. If we allow to introduce additional variables, i.e., we lift the formulation of X to an extended space, we will see below that X always admits a 3-sparse formulation. In fact, this result does not exclusively hold for binary sets but for arbitrary polyhedrally representable sets. A mixed-integer set $X \subseteq \mathbb{R}^n \times \mathbb{Z}^q$ is *polyhedrally representable* if there exists a polyhedron $P \subseteq \mathbb{R}^{n+q}$ with $X = P \cap (\mathbb{R}^n \times \mathbb{Z}^q)$. An inequality description $Ax \leq b$ of P is called a *mixed-integer formulation* of X.

Proposition 16. *Let $X \subseteq \mathbb{R}^n \times \mathbb{Z}^q$ be polyhedrally representable and let $Ax \leq b$ be a mixed-integer formulation of X with m inequalities. Then there exists a mixed-integer formulation of X all of whose inequalities are 3-sparse with $O(m(n+q))$ variables and constraints.*

Proof. Consider an inequality $a^\top x \leq \beta$ in the given formulation with $s \geq 4$ non-zero coefficients and assume w.l.o.g. that the first s coefficients are non-zero. We introduce artificial variables $y_1, \ldots, y_{s-2}$ and the following $s-1$ additional constraints: Define $y_1 \coloneqq a_1x_1 + a_2x_2$ and $y_k = y_{k-1} + a_{k+1}x_{k+1}$ for each $k \in \{2, \ldots, s-2\}$, as well as $y_{s-2} + a_sx_s \leq \beta$. Then the additional constraints have at most three non-zero coefficients, and a point (x, y) is feasible for this set of constraints if and only if x fulfills $a^\top x \leq \beta$. □

If X admits a mixed-integer formulation (in its original or in an extended space) of polynomial size, an immediate consequence of Proposition 16 is that X admits a 3-sparse mixed-integer formulation of polynomial size in an extended space. Since the feasible region $X^I \subseteq \{0,1\}^{n(I)}$ for each instance I of a 0/1-problem whose membership problem is in $\mathcal{NP}$ admits an extended mixed-integer formulation of size polynomial in $n(I)$, see Kaibel and Weltge [7, Proposition 2], every such problem admits a 3-sparse mixed-integer formulation of polynomial size. Thus, many combinatorial problems including the traveling salesman or max-cut problem are sparsely representable.

6 Conclusion and Outlook

We exactly characterized the maximum sparsity of inequalities that allows to define an integer formulation of $X \subseteq \{0,1\}^n$ and we proved that sparsest formulations are $\mathcal{NP}$-hard to separate in general. Complementing this result, we showed that the densest possible formulation via infeasibility cuts is always separable in polynomial time, provided the membership problem for X is polynomial time solvable. Moreover, it is possible to derive sparse formulations in an extended space.

Another interesting topic is to investigate the size of sparse integer formulations. Knapsack problems, for example, admit an integer formulation consisting of box constraints and an additional completely dense (knapsack) inequality. Replacing the knapsack inequality by a family of sparse inequalities typically increases the size of an integer formulation drastically. An upper bound on the number of non-trivial inequalities in a sparsest formulation is given by the number of maximally infeasible faces of $[0,1]^n$ w.r.t. X, but this bound may not be tight. The investigation of this topic will be left for future research.

Acknowledgments We thank Tobias Fischer, Tristan Gally, Oliver Habeck, Imke Joormann, Andreas Paffenholz and Andreas M. Tillmann for fruitful discussions. We also thank four anonymous referees for valuable suggestions on an earlier version of this paper. The last two authors were partly supported by the German Research Foundation within the CRC 666.

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