Lifting Grothendieck Universes

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Both in set theory and constructive type theory *universes* are a useful and necessary tool for formulating abstract mathematics, e.g. when one wants to quantify over *all* structures of a certain kind. Structures of a certain kind living in universe $\mathcal{U}$ are usually referred to as “small structures” (of that kind). Prominent examples are “small monoids”, “small groups” ... and last, but not least “small sets”.

For (classical) set theory an appropriate notion of universe was introduced by A. Grothendieck for the purposes of his development of Grothendieck toposes (of sheaves over a (small) site). Most concisely, a *Grothendieck universe* can be defined as a *transitive set* $\mathcal{U}$ such that $(\mathcal{U}, \in_{\restriction \mathcal{U}\times\mathcal{U}})$ itself constitutes a model of set theory.[1]

In (constructive) type theory a universe (in the sense of Martin–Löf) is a type $\mathcal{U}$ of types that is closed under the usual type forming operations as e.g. Π, Σ and $\mathsf{I}d$. More precisely, it is a type $\mathcal{U}$ *together* with a family of types $(\, \mathsf{E}l(A) \mid A \in \mathcal{U}\,)$ assigning its type $\mathsf{E}l(A)$ to every $A \in \mathcal{U}$. Of course, $\mathsf{E}l(A) = \mathsf{E}l(B)$ iff $A = B \in \mathcal{U}$.

For the purposes of Synthetic Domain Theory (see [6, 3]) or Semantic Normalisations Proofs (see [1]) it turns out to be necessary to organise (pre)sheaf toposes into models of type theory admitting a universe. In this note we show how a *Grothendieck universe* $\mathcal{U}$ gives rise to a *type-theoretic universe* in the presheaf topos $\widehat{\mathbb{C}}$ where $\mathbb{C}$ is a small category (i.e. $\mathbb{C}$ lives in $\mathcal{U}$). The problems of extending this construction to toposes of sheaves will be discussed later.

A presheaf $F \in \widehat{\mathbb{C}}$ is called *small* iff $F(I) \in \mathcal{U}$ for all $I \in \mathbb{C}$.

Before defining the notion of family of small presheaves over a given (not necessarily small) presheaf F recall the following equivalence

$$\widehat{\mathbb{C}}{\downarrow}F \;\simeq\; \widehat{\mathsf{E}lts(F)}$$

where $\mathsf{E}lts(F)$, the category of generalised elements of F, is given by the comma category $\mathsf{Y}_{\mathbb{C}}{\downarrow}F$, see e.g. [5]. Accordingly, a family of small presheaves over

[1] In the slang of set theory this is called a *small inner model* as in set theory *inner models* are defined as transitive classes $\mathcal{M}$ such that $(\mathcal{M}, \in_{\restriction \mathcal{M}\times\mathcal{M}})$ constitutes a model of set theory.

$F \in \widehat{\mathbb{C}}$ is defined as a small presheaf over $\mathsf{E}lts(F)$. Instead of "family of small presheaves" we often simply say "small family". We write $\mathsf{s}f(F)$ for the collection of all small families over F.

A type-theoretic universe in $\widehat{\mathbb{C}}$ is now given by a small family $\mathsf{E}l$ over a (non-small) presheaf U such that $\mathsf{E}l$ *classifies small families*. More explicitely, this means that for every $F \in \widehat{\mathbb{C}}$ we have

$$\mathsf{s}f(F) \cong \widehat{\mathbb{C}}(F, U)$$

naturally in F. Thus, by the Yoneda lemma we have

$$U(I) \cong \widehat{\mathbb{C}}(\mathsf{Y}_{\mathbb{C}}(I), U) \cong \mathsf{s}f(\mathsf{Y}_{\mathbb{C}}(I)) \cong \mathcal{U}^{(\mathbb{C}\downarrow I)^{op}}$$

suggesting the following definition.

The presheaf U is defined as

$$U(I) := \widehat{\mathbb{C}\downarrow I} \qquad\qquad U(f)(A) := A \circ (\mathbb{C}\downarrow f)^{op}$$

and $\mathsf{E}l \in \widehat{\mathsf{E}lts(U)}$ is defined as

$$\mathsf{E}l(\langle I, A\rangle) := A(\mathsf{id}_I) \qquad \mathsf{E}l(f : \langle J, U(f)(A)\rangle \to \langle I, A\rangle)(a) := A(f : f \to \mathsf{id}_I)(a)\ .$$

A small family $G \in \mathsf{s}f(F)$, i.e. $G \in \widehat{\mathsf{E}lts(F)}$, is classified by the map $\chi : F \to U$ in $\widehat{\mathbb{C}}$ where for $x \in F(I)$ the presheaf $\chi_I(x)$ over $\mathbb{C}\downarrow I$ is given by

$$\chi_I(x)(f : J \to I) := G(\langle J, F(f)(x)\rangle)$$

and

$$\chi_I(x)(g : f \circ g \to f) := G(g : \langle K, F(f \circ g)(x)\rangle \to \langle J, F(f)(x)\rangle)$$

for $g : K \to J$.

On the other hand a $\chi : F \to U$ classifies the $G \in \mathsf{s}f(F)$ where

$$G(\langle I, x\rangle) := \chi_I(x)(\mathsf{id}_I)$$

and

$$G(f : \langle J, F(f)(x)\rangle \to \langle I, x\rangle) := \chi_I(x)(f : f \to \mathsf{id}_I)$$

for $f : J \to I$.

It is straightforward to check that these correspondences are inverse to each other.

But, alas, it is not clear how to perform a similar construction for proper sheaf toposes $\mathsf{s}h_{\mathcal{J}}(\mathbb{C})$ where $\mathcal{J}$ is a Grothendieck toplogy on $\mathbb{C}$. Of course, one may restrict U to the sub-presheaf $U_{\mathcal{J}}$ where $U_{\mathcal{J}}(I)$ consists of all $F : (\mathbb{C} \downarrow I)^{op} \to \mathcal{U}$ which are sheaves w.r.t. the topology $\mathcal{J}_I := \Sigma_I^{-1}(\mathcal{J})$ where $\Sigma_I = \partial_0 : \mathbb{C} \downarrow I \to \mathbb{C}$. But, the problem is that in general $U_{\mathcal{J}}$ will not be a $\mathcal{J}$-sheaf. Alas, this would be necessary for considering the sheaf models for SDT of [3] as models of SDT in the sense of [6].

Another open question is how our universe construction for presheaf toposes is related to the the universes constructed in Joyal and Moerdijk's work on Algebraic Set Theory (see [4]).

Finally we want to explain how our construction of U and $\mathsf{E}l$ is related to a somewhat more general construction of [2] where J. Bénabou introduced the fibration $\mathsf{F}ib(\mathbb{C}) \to \mathbb{C}$ where a morphism over $f : J \to I$ is a commuting square

$$\begin{tikzcd} \mathbb{Y} \arrow[r, "F"] \arrow[d, "Q"'] & \mathbb{X} \arrow[d, "P"] \\ \mathbb{C} \downarrow J \arrow[r, "\Sigma_f"'] & \mathbb{C} \downarrow I \end{tikzcd}$$

where P and Q are fibrations and F is cartesian, i.e. F maps Q-cartesian arrows to P-cartesian arrows.

Our U appears as a "split" version of the restriction of $\mathsf{F}ib(\mathbb{C}) \to \mathbb{C}$ to those squares of the above form which are are pullbacks in $\mathsf{C}at$ and where P and Q are discrete fibrations whose fibers live in $\mathcal{U}$.

Finally, our $\mathsf{E}l$ appears as the restriction to base U of the fibration of "pointed objects" of $\mathsf{F}ib(\mathbb{C}) \to \mathbb{C}$ whose arrows over morphism

$$\begin{tikzcd} \mathbb{Y} \arrow[r, "F"] \arrow[d, "Q"'] & \mathbb{X} \arrow[d, "P"] \\ \mathbb{C} \downarrow J \arrow[r, "\Sigma_f"'] & \mathbb{C} \downarrow I \end{tikzcd}$$

in $\mathsf{F}ib(\mathbb{C}) \to \mathbb{C}$ are commuting squares of the form

$$\begin{tikzcd} \mathbb{C} \downarrow J \arrow[r, "\Sigma_f"] \arrow[d, "T"'] & \mathbb{C} \downarrow I \arrow[d, "S"] \\ \mathbb{Y} \arrow[r, "F"'] & \mathbb{X} \end{tikzcd}$$

where S and T are cartesian sections of P and Q, respectively. The reason is that cartesian sections of a discrete fibration P over $\mathbb{C} \downarrow I$ are in canonical correspondance with $P(\mathsf{id}_I)$.

References

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