ANALYSIS I

Lecture Notes

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February 2001
last modified August 27, 2017

Contents

1 Fundamental notions

1.1 Sets

The basic notion of a set was introduced by Georg Cantor (1845 – 1918). By the definition of Cantor, a set is a

Combination of well determined and well distinguished objects to an entity.

These objects are called elements of the set. Sets can be specified in two different ways:

1. One lists all its elements between curly brackets

$$\begin{aligned} M_1 &= \{1, 2, 3, 4\}\,, \\ M_2 &= \{-1, 2, 4, 6, 7.5\}\,, \\ M_3 &= \{2, 4, 6, 8, 10\}\,. \end{aligned}$$

2. Alternatively, one can give a defining property:

$$\begin{aligned} M_4 &= \{x \mid x \text{ is an even number between 1 and 10}\}\,, \\ M_5 &= \{a \mid a^2 = 1\}\,, \\ M_6 &= \{x \mid x \text{ is a prime number}\}\,. \end{aligned}$$

The order of the elements does not matter. Hence,

$$M_1 = \{1, 2, 3, 4\} = \{4, 3, 1, 2\}.$$

The formulation of the definition of the set must allow to decide for every object whatsoever, whether it belongs to the set or not (every element must be "well determined"). In a set every element may be contained only once (every element must be "well distinguished"). One says that two sets are equal,

$$A = B,$$

if both contain the same elements. Thus,

$$M_3 = \{2, 4, 6, 8, 10\} = M_4 = \{x \mid x \text{ is an even number between 1 and 10}\}.$$

One says that A is a subset of B,

$$A \subseteq B,$$

if every element of A is an element of B:

$$\begin{aligned} \{1,4\} &\subseteq \{1,2,3,4\}\,, \\ \{1,2,3,4\} &\subseteq \{a \mid a \text{ is an integer}\}\,. \end{aligned}$$

If, in addition, there is an element of B which is not in A, then A is said to be a proper subset of B. For the statement "m is element of M" one introduces the symbol

$$m \in M.$$

The negation of this statement is

$$m \notin M.$$

This symbol thus means "m is not element of M".

The set which contains no element is called the empty set and is denotes by $\emptyset$. It is efficient to allow the empty set, since otherwise in many statements different cases must be distinguished. The empty set is a subset of every set.

For several sets standard notations are used:

$$\begin{aligned} \mathbb{N} &= \{1,2,3,\ldots\} \quad \text{(the set of natural numbers)}, \\ \mathbb{Z} &= \{0,1,-1,2,-2,\ldots\} \quad \text{(the set of integers)}, \\ \mathbb{Q} &= \{q \mid q \text{ is a rational number}\}, \\ \mathbb{R} &= \{r \mid r \text{ is a real number}\}. \end{aligned}$$

Let $M_1 \subseteq M$. Then the set

$$C = \{m \mid m \text{ belongs to}\, M \text{ but not to } M_1\}$$

is called the complement of M_1 in M. One writes

$$C = M \backslash M_1.$$

The set of all subsets of a set M is denoted by $\mathcal{P}(M)$ or, for reasons which will become clear later, by 2^M. In particular,

$$\emptyset \in \mathcal{P}(M), \quad M \in \mathcal{P}(M).$$

(M is a subset of itself.)

Example: The set of all subsets of $\{1,2,3,4\}$ is

$$\begin{aligned}\mathcal{P}(\{1,2,3,4\}) = \big\{&\emptyset,\\ &\{1\},\{2\},\{3\},\{4\}\\ &\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\\ &\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\\ &\{1,2,3,4\}\big\}.\end{aligned}$$

This set contains $16 = 2^4$ elements.

The notion of a set is not defined with the precision one is used to in mathematics. It is a fundamental notion upon which mathematics is based. After all, also mathematics must start somewhere. Caution is necessary, however, since certain definitions of sets lead to contradictions, thence cannot be allowed:

Example: "Russel's antinomy" (Bertrand Russel, 1872 – 1970): Let M be the set of all sets, which do not contain themselves as an element. Does $M \in M$ hold?

From $M \in M$ it follows that $M \notin M$. Conversely, $M \notin M$ implies $M \in M$. Therefore neither $M \in M$ nor $M \notin M$ can be true. Since one of these statements must hold, a contradiction results. Therefore this definition of a set M is forbidden.

To avoid such contradictions, in the *axiomatic set theory* one defines the relations valid between sets by a system of formal *axioms*. However, as usual in calculus courses, we take the view of the *naive set theory* and use Cantor's definition of a set, but avoid definitions of sets leading to contradictions.

Further definitions and notations: Let M_1 and M_2 be sets. Then the set $M_1 \cup M_2$ defined by

$$M_1 \cup M_2 = \{x \mid x \in M_1 \text{ or } x \in M_2\}$$

is called union of the sets M_1 and M_2. Here, as usual in mathematics, the word "or" is not used as "exclusive or". Thus, an element x may belong to both of the sets M_1 and M_2.

For an arbitrary finite or infinite system S of sets one writes

$$\bigcup_{M \in S} M = \{x \mid \text{there is } M \in S \text{ with } x \in M\}.$$

One also uses the notations

$$\bigcup_{i=1}^{n} M_i = M_1 \cup M_2 \cup \ldots \cup M_n\,; \quad \bigcup_{i=1}^{\infty} M_i = M_1 \cup M_2 \cup M_3 \cup \ldots .$$

Similarly, one defines

$$M_1 \cap M_2 = \{x \mid x \in M_1 \text{ and } x \in M_2\},$$

and calls $M_1 \cap M_2$ the intersection of the sets M_1 and M_2. For an arbitrary system S of sets one writes

$$\bigcap_{M \in S} M = \{x \mid x \in M \text{ for all } M \in S\},$$

and one also uses the notations

$$\bigcap_{i=1}^{n} M_i = M_1 \cap M_2 \cap \ldots \cap M_n\,; \quad \bigcap_{i=1}^{\infty} M_i = M_1 \cap M_2 \cap M_3 \cap \ldots\,.$$

If $M_1 \cap M_2 = \emptyset$ holds, the sets M_1 and M_2 are said to be disjoint.

Examples. 1.) Let $M_k = \{1, 2, \ldots, k\}$. Then

$$\begin{aligned} \bigcup_{k=1}^{n} M_k &= M_n\,, \\ \bigcup_{k=1}^{\infty} M_k &= \mathbb{N}\,, \\ \bigcap_{k=1}^{n} M_k &= \bigcap_{k=1}^{\infty} M_k = \{1\}. \end{aligned}$$

2.) Let a be a positve real number and let $M_a = \{x \mid x \text{ is a real number with } 0 < x < a\}$. For

$$S = \{M_a \mid a > 0\}$$

we then have

$$\begin{aligned} \bigcap_{M \in S} M &= \emptyset, \\ \bigcup_{M \in S} M &= \{x \mid x \text{ is a positive real number}\}. \end{aligned}$$

3.) Let a be a positive real number and let

$$M'_a = \{x \mid x \text{ is a real number with } 0 \leq x < a\}.$$

For

$$S' = \{M'_a \mid a > 0\}$$

we then have

$$\bigcap_{M\in S'} M = \{0\},$$

$$\bigcup_{M\in S'} M = \{x \mid x \text{ is a real number with } x \geq 0\}.$$

Rules of de Morgan (Augustus de Morgan, 1806-1871). Let K be a set and let S be a family of subsets of K. The complement of M in K is denoted by M'. Then

$$(\bigcup_{M\in S} M)' = \bigcap_{M\in S} M',$$

$$(\bigcap_{M\in S} M)' = \bigcup_{M\in S} M'.$$

1.2 Product sets, relations

Cartesian product (René Descartes 1596 – 1650). Let A, B be sets. The cartesian product of A and B is the set of all ordered pairs (x, y) with $x \in A$ and $y \in B$:

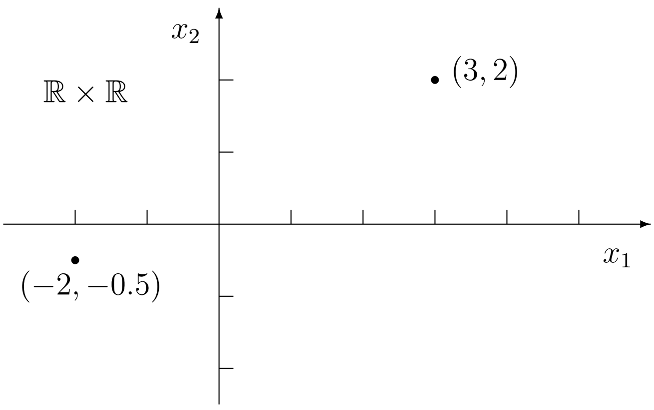
$$A \times B = \{z \mid z = (x, y),\ x \in A,\ y \in B\}.$$

The ordering of the pair (x, y) matters. Two ordered pairs (x, y) and (x', y') are said to be equal if and only if $x = x'$ and $y = y'$.

A set theoretic definition of an ordered pair is

$$(x, y) := \{x, \{x, y\}\}.$$

Example. Let $A = B = \mathbb{R}$. Every point in the plane can be specified by a pair of coordinates $(x_1, x_2) \in \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$; every point in the space can be specified by a triple of coordinates $(x_1, x_2, x_3) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$.



Relations. On the set of the real numbers the relation $<$ "less than" is defined. For example, $1 < 2$ holds. This can also be expressed by saying: The relation $<$ holds for the ordered pair $(1, 2)$. Clearly, the relation $<$ does not hold for the ordered pair $(2, 1)$. The relation $<$ is thus determined by the set of all ordered pairs from $\mathbb{R} \times \mathbb{R}$, for which the relation $<$ holds. This suggests to identify the relation $<$ with the set of all ordered pairs

$$\{(x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R} \text{ and } x < y\} \subseteq \mathbb{R} \times \mathbb{R}.$$

Moving further along this line of reasoning, one arrives at the

Definition 1.1 *Let M be a set. Every subset R of $M \times M$ is called a binary relation on M. This relation holds for the pair (x, y) if and only if $(x, y) \in R$.*

Example. Let M be a set. A finite or infinite set P of subsets of M is called partition of M if $S \cap T = \emptyset$ for all $S, T \in P$ with $S \neq T$ and if

$$M = \bigcup_{T \in P} T.$$

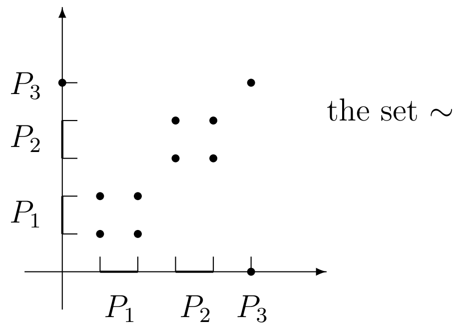
Therefore a set P of subsets of M is a partition of M if every element of M is contained in exactly one of the sets of P.

Let P be a partition of M. Two elements $x, y \in M$ are called equivalent, and one writes $x \sim y$, if and only if x and y belong to the same set of P. This defines a binary relation $\sim$ on M :

$$\begin{aligned} \sim \;&=\; \{(x, y) \mid (x, y) \in M \times M \text{ and } x \sim y\} \\ &=\; \{(x, y) \mid x \text{ and } y \text{ belong to the same set of } P\}. \end{aligned}$$

To be concrete, let $M = \{1, 2, 3, 4, 5\}$ and $P_1 = \{1, 2\}$, $P_2 = \{3, 4\}$, $P_3 = \{5\}$. Then $P = \{P_1, P_2, P_3\}$ is a partition of M. The relation $\sim$ defined by P is

$$\sim = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)\} \subseteq M \times M.$$



$\sim$ has the following properties:

(i) $x \sim x$ $\quad$ ($\sim$ is reflexive),

(ii) $x \sim y$ implies $y \sim x$ $\quad$ ($\sim$ is symmetric),

(iii) $x \sim y$ and $y \sim z$ implies $x \sim z$ $\quad$ ($\sim$ is transitive).

Every relation with the properties (i) – (iii) is called an equivalence relation. The most simple example for an equivalence relation is "=" $\quad$ (equalitiy).

1.3 Composition of statements

A statement can be true (T) or false (F). By logical composition of statements new statements are obtained, whose values (T or F) only depend on the values of the original statements. Here I present the truth tables for the four composition operations $\wedge$ ("and"), $\vee$ ("or"), $\Longrightarrow$ ("implication"), $\Longleftrightarrow$ ("equivalence"). (In fact, these operations are defined by these tables.) In the tables, A and B denote statements.

1.) "And" $\wedge$

A	B	$A \wedge B$
T	T	T
F	T	F
T	F	F
F	F	F

For A "and" B to be true, both A and B must be true. We give an example for the application of this operation: The definition of a set

$$M = \{m \mid \text{defining statement}\}$$

can be understood in the following sense: the statement $m \in M$ shall be true (takes on the value T), if the defining statement applied to the element m is true (takes on the value T), and $m \in M$ shall be false, if the defining statement is false for m. The intersection of two sets M_1 and M_2 can then be defind by

$$M_1 \cap M_2 := \{m \mid (m \in M_1) \wedge (m \in M_2)\}.$$

2.) “Or” $\vee$

A	B	$A \vee B$
T	T	T
F	T	T
T	F	T
F	F	F

For A “or” B to be true, it suffices that one of the statements A, B is true. This operation can be used to define the union of two sets:

$$M_1 \cup M_2 := \{m \mid (m \in M_1) \vee (m \in M_2)\}.$$

3.) “Implication” $\Longrightarrow$ (A implies B, from A it follows that B, if A then B)

A	B	$A \Longrightarrow B$
T	T	T
F	T	T
T	F	F
F	F	T

Saying “if A then B” in the colloquial language, one is only interested in the case where A is true. Only in this case an assertion for B is made. What happens when A is not true, is left open:

Example. “If it is raining, the street becomes wet.” “If it is raining, it follows that the street becomes wet.” It is left open what happens if it does not rain. In this case the street can stay dry, or can become wet for other reasons. By definiton of $\Longrightarrow$ in the above truth table, the statement $A \Longrightarrow B$ is always true if A is false. Therefore, in formal logic $A \Longrightarrow B$ is the analogue of the statement “if A then B” in the colloquial language.

Examples for the application of $\Longrightarrow$.

$$\begin{aligned} N &:= \Big\{m \mid m \in M \wedge (m \in M_1 \Longrightarrow m \in M_2)\Big\} \\ &= M \cap ((M \backslash M_1) \cup (M_1 \cap M_2)) = \\ &= M \cap (M \backslash M_1) \cup (M \cap M_1 \cap M_2) = (M \backslash M_1) \cup (M \cap M_1 \cap M_2), \end{aligned}$$

$$\begin{aligned} L &:= \Big\{m \mid m \in M \wedge [(m \in M_1 \Longrightarrow m \in M_2) \wedge (m \in M_2 \Longrightarrow m \in M_1)]\Big\} \\ &= M \cap [(M \backslash M_1) \cup (M_1 \cap M_2)] \cap [(M \backslash M_2) \cup (M_1 \cap M_2)] \\ &= (M \cap M_1 \cap M_2) \cup [M \backslash (M_1 \cup M_2)]. \end{aligned}$$

4.) "Equivalence" $\Longleftrightarrow$ (if and only if)

A	B	$A \Longleftrightarrow B$
T	T	T
F	T	F
T	F	F
F	F	T

Example.

$$\{a \mid (a \in \mathbb{R}) \wedge (a^2 > 1 \Longleftrightarrow a > 1)\} = \{a \mid a \geq -1\}.$$

To prove for any real number a that $a^2 > 1$ is true if and only if $|a| > 1$, requires the same action as proving that the statement $(a^2 > 1) \Longleftrightarrow (|a| > 1)$ is true for any real number a. Namely, one has to show that

1. if $a^2 > 1$ then $|a| > 1$,
2. if $|a| > 1$ then $a^2 > 1$.

Equivalent to 1. is the proof that the statement $(a^2 > 1) \Longrightarrow (|a| > 1)$ is true for any real number a. Equivalent to 2. is the proof that the statement $(|a| > 1) \Longrightarrow (a^2 > 1)$ is true for any real number a. Therefore $\Longleftrightarrow$ is a "two sided" operation, whereas $\Longrightarrow$ is a "one sided" logical operation.

1.4 Quantifiers, negation of statements

Statements can be expressed in a concise way using the quantifiers $\forall$, $\exists$.
All quantifier: $\forall$ "for all"
Existence quantifier: $\exists$ "there is"
Examples. 1.)

$$\forall_{a \in \mathbb{R}} \, \exists_{n \in \mathbb{N}} \, : \, n > a \, .$$

"For every real number a there is a (at least one) natural number n with $n > a$."
2.)

$$\forall_{a \in \mathbb{R}, \, a > 1} \, \forall_{n \in \mathbb{N}} \, \exists_{m \in \mathbb{N}} \, : \, n \leq a^m .$$

"For all real numbers a greater than 1 and for all natural numbers n there is a natural number m such that $n \leq a^m$ holds."

The negation of a statement written with quantifies can be obtained applying formal rules. The negation of the statement in the first example is

$$\exists_{a \in \mathbb{R}} \, \forall_{n \in \mathbb{N}} \, : \, n \leq a \, .$$

"There is $a \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ the inequality $n \leq a$ holds." The negation of the second statement is

$$\exists_{a\in\mathbb{R},\, a>1}\, \exists_{n\in\mathbb{N}}\, \forall_{m\in\mathbb{N}} \;:\; n > a^m.$$

"There is a real number a greater than 1 and there is a natural number n such that for all real numbers m the inequality $n > a^m$ holds."

2 Real numbers

$$\begin{aligned} \frac{10}{7} &= 1.428571428571\ldots \\ \sqrt{2} &= 1.4142135\ldots \\ \pi &= 3.1415926\ldots \end{aligned}$$

are real numbers. In this chapter we shall clarify the meaning of these infinite fractions. This makes it necessary to define precisely how one wants to interpret the notion of "the real numbers" and what properties the real numbers should have. First I want to sketch how I shall proceed.

One first notes that it is not important what the real numbers really are, but how one computes with them. Therefore one first lists the rules of computing and the properties which should be satisfied by the real numbers. Then one tries to reduce all these rules and properties to as few as possible basic laws, called axioms, from which all other rules can be deduced. After these axioms have been established, one forgets the idea of a real number, which one has from experience, and assumes that a set and operations of computing (addition and multiplication) on this set are given, which satisfy these axioms. Then one shows that if a second such set is given, it can be "mapped" onto the former set, hence is equivalent to the former set in a certain sense. Thus, in this sense there is only one such set, and this set (together with the operations of computing satisfying the axioms) is called "the real numbers". Finally, one has to show that the set of decimal fractions with the ordinary operations $+, \cdot, <$ etc., is just such a set, a so called model for the real numbers.

This procedure is called the axiomatic definition of the real numbers. Thus, one reduces the properties of the real numbers to some simple axioms, i.e. basic assumptions. These axioms are not reduced further, hence are not deduced from still simpler assumptions. The axioms must not contradict themselves, and it must be possible to deduce from these axioms all properties the real numbers should have. Within the frame set by these two requirements one is free to choose different systems of axioms. In the next section such a system of axioms will be given. It is worth mentioning, however, that it is also possible to choose a system of axioms which only specifies properties of the natural numbers $\{1, 2, 3, ...\}$. This defines the set of natural numbers. From these natural numbers one can construct the rational numbers, and in a second step, from the rational numbers one can construct the real numbers. This is called the constructive definition of the real numbers. The result is the same. The constructive definition is important, since it seems to be simpler, more natural and more convincing to make assumptions only for the natural numbers, and not for the real numbers immediately. The constructive definition needs much work to be done, however. Therefore one does not present the constructive definition in an introductory course, but starts with a system of axioms for the real numbers.

2.1 Field axioms

We assume that to any two real numbers a, b there is associated a unique real number $a + b$, their sum, and another real number $a \cdot b$, their product. For these relationships, which one calls addition and multiplication, the following axioms should hold:

Addition	A1.	$a + b = b + a$.	commutative law
	A2.	$a + (b + c) = (a + b) + c$.	associative law
	A3.	there is exactly one number 0 such that for all a one has $a + 0 = a$.	existence and uniqueness of 0
	A4.	to every a there is exactly one x such that $a + x = 0$.	unique solvability of the equation $a + x = 0$
Multiplication	A5.	$a \cdot b = b \cdot a$.	commutative law
	A6.	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$.	associative law
	A7.	there is exactly one number 1, different from 0, such that for all a one has $a \cdot 1 = a$.	existence and uniqueness of 1
	A8.	to every $a \neq 0$ there is exactly one x such that $a \cdot x = 1$.	unique solvability of the equation $a \cdot x = 1$
	A9.	$a \cdot (b + c) = a \cdot b + a \cdot c$.	distributive law

A set with an addition operation and a multiplication operation statisfying the axioms A1 – A9 is called a field. For the product one can also write ab instead of $a \cdot b$.

2.2 Consequences of the field axioms

We note first that because of the associative laws A2 and A6 one can drop the brackets:

$$\begin{aligned} a + (b + c) &= (a + b) + c =: a + b + c \\ a(bc) &= (ab)c =: abc. \end{aligned}$$

1.) One has

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \text{(binomial theorem)}$$

We set $c^2 := cc,\ 2 := 1+1$.

Proof:

$$\begin{aligned}(a+b)(a+b) &= (a+b)a + (a+b)b \\ &= a(a+b) + b(a+b) \\ &= (aa+ab) + (ba+bb) \\ &= aa + ab + ba + bb \\ &= aa + ab(1+1) + bb \\ &= a^2 + 2ab + b^2.\end{aligned}$$

■

By A4, there is exactly one solution x of the equation $a + x = 0$. This solution is denoted by $-a$.

2.) For all real numbers a, b the equation $a + x = b$ possesses exactly one solution.

Proof: Assume that a solution x of this equation exists. Addition of the number $-a$ to this equation yields

$$x = (-a) + a + x = (-a) + b = b + (-a).$$

Therefore the only possible solution is $b + (-a)$. In fact, this is the solution, since

$$a + (b + (-a)) = a + (-a) + b = 0 + b = b.$$

■

One sets

$$b - a := b + (-a) \qquad \text{(difference)}.$$

Hence, the *subtraction* is defined.

By axiom A8, to every $a \neq 0$ there is a unique number a^{-1} statisfying $aa^{-1} = 1$; one also writes $\frac{1}{a} = a^{-1}$.

3.) For $a \neq 0$ the equation $ax = b$ possesses a unique solution

$$x = b\,a^{-1} =: \frac{b}{a} \qquad \text{(quotient)}.$$

This defines the *division*.

The **proof** is left as an exercise.

The following result shows that in axiom A8 it must be assumed that $a \neq 0$:

4a.) For every a the axioms A1 – A4 and A9 imply $a \cdot 0 = 0$.

Proof:

$$a \cdot 1 = a \cdot (1 + 0) = a \cdot 1 + a \cdot 0 \,.$$

Subtraction of $a \cdot 1$ from both sides of this equation yields

$$0 = a \cdot 0.$$

■

Thus, if one of the factors in a product is 0, then the product is 0. Also the converse statement is true:

4b.) If a product is 0*, then at least one of the factors is* 0.

Proof: For, assume that $ab = 0$ and $a \neq 0$. Then the preceding statement implies

$$0 = \frac{1}{a} \cdot 0 = \frac{1}{a}\,(ab) = (\frac{1}{a}\,a)\,b = 1 \cdot b = b \,.$$

■

In summary, a product is 0 if and only if (at least) one of the factors is 0.

5.) For all a,b we have

$$(-a)b = -(ab).$$

Here $-(ab)$ is the unique solution of the equation $ab + x = 0$.

Proof: We have

$$ab + (-a)b = (a + (-a))b = 0\,b = 0.$$

Therefore $(-a)b$ is a solution of the equation $ab + x = 0$. Since this solution is unique by axiom A4, it follows that $(-a)b = -(ab) =: -ab$. ■

2.3 Axioms of order

On the set of real numbers a relation is defined, which should statisfy the following axioms:

A10.)	For all real numbers a, b exactly one of the relations $a < b$, $a = b$ or $b < a$ holds.	trichotomy
A11.)	$a < b$ and $b < c$ imply $a < c$.	transitivity
A12.)	$a < b$ implies $a + c < b + c$ for every c.	monotonicity of addition
A13.)	$a < b$ and $0 < c$ imply $ac < bc$.	monotonicity of multiplication

a is called positive if $0 < a$ holds. If $a < 0$ holds, a is called negative. The expression $a < b$ is called inequality or estimate. A relation $<$ statisfying the axioms A10 and A11 is called a (strict) order relation. A field with an order relation satisfying the axioms A12 and A13 is called an ordered field. The axioms A12 and A13 connect the addition and multiplication operation with the order relation.

2.4 Consequences of the axioms of order

1.) From $a < b$ and $c < d$ it follows that $a + c < b + d$.

Proof: From $a < b$ it follows that

$$a + c < b + c,$$

by axiom A12. Analogously, $c < d$ implies

$$b + c < b + d.$$

Axiom A11 thus yields

$$a + c < b + d.$$

■

2.) From $a < b$ and $c < 0$ it follows that $bc < ac$.

Proof: First it is shown that

$$c < 0 \Longleftrightarrow 0 < -c \,. \qquad (*)$$

For, addition of $-c$ to the inequality $c < 0$ yields $-c+c < -c$, whence $0 < -c$. Conversely, addition of c to the inequaltity $0 < -c$ yields $c < -c + c = 0$. This proves $(*)$.

Now assume that $a < b$ and $c < 0$. Then $(*)$ implies $0 < -c$, hence A13 yields

$$-ca < -cb,$$

consequently $-(ca) < -(cb)$, by *5.)* of Section 2.2. Addition of $ac + bc$ to both sides of this inequality results in

$$bc < ac.$$

■

3.) $a \neq 0$ implies $0 < a^2$.

Proof: Let $0 < a$. Then

$$0 = 0a < aa = a^2.$$

Let $a < 0$. Then we obtain $0 < -a$, hence

$$0 = 0(-a) < (-a)(-a) = -(a(-a)) = -(-aa) = -(-(aa)) = aa.$$

■

4.) $0 < a$ implies $0 < a^{-1} = \frac{1}{a}$.

Proof: We have $a^{-1} \neq 0$. For, otherwise $1 = aa^{-1} = 0$, which contradicts axiom A7. Moreover, it cannot be that $a^{-1} < 0$. For, this inequality together with $0 < a$ would yield

$$1 = a^{-1}a < 0a = 0,$$

which contradicts the inequality $0 < 1 \cdot 1 = 1$ following from *3.)*. Consequently, axiom A10 yields $0 < a^{-1}$. ■

5.) Form $a < b$ and $0 < ab$ it follows that $\frac{1}{b} < \frac{1}{a}$. From $a < b$ and $ab < 0$ it follows that $\frac{1}{a} < \frac{1}{b}$.

Proof: Let $0 < ab$. Then $0 < \frac{1}{ab}$, by *4.)*. From $a < b$ we thus obtain

$$\frac{1}{b} = (a\frac{1}{a})\frac{1}{b} = a\frac{1}{ab} < b\frac{1}{ab} = \frac{1}{a}\frac{b}{b} = \frac{1}{a}.$$

If $ab < 0$, then $(*)$ yields $0 < -ab$, whence

$$0 < \frac{1}{-ab} = -\frac{1}{ab},$$

by *4.)*. Using $(*)$ again, we obtain $\frac{1}{ab} < 0$. From $a < b$ and *2.)* we thus conclude

$$\frac{1}{a} = \frac{b}{b}\frac{1}{a} = b\frac{1}{ab} < a\frac{1}{ab} = \frac{a}{a}\frac{1}{b} = \frac{1}{b}.$$ ■

One writes $a > b$ if and only if $b < a$. The relation $a \leq b$ holds if and only if $a < b$ or $a = b$.

6.) For all a, b with $a \leq b$

$$a \leq \frac{a+b}{2} \leq b.$$

Proof: $0 < 1$ implies $1 < 2$, whence $0 < 2$, whence $0 < \frac{1}{2}$, hence $\frac{a}{2} < \frac{b}{2}$, thus

$$a = \left(\frac{2}{2}\right)a = \left(\frac{1}{2} + \frac{1}{2}\right)a = \frac{1}{2}a + \frac{1}{2}a \leq \frac{b}{2} + \frac{a}{2} \leq \frac{b}{2} + \frac{b}{2} = b.$$

■

We can now define the notion of an interval: Let $a \leq b$

$$\begin{aligned} [a,b] &:= \{x \mid a \leq x \leq b\} && \text{closed interval} \\ [a,b) &:= \{x \mid a \leq x < b\} && \text{half open intervals} \\ (a,b] &:= \{x \mid a < x \leq b\} && \\ (a,b) &:= \{x \mid a < x < b\} && \text{open interval.} \end{aligned}$$

Also the following sets are intervals:

$$\begin{aligned} [a,\infty) &:= \{x \mid a \leq x\} \\ (a,\infty) &:= \{x \mid a < x\} \\ (-\infty,a] &:= \{x \mid x \leq a\} \\ (-\infty,a) &:= \{x \mid x < a\} \\ (-\infty,\infty) &:= \mathbb{R}. \end{aligned}$$

$[a,a]$ is an interval containing one element and (a,a) is the empty set.

Definition 2.1 *The absolut value $|a|$ of the number a is defined by*

$$|a| := \begin{cases} a, & a \geq 0 \\ -a, & a < 0\,. \end{cases}$$

This definition implies $|a| \geq 0$ and $-|a| \leq a \leq |a|$.

7.) We have

a.) $|a| = 0$ *if and only if* $a = 0$,

b.) $|ab| = |a|\,|b|$,

c.) $|a+b| \leq |a| + |b|$ *(triangle inequality).*

Proof: a.) is evident.

b.) In the proof one has to distinguish the four cases

$$a \geq 0, \quad b \geq 0$$

$$a \geq 0, \quad b < 0$$

$$a < 0, \quad b \geq 0$$

$$a < 0, \quad b < 0.$$

I shall consider only the last case: If $a < 0$, then multiplication of both sides of the inequality $b < 0$ with a yields

$$ab > 0,$$

by *2.)*. Hence $|ab| = ab$. On the other hand, the inequalities $a < 0,\ b < 0$ imply $|a| = -a$, $|b| = -b$, whence

$$|a|\,|b| = (-a)(-b) = -(a(-b)) = -(-(ab)) = ab.$$

Together we obtain $|ab| = ab = |a|\,|b|$.

c.) From $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$ it follows

$$-(|a| + |b|) = (-|a|) + (-|b|) \leq a + b \leq |a| + |b|.$$

If $a + b \geq 0$, then the statement follows from the right hand side of this inequality. If $a + b < 0$, then $|a + b| = -(a + b)$. Thus, multiplication of the left hand side of the above inequality with -1 results in

$$|a + b| = -(a + b) \leq |a| + |b|.$$

■

8.) If $b \neq 0$, *then*

$$|\frac{a}{b}| = \frac{|a|}{|b|}.$$

Proof: Because of statement *7 b.)* we have $|\frac{a}{b}|\,|b| = |\frac{a}{b}\,b| = |a|$. Division by $|b|$ yields the statement. ■

9.) $\big||a| - |b|\big| \leq |a + b|$ *(inverse triangle inequality).*

Proof: Let $c := a + b$. Then $b = c - a$. The triangle inequality yields

$$|b| = |c - a| = |c + (-a)| \leq |c| + |-a| = |c| + |a| = |a + b| + |a|.$$

Thus

$$|b| - |a| \leq |a+b|.$$

Interchanging of a and b yields

$$-(|b| - |a|) = |a| - |b| \leq |a+b|.$$

Together it follows

$$\big| |a| - |b| \big| \leq |a+b|.$$

■

2.5 Natural numbers, the principle of induction

It is suggestive to say that the natural numbers are obtained by continual addition of 1, starting from 1. However, it is better to define the natural numbers as follows:

Definition 2.2 *A set M of real numbers is called inductive, if*

(a) $1 \in M$

(b) if $x \in M$, then also $x + 1 \in M$.

Inductive sets exist; examples are the set of all real numbers and the set of positive real numbers.

Assume that an arbitrary system of inductive sets is given. Then also the intersection of all these sets is an inductive set. For, 1 belongs to every inductive set, hence it belongs to the intersection. If x belongs to the intersection, then x belongs to every set of the system. Since all these sets are inductive, (b) implies that also $x + 1$ belongs to everyone of these sets, hence $x + 1$ belongs to the intersection.

Definition 2.3 *The intersection of all inductive sets of real numbers is called the set of natural numbers. It is denoted by* $\mathbb{N}$.

1 is the smallest natural number, since the set of all real numbers greater or equal to 1 is an inductive set. Consequently, the natural numbers must be a subset of this set. On the other hand, there is no greatest natural number. For, to every $n \in \mathbb{N}$ there is $n + 1 \in \mathbb{N}$, and because of $0 < 1$ we habe $n < n + 1$.

We have the following important

Theorem 2.4 (Induction) *Let $W \subseteq \mathbb{N}$, and assume that*

(i) $1 \in W$

(ii) *if* $n \in W$, *then also* $n + 1 \in W$.

Then $W = \mathbb{N}$.

Proof: By assumption we have $W \subseteq \mathbb{N}$. On the other hand, W is an inductive set. By definition, $\mathbb{N}$ is a subset of every inductive set, hence $\mathbb{N} \subset W$. Consequently, $W = \mathbb{N}$. ■

The method of proof by induction rests on this theorem. Proofs based on this method proceed as follows:

Let $A(n)$ be a statement about the natural number n. To show that $A(n)$ is true for every $n \in \mathbb{N}$, prove

a.) Start of the induction: $A(1)$ is true.

b.) Induction step: From the assumption, that $A(n)$ is true (induction hypothesis), it follows that $A(n + 1)$ is true.

Then $A(n)$ is true for every natural number n. For, let W be the set of natural numbers, for which $A(n)$ holds. (a) and (b) imply that W satisfies the hypotheses (i) and (ii) of the theorem of induction, hence $W = \mathbb{N}$.

Examples for proofs by induction: In these examples we use sums of the form

$$a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$$

and products of the form

$$a_1 a_2 a_3 \ldots a_{n-1} a_n.$$

At first, the meaning of these expressions is undefined, since it is not clear, in what order the summation or the multiplication is to be performed. However, because of the axioms of associativity and commutativity the value of these expressions is independent of the order of summation or multiplication. This is not clear by itself, but must be proved by induction with respect to n. I skip this proof.

Since these expressions are independet of the order, the following symbols can be introduced:

$$\begin{aligned} \sum_{i=1}^{n} a_i &:= a_1 + \ldots + a_n \\ \prod_{i=1}^{n} a_i &:= a_1 \ldots a_n. \end{aligned}$$

Clearly, the index of summation or of multiplication can be renamed arbitrarily. Also, one defines

$$\begin{aligned} a^0 &:= 1 \qquad (0^0 := 1) \\ a^1 &:= a \\ a^2 &:= aa \\ &\vdots \\ a^n &:= \underbrace{aa\ldots a}_{\text{n factors}} . \end{aligned}$$

Example 1. Let $a > -1$. Then the *Bernoulli inequality*

$$(1+a)^n \geq 1 + an$$

holds for all natural numbers n.

Proof: a.) Start of the induction: For $n = 1$ the statement is true, since

$$(1+a)^1 = 1 + a.$$

b.) Induction step: Let n be a natural number and assume that the statement is true for this n. Thus, we assume that

$$(1+a)^n \geq 1 + na.$$

It must be shown that this assumption implies the statement for $n+1$. Now, this assumption and $1 + a > 0$ imply

$$\begin{aligned} (1+a)^{n+1} &= (1+a)^n(1+a) \geq (1+na)(1+a) \\ &= 1 + na + a + na^2 = 1 + (n+1)a + na^2 \\ &\geq 1 + (n+1)a, \end{aligned}$$

since $na^2 \geq 0$. ■

Example 2. For all natural numbers n the equation

$$1 + 3 + 5 + \ldots + (2n-3) + (2n-1) = n^2$$

holds. This equation can also be written in the form

$$\sum_{j=1}^{n} (2j-1) = n^2.$$

Proof: a.) Start of the induction: for $n = 1$ the statement is obviously true.
b.) Induction step: Assume that the statement is true for a natural n. Thus, assume that

$$\sum_{j=1}^{n} (2j - 1) = n^2.$$

It must be shown that this implies $\sum_{j=1}^{n+1} (2j-1) = (n+1)^2$. This equation is in fact true, since by this assumption

$$\sum_{j=1}^{n+1} (2j - 1) = \sum_{j=1}^{n} (2j - 1) + \big(2(n + 1) - 1\big) = n^2 + 2(n + 1) - 1 = (n + 1)^2.$$

■

In the next example we need new notations: For $n \in (\{0\} \cup \mathbb{N})$ set

$$n! = \begin{cases} 1, & \text{if } n = 0, \\ 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n, & \text{if } n \in \mathbb{N}. \end{cases}$$

$n!$ is spoken as "n factorial". If n is a natural number and $k \in \{0, 1, \ldots, n\}$, then one defines the *binominal coefficient*

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}\,.$$

$\binom{n}{k}$ is spoken as "n over k". For all natural numbers k and n with $1 \le k \le n$ the binomial coefficients satisfy the equation

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

Here we use that $k \in \mathbb{N} \wedge k \geq 2 \Longrightarrow k - 1 \in \mathbb{N}$, which is proved below.

Proof: This formula can be proved by computation. The principle of induction is not needed for the proof:

$$\begin{aligned} \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} &= \frac{n!k}{k!(n-k+1)!} + \frac{n!(n+1-k)}{k!(n+1-k)!} \\ &= \frac{n!(k+n+1-k)}{k!(n+1-k)!} = \frac{(n+1)!}{k!(n+1-k)!} \end{aligned}$$

■

This result can be used to proof the *binomial theorem*:

Example 3. Let a, b be real numbers. Then for all natural numbers n

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

For $n = 2$ the formula takes the form

$$(a+b)^2 = \sum_{k=0}^{2} \binom{n}{k} a^{n-k} b^k = a^2 + 2ab + b^2 \,.$$

Proof: a.) Start of the induction: Let $n = 1$.

$$\sum_{k=0}^{1} \binom{1}{k} a^{1-k} b^k = \binom{1}{0} a + \binom{1}{1} b = a + b = (a+b)^1.$$

b.) Induction step: Assume that $n \in \mathbb{N}$ and assume that the formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

is valid. Then

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)\cdot(a+b)^n = (a+b)\cdot \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\
&= \sum_{k=0}^{n} \binom{n}{k} a^{n-k+1} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1} \\
&= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^{n-1} \binom{n}{k} a^{n-k} b^{k+1} + b^{n+1} \,.
\end{aligned}$$

In the second sum on the right we replace the summation index k by $j = k + 1$. Since $k = j - 1$, we obtain with the formula for the binomial coefficients proved above that

$$\begin{aligned}
(a+b)^{n+1} &= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{j=1}^{n} \binom{n}{j-1} a^{n+1-j} b^j + b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1} \right) a^{n+1-k} b^k + b^{n+1} \\
&= a^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^{n+1-k} b^k + b^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k \,.
\end{aligned}$$

■

Next we derive some properties of the natural numbers.

Theorem 2.5 *If $n \geq 2$ is a natural number, then also $n - 1$.*

Proof: If this were incorrect, there would exist $n_0 \in \mathbb{N}$, $n_0 \geq 2$, with $n_0 - 1$ not being a natural number. It is immediately seen that in this case $\mathbb{N}\backslash\{n_0\}$ would be an inductive set, which is a proper subset of $\mathbb{N}$. This is impossible, since $\mathbb{N}$ is the smallest inductive set. ■

Theorem 2.6 *Every natural number n has the following property: There is no natural number m with $n < m < n+1$.*

Proof by induction: Start of the induction: The statement is true for $n = 1$. For, $\{1\} \cup \{x \,|\, x \geq 2\}$ is an inductive set, whence $\mathbb{N}$ is a subset of this set.

Induction step: Let $n \in \mathbb{N}$ and assume there is no natural number m with $n < m < n+1$. Then there is no natural number m' with $n+1 < m' < n+2$, since otherwise the preceding theorem would imply that $m' - 1$ is a natural number satisfying $n < m' - 1 < n + 1$. ■

Theorem 2.7 *In every non-empty subset A of natural numbers there is a smallest element.*

Proof: Consider the set

$$W = \{n \mid n \in \mathbb{N} \text{ and } n \leq a \text{ for all } a \in A\}$$

of natural numbers. 1 belongs to W. If to every $n \in W$ also $n+1$ would belong to W, it would follow that $W = \mathbb{N}$. This is impossible, since A is non-empty. For, if $a \in A$, then $a+1$ is a natural number, which by definition of W does not belong to W.

Consequently, there is a number $k \in W$ with $k+1 \notin W$. This k is the smallest element of A. To prove this, it must be shown that $k \in A$ and

$$\forall_{a \in A} : k \leq a.$$

It is evident that $k \leq a$ holds for all $a \in A$, since $k \in W$. It thus remains to show that $k \in A$.

Since $k+1 \notin W$, there is $a_0 \in A$ with $k+1 > a_0$. This implies $a_0 \leq k$, because there is no natural number between k and $k+1$. We thus have

$$a_0 \leq k \text{ and } k \leq a_0$$

wich together imply $k = a_0 \in A$. ■

A set, on which an order relation $<$ is defined satisfying the axioms A10 and A11, is called an ordered set. An ordered set, for which every non-empty subset has a smallest element, is called well-ordered. The preceding theorem can therefore be rephrased as

Theorem 2.8 *The set of natural numbers is well ordered.*

In proofs by induction one often wants to start not with 1 but with another natural number n_0. This is allowed, as is shown by the following

Theorem 2.9 *Let* $W \subseteq \mathbb{N}$ *and assume that*

(a.) $n_0 \in W$.

(b.) if $n \geq n_0$ *and* $n \in W$*, then* $n+1 \in W$.

Then we have $\{n \,|\, n \in \mathbb{N} \wedge n \geq n_0\} \subseteq W$.

Proof: The set

$$W' = \{k \,|\, k \in \mathbb{N} \wedge k \leq n_0 - 1\} \cup W$$

is inductive. For, we obviously have $1 \in W'$. Moreover, if $n \in W'$, then also $n+1 \in W'$. To prove this last property, we distinguish three cases:

(i) $n < n_0 - 1$

(ii) $n = n_0 - 1$

(iii) $n > n_0 - 1$.

In case (i) we have $n + 1 < n_0$. Since there is no natural number between $n_0 - 1$ and n_0, this implies $n+1 \leq n_0 - 1$, whence $n+1 \in W'$. In case (ii) we have $n+1 = n_0 \in W \subseteq W'$. In case (iii) we have $n \geq n_0$, since there is no natural number between $n_0 - 1$ and n_0. By definition of W' we therefore have $n \in W$, which implies $n + 1 \in W \subseteq W'$, since W is inductive.

Thus, W' is an inductive set, hence $\mathbb{N} \subseteq W'$. The statement of the theorem is a consequence of this relation. ■

The proof of the following two theorems is left as an excercise:

Theorem 2.10 *Let* m *and* n *be natural numbers. Then also* $n+m$ *and* $n \cdot m$ *are natural numbers.*

Theorem 2.11 *Let* m, n *be natural numbers with* $m > n$*. Then also* $m - n$ *is a natural number.*

Having introduced the natural numbers, we can now define the set of integers and the set of rationals:

Definition 2.12 *The set*

$$\mathbb{Z} := \{m \mid m = 0 \text{ or } m \in \mathbb{N} \text{ or } -m \in \mathbb{N}\}$$

is called the set of integers, and the set

$$\mathbb{Q} := \{q \,\big|\, q = \frac{m}{n},\ m \in \mathbb{Z},\ n \in \mathbb{N}\}$$

is called the set of rational numbers.

Observe that the representation $q = \frac{m}{n}$ of a rational number is not unique. For example, one has

$$q = \frac{m}{n} = \frac{2m}{2n} = \frac{3m}{3n} = \dots .$$

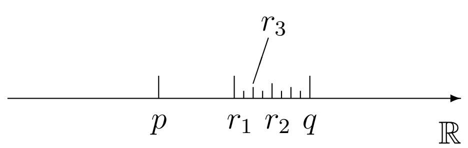
It is obvious that the sum and the product of two rational numbers are again rational numbers.

Theorem 2.13 *For any two rational numbers p, q with $p < q$ there is a third rational number r with $p < r < q$.*

Proof: Set $r = \frac{p+q}{2}$. ∎

2.6 Completeness axiom, Dedekind cut

Since the sum and the product of two rational numbers are rational numbers, the operations of addition and multiplication are defined on the set $\mathbb{Q}$. Also, 0 and 1 are rational numbers. Therefore the usual operations of addition and multiplication and the order relation $<$ satisfy all axioms A1 – A13 on $\mathbb{Q}$. Thus $\mathbb{Q}$ is an ordered field. Moreover, the last theorem shows that the rational numbers are densely distributed along the line of numbers:



However, the rational numbers do not cover the line of numbers completely. Instead, the irrational numbers are lying between the rational numbers. The existence of irrational numbers was discovered by the Greeks in the second half of the fifth century B. C., when they found that the length of the diagonal of a square is not a rational multiple of the length of the sides.

To study the irrational numbers, we need to introduce several fundamental notions:

Definition 2.14 *A set M of real numbers is said to be bounded above, if*

$$\exists_{s\in\mathbb{R}} \quad \forall_{x\in M} \ : \ x \leq s.$$

M is said to be bounded below, if

$$\exists_{s\in\mathbb{R}} \quad \forall_{x\in M} \ : \ s \leq x.$$

s is called upper bound or lower bound, respectively. A set is said to be bounded, if it is bounded above and below.

We note that this definition implies that the empty set is bounded.

Definition 2.15 *An upper or lower bound of M, respectively, which belongs to M, is called maximum of M or minimum of M, respectively (greatest or smallest number of M).*

Every non-empty finite set of real numbers is bounded and has a maximum and a minimum. This is proved by induction with respect to the number n of elements. We already proved that every set of natural numbers is bounded below and has a minimum. However, later we shall prove that the natural numbers are not bounded above.

The intervals

$$[a,b], \ (a,b), \ [a,b), \ (a,b]$$

are bounded, whereas the intervals

$$[a,\infty), \ (a,\infty), \ (-\infty,a), \ (-\infty,a], \ (-\infty,\infty) = \mathbb{R}$$

are unbounded.

A bounded set does not need to have a maximum or a minimum. This is shown by the example of the intervall $(0,1)$, which is bounded, but does neither have a maximum nor a minimum. For,

$$(0,1) = \{x \,\big|\, 0 < x < 1\}.$$

Would $m \in (0,1)$ be the maximum, then we were to have

$$0 < m < 1$$

and

$$\forall_{x\in(0,1)} : x \leq m.$$

But this cannot be, since

$$m < \frac{m+1}{2} < 1,$$

hence $\frac{m+1}{2} \in (0,1)$ would be a number larger than m.

Definition 2.16 *Let M be a nonempty set of real numbers bounded above. The upper bound s_0 is called supremum or least upper bound of M, if for every upper bound s of M the inequality*

$$s_0 \leq s$$

holds. The supremum is denoted by $s_0 = \sup M = \operatorname{lub} M$.

Let M be a nonempty set bounded below. The lower bound s_0 is called infimum or greatest lower bound of M, if for every lower bound s the inequality

$$s \leq s_0$$

holds. The infimum is denoted by $s_0 = \inf M = \operatorname{glb} M$.

The set $(0, 1)$ possesses a supremum and an infimum:

$$\begin{aligned} \sup (0, 1) &= 1 \\ \inf (0, 1) &= 0. \end{aligned}$$

For, 1 is an upper bound, and if a smaller upper bound would exist, we could derive a contradiction as above.

Now, one would like to have that every non-empty set bounded above has a supremum (and correspondingly, that every non-empty set bounded below has an infimum). Later it will be seen that this is an exceedingly important property in analysis. However, the existence of the supremum cannot be infered from the axioms A1 – A13. To see this, note that the rational numbers satisfy the axioms A1 – A13. Yet, I show that in $\mathbb{Q}$ not every non-empty subset bounded above has a supremum.

Such a set is $M = \{x \mid x^2 \leq 2\}$. Since $1 \in M$, this set is non-empty. Moreover, it is bounded above. An upper bound is $s = \frac{3}{2}$, for example. To see this, note that if $x \in M$ would exist with $x > \frac{3}{2}$, then by the axioms of order we would have $x^2 > (\frac{3}{2})^2 = \frac{9}{4} > 2$, which contradicts the assumption $x \in M$.

The set M cannot have a rational number as supremum. To see this, note first that the supremum s_0, if it exists, must be greater oder equal to 1 since $1 \in M$, and it must satisfy the equation $s_0^2 = 2$. To prove this, I exclude the cases

a.) $s_0^2 > 2$,

b.) $s_0^2 < 2$.

Below we show that if $s_0^2 > 2$ would hold, then a positive number h could be found satisfying $(s_0-h)^2>2$ and $s_0-h>0$. Because of the first inequality s_0-h would not belong to M. Together with the second inequality it follows that $s_0-h>x$ for all $x\in M$, whence s_0-h would be an upper bound for M, which contradicts the assumption, that s_0 is the least upper bound. Such a number is $h=\frac{s_0^2-2}{2s_0}$. For, we have

$$s_0-h=\frac{s_0}{2}+\frac{1}{s_0}>0 \quad \text{and} \quad (s_0-h)^2=s_0^2-2s_0h+h^2>s_0^2-2s_0h=2\,.$$

It thus follows that $s_0^2\le 2$. If $s_0^2<2$, would hold, then a positive number h could be determined such that the inequality $(s_0+h)^2\le 2$ would be satisfied, which implies that $s_0+h\in M$. Because of $s_0+h>s_0$ the number s_0 could not be an upper bound of M, so also not equal to the supremum, in contradiction to the assumption. Such a number is $h=\frac{2-s_0^2}{2s_0+1}>0$. For, because of $s_0\ge 1$ it follows that $h<1$, hence

$$(s_0+h)^2=s_0^2+h(2s_0+h)\le s_0^2+h(2s_0+1)=2\,.$$

Consequently, the supremum s_0 must satisfy $s_0^2=2$. However, no rational number can satisfy this equation. For, assume that $s_0=\frac{m}{n}$ with $m,n\in\mathbb{N}$ is a solution, where we can assume that the fraction has been maximally simplified so that m and n do not contain a common factor. Then

$$\frac{m^2}{n^2}=s_0^2=2$$

must hold, or

$$m^2=2n^2.$$

Thus, m^2 is even, which implies that also m must be an even number, since otherwise $m=2m'+1$ would hold with a suitable integer number m', which yields $m^2=4m'^2+4m'+1$, whence m^2 would be odd. Consequently, $m=2m'$ must hold, hence $2m'^2=n^2$, from which we conclude that also n must be even, and thus contains the factor 2 just as m does. But this contradicts the choice of n and m, which do not contain a common factor.

Therefore the hypothesis, that s_0 is rational, has led to a contradiction, and must be false. Consequently, M does not have a supremum in the set of rational numbers, which demonstrates that the axioms A1 – A13 do not guarantee the existence of the supremum. To ensure the existence, we must require it in an additional axiom:

A 14.) **Axiom of completeness:** Every non-empty set of real numbers bounded above has a least upper bound.

A 1 – A 14 are all the axioms required for the real numbers $\mathbb{R}$. Every set with operations of addition und multiplication and with an order relation satisfying A1 – A14 is called a complete ordered field, hence $\mathbb{R}$ is such a field.

Dedekind cut. The question arises whether the completeness axiom makes sense, that is to say, whether an ordered field exists, which satisfies the axiom A14. To answer this question I shall shortly touch upon the constructive definition of the real numbers and show, that from the rational numbers one can construct a larger set, the set of the real numbers, which satisfies the completeness axiom. I shall show, how this set can be constructed with the Dedekind cut. Besides the Dedekind cut other methods exist to construct $\mathbb{R}$ from $\mathbb{Q}$. For example, one can use Cauchy sequences or nested intervals.

To construct the real numbers from the rationals, one has to supplement the rationals with new numbers, called the irrational numbers. For, as we showed, if the completeness axiom is satisfied, the real numbers must contain a number, whose square is 2. These new numbers are constructed by adding new "objects" to the set of rational numbers, and by defining the rules of computing on these new objects such that the axioms are satisfied.

Definition 2.17 *A pair* $(\underline{U}, \overline{U})$ *of subsets of* $\mathbb{Q}$ *is called Dedekind cut in* $\mathbb{Q}$, *if*

1. $\mathbb{Q} = \underline{U} \cup \overline{U}$,
2. $\underline{U} \neq \emptyset, \ \overline{U} \neq \emptyset$,
3. *if* $\underline{q} \in \underline{U}, \ \overline{q} \in \overline{U}$, *then* $\underline{q} < \overline{q}$,
4. $\overline{U}$ *does not have a minimum.*

$\underline{U}$ *is called the subclass,* $\overline{U}$ *the superclass of the cut. (Richard Dedekind, 1831 – 1916).*

Every Dedekind cut is called a real number. A real number $(\underline{U}, \overline{U})$, whose subclass $\underline{U}$ contains a maximal element p, is identified with this rational number p. This makes $\mathbb{Q}$ a subset of the set $\mathbb{R}$ of real numbers (= the set of all cuts).

On the set of real numbers thus constructed the addition and the order relation are defined as follows:

Let $r_1 = (\underline{U}_1, \overline{U}_1), \ \ r_2 = (\underline{U}_2, \overline{U}_2)$. Then $r_1 + r_2$ is the real number $(\underline{V}, \overline{V})$ with

$$\begin{aligned} \underline{V} &:= \underline{U}_1 + \underline{U}_2 := \{q_1 + q_2 \,|\, q_1 \in \underline{U}_1, \, q_2 \in \underline{U}_2\}, \\ \overline{V} &:= \overline{U}_1 + \overline{U}_2 := \{q_1 + q_2 \,|\, q_1 \in \overline{U}_1, \, q_2 \in \overline{U}_2\}. \end{aligned}$$

The relation $r_1 < r_2$ holds if and only if $\underline{U}_1 \subseteq \underline{U}_2, \ \overline{U}_1 \supseteq \overline{U}_2$.

To define the multiplication we first assume that $r_1 \geq 0$ and $r_2 \geq 0$. In this case we set $r_1 \cdot r_2 = (\underline{V}, \overline{V})$ with

$$\underline{V} = \{q_1 q_2 \,|\, q_1 \in \underline{U}_1, \, q_2 \in \underline{U}_2, \, q_1 \geq 0, \, q_2 \geq 0\} \cup \{q \,|\, q < 0\}.$$

Otherwise we define

$$r_1 \cdot r_2 = \begin{cases} -(|r_1|\,|r_2|), & \text{if} \quad r_1 < 0, r_2 \geq 0 \quad \text{or} \quad r_1 \geq 0, r_2 < 0 \\ |r_1|\,|r_2|, & \text{if} \quad r_1 < 0, r_2 < 0. \end{cases}$$

It is easy to see that with this definition of the addition, the multiplication and the order relation the field and order axioms are satisfied. Furthermore, the completeness axiom is satisfied. For, let M be a non-empty set of real numbers bounded above. This set has the supremum $s_0 = (\underline{V}, \overline{V})$ with

$$\begin{aligned} \underline{V} &= \bigcap_{(\underline{U},\overline{U})\in M'} \underline{U}, \\ \overline{V} &= \bigcup_{(\underline{U},\overline{U})\in M'} \overline{U}, \end{aligned}$$

where $M' = \{r \,|\, r \text{ is upper bound of } M\}$. On the set of Dedekind cuts the axioms A 1 – A 14 are therefore satisfied. This completes the construction of the real numbers.

2.7 Consequences of the completeness axiom

To derive consequences of the completeness axiom we need a characterising property of the supremum, which is an indispensable tool when working with the supremum. We state and prove this property first.

Theorem 2.18 *The number s is the supremum of the set M if and only if the following two conditions are satisfied:*

(i) $x \leq s$ *for every* $x \in M$

(ii) *to every* $\varepsilon > 0$ *there is* $x \in M$ *with* $s - \varepsilon \leq x \leq s$.

Proof: "$\Longrightarrow$" Let s be the supremum of M. Then (i) holds, since s is an upper bound of M. If (ii) would not be satisfied, then there would be $\varepsilon > 0$ with $x < s - \varepsilon$ for all $x \in M$. Consequently $s - \varepsilon$ would be an upper bound of M smaller than s, which contradicts the assumption.
"$\Longleftarrow$" Let (i) and (ii) be satisfied. (i) implies that s is an upper bound, and because of (ii), no upper bound smaller than s can exist, hence s is the supremum. ∎

With this result we can now study consequences of the completness axiom.

2.7.1 Existence of the infimum and the square root

Theorem 2.19 *Every non-empty set of real numbers bounded below has an infimum.*

Proof: The set $M' = \{x \,\big|\, (-x) \in M\}$ is bounded above. Thus, M' has the supremum s_0', and $u_0 = -s_0'$ is the infimum of M. ■

Theorem 2.20 *To every positive real number x there is a unique positive real number s_0, which satisfies the equation $s_0^2 = x$. This number is denoted by $\sqrt{x}$.*

Proof: Set $M_x = \{y \in \mathbb{R} \mid y^2 \leq x\}$. We show that $s_0 = \sup M_x$ satisfies the equation $s_0^2 = x$. The proof runs along the same lines as above for the case $x = 2$. Note first that $0 \in M_x$ implies $s_0 \geq 0$, by Theorem 2.18 (i). If $s_0^2 > x$ would hold, we would have $s_0 > 0$ and $\varepsilon = \frac{s_0^2 - x}{2s_0} > 0$. Consequently, by Theorem 2.18 (ii) there is a number $z \in M_x$ satisfying $s_0 - \varepsilon \leq z \leq s_0$. This yields $x \geq z^2 \geq (s_0 - \varepsilon)^2 = x + \varepsilon^2 > x$, in contradiction to the axioms of order. We therefore have $s_0^2 \leq x$. If $s_0^2 < x$ would hold, then for $\varepsilon = \min\{\frac{x - s_0^2}{2s_0 + 1}, 1\} > 0$ we have $(s_0 + \varepsilon)^2 = s_0^2 + 2\varepsilon s_0 + \varepsilon^2 \leq s_0^2 + \varepsilon(2s_0 + 1) \leq x$, whence $s_0 + \varepsilon \in M_x$, which contradicts Theorem 2.18 (i). Therefore we indeed have $s_0^2 = x$.

To see that there is no other number y, which satisfies the equation, note that from the axioms of order we obtain for $0 < y < s_0$ the chain of inequalities $y^2 < s_0 \cdot y < s_0^2 = x$, and for $s_0 < y$ the chain of inequalities $x = s_0^2 < s_0 \cdot y < y^2$. ■

2.7.2 Archimedian ordering of the real numbers

Theorem 2.21 *The set of natural numbers is unbounded.*

Proof: If $\mathbb{N}$ would be bounded above, then a least upper bound s_0 would exist. By Theorem 2.18 the relation

$$\forall_{n \in \mathbb{N}} : n \leq s_0$$

would hold, and a number $n_0 \in \mathbb{N}$ would exist with $n_0 > s_0 - 1$. This implies $n_0 + 1 > s_0$. Since $n_0 + 1 \in \mathbb{N}$, the number s_0 could not be an upper bound of $\mathbb{N}$. ■

Theorem 2.22 *To every $a > 0$ and to every $b \in \mathbb{R}$ there is a natural number n such that*

$$na > b$$

holds. (This is sometimes expressed by saying that $\mathbb{R}$ carries an Archimedian ordering.)

Proof: If such a number n would not exist, then for all $n \in \mathbb{N}$ the inequality $na \leq b$ would be satisfied, hence

$$n \leq \frac{b}{a}.$$

Thus, $\mathbb{N}$ would be bounded. ■

Theorem 2.23 *If $a \geq 0$ holds and if for every natural number n the inequality $a \leq \frac{1}{n}$ is satisfied, then $a = 0$.*

Proof: $a > 0$ would imply

$$\forall_{n\in\mathbb{N}} : n \leq \frac{1}{a},$$

i.e. $\mathbb{N}$ would be bounded above. ■

2.7.3 Density of the real numbers

Theorem 2.24 (i) *If a, b are real numbers with $a < b$, then there exists at least one rational number r such that $a < r < b$.*
(ii) *If a, b are rational numbers with $a < b$, then there exists at least one irrational number r such that $a < r < b$.*

Proof: (i) Let $a \geq 1$. Because of $b > a$ there is $n \in \mathbb{N}$ with $n(b - a) > 1$, hence $0 < \frac{1}{n} < b - a$. Fix n and consider the set

$$M = \{m \,\big|\, m \in \mathbb{N} \wedge \frac{m}{n} > a\}.$$

This set is non-empty, since at least one $m \in \mathbb{N}$ exists with $m\frac{1}{n} > a$. Furthermore, $a \geq 1$ implies $m \geq 2$ for all $m \in M$. Finally, M is a subset of the natural numbers, hence contains a smallest element k. As an element of M, this k satisfies $k \geq 2$ and

$$a < \frac{k}{n}.$$

Consequently, since $\frac{k}{n}$ is rational, to prove (i) it suffices to show that $\frac{k}{n} < b$.

If $\frac{k}{n} \geq b$ would hold, then

$$\frac{k-1}{n} = \frac{k}{n} - \frac{1}{n} \geq b - \frac{1}{n} > b - (b - a) = a,$$

which implies $k - 1 \in M$. Thus, k could not be the minimum of M, whence

$$a < \frac{k}{n} < b$$

holds, which proves (i) for $a \geq 1$. If $a < 1$, we shift the interval $[a, b]$ to the half line $\{x \mid x \geq 1\}$ by adding a suitable rational number q to every element of $[a, b]$, and construct a rational number p in the interior of the shifted interval as above. Then $a < p - q < b$.

(ii) Assume the statement is false. Then rational numbers a, b with $a < b$ exist so that the interval $[a, b]$ totally consists of rational numbers. The sum and the product of two rational numbers are rational, since $\mathbb{Q}$ is a field. Hence $x - \frac{1}{2}(a + b)$ is rational for every rational number x, and consequently the interval

$$\Big[-\frac{b-a}{2}, \frac{b-a}{2}\Big] = \Big\{x - \frac{a+b}{2} \;\Big|\; x \in [a, b]\Big\}$$

totally consists of rational numbers. Moreover, also the interval

$$\Big[-n\,\frac{b-a}{2}\,,\, n\,\frac{b-a}{2}\Big] = \Big\{nx \;\Big|\; x \in \Big[-\frac{1}{2}\,(b-a)\,,\, \frac{1}{2}\,(b+a)\Big]\Big\}$$

totally consists of rational numbers.

Now let r be an arbitrary real number. Since $\frac{b-a}{2} > 0$, there is $n \in \mathbb{N}$ with

$$r \in \Big[-n\,\frac{b-a}{2}\,,\, n\,\frac{b-a}{2}\Big],$$

hence r is rational. From this we conclude that every real number is rational. But we already proved that $\sqrt{2}$ is irrational, and we arrive at a contradiction. Thus, the statement of the theorem must be true. ■

Corollary 2.25 *Let a, b be real numbers with $a < b$. Then there is at least one irrational number r with $a < r < b$.*

3 Functions

3.1 Elementary notions

Let A, B be sets. We assume that with every $x \in A$ there is associated a unique element from B, which is denoted by $f(x)$. Then f is said to be a *mapping* from A to B. Mappings are denoted by

$$f : A \to B$$

and, if the mapping of the elements is emphasized, by

$$x \mapsto f(x).$$

Besides the word mapping one also uses *function*. Note that the name of the function is f and not $f(x)$. The latter symbol denotes the unique element $y = f(x) \in B$ associated with $x \in A$.

With every element $x \in A$ only one element $y \in B$ may be associated, but it is allowed that with different elements $x_1, x_2 \in B$ the same element $y \in B$ is associated. Thus, we may have

$$f(x_1) = f(x_2)$$

even if $x_1 \neq x_2$.

Examples:

1.) Let a function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := 1$$

for every $x \in \mathbb{R}$. Then f is called a constant function.

2.) Let A be a nonempty set. The function $f : A \to A$ defined by

$$f(x) := x$$

for every $x \in A$ is called the identity mapping on A.

3.) Three simple, but important functions are

$$\begin{aligned}
&g : \mathbb{R} \to \mathbb{R}, && x \mapsto g(x) := x^2 \\
&h_1 : [0, \infty) \to \mathbb{R}, && x \mapsto h_1(x) := \sqrt{x} \\
&h_2 : [0, \infty) \to \mathbb{R}, && x \mapsto h_2(x) := -\sqrt{x}.
\end{aligned}$$

4.) The distance s travelled by a free falling body depends on the falling time t. One says that s is a function of t. The dependence of s on t is

$$s = \frac{1}{2}\, g\, t^2,$$

where the positive constant g measures the acceleration by the gravity of the earth. This dependence defines a function

$$S : [0, \infty) \mapsto [0, \infty), \quad t \mapsto S(t) := \frac{1}{2}\, g\, t^2.$$

5.) With the set $\mathcal{P}(\mathbb{R})$ of all subsets of $\mathbb{R}$ let the function $Q : [0, \infty) \to \mathcal{P}(\mathbb{R})$ be defined by

$$x \mapsto Q(x) := \big\{\sqrt{x}, -\sqrt{x}\big\}.$$

6.) For $x \in \mathbb{R}$ let the symbol $[x]$ denote the greatest integer number less or equal to x. Then

$$G(x) := [x]$$

defines a function $G : \mathbb{R} \to \mathbb{Z}$.

Definition 3.1 *Let $f : A \to B$ be a mapping. The set A is called the domain of definition of f and the set B is called the target of f. For the domain of definition we also write $D(f)$. The set*

$$W(f) = \big\{y \,\big|\, \exists_{x\in A} : y = f(x)\big\} \subseteq B$$

is called the range of f. If C is a subset of A, then the set

$$f(C) = \big\{y \in B \,\big|\, \exists_{x\in C} : y = f(x)\big\}$$

is called the image of C under the mapping f. Also, if D is a subset of B, then

$$f^{-1}(D) = \big\{x \in A \,\big|\, f(x) \in D\big\}$$

is called the inverse image of D under the mapping f.

One says that the function f depends on the *argument* x. For $x \in A$ the element $f(x)$ is called the value of f at x or the image of x under f. Thus, the range $W(f) = f(A)$ is the set of values of f. Note that

$$C \subseteq f^{-1}\big(f(C)\big), \quad f\big(f^{-1}(D)\big) \subseteq D$$

for every $C \subseteq A$ and $D \subseteq B$.

Definition 3.2 *Two mappings f and g are said to be equal, and one writes $f = g$, if and only if the domains of definition $D(f)$ and $D(g)$, the target sets of f and g and the values coincide:*

$$f(x) = g(x),$$

for all $x \in D(f) = D(g)$.

If the domain of definition $D(f)$ of f is contained in the domain of definition $D(g)$ of g, and if

$$f(x) = g(x)$$

for all $x \in D(f) \subseteq D(g)$, then g is said to be an extension of f and f is said to be a restriction of g. If $A = D(f)$, then one writes

$$f = g|_A.$$

$g|_A$ *denotes the* restriction of g to A.

Definition 3.3 *If $f : A \to B_1$ and $g : B_2 \to C$ are mappings with $f(A) \subseteq B_2$, then the mapping $g \circ f : A \to C$ is defined by*

$$(g \circ f)(x) = g\big(f(x)\big).$$

$g \circ f$ is called the composition mapping of f and g.

It is clear from this definition that for two mappings f, g whatsoever the composition $g \circ f$ is defined if and only if $W(f) \subseteq D(g)$.

Theorem 3.4 *Let f, g, h be mappings such that the compositions $g \circ f$ and $h \circ g$ are defined. Then also the mappings $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are defined, and*

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(The composition is an associative operation on functions.)

Proof: $h \circ (g \circ f)$ is defined if and only if $W(g \circ f) \subseteq D(h)$. The latter relation is true, since by assumption $h \circ g$ is defined, whence $W(g) \subseteq D(h)$, and so

$$W(g \circ f) \subseteq W(g) \subseteq D(h).$$

Similarly, $(h \circ g) \circ f$ is defined if and only if $W(f) \subseteq D(h \circ g)$. This is true, since by assumption $g \circ f$ is defined, hence

$$W(f) \subseteq D(g) = D(h \circ g).$$

The domains of definition of $h \circ (g \circ f)$ and $(h \circ g) \circ f$ coincide, since

$$D\big(h \circ (g \circ f)\big) = D(g \circ f) = D(f) = D\big((h \circ g) \circ f\big).$$

The target sets of $h \circ (g \circ f)$ and $(h \circ g) \circ f$ agree, since both are equal to the target set of h. Moreover, for $x \in D(f)$,

$$\begin{aligned} \big[h \circ (g \circ f)\big](x) &= h\big((g \circ f)(x)\big) = h\big(g(f(x))\big) \\ &= (h \circ g)\big(f(x)\big) = \big[(h \circ g) \circ f\big](x), \end{aligned}$$

whence $h \circ (g \circ f) = (h \circ g) \circ f$. ■

Let A, B be sets and id_A, id_B the identity mappings on A and B, respectively. For $f : A \to B$ we then have

$$f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f.$$

Therefore, on the set of all functions $f : A \to B$ the function id_A is a right neutral element and id_B is a left neutral element with respect to the composition operation.

Definition 3.5 *Let A, B be sets. A function $f : A \to B$ is called*

(i) *injective, or one–to–one, if $f(x_1) = f(x_2)$ implies $x_1 = x_2$,*

(ii) *surjective, or onto, if $W(f) = B$,*

(iii) *bijective, if f is injective and surjective.*

If $f : A \to B$ is bijective, then the mapping $f^{-1} : B \to A$ inverse to f exists. This inverse mapping is defined as follows: For all $y \in B$ there exists a unique $x \in A$ with $f(x) = y$. Now set

$$f^{-1}(y) := x.$$

Thus

$$f^{-1}(y) = x \iff y = f(x)$$

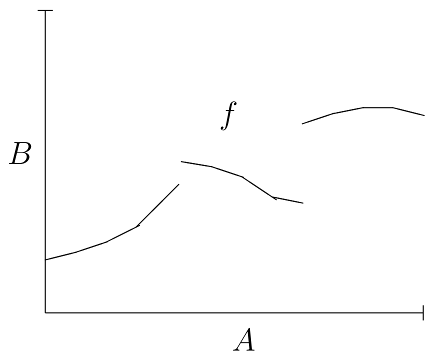
for all $y \in B$. The compositions $f^{-1} \circ f : A \to A$ and $f \circ f^{-1} : B \to B$ are obviously defined, and one has

$$f^{-1} \circ f = \mathrm{id}_A, \quad f \circ f^{-1} = \mathrm{id}_B.$$

Just as the notion of a set, the notion of a function is not precisely defined. However, one can reduce the notion of a function in a precise way to the notion of a set:

Definition 3.6 *Let A and B be sets. A function f from A to B is a subset of $A \times B$ with the following property: To every $x \in A$ there is a unique $y \in B$ with $(x, y) \in f$.*

In this definition, a function is identified with a subset of $A \times B$. Usually this subset is called the *graph* of f.



3.2 Real functions

A function, whose range is a subset of the real numbers, is called a real valued function. If also the domain of definition is a subset of the real numbers, then it is called a real function. We discuss some examples of real functions:

1. Polynomials

Let $a_0, \ldots, a_n \in \mathbb{R}$ with $a_n \neq 0$. An expression of the form

$$\sum_{m=0}^{n} a_m x^m$$

is called a *polynomial of degree* n. The numbers $a_0, \ldots, a_n$ are called the coefficients of the polynomial and a_n is said to be the leading coefficient. This polynomial defines a real function $p : \mathbb{R} \to \mathbb{R}$ by

$$x \mapsto p(x) = \sum_{m=0}^{n} a_m x^m,$$

which we call a *polynomial function*. As usual, I identify the polynomial with the associated polynomial function p and denote also the polynomial with the function name p. It will be shown presently that this identification is justified. If p_1 and p_2 are polynomials, then also the sum $p_1 + p_2$ and the product $p_1 \cdot p_2$ defined by

$$(p_1 + p_2)(x) = p_1(x) + p_2(x), \quad (p_1 \cdot p_2)(x) = p_1(x)p_2(x)$$

are polynomials with

$$\begin{aligned} \text{degree}\,(p_1 + p_2) &\leq \max\big\{\text{degree}\,(p_1), \text{degree}\,(p_2)\big\} \\ \text{degree}\,(p_1 \cdot p_2) &= \text{degree}\,(p_1) + \text{degree}\,(p_2). \end{aligned}$$

A polynomial function, whose degree is not greater than one, is called affine function:

$$x \mapsto a_1 x + a_0.$$

Also the linear functions $x \mapsto a_1 x$ and the constant functions $x \mapsto a_0$ are affine functions. The constant functions are polynomials of degree 0.

Of course, the polynomial determines uniquely the associated polynomial function. This association defines therefore a mapping from the set of polynomials to the set of real functions. Could it be that one and the same polynomial function is associated to two different polynomials? In this case this mapping would not be injective. The next theorem shows that this cannot happen, the mapping is injective. Therefore the identification of the polynomial and the associated polynomial function is justified.

Theorem 3.7 *Let*

$$p_1(x) = \sum_{m=0}^{n} a_m x^m, \quad p_2(x) = \sum_{m=0}^{\ell} b_m x^m$$

be polynomials with $p_1(x) = p_2(x)$ *for all* $x \in \mathbb{R}$. *Then* $n = \ell$ *and* $a_m = b_m$ *for all* $m = 0, \ldots, n$.

Proof: Without restriction of generality we assume that $n \geq \ell$. Set $b_m = 0$ for $m = \ell + 1, \ldots, n$. Then the function $q = p_1 - p_2$ satisfies the equation

$$q(x) = \sum_{m=0}^{n} c_m x^m = 0$$

for all $x \in \mathbb{R}$, where $c_m = a_m - b_m$ for $m = 0, 1, \ldots, n$. For all x different from zero q can be written in the form

$$q(x) = x^n \Big(c_n + \frac{c_{n-1}}{x} + \ldots + \frac{c_0}{x^n} \Big).$$

If $x > 1$ holds, then the inverse triangle inequality yields because of $x^k \geq x$ for $k \in \mathbb{N}$ that

$$\begin{aligned} |q(x)| &\geq x^n \Big(|c_n| - \Big| \frac{c_{n-1}}{x} + \ldots + \frac{c_0}{x^n} \Big| \Big) \\ &\geq x^n \Big(|c_n| - \Big| \frac{c_{n-1}}{x} \Big| - \ldots - \Big| \frac{c_0}{x^n} \Big| \Big) \geq x^n \Big(|c_n| - \frac{|c_{n-1}| + \ldots + |c_0|}{x} \Big). \end{aligned}$$

Now, if c_n would differ from zero, we could choose a number x sufficiently large such that

$$\frac{|c_{n-1}| + \ldots + |c_0|}{x} < \frac{1}{2}\,|c_n|,$$

whence, for this x,

$$0 = |q(x)| \geq x^n \Big(|c_n| - \frac{1}{2}\,|c_n|\Big) \geq \frac{1}{2}\,|c_n|,$$

which contradicts $|c_n| > 0$. Hence $c_n = 0$, and therefore $q(x) = \sum_{m=0}^{n-1} c_m x^m$. The same arguments can be applied to prove successively that $c_{n-1} = c_{n-2} = \ldots = c_0 = 0$. Consequently, $a_m = b_m$ for $m = 0, \ldots, n$, which also implies $\ell = n$. ■

2. Rational functions

Let g and h be polynomials and let M be the set of zeros of h :

$$M = \{x \in \mathbb{R} \mid h(x) = 0\}.$$

Define the function $r : \mathbb{R}\backslash M \to \mathbb{R}$ by

$$x \mapsto r(x) := \frac{g(x)}{h(x)}\,.$$

r is called a rational function. Every rational function can be written in the form

$$r = \frac{g}{h} = p + \frac{s}{h}\,,$$

where p and s are polynomials with degree$(s) <$ degree(h) and with

$$\text{degree}(p) = \text{degree}(g) - \text{degree}(h)\,,$$

provided that degree$(g) \geq$ degree(h). Otherwise one has $p = 0$.

This representation of r can be obtained with the algorithm of polynomial division. The following example is self explanatory:

Let $g(x) = x^4 + x^3 - 2$ and $h(x) = x^2 - 1$. Then

$$\begin{array}{l} (x^4 + x^3 + 0\,x^2 + 0\,x - 2) : (x^2 - 1) \;=\; x^2 + x + 1 + \dfrac{x-1}{x^2-1} \\ -\underline{(x^4 \qquad - \; x^2)} \\ \qquad x^3 + \; x^2 + 0\,x \\ \quad -\underline{(x^3 \qquad - \; x)} \\ \qquad\qquad x^2 + \; x - 2 \\ \qquad -\underline{(x^2 \qquad - 1)} \\ \qquad\qquad\qquad x - 1 \end{array}$$

hence

$$\frac{x^4+x^3-2}{x^2-1} = (x^2+x+1) + \frac{x-1}{x^2-1}.$$

Unlike polynomial functions, rational functions can be represented in different ways. This is shown by the function $\frac{s}{h}$ from the example above:

$$\frac{s(x)}{h(x)} = \frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1}.$$

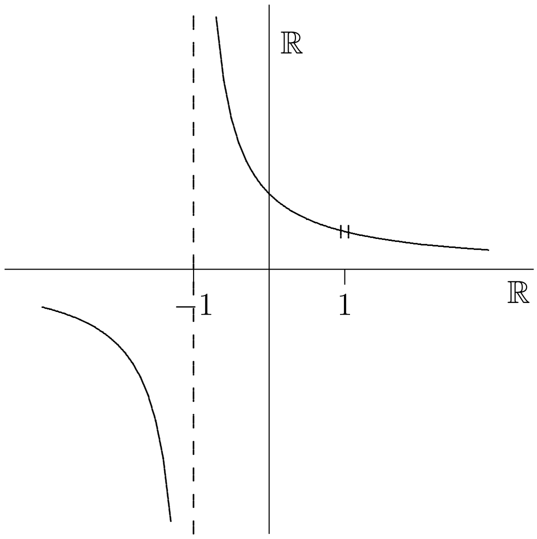
Note however, that x^2-1 has the zeros $x=\pm 1$. Therefore, $f_1(x) := \frac{x-1}{x^2-1}$ is defined on $\mathbb{R}\backslash\{-1,1\}$:

$$f_1 : \mathbb{R}\backslash\{-1,1\} \to \mathbb{R}.$$

On the other hand, $x+1$ has only one zero at $x=-1$. Consequently, $f_2(x) := \frac{1}{x+1}$ is defined on $\mathbb{R}\backslash\{-1\}$:

$$f_2 : \mathbb{R}\backslash\{-1\} \to \mathbb{R}.$$

Thus $f_1 \neq f_2$. More precisely, since $D(f_1) \subseteq D(f_2)$ and $f_1(x) = f_2(x)$ for all $x \in D(f_1)$, the function f_2 is an extension of f_1. The numerator $x-1$ of f_1 has a zero at the point $x=1$, which compensates the zero $x=1$ of the denominator x^2-1. The function f_1 can be extented continuously to the function f_2.



If g is a polynomial with degree$(g) > 0$ and if one chooses $h(x) = x - x_1$ with a given fixed number x_1, then polynomial division yields the equation $g(x)/h(x) = p(x) + s(x)/(x-x_1)$, where s is a polynomial of degree 0, hence a constant b. If one writes this equation in the form

$$g(x) = (x-x_1)p(x) + b$$

and inserts $x = x_1$, then one sees that $b = g(x_1)$ holds. Thus, if x_1 is a zero of g, then the decomposition

$$g(x) = (x - x_1)p_1(x)$$

results, where p_1 is a polynomial of degree $n - 1$. If x_2 is a zero of p_1, then by the same procedure we can split off the factor $x - x_2$ from p_1. After k steps one obtains the representation

$$g(x) = (x - x_1)(x - x_2)\dots(x - x_k)p_k(x), \tag{3.1}$$

of g, in which p_k is a polynomial of degree $n - k$. The procedure ends if p_k does not have a zero, hence it ends at the latest after n steps, since $\text{degree}(p_n) = 0$ holds, which means that p_n is a constant. This constant is different from zero, since $\text{degree}(g) = n > 0$ was assumed. Therefore the inequality $k \leq \text{degree}(g)$ holds.

It is possible that some or all of the numbers $x_1, \dots, x_k$ are equal. If the factor $x - x_i$ appears ℓ–times in the equation (3.1), then x_i is said to be a zero of the polynomial g with multiplicity ℓ or of order ℓ. If $y_1, \dots, y_m$ are pairwise different zeros of g and if ℓ_j is the order of the zero y_j, then the relation

$$\sum_{j=1}^{m} \ell_j = k \leq \text{degree}(g)$$

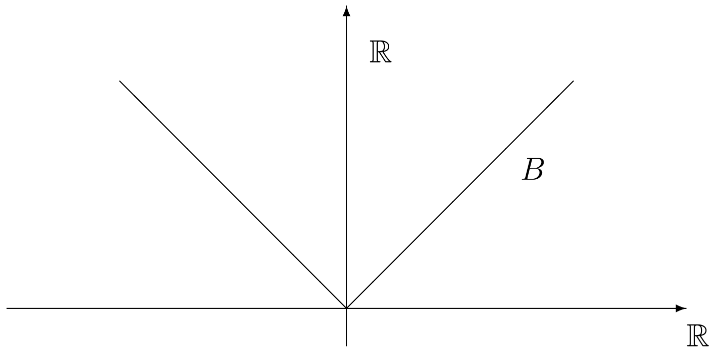
results.

Corollary 3.8 *The number of zeros, counted with multiplicity, of a polynomial, which differs from zero, is not greater than the degree of the polynomial.*

3. The absolute value function

Let $B : \mathbb{R} \to \mathbb{R}$ be defined by

$$x \mapsto B(x) := |x| = \begin{cases} x, & \text{für } x \geq 0 \\ -x, & \text{für } x < 0. \end{cases}$$

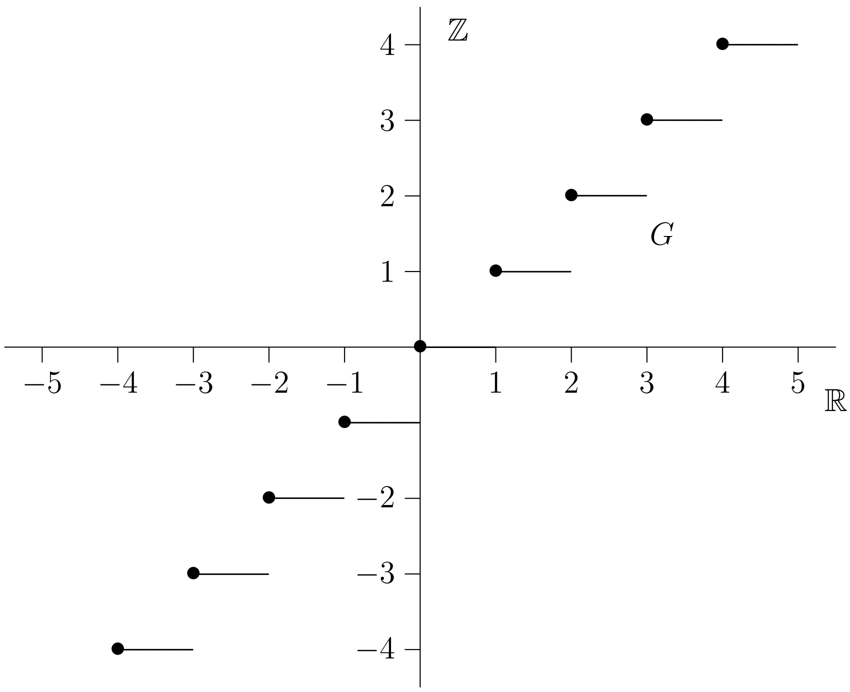


This function is neither injective nor surjective. However, if we change the target set to $[0, \infty)$, the function $\hat{B} : \mathbb{R} \to [0, \infty)$ with $\hat{B}(x) = B(x)$ results, which is surjective, but not injective.

4. Integer part function

In one of the examples at the beginning of this section I introduced the "integer part function" $G : \mathbb{R} \to \mathbb{R}$ defined by

$$x \mapsto G(x) := [x] := \text{ greatest integer less or equal to } x = \text{ integer part of } x.$$



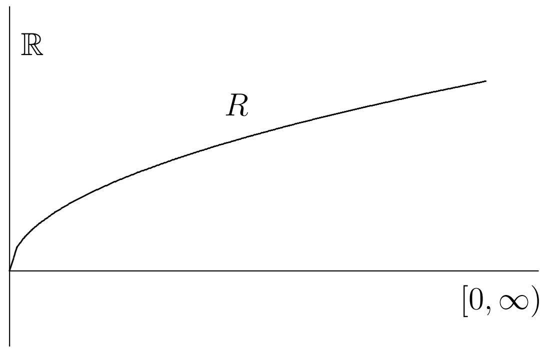
The range of this function is $W(G) = \mathbb{Z}$. For any integer n the inverse image of $\{n\}$ under

G is $G^{-1}(\{n\}) = [n, n+1)$. Therefore also the function $G : \mathbb{R} \to \mathbb{R}$ is neither surjective nor injective.

5. Square root function

In Section 2 it was shown that because of the completeness of the real numbers to every nonnegative x a unique square root $\sqrt{x}$ exists. Therefore we can define the square root function

$$R : [0, \infty) \to \mathbb{R}, \qquad x \mapsto R(x) := \sqrt{x}.$$



The function R is injective, since $R(x) = R(y)$ implies $x = R(x)^2 = R(y)^2 = y$. The range of R is $W(R) = [0, \infty)$. To prove this, note that for every nonnegative number a the number a^2 is also nonnegative and thus belongs to the domain of definition of R. Therefore the function R can be applied to a^2 and we obtain $R(a^2) = \sqrt{a^2} = a$, whence $a \in W(R)$. This yields $[0, \infty) \subseteq W(R) \subseteq [0, \infty)$, which means that $W(R) = [0, \infty)$.

Thus, the function R is injective but not surjective. However, if we change the target set to $[0, \infty)$, we obtain the function $\hat{R} : [0, \infty) \to [0, \infty)$ with $\hat{R}(x) = R(x) = \sqrt{x}$. This function is surjective and injective, hence it is bijective and has an inverse $\hat{R}^{-1} : [0, \infty) \to [0, \infty)$. Of course, the inverse is $x \mapsto \hat{R}^{-1}(x) := x^2$. This means in particular that the range of the function $x \mapsto x^2 : [0, \infty) \to [0, \infty)$ is the interval $[0, \infty)$.

6. Dirichlet function

The Dirichlet function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \backslash \mathbb{Q}\,, \end{cases}$$

hence $W(f) = \{0, 1\}$. It is not possible to draw the graph of this function, since between any two rational numbers there is an irrational number, and between any two irrational numbers there is a rational number. (Peter Gustav Lejeune Dirichlet, 1805 – 1859)

7. Step function

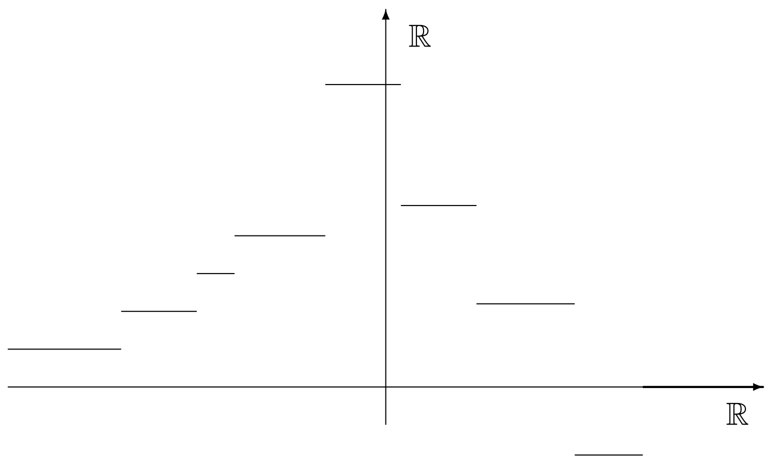
The *characteristic function* $\chi_M : \mathbb{R} \to \mathbb{R}$ *of a set* $M \subseteq \mathbb{R}$ is defined by

$$\chi_M(x) = \begin{cases} 1, & x \in M, \\ 0, & x \in \mathbb{R} \setminus M. \end{cases}$$

If $I_1, \ldots, I_m$ are intervals and $a_1, \ldots, a_m$ are given numbers, then the function

$$t : \mathbb{R} \to \mathbb{R}, \quad \text{with } t(x) = \sum_{k=1}^{m} a_k \chi_{I_k}(x),$$

is called a *step function.* For a step function the range $W(t)$ is a finite set and for every $y \in W(t)$ the inverse image $t^{-1}(\{y\})$ is a union of finitely many intervals.



3.3 Sequences, countable sets

Definition 3.9 *A mapping, whose domain of definition is* $\mathbb{N}$*, is called a sequence.*

The target set of a sequence can be any set. If the range of a sequence is a subset of the real numbers, then one speaks of a sequence of numbers or a numerical sequence. Sequences are denoted by

$$\{a_1, a_2, a_3, \ldots\} = \{a_n\}_{n=1}^{\infty}.$$

Here $a_n = a(n)$ is the image of the number $n \in \mathbb{N}$. We call a_n the n–th term of the sequence $\{a_n\}_{n=1}^{\infty}$.

The sequence $\{a_n\}_{n=1}^{\infty}$, a function, should not be mistaken with the set $\{a_n \mid n \in \mathbb{N}\}$, the range of this function.

Examples of numerical sequences are

$$\{0,0,0,\ldots\}, \quad \{0,1,0,1,0,1,\ldots\}, \quad \{n\}_{n=1}^{\infty}, \quad \left\{\frac{1}{n}\right\}_{n=1}^{\infty}, \quad \left\{\frac{-1}{n^2+1}\right\}_{n=1}^{\infty}.$$

An example for a sequence $\{x_n\}_{n=1}^{\infty}$ *defined recursively* is as follows:

1.) $x_1 = 1$

2.) $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$, for all $n \in \mathbb{N}$.

Intuitively it is clear that there is only one sequence satisfying both requirements. However, this must be proved using properties of the natural numbers. We avoid this proof.

The notion of a countable set is defined using sequences. In this definition and later it is convenient to use the notation

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Definition 3.10 *For $n \in \mathbb{N}_0$ let $A_n = \{k \mid k \in \mathbb{N}_0, k < n\}$ be the segment of $\mathbb{N}_0$ belonging to n.*

We remark that $A_0 = \emptyset$.

Definition 3.11 *A set M is called finite, if there is a number $n \in \mathbb{N}_0$ such that a bijective mapping of A_n onto M exists. n is called the number of elements of M.*

A set M is called countably infinite, if a bijective mapping of $\mathbb{N}_0$ onto M exists.

Two sets M_1 and M_2 are said to be of the same cardinality, if a bijective mapping of M_1 onto M_2 exists.

For the moment I write $M_1 \sim M_2$ if the two sets M_1 and M_2 are of the same cardinality. The relation $\sim$ between sets thus defined satisfies

(i) $M \sim M$

(ii) $M_1 \sim M_2$ implies $M_2 \sim M_1$

(iii) $M_1 \sim M_2$ and $M_2 \sim M_3$ imply $M_1 \sim M_3$.

Thus, $\sim$ is an equivalence relation.

The sets $\mathbb{N}_0$ and $\mathbb{N}$ are of the same cardinality, since

$$n \mapsto n+1 : \mathbb{N}_0 \to \mathbb{N}$$

is a bijective mapping. Therefore, since $\sim$ is an equivalence relation, a set M is countably infinite if it is of the same cardinality as the set $\mathbb{N}$ of natural numbers.

Definition 3.12 *A set is said to be countable if it is finite or countably infinite. Otherwise it is called uncountable.*

The definition of the number of elements of a set makes sense because of the following

Theorem 3.13 *Let $n, m \in \mathbb{N}_0$ with $n > m$. Then there is no injective mapping $f : A_n \to A_m$.*

From this theorem it also follows that a set M cannot be finite and countably infinite simultaneously. For, otherwise $m \in \mathbb{N}_0$ and bijective mappings $f : A_m \to M,\ g : \mathbb{N}_0 \to M$ would exist, hence $f^{-1} \circ g : \mathbb{N}_0 \to A_m$ would be a bijective mapping. For $n > m$ the restriction

$$(f^{-1} \circ g)\big|_{A_n} : A_n \to A_m$$

would be an injective mapping, contradicting the theorem.

Proof of the theorem: Assume that the statement is false. Then there are segments A_n and A_m with $n > m$ such that A_n can be mapped one–to–one into A_m. Obviously, A_n can then be mapped one–to–one into A_{n-1}. From $n - 1 \in \mathbb{N}_0$ it follows that $n \in \mathbb{N}$. Therefore the set of all $n \in \mathbb{N}$, for which A_n can be mapped one–to–one into A_{n-1} is not empty. Thence, this set contains a smallest element k. As we show next, from this we can conclude that $k \geq 2$ and that A_{k-1} can be mapped one–to–one into A_{k-2}, an obvious contraction to the definition of k.

Let $f : A_k \to A_{k-1}$ be the injective mapping. It is clear that A_1 cannot be mapped into $A_0 = \emptyset$, hence $k \geq 2$. Now three cases are possible:

(i) $k - 2$ does not belong to the range of f.

(ii) $k - 2$ is the image of $k - 1$ under the mapping f.

(iii) $k - 2$ is the image of a number j with $j < k - 1$ under f.

Since $A_k = A_{k-1} \cup \{k-1\},\ A_{k-1} = A_{k-2} \cup \{k-2\}$, in the first two cases $f\big|_{A_{k-1}} : A_{k-1} \to A_{k-2}$ is injective. In the last case an injective mapping $g : A_{k-1} \to A_{k-2}$ is obtained by

$$g(\ell) = \begin{cases} f(\ell), & \text{if } \ell \in A_{k-1},\ \ell \neq j \\ f(k-1), & \text{if } \ell = j. \end{cases}$$

Thus, we constructed a contradiction and the statement of the theorem must be true. ■

Theorem 3.14 *The set $\mathbb{Q}$ of rational numbers is countable.*

Proof: Arrange the positive rational numbers in a doubly infinite scheme:

$$\begin{array}{ccccccccc}
\frac{1}{1} & \rightarrow & \frac{2}{1} & & \frac{3}{1} & & \frac{4}{1} & \cdots \\
& \swarrow & & \swarrow & & \swarrow & & \swarrow \\
\frac{1}{2} & & \frac{2}{2} & & \frac{3}{2} & & \frac{4}{2} & \cdots \\
& \swarrow & & \swarrow & & \swarrow & & \swarrow \\
\frac{1}{3} & & \frac{2}{3} & & \frac{3}{3} & & \frac{4}{3} & \cdots \\
& \swarrow & & \swarrow & & \swarrow & & \swarrow \\
\frac{1}{4} & & \frac{2}{4} & & \frac{3}{4} & & \frac{4}{4} & \cdots \\
& \swarrow & & \swarrow & & \swarrow & & \\
\vdots & & \vdots & & \vdots & & \vdots &
\end{array}$$

We count the terms in the order indicated by the arrows, omiting fractions which represent numbers counted before. This associates a natural number to any positive rational number and arranges the positive rational numbers in a sequence $\{q_n\}_{n=1}^{\infty}$. All rational numbers are contained in the sequence $\{0, q_1, -q_1, q_2, -q_2, \ldots\}$. ∎

Theorem 3.15 *The set* $\mathbb{R}$ *of all real numbers is uncountable.*

Proof: Assume that the set $\mathbb{R}$ is countable. Then all its elements can be arranged in a sequence (can be numbered)

$$\{x_1, x_2, x_3, \ldots\}.$$

Now I construct a number which certainly does not belong to this sequence. This is a contradiction, hence the assumption must be false and the theorem must be true.

To construct such a number, choose an intervall $I_1 = [a_1, b_1]$ with $a_1 < b_1$, which does not contain x_1. Subdivide I_1 into three closed subintervals of equal length. x_2 cannot be contained in all three subintervals. Choose one of the subintervals not containing x_2 and denote it by $I_2 = [a_2, b_2]$. Repeating this construction indefinitely yields a sequence $\{I_n\}_{n=1}^{\infty}$ of nested intervals:

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots .$$

Every interval is of the form $I_n = [a_n, b_n]$. Moreover, $x_n \notin I_n$ for every n. The set $\{a_n \mid n \in \mathbb{N}\}$ is bounded above; every b_n is an upper bound. Therefore

$$s = \sup \{a_n \bigm| n \in \mathbb{N}\}$$

exists. s belongs to everyone of the intervals I_n. If this would not be true, then $n_0 \in \mathbb{N}$ would exist with $s \notin I_{n_0}$, whence $s > b_{n_0}$, since by definition $s \geq a_{n_0}$. Consequently, s could not be the supremum, since every b_n is an upper bound. Thus we proved $s \in I_n$ and $x_n \notin I_n$ for all $n \in \mathbb{N}$, hence $s \neq x_n$ for all n. ∎

3.4 Vector spaces of real valued functions

Let $D \neq \emptyset$ be a set and let $F(D) = F(D, \mathbb{R})$ be the set of all real valued functions $f : D \to \mathbb{R}$. On the set $F(D)$ an addition operation is defined, which assigns to every pair of functions $f, g \in F(D)$ the function $f + g \in F(D)$ defined by

$$(f + g)(x) = f(x) + g(x), \quad x \in D.$$

This operation satisfies the axioms A1 – A4 of section 2:

V1. $f + g = g + f$

V2. $f + (g + h) = (f + g) + h$

V3. there is exactly one neutral element, namely the function $f : D \to \mathbb{R},\ x \mapsto f(x) := 0$. This function is denoted by 0.

V4. to every f there is exactly one g such that $f + g = 0$, namely the function $g : D \to \mathbb{R}$, $x \mapsto g(x) := -f(x)$. This function ist denoted by $-f$.

We note that every set with an operation satisfying these four axioms is called a commutative group.

Every function $f \in F(D)$ can be multiplied by a number $a \in \mathbb{R}$. The product af is a function from $F(D)$ and is defined by

$$(af)(x) = a \cdot f(x),\ x \in D,$$

where $a \cdot f(x)$ is the multiplication of real numbers. For this multiplication operation the following rules hold:
Let $a, b \in \mathbb{R}$ and $f, g \in F(D)$. Then

V5. $(a + b)f = af + ag$

V6. $a(f + g) = af + ag$

V7. $a(bf) = (ab)f$

V8. $1f = f$

A set with an addition operation and with a multiplication by real numbers (scalars), such that V1. – V8. are satisfied, is called a real vector space or a vector space over $\mathbb{R}$.

Therefore a vector space consists of a set V and of two operations $+$ and $\cdot$. Hence, a vector space is a triple $(V, +, \cdot)$. However, if the operations $+$ und $\cdot$ are understood, one often speaks of the vector space V for simplicity. The triple $(F(D), +, \cdot)$ with the addition and multiplication defined above is a real vector space.

Definition 3.16 *$(U, +, \cdot)$ is a subspace of the vector space $(V, +, \cdot)$, if U is a subset of V and if U with the operations $+$ and $\cdot$ induced on U by V is a vector space.*

If $(V, +, \cdot)$ is a vector space and if U is a non–empty subset of V, then $(U, +, \cdot)$ is a subspace of $(V, +, \cdot)$, if for all $f, g \in U$ and all $a \in \mathbb{R}$ the elements $f + g$ and af also belong to U. For, the axioms V3. und V4. are satisfied, since $0 = 0 \cdot f \in U$ and $-f = (-1) \cdot f \in U$. The axioms V1., V2. and V5. – V8. are satisfied automatically on U, because $+$ und $\cdot$ satisfy these axioms on V.

A number of subspaces of $F(D)$ play an important part in analysis. I consider some examples. Further examples are introduced later:

1.) The set of all bounded functions $f : D \to \mathbb{R}$ is a subspace of $F(D)$. A function f is said to be bounded, if

$$\exists_{C>0} \ \forall_{x \in D} \ : |f(x)| \le C.$$

This set is a subspace, since with bounded functions f and g also $f+g$ and af are bounded functions for all $a \in \mathbb{R}$. To see this, let C_1 und C_2 be constants with $|f(x)| \le C_1$ and $|g(x)| \le C_2$ for all $x \in D$. Then

$$|(f + g)(x)| = |f(x) + g(x)| \le |f(x)| + |g(x)| \le C_1 + C_2$$

and

$$|(af)(x)| = |a \cdot f(x)| \le |a| \ |f(x)| \le |a| C_1$$

for all $x \in D$.

For $D = \mathbb{N}$, the vector space of bounded numerical sequences is obtained.

If the set D is finite, then the dimension of the subspace of bounded functions is equal to the number of elements of D. If D is an infinite set, then also the dimension of this subspace is infinite. (The dimension of a vector space is defined in the linear algebra course.)

2.) The set of polynomials is a subspace of $F(\mathbb{R})$. For, if p_1 and p_2 are polynomials, then also the sum $p_1 + p_2$ is a polynomial and for every real number a also $a \cdot p_1$ is a polynomial. The dimension of the subspace of all polynomials is infinite.

3.) Let $n \in \mathbb{N}_0$ be a given number. The set of all polynomials, whose degree does not exceed n, is a finite dimensional subspace of $F(\mathbb{R})$. The dimension of this subspace is $n + 1$.

4.) The set of step functions is a subspace of $F(\mathbb{R})$. For, by definition a step function is a linear combination of characteristic functions of intervals. Thus, if $t_1 = \sum_{k=1}^{m} a_k \chi_{I_k}$ and $t_2 = \sum_{\ell=1}^{n} b_\ell \chi_{J_\ell}$ are step functions, then also $t_1 + t_2 = \sum_{k=1}^{m} a_k \chi_{I_k} + \sum_{\ell=1}^{n} b_\ell \chi_{J_\ell}$ is such a linear combination, hence it is a step function. Likewise, for every step function $t = \sum_{k=1}^{m} a_k \chi_{I_k}$ and for every real number a the product $at = \sum_{k=1}^{m} a a_k \chi_{I_k}$ is a step function.

The dimension of the subspace of step functions is infinite.

On the vector space $F(D, \mathbb{R})$ a multiplication can be defined which assigns to every pair of functions $f, g \in F(D, \mathbb{R})$ a function $fg \in F(D, \mathbb{R})$, the product. This product is given by

$$(fg)(x) := f(x) \cdot g(x), \; x \in D.$$

This multiplication of functions is associative and commutative. The constant function $x \mapsto 1$ is the unique neutral element. Furthermore, the distributive law holds for the multiplication and addition of functions. However, to an element $f \in F(D, \mathbb{R})$ not necessarily an inverse element exists: an inverse element exists if and only if f does not have zeros. In this case the inverse is $x \mapsto \frac{1}{f(x)}$.

Therefore, $F(D, \mathbb{R})$ with the addition and multiplication is not a field, but it is a commutative, associative algebra with unit. This algebra contains divisors of zero, since it is easy to construct functions $f \neq 0$, $g \neq 0$ with $fg = 0$.

All of the examples given above are subalgebras of $F(D)$ save the subspace of polynomials with degree at most n. (The notions “algebra” and “divisor of zero” are introduced in algebra courses.)

3.5 Simple properties of real functions

By the definition given in the previous section, a real function f is bounded if and only if the range $W(f)$ is a bounded subset of $\mathbb{R}$.

Definition 3.17 *A real function f is said to be bounded above if the range $W(f) \subseteq \mathbb{R}$ is bounded above. f is said to be bounded below if $W(f)$ is bounded below.*

Obviously, f is bounded if and only if it is bounded below and above.

Definition 3.18 *A real function $f : D \to \mathbb{R}$ is said to be increasing (decreasing), if and only if*

$$\forall_{x_1,x_2\in D} : x_1 \le x_2 \Longrightarrow f(x_1) \le f(x_2) \ \big(f(x_1) \ge f(x_2)\big).$$

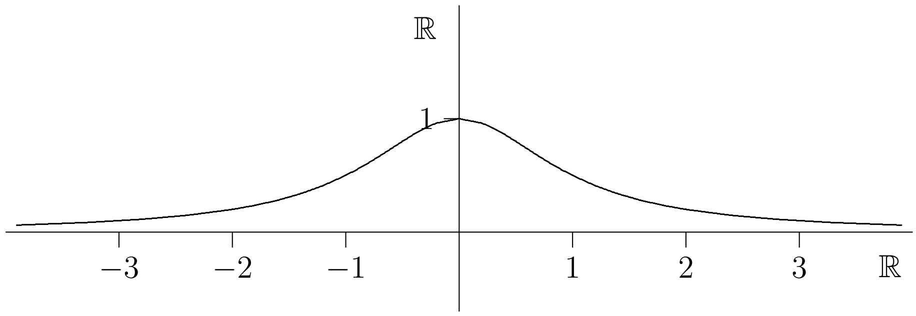
f is said to be strictly increasing (strictly decreasing), if and only if

$$\forall_{x_1,x_2\in D} : x_1 < x_2 \Longrightarrow f(x_1) < f(x_2) \ \big(f(x_1) > f(x_2)\big).$$

A function is called monotone, if it is increasing or decreasing. It is called strictly monotone, if it is strictly increasing or strictly decreasing.

Examples.

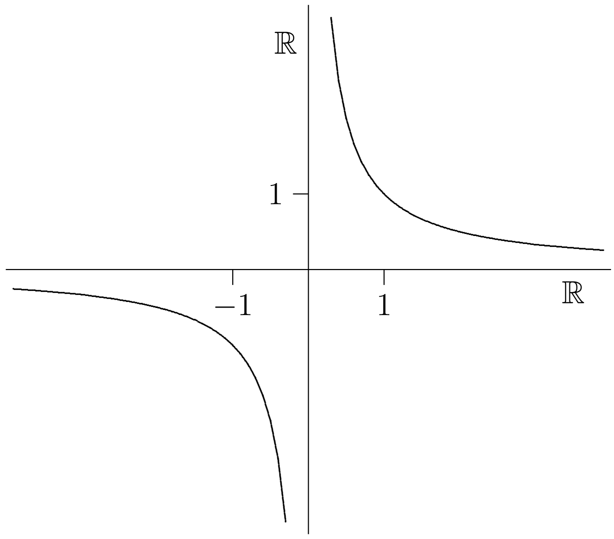
1.) $x \mapsto \dfrac{1}{1+x^2} : \mathbb{R} \to \mathbb{R}$.



This function is bounded, but not monotone. The restriction of this function to $[0,\infty)$ is however strictly decreasing. For, let $0 \le x_1 < x_2$. Then $x_1^2 < x_2^2$, hence $1 + x_1^2 < 1 + x_2^2$, and so

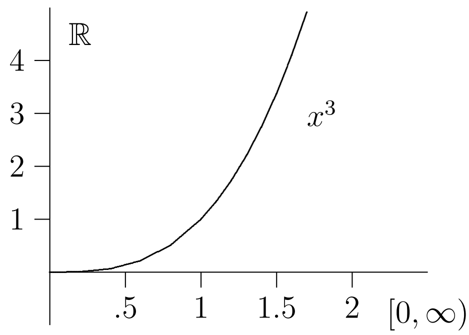
$$\frac{1}{1+x_1^2} > \frac{1}{1+x_2^2}.$$

2.) $x \mapsto \dfrac{1}{x} : \mathbb{R}\backslash\{0\} \to \mathbb{R}$.



This function is not bounded and not monotone. However, the restrictions of this function to the intervals $(-\infty, 0)$ and $(0, \infty)$ both are strictly decreasing.

3.) $x \mapsto x^n : [0, \infty) \to \mathbb{R}, \;\; n \in \mathbb{N}$.



This function is not bounded, but strictly increasing. This can be proved by induction with respect to n.

4.) $x \mapsto [x] : \mathbb{R} \to \mathbb{Z}$. The graph of this integer part function is shown in Section 3.2. It is unbounded, increasing, but not strictly increasing. In Section 3.2 we already mentioned that this function is not injective.

However, strictly monotone functions are injective:

Theorem 3.19 *Let f be a real, strictly monotone function. Then f is one–to–one, whence on the set $W(f)$ the inverse function $f^{-1} : W(f) \to D(f)$ exists. f^{-1} is strictly monotone as well.*

Proof: Without restriction of generality assume that f is strictly increasing. If $x_1 \neq x_2$, then $x_1 < x_2$ or $x_2 < x_1$ holds. Since f is strictly increasing, this implies $f(x_1) < f(x_2)$ or $f(x_2) < f(x_1)$, whence $f(x_1) \neq f(x_2)$. Therefore f is one–to–one. Consequently, the inverse $f^{-1} : W(f) \to D(f)$ exists. f^{-1} is strictly increasing. For otherwise $y_1, y_2 \in W(f)$ would exist with $y_1 < y_2$ and with $f^{-1}(y_1) \geq f^{-1}(y_2)$. Because f is increasing, this would result in

$$y_1 = f\big(f^{-1}(y_1)\big) \geq f\big(f^{-1}(y_2)\big) = y_2,$$

an obvious contradiction. ■

The strict monotonicity is a sufficient, but not a necessary condition for the invertibility of a real function. This can be seen from the following

Example: Let $D = [0, 1]$, and let the function $f : D \to \mathbb{R}$ be defined by

$$x \mapsto f(x) := \begin{cases} x, & x \in \mathbb{Q} \\ 1 - x, & x \notin \mathbb{Q}. \end{cases}$$

This function maps the interval $[0, 1]$ onto itself in a one–to–one way, but in no subinterval of $[0, 1]$ it is monotone.

Powers with rational exponent. In the section on continuous functions it will be shown that for every $n \in \mathbb{N}$ the range of the function $x \mapsto x^n : [0, \infty) \to [0, \infty)$ is $[0, \infty)$, hence this function is surjective. For $n = 2$ this was proved in Section 3.2. As in that proof, we need the completeness of the real numbers also in the case of general n. Since this function is also strictly increasing, we conclude from Theorem 3.19 that the inverse function

$$x \mapsto x^{\frac{1}{n}} : [0, \infty) \to [0, \infty)$$

exists, which is strictly increasing as well. This defines the power function to the exponent $\frac{1}{n}$. For a rational number $q = \frac{m}{n} > 0$ and for $x \geq 0$ the power x^q can be explained in two ways:

$$x^{\frac{m}{n}} = \big(x^{\frac{1}{n}}\big)^m \quad \text{or} \quad x^{\frac{m}{n}} = \big(x^m\big)^{\frac{1}{n}}.$$

Both definitions agree. For, we have

$$x^m = \Big[\big(x^{\frac{1}{n}}\big)^n\Big]^m = \big(x^{\frac{1}{n}}\big)^{n \cdot m} = \Big[\big(x^{\frac{1}{n}}\big)^m\Big]^n,$$

whence

$$\big(x^m\big)^{\frac{1}{n}} = \big(x^{\frac{1}{n}}\big)^m.$$

Thus, x^q is defined for all $x \geq 0$ and all positive rational q. For negative rational q we set

$$x^q := \frac{1}{x^{-q}}, \quad x > 0.$$

Moreover, we set

$$x^0 := 1, \quad x \geq 0.$$

With these definitions one has

$$\begin{aligned} x^r \cdot x^s &= x^{r+s} \\ x^r \cdot y^r &= (xy)^r \\ (x^r)^s &= x^{rs}. \end{aligned}$$

4 Convergent sequences

4.1 Fundamental definitions and properties

All the terms of the sequence

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$$

are positive, but they approach zero for growing n. In contrast, the terms of the sequence

$$\left\{(-1)^n \frac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, \ldots\right\}$$

do not approach a number for large n. Thus, intuitively one observes that the first sequence has a property, which the second does not have. This property is stated precisely in the following.

Definition 4.1 *A numerical sequence $\{x_n\}_{n=1}^{\infty}$ is said to converge to the number a, if to every number $\varepsilon > 0$ there is a number $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ with $n \geq n_0$ the inequality*

$$|x_n - a| < \varepsilon$$

holds. a is called limit of the sequence $\{x_n\}_{n=1}^{\infty}$.

Using quantifiers this can be written in the following form: $\{x_n\}_{n=1}^{\infty}$ converges to $a \in \mathbb{R}$, if

$$\forall_{\varepsilon>0} \; \exists_{n_0 \in \mathbb{N}} \; \forall_{\substack{n \in \mathbb{N} \\ n \geq n_0}} \; : \; |x_n - a| < \varepsilon.$$

Definition 4.2 *Let $\varepsilon > 0$ and $a \in \mathbb{R}$. The set*

$$U_\varepsilon(a) := \{x \in \mathbb{R} \,\big|\, |x - a| < \varepsilon\}$$

is called ε–neighborhood of a. The set $U \subseteq \mathbb{R}$ is called neighborhood of a, if it contains an ε–neighborhood of a as a subset.

With these notions the property of convergence can be stated in a third way:
The sequence $\{x_n\}_{n=1}^{\infty}$ converges to $a \in \mathbb{R}$, if to every ε–neighborhood U_ε there is a number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$x_n \in U_\varepsilon.$$

Equivalently, this can also be formulated as follows:
$\{x_n\}_{n=1}^{\infty}$ converges to $a \in \mathbb{R}$, if to every neighborhood U there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$x_n \in U.$$

If a sequence $\{x_n\}_{n=1}^{\infty}$ converges to the limit a, one writes

$$\lim_{n \to \infty} x_n = a.$$

A numerical sequence $\{x_n\}_{n=1}^{\infty}$ with

$$\lim_{n \to \infty} x_n = 0$$

is called a *null sequence*. The definition of convergence immediately yields the following

Lemma 4.3 *A sequence $\{x_n\}_{n=1}^{\infty}$ converges to $a \in \mathbb{R}$, if and only if $\{x_n - a\}_{n=1}^{\infty}$ is a null sequence.*

Theorem 4.4 *Every sequence possesses at most one limit.*

Proof: Let a, b be limits of $\{x_n\}_{n=1}^{\infty}$, and let $\varepsilon > 0$ be an arbitrary number. Since a and b are limits, there exist numbers $n_0, n_1 \in \mathbb{N}$ with $|x_n - a| < \varepsilon$ for $n \geq n_0$ and $|x_n - b| < \varepsilon$ for $n \geq n_1$. For $n \geq \max(n_0, n_1)$ one thus has $|x_n - a| < \varepsilon$ and $|x_n - b| < \varepsilon$, whence

$$|a - b| = |(a - x_n) - (b - x_n)| \leq |x_n - a| + |x_n - b| < 2\varepsilon.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, this implies $|a - b| = 0$, hence $a = b$. ■

A sequence is said to be *convergent*, if it has a limit. Otherwise, it is said to be divergent.

The convergence behavior of a sequence is not modified, if finitely many terms of the sequence are modified. That is, if the sequence has the limit a, then also the modified sequence has this limit; if the sequence diverges, then also the modified sequence diverges. Before studying some properties of convergent sequences, I consider the following

Example: $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to zero, hence it is a null sequence.

Proof: To every $\varepsilon > 0$ we must find a number $n_0 \in \mathbb{N}$ such that $|\frac{1}{n} - 0| = \frac{1}{n} < \varepsilon$ holds for all $n \geq n_0$.

To find such a number n_0, note that

$$\Big\{ n \,\Big|\, n \in \mathbb{N} \wedge n > \frac{1}{\varepsilon} \Big\} \neq \emptyset,$$

because of the Archimedian order of the real numbers. Therefore, as a subset of the natural numbers, this set has a smallest element, which we denote by n_0. For all $n \geq n_0$ we thus have $n > \frac{1}{\varepsilon}$, hence $\frac{1}{n} < \varepsilon$. Consequently, n_0 is the number sought. ■

Other examples will be considered later.

Theorem 4.5 *Every convergent sequence is bounded.*

Proof: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence converging to $a \in \mathbb{R}$. Then there exists $n_0 \in \mathbb{N}$ with $|x_n - a| < 1$ for all $n \geq n_0$, thus

$$|x_n| = |x_n - a + a| \leq |x_n - a| + |a| < |a| + 1$$

for all $n \geq n_0$. Therfore

$$\forall_{n\in\mathbb{N}} : |x_n| \leq \max \Big(\big\{x_n \,\big|\, n < n_0\big\} \cup \big\{|a|+1\big\}\Big).$$

■

Theorem 4.6 *Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ be convergent sequences with $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} y_n = b$. Then also the sequences $\{x_n + y_n\}_{n=1}^{\infty}$, $\{x_n - y_n\}_{n=1}^{\infty}$ and $\{x_n y_n\}_{n=1}^{\infty}$ converge, and*

$$\lim_{n\to\infty} (x_n \pm y_n) = a \pm b, \quad \lim_{n\to\infty} x_n y_n = ab.$$

If all y_n and the limit b are different from 0, then also the sequence of quotients $\{\frac{x_n}{y_n}\}_{n=1}^{\infty}$ converges, and

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{a}{b}.$$

This theorem implies that the sequence $\{cx_n\}_{n=1}^{\infty}$ converges to ca if c is a real number and $\lim_{n\to\infty} x_n = a$ holds. For, the constant sequence $\{c\}_{n=1}^{\infty}$ obviously converges to c.

Proof of the theorem: Let $\varepsilon > 0$ be given. There is $n_0, n_1 \in \mathbb{N}$ with $|x_n - a| < \frac{\varepsilon}{2}$ for all $n \geq n_0$ and with $|y_n - b| < \frac{\varepsilon}{2}$ for all $n \geq n_1$. For $n \geq n_2 := \max(n_0, n_1)$ this implies

$$|(x_n \pm y_m) - (a \pm b)| = |(x_n - a) \pm (y_n - b)| \leq |x_n - a| + |y_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves the statement for the sum and the difference. To prove the statement for the product we use that, as a convergent sequence, the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. Let $c > 0$ denote an upper bound for the set $\{|x_n| \,\big|\, n \in \mathbb{N}\}$, and let $\varepsilon > 0$ be given. By assumption, $n_0 \in \mathbb{N}$ and $n_1 \in \mathbb{N}$ can be chosen with $|x_n - a| < \varepsilon/(2|b|+1)$ for all $n \geq n_0$ and with $|y_n - b| < \varepsilon/(2c)$ for all $n \geq n_1$. From these relations we infer that for $n \geq n_2 := \max(n_0, n_1)$

$$\begin{aligned}
|x_n y_n - ab| &= |x_n y_n - x_n b + x_n b - ab| \\
&= |x_n(y_n - b) + b(x_n - a)| \leq |x_n|\,|y_n - b| + |b|\,|x_n - a| \\
&\leq c\frac{\varepsilon}{2c} + |b|\frac{\varepsilon}{2|b|+1} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This concludes the proof for the product. To prove the statement for the quotient, it suffices to verify that

$$\lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{b},$$

since the sequence of quotients $\{\frac{x_n}{y_n}\}_{n=1}^{\infty}$ can be written as the sequence $\{x_n \frac{1}{y_n}\}_{n=1}^{\infty}$ of products.

The sequence $\{\frac{1}{y_n}\}_{n=1}^{\infty}$ is bounded. For, $\lim_{n\to\infty} y_n = b$ implies that $n_0 \in \mathbb{N}$ exists with $|b - y_n| < |b|/2$ for $n \geq n_0$, whence

$$|y_n| = |y_n - b + b| \geq |b| - |y_n - b| > \frac{|b|}{2},$$

by the inverse triangle inequality. This yields

$$\frac{1}{|y_n|} \leq c := \max\left(\left\{|\frac{1}{y_n}| \,\middle|\, n < n_0\right\} \cup \left\{\frac{2}{|b|}\right\}\right)$$

for all $n \in \mathbb{N}$, thus

$$|\frac{1}{y_n} - \frac{1}{b}| = |\frac{b - y_n}{y_n b}| \leq \frac{c}{|b|}\, |b - y_n|.$$

Now let $\varepsilon > 0$ be given. Then there is $n_1 \in \mathbb{N}$ such that $|b - y_n| < \frac{|b|}{c}\,\varepsilon$ for $n \geq n_1$, so

$$|\frac{1}{y_n} - \frac{1}{b}| \leq \frac{c}{|b|}\, |b - y_n| < \varepsilon$$

for all $n \geq n_1$. ■

The set of numerical sequences is equal to the space $F(\mathbb{N}, \mathbb{R})$ of real functions with domain of definition $\mathbb{N}$. Therefore the sum and the product of two sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ are defined by $\{x_n + y_n\}_{n=1}^{\infty}$ and by $\{x_n y_n\}_{n=1}^{\infty}$. As discussed in the last section, with this addition and with the multiplication by scalars $F(\mathbb{N}, \mathbb{R})$ is a vector space. Since by the last theorem the sum of two convergent sequences is a convergent sequence and the multiplication of a convergent sequence by a scalar yields a convergent sequence, we obtain

Corollary 4.7 *The set of convergent sequences and the set of null sequences are subspaces of* $F(\mathbb{N}, \mathbb{R})$.

Theorem 4.8 *Let* F *be a rational function with domain of definition* D, *and let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence with* $x_n \in D$ *for all* $n \in \mathbb{N}$ *and with* $\lim_{n\to\infty} x_n = a \in D$. *Then the sequence* $\{F(x_n)\}_{n=1}^{\infty}$ *converges with*

$$\lim_{n\to\infty} F(x_n) = F(a).$$

Note that the statement can also be written as

$$\lim_{n\to\infty} F(x_n) = F\big(\lim_{n\to\infty} x_n\big)\,.$$

"Rational functions and limits can be interchanged."

Proof: With suitable coefficients one has $F(x) = \frac{a_0+\ldots+a_r x^r}{b_0+\ldots+b_s x^s}$. The theorem proved above implies that the sequences $\{a_0 + \ldots + a_r x_n^r\}_{n=1}^\infty$ and $\{b_0 + \ldots + b_s x_n^s\}_{n=1}^\infty$ converge with limits $a_0 + \ldots + a_r a^r$ and $b_0 + \ldots b_s a^s$. By definition of rational functions, for $x_n, a \in D$ the polynomials $b_0 + \ldots + b_s x_n^s$ and $b_0 + \ldots + b_s a^s$ differ from zero, hence the last theorem implies that the sequence of quotients

$$F(x_n) = \frac{a_0 + \ldots + a_r x_n^r}{b_0 + \ldots + b_s x_n^s}$$

converges to $F(a)$. ■

Theorem 4.9 *Let $\{x_n\}_{n=1}^\infty$ be a null sequence and $\{y_n\}_{n=1}^\infty$ be a bounded sequence. Then $\{x_n y_n\}_{n=1}^\infty$ is a null sequence.*

Proof: Let $c > 0$ be an upper bound for $\{|y_n|\}_{n=1}^\infty$. Let $\varepsilon > 0$. Then there is $n_0 \in \mathbb{N}$ with $|x_n - 0| = |x_n| < \frac{\varepsilon}{c}$ for all $n \geq n_0$, whence

$$|x_n y_n - 0| = |x_n y_n| = |x_n|\,|y_n| < c\frac{\varepsilon}{c} = \varepsilon$$

for all $n \geq n_0$. ■

Theorem 4.10 *Let $\{x_n\}_{n=1}^\infty$ be a sequence with $\lim_{n\to\infty} x_n = a$. Then $\{|x_n|\}_{n=1}^\infty$ converges with $\lim_{n\to\infty} |x_n| = |a|$.*

Proof: Let $\varepsilon > 0$ be given. Then there is $n_0 \in \mathbb{N}$ with $|x_n - a| < \varepsilon$ for $n \geq n_0$, whence, by the inverse triangle inequality,

$$\big|\,|x_n| - |a|\,\big| \leq |x_n - a| < \varepsilon.$$ ■

It is easy to construct an example which shows that the converse statement is false.

Theorem 4.11 *Let $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ be sequences.*

(i) If $\lim_{n\to\infty} x_n = a$, $\lim_{n\to\infty} y_n = b$ and $x_n \leq y_n$ for all n, then $a \leq b$.

(ii) If $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = a$ and $x_n \leq z_n \leq y_n$ for all n, then

$$\lim_{n\to\infty} z_n = a.$$

Proof: (i) Let $\varepsilon > 0$. Then there are $n_0, n_1 \in \mathbb{N}$ with $|x_n - a| < \varepsilon$ for all $n \geq n_0$ and $|y_n - b| < \varepsilon$ for all $n \geq n_1$. Then for $n \geq n_2 = \max(n_0, n_1)$

$$\begin{aligned} a &= a - x_n + x_n \leq |a - x_n| + x_n \leq x_n + \varepsilon \\ &\leq y_n + \varepsilon = y_n - b + b + \varepsilon \leq |y_n - b| + b + \varepsilon \leq b + 2\varepsilon. \end{aligned}$$

Thus

$$a \leq b + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $a \leq b$. ■

(ii) Let $\varepsilon > 0$. There exist $n_0, n_1 \in \mathbb{N}$ with $|x_n - a| < \varepsilon$ for all $n \geq n_0$ and $|y_n - a| < \varepsilon$ for all $n \geq n_1$. For $n \geq n_2 = \max(n_0, n_1)$ we thus have

$$-\varepsilon < -|a - x_n| \leq -(a - x_n) = x_n - a \leq z_n - a \leq y_n - a \leq |y_n - a| < \varepsilon,$$

hence

$$|z_n - a| < \varepsilon.$$ ■

Definition 4.12 *Let A be a set, let $\{x_n\}_{n=1}^{\infty}$ be a sequence and let $\varphi : \mathbb{N} \to \mathbb{N}$ be a strictly increasing mapping. Then the sequence*

$$(n \mapsto x_{\varphi(n)}) : \mathbb{N} \to A$$

is called a subsequence of $\{x_n\}_{n=1}^{\infty}$. This subsequence is denoted by $\{x_{\varphi(n)}\}_{n=1}^{\infty}$ or, since φ itself is a sequence, by $\{x_{j_n}\}_{n=1}^{\infty}$, where we set $j_n := \varphi(n)$.

Theorem 4.13 *Let $\{x_n\}_{n=1}^{\infty}$ be a numerical sequence converging to a. Then every subsequence $\{x_{\varphi(n)}\}_{n=1}^{\infty}$ converges to a.*

Proof: φ is strictly increasing, which implies

$$\varphi(n) \geq n$$

for all $n \in \mathbb{N}$. We prove this by induction:
a) For $n = 1$ the inequality is obviously satisfied, since $\varphi(1) \in \mathbb{N}$ implies $\varphi(1) \geq 1$.
b.) Let n be a natural number and assume that the inequality is fulfilled for n. Then, since φ is strictly increasing,

$$\varphi(n+1) > \varphi(n) \geq n,$$

hence $\varphi(n+1) \geq n + 1$. Consequently, the inequality is satisfied for all $n \in \mathbb{N}$.

The theorem can now be proved as follows: Since $\lim_{n\to\infty} x_n = a$, to $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$ for all $n \geq n_0$. Whence

$$|x_{\varphi(n)} - a| < \varepsilon$$

for all $n \geq n_0$, since for such n one has $\varphi(n) \geq n \geq n_0$. ■

4.2 Examples for converging and diverging sequences

1.) For $p \in \mathbb{N}$ the sequence $\{n^{-\frac{1}{p}}\}_{n=1}^{\infty}$ is a null sequence:

$$\lim_{n\to\infty} n^{-\frac{1}{p}} = \lim_{n\to\infty} \frac{1}{n^{1/p}} = 0.$$

Proof: Let $\varepsilon > 0$ be given. Then

$$|n^{-\frac{1}{p}} - 0| = n^{-\frac{1}{p}} = \frac{1}{n^{1/p}} < \varepsilon \Longleftrightarrow n > \frac{1}{\varepsilon^p} = \varepsilon^{-p}.$$

The set $\{n \,\big|\, n \in \mathbb{N} \wedge n > \varepsilon^{-p}\}$ is not empty, hence it has a smallest element n_0. For all $n \geq n_0$ we thus have $n > \varepsilon^{-p}$, whence $n^{-\frac{1}{p}} < \varepsilon$. ■

2.) **Corollary** *Let $q > 0$ be rational. Then*

$$\lim_{n\to\infty} n^{-q} = 0.$$

Proof: Let $q = \frac{r}{s}$ with $r, s \in \mathbb{N}$, and let the rational function F be defined by

$$x \mapsto F(x) := x^r : \mathbb{R} \to \mathbb{R}.$$

Since $\lim_{n\to\infty} n^{-\frac{1}{s}} = 0$ and since rational functions and limits can be interchanged, we obtain

$$\lim_{n\to\infty} n^{-\frac{r}{s}} = \lim_{n\to\infty} F(n^{-\frac{1}{s}}) = F(\lim_{n\to\infty} n^{-\frac{1}{s}}) = F(0) = 0.$$ ■

3.) Let $q \in \mathbb{R}$. For the sequence $\{q^n\}_{n=1}^{\infty}$ we must distinguish the following cases:

(i) If $|q| < 1$, then $\lim_{n\to\infty} q^n = 0$.

(ii) If $q = 1$, then $\lim_{n\to\infty} q^n = 1$.

(iii) If $q = -1$, then $\{q^n\}_{n=1}^{\infty}$ is bounded and diverges.

(iv) If $|q| > 1$, then $\{q^n\}_{n=1}^{\infty}$ is unbounded, hence diverges.

Proof: (i) For $q = 0$ the statement is true. So let $0 < |q| < 1$. We have to show that $n_0 \in \mathbb{N}$ exists such that

$$|q^n| = |q|^n < \varepsilon$$

holds for all $n \geq n_0$. To this end we set $h := \frac{1}{|q|} - 1$. Since $h > 0$, the Bernoulli inequality yields

$$\frac{1}{|q|^n} = (1+h)^n \geq 1 + nh > nh = n\Big(\frac{1}{|q|} - 1\Big),$$

thus

$$|q|^n < \frac{1}{n}\,\frac{|q|}{1-|q|}.$$

Now choose

$$n_0 := \min\Big\{n \,\Big|\, n \in \mathbb{N} \wedge \Big(\frac{1}{n}\,\frac{|q|}{1-|q|} < \varepsilon\Big)\Big\}.$$

(ii) Evident.

(iii) For $q = -1$ one has

$$\{q^n\}_{n=1}^{\infty} = \big\{(-1)^n\big\}_{n=1}^{\infty} = \{-1, 1, -1, 1, \ldots\}.$$

This sequence contains as subsequences the constant sequences $\{-1, -1, -1, \ldots\}$ and $\{1, 1, 1, \ldots\}$, which converge to -1 and 1, respectively. If $\{q^n\}_{n=1}^{\infty}$ would converge, then both subsequences would have to converge to the same limit, namely to the uniquely determined limit of $\{q^n\}_{n=1}^{\infty}$. Therefore $\{q^n\}_{n=1}^{\infty}$ must diverge.

(iv) For $q \in \mathbb{R}$ with $|q| > 1$ let $h := |q| - 1$. Then the Bernoulli inequality gives

$$|q|^n = (1+h)^n \geq 1 + nh > nh = n(|q| - 1).$$

Since $|q| - 1 > 0$, the set $\{n(|q|-1) \,\big|\, n \in \mathbb{N}\}$ is unbounded, hence $\{q^n\}_{n=1}^{\infty}$ is unbounded and cannot converge. ■

4.) The sequence

$$\Big\{(-1)^n \frac{n}{n+1}\Big\}_{n=1}^{\infty}$$

is divergent.

Proof: With $n \mapsto \varphi(n) := 2n : \mathbb{N} \to \mathbb{N}$ and $n \mapsto \chi(n) := 2n+1 : \mathbb{N} \to \mathbb{N}$ we obtain the subsequences

$$\Big\{(-1)^{\varphi(n)} \frac{\varphi(n)}{\varphi(n)+1}\Big\}_{n=1}^{\infty} = \Big\{\frac{2n}{2n+1}\Big\}_{n=1}^{\infty}$$

and

$$\Big\{(-1)^{\chi(n)} \frac{\chi(n)}{\chi(n)+1}\Big\}_{n=1}^{\infty} = \Big\{-\frac{2n+1}{2n+2}\Big\}_{n=1}^{\infty}.$$

We prove that the first subsequence converges to 1, the second to -1, As a consequence, the original sequence must diverge.

To prove that the limit of the first subsequence is 1, observe that

$$\frac{2n}{2n+1} = \frac{2n+1}{2n+1} - \frac{1}{2n+1} = 1 - \frac{1}{2n+1}.$$

Since $\{1, 1, 1, \ldots\}$ converges to 1 and $\{\frac{1}{2n+1}\}_{n=1}^{\infty}$ is a null sequence, we obtain

$$\lim_{n\to\infty}\Big(1 - \frac{1}{2n+1}\Big) = \lim_{n\to 1} 1 - \lim_{n\to\infty}\frac{1}{2n+1} = 1 - 0 = 1.$$

Similarly

$$\lim_{n\to\infty}\Big(-\frac{2n+1}{2n+2}\Big) = \lim_{n\to\infty}\Big(-1 + \frac{1}{2n+2}\Big) = -1 + 0 = -1.$$

■

5.) For a, $b > 0$ let the sequence $\{x_n\}_{n=1}^{\infty}$ be defined recursively by

$$\begin{aligned} x_1 &:= b \\ x_{n+1} &:= \frac{1}{2}\Big(x_n + \frac{a}{x_n}\Big). \end{aligned}$$

If this sequence converges, then

$$\lim_{n\to\infty} x_n = \sqrt{a}.$$

Proof: First I show that

$$x_n \geq \sqrt{a} \qquad (*)$$

for all $n = 2, 3, 4, \ldots$. To this end note that for $x_n > 0$ the relation

$$0 \leq \Big(\sqrt{x_n} - \sqrt{\frac{a}{x_n}}\Big)^2 = x_n - 2\sqrt{a} + \frac{a}{x_n}$$

implies

$$\sqrt{a} \leq \frac{1}{2}\Big(x_n + \frac{a}{x_n}\Big) = x_{n+1}.$$

Therefore, if $(*)$ holds for a number $n \in \mathbb{N}$, then $x_n > 0$, and so $x_{n+1} \geq \sqrt{a}$. Furthermore, by definition we have $x_1 = b > 0$, and so $x_2 \geq \sqrt{a}$. This proves $(*)$ by induction.

From $(*)$ we obtain $c = \lim_{n\to\infty} x_n \geq \sqrt{a}$. Since $F(x) = \frac{1}{2}\,(x + \frac{a}{x})$ is a rational function with $(0, \infty) \subseteq D(F)$, and since rational functions can be interchanged with limits, we find

$$\begin{aligned} c &= \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}\Big(x_n + \frac{a}{x_n}\Big) \\ &= \lim_{n\to\infty} F(x_n) = F(\lim_{n\to\infty} x_n) = F(c) = \frac{1}{2}\Big(c + \frac{a}{c}\Big). \end{aligned}$$

This equation yields $c^2 = a$, hence $c = \sqrt{a}$. ■

Therefore, if this recursive sequence converges, it can be used to compute $\sqrt{a}$ approximately. Below we show that the sequence actually converges.

4.3 Accumulation point, Bolzano-Weierstraß theorem, Cauchy sequence

Since sequences are real functions, the notions "monotone", "decreasing" and "increasing" are defined for sequences.

Theorem 4.14 *A monotone bounded sequence is convergent.*

Proof: Let $\{x_n\}_{n=1}^{\infty}$ be an increasing, bounded sequence, and let $c = \sup\{x_n \mid n \in \mathbb{N}\}$. Then to every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ with

$$c - \varepsilon \le x_{n_0} \le c,$$

hence

$$c - \varepsilon \le x_n \le c,$$

for all $n \ge n_0$, since the sequence is increasing. This implies $|x_n - c| \le \varepsilon$ for all $n \ge n_0$. Thus, the sequence converges to c. In the same way it is shown that a decreasing bounded sequence converges. ∎

Corollary 4.15 *An increasing (decreasing) bounded sequence converges to its supremum (infimum).*

Example: With this result it can be shown that the recursively defined sequence $\{x_n\}_{n=1}^{\infty}$ from example 5 converges. It was already shown that $x_n \ge \sqrt{a}$ for $n \ge 2$, hence $\frac{a}{x_n} \le x_n$, consequently

$$x_{n+1} = \frac{1}{2}\Big(x_n + \frac{a}{x_n}\Big) \le \frac{1}{2}\,x_n + \frac{1}{2}x_n = x_n.$$

Therefore, this sequence decreases for $n \ge 2$ and is bounded below by $\min\{\sqrt{a}, b\}$, hence it converges.

Theorem 4.16 (Nested intervals) *Let $\{J_n\}_{n=1}^{\infty}$ be a sequence of closed intervals $J_n = [a_n, b_n]$ with*

$$J_1 \supseteq J_2 \supseteq J_3 \supseteq \dots .$$

If $b_n - a_n$ is a null sequence, then the intersection $\bigcap_{n=1}^{\infty} J_n$ contains exactly one point.

Proof: The sequence $\{a_n\}_{n=1}^{\infty}$ is increasing and bounded, since b_1 is an upper bound. Analogously, $\{b_n\}_{n=1}^{\infty}$ is decreasing and bounded, whence both sequences converge. Thus, by assumption

$$\lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = \lim_{n\to\infty} (a_n - b_n) = 0,$$

which yields $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = a$.

Since $\{a_n\}_{n=1}^{\infty}$ is increasing and $\{b_n\}_{n=1}^{\infty}$ is decreasing, it follows that a is the supremum of $\{a_n\}_{n=1}^{\infty}$ and the infimum of $\{b_n\}_{n=1}^{\infty}$, whence

$$a_n \leq a \leq b_n$$

for all $n \in \mathbb{N}$, which implies

$$a \in \bigcap_{n=1}^{\infty} J_n.$$

Clearly, a is the only point belonging to all the intervals J_n. ■

Note that the statement becomes false if the closed intervals are replaced by open intervals.

Definition 4.17 *A number a is called accumulation point of the sequence $\{x_n\}_{n=1}^{\infty}$, if to every neighborhood U of a there exist infinitely many $n \in \mathbb{N}$ with $x_n \in U$.*

Equivalently, this can be stated as follows: a is an accumulation point of $\{x_n\}_{n=1}^{\infty}$, if and only if

$$\forall_{\varepsilon>0} \forall_{n\in\mathbb{N}} \exists_{\substack{m\in\mathbb{N} \\ m\geq n}} \ : \ |x_m - a| < \varepsilon.$$

Theorem 4.18 *If a is an accumulation point of $\{x_n\}_{n=1}^{\infty}$, then there is a subsequence $\{x_{j_n}\}_{n=1}^{\infty}$ converging to a.*

Proof: It suffices to prove that there is a sequence $\{j_n\}_{n=1}^{\infty}$ of numbers $j_n \in \mathbb{N}$ satisfying $j_n > j_{n-1}$ for $n \geq 2$ and

$$|x_{j_n} - a| < \frac{1}{n}$$

for all n. Obviously, this implies that the subsequence $\{x_{j_n}\}_{n=1}^{\infty}$ converges to a.

We construct $\{j_n\}_{n=1}^{\infty}$ by induction:

a.) Start of the induction: Since a is an accumulation point of $\{x_n\}_{n=1}^{\infty}$, there is n_0 with

$$|x_{n_0} - a| < 1.$$

Set $j_1 := n_0$.

b.) Induction step: Let $n \in \mathbb{N}$ and assume that j_n is constructed. Using again that a is an accumulation point, we conclude that there exist infinitely many $m \in \mathbb{N}$ with

$$|x_m - a| < \frac{1}{n+1}\,.$$

Since infinitely many of these numbers are greater than j_n, we can choose such an m_0 with $m_0 > j_n$. Set $j_{n+1} := m_0$. ■

Theorem 4.19 (of Bolzano and Weierstraß) *Every bounded numerical sequence possesses an accumulation point. (Bernhard Bolzano 1781 – 1848, Karl Weierstraß 1815 – 1897)*

Proof: Let a_1 be a lower bound and b_1 be an upper bound for the numerical sequence $\{x_n\}$. If the interval $J_1 = [a_1, b_1]$ is divided into two closed intervals of equal length, then at least one of these intervals contains infinitely many of the terms of the sequence. Let J_2 be this interval. I divide J_2 into two intervals of equal length and proceed in the same way to construct a sequence $\{J_n\}_{n=1}^{\infty}$ of nested intervals $J_n = [a_n, b_n]$ with the property that any of these intervals contains infinitely many terms of $\{x_n\}_{n=1}^{\infty}$. Since $b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} \to 0$ for $n \to \infty$, the intersection $\bigcap_{n=1}^{\infty} J_n$ contains exactly one point c.

c is an accumulation point. For, to $\varepsilon > 0$ choose $n \in \mathbb{N}$ large enough such that $b_n - a_n < \varepsilon$. From $c \in J_n$ we then obtain

$$\begin{aligned} a_n &\geq c - (b_n - a_n) > c - \varepsilon \\ b_n &\leq c + (b_n - a_n) < c + \varepsilon, \end{aligned}$$

thus

$$J_n = [a_n, b_n] \subseteq (c - \varepsilon, c + \varepsilon),$$

and therefore infinitely many of the terms of $\{x_n\}_{n=1}^{\infty}$ belong to the interval $(c - \varepsilon, c + \varepsilon)$ ■

Corollary 4.20 *Every bounded sequence possesses a convergent subsequence.*

Proof: Every bounded sequence possesses an accumulation point. Therefore a subsequence can be selected, which converges to the accumulation point. ■

The criterion stated in the next theorem allows to investigate a sequence for convergence without knowing the limit. It can be applied to any sequence. To formulate it, we need the following

Definition 4.21 *A numerical sequence $\{x_n\}_{n=1}^{\infty}$ is called Cauchy sequence, if to every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$*

$$|x_n - x_m| < \varepsilon$$

holds. (Augustin Louis Cauchy 1789 – 1857)

Written with quantifiers, this condition is:

$$\forall_{\varepsilon>0}\, \exists_{n_0\in\mathbb{N}}\, \forall_{\substack{n\in\mathbb{N}\\ n\geq n_0}}\, \forall_{\substack{m\in\mathbb{N}\\ m\geq n_0}} \;:\; |x_n - x_m| < \varepsilon.$$

Theorem 4.22 *A sequence converges if and only if it is a Cauchy sequence.*

Proof: "$\Longrightarrow$" Assume that the sequence $\{x_n\}_{n=1}^{\infty}$ converges. To show that it is a Cauchy sequence, let a be the limit and let $\varepsilon > 0$. Then there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$|x_n - a| < \frac{\varepsilon}{2}$$

holds. Therefore, if $n, m \in \mathbb{N}$ satisfy $n, m \geq n_0$, we obtain

$$|x_n - x_m| = |x_n - a + a - x_m| \leq |x_n - a| + |x_m - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

"$\Longleftarrow$" Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence. To show that this sequence converges, I show first that it is bounded. From the condition defining a Cauchy sequence it follows that there exists $n_0 \in \mathbb{N}$ with

$$|x_n - x_{n_0}| < 1$$

for all $n \geq n_0$. Thus, for these n

$$|x_n| = |x_n - x_{n_0} + x_{n_0}| \leq |x_n - x_{n_0}| + |x_{n_0}| < 1 + |x_{n_0}|.$$

Therefore we obtain

$$|x_n| \leq \max\{|x_1|, \ldots, |x_{n_0}|\} + 1$$

for all $n \in \mathbb{N}$, hence $\{x_n\}_{n=1}^{\infty}$ is bounded and thus has an accumulation point a, by the Bolzano–Weierstraß theorem.

Next I show that a is the limit of $\{x_n\}_{n=1}^{\infty}$. Let $\varepsilon > 0$. Since $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, there is $n_0 \in \mathbb{N}$ with

$$|x_n - x_m| < \frac{\varepsilon}{2}$$

for all $n, m \geq n_0$. By definition of an accumulation point, there exist inifinitely many $n \in \mathbb{N}$ with $|x_n - a| < \varepsilon/2$. Of these infinitely many we can choose one, denoted by n_1, with $n_1 \geq n_0$. For $n \geq n_0$ we thus obtain

$$|x_n - a| = |x_n - x_{n_1} + x_{n_1} - a| \leq |x_n - x_{n_1}| + |x_{n_1} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Consequently, a is the limit of $\{x_n\}_{n=1}^{\infty}$, hence $\{x_n\}_{n=1}^{\infty}$ converges. ■

Example: Let $\{x_n\}_{n=1}^{\infty}$ be defined recursively by

$$\begin{aligned} x_1 &= 1 \\ x_{n+1} &= \frac{1}{1+x_n}, \quad n \geq 1. \end{aligned}$$

This sequence converges. To show this, I first prove by induction that the estimate

$$\frac{1}{2} \leq x_n \leq 1$$

holds for all $n \in \mathbb{N}$. For $n = 1$ the estimate is obviously true. Assume that the statement is true for a number $n \in \mathbb{N}$. Then

$$\frac{1}{2} \leq \frac{1}{1+x_n} \leq \frac{2}{3},$$

whence

$$\frac{1}{2} \leq x_{n+1} \leq \frac{2}{3} \leq 1,$$

which shows that the estimate holds for all n.

Now

$$x_{n+1+k} - x_{n+1} = \frac{1}{1+x_{n+k}} - \frac{1}{1+x_n} = -\frac{x_{n+k} - x_n}{(1+x_{n+k})(1+x_n)},$$

and so the estimate yields

$$\begin{aligned} |x_{n+1+k} - x_{n+1}| &= \frac{|x_{n+k} - x_n|}{(1+x_{n+k})(1+x_n)} \leq \frac{1}{(1+\frac{1}{2})^2} |x_{n+k} - x_n| \\ &= \frac{4}{9} |x_{n+k} - x_n|. \end{aligned}$$

Using this estimate it follows by induction with respect to n that for n and k

$$|x_{n+k} - x_n| \leq \Big(\frac{4}{9}\Big)^{n-1} |x_{k+1} - x_1| \leq 2\Big(\frac{4}{9}\Big)^{n-1}.$$

Since $\left(\frac{4}{9}\right)^{n-1} = \frac{9}{4}\left(\frac{4}{9}\right)^{n} \to 0$ for $n \to \infty$, the Cauchy criterion can be applied: Let $\varepsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that $\left(\frac{4}{9}\right)^{n-1} < \varepsilon/2$ for $n \geq n_0$. Then we obtain for all $m > n \geq n_0$

$$|x_m - x_n| = |x_{n+(m-n)} - x_n| \leq 2\Big(\frac{4}{9}\Big)^{n-1} < 2\,\frac{\varepsilon}{2} = \varepsilon,$$

whence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, and so it converges.

Also the limit can be determined. We denote this limit by a. Since $\frac{1}{2} \leq x_n \leq 1$, it follows that $\frac{1}{2} \leq a \leq 1$, hence all the x_n and a belong to the domain of definition of the rational function $x \mapsto \frac{1}{1+x}$. Consequently, $\{\frac{1}{1+x_n}\}_{n=1}^{\infty}$ converges to $\frac{1}{1+a}$. Using the definition of the sequnce, we obtain

$$a = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{1+x_n} = \frac{1}{1+a},$$

thus $(1+a)a = a + a^2 = 1$, hence $a = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$.

4.4 Limes superior and limes inferior

Theorem 4.23 *Assume that the set M of accumulation points of a sequence $\{x_n\}_{n=1}^{\infty}$ bounded above (bounded below) is not empty. Then M has a maximum (a minimum).*

Proof: Let $\{x_n\}_{n=1}^{\infty}$ be bounded above. Then also M is bounded above, hence $s_0 = \sup M$ exists. The supremum has the property that to every $\varepsilon > 0$ there is $a \in M$ with $s_0 - \varepsilon/2 \leq a \leq s_0$. Since a is accumulation point of $\{x_n\}_{n=1}^{\infty}$, there are infinitely many $n \in \mathbb{N}$ with

$$|x_n - a| < \frac{\varepsilon}{2},$$

and so for these infinitely many n,

$$|x_n - s_0| = |x_n - a + a - s_0| \leq |x_n - a| + |a - s_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, s_0 itself is an accumulation point of $\{x_n\}_{n=1}^{\infty}$, hence $s_0 \in M$, which implies that

$$s_0 = \max M.$$

For a sequence bounded below the statement is proved with analogous arguments. ■

Definition 4.24 *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence with non-empty set M of accumulation points. If the sequence is bounded above, then the maximum of M is called limes superior or upper limit of $\{x_n\}_{n=1}^{\infty}$. If the sequence is bounded below, then the minimum of M is called limes inferior or lower limit of $\{x_n\}_{n=1}^{\infty}$. The limes superior is denoted by*

$$\overline{\lim}_{n\to\infty} x_n \quad \text{or} \quad \limsup_{n\to\infty} x_n,$$

whereas

$$\underline{\lim}_{n\to\infty} x_n \quad \text{or} \quad \liminf_{n\to\infty} x_n$$

denotes the limes inferior.

Theorem 4.25 *(i) The number a satisfies $a = \overline{\lim}_{n\to\infty} x_n$, if and only if for every $\varepsilon > 0$ the follwing two conditions hold:*
a.) $x_n > a - \varepsilon$ holds for infinitely many n,
b.) $x_n < a + \varepsilon$ holds for all but finitely many n.
(ii) b satisfies $b = \underline{\lim}_{n\to\infty} x_n$, if and only if for every $\varepsilon > 0$ the following two conditions hold:
a.) $x_n < b + \varepsilon$ holds for infinitely many n,
b.) $x_n > b - \varepsilon$ holds for all but finitely many n.

"For all but finitely many n" includes the case "for all n". For brevity, instead of "for all but finitely many n" one sometimes writes "for almost all n".

Proof: (i) "$\Longrightarrow$" Let $a = \overline{\lim}_{n\to\infty} x_n$ and let $\varepsilon > 0$. Since a is an accumulation point of $\{x_n\}_{n=1}^{\infty}$, this implies $|x_n - a| < \varepsilon$ for infinitely many n, hence, in particular, $x_n > a - \varepsilon$ for infinitely many n. This proves a.). The inequality $x_n \geq a + \varepsilon$ can be satiesfied for at most finitely many n. For, otherwise we could select a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with

$$x_{n_k} \geq a + \varepsilon$$

for all $k \in \mathbb{N}$. Since by assumption $\{x_n\}_{n=1}^{\infty}$ is bounded above, also the subsequence would be bounded above, hence it would be bounded. Consequently, the subsequence would have an accumulation point c, which would also be an accummulation point of $\{x_n\}_{n=1}^{\infty}$. Since the subsequence is bounded below by $a + \varepsilon$, the accumulation point would satisfy $c \geq a + \varepsilon$, hence a could not be the largest accumulation point. Thus, also b.) is satisfied.

"$\Longleftarrow$" Assume that a.) and b.) are satiesfied for a number a. Then a.) and b.) together imply that a is an accumulation point of $\{x_n\}_{n=1}^{\infty}$, whereas b.) implies that no accumulation point of $\{x_n\}_{n=1}^{\infty}$ larger than a can exist. Thus, $a = \overline{\lim}_{n\to\infty} x_n$.
(ii) is proved analogously. ■

For a bounded sequence $\overline{\lim}_{n\to\infty} x_n$ and $\underline{\lim}_{n\to\infty} x_n$ always exist, since every bounded sequence has an accumulation point. Of course, by definition

$$\underline{\lim}_{n\to\infty} x_n \leq \overline{\lim}_{n\to\infty} x_n.$$

Theorem 4.26 *For a sequence $\{x_n\}_{n=1}^{\infty}$ the limes superior and the limes inferior exist and are equal, if and only if the sequence converges. In this case one has*

$$\underline{\lim}_{n\to\infty} x_n = \lim_{n\to\infty} x_n = \overline{\lim}_{n\to\infty} x_n.$$

Proof: Assume that the limes superior and the limes inferior exist and satisfy $a = \underline{\lim}_{n\to\infty} x_n = \overline{\lim}_{n\to\infty} x_n$. Then the preceding theorem implies for all $\varepsilon > 0$ that

$$a - \varepsilon < x_n < a + \varepsilon$$

for all but finitely many n, hence there is $n_0 \in \mathbb{N}$ such that

$$|x_n - a| < \varepsilon$$

for all $n \geq n_0$, hence $\lim_{n\to\infty} x_n = a$.

On the other hand, if $a = \lim_{n\to\infty} x_n$ holds, then

$$a - \varepsilon < x_n < a + \varepsilon$$

holds for all but finitely many n. Hence, $a = \underline{\lim}_{n\to\infty} x_n = \overline{\lim}_{n\to\infty} x_n$, by the preceding theorem. ■

5 Series

5.1 Fundamental definitions

In this section we study infinite sums of the form

$$\sum_{k=1}^{\infty} a_k .$$

First a precise meaning must be given to this symbol.

Definition 5.1 *Let $\{a_k\}_{k=1}^{\infty}$ be a numerical sequence and let $s_n = \sum_{k=1}^{n} a_k$. The numerical sequence $\{s_n\}_{n=1}^{\infty} = \{\sum_{k=1}^{n} a_k\}_{k=1}^{\infty}$ is called the infinite series belonging to $\{a_k\}_{k=1}^{\infty}$. We call a_k the k–th term and s_n the n–th partial sum of the series. Instead of $\{\sum_{k=1}^{n} a_k\}_{k=1}^{\infty}$, one uses the symbol $\sum_{k=1}^{\infty} a_k$.*

If $\{s_n\}_{n=1}^{\infty}$ converges with the limit s, then s is called the sum of the infinite series, and one writes

$$s = \sum_{k=1}^{\infty} a_k .$$

Example: Let $q \in \mathbb{R}$. The series

$$\sum_{k=0}^{\infty} q^k$$

is called geometrical series. (For convenience, one often starts with the summation at $k = 0$.) For $|q| < 1$, this series has a finite sum. To show this, let $s_n = \sum_{k=0}^{n} q^k$. Then

$$(1-q)s_n = \sum_{k=0}^{n} q^k - \sum_{k=1}^{n+1} q^k = 1 - q^{n+1},$$

hence

$$s_n = \frac{1-q^{n+1}}{1-q} = \frac{1}{1-q} - \frac{q^{n+1}}{1-q}.$$

From $\lim_{n\to\infty} q^{n+1} = 0$ we thus obtain

$$\sum_{k=0}^{\infty} q^k = \lim_{n\to\infty} s_n = \frac{1}{1-q} - \frac{1}{1-q} \lim_{n\to\infty} q^{n+1} = \frac{1}{1-q}.$$

From the theory of convergent sequences we immediately obtain the following

Theorem 5.2 *Let $\sum_{n=1}^{\infty} a_n = a$ and $\sum_{n=1}^{\infty} b_n = b$. Then the series $\sum_{n=1}^{\infty} (a_n + b_n)$ and $\sum_{n=1}^{\infty} c a_n$ converge for every $c \in \mathbb{R}$ with the sums*

$$\sum_{n=1}^{\infty} (a_n + b_n) = a + b, \quad \sum_{n=1}^{\infty} c a_n = ca.$$

Furthermore, if $a_n \le b_n$ holds for all n, then $a \le b$.

5.2 Criteria for convergence

The Cauchy convergence criterion for sequences immediately yields the following convergence criterion for series:

Theorem 5.3 *The series $\sum_{k=1}^{\infty} a_k$ converges if and only if to every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $m \geq n$*

$$\Big|\sum_{k=n}^{m} a_k\Big| < \varepsilon.$$

If we choose $m = n$ in this condition, we obtain the following

Corollary 5.4 *If the series $\sum_{k=1}^{\infty} a_k$ converges, then $\{a_k\}_{k=1}^{\infty}$ is a null sequence.*

Thus, for the series $\sum_{k=1}^{\infty} a_k$ to converge it is necessary that $\{a_k\}_{k=1}^{\infty}$ is a null sequence. However, this is not sufficient for convergence of the series. This is shown by the following

Example: The *harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. To see this, note that

$$\begin{aligned} s_{2^m} &= \sum_{k=1}^{2^m} \frac{1}{k} \\ &= 1 + \frac{1}{2} + \Big(\frac{1}{3} + \frac{1}{4}\Big) + \Big(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\Big) \\ &\quad + \Big(\frac{1}{2^{i-1}+1} + \ldots + \frac{1}{2^i}\Big) + \ldots + \Big(\frac{1}{2^{m-1}+1} + \ldots + \frac{1}{2^m}\Big). \end{aligned}$$

The i–th bracket contains 2^{i-1} terms, which all are greater or equal to $\frac{1}{2^i}$, and therefore the sum of this bracket is greater or equal to $2^{i-1}2^{-i} = \frac{1}{2}$. Consequently,

$$s_{2^m} \geq 1 + m\frac{1}{2},$$

which shows that the subsequence $\{s_{2^m}\}_{m=1}^{\infty}$ of the sequence $\{s_{2^m}\}_{m=1}^{\infty}$ is unbounded. Hence $\{s_m\}_{m=1}^{\infty}$ is unbounded and thus diverges.

Definition 5.5 *A series is said to be alternating, if the signs of the terms alternate.*

Theorem 5.6 *(Convergence criterion of Leibniz) Let $\sum_{n=1}^{\infty} a_n$ be an alternating series and let $\{|a_n|\}_{n=1}^{\infty}$ be a decreasing null sequence. Then the series $\sum_{n=1}^{\infty} a_n$ converges. (Gottfried Wilhelm Leibniz, 1646 – 1716.)*

Proof: Without restriction of generality we can assume that $a_1 \geq 0$, hence all the terms a_k with k odd are non–negative. Let $s_n = \sum_{k=1}^{n} a_k$. The subsequence $\{s_{2n}\}_{n=1}^{\infty}$ of $\{s_n\}_{n=1}^{\infty}$ is increasing, since by assumption

$$a_{2n+1} + a_{2n+2} = |a_{2n+1}| - |a_{2n+2}| \geq 0,$$

hence

$$s_{2(n+1)} = s_{2n} + (a_{2n+1} + a_{2n+2}) \geq s_{2n}.$$

Moreover, this subsequence is bounded above by a_1, since

$$\begin{aligned} s_{2n} &= a_1 + (-|a_2| + |a_3|) + (-|a_4| + |a_5|) \\ &\quad + \ldots + (-|a_{2n-2}| + |a_{2n-1}|) - |a_{2n}| \\ &\leq a_1. \end{aligned}$$

In the same way it is seen that the subsequence $\{s_{2n-1}\}_{n=1}^{\infty}$ is decreasing and bounded below by $s_2 = a_1 + a_2$, hence both subsequences are convergent. Since

$$s_{2n} - s_{2n-1} = a_{2n},$$

and since $\{a_{2n}\}_{n=1}^{\infty}$ is a null sequence, the limits of both subsequences concide. Clearly, this implies that the sequence $\{s_n\}_{n=1}^{\infty}$ converges to the same limit. ■

This proof implies for the limit s that

$$s_{2n} \leq s \leq s_{2n-1}, \quad s_{2n} \leq s \leq s_{2n+1},$$

hence

$$|s_n - s| \leq |a_{n+1}|,$$

beause of $s_{2n} - s_{2n-1} = a_{2n}$ and $s_{2n+1} - s_{2n} = a_{2n+1}$. This inequality is called an *error–estimate*.

Example: The series

$$-1 + \frac{1}{2} - \frac{1}{3} + \ldots = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

and the Leibniz series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$$

satisfy the criterion of the theorem and thus converge.

5.3 Rearrangement of series, absolutely convergent series

The value of a finite sum $\sum_{k=1}^{n} a_k$ is independent of the order of summation. In the following it will be seen that this is different for infinite series. There are series, the absolutely convergent series, whose sum is independent of the order of summation, but there are also series whose sum depends on this order. This is shown by the following example of a series with sum depending on the order of summation:

We have just shown that the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$ converges. Let the sum be s. The inequalities proved above yield $\frac{1}{2} \le s \le 1$. Clearly,

$$\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n} = \frac{1}{2}\,s.$$

We add both series

$$\begin{array}{l} 1 \;-\frac{1}{2} +\frac{1}{3} -\frac{1}{4} +\frac{1}{5} -\frac{1}{6} +\frac{1}{7} -\frac{1}{8} +\frac{1}{9} -\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\ldots \;=\; s \\ 0 \;+\frac{1}{2} +0 \;-\frac{1}{4} +0 \;+\frac{1}{6} +0 \;-\frac{1}{8} +0 \;+\frac{1}{10}+0 \;-\frac{1}{12}+\ldots \;=\; \frac{1}{2}\,s \end{array}$$

and obtain the series

$$1 \quad +\frac{1}{3}-\frac{1}{2}+\frac{1}{5} \quad +\frac{1}{7}-\frac{1}{4}+\frac{1}{9} \quad +\frac{1}{11}-\frac{1}{6}+\frac{1}{13}+\ldots = \frac{3}{2}\,s\,,$$

a rearrangement of the sequence $\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$. Also the rearranged series converges, but to the limit $\frac{3}{2}\,s \neq s$, since $s \neq 0$.

Definition 5.7 *Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijective mapping. Then the series*

$$\sum_{k=1}^{\infty} a_{\sigma(k)}$$

is called a rearrangement of the series $\sum_{k=1}^{\infty} a_k$.

Theorem 5.8 *Assume that $a_k \ge 0$ for all $k \in \mathbb{N}$, and assume that the series $\sum_{k=1}^{\infty} a_k$ converges to s. Then every rearrangement of this series also converges to s.*

Proof: Let $\sum_{k=1}^{\infty} a_{\sigma(k)}$ be a rearrangement. We denote the partial sums by $s_n = \sum_{k=1}^{n} a_k$ and $s_n' = \sum_{k=1}^{n} a_{\sigma(k)}$. The sequence $\{s_n\}_{n=1}^{\infty}$ is increasing, which implies

$$s = \lim_{n\to\infty} s_n = \sup\{s_n \big| n \in \mathbb{N}\}.$$

We show that $s_n' \le s$ for every $n \in \mathbb{N}$. To this end we set

$$m := \max\{\sigma(1), \ldots, \sigma(n)\},$$

then $\{\sigma(1), \ldots, \sigma(n)\} \subseteq \{1, \ldots, m\}$, hence

$$s'_n = \sum_{k=1}^{n} a_{\sigma(k)} \leq \sum_{k=1}^{m} a_k = s_m \leq s.$$

Thus, the increasing sequence $\{s'_n\}_{n=1}^{\infty}$ is bounded above by s, and thus converges to a limit s' with $s' \leq s$.

Since $\sum_{k=1}^{\infty} a_k$ is a rearrangement of $\sum_{k=1}^{\infty} a_{\sigma(k)}$, in this argument the roles of $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} a_{\sigma(k)}$ can be interchanged. This yields $s \leq s'$, hence $s = s'$. ■

Definition 5.9 *A series $\sum_{k=1}^{\infty} a_k$ is said to be absolutely convergent, if the series $\sum_{k=1}^{\infty} |a_k|$ converges.*

Theorem 5.10 *An absolutely convergent series converges.*

Proof: We use the Cauchy criterion to show that $\sum_{k=1}^{\infty} a_k$ converges. Let $\varepsilon > 0$. Since this series converges absolutely, there is $n_0 \in \mathbb{N}$ such that for all $m \geq n \geq n_0$

$$\sum_{k=n}^{m} |a_k| < \varepsilon,$$

whence

$$\Big|\sum_{k=n}^{m} a_k\Big| \leq \sum_{k=n}^{m} |a_k| < \varepsilon,$$

by repeated application of the triangle inequality. Therefore $\sum_{k=1}^{\infty} a_k$ converges. ■

Lemma 5.11 *A series $\sum_{k=1}^{\infty} a_k$ converges absolutely if and only if $\{\sum_{k=1}^{n} |a_k|\}_{n=1}^{\infty}$ is a bounded sequence.*

The **proof** is obvious.

Theorem 5.12 *Let $\sum_{k=1}^{\infty} a_k$ be absolutely convergent. Then every rearrangement $\sum_{k=1}^{\infty} a_{\sigma(k)}$ is absolutely convergent and*

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{\sigma(k)}.$$

Proof: The sequence $\sum_{k=1}^{\infty} |a_{\sigma(k)}|$ is a rearrangement of the convergent series $\sum_{k=1}^{\infty} |a_k|$ with non–negative terms. Therefore $\sum_{k=1}^{\infty} |a_{\sigma(k)}|$ converges, hence $\sum_{k=1}^{\infty} a_{\sigma(k)}$ is absolutely convergent.

In particular, this means that $\sum_{k=1}^{\infty} a_{\sigma(k)}$ converges. Let

$$s'_n = \sum_{k=1}^{n} a_{\sigma(k)}, \quad s_n = \sum_{k=1}^{n} a_k,$$

and set $s' = \lim_{n\to\infty} s'_n$, $s = \lim_{n\to\infty} s_n$. We must show that $s' = s$.

To this end I choose a sequence $\{m_n\}_{n=1}^{\infty}$ of numbers $m_n \in \mathbb{N}$ with $m_{n+1} > m_n$ and with

$$\sigma^{-1}\big(\{1,\dots,n\}\big) \subseteq \{1,\dots,m_n\}.$$

The former condition means that $\{s'_{m_n}\}_{n=1}^{\infty}$ is a subsequence of $\{s'_n\}_{n=1}^{\infty}$, and the latter condition implies that

$$\{1,\dots,n\} \subseteq \sigma\big(\{1,\dots,m_n\}\big) = \{\sigma(1),\dots,\sigma(m_n)\},$$

hence, with $M_n = \max\{\sigma(1),\dots,\sigma(m_n)\}$,

$$|s'_{m_n} - s_n| = \Big|\sum_{k=1}^{m_n} a_{\sigma(k)} - \sum_{k=1}^{n} a_k\Big| = \Big|\sum_{\ell\in\sigma(\{1,\dots,m_n\})\setminus\{1,\dots,n\}} a_\ell\Big| \le \sum_{\ell=n+1}^{M_n} |a_\ell|.$$

This inequality implies that $\{s'_{m_n} - s_n\}_{n=1}^{\infty}$ is a null sequence. To see this, let $\varepsilon > 0$. Since $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, there is $n_0 \in \mathbb{N}$ such that for all $m > n \ge n_0$ $\sum_{k=n+1}^{m} |a_k| < \varepsilon$. Hence, for $n \ge n_0$

$$|s'_{m_n} - s_n| \le \sum_{\ell=n+1}^{M_n} |a_\ell| < \varepsilon.$$

This means that $\{s'_{m_n} - s_n\}_{n=1}^{\infty}$ converges to zero, whence

$$s' = \lim_{n\to\infty} s'_n = \lim_{n\to\infty} s'_{m_n} = \lim_{n\to\infty} \Big(s'_{m_n} - (s'_{m_n} - s_n)\Big) = \lim_{n\to\infty} s_n = s.$$

■

Therefore every rearrangement of an absolutely convergent series converges to the same sum. In fact, also the converse of this statement holds:

Theorem 5.13 *A series is absolutely convergent if and only if every rearrangement converges.*

Proof: Since we already proved that every rearrangement of an absolutely converging series converges, it suffices to show that every series, which is not absolutely convergent, has a divergent rearrangement.

For this purpose let $\sum_{k=1}^{\infty} a_k$ be a series which converges, but is not absolutely convergent. Set

$$b_k = \frac{a_k + |a_k|}{2} = \begin{cases} a_k, & \text{if} \quad a_k \geq 0 \\ 0, & \text{if} \quad a_k < 0 \end{cases}$$

$$c_k = \frac{a_k - |a_k|}{2} = \begin{cases} 0, & \text{if} \quad a_k > 0 \\ a_k, & \text{if} \quad a_k \leq 0. \end{cases}$$

We have $a_k = b_k + c_k$. Up to additional zeros, the series $\sum_{k=1}^{\infty} b_k$ and $\sum_{k=1}^{\infty} c_k$ are the series of all positive or all negative terms of $\sum_{k=1}^{\infty} a_k$, respectively.

Since $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}(b_k + c_k)$ and since $\sum_{k=1}^{\infty} a_k$ converges, the series $\sum_{k=1}^{\infty} b_k$ and $\sum_{k=1}^{\infty} c_k$ can only converge or diverge simultaneously. In fact, because of $|a_k| = b_k - c_k$ both must diverge, for otherwise $\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} b_k - \sum_{k=1}^{\infty} c_k$ would converge, in contradiction to the assumption.

Since $\sum_{k=1}^{\infty} b_k$ consists of non-negative terms and diverges, to every $C > 0$ and to every $n_0 \in \mathbb{N}$ a number $m \geq n_0$ can be found such that

$$\sum_{k=n_0}^{m} b_k \geq C \,.$$

For, otherwise the sequence $\{\sum_{k=1}^{n} b_k\}_{n=1}^{\infty}$ would be bounded above and increasing, hence it would converge. Analogously, a number $\ell \geq n_0$ can be found such that

$$\sum_{k=n_0}^{\ell} c_k \leq -C \,.$$

A diverging rearrangement can now be constructed as follows: Let k_0 be the smallest natural number with

$$\sum_{k=1}^{k_0} b_k \geq 1 \,,$$

let k_1 be the smallest natural number with

$$\sum_{k=1}^{k_0} b_k + \sum_{k=1}^{k_1} c_k \leq -1 \,,$$

and let k_2 be the smallest natural number with

$$\sum_{k=1}^{k_0} b_k + \sum_{k=1}^{k_1} c_k + \sum_{k=k_0+1}^{k_2} b_k \geq 1 \,.$$

We continue this procedure and obtain a series $\sum_{k=1}^{\infty} d_k$ with

$$d_k = \begin{cases} b_k\,, & 1 \le k \le k_0 \\ c_{k-k_0}\,, & k_0+1 \le k \le k_0+k_1 \\ b_{k-k_1}\,, & k_0+k_1+1 \le k \le k_0+k_1+k_2 \\ \vdots & \end{cases}$$

By droping additional zeros, $\sum_{k=1}^{\infty} d_k$ becomes a rearrangement of $\sum_{k=1}^{\infty} a_k$. The rearranged series satisfies $\overline{\lim}_{n\to\infty} \sum_{k=1}^{n} d_k \ge 1$ and $\underline{\lim}_{n\to\infty} \sum_{k=1}^{n} d_k \le -1$, hence the limes superior and the limes inferior are different. Consequently $\sum_{k=1}^{\infty} d_k$ diverges. ■

The set $\mathbb{N}\times\mathbb{N}$ is countable, i.e. there exists at least one bijective map $\sigma : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$. We call such a map a numbering of $\mathbb{N}\times\mathbb{N}$. To construct such a map, arrange the elements $(k,\ell) \in \mathbb{N}\times\mathbb{N}$ in a doubly infinite scheme and count along diagonals:

$$\begin{array}{ccccccc} (1,1) & & (1,2) & & (1,3) & \dots \\ & \swarrow & & \swarrow & & \swarrow \\ (2,1) & & (2,2) & & (2,3) & \dots \\ & \swarrow & & \swarrow & & \swarrow \\ (3,1) & & (3,2) & & (3,3) & \dots \\ \vdots & \swarrow & \vdots & \swarrow & \vdots & \swarrow \end{array}$$

Of course, σ is not uniquely determined.

Theorem 5.14 ("Großer Umordnungssatz") *Let a double series $\sum_{k,\ell=1}^{\infty} a_{k\ell}$ be given. Suppose that there is a number $M \in \mathbb{R}$ with*

$$\sum_{k,\ell=1}^{m} |a_{k\ell}| \le M$$

for all $m \in \mathbb{N}$. Then

(i) *For every numbering σ of $\mathbb{N}\times\mathbb{N}$ the series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ is absolutely convergent with the same sum s. (We call $\sum_{n=1}^{\infty} a_{\sigma(n)}$ an arrangement of the double series into a series.)*

(ii) *The series $\sum_{\ell=1}^{\infty} a_{k\ell}$ and $\sum_{k=1}^{\infty} a_{k\ell}$ are absolutely convergent.*

(iii) *Let $s_k = \sum_{\ell=1}^{\infty} a_{k\ell}$ and $s'_\ell = \sum_{k=1}^{\infty} a_{k\ell}$. Then the series $\sum_{k=1}^{\infty} s_k$ and $\sum_{\ell=1}^{\infty} s'_\ell$ are absolutely convergent with the sum s.*

Remark: (iii) means

$$\begin{aligned}
\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} a_{k\ell} &= \lim_{m\to\infty}\sum_{k=1}^{m}\Big(\lim_{j\to\infty}\sum_{\ell=1}^{j} a_{k\ell}\Big)\\
&= \lim_{m\to\infty}\lim_{j\to\infty}\sum_{k=1}^{m}\sum_{\ell=1}^{j} a_{k\ell} = \lim_{j\to\infty}\lim_{m\to\infty}\sum_{\ell=1}^{j}\sum_{k=1}^{m} a_{k\ell}\\
&= \lim_{j\to\infty}\sum_{\ell=1}^{j}\Big(\lim_{m\to\infty}\sum_{k=1}^{m} a_{k\ell}\Big) = \sum_{\ell=1}^{\infty}\sum_{k=1}^{\infty} a_{k\ell}.
\end{aligned}$$

Therefore (iii) says that under the condition of the theorem both limits can be interchanged.

Proof: (i) Let σ be a numbering of $\mathbb{N}\times\mathbb{N}$, and let m be the greatest natural number occuring in the pairs $\sigma(1),\sigma(2),\ldots,\sigma(n)$. Then

$$\sum_{j=1}^{n}|a_{\sigma(j)}| \le \sum_{k,\ell=1}^{m}|a_{k\ell}| \le M,$$

hence $\sum_{j=1}^{\infty} a_{\sigma(j)}$ converges absolutely. If σ' is a second numbering of $\mathbb{N}\times\mathbb{N}$, then $\sum_{j=1}^{\infty} a_{\sigma'(j)}$ is a rearrangement of $\sum_{j=1}^{\infty} a_{\sigma(j)}$ and thus converges absolutely with the same sum.

(ii) To $k,n\in\mathbb{N}$ set $m=\max(k,n)$. Then

$$\sum_{\ell=1}^{n}|a_{k\ell}| \le \sum_{k,\ell=1}^{m}|a_{k\ell}| \le M,$$

consequently $\sum_{\ell=1}^{\infty} a_{k\ell}$ is absolutely convergent. It follows in the same way that $\sum_{k=1}^{\infty} a_{k\ell}$ converges absolutely.

(iii) We have

$$\begin{aligned}
\sum_{k=1}^{m}|s_k| &= \sum_{k=1}^{m}\Big|\sum_{\ell=1}^{\infty} a_{k\ell}\Big| = \sum_{k=1}^{m}\Big|\lim_{n\to\infty}\sum_{\ell=1}^{n} a_{k\ell}\Big|\\
&= \sum_{k=1}^{m}\lim_{n\to\infty}\Big|\sum_{\ell=1}^{n} a_{k\ell}\Big| \le \sum_{k=1}^{m}\lim_{n\to\infty}\sum_{\ell=1}^{n}|a_{k\ell}|\\
&= \lim_{n\to\infty}\sum_{k=1}^{m}\sum_{\ell=1}^{n}|a_{k\ell}| \le \lim_{n\to\infty}\sum_{k,\ell=1}^{n}|a_{k\ell}| \le M,
\end{aligned}$$

whence $\sum_{k=1}^{\infty} s_k$ converges absolutely. It follows in the same way that $\sum_{\ell=1}^{\infty} s'_\ell$ converges absolutely. Thus, it remains to show that both sums coincide with s.

For this purpose let σ be a numbering of $\mathbb{N}\times\mathbb{N}$ and let $\varepsilon > 0$. Since $\sum_{k=1}^{\infty} a_{\sigma(k)}$ converges absolutely, there is $n_0 \in \mathbb{N}$ with

$$\sum_{k=n+1}^{\infty} |a_{\sigma(k)}| = \Big|\sum_{k=1}^{\infty} |a_{\sigma(k)}| - \sum_{k=1}^{n} |a_{\sigma(k)}|\Big| < \varepsilon$$

for all $n \geq n_0$. For such an n with $n \geq n_0$ let μ be the greatest natural number occuring in the pairs $\sigma(1), \ldots, \sigma(n)$. Then for $m \geq \mu$ and $j \geq \mu$ the sum $\sum_{k=1}^{m} \sum_{\ell=1}^{j} a_{k\ell} - \sum_{\nu=1}^{n} a_{\sigma(\nu)}$ only contains terms $a_{k\ell}$ with $(k,\ell) \neq \sigma(\nu)$ for all $\nu = 1, \ldots, n$. Thus,

$$\Big|\sum_{k=1}^{m}\sum_{\ell=1}^{j} a_{k\ell} - \sum_{\nu=1}^{n} a_{\sigma(\nu)}\Big| \leq \sum_{\nu=n+1}^{\infty} |a_{\sigma(\nu)}| < \varepsilon,$$

hence

$$\begin{aligned}
\Big|\sum_{k=1}^{\infty}\Big(\sum_{\ell=1}^{\infty} a_{k\ell}\Big) - \sum_{\nu=1}^{n} a_{\sigma(\nu)}\Big| &= \Big|\lim_{m\to\infty}\big(\lim_{j\to\infty}\sum_{k=1}^{m}\sum_{\ell=1}^{j} a_{k\ell}\big) - \sum_{\nu=1}^{n} a_{\sigma(\nu)}\Big| \\
&= \Big|\lim_{m\to\infty}\Big(\lim_{j\to\infty}\Big[\sum_{k=1}^{m}\sum_{\ell=1}^{j} a_{k\ell} - \sum_{\nu=1}^{n} a_{\sigma(\nu)}\Big]\Big)\Big| \\
&= \lim_{m\to\infty}\Big|\lim_{j=\infty}\Big[\sum_{k=1}^{m}\sum_{\ell=1}^{j} a_{k\ell} - \sum_{\nu=1}^{n} a_{\sigma(\nu)}\Big]\Big| \\
&= \lim_{m\to\infty}\lim_{j\to\infty}\Big|\sum_{k=1}^{m}\sum_{\ell=1}^{j} a_{k\ell} - \sum_{\nu=1}^{n} a_{\sigma(\nu)}\Big| \\
&\leq \lim_{m\to\infty}\lim_{j\to\infty} \varepsilon = \varepsilon.
\end{aligned}$$

Since this estimate holds for all $n \geq n_0$, it follows that $|\sum_{k=1}^{\infty}(\sum_{\ell=1}^{\infty} a_{k\ell}) - \sum_{\nu=1}^{\infty} a_{\sigma(\nu)}| \leq \varepsilon$. Because $\varepsilon > 0$ was chosen arbitrarily, we infer that

$$\sum_{k=1}^{\infty}\Big(\sum_{\ell=1}^{\infty} a_{k\ell}\Big) = \sum_{\nu=1}^{\infty} a_{\sigma(\nu)} = s.$$

The equation

$$\sum_{\ell=1}^{\infty}\Big(\sum_{k=1}^{\infty} a_{k\ell}\Big) = s$$

is shown in the same way. ■

This theorem can be applied to products of series. The distributive law yields for the product of finite sums

$$\Big(\sum_{k=1}^{n} a_k\Big)\Big(\sum_{\ell=1}^{n} b_\ell\Big) = \sum_{k,\ell=1}^{n} (a_k b_\ell).$$

The order of summation is arbitrary. Such a result also holds for products of absolutely convergent series:

Theorem 5.15 *Every product of two absolutely convergent series* $\sum_{k=1}^\infty b_k$ *and* $\sum_{\ell=1}^\infty c_\ell$ *is absolutely convergent with the same sum:*

$$\sum_{k,\ell=1}^\infty b_k c_\ell = \Big(\sum_{k=1}^\infty b_k\Big)\Big(\sum_{\ell=1}^\infty c_\ell\Big).$$

A more precise formulation is as follows: Let $\sigma : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a numbering with $\sigma(k) = (\sigma_1(k), \sigma_2(k)) \in \mathbb{N} \times \mathbb{N}$. Then $\sum_{k=1}^\infty b_{\sigma_1(k)} c_{\sigma_2(k)}$ is absolutely convergent and

$$\sum_{k=1}^\infty b_{\sigma_1(k)} c_{\sigma_2(k)} = \Big(\sum_{k=1}^\infty b_k\Big)\Big(\sum_{\ell=1}^\infty c_\ell\Big).$$

Proof: This is an immediate consequence of the *Großer Umordnungssatz* if we set $a_{k\ell} = b_k c_\ell$. The hypothesis of the Umordnungssatz is satisfied, since

$$\begin{aligned}\sum_{k,\ell=1}^m |a_{k\ell}| &= \sum_{k,\ell=1}^m |b_k c_\ell| = \sum_{k,\ell=1}^m |b_k|\,|c_\ell| \\ &= \sum_{k=1}^m |b_k| \sum_{\ell=1}^m |c_\ell| \le \sum_{k=1}^\infty |b_k| \sum_{\ell=1}^\infty |c_\ell| =: M.\end{aligned}$$

Therefore statement (iii) of the Umordnungssatz yields

$$\begin{aligned}\sum_{k=1}^\infty b_{\sigma_1(k)} c_{\sigma_2(k)} &= \sum_{k=1}^\infty a_{\sigma(k)} = \sum_{k=1}^\infty \Big(\sum_{\ell=1}^\infty a_{k\ell}\Big) \\ &= \sum_{k=1}^\infty \Big(\sum_{\ell=1}^\infty b_k c_\ell\Big) = \sum_{k=1}^\infty \Big(b_k \sum_{\ell=1}^\infty c_\ell\Big) = \Big(\sum_{k=1}^\infty b_k\Big)\Big(\sum_{\ell=1}^\infty c_\ell\Big).\end{aligned}$$

■

Let $\sigma : \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0$ be the numbering by diagonals

$$\begin{array}{ccccccc} (0,0) & & (0,1) & & (0,2) & \dots \\ & \swarrow & & \swarrow & & \swarrow \\ (1,0) & & (1,1) & & (1,2) & \dots \\ & \swarrow & & \swarrow & & \swarrow \\ (2,0) & & (2,1) & & (2,2) & \dots \\ \vdots & \swarrow & \vdots & \swarrow & \vdots & \swarrow \end{array}$$

Using this numbering in the preceding theorem yields the *Cauchy product*

$$\sum_{k=0}^\infty \sum_{\ell=0}^k b_\ell c_{k-\ell}\,.$$

This theorem thus implies

Corollary 5.16 *The Cauchy product of two absolutely convergent series* $\sum_{k=0}^{\infty} b_k$ *and* $\sum_{k=0}^{\infty} c_k$ *is absolutely convergent and*

$$\sum_{k=0}^{\infty}\left(\sum_{\ell=0}^{k} b_\ell c_{k-\ell}\right) = \left(\sum_{k=0}^{\infty} b_k\right)\left(\sum_{k=0}^{\infty} c_k\right).$$

Example (Exponential function): For every $x \in \mathbb{R}$ the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges absolutely. For, let k_0 be the smallest natural number satisfying $\frac{|x|}{k_0} \leq \frac{1}{2}$. Then we obtain for $m \geq k_0$

$$\begin{aligned}\sum_{k=k_0}^{m} \frac{|x|^k}{k!} &\leq \sum_{k=k_0}^{m} \frac{|x|^k}{k_0^{k-k_0+1}} = \sum_{k=k_0}^{m} \frac{|x|^{k_0}}{k_0}\left(\frac{|x|}{k_0}\right)^{k-k_0} \\ &\leq \frac{|x|^{k_0}}{k_0}\sum_{k=k_0}^{m}\left(\frac{1}{2}\right)^{k-k_0} = \frac{|x|^{k_0}}{k_0}\sum_{j=0}^{m-k_0}\left(\frac{1}{2}\right)^{j} \\ &\leq \frac{|x|^{k_0}}{k_0}\sum_{j=0}^{\infty}\left(\frac{1}{2}\right)^{j} = 2\,\frac{|x|^{k_0}}{k_0},\end{aligned}$$

hence

$$\sum_{k=0}^{m} \frac{|x|^k}{k!} \leq \sum_{k=0}^{k_0-1} \frac{|x|^k}{k!} + 2\,\frac{|x|^{k_0}}{k_0} =: M.$$

Therefore the sequence $\left\{\sum_{k=0}^{m} \frac{|x|^k}{k!}\right\}_{m=1}^{\infty}$ is bounded, which proves that $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges absolutely. The preceding corollary and the binomial theorem thus yield for $x, y \in \mathbb{R}$

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{\ell=0}^{\infty} \frac{y^\ell}{\ell!} &= \sum_{k=0}^{\infty}\left[\sum_{\ell=0}^{k}\left(\frac{x^\ell}{\ell!}\,\frac{y^{k-\ell}}{(k-\ell)!}\right)\right] \\ &= \sum_{k=0}^{\infty}\left[\frac{1}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell} x^\ell y^{k-\ell}\right] = \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!}.\end{aligned}$$

If one defines the *exponential function* $\exp : \mathbb{R} \to \mathbb{R}$ by

$$x \mapsto \exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

then this equation can be written as

$$\exp(x+y) = \exp(x)\,\exp(y). \tag{5.1}$$

In particular, since $\exp(0) = 1$ we obtain for $y = -x$

$$\exp(x)\,\exp(-x) = 1,$$

whence

$$\exp(-x) = \frac{1}{\exp(x)}\,. \tag{5.2}$$

Leonhard Euler (1707–1783) introduced the symbol e for the number $\exp(1)$:

$$e = \exp(1) = \sum_{k=0}^{\infty} \frac{1}{k!}\,.$$

By induction we obtain from (5.1) for all $n \in \mathbb{N}_0$

$$\exp(nx) = [\exp(x)]^n.$$

If we set $x = \frac{1}{m}$ with $m \in \mathbb{N}$, then this equation yields

$$\exp\left(\frac{1}{m}\right)^m = \exp(1) = e.$$

By our definition of powers with rational exponents at the end of Section 3 this implies $\exp\left(\frac{1}{m}\right) = e^{\frac{1}{m}}$, hence $e^{\frac{n}{m}} = (e^{\frac{1}{m}})^n = [\exp\left(\frac{1}{m}\right)]^n = \exp\left(\frac{n}{m}\right)$, for every $n \in \mathbb{N}_0$. Together with (5.2) this yields

$$\exp(q) = e^q$$

for every rational q. We generalize this formula and *define* for every $x \in \mathbb{R}$

$$e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

With this definition the equations (5.1) and (5.2) can be written as

$$e^{x+y} = e^x e^y, \quad e^{-x} = \frac{1}{e^x}.$$

5.4 Criteria for absolute convergence

Definition 5.17 *Let $\sum_{k=1}^{\infty} c_k$ be a series of non–negative terms. This series is called a majorant of the series $\sum_{k=1}^{\infty} a_k$, if $\sum_{k=1}^{\infty} c_k$ converges and satisfies*

$$|a_k| \le c_k$$

for all k, and it is called a minorant of the series $\sum_{k=1}^{\infty} a_k$, if $\sum_{k=1}^{\infty} c_k$ diverges and satisfies

$$c_k \le |a_k|$$

for all k.

Theorem 5.18 *The series* $\sum_{k=1}^{\infty} a_k$ *converges absolutely if and only if it has a majorant. It is not absolutely convergent if and only if it has a minorant.*

Proof: If a majorant $\sum_{k=1}^{\infty} c_k$ exists, then

$$\sum_{k=1}^{n} |a_k| \leq \sum_{k=1}^{n} c_k \leq \sum_{k=1}^{\infty} c_k < \infty,$$

whence $\sum_{k=1}^{\infty} a_k$ converges absolutely. Conversely, if $\sum_{k=1}^{\infty} a_k$ converges absolutely, then $\sum_{k=1}^{\infty} |a_k|$ is a majorant.

If a minorant $\sum_{k=1}^{\infty} c_k$ exists, then to every $M \in \mathbb{R}$ there is $n \in \mathbb{N}$ with

$$\sum_{k=1}^{n} |a_k| \geq \sum_{k=1}^{n} c_k \geq M,$$

whence $\sum_{k=1}^{\infty} a_k$ does not converge absolutely. Conversely, if $\sum_{k=1}^{\infty} a_k$ is not absolutely convergent, then $\sum_{k=1}^{\infty} |a_k|$ is a minorant. ■

Theorem 5.19 (Root test for absolute convergence.) *Suppose that*

$$\overline{\lim}_{k\to\infty} \sqrt[k]{|a_k|} = a.$$

Then the series $\sum_{k=1}^{\infty} a_k$ *converges absolutely if* $a < 1$, *and diverges if* $a > 1$. *If the limes superior does not exist, then* $\sum_{k=1}^{\infty} a_k$ *diverges.*

Remark: Of course, the *k–th root* is defined by $\sqrt[k]{|a_k|} = |a_k|^{\frac{1}{k}}$.

Proof: If $a < 1$ then $\theta = \frac{a+1}{2}$ satisfies $a < \theta < 1$. Hence, by Theorem 4.25 there is $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$

$$\sqrt[k]{|a_k|} \leq \theta,$$

thus $|a_k| \leq \theta^k$. Therefore $\sum_{k=1}^{\infty} \theta^k$ is a majorant, possibly after modification of finitely many terms.

If $a > 1$ then Theorem 4.25 implies that there are infinitely many k with $\sqrt[k]{|a_k|} \geq 1$, whence $|a_k| \geq 1$. Consequently, $\{a_k\}_{k=1}^{\infty}$ is not a null sequence, and therefore $\sum_{k=1}^{\infty} a_k$ diverges.

If $\overline{\lim}_{k\to\infty} \sqrt[k]{|a_k|}$ does not exist, then the sequence $\{|a_k|\}_{k=1}^{\infty}$ is unbounded, hence again $\{a_k\}_{k=1}^{\infty}$ is not a null sequence, and $\sum_{k=1}^{\infty} a_k$ diverges. ■

Theorem 5.20 (Ratio test) *(i) Suppose that $a_k \neq 0$ for all but finitely many k, and that*

$$\overline{\lim}_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = b.$$

If $b < 1$, then the series $\sum_{k=1}^{\infty} a_k$ converges absolutely.
(ii) Suppose that $a_k \neq 0$ and

$$\left| \frac{a_{k+1}}{a_k} \right| \geq 1$$

for all but finitely many k. Then $\sum_{k=1}^{\infty} a_k$ diverges.

Proof: (i) Let $b < 1$, and let $\theta = \frac{b+1}{2}$. Since $b < \theta < 1$, we can again use Theorem 4.25 to conclude that there is $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$

$$\frac{|a_{k+1}|}{|a_k|} \leq \theta,$$

whence $|a_{k+1|} \leq \theta |a_k|$. By induction it follows from this inequality that $|a_k| \leq \theta^{k-k_0} |a_{k_0}|$, hence

$$\frac{|a_{k_0|}}{\theta^{k_0}} \sum_{k=1}^{\infty} \theta^k$$

is a majorant of $\sum_{k=1}^{\infty} a_k$ (possibly after enlargement of the first $k_0 - 1$ terms).
(ii) Since $|a_{k+1}| \geq |a_k|$ and $|a_k| \neq 0$ for all but finitely many k, the sequence $\{a_k\}_{k=1}^{\infty}$ is not a null sequence, hence $\sum_{k=1}^{\infty} a_k$ diverges. ■

Examples: 1.) The series $\sum_{\nu=1}^{\infty} (\frac{\nu}{\nu+1})^{\nu^2}$ is absolutely convergent. For, the root test gives

$$\begin{aligned} \overline{\lim}_{\nu\to\infty} \sqrt[\nu]{\left(\frac{\nu}{\nu+1}\right)^{\nu^2}} &= \overline{\lim}_{\nu\to\infty} \left(\frac{\nu}{\nu+1}\right)^{\nu} = \overline{\lim}_{\nu\to\infty} \frac{1}{(1+\frac{1}{\nu})^{\nu}} \\ &\leq \overline{\lim}_{\nu\to\infty} \frac{1}{1+\nu\frac{1}{\nu}} = \frac{1}{2}, \end{aligned}$$

by the Bernoulli inequality.
2.) The series $\sum_{\nu=1}^{\infty} (\frac{\nu}{\nu+1})^{\nu}$ diverges. To prove this, the root test cannot be applied, since

$$\overline{\lim}_{\nu\to\infty} \sqrt[\nu]{\left(\frac{\nu}{\nu+1}\right)^{\nu}} = \overline{\lim}_{\nu\to\infty} \frac{\nu}{\nu+1} = 1.$$

However, the binomial theorem yields

$$\left(\frac{\nu}{\nu+1}\right)^{\nu} = \frac{1}{(1+\frac{1}{\nu})^{\nu}} = \frac{1}{\sum_{k=0}^{\nu} \binom{\nu}{k} \frac{1}{\nu^k}}.$$

Since

$$\sum_{k=0}^{\nu} \binom{\nu}{k} \frac{1}{\nu^k} = \sum_{k=0}^{\nu} \frac{1}{k!} \frac{\nu!}{(\nu-k)!\nu^k} \leq \sum_{k=0}^{\nu} \frac{1}{k!} \leq e,$$

we obtain

$$\left(\frac{\nu}{\nu+1}\right)^{\nu} \geq \frac{1}{e},$$

and therefore $\{(\frac{\nu}{\nu+1})^{\nu}\}_{\nu=1}^{\infty}$ is not a null sequence.

6 Continuous functions

6.1 Topology of $\mathbb{R}$

We already introduced the notions *neighborhood* and *open, closed, half open interval.* In this section these topological notions are put into a general framework.

Definition 6.1 *Let M be a set of real numbers. $x \in M$ is said to be an interior point of M, if M contains a neighborhood of M as a subset (equivalently, if M is a neighborhood of x).*

Definition 6.2 *Let M be a set of real numbers. $x \in \mathbb{R}$ is said to be an accumulation point of M, if every neighborhood of x contains a point $y \in M$ with $y \neq x$.*

Note that whereas an interior point of M always belongs to M, an accumulation point of M is not necessarily a point of M. The notions of an accumulation point of a sequence and of an accumulation point of a set are different. In particular, an accumulation point of a sequence is not necessarily an accumulation point of the range of the sequence. An interior point of M is always an accumulation of M.

Examples: 1.) Let $M = (a, b)$ with $a < b$. Then all points of M are interior points. a and b are accumulation points of M, which do not belong to M. In particular, a and b are not interior points of M. Also for $M = [a, b]$ the numbers a and b are accumulation points, which this time belong to M. Still, a and b are not interior points of M.
2.) Since between any two real numers x, y with $x \neq y$ there lies a point of $\mathbb{Q}$, every real number is an accumulation point of $\mathbb{Q}$. However, since between any two rational numbers there lies an irrational number, $\mathbb{Q}$ does not have interior points.

Definition 6.3 *A set $M \subseteq \mathbb{R}$ is called open, if every point of M is an interior point.*

Definition 6.4 *A set $M \subseteq \mathbb{R}$ is called closed, if it contains all its accumulation points.*

Examples: Every open interval (a, b) is an open set. Also the sets $\mathbb{R}, \emptyset, (a, b) \cup (c, d)$ are open. The sets $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ are not open.

Every closed interval $[a, b]$ is a closed set. In particular, the set $[a, a] = \{a\}$ containing one point is closed. Also the sets $(-\infty, b]$, $[a, \infty)$, $(-\infty, \infty) = \mathbb{R}$, $[a, b] \cup [c, d], \mathbb{N}, \mathbb{Z}, \emptyset$ are closed. The set $\mathbb{Q}$ is not closed.

Theorem 6.5 *A set M is closed if and only if its complementary set $\mathbb{R} \backslash M$ is open.*

Proof: Let M be closed and let $x \in \mathbb{R} \backslash M$. Then x is not an accumulation point of M, hence there exists a neighborhood U of x with $U \cap M = \emptyset$, whence $U \subseteq \mathbb{R} \backslash M$. Therefore x is interior point of $\mathbb{R} \backslash M$, hence $\mathbb{R} \backslash M$ is open. Conversely, let $\mathbb{R} \backslash M$ be open and let x be an accumulation point of M. Then every neighborhood of x contains an element of M, hence there is no neighborhood of x contained in $\mathbb{R} \backslash M$, whence x is not an interior point of $\mathbb{R} \backslash M$. Since $\mathbb{R} \backslash M$ totally consists of interior points, it follows that $x \notin \mathbb{R} \backslash M$, hence $x \in M$. Thus, M is closed. ■

Theorem 6.6 *The union of an arbitrary system of open sets is an open set. The intersection of finitely many open sets is an open set.*

The intersection of an arbitrary system of closed sets is a closed set. The union of finitely many closed sets is a closed set.

Proof: Let S be a system of open sets and suppose that $x \in \bigcup_{M \in S} M$. Then there is $N \in S$ with $x \in N$. Since N is open, N itself is a neighborhood of x, hence the superset $\bigcup_{M \in S} M$ is a neighborhood of x. Therefore $\bigcup_{M \in S} M$ is open, since x was chosen arbitrarily from this set.

Let $M_1, \ldots, M_n$ be open sets and suppose that $x \in \bigcap_{i=1}^n M_i$. Then $x \in M_i$ for all $i = 1, \ldots, n$, and since M_i is open, there are numbers $\varepsilon_1, \ldots, \varepsilon_n > 0$ with

$$U_{\varepsilon_i}(x) = \{y \in \mathbb{R} \mid |y - x| < \varepsilon_i\} \subseteq M_i$$

for $i = 1, \ldots, n$. Set $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$. Then $\varepsilon > 0$ and $U_\varepsilon(x) \subseteq \bigcap_{i=1}^n U_{\varepsilon_i}(x) \subseteq \bigcap_{i=1}^n M_i$. Consequently, $\bigcap_{i=1}^n M_i$ is open.

Using these statements for open sets, the statements for the closed sets are implied by the de Morgan rules

$$\mathbb{R} \backslash \Big(\bigcap_{M \in S} M \Big) = \bigcup_{M \in S} (\mathbb{R} \backslash M), \quad \mathbb{R} \backslash (\bigcup_{M \in S} M) = \bigcap_{M \in S} (\mathbb{R} \backslash M),$$

and by the preceding theorem. ■

The intersection of an infinite system of open sets need not be open and that the union of an infinite system of closed sets need not be closed. This is shown by the examples

$$\bigcap_{a>0} (-a, a) = \{0\}$$

$$\bigcup_{0<a<1} [0, a] = [0, 1).$$

Let M be a subset of $\mathbb{R}$. The system S of all closed sets A of real numbers with $M \subseteq A$ is not empty, since S contains $\mathbb{R}$. Therefore

$$\overline{M} := \bigcap_{A \in S} A$$

is a closed set with $M \subseteq \overline{M}$.

Definition 6.7 *$\overline{M}$ is called closed hull of M.*

Theorem 6.8 *The closed hull $\overline{M}$ consists of all the points of M and of all the accumulation points of M.*

Proof: $M \subseteq \overline{M}$ implies that every accumulation point of M also is an accumulation point of $\overline{M}$. Therefore, since $\overline{M}$ is closed, it must contain all the accumulation points of M. On the other hand, if x does not belong to M and is not an accumulation point of M, then there exists an open neighborhood U of x (for example an ε–neighborhood $U = (x - \varepsilon\,,\, x + \varepsilon)$) with $U \cap M \neq \emptyset$. Since U is open, the complement $\mathbb{R}\backslash U$ is a closed set with $M \subseteq \mathbb{R}\backslash U$, hence $\mathbb{R}\backslash U$ belongs to the system of all closed sets containing M as a subset, hence $\overline{M} \subseteq \mathbb{R}\backslash U$. This implies $x \notin \overline{M}$. ∎

Definition 6.9 *x is called point of contact of M, if every neighborhood of x contains a point of M.*

x is called isolated point of M, if $x \in M$ and if there exists a neighborhood of x which besides x contains no other point of M.

x is called boundary point of M, if every neighborhood of x contains a point of M and a point of the complement $\mathbb{R}\backslash M$.

Lemma 6.10 *x is an accumulation point of a set M, if and only if there is a sequence $\{x_n\}_{n=1}^{\infty}$, which consists of numbers $x_n \in M$ that all differ, and which converges to x.*

Proof: Suppose that $\{x_n\}_{n=1}^{\infty}$ is a sequence as in the lemma, which converges to x. Then every neighborhood U of x contains all but finitely many terms of $\{x_n\}_{n=1}^{\infty}$. Since all the numbers x_n differ and belong to M, the neighborhood U contains at least one $x_n \in M$ with $x_n \neq x$, hence x is an accumulation point of M.

If x is an accumulation point of M, then a sequence $\{x_n\}_{n=1}^{\infty}$, whose terms all belong to M and differ, and which converges to x, can be constructed similarly as in the proof given in Section 4 of the corresponding Theorem 4.18 for accumulation points of sequences. We leave it as an exercise to make the necessary modifications. ∎

Theorem 6.11 (of Bolzano and Weierstraß for sets) *Every bounded infinite set has an accumulation point.*

Proof: Since M is infinite, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of numbers $x_n \in M$ that all differ. $\{x_n\}_{n=1}^{\infty}$ is bounded and thus has a convergent subsequence, by Corollary 4.20. Since all the terms of the subsequence differ, the limit is an accumulation point of M, by the preceding lemma. ■

Definition 6.12 *A set $M \subseteq \mathbb{R}$ is called compact, if it is bounded and closed.*

Examples: All closed intervals $[a, b]$ $(a, b \neq \infty)$ and all finite sets are compact. A finite union of compact sets is compact. The set $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ is not compact, but $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$ is compact. $\emptyset$ is compact.

Theorem 6.13 *A non-empty compact set possesses a maximum and a minimum.*

Proof. Since M is non-empty and bounded, $s = \sup M$ and $u = \inf M$ exist. It must be shown that $s, u \in M$. But, as a consequence of Theorem 2.18, the supremum and the infimum of a set belong to this set or are accumulation points of the set. Thus, for compact sets the supremum and the infimum always belong to the set. ■

The theorem of Bolzano and Weierstraß immediately yields the following important

Theorem 6.14 *Every infinite subset of a compact set has an accumulation point belonging to M.*

The following theorems give characterizations of compact sets:

Theorem 6.15 *A set M is compact if and only if every sequence with values in M has a convergent subsequence with limit in M.*

Proof: As a corollary of the Bolzano-Weierstraß theorem for sequences, a sequence with values in a compact set M has a convergent subsequence. The limit is a point of contact of the range of the subsequence; hence it is a point of contact of M. If a point of contact of a set does not belong to the set, it must be an accumulation point. Therefore, for a compact set the points of contact always belong to this set. Hence, the limit belongs to M.

To prove the converse statement, suppose that every sequence with values in M has a convergent subsequence with limit in M. This implies that M is bounded. For, otherwise to every $n \in \mathbb{N}$ there would exist $x_n \in M$ with $|x_n| > n$. For a subsequence $\{x_{n_m}\}_{m=1}^{\infty}$

of the sequence $\{x_n\}_{n=1}^{\infty}$ the inequality $n_m \geq m$ must hold, which would imply $|x_{n_m}| > n_m \geq m$. Therefore the subsequence would be unbounded, hence would not converge. Thus, $\{x_n\}_{n=1}^{\infty}$ would be a sequence in M without convergent subsequence.

Furthermore, M is closed. For, let x be an accumulation point of M. Then a sequence $\{x_n\}_{n=1}^{\infty}$ with values in M exists, which converges to x. By assumption, this sequence has a subsequence converging to $y \in M$. However, since $\{x_n\}_{n=1}^{\infty}$ converges to x, every subsequence converges to x, which implies $x = y \in M$. Hence M is closed. ■

To state the theorem with the second characterization of compact sets, I need the following

Definition 6.16 *Let M be a subset of $\mathbb{R}$. A system $\mathcal{U}$ of open subsets of $\mathbb{R}$ is called an open covering of M if*

$$M \subseteq \bigcup_{U \in \mathcal{U}} U .$$

Theorem 6.17 (of Heine and Borel) *A subset M of $\mathbb{R}$ is compact if and only if every open covering $\mathcal{U}$ of M contains finitely many sets $U_1, \ldots, U_n \in \mathcal{U}$ which already cover M:*

$$M \subseteq \bigcup_{i=1}^{n} U_i$$

(Eduard Heine 1821–1881, Emile Borel 1871–1956).

Proof: Suppose that M is compact, but does not have the Heine-Borel covering property. Then there is an open covering $\mathcal{U}$ of M which does not contain a finite collection of sets covering M. Since M is bounded, there are numbers $a < b$ with $J_0 := [a, b] \supseteq M$. Let $m = \frac{a+b}{2}$. For at least one of the sets $[a, m] \cap M$ or $[m, b] \cap M$ infinitely many sets from $\mathcal{U}$ are needed to cover it, since otherwise finitely many sets from $\mathcal{U}$ suffice to cover M, contrary to the assumption. Let this be $[a, m] \cap M$, say. We set $J_1 = [a, m]$ and bisect this interval. Repeating this procedure, we construct a sequence of closed intervals $\{J_n\}_{n=0}^{\infty}$ with

$$J_0 \supseteq J_1 \supseteq J_2 \supseteq \ldots ,$$

such that infinitely many sets from $\mathcal{U}$ are needed to cover $J_n \cap M$ for every n. The length $2^{-n}(b - a)$ of J_n tends to zero for $n \to \infty$.

The intersection of all these nested intervals contains exactly one element s:

$$s \in \bigcap_{n=0}^{\infty} J_n$$

s belongs to the set M. For, every neighborhood of s contains an ε–neighborhood $U_\varepsilon(x) = (s-\varepsilon, s+\varepsilon)$. Since $s \in J_n$ for all n and since the length of J_n converges to zero, the intervals J_n are contained in $(s-\varepsilon, s+\varepsilon)$ for all sufficiently large n. Therefore also the non–empty sets $J_n \cap M$ are contained in this ε–neighborhood, and consequently every neighborhood of s contains an element of M. Thus, s is a point of contact of M, hence belongs to the compact set M.

Since $\mathcal{U}$ is a covering of M, it follows that there exists an open set $U \in \mathcal{U}$ with $s \in U$. As an open set, U contains an ε–neighborhood $(s-\varepsilon, s+\varepsilon)$ of s as a subset. Since for sufficiently large n the set $J_n \cap M$ is contained in $(s-\varepsilon, s+\varepsilon)$, we obtain

$$J_n \cap M \subseteq (s-\varepsilon, s+\varepsilon) \subseteq U,$$

which means that one set of $\mathcal{U}$ suffices to cover $J_n \cap M$, contradicting the construction of J_n. Therefore, the compact set M must have the Heine–Borel property.

Conversely, suppose that M has the Heine–Borel covering property. It must be shown that M is compact.

Note first that the system

$$\mathcal{U} = \{U_1(x) \,\big|\, x \in M\}$$

of all open 1–neighborhoods is an open covering of M. By the Heine–Borel property, finitely many of these suffice to cover M, hence M is contained in the union of finitely many bounded sets, and thus is bounded.

If M would not be closed, then an accumulation point a of M would exist with $a \notin M$. For $x \in M$ we set $\varepsilon(x) := \frac{1}{2}|x-a| > 0$. Then

$$\mathcal{U} = \{U_{\varepsilon(x)}(x) \,\big|\, x \in M\}$$

is an open covering of M, but it contains no finite collection of sets which cover M. To see this, choose finitely many sets $U_{\varepsilon(x_1)}(x_1), \ldots, U_{\varepsilon(x_n)}(x_n)$ from $\mathcal{U}$ and define

$$\varepsilon = \min\{\varepsilon(x_1), \ldots, \varepsilon(x_n)\} > 0.$$

Then

$$U_\varepsilon(a) \cap U_{\varepsilon(x_i)}(x_i) = \emptyset$$

for all $i = 1, \ldots, n$, hence $U_\varepsilon(a)$ is disjoint from the union $U_{\varepsilon(x_1)}(x_1) \cup \ldots \cup U_{\varepsilon(x_n)}(x_n)$. Yet, there exists $x \in U_\varepsilon(a) \cap M$, since a is an accumulation point M. Thus, this $x \in M$ does not belong to $\bigcup_{i=1}^n U_{\varepsilon(x_i)}(x_i)$, hence M is not covered by finitely many sets of $\mathcal{U}$. This contradicts the Heine–Borel property, whence M is closed. Thus, M is compact. ■

6.2 Continuity

There are many important functions in mathematics and in the applications, which have the property that the function value changes little if the argument is only changed by a small amount. A function with this property is said to be continuous. First I give a precise definition of continuity:

Definition 6.18 *Let $D \subseteq \mathbb{R}$. The function $f : D \to \mathbb{R}$ is said to be continuous at the point $a \in D$, if to every $\varepsilon > 0$ there is $\delta > 0$ such that for all $x \in D$ with $|x - a| < \delta$ the inequality*

$$|f(x) - f(a)| < \varepsilon$$

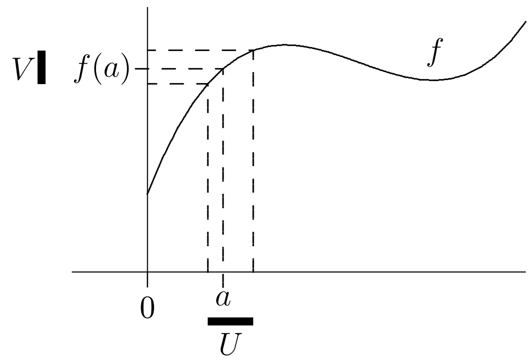
holds.

Using quantifiers, this property can be written as

$$\underset{\varepsilon>0}{\forall} \ \underset{\delta>0}{\exists} \ \underset{\substack{x\in D \\ |x-a|<\delta}}{\forall} : |f(x) - f(a)| < \varepsilon.$$

Since every neighborhood of a point contains an ε–neighborhood, we immediately obtain the following result:

Theorem 6.19 *The function $f : D \to \mathbb{R}$ is continuous at $a \in D$, if and only if to every neighborhood V of $f(a)$ there is a neighborhood U of a such that $f(U \cap D) \subseteq V$.*



Examples: 1.) The identity mapping $x \mapsto \mathrm{id}\,(x) = x : \mathbb{R} \to \mathbb{R}$ is continuous at every point $a \in \mathbb{R}$.

Proof: Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Then we trivially obtain for $x \in \mathbb{R}$ with $|x - a| < \delta$ that

$$|\mathrm{id}\,(x) - \mathrm{id}\,(a)| = |x - a| < \delta = \varepsilon.$$

∎

2.) The square root function $Q : [0, \infty) \to \mathbb{R}$,

$$x \mapsto Q(x) := \sqrt{x},$$

is continuous at every point $a \in [0, \infty)$.

Proof: First we consider $a = 0$. Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon^2$. Then we obtain for all $x \in [0, \infty)$ with $|x - 0| = x < \delta$ that

$$|Q(x) - Q(0)| = Q(x) = \sqrt{x} < \sqrt{\delta} = \varepsilon,$$

since the square root function is strictly increasing. Therefore Q is continuous at 0.

Next, let $a > 0$ and let $\varepsilon > 0$ be given. To find $\delta > 0$, consider the following computation

$$\sqrt{x} - \sqrt{a} = \frac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{\sqrt{x} + \sqrt{a}} = \frac{x - a}{\sqrt{x} + \sqrt{a}}.$$

This equation shows that we can choose $\delta = \sqrt{a}\,\varepsilon$. For, $x \in [0, \infty)$ and $|x - a| < \delta$ imply

$$|\sqrt{x} - \sqrt{a}| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}} \le \frac{|x - a|}{\sqrt{a}} < \frac{\delta}{\sqrt{a}} = \frac{\sqrt{a}}{\sqrt{a}}\varepsilon = \varepsilon.$$

Consequently, Q is continuous at a. ∎

3.) Let $f : (0, \infty) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ \frac{1}{p+q}, & \text{if } x = \frac{p}{q} \text{ with natural numbers } p, q, \text{ which are relatively prime.} \end{cases}$$

Then f is discontinuous at every rational $x > 0$, but continuous at every irrational $x > 0$.

Proof: Let $x = \frac{p}{q}$ be rational. To show that f is discontinuous at x means to prove that the statement in the definition of continuity is false for this function f and for this point x. Hence, it must be shown that the negation of this statement is true. This negation is

$$\underset{\varepsilon>0}{\exists}\ \underset{\delta>0}{\forall}\ \underset{\substack{y\in(0,\infty)\\ |y-x|<\delta}}{\exists} : |f(y) - f(x)| \ge \varepsilon. \tag{6.1}$$

We have $f(x) = \frac{1}{p+q} > 0$. Choose $\varepsilon = f(x)$. Then to every $\delta > 0$ there is an irrational number y with $x < y < x + \delta$, hence $|x - y| < \delta$, since between two real numbers there always lies an irrational number. For this y

$$|f(x) - f(y)| = |f(x) - 0| = f(x) = \varepsilon$$

Thus, (6.1) is true and f is not continuous at rational x.

Next, suppose that x is irrational. Then $f(x) = 0$. Let $\varepsilon > 0$ be given. We are looking for a suitable positive δ. There are at most finitely many pairs (p, q) of natural numbers with $p + q \le \frac{1}{\varepsilon}$, hence with $\frac{1}{p+q} \ge \varepsilon$. If no such pair exists, we set $\delta = 1$. Otherwise we set

$$\delta = \min \left\{ |\frac{p}{q} - x| \,\Big|\, p, q \in \mathbb{N} \wedge \frac{1}{p+q} \ge \varepsilon \right\} > 0.$$

Now let $y = \frac{p}{q} \in (0, \infty)$ be a rational number with $|y - x| < \delta$. Then we must have $\frac{1}{p+q} < \varepsilon$, by definition of δ, whence

$$|f(y) - f(x)| = f(y) = \frac{1}{p+q} < \varepsilon.$$

Because for irrational y with $|y - x| < \delta$ we anyway have $|f(y) - f(x)| = 0 < \varepsilon$, it follows that f is continuous at irrational x. ■

4.) The Dirichlet function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$$

is obviously nowhere continuous.

Definition 6.20 *A real function is called continuous, if it is continuous at every point of its domain of defintion.*

Theorem 6.21 *Let $D \subseteq \mathbb{R}$. A function $f : D \to \mathbb{R}$ is continuous at $a \in D$ if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in D$ and with $\lim_{n\to\infty} x_n = a$ the sequence $\{f(x_n)\}_{n=1}^{\infty}$ satisfies*

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(a).$$

This means that for a continuous function the function symbol can be interchanged with the limit symbol.

Proof: Suppose that f is continuous at $a \in D$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence with $x_n \in D$ and with $\lim_{n\to\infty} x_n = a$. To prove that $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(a)$, let $\varepsilon > 0$. Since f is continuous at a, there is $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ for all $x \in D$ with $|x - a| < \delta$. Furthermore, since $\{x_n\}_{n=1}^{\infty}$ converges to a, there exists $n_0 \in \mathbb{N}$ such that the estimate $|x_n - a| < \delta$ is satisfied for all $n \ge n_0$. Thus, for this n_0 and for all $n \ge n_0$

$$|f(x_n) - f(a)| < \varepsilon,$$

which means that $\lim_{n\to\infty} f(x_n) = f(a)$.

Conversely, assume that f is not continuous at $a \in D$. It must be shown that there is a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in D$ and with $\lim_{n\to\infty} x_n = a$, for which $\{f(x_n)\}_{n=1}^{\infty}$ is not convergent to $f(a)$. To this end we note that

$$\underset{\varepsilon>0}{\exists}\ \underset{\delta>0}{\forall}\ \underset{\substack{x\in D\\ |x-a|<\delta}}{\exists} : \ |f(x) - f(a)| \geq \varepsilon$$

holds, since f is not continuous at a. This statement asserts the existence of a number $\varepsilon > 0$, such that for every $n \in \mathbb{N}$ to $\delta = \frac{1}{n}$ we can choose a number $x_n \in D$ with $|x_n - a| < \frac{1}{n}$ and with $|f(x_n) - f(a)| \geq \varepsilon$. The former inequality implies that $\{x_n\}_{n=1}^{\infty}$ converges to a, yet the latter implies that $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to $f(a)$. ∎

Corollary 6.22 *All rational functions are continuous.*

Proof: Let $F : D \to \mathbb{R}$ be a rational function. In Section 4 we showed that

$$\lim_{n\to\infty} F(x_n) = F(\lim_{n\to\infty} x_n)$$

holds for all sequences $\{x_n\}_{n=1}^{\infty}$ with $x_n \in D$ and with $\lim_{n\to\infty} x_n \in D$. The preceding theorem thus implies that F is continuous at every $x \in D$. ∎

Theorem 6.23 *Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ both be continuous at $a \in D$. Then also the functions $f + g, cf, f \cdot g, |f|$ defined on D are continuous at a. Here c is a real number. If $f(a) \neq 0$, then also the function $\frac{1}{f}$ is continuous at a.*

Proof of the theorem: For every sequence $\{x_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} x_n = a$ we have $\lim_{n\to\infty} f(x_n) = f(a)$ and $\lim_{n\to\infty} g(x_n) = g(a)$, whence

$$\lim_{n\to\infty} (f + g)(x_n) = \lim_{n\to\infty} \Big(f(x_n) + g(x_n)\Big) = f(a) + g(a) = (f + g)(a)$$

and

$$\lim_{n\to\infty} (cf)(x_n) = c \lim_{n\to\infty} f(x_n) = cf(a) = (cf)(a),$$

and therefore $f + g$ and cf are continuous at a. In the same way this follows for $f \cdot g$ and $|f|$.

Let $f(a) \neq 0$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence with $x_n \in D(\frac{1}{f}) = \{x \in D \,\big|\, f(x) \neq 0\}$ and with $\lim_{n\to\infty} x_n = a$. Then $f(x_n) \neq 0$ for all n, hence

$$\lim_{n\to\infty} \Big(\frac{1}{f}\Big)(x_n) = \lim_{n\to\infty} \frac{1}{f(x_n)} = \frac{1}{\lim_{n\to\infty} f(x_n)} = \frac{1}{f(a)} = \frac{1}{f}(a).$$

Thus, $\frac{1}{f}$ is continuous at a. ∎

Corollary 6.24 *Let $D \subseteq \mathbb{R}$ and let $a \in D$. The set of functions $f : D \to \mathbb{R}$, which are continuous at $a \in D$, is a vector space over $\mathbb{R}$. Also, the set $C(D, \mathbb{R})$ of all continuous functions $f : D \to \mathbb{R}$ is a vector space over $\mathbb{R}$.*

Theorem 6.25 *Let $D_1, D_2 \subseteq \mathbb{R}$, let $f : D_1 \to \mathbb{R}$, $g : D_2 \to \mathbb{R}$ and let $g \circ f$ be defined. Suppose that f is continuous at $a \in D_1$ and g is continuous at $b = f(a)$. Then $g \circ f$ is continuous at a.*

Proof: Let $\{x_n\}_{n=1}^{\infty}$ be a sequence with $x_n \in D_1$ and with $\lim_{n\to\infty} x_n = a$. Then $\{f(x_n)\}_{n=1}^{\infty}$ is a sequence with $f(x_n) \in D_2$ and with $\lim_{n\to\infty} f(x_n) = f(a) = b$, hence $\lim_{n\to\infty} (g \circ f)(x_n) = \lim_{n\to\infty} g(f(x_n)) = g(f(a)) = (g \circ f)(a)$. This proves that $g \circ f$ is continuous at a. ■

6.3 Mapping properties of continuous functions

Let D be an open set and $f : D \to \mathbb{R}$ be continuous. Then $f(D)$ is not necessarily open. For example, a constant function is continuous and maps D to a set containing one element. Such a set is not open. Another example is the function $x \mapsto x^2$, which maps $(-1, 1)$ onto the half open set $[0, 1)$.

Similarly, if D is closed, then $f(D)$ is not necessarily closed. For example, the function $x \mapsto \frac{1}{1+x^2}$ maps $\mathbb{R}$ onto $(0, 1]$.

This is different for compact sets:

Theorem 6.26 *If $D \subseteq \mathbb{R}$ is compact and $f : D \to \mathbb{R}$ is continuous, then $f(D)$ is compact.*

Proof: We prove that with D also $f(D)$ has the Heine–Borel covering property. Let $\mathcal{U}$ be an open covering of $f(D)$. Then the set system $\{f^{-1}(U)\}_{U\in\mathcal{U}}$ is a covering of D, but the sets $f^{-1}(U)$ are not necessarily open. Yet, we can construct an open covering of D as follows:

From $f(D) \subseteq \bigcup_{U\in\mathcal{U}} U$ it follows that to every $x \in D$ there is $U(x) \in \mathcal{U}$ with $x \in f^{-1}(U)$. Since $U(x)$ is open, $U(x)$ is a neighborhood of $f(x)$. Hence, because of the continuity of f, there is a neighborhood $V(x)$ of x with $f(V(x) \cap D) \subseteq U(x)$.

$\{V(x) \,|\, x \in D\}$ is an open covering of the compact set D, hence finitely many sets $V(x_1), \ldots, V(x_n)$ can be selected such that $D \subseteq V(x_1) \cup \ldots \cup V(x_n)$. Thus

$$\begin{aligned} f(D) &= f\Big(D \cap \big(V(x_1) \cup \ldots \cup V(x_n)\big)\Big) = f\Big(\big(D \cap V(x_1)\big) \cup \ldots \cup \big(D \cap V(x_n)\big)\Big) \\ &= f\big(D \cap V(x_1)\big) \cup \ldots \cup f\big(D \cap V(x_n)\big) \subseteq U(x_1) \cup \ldots \cup U(x_n). \end{aligned}$$

This proves that $f(D)$ has the Heine–Borel property, hence is compact. ■

Corollary 6.27 *If D is a non–empty compact subset of $\mathbb{R}$ and if $f : D \to \mathbb{R}$ is continuous, then f assumes the minimum and maximum on D, i.e. there are $x_1, x_2 \in D$ with $f(x_1) = \min f(D)$ and $f(x_2) = \max f(D)$.*

Proof: This is obvious, since the non–empty compact set $f(D)$ has a minimum und a maximum. ■

Below we present an example which shows that the inverse function of a continuous function is not necessarily continuous. This is different for a compact domain of definition:

Theorem 6.28 *Let $D \subseteq \mathbb{R}$ be compact and suppose that $f : D \to \mathbb{R}$ is injective and continuous. Then*

$$f^{-1} : f(D) \to \mathbb{R}$$

is continuous.

Proof: Let $b \in f(D)$. To show that f^{-1} is continuous at b, we must prove that if $\{y_n\}_{n=1}^{\infty}$ is a sequence with $y_n \in f(D)$ and with $\lim_{n\to\infty} y_n = b$, then $\{x_n = f^{-1}(y_n)\}_{n=1}^{\infty}$ converges to $a := f^{-1}(b)$. The sequence $\{x_n\}_{n=1}^{\infty}$ is bounded and thus has accumulation points in D. If $a' \in D$ is an accumulation point, we can choose a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ converging to a'. The continuity of f implies

$$f(a') = f(\lim_{j\to\infty} x_{n_j}) = \lim_{j\to\infty} f(x_{n_j}) = \lim_{j\to\infty} y_{n_j} = b,$$

which yields $a' = f^{-1}(b) = a$. Consequently, a is the only accumulation point of $\{x_n\}_{n=1}^{\infty}$, hence

$$\overline{\lim}_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n = a,$$

and this implies $\lim_{n\to\infty} x_n = a$. Therefore f^{-1} is continuous in every point of $f(D)$. ■

Lemma 6.29 *Let $D \subseteq \mathbb{R}$, let $a \in D$ and $y \in \mathbb{R}$. If $g : D \to \mathbb{R}$ is a continuous function with $g(a) > y$ ($g(a) < y$, respectively), then there is a neighborhood U of a with $g(x) > y$ ($g(x) < y$, respectively) for all $x \in U \cap D$.*

Proof: Suppose that $g(a) > y$. Then to $\varepsilon = g(a) - y > 0$ there is $\delta > 0$ such that $|g(x) - g(a)| < \varepsilon$ for all $x \in D$ with $|x - a| < \delta$. With $U = U_\delta(a) = (a - \delta, a + \delta)$ we thus obtain for $x \in U \cap D$ that

$$g(x) = g(x) - g(a) + g(a) \geq g(a) - |g(x) - g(a)| > g(a) - (g(a) - y) = y.$$

If $g(a) < y$, then the function $-g$ is greater than $-y$ in a neighborhood of a, hence g is smaller than y in this neighborhood. ■

Theorem 6.30 (Intermediate Value Theorem) *Let $a < b$ and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then to each number y between $f(a)$ and $f(b)$ there is a z with $a < z < b$ such that $f(z) = y$.*

This can also be formulated differently: If $f(a) \leq f(b)$ ($f(b) \leq f(a)$, respectively), then the interval $[f(a), f(b)]$ ($[f(b), f(a)]$, respectively) is contained in the range of f.

Proof: If $f(a) = f(b)$ nothing needs to be shown. So, assume first that $f(a) < f(b)$ and $f(a) < y < f(b)$. To construct an inverse image z of y, note that the set $M_y = \{x \mid f(x) \leq y,\ a \leq x \leq b\}$ is bounded above by b and is non-empty, since it contains a. Therefore the supremum $z = \sup M_y$ exists. We show that z satisfies $f(z) = y$, hence it is an inverse image of y. To this end we exclude the cases

1.) $f(z) > y$,

2.) $f(z) < y$.

Assume first that $f(z) > y$ holds. Then by Lemma 6.29 there is a neighborhood U of z with $f(x) > y$ for all $x \in U \cap [a, b]$. By Theorem 2.18 every neighborhood of the supremum z contains a point of M_y, hence there is a point $x \in U \cap M_y$. This point satisfies $f(x) > y$, since $x \in U$, and $f(x) \leq y$, since $x \in M_y$, which is impossible. We conclude that $f(z) \leq y$ must hold. Assume next that $f(z) < y$ holds. Then there is a neighborhood U of z with $f(x) < y$ for all $x \in U \cap [a, b]$, which implies that $(U \cap [a, b]) \subseteq M_y$. Since $z < b$, there is a point $x \in U \cap [a, b]$ satisfying $z < x$. Since $x \in M_y$, this means that z is not an upper bound of M_y, contradicting the assumption that z is the supremum of M_y. We conclude that indeed $f(z) = y$ holds.

Finally, if $f(a) > f(b)$ we apply the preceding result to the function $-f$. ■

Corollary 6.31 *For every $n \in \mathbb{N}$ the range of the power function $x \mapsto f_n(x) := x^n : [0, \infty) \to \mathbb{R}$ is the interval $[0, \infty)$.*

Proof: Since f_n is continuous and satisfies $f_n(0) = 0$, it follows that for every natural number m the interval $[f_n(0), f_n(m)] = [0, f_n(m)]$ belongs to the range $W(f_n)$. The function f_n is increasing and unbounded, hence to every $y \in [0, \infty)$ there is a number $m \in \mathbb{N}$ such that $y \in [0, f_n(m)]$, which implies that $[0, \infty) \subseteq \bigcup_{m \in \mathbb{N}} [0, f_n(m)] = W(f_n)$. ■

Corollary 6.32 *If f is continuous in $[a, b]$ and satisfies $f(a) \cdot f(b) < 0$, then f has a zero. Every polynomial with odd degree has a zero.*

Proof: Without restriction of generality assume that $f(a) < 0$, $f(b) > 0$. Then $0 \in [f(a), f(b)]$, hence the first assertion is implied by the intermediate value theorem.

If $p(x) = a_{2n+1}x^{2n+1} + \ldots + a_1 x + a_0$ is a polynomial with $a_{2n+1} > 0$, then

$$p(x) = x^{2n+1}\Big[a_{2n+1} + \frac{a_{2n}}{x} + \ldots + \frac{a_0}{x^{2n+1}}\Big].$$

Since the bracketed expression is positive for $|x|$ sufficiently large, it follows from this equation that $p(x) > 0$ for $x > 0$ sufficiently large and $p(x) < 0$ for $x < 0$ sufficiently small. Hence p has a zero. ■

Theorem 6.33 *The continuous image of a compact interval is a compact interval.*

Proof: A continuous function $f : [a, b] \to \mathbb{R}$ on the compact interval $[a, b]$ takes on its minimum value $c = \min f([a, b])$ and its maximum value $d = \max f([a, b])$. Then obviously $f([a, b]) \subseteq [c, d]$. On the other hand, the intermediate value theorem yields $[c, d] \subseteq f([a, b])$, whence $f([a, b]) = [c, d]$. ■

Theorem 6.34 *Let $f : [a, b] \to \mathbb{R}$ be continuous and injective. Then f is strictly monotone in $[a, b]$.*

Proof: Since f is injective either $f(a) < f(b)$ or $f(a) > f(b)$ holds. Without restriction of generality we assume that $f(a) < f(b)$. The function f takes on the minimum value at the single point a. For, if f would take on the minimum at a point $x > a$, then because of the injectivity of f the minimum would satisfy $f(x) < f(a)$. By the intermediate value theorem we would conclude that $y \in (x, b)$ exists with $f(y) = f(a)$, which contradicts the injectivity of f.

In the same way it follows that f takes on the maximum at the single point b.

If f would not be strictly increasing, then x_1, x_2 would exist with $a < x_1 < x_2 < b$ and $f(x_1) > f(x_2)$. Because of

$$f(a) < f(x_2) < f(x_1),$$

we would conclude from the intermediate value theorem that $y \in (a, x_1)$ exists satisfying $f(y) = f(x_2)$. Again this contradicts the injectivity of f, hence f must be strictly increasing. ■

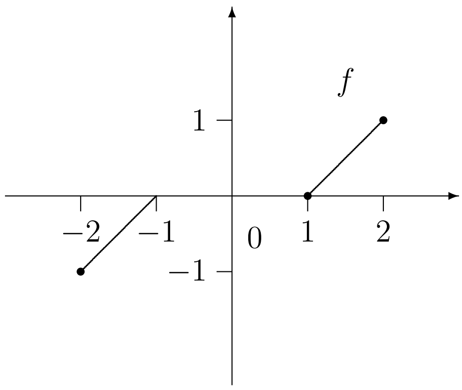
Theorem 6.35 *Let D be open and $f : D \to \mathbb{R}$ be continuous and injective. Then $f^{-1} : f(D) \to \mathbb{R}$ is continuous.*

Proof: Let $y \in f(D)$ and let U be a neighborhood of $x = f^{-1}(y)$. Since D is open, we can assume that $U \subseteq D$. Choose a compact interval $[a,b] \subseteq U$, which contains x as an interior point. Since f is continuous and injective, it follows that f is strictly monotone on $[a,b]$. Therefore the interval $V = f([a,b])$ contains $y = f(x)$ as an interior point, hence V is a neighborhood of y with $f^{-1}(V) = f^{-1}\big(f([a,b])\big) = [a,b] \subseteq U$. This means that f^{-1} is continuous at every $y \in f(D)$. ■

In this theorem the assumption that D be open cannot be dropped. This is shown by the following

Example. Let $D = [-2,-1) \cup [1,2]$ and

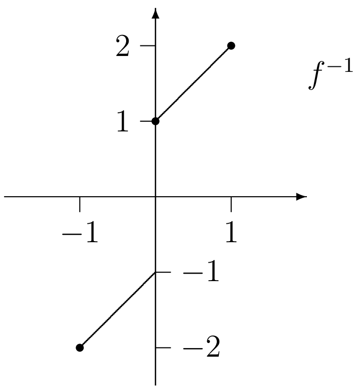
$$f(x) = \begin{cases} x+1\,, & \text{for } x \in [-2,-1) \\ x-1\,, & \text{for } x \in [1,2]\,. \end{cases}$$



$f : D \to [-1,1]$ is continuous and bijective. Yet, the inverse $f^{-1} : [-1,1] \to D$,

$$f^{-1}(y) = \begin{cases} y-1\,, & \text{for } y \in [-1,0) \\ y+1\,, & \text{for } x \in [0,1] \end{cases}$$

is not continuous.



6.4 Limits of functions, uniform continuity

Let $f : D \to \mathbb{R}$ be continuous and let a be an accumulation point, which does not belong to D. When can f be extended to a continuous function on $D \cup \{a\}$? To study this question, we consider the following examples of three functions $f_1, f_2, f_3 : D \to \mathbb{R}$ with $D = [-1, 0) \cup (0, 1]$ and $a = 0$:

1. $f_1(x) = \dfrac{1}{|x|}$
2. $f_2(x) = \dfrac{x}{|x|}$
3. $f_3(x) = \dfrac{x^2}{|x|}$.

All three functions are continuous, and f_3 has the continuous extension $\hat{f}_3 : [-1, 1] \to \mathbb{R}$,

$$\hat{f}_3(x) := |x| .$$

However, the functions f_1 and f_2 do not have continuous extensions to $D \cup \{a\} = [-1, 1]$. This is immediately clear for the unbounded function f_1, since $[-1, 1]$ is compact and therefore every continuous extension $\hat{f}_1 : [-1, 1] \to \mathbb{R}$ must be bounded. Thus, also $f_1 = \hat{f}_{1|D}$ had to be bounded.

To prove that f_2 does not have a continuous extension, suppose that $\hat{f}_2$ would be such an extension. Then for every sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in D$ and $\lim_{n\to\infty} x_n = a = 0$ the relation

$$\lim_{n\to\infty} f_2(x_n) = \lim_{n\to\infty} \hat{f}_2(x_n) = \hat{f}_2(0)$$

would hold. However, $\big\{(-1)^n \frac{1}{n}\big\}_{n=1}^{\infty}$ is a sequence with $\lim_{n\to\infty}(-1)^n \frac{1}{n} = 0$, but $\big\{f_2\big((-1)^n \frac{1}{n}\big)\big\}_{n=1}^{\infty} = \big\{(-1)^n\big\}_{n=1}^{\infty}$ does not have a limit.

Lemma 6.36 *Let a be an accumulation point of $D \subseteq \mathbb{R}$ and let $f : D \to \mathbb{R}$. If for every sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in D$ and $\lim_{n\to\infty} x_n = a$ the sequence $\{f(x_n)\}_{n=1}^{\infty}$ of images converges, then all the sequences of images converge to the same limit.*

Proof: Let $\{x_n'\}_{n=1}^{\infty}$ and $\{x_n''\}_{n=1}^{\infty}$ be sequences with $x_n', x_n'' \in D$ such that $\lim_{n\to\infty} x_n' = \lim_{n\to\infty} x_n'' = a$. Then also the sequence $\{x_n\}_{n=1}^{\infty} = \{x_1', x_1'', x_2', x_2'', x_3', x_3'', \ldots\}$ converges to a, hence $\{f(x_n)\}_{n=1}^{\infty}$ converges. This sequence contains the subsequences $\{f(x_n')\}_{n=1}^{\infty}$ and $\{f(x_n'')\}_{n=1}^{\infty}$, which both must therefore converge to the limit of $\{f(x_n)\}_{n=1}^{\infty}$. Thus, the limits of $\{f(x_n')\}_{n=1}^{\infty}$ and $\{f(x_n'')\}_{n=1}^{\infty}$ must coincide. ■

Definition 6.37 *Let a be an accumulation point of $D \subseteq \mathbb{R}$ and let $f : D \to \mathbb{R}$. If for every sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in D \setminus \{a\}$ and $\lim_{n\to\infty} x_n = a$ the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges, then it is said that the function f has a limit at a. The common limit b of all the sequences $\{f(x_n)\}_{n=1}^{\infty}$ is called the limit of f at a and one writes*

$$\lim_{x\to a} f(x) = b.$$

Note that f need not be defined at a. If f is defined at a and has a limit at a, we may have $f(a) \neq \lim_{x\to a} f(x)$.

From this definition and from Theorem 6.21 we immediately obtain

Corollary 6.38 *Let a be an accumulation point of D, which belongs to D. Then $f : D \to \mathbb{R}$ is continuous at a if and only if f has a limit at a, which satisfies*

$$\lim_{x\to a} f(x) = f(a).$$

Theorem 6.39 *Let a be an accumulation point of D, which does not belong to D, and let $f : D \to \mathbb{R}$. Then there exists an extension g of f to $D \cup \{a\}$, which is continuous at a, if and only if f has a limit at a. The value $g(a)$ is uniquely determined by*

$$g(a) = \lim_{x\to a} f(x).$$

Proof: Let g be an extension of f to the point a. The definition of function limits implies that g has a limit at a if and only if f has a limit at a, and the limits satisfy

$$\lim_{x\to a} g(x) = \lim_{x\to a} f(x).$$

It thus follows from this equation and from Corollary 6.38 that g is continuous at a if and only if f has a limit at a and the value of g at a satisfies $g(a) = \lim_{x\to a} f(x)$. ■

An equivalent condition for the existence of a limit is given in the following

Theorem 6.40 *Let a be an accumulation point of D, let $b \in \mathbb{R}$ and let $f : D \to \mathbb{R}$. The following statements are equivalent:*

(i) $\lim_{x\to a} f(x) = b$

(ii) *To every $\varepsilon > 0$ there is $\delta > 0$ such that for all $x \in D \setminus \{a\}$ with $|x - a| < \delta$*

$$|f(x) - b| < \varepsilon.$$

With quantifiers, the second statement can be reformulated as

$$\underset{\varepsilon>0}{\forall} \quad \underset{\delta>0}{\exists} \quad \underset{\substack{x\in D\setminus\{a\} \\ |x-a|<\delta}}{\forall} : \ |f(x)-b|<\varepsilon\,.$$

With neighborhoods, this statement reads as follows: To every neighborhood U of b there is a neighborhood V of a such that $f\big((V\cap D)\setminus\{a\}\big)\subseteq U$.

Proof of the theorem: Define a function $g : D\cup\{a\}\to\mathbb{R}$ by

$$g(x)=\begin{cases} f(x), & x\in D\setminus\{a\}\,,\\ b, & x=a. \end{cases}$$

This function satisfies $\lim_{x\to a} g(x) = \lim_{x\to a} f(x)$. Hence statement (i) is equivalent to the equation $\lim_{x\to a} g(x) = g(a)$, and by Corollary 6.38 this equation is equivalent to continuity of g at a.

Statement (ii) holds if and only if to $\varepsilon>0$ there is $\delta>0$ such that for all $x\in D\setminus\{a\}$ with $|x-a|<\delta$

$$|g(x)-g(a)|=|f(x)-b|<\varepsilon\,,$$

and this statement is equivalent to continuity of g at a. Whence, (i) and (ii) are equivalent statements. ∎

We can formulate the Cauchy criterion for the existence of a limit of a function:

Theorem 6.41 *Let a be an accumulation point of D and let $f : D\to\mathbb{R}$. The function f has a limit at a if and only if to $\varepsilon>0$ there is $\delta>0$ such that*

$$|f(x)-f(y)|<\varepsilon$$

for all $x,y\in D\setminus\{a\}$ with $|x-a|<\delta$ and $|y-a|<\delta$.

With quantifiers the Cauchy criterion for the existence of a limit can be reformulated as:

$$\underset{\varepsilon>0}{\forall} \quad \underset{\delta>0}{\exists} \quad \underset{\substack{x,y\in D\setminus\{a\} \\ |x-a|<\delta \\ |y-a|<\delta}}{\forall} : \ |f(x)-f(y)|<\varepsilon\,.$$

Proof of the theorem: If $\lim_{x\to a} f(x) = b$ holds, then to $\varepsilon>0$ there is $\delta>0$ such that for all $x\in D\setminus\{a\}$ with $|x-a|<\delta$

$$|f(x)-b|<\frac{\varepsilon}{2}\,.$$

Thus, for all $x, y \in D \setminus \{a\}$ with $|x-a| < \delta$ and $|y-a| < \delta$

$$|f(x) - f(y)| = |f(x) - b + b - f(y)| \leq |f(x) - b| + |f(y) - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,,$$

hence the Cauchy criterion for the function limit is satisfied.

To prove the converse, assume that the Cauchy criterion is satisfied. We must show that for any sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in D \setminus \{a\}$ and with $\lim_{n\to\infty} x_n = a$ the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges. Thus, let $\{x_n\}_{n=1}^{\infty}$ be such a sequence. We show that $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence. To this end let $\varepsilon > 0$. By assumption we can choose $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in D \setminus \{a\}$ with $|x-a| < \delta$, $|y-a| < \delta$. Since $\lim_{n\to\infty} x_n = a$, there is n_0 such that $|x_n - a| < \delta$ for all $n \geq n_0$. Hence, for $n, m \geq n_0$

$$|f(x_n) - f(x_m)| < \varepsilon\,,$$

and so $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence. ∎

Let $f : D \to \mathbb{R}$ be continuous. By Theorem 6.38 the function f has a unique continuous extension g to the closed hull $\overline{D}$, if f has a limit at every accumulation point of D. A condition sufficient for f to have this property is uniform continuity:

Definition 6.42 *The function $f : D \to \mathbb{R}$ is said to be uniformly continuous, if to every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$|f(x) - f(y)| < \varepsilon$$

for all $x, y \in D$ with $|x - y| < \delta$.

With quantifiers this condition reads

$$\underset{\varepsilon>0}{\forall} \quad \underset{\delta>0}{\exists} \quad \underset{x\in D}{\forall} \quad \underset{\substack{y\in D\\|x-y|<\delta}}{\forall} \ : \ |f(x) - f(y)| < \varepsilon\,.$$

Uniform continuity is a condition stronger than continuity, hence every uniformly continuous function is continuous: For a uniformly continuous function to given $\varepsilon > 0$ the number δ can be chosen independently of $x \in D$, whereas for a continuous function δ may depend on x.

Theorem 6.43 *Let D be a subset of $\mathbb{R}$ and let $f : D \to \mathbb{R}$ be uniformly continuous. Then f has a unique continuous extension to $\overline{D}$.*

Proof: It suffices to verify that f has a limit at every point $a \in \overline{D}$. The uniform continuity of f implies that to $\varepsilon > 0$ there is $\delta > 0$ such that for $x, y \in D$ with $|x - y| < \delta$ the inequality $|f(x) - f(y)| < \varepsilon$ is satisfied. Thus, since for $x, y \in D$ with $|x - a| < \delta/2$ and $|y - a| < \delta/2$ the inequality

$$|x - y| = |x - a + a - y| \le |x - a| + |y - a| < \delta$$

holds, we obtain $|f(x) - f(y)| < \varepsilon$. Therefore, by the Cauchy criterion f has a limit at a.

■

Example: 1.) Let $f : D = [-1, 0) \cup (0, 1] \to \mathbb{R}$ be the function $x \mapsto f(x) = \frac{x^2}{|x|} = |x|$. This function is uniformly continuous. For, to $\varepsilon > 0$ choose $\delta = \varepsilon$. Then the inverse triangle inequality implies for all $x, y \in D$ with $|x - y| < \delta$ that

$$|f(x) - f(y)| = \big||x| - |y|\big| \le |x - y| < \varepsilon.$$

Therefore this function has a unique continuous extension to $\overline{D} = [-1, 1]$. As already mentioned, this extension is $x \mapsto g(x) = |x|$.

2.) The function $x \mapsto f(x) = \frac{1}{x}$ is not uniformly continuous on $D = (0, 1]$. To prove this we show that for this function the negation of the statement in the definition of uniform continuity is true:

$$\underset{\varepsilon>0}{\exists} \quad \underset{\delta>0}{\forall} \quad \underset{x\in D}{\exists} \quad \underset{\substack{y\in D \\ |x-y|<\delta}}{\exists} \; : \; |f(x) - f(y)| \ge \varepsilon. \tag{6.2}$$

To this end consider $\varepsilon = 1$ and let $\delta > 0$. For $x > 0$ and $y = \frac{1}{2}x$ we have $|x - y| = \frac{1}{2}x$ and

$$|f(x) - f(y)| = \Big|\frac{1}{x} - \frac{1}{y}\Big| = \Big|\frac{1}{x} - \frac{2}{x}\Big| = \frac{1}{x}.$$

Thus, choosing $x = \min(\delta, 1)$ and $y = \frac{1}{2}x = \frac{1}{2}\min(\delta, 1)$ yields $|x - y| = \frac{1}{2}\min(\delta, 1) < \delta$ and

$$|f(x) - f(y)| = \frac{1}{x} = \max\big(\frac{1}{\delta}, 1\big) \ge 1 = \varepsilon.$$

This shows that (6.2) is satisfied.

Theorem 6.44 *Let $D \subseteq \mathbb{R}$ be compact and let $f : D \to \mathbb{R}$ be continuous. Then f is uniformly continuous.*

Proof: Suppose that f is not uniformly continuous. Then there is $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ numbers $x_n, y_n \in D$ can be found with $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \ge \varepsilon$. Since D is compact, $\{x_n\}_{n=1}^{\infty}$ possesses a convergent subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ with limit $x_0 \in D$.

Also the sequence $\{y_{n_j}\}_{j=1}^\infty$ converges to x_0. To see this, let $\eta > 0$ and let j_0 be the smallest natural number with $\frac{1}{j_0} < \frac{\eta}{2}$. There is j_1 such that

$$|x_{n_j} - x_0| < \eta/2$$

for all $j \geq j_1$, whence, for $j \geq \max(j_0, j_1)$

$$|x_0 - y_{n_j}| \leq |x_0 - x_{n_j}| + |x_{n_j} - y_{n_j}| < \frac{\eta}{2} + \frac{1}{n_j} \leq \frac{\eta}{2} + \frac{1}{j} < \eta.$$

This proves that $\lim_{j\to\infty} y_{n_j} = x_0$.

Thus, since f is continuous,

$$\begin{aligned} 0 = |f(x_0) - f(x_0)| &= \big| \lim_{j\to\infty} f(x_{n_j}) - \lim_{j\to\infty} f(y_{n_j}) \big| \\ &= \big| \lim_{j\to\infty} \big(f(x_{n_j}) - f(y_{n_j}) \big) \big| = \lim_{j\to\infty} |f(x_{n_j}) - f(y_{n_j})| \geq \lim_{j\to\infty} \varepsilon = \varepsilon > 0, \end{aligned}$$

which is an obvious contradiction. Therefore the assumption must be false, and f must be uniformly continuous. ∎

Example: 1.) Since the square root function $Q : [0,1] \to \mathbb{R}$, $x \mapsto Q(x) := \sqrt{x}$ is continuous and since $[0,1]$ is compact, Q is uniformly continuous.

2.) Also $Q : [0,\infty) \to \mathbb{R}$, $Q(x) = \sqrt{x}$, is uniformly continuous. The proof is left as an exercise.

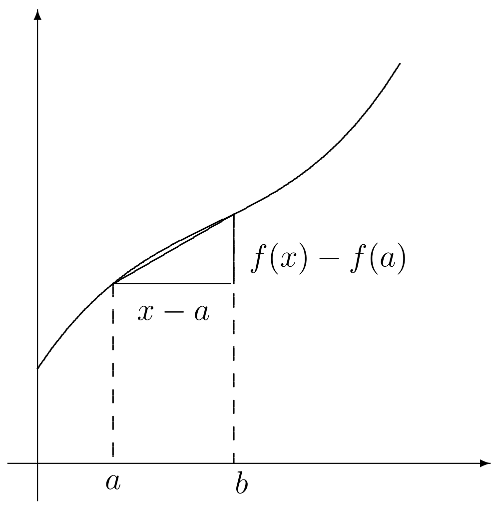
7 Differentiable functions

7.1 Definitions and calculus

Let J be an interval, let $a \in J$ and $f : J \to \mathbb{R}$. The function

$$x \mapsto \frac{f(x) - f(a)}{x - a}$$

is defined in all points of J with the exception of a. This function is called the *difference quotient*. The value of this function at x is the slope of the secant of the graph of f through the points $\big(a, f(a)\big)$ and $\big(x, f(x)\big)$.



If x converges to a, then the secant approaches the tangent to the graph of f at $\big(a, f(a)\big)$, if this tangent exists, and the difference quotient converges to the slope of the tangent. One can interpret the tangent as graph of that affine function, which best approximates the function f in a neighborhood of the point a. Therefore differentiation, which is a method to find the tangent to a function, can be put in a more general frame and be considered as a method to approximate complicated functions by simpler functions.

Definition 7.1 *Let D be a subset of $\mathbb{R}$ and let $a \in D$ be an accumulation point of D. A function $f : D \to \mathbb{R}$ is said to be differentiable at a if the limit*

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists.

Theorem 7.2 *$f : D \to \mathbb{R}$ is differentiable at $a \in D$, if and only if there is a number m and a function $r : D \to \mathbb{R}$, which is continuous at a and satisfies $r(a) = 0$, such that*

$$f(x) = f(a) + m(x-a) + r(x)(x-a)$$

holds for all $x \in D$. The number m is uniquely determined.

Proof: Suppose that the limit

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. Define the function r by

$$r(x) = \begin{cases} \frac{f(x)-f(a)}{x-a} - m\,, & x \neq a \\ 0\,, & x = a\,. \end{cases}$$

Then the equation

$$f(x) = f(a) + m(x-a) + r(x)(x-a)$$

holds for all $x \in D$, and because of

$$\lim_{x \to a} r(x) = \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} - m \right] = 0 = r(a)\,,$$

the function r is continuous at a.

Conversely, suppose that m and r as in the theorem exist. Then the continuity of r at a implies that the limit of the difference quotient at a exists and satisfies

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = m + \lim_{r \to a} r(x) = m\,.$$

This equation also shows that m must be equal to the unique limit on the left hand side. Hence m is uniquely determined. ■

Definition 7.3 *Let D be a subset of $\mathbb{R}$, let $a \in D$ be an accumulation point of D, and let the function $f : D \to \mathbb{R}$ be differentiable at a. Then the unique number*

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

is called derivative of f at the point a. One writes $f'(a) := m$.

Theorem 7.4 *Let $f : D \to \mathbb{R}$ be differentiable at $a \in D$. Then f is continuous at a.*

Proof: f can be represented in the form

$$f(x) = f(a) + f'(a)(x-a) + r(x)(x-a)\,,$$

with a function r continuous at a. Thus, the right hand side is a function continuous at a, and so is f. ■

Before we turn to examples, we prove some rules for computation with differentiable functions:

Lemma 7.5 *Let a be an accumulation point of $D \subseteq \mathbb{R}$ and let $f : D \to \mathbb{R}$ be differentiable at a with derivative $f'(a) > 0$. Then there is a neighborhood U of a such that $f(x) < f(a)$ for all $x \in U \cap D$ with $x < a$ and $f(x) > f(a)$ for all $x \in U \cap D$ with $x > a$.*

Proof: f can be represented in the form

$$f(x) = f(a) + \Big(f'(a) + r(x)\Big)(x-a),$$

where the function $x \mapsto f'(a) + r(x)$ is continuous and positive at $x = a$. Thus, there is a neighborhood U of a such that this function is positive in $U \cap D$, which implies that $f(x) - (a) = (f'(a) + r(x))(x-a)$ is positive for all $x \in U \cap D$ with $x > a$ and negative for $x < a$. ■

Note however, that from $f'(a) \neq 0$ one cannot conclude that f is one-to-one in a neighborhood of a.

Theorem 7.6 *Let $a \in D \subseteq \mathbb{R}$ and let $f, g : D \to \mathbb{R}$ be differentiable at a. Then also $f+g$ $f-g$, cf and $f \cdot g$ are differentiable at a, and one has*

$$\begin{aligned} (f \pm g)'(a) &= f'(a) \pm f'(b) \\ (fg)'(a) &= f'(a)g(a) + f(a)g'(a) \quad \textit{(product rule)} \\ (cf)'(a) &= cf'(a)\,. \end{aligned}$$

If $g(a) \neq 0$, then $\frac{f}{g}$ is differentiable in a, and one has

$$\Big(\frac{f}{g}\Big)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{\big(g(a)\big)^2}$$

(quotient rule).

The quotient rule implies

$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{\big(g(a)\big)^2}\,.$$

Proof:

$$\lim_{x\to a}\frac{\big(f(x)\pm g(x)\big)-\big(f(a)\pm g(a)\big)}{x-a} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \pm \lim_{x\to a}\frac{g(x)-g(a)}{x-a} = f'(a)\pm g'(a)\,.$$

The statement for cf is proved similarly. For the product fg we have

$$\begin{aligned}
&\lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a}\\
&\quad=\lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}g(x)+\frac{g(x)-g(a)}{x-a}f(a)\right]\\
&\quad=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\lim_{x\to a}g(x)+\lim_{x\to a}\frac{g(x)-g(a)}{x-a}f(a)\\
&\quad=f'(a)g(a)+g'(a)f(a)\,,
\end{aligned}$$

since g is continuous at a.

To prove the quotient rule note first that if $g(a)\neq 0$, then Lemma 6.29 implies that $g(x)\neq 0$ for all x from a neighborhood of a. Here we again used the continuity of g at a. Therefore the function $\frac{1}{g}$ is defined for all $x\in D$ belonging to a neighborhood of a. Thus

$$\begin{aligned}
\lim_{x\to a}\frac{\frac{1}{g(x)}-\frac{1}{g(a)}}{x-a} &= \lim_{x\to a}\left[-\frac{1}{g(x)g(a)}\frac{g(x)-g(a)}{x-a}\right]\\
&= -\lim_{x\to a}\frac{1}{g(x)g(a)}\lim_{x\to a}\frac{g(x)-g(a)}{x-a} = -\frac{g'(a)}{\big(g(a)\big)^2}\,.
\end{aligned}$$

The formula for $\left(\frac{f}{g}\right)'$ follows from this equation and from the product rule. ■

Corollary 7.7 *The set of functions $f: D\to\mathbb{R}$, which are differentiable at $a\in D$, is a vector space.*

Theorem 7.8 *Let $D_1, D_2\subseteq\mathbb{R}$, let $f: D_1\to\mathbb{R}$, $g: D_2\to\mathbb{R}$, and assume that $g\circ f$ is defined. Moreover, assume that f is differentiable in $a\in D_1$ and g is differentiable in $b=f(a)\in D_2$. Then $g\circ f$ is differentiable, and*

$$\big(g\circ f\big)'(a) = g'\big(f(a)\big)\cdot f'(a)$$

(chain rule).

Proof: Since g is differentiable, there exists a function $r : D_2 \to \mathbb{R}$, which is continuous at b and satisfies $r(b) = 0$, such that for $x \in D_2$

$$g(y) - g(b) = g'(b)(y - b) + r(y)(y - b)\,.$$

Hence,

$$\begin{aligned}
&\lim_{x\to a} \frac{g\big(f(x)\big) - g\big(f(a)\big)}{x - a} \\
&\quad = \lim_{x\to a} \Big[g'\big(f(a)\big) \frac{f(x) - f(a)}{x - a} + r\big(f(x)\big) \frac{f(x) - f(a)}{x - a} \Big] \\
&\quad = \Big[g'\big(f(a)\big) + \lim_{x\to a} r\big(f(x)\big) \Big] \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = g'\big(f(a)\big) f'(a)\,.
\end{aligned}$$

Here we used that $r \circ f$ is continuous at a, which implies $\lim_{x\to a} r\big(f(x)\big) = r\big(f(a)\big) = r(b) = 0$. ■

Theorem 7.9 *Let $f : D \to \mathbb{R}$ be one-to-one and differentiable at $a \in D$ with $f'(a) \neq 0$. If the inverse function g of f is continuous at $b = f(a)$, then g is even differentiable at b with derivative*

$$g'(b) = \frac{1}{f'(a)} = \frac{1}{f'\big(g(b)\big)}\,.$$

Proof: Let $h : D \to \mathbb{R}$ be defined by

$$h(x) = \begin{cases} \frac{f(x) - f(a)}{x - a}\,, & x \neq a \\ f'(a)\,, & x = a\,. \end{cases}$$

Since f is differentiable at a, we obtain

$$\lim_{x\to a} h(x) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a) = h(a)\,,$$

hence h is continuous at a. By assumption, g is continuous at b, and so from $a = g(b)$ we infer that $h \circ g$ is continuous at b. Thus

$$\lim_{y\to b} (h \circ g)(y) = (h \circ g)(b) = h\big(g(b)\big) = h(a) = f'(a)\,.$$

Since by assumption $(h \circ g)(b) = f'(a) \neq 0$, we obtain from this equation $\lim_{y\to b} \frac{1}{(h\circ g)(y)} = \frac{1}{f'(a)}$. Using this relation, it follows

$$\begin{aligned}
\lim_{y\to b} \frac{g(y) - g(b)}{y - b} &= \lim_{y\to b} \frac{g(y) - a}{f\,(g(y)) - f(a)} \\
&= \lim_{y\to b} \frac{1}{\frac{f(g(y)) - f(a)}{g(y) - a}} = \lim_{y\to b} \frac{1}{(h \circ g)(y)} = \frac{1}{f'(a)}\,.
\end{aligned}$$

■

Definition 7.10 *A function is called differentiable, if it is differentiable at every point of the domain of definition. To a differentiable function $f : D \to \mathbb{R}$ a new function f' can be defined by*

$$x \mapsto f'(x) : D \to \mathbb{R}.$$

f' is called the derivative of f. If f' is continuous, then f is said to be continuously differentiable.

Often the notation

$$\frac{d}{dx} f(x) := f'(x)$$

is used. This notation dates back the Leibniz. It is particularly useful if the function f depends on several variables, since it indicates the variable with respect to which the derivative is taken.

The preceding theorems immediately yield the

Theorem 7.11 *Let D be a subset of $\mathbb{R}$. The set of all differentiable functions $f : D \to \mathbb{R}$ is a vector space, and the set of all continuously differentiable functions $f : D \to \mathbb{R}$ is a vector space.*

The set of all continuously differentiable functions is denoted by $C^1(D, \mathbb{R})$.

Also the product of two differentiable functions (continuously differentiable functions, respectively) is differentiable (continuously differentiable). Therefore $C^1(D, \mathbb{R})$ is in fact an algebra.

If f' is differentiable, then the derivative of f' is denoted by f'' or by $f^{(2)}$, and called the second derivative of f. The kth derivative $f^{(k)}$ of f is defined recursively by

$$f^{(k+1)} = (f^{(k)})'.$$

If f is k–times differentiable, then the derivatives $f^{(0)} = f, f^{(1)}, \ldots, f^{(k-1)}$ are continuous. If also $f^{(k)}$ is continuous, one says that f is k–times continuously differentiable. The set of all k–times continuously differentiable functions $f : D \to \mathbb{R}$ is a vector space over $\mathbb{R}$. This set is denoted by $C^k(D, \mathbb{R})$. For $k = 0$ one defines $C^0(D, \mathbb{R}) = C(D, \mathbb{R})$, where $C(D, \mathbb{R})$ denotes the vector space of continuous functions. If $f^{(k)}$ exists for all k, then f is said to be infinitely often differentiable. The set of infinitely differentiable functions $f : D \to \mathbb{R}$ is denoted by $C^\infty(D, \mathbb{R})$ Also this set is a vector space. One uses the notation

$$\frac{d^k}{dx^k} f(x) = f^{(k)}(x).$$

7.2 Examples and applications

7.2.1 Derivatives of polynomials and rational functions

The constant function $f : \mathbb{R} \to \mathbb{R},\ x \mapsto f(x) := c$ is differentiable at every $a \in \mathbb{R}$ with derivative $f'(a) = 0$.

The identity $x \mapsto \mathrm{id}\,(x) := x$ is differentialbe at every $a \in \mathbb{R}$ with derivative

$$\mathrm{id}'(x) = \lim_{x\to a} \frac{\mathrm{id}\,(x) - \mathrm{id}\,(a)}{x-a} = \lim_{x\to a} \frac{x-a}{x-a} = 1.$$

This implies that all polynomials $p(x) = \sum_{n=0}^{m} a_n x^n$ are differentiable and that all rational functions are differentiable. To show this, we first prove by induction that

$$(x^n)' = \begin{cases} 0, & n = 0 \\ nx^{n-1}, & n \geq 1. \end{cases}$$

For $n = 0$ this relation is true since $x^0 = 1$ is constant. Suppose that n is a number from $\mathbb{N}_0$, for which $(x^n)' = nx^{n-1}$ is valid. Then the product rule yields

$$(x^{n+1})' = (xx^n)' = x'x^n + x(x^n)' = x^n + x(nx^{n-1}) = (n+1)x^n.$$

Consequently, the induction principle assures that $(x^n)' = nx^{n-1}$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}_0$.

With this result we obtain that the polynomial p is differentiable with derivative

$$p'(x) = \Big(\sum_{n=0}^{m} a_n x^n\Big)' = \sum_{n=0}^{m} a_n (x^n)' = \sum_{n=1}^{m} n a_n x^{n-1}.$$

Thus, the derivative of a polynomial is a polynomial with degree reduced by one. Together with the quotient rule this implies that every rational function $r(x) = \frac{p(x)}{q(x)}$ is differentiable and the derivative is again a rational function:

$$r'(x) = \frac{p'(x)q(x) - p(x)q'(x)}{q(x)^2}.$$

In particular, for every $n \in \mathbb{N}$

$$(x^{-n})' = \Big(\frac{1}{x^n}\Big)' = -\frac{nx^{n-1}}{x^{2n}} = -nx^{-n-1}.$$

7.2.2 Derivative of the exponential function

In Section 5 we defined the exponential function $\exp : \mathbb{R} \to \mathbb{R}$ by

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} =: e^x.$$

We have

$$\begin{aligned} \exp'(0) &= \lim_{x\to 0} \frac{e^x - e^0}{x - 0} = \lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \Big(1 + \sum_{k=2}^{\infty} \frac{x^{k-1}}{k!}\Big) \\ &= 1 + \lim_{x\to 0} \sum_{k=2}^{\infty} \frac{x^{k-1}}{k!} = 1, \end{aligned}$$

since

$$\begin{aligned} \Big|\lim_{x\to 0} \sum_{k=2}^{\infty} \frac{x^{k-1}}{k!}\Big| &= \lim_{x\to 0} \Big|x \sum_{k=0}^{\infty} \frac{x^k}{(k+2)!}\Big| \le \lim_{x\to 0} |x| \sum_{k=0}^{\infty} \frac{|x|^k}{k!} \\ &\le \lim_{x\to 0} |x| \sum_{k=0}^{\infty} \frac{1}{k!} = 0. \end{aligned}$$

The addition theorem $e^{x+h} = e^x e^h$ thus yields for $x \in \mathbb{R}$ that

$$\exp'(x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \Big(e^x \frac{e^h - 1}{h}\Big) = e^x,$$

hence the exponential function is differentiable with derivative

$$\exp'(x) = \exp(x).$$

This can also be written as

$$(e^x)' = e^x.$$

As a differentiable function, exp is continuous.

7.2.3 The natural logarithm

The relation $e^x e^{-x} = 1$ implies $e^x \neq 0$ for all $x \in \mathbb{R}$. From the intermediate value theorem we therefore conclude that either $e^x > 0$ for all x or $e^x < 0$ for all x. Since $e^0 = 1$, the first alternative must be true, hence $e^x > 0$ for all $x \in \mathbb{R}$.

If $x > 0$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} > 1 + x > 1.$$

Therefore $x_2 > x_1$ yields

$$e^{x_2} = e^{x_2 - x_1 + x_1} = e^{x_2 - x_1} e^{x_1} > e^{x_1},$$

which proves that e^x is strictly increasing on all of $\mathbb{R}$. Consequently, e^x has an inverse, which is denoted by $\log$.

The range of exp is $\mathbb{R}^+ = (0, \infty)$. To see this, observe that $e^x > x+1$ for $x > 0$ implies that e^x takes on arbitrarily large values for x sufficiently large. Thus, using $e^0 = 1$, we obtain from the intermediate value theorem that $[1, \infty) \subseteq \exp(\mathbb{R})$. Because of $e^{-x} = \frac{1}{e^x}$, we conclude from this that also $(0, 1) \subseteq \exp(\mathbb{R})$, whence $\exp(\mathbb{R}) = \mathbb{R}^+$. Consequently

$$\exp : \mathbb{R} \to \mathbb{R}^+, \quad \log : \mathbb{R}^+ \to \mathbb{R}$$

are bijective mappings.

Since $\exp : \mathbb{R} \to \mathbb{R}^+$ is continuous with the domain $\mathbb{R}$ being an open set, we conclude from Theorem 6.34 that the inverse function log is continuous. Since $\exp'(x) = \exp(x) \neq 0$ for all x, we conclude from Theorem 7.9 that log even is differentiable and has the derivative

$$\log' x = \frac{1}{\exp'(\log x)} = \frac{1}{\exp(\log x)} = \frac{1}{x},$$

for $x > 0$.

From the chain rule we obtain that the function $(x \mapsto \log(-x)) : \mathbb{R}^- \to \mathbb{R}$ has the derivative

$$\big(\log(-x)\big)' = \big(\log'(-x)\big)(-x)' = \frac{-1}{-x} = \frac{1}{x}.$$

7.2.4 Definition of the general power

The general power can be defined using exp and log : In Section 5 we showed that $\exp(ny) = \exp(y)^n$ holds for every $y \in \mathbb{R}$ and all $n \in \mathbb{N}_0$. From this we conclude for $m \in \mathbb{N}$ and $y \in \mathbb{R}$ that

$$\exp\Big(\frac{y}{m}\Big)^m = \exp\Big(m\frac{y}{m}\Big) = \exp(y).$$

By definition, $z \mapsto z^{\frac{1}{m}} : [0, \infty) \to [0, \infty)$ is the inverse function of $x \mapsto x^m : [0, \infty) \to [0, \infty)$. We apply this inverse function to the above equation and obtain

$$\exp\Big(\frac{y}{m}\Big) = \Big[\exp\Big(\frac{y}{m}\Big)^m\Big]^{\frac{1}{m}} = \exp(y)^{\frac{1}{m}},$$

hence, for $y \in \mathbb{R}$ and $q = \frac{n}{m}$ with $n \in \mathbb{N}_0$, $m \in \mathbb{N}$

$$\exp(qy) = \exp\Big(\frac{n}{m}\, y\Big) = \exp(ny)^{\frac{1}{m}} = \Big[\exp(y)^n\Big]^{\frac{1}{m}} = \exp(y)^q.$$

From this we infer for negative $q \in \mathbb{Q}$ that

$$\exp(qy) = \exp\Big((-q)(-y)\Big) = \exp(-y)^{-q} = \Big(\frac{1}{\exp(y)}\Big)^{-q} = \exp(y)^q,$$

hence the equation

$$(e^y)^q = \exp(qy)$$

holds for all $y \in \mathbb{R}$ and all $q \in \mathbb{Q}$.

We now extend the validity of this equation from the set $\mathbb{Q}$ to the set $\mathbb{R}$ by *defining* the expression $(e^y)^x$ for all $x, y \in \mathbb{R}$ by

$$(e^y)^x := e^{xy} = \exp(xy).$$

Let $a > 0$. Then $a = \exp(\log a) = e^{\log a}$, hence this definition implies for all $x \in \mathbb{R}$

$$a^x = \big(e^{\log a}\big)^x = e^{x \log a} = \exp(x \log a).$$

This equation defines a^x for all $a > 0$ and all $x \in \mathbb{R}$. From this definition we obtain for $\log x$ and a^x

1.) $\log(xy) = \log(e^{\log x} e^{\log y}) = \log e^{\log x + \log y} = \log x + \log y$

2.) $\log a^x = \log(e^{x \log a}) = x \log a$

3.) $a^x a^y = e^{x \log a} e^{y \log a} = e^{(x+y) \log a} = a^{x+y}$

4.) $(a^x)^y = (e^{x \log a})^y = e^{xy \log a} = a^{xy}$

5.) $a^x b^x = e^{x \log a} e^{x \log b} = e^{x(\log a + \log b)} = e^{x \log(ab)} = (ab)^x$.

Since exp is differentiable, the chain rule implies that $x \mapsto a^x$ is differentiable and satisfies

$$\frac{d}{dx}\,(a^x) = \big(e^{x \log a}\big)' = \exp'(x \log a)(\log a) = (\log a) e^{x \log a} = (\log a) a^x.$$

For $c \in \mathbb{Z}$ we showed that $a \mapsto a^c$ is differentiable and $(a^c)' = c a^{c-1}$. The function $a \mapsto a^c : \mathbb{R}^+ \to \mathbb{R}^+$ is differentiable also for arbitrary $c \in \mathbb{R}$ and

$$\begin{aligned} \frac{d}{da}\Big(a^c\Big) &= \frac{d}{da}\, e^{c \log a} = \frac{d}{da}\, \exp(c \log a) \\ &= \exp'(c \log a) \frac{d}{da}\,(c \log a) = a^c \frac{c}{a} = c a^{c-1}. \end{aligned}$$

7.2.5 Derivative of the square root

In particular, this implies that the square root

$$x \mapsto Q(x) = \sqrt{x} : \mathbb{R}_0^+ \to \mathbb{R}_0^+$$

is differentiable for all $x > 0$ with derivative

$$Q'(x) = (x^{1/2})' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}\,.$$

However, Q is not differentiable at $x = 0$, since for $x > 0$

$$\frac{Q(x) - Q(0)}{x - 0} = \frac{1}{\sqrt{x}}$$

is unbounded in every neighborhood of 0, hence the difference quotient does not have a limit at 0.

7.2.6 Eulerian limit formula

We have

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = \frac{d}{dx}\log(x)\big|_{x=1} = \big(\frac{1}{x}\big)\big|_{x=1} = 1\,.$$

For $a \in \mathbb{R}$, $a \neq 0$ and $n \in \mathbb{N}$ this yields

$$\lim_{n\to\infty} \log\Big[\big(1+\frac{a}{n}\big)^n\Big] = \lim_{n\to\infty} n\log\big(1+\frac{a}{n}\big) = \lim_{n\to\infty} a\frac{\log(1+\frac{a}{n})}{\frac{a}{n}} = a\,.$$

Using the continuity of the exponential function we thus obtain the *Eulerian limit formula*

$$\begin{aligned} \lim_{n\to\infty} \big(1+\frac{a}{n}\big)^n &= \lim_{n\to\infty} \exp\Big(\log\big[\big(1+\frac{a}{n}\big)^n\big]\Big) \\ &= \exp\big(\lim_{n\to\infty} \log\big[\big(1+\frac{a}{n}\big)^n\big]\big) = \exp(a) = e^a\,. \end{aligned}$$

7.2.7 One-sided derivatives

The function $x \mapsto |x| : \mathbb{R} \to [0,\infty)$ is everywhere differentiable with the exception of zero. We have

$$\frac{d}{dx}|x| = \begin{cases} -1\,, & x < 0 \\ \ \ 1\,, & x > 0\,. \end{cases}$$

However, at zero the left sided and right sided limits

$$\lim_{\substack{x\to 0\\ x<0}} \frac{|x| - |0|}{x - 0} = -1\,, \quad \lim_{\substack{x\to 0\\ x>0}} \frac{|x| - |0|}{x - 0} = 1$$

exist. One says that $|x|$ has left sided and right sided derivatives at 0.

Remark: Every differentiable function is continuous, but there are continuous functions, which are nowhere differentiable. An example can be found in the book of Barner & Flohr, Analysis I, pp. 261 (in German).

7.2.8 Trigonometric functions

The series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ and $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ are absolutely convergent for every $x \in \mathbb{R}$, since

$$\sum_{n=0}^{\infty}\Big|\frac{x^{2n+1}}{(2n+1)!}\Big| + \sum_{n=0}^{\infty}\Big|\frac{x^{2n}}{(2n)!}\Big| = \sum_{n=0}^{\infty}\frac{|x|^n}{n!} = e^{|x|}\,,$$

which shows that the exponential series is a majorant. Therefore the **sine** and **cosine** functions

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\,, \quad \cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$

are defined on the whole real line.

Theorem 7.12 (Addition theorems) *For all* $x, y \in \mathbb{R}$

$$\begin{aligned} \sin(x+y) &= \sin x\, \cos y + \sin y\, \cos x \\ \cos(x+y) &= \cos x\, \cos y - \sin x\, \sin y\,. \end{aligned}$$

Proof: The Binomial Theorem and Corollary 5.16 for the Cauchy product of two series yield

$$\begin{aligned} \sin(x+y) &= \sum_{n=0}^{\infty}(-1)^n \frac{(x+y)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \sum_{\ell=0}^{2n+1} \frac{x^{2n+1-\ell}}{(2n+1-\ell)!}\,\frac{y^{\ell}}{\ell!} \\ &= \sum_{n=0}^{\infty}\Big[\sum_{\ell=0}^{n}(-1)^{n-\ell}\frac{x^{2n+1-2\ell}}{(2n+1-2\ell)!}(-1)^{\ell}\frac{y^{2\ell}}{(2\ell)!} + \sum_{\ell=0}^{n}(-1)^{n-\ell}\frac{x^{2n-2\ell}}{(2n-2\ell)!}(-1)^{\ell}\frac{y^{2\ell+1}}{(2\ell+1)!}\Big] \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}\cdot\sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{y^{2\ell}}{(2\ell)!} + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}\sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{y^{2\ell+1}}{(2\ell+1)!} \\ &= \sin x\, \cos y + \cos x\, \sin y\,. \end{aligned}$$

The addition theorem for cosine is proved in the same way. ■

With these addition theorems we can show that sine and cosine are differentiable functions. Note first that sine is differentiable at $x = 0$ with derivative $\sin'(0) = 1$, since

$$\sin'(0) = \lim_{x\to 0}\frac{\sin x - \sin 0}{x - 0} = \lim_{x\to 0}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n+1)!} = 1 + \lim_{x\to 0}\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n}}{(2n+1)!} = 1\,,$$

where we used that

$$\Big|\lim_{x\to 0}\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n}}{(2n+1)!}\Big| \;\le\; \lim_{x\to 0}\Big(|x|^2\sum_{n=1}^{\infty}\frac{|x|^{2n-2}}{(2n+1)!}\Big)$$

$$\le\; \lim_{x\to 0}\Big(|x|^2\sum_{n=0}^{\infty}\frac{|x|^{n}}{n!}\Big) = \lim_{x\to 0}|x|^2\, e^{|x|} = 0\,.$$

(This result suggests that interchanging the limits in

$$\lim_{x\to 0}\lim_{m\to\infty}\sum_{n=0}^{m}(-1)^n\frac{x^{2n}}{(2n+1)!} = \lim_{m\to\infty}\lim_{x\to 0}\sum_{n=0}^{m}(-1)^n\frac{x^{2n}}{(2n+1)!} = 1$$

is allowed. However, we have not yet studied this problem.)

In the same way it follows that cosine is differentiable at $x=0$ with derivative $\cos'(0) = 0$. To show that sine is differentiable at an arbitrary point $x\in\mathbb{R}$, note that the addition theorem for sine yields

$$\begin{aligned}
\sin'(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \frac{d}{dh}\,\sin(x+h)\big|_{h=0}\\
&= \frac{d}{dh}\big(\sin x\;\cosh h + \cos x\;\sin h\big)\big|_{h=0} = \sin x\;\cos'(0) + \cos x\;\sin'(0)\\
&= \cos x\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\cos'(x) &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \frac{d}{dh}\,\cos(x+h)\big|_{h=0}\\
&= \frac{d}{dh}\big(\cos x\;\cos h - \sin x\;\sin h\big)\big|_{h=0} = \cos x\;\cos'(0) - \sin x\;\sin'(0)\\
&= -\sin x\,.
\end{aligned}$$

From the addition theorems we can derive some more formulas. We first note that sine is an odd function, i.e.

$$\sin(-x) = \sum_{n=0}^{\infty}(-1)^n\frac{(-x)^{2n+1}}{(2n+1)!} = -\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = -\sin x\,.$$

We conclude in the same way that cosine is an even function:

$$\cos(-x) = \cos x\,.$$

The addition theorems therefore yield

$$\begin{aligned}\sin(x-y) &= \sin\big(x+(-y)\big) = \sin x\ \cos(-y) + \sin(-y)\cos x \\ &= \sin x\ \cos y - \sin y\ \cos x\end{aligned}$$

and

$$\cos(x-y) = \cos x\ \cos(-y) - \sin x\ \sin(-y) = \cos x\ \cos y + \sin x\ \sin y\,.$$

For $x=y$ the last formula yields

$$(\cos x)^2 + (\sin x)^2 = 1\,,$$

since $\cos(0)=1$. This implies $|\cos x| \le 1$ and $|\sin x| \le 1$.

Definition of π: The series $\sum_{n=2}^{\infty}(-1)^n \frac{2^{2n}}{(2n)!}$ is alternating and the terms form a decreasing null sequence $\left\{\frac{2^{2n}}{(2n)!}\right\}_{n=2}^{\infty}$. To see this, note that for $n \ge 2$

$$\frac{2^{2(n+1)}}{(2(n+1))!} = \frac{4}{(2n+2)(2n+1)}\,\frac{2^{2n}}{(2n)!} \le \frac{4}{30}\,\frac{2^{2n}}{(2n)!} < \frac{2^{2n}}{(2n)!}\,.$$

Therefore the error estimate derived in the proof of the convergence criterion of Leibniz (Theorem 5.6) yields

$$\sum_{n=2}^{\infty}(-1)^n\,\frac{2^{2n}}{(2n)!} = \frac{2^4}{4!} - \frac{2^6}{6!} + \ldots \le \frac{2^4}{4!} = \frac{2}{3}\,.$$

Thus,

$$\cos 2 = \sum_{n=0}^{\infty}(-1)^n\,\frac{2^{2n}}{(2n)!} = 1 - 2 + \sum_{n=2}^{\infty}(-1)^n\,\frac{2^{2n}}{(2n)!} \le -1 + \frac{2}{3} = -\frac{1}{3}\,.$$

Since $\cos 0 = 1$, we conclude from the intermediate value theorem that cosine has at least one, but possibly many zeros between 0 and 2. We denote the infimum of the set of zeros of cosine between 0 and 2 by the symbol $\frac{\pi}{2}$. Since as a differentiable function cosine is continuous, the infimum $\frac{\pi}{2}$ is itself a zero. To see this, choose a sequence $\{x_n\}_{n=1}^{\infty}$ of zeros of cosine with $\lim_{n\to\infty} x_n = \frac{\pi}{2}$. Then

$$\cos\frac{\pi}{2} = \cos(\lim_{n\to\infty} x_n) = \lim_{n\to\infty}\cos x_n = 0\,.$$

Thus, the number π is defined to be twice the smallest positive zero of cosine.

From $(\cos x)^2 + (\sin x)^2 = 1$ it follows that

$$|\sin\frac{\pi}{2}| = 1\,.$$

Since cosine is positive between 0 and $\frac{\pi}{2}$, we obtain

$$\sin' x = \cos x > 0$$

for all $0 \le x < \frac{\pi}{2}$. Later we show that this implies that sine is strictly increasing on the interval $[0, \frac{\pi}{2}]$. From $\sin 0 = 0$ we thus conclude that $\sin \frac{\pi}{2} > 0$, hence

$$\sin \frac{\pi}{2} = 1 .$$

The addition theorem yields now

$$\sin\left(\frac{\pi}{2} + x\right) = \sin \frac{\pi}{2} \cos x + \cos \frac{\pi}{2} \sin x = \cos x ,$$

whence

$$\sin\left(\frac{\pi}{2} - x\right) = \cos(-x) = \cos x = \sin\left(\frac{\pi}{2} + x\right) . \qquad (*)$$

This formula gives the values of sin in the interval $\left[\frac{\pi}{2}, \pi\right]$ from the values in the interval $\left[0, \frac{\pi}{2}\right]$. From $\sin(-x) = -\sin x$ we can subsequently compute the values in the interval $[-\pi, \pi]$. The extension to all of $\mathbb{R}$ is finally obtained from the periodicity

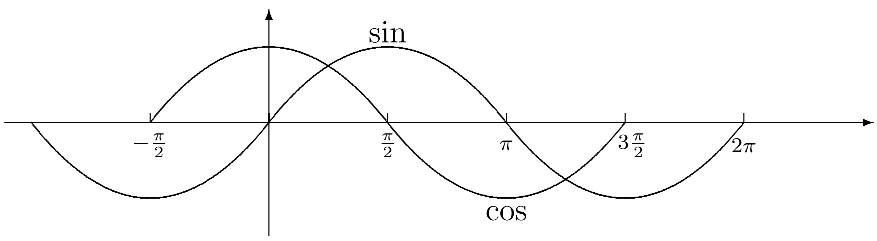
$$\sin(x + 2\pi) = \sin x .$$

This formula results as follows: $(*)$ implies

$$\sin(x + \pi) = \sin\left(\frac{\pi}{2} + \frac{\pi}{2} + x\right) = \sin\left(\frac{\pi}{2} - \left(\frac{\pi}{2} + x\right)\right) = \sin(-x) = -\sin x .$$

Consequently

$$\sin(x + 2\pi) = \sin(x + \pi + \pi) = -\sin(x + \pi) = \sin x .$$



sine and cosine are used to define **tangent** and **cotangent** by

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}.$$

These functions will be discussed in Section 8.

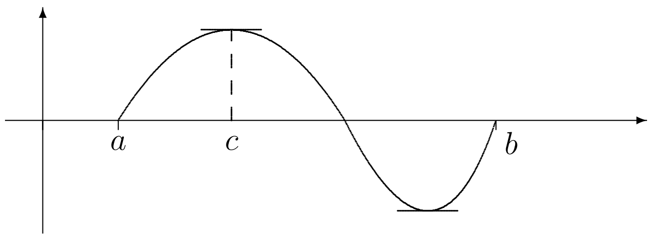
7.3 Mean value theorem

Theorem 7.13 (Theorem of Rolle) *Let* $-\infty < a < b < \infty$*. If* $f : [a,b] \to \mathbb{R}$ *is continuous with* $f(a) = f(b) = 0$*, and if* f *is differentiable in* (a,b)*, then there is* $c \in (a,b)$ *such that*

$$f'(c) = 0.$$

(Michel Rolle 1652 – 1719)

Proof: If $f = 0$, every $c \in (a,b)$ is suitable. Therefore assume that $f \neq 0$. Without restriction of generality we assume that there is $x \in (a,b)$ such that $f(x) > 0$. Otherwise consider the function $-f$.



f attains the maximum in a point $c \in (a,b)$. We have $f'(c) = 0$. To see this, observe that the inequality $f(x) \leq f(c)$, which holds for every $x \in [a,b]$, implies

$$\frac{f(x) - f(c)}{x - c} \begin{cases} \leq 0, & \text{if } x > c \\ \geq 0, & \text{if } x < c. \end{cases}$$

Consequently,

$$f'(c) = \lim_{\substack{x \to c \\ x < c}} \frac{f(x) - f(c)}{x - c} \geq 0$$

and

$$f'(c) = \lim_{\substack{x \to c \\ x > c}} \frac{f(x) - f(c)}{x - c} \leq 0.$$

Both relations can only be true if $f'(c) = 0$ ■

From this theorem we obtain

Theorem 7.14 (First mean value theorem) *Let* $f : [a,b] \to \mathbb{R}$ *be a continuous function, which is differentiable in* (a,b)*. Then there is* $c \in (a,b)$ *such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Proof: Let $g : [a,b] \to \mathbb{R}$ be defined by

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a).$$

This function is continuous in $[a,b]$, differentiable in (a,b), and satisfies $g(a) = g(b) = 0$. Therefore the assumptions of the Theorem of Rolle are satisfied, and so there is $c \in (a,b)$ with $g'(c) = 0$. The statement of the theorem follows from

$$f'(c) = g'(c) + \frac{f(b) - f(a)}{b - a} = \frac{f(b) - f(a)}{b - a}.$$

■

It is immediately clear that the mean value theorem can also be formulated as follows: Let $x, x+h \in [a,b]$. Then there is θ with $0 < \theta < 1$ such that

$$f(x+h) = f(x) + f'(x + \theta h)h.$$

(Here h can also be negative.)

We discuss several simple applications of the mean value theorem:

If $f : D \to \mathbb{R}$ is a constant function, then $f'(x) = 0$ for all $x \in D$. The converse is not true for general domains D, but it is true if D is an interval:

Theorem 7.15 *Let J be an interval and let $f : J \to \mathbb{R}$ with $f'(x) = 0$ for all $x \in J$. Then f is constant.*

Proof: Let $a \in J$. For every $x \in J$ we have $[a,x] \subseteq J$ if $x > a$ and $[x,a] \subseteq J$ if $x < a$, hence the mean value theorem implies that there exists a number c between a and x with

$$f(x) = f(a) + f'(c)(x - a) = f(a),$$

thence $f \equiv f(a)$. ■

Definition 7.16 *Let $f : D \to \mathbb{R}$ be a function. A differentiable function $F : D \to \mathbb{R}$ satisfying $F' = f$ is called antiderivative or primitive of f. ("Stammfunktion")*

Theorem 7.17 *Let J be an interval and let $f : J \to \mathbb{R}$. If F, G are antiderivatives of f, then $F - G = \text{const}$.*

Proof: $(F - G)' = F' - G' = f - f = 0$, hence $F - G = \text{const}$ on the interval J. ■

Application to differential equations. A *differential equation* is an equation, which involves an unknown function and the first or higher derivatives of this function. A *solution* is a function, which satisfies the differential equation. An example for a differential equation is

$$f'(x) = f(x).$$

We want to find a solution, which fulfills this equation for all $x \in \mathbb{R}$. The exponential function is a solution, but there can be other solutions. The following theorem yields all solutions:

Theorem 7.18 *A differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfies the differential equation $f' = f$, if and only if it is of the form $f(x) = ce^x$ for all $x \in \mathbb{R}$ with a constant $c \in \mathbb{R}$.*

Proof: It is clear that $f(x) = ce^x$ solves the differential equation for all $x \in \mathbb{R}$. Conversely, if f is a differentiable function satisfying the differential equation on $\mathbb{R}$, then the function $g(x) = f(x)e^{-x}$ is differentiable on $\mathbb{R}$ and satisfies

$$\Big(f(x)e^{-x}\Big)' = f'(x)e^{-x} - f(x)e^{-x} = f(x)e^{-x} - f(x)e^{-x} = 0.$$

Therefore there exists a constant c satisfying $f(x)e^{-x} = c$ for all $x \in \mathbb{R}$, hence $f(x) = ce^x$.

■

The second mean value theorem contains the first one as a special case. We use the second mean value theorem later to determine limits:

Theorem 7.19 (Second mean value theorem) *Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions. If these functions are differentiable in (a, b), then there is $c \in (a, b)$ such that*

$$g'(c)\Big(f(b) - f(a)\Big) = f'(c)\Big(g(b) - g(a)\Big).$$

Proof: Let

$$h(x) = \Big(f(b) - f(a)\Big)\Big(g(x) - g(a)\Big) - \Big(g(b) - g(a)\Big)\Big(f(x) - f(a)\Big).$$

Then $h(a) = h(b) = 0$. From the theorem of Rolle we thus infer that $c \in (a, b)$ exists satisfying $h'(c) = 0$, whence

$$0 = h'(c) = \Big(f(b) - f(a)\Big)g'(c) - \Big(g(b) - g(a)\Big)f'(c).$$

This equation implies the statement of the theorem. ■

The first mean value theorem is obtained from this theorem with $g(x) = x$. If $g'(x) \neq 0$ for all $x \in (a, b)$, then the first mean value theorem implies $g(b) \neq g(a)$, hence the formula in the second mean value theorem can be written in the form

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

7.4 Taylor formula and Taylor series

Let

$$f(x) = \sum_{k=0}^{n} c_k x^k$$

be a polynomial. Since for large k the values of $|x^k|$ are small if x varies in a neighborhood of zero, the behavior of f in such a neighborhood is determined by the behavior of the terms $c_k x^k$ with small k. This behavior is known if the coefficients c_k are known. These coefficients can be computed from the derivatives of f at zero. For, using

$$f^{(\ell)}(x) = \begin{cases} \displaystyle\sum_{k=\ell}^{n} c_k k(k-1)\ldots(k-\ell+1)x^{k-\ell}, & \ell \leq n \\ 0, & \ell > n \end{cases}$$

we obtain by setting $x = 0$ that

$$c_\ell = \frac{f^{(\ell)}(0)}{\ell!}, \quad \ell = 1, \ldots, n.$$

This shows that the behavior of f close to $x = 0$ is determined by the derivatives of f at $x = 0$.

To determine the behavior of this polynomial in the neighborhood of an arbitrary point $a \in \mathbb{R}$, it is useful to represent f in the form

$$f(x) = \sum_{k=0}^{n} b_k (x-a)^k$$

with suitable coefficients b_k, since the behavior of the terms $b_k(x-a)^k$ is known for x near to a, and since $|(x-a)^k|$ is small for such x and large k. Below we show that such a representation is possible. If this respresentation exists, then we obtain just as above that $b_\ell = \frac{f^{(\ell)}(a)}{\ell!}$ for $\ell \leq n$ and $b_\ell = 0$ for $\ell > n$, whence

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Clearly, the right hand side is a polynomial, so only polynomial functions f can be represented in this form. However, if f is a general, at least n–times differentiable function, then the expression on the right hand side is defined, and one can hope that it yields a good approximation to f in a neighborhood of $x = a$. The error, which is made by replacing f by the term on the right hand side can be estimated with the Taylor formula:

Theorem 7.20 (Taylor formula) *Let $f : [a, b] \to \mathbb{R}$ be n–times continuously differentiable, and assume that f has $n + 1$ derivatives in the open interval (a, b). Then there is a number $c \in (a, b)$ with*

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

(Brook Taylor, 1685 – 1731.)

Proof: Define the function $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(b-x)^k. \tag{7.1}$$

F is continuous and differentiable in the interval (a, b). Therefore the second mean value theorem can be applied to F and to the function $g : [a, b] \to \mathbb{R}$ defined by $g(x) = (b-x)^{n+1}$. Because of $g'(x) \neq 0$ for all $x \in (a, b)$, we obtain with suitable $c \in (a, b)$ that

$$F(b) - F(a) = F'(c)\,\frac{g(b) - g(a)}{g'(c)} = F'(c)\,\frac{(b-a)^{n+1}}{(n+1)(b-c)^n}\,. \tag{7.2}$$

Equation (7.1) implies

$$F'(c) = \sum_{k=0}^{n} \frac{f^{(k+1)}(c)}{k!}(b-c)^k - \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}\,k(b-c)^{k-1} = \frac{f^{(n+1)}(c)}{n!}(b-c)^n\,.$$

If we insert this equation into (7.2) and note that (7.1) implies $F(b) = f(b)$, we obtain

$$f(b) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k = \frac{f^{(n+1)}(c)}{n!}(b-c)^n\,\frac{(b-a)^{n+1}}{(n+1)(b-c)^n} = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

This is the Taylor formula. ■

Corollary 7.21 *Let f be a polynomial of degree at most n. Then for every $a \in \mathbb{R}$*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Proof: This follows from the Taylor formula since $f^{(n+1)} = 0$. ■

Corollary 7.22 *Let $f : [a,b] \to \mathbb{R}$ be a $(n+1)$–times differentiable function with $f^{(n+1)} = 0$. Then f is a polynomial of degree not greater that n.*

Proof: The Taylor formula yields that $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ for all $x \in [a,b]$, and the right hand side is a polynomial of degree not greater than n. ∎

The theorem for the Taylor formula can also be formulated as follows: Let $x, x+h \in [a,b]$. Then there exists a number θ with $0 < \theta < 1$ such that

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + \frac{f^{(n+1)}(x+\theta h)}{(n+1)!} h^{n+1}.$$

With $n = 1$ the Taylor formula yields the mean value theorem. The function

$$x \mapsto \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is called n–th Taylor polynomial of the function f at the point a,

$$R_n(b) = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

is called n–th remainder term in Lagrange representation. Note that $c = c(b)$ is a function of b. The size of this remainder term determines how good the n–th Taylor polynomial approximates the function f. It is also possible to represent the remainder term in a different form using integration. If the remainder term is small, the Taylor formula can be used to compute the value of a function approximately.

Examples: 1.) Computation of $\log x$ in a neighborhood of the point $x = 1$: We have

$$\frac{d}{dx} \log x = \frac{1}{x}$$

and

$$\frac{d^n}{dx^n} \log x = \frac{d^{n-1}}{dx^{n-1}} \frac{1}{x} = (n-1)!(-1)^{n-1} \frac{1}{x^n},$$

and so the Taylor formula yields

$$\begin{aligned} \log(1+h) &= \sum_{k=0}^{n} \frac{\log^{(k)}(1)}{k!} h^k + \frac{\log^{(n+1)}(1+\theta h)}{(n+1)!} h^{n+1} \\ &= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} h^k + \frac{(-1)^n}{n+1} \frac{h^{n+1}}{(1+\theta h)^{n+1}} \qquad (*) \end{aligned}$$

with a suitable $\theta = \theta(h)$, $0 < \theta < 1$. For $0 \le h \le 1$ the remainder

$$R_n(h) = (-1)^n \frac{1}{n+1} \frac{h^{n+1}}{(1+\theta h)^{n+1}}$$

can be estimated by

$$|R_n(h)| = \frac{1}{n+1}\left|\frac{h}{(1+\theta h)}\right|^{n+1} \le \frac{h^{n+1}}{n+1}\,, \qquad (**)$$

and for $-1 < h \le 0$ by

$$|R_n(h)| \le \frac{1}{n+1}\,\frac{|h|^{n+1}}{(1-|h|)^{n+1}}\,.$$

For $-\frac{1}{2} \le h \le 1$ these estimates yield

$$|R_n(h)| \le \frac{1}{n+1}\,,$$

whence $\lim_{n\to\infty} R_n(h) = 0$. Therefore $(*)$ yields

$$\begin{aligned}\log(1+h) &= \lim_{n\to\infty}\Big(\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\,h^k + R_n(h)\Big)\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{h^k}{k} + \lim_{n\to\infty}R_n(h) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{h^k}{k}\,,\end{aligned}$$

which can also be written in the form

$$\log x = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\,(x-1)^k\,,$$

$\frac{1}{2} \le x \le 2$. As an example, for $x = 2$ we obtain

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\,.$$

This series does however converge only very slowly, and is therefore not suitable for numerical computations. Much better results are obtained for values of x nearer to 1 : For example, for $x = \frac{3}{2}\,(h = \frac{1}{2})$ the inequality $(**)$ yields for the fifth remainder term in the Taylor formula

$$\Big|R_5\Big(\frac{1}{2}\Big)\Big| \le \frac{1}{6\cdot 2^6} = \frac{1}{384} < 0.0027\,,$$

and the sum of the first five terms in the Taylor series is

$$\sum_{k=1}^{5}(-1)^{k-1}\frac{1}{k2^k} = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \frac{1}{160} \approx 0.40729.$$

In fact, $\log\frac{3}{2} \sim 0.405465108$.

2.) For $s \in \mathbb{R}$ consider the function

$$x \mapsto x^s : \mathbb{R}^+ \to \mathbb{R}\,.$$

Let $a > 0$ and let $h \in \mathbb{R}$ with $|h| < a$. Then the Taylor formula yields with

$$\frac{d^k}{dx^k}\, x^s = s(s-1)(s-2)\dots(s-k+1)x^{s-k}$$

that

$$(a+h)^s = \sum_{k=0}^{n} \binom{s}{k} a^{s-k}h^k + \binom{s}{n+1}(a+\theta h)^{s-n-1}h^{n+1}\,,$$

where $0 < \theta < 1$ is suitable. Here

$$\binom{s}{k} = \frac{s(s-1)\dots(s-k+1)}{k!}\,,$$

for every $k \in \mathbb{N}_0$. We leave it as an exercise to show that for $0 \le h < a$

$$\lim_{n\to\infty} R_n(h) = \lim_{n\to\infty} \binom{s}{n+1} \frac{h^{n+1}}{(a+\theta h)^{n+1-s}} = 0\,.$$

This implies for $0 \le h < a$ that

$$(a+h)^s = \sum_{k=0}^{\infty} \binom{s}{k} a^{s-k}h^k\,.$$

Compare this formula with the binomial theorem!

In both of these examples $\lim_{n\to\infty} R_n = 0$ holds. This implies that both of the infinitely often differentiable functions $\log x$ and x^s can be expanded into series in certain intervals (*Taylor series*). It is not true, however, that for every infinitely often differentiable function the remainder term in the Taylor formula tends to zero. If the remainder does not tend to zero, then it can happen that the Taylor series does not converge, or for other functions it can even happen that the Taylor series converges, but to a limit different from the value of the function. Functions, which can be expanded into a Taylor series, are called analytic functions. Analytic functions are studied in *the theory of functions of complex variables.*

7.5 Monotonicity, extreme values, rules of de l'Hospital

Theorem 7.23 *Let J be an intervall and let $f : J \to \mathbb{R}$ be differentiable. f is increasing if and only if*

$$f'(x) \ge 0$$

for all $x \in J$.

Proof: If f is increasing, then for $x \neq y$

$$\frac{f(y) - f(x)}{y - x} \geq 0\,,$$

hence

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} \geq 0\,.$$

Conversely, if $f'(x) \geq 0$ for all x, then it follows from the mean value theorem for $y > x$ with a suitable z between x and y that

$$f(y) - f(x) = f'(z)(y - x) \geq 0\,.$$

■

Remark: This proof shows that if $f'(x) > 0$ for all $x \in J$, then$f$ is strictly increasing. However, this is only a sufficient condition. For example, $x \mapsto x^3$ is strictly increasing, but $(x^3)'\big|_{x=0} = 3x^2\big|_{x=0} = 0$.

Definition 7.24 *Let $D \subseteq \mathbb{R}$, $a \in D$, and let $f : D \to \mathbb{R}$. If there is a neighborhood U of a such that for all $x \in D \cap U$*

$$f(x) \geq f(a)$$

holds, then f is said to have a local minimum at a. If for every $x \in D \cap U$

$$f(x) \leq f(a)$$

holds then f is said to have a local maximum at a. If f has a local maximum or a local minimum at a, then it is said to have a local extreme value.

Theorem 7.25 *Let a be an interior point of D, let $f : D \to \mathbb{R}$ be differentiable, and assume that f attains a local extreme value at a. Then*

$$f'(a) = 0\,.$$

The **proof** is the same as the proof of the theorem of Rolle.

$f'(a) = 0$ is a condition necessary for a local extreme value, but not sufficient. For example, $f(x) = x^3$ satisfies

$$f'(0) = 0\,,$$

but f does not have a local extreme value at 0. A sufficient condition is given in the following

Theorem 7.26 *Let D be a subset of $\mathbb{R}$, let a be an interior point of D, let $f : D \to \mathbb{R}$ be n–times continuously differentiable and satisfy*

$$f'(a) = f''(a) = \ldots = f^{n-1}(a) = 0\,,$$

but $f^{(n)}(a) \neq 0$.

If n is odd, then f does not have a local extreme value at a. If $n \geq 2$ is even, then f has a local minimum at a if $f^{(n)}(a) > 0$ and a local maximum if $f^{(n)}(a) < 0$.

Proof: Since $f^{(n)}$ is continuous and satisfies $f^{(n)}(a) \neq 0$, there is an ε–neighborhood $(a-\varepsilon, a+\varepsilon)$ of a such that $f^{(n)}(x)$ differs from zero and has the same sign as $f^{(n)}(a)$ for all $x \in (a-\varepsilon, a+\varepsilon)$. Furthermore, since $f'(a) = \ldots = f^{(n-1)}(a) = 0$, Taylor's theorem yields for all $x \in (a-\varepsilon, a+\varepsilon)$ that

$$f(x) - f(a) = \frac{f^{(n)}(y)}{n!}(x-a)^n \tag{$*$}$$

with a suitable number y between a and x, hence $y \in (a-\varepsilon, a+\varepsilon)$. From this we conclude that if n is odd, then the right hand side of $(*)$ has different signs for $x < a$ and $x > a$. Thus, $f(x) - f(a)$ has different signs, and therefore f does not have a local extreme value at a. If n is even, then $(*)$ implies for all $x \in (a-\varepsilon, a+\varepsilon)$ that

$$\begin{aligned} f(x) - f(a) &\geq 0\,, \qquad \text{if } f^{(n)}(a) > 0 \\ f(x) - f(a) &\leq 0\,, \qquad \text{if } f^{(n)}(a) < 0\,, \end{aligned}$$

which means that f as a local minimum at a if $f^{(n)}(a) > 0$ and a local maximum if $f^{(n)}(a) < 0$. ■

Determination of limits by differentiation. We have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}\,,$$

if the limits of f and g exist and the limit of g is different from zero. If the limit of f differs from zero and the limit of g is equal to zero, then $\frac{f}{g}$ does not have a limit, but it is possible that the limit of $\frac{f}{g}$ exists if the limits of f and g are both equal to zero. The **rules of de l'Hospital** deal with this situation:

Theorem 7.27 *Suppose that the functions f and g are defined for $x > a$ and differentiable. Moreover, suppose that $g(x) \neq 0$ and $g'(x) \neq 0$ for $x > a$ and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$.*

Then, if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ *exists, also the limit* $\lim_{x\to a} \frac{f(x)}{g(x)}$ *exists, and*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} .$$

Proof: Extend the functions f and g continuously to $x = a$ by setting $f(a) = g(a) = 0$. We can apply the second mean value theorem to the extended functions and obtain for $x > a$ that $y \in (a, x)$ exists with

$$\big(f(x) - f(a)\big)\, g'(y) = \big(g(x) - g(a)\big)\, f'(y) ,$$

hence

$$\frac{f(x)}{g(x)} = \frac{f'(y)}{g'(y)} . \tag{$*$}$$

To prove that $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists and is equal to $c = \lim_{y\to a} \frac{f'(y)}{g'(y)}$, let $\varepsilon > 0$. Theorem 6.39 implies that $\delta > 0$ exists with

$$\Big|\frac{f'(y)}{g'(y)} - c\Big| < \varepsilon$$

for all $x \in (a, a+\delta)$. Thus, since $0 < y < x$ in ($*$), we obtain for $x \in (a, a+\delta)$ that

$$\Big|\frac{f(x)}{g(x)} - c\Big| = \Big|\frac{f'(y)}{g'(y)} - c\Big| < \varepsilon ,$$

whence $\lim_{x\to a} \frac{f(x)}{g(x)} = c$. ■

Theorem 7.28 *Suppose that the functions* f *and* g *are defined for* $x > a$ *and* n*– times differentiable. Moreover, suppose that* $g(x) \neq 0, \ldots, g^{(n)}(x) \neq 0$ *for* $x > a$ *and* $\lim_{x\to a} f^{(k)}(x) = \lim_{x\to a} g^{(k)}(x) = 0$, $k = 0, 1, \ldots, n-1$.

Then, if $\lim_{x\to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}$ *exists, also the limit* $\lim_{x\to a} \frac{f(x)}{g(x)}$ *exists and*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f^{(n)}(x)}{g^{(n)}(x)} .$$

Proof: Apply the preceding theorem repeatedly. ■

Let $f : D \to \mathbb{R}$ with a domain of definition D, which is not bounded above. One defines *the limit of* f *at infinity* by

$$\lim_{x\to\infty} f(x) = \lim_{y\to 0} f\Big(\frac{1}{y}\Big) .$$

With this definition the investigation of $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ is reduced to the investigation of $\lim_{y\to 0} \frac{f(1/y)}{g(1/y)}$. Application of the preceding theorems to this last limit shows that they remain valid if a is replaced by ∞ and $x > a$ by $x < \infty$. In particular,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} .$$

Also the case of functions f and g unbounded in a neighborhood of a can be treated:

Theorem 7.29 *Suppose that the functions f and g are defined in $x > a$ and differentiable. Suppose moreover that $\lim_{x\to a} \frac{1}{f(x)} = \lim_{x\to a} \frac{1}{g(x)} = 0$.*

Then, if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, also $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}\,.$$

Proof: The proof is more complicated, since f and g cannot be extended continuously to a. Let $c = \lim_{x\to a} \frac{f'(x)}{g'(x)}$. Then to $\varepsilon > 0$ there is a number $b > a$ such that $\left|\frac{f'(y)}{g'(y)} - c\right| < \varepsilon$ for all $a < y \leq b$. We have

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(b)}{g(x)-g(b)}\,\frac{f(x)}{f(x)-f(b)}\,\frac{g(x)-g(b)}{g(x)}\,.$$

If x is sufficiently close to a, all denominators in this equation are different from zero, since $f(x)$ and $g(x)$ "tend to infinity" for $x \to a$. For these x we obtain from this equation

$$\begin{aligned}\left|\frac{f(x)}{g(x)} - c\right| \;\leq\; & \left|\frac{f(x)-f(b)}{g(x)-g(b)} - c\right| \left|\frac{f(x)}{f(x)-f(b)}\right| \left|\frac{g(x)-g(b)}{g(x)}\right| \\ & +|c| \left|\frac{f(x)}{f(x)-f(b)}\,\frac{g(x)-g(b)}{g(x)} - 1\right|.\end{aligned}$$

Using the second mean value theorem, we obtain with a suitable $y \in (x,b)$ that

$$\left|\frac{f(x)-f(b)}{g(x)-g(b)} - c\right| = \left|\frac{f'(y)}{g'(y)} - c\right| < \varepsilon\,,$$

hence

$$\left|\frac{f(x)}{g(x)} - c\right| \leq \varepsilon \left|\frac{1}{1-\frac{f(b)}{f(x)}}\right| \left|1 - \frac{g(b)}{g(x)}\right| + |c| \left|\frac{1-\frac{g(b)}{g(x)}}{1-\frac{f(b)}{f(x)}} - 1\right|.$$

$\lim_{x\to a} \frac{1}{f(x)} = 0$ and $\lim_{x\to a} \frac{1}{g(x)} = 0$ imply $\lim_{x\to a} \big(1 - \frac{f(b)}{f(x)}\big) = \lim_{x\to a} \big(1 - \frac{g(b)}{g(x)}\big) = 1$, hence

$$\lim_{x\to a} \left(\frac{1-\frac{g(b)}{g(x)}}{1-\frac{f(b)}{f(x)}} - 1\right) = 0\,.$$

Consequently, there is $a < \delta \leq b$ such that for all $x \in (a,\delta)$

$$\left|\frac{f(x)}{g(x)} - c\right| \leq 2\varepsilon + |c|\varepsilon = (2+|c|)\varepsilon\,.$$

By Theorem 6.39, this means that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = c = \lim_{x\to a} \frac{f'(x)}{g'(x)}\,.$$

■

By repeated application of this theorem we obtain

Theorem 7.30 *Let the functions f and g be defined in $x > a$ and n–times differentiable. Let* $\lim_{x\to a} \frac{1}{f^{(k)}(x)} = \lim_{x\to a} \frac{1}{g^{(k)}(x)} = 0$ *for* $k = 0, 1, \ldots, n-1$. *Then, if* $\lim_{x\to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}$ *exists, also* $\lim_{x\to a} \frac{f(x)}{g(x)}$ *exists and*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f^{(n)}(x)}{g^{(n)}(x)} .$$

The last two theorems remain valid if a is replaced by ∞ and $x > a$ by $x < \infty$.

Examples: 1.) Let $s > 0$ and $f(x) = x^s$, $g(x) = e^x$. We want to study whether the limit

$$\lim_{x\to\infty} \frac{x^s}{e^x}$$

exists. To this end let n be the unique natural number $n \in [s, s+1)$. Then $n - 1 < s$, hence

$$\lim_{x\to\infty} \frac{1}{f^{(k)}(x)} = \lim_{x\to\infty} \frac{1}{(x^s)^{(k)}} = \lim_{x\to\infty} \frac{1}{s(s-1)\ldots(s-k+1)x^{s-k}} = 0$$

and $\lim_{x\to\infty} \frac{1}{g^{(k)}(x)} = \lim_{x\to\infty} e^{-x} = 0$ for $k = 0, 1, \ldots, n-1$. Moreover,

$$\lim_{x\to\infty} \frac{f^{(n)}(x)}{g^{(n)}(x)} = \lim_{x\to\infty} \frac{(x^s)^{(n)}}{(e^x)^{(n)}} = \lim_{x\to\infty} \frac{s(s-1)\ldots(s-n+1)}{x^{n-s}e^x} = 0 ,$$

and therefore the preceding theorem yields with $a = \infty$ that

$$\lim_{x\to\infty} \frac{x^s}{e^x} = 0 ,$$

i.e. the exponential function grows faster than any power.

2.) Let $s > 0$. Since $\lim_{x\to\infty} \frac{1}{\log x} = \lim_{x\to\infty} \frac{1}{x^s} = 0$, Theroem 7.29 yields with $f(x) = \log x$, $g(x) = x^s$ and $a = \infty$ that

$$\lim_{x\to\infty} \frac{\log x}{x^s} = \lim_{x\to\infty} \frac{\log' x}{(x^s)'} = \lim_{x\to\infty} \frac{1}{sx^s} = 0 .$$

Similarly, because of $\lim_{x\to 0} \frac{1}{\log x} = \lim_{x\to 0} \frac{1}{x^{-s}} = 0$, this theorem yields with $a = 0$ that

$$\lim_{x\to 0} x^s \log x = \lim_{x\to 0} \frac{\log x}{x^{-s}} = \lim_{x\to 0} \frac{\log' x}{(x^{-s})'} = \lim_{x\to 0} \Big(-\frac{x^s}{s} \Big) = 0 .$$

The logarithm grows slower for $x \to \infty$ and for $x \to 0$ than any power.

Postface

The lecture notes Analysis I and II originated from my handwritten notes in German for the course Infinitesimalrechnung I – IV, which I gave at the Universität Bonn during the years 1982 – 1984. In writing the notes I drew from several textbooks and reference works. Those books, which I mainly used, are listed below. As a guideline for the course I used the two volume work of Barner and Flohr. Therefore these lecture notes owe much to this excellent work.

The notes have been revised and typed in latex when I gave the course Analysis I – IV several times during the years 1990 – 2008 at the Technische Universität Darmstadt. The English version, which covers the material of a two semester introductory course, was prepared in 2001.

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