ANALYSIS II

Lecture Notes

Hans-Dieter Alber

with an Appendix: German Translation of Section 7
mit einem Anhang: Deutsche Übersetzung von Kapitel 7

June 2001
last modified August 2, 2017

Contents

1 Sequences of functions, uniform convergence, power series

1.1 Pointwise convergence

In section 4 of the lecture notes to the Analysis I course we introduced the exponential function

$$x \mapsto \exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\,.$$

For every $n \in \mathbb{N}$ we define the polynomial function $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) := \sum_{k=0}^{n} \frac{x^k}{k!}\,.$$

Then $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions with the property that

$$\exp(x) = \lim_{n\to\infty} f_n(x)$$

for every $x \in \mathbb{R}$. We say that the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the exponential function.

Definition 1.1 *Let D be a set (not necessarily a set of real numbers), and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $f_n : D \to \mathbb{R}$. This sequence is said to converge pointwise, if a function $f : D \to \mathbb{R}$ exists such that*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for all $x \in D$. We call f the pointwise limit function of $\{f_n\}_{n=1}^{\infty}$.

The sequence $\{f_n\}_{n=1}^{\infty}$ of functions converges pointwise if and only if the numerical sequence $\{f_n(x)\}_{n=1}^{\infty}$ converges for every $x \in D$. For, if $\{f_n\}_{n=1}^{\infty}$ converges pointwise, then $\{f_n(x)\}_{n=1}^{\infty}$ converges by definition. On the other hand, if $\{f_n(x)\}_{n=1}^{\infty}$ converges for every $x \in D$, then a function $f : D \to \mathbb{R}$ is defined by

$$f(x) := \lim_{n\to\infty} f_n(x)\,,$$

and so $\{f_n\}_{n=1}^{\infty}$ converges pointwise.

Clearly, this shows that the limit function of a pointwise convergent function sequence is uniquely determined. Moreover, together with the Cauchy convergence criterion for numerical sequences it immediately yields the following

Theorem 1.2 *A sequence* $\{f_n\}_{n=1}^{\infty}$ *of functions* $f_n : D \to \mathbb{R}$ *converges pointwise, if and only if to every* $x \in D$ *and to every* $\varepsilon > 0$ *there is a number* $n_0 \in \mathbb{N}$ *such that*

$$|f_n(x) - f_m(x)| < \varepsilon$$

for all $n, m \geq n_0$.

With quantifiers this can be written as

$$\underset{x>0}{\forall} \ \underset{\varepsilon>0}{\forall} \ \underset{n_0\in\mathbb{N}}{\exists} \ \underset{n,m\geq n_0}{\forall} \ : \ |f_n(x) - f_m(x)| < \varepsilon \, .$$

Examples

1. Let $D = [0,1]$ and $x \mapsto f_n(x) := x^n$. Since for $x \in [0,1)$ we have $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0$, and since $\lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} 1^n = 1$, the function sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the limit function $f : [0,1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 0\,, & 0 \leq x < 1 \\ 1\,, & x = 1\,. \end{cases}$$

2. Above we considered the sequence of polynomial functions $\{f_n\}_{n=1}^{\infty}$ with $f_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$, which converges pointwise to the exponential function. This sequence $\left\{\sum_{k=0}^{n} \frac{x^k}{k!}\right\}_{n=1}^{\infty}$ can also be called a *function series.*

3. Let $D = [0,2]$ and

$$f_n(x) = \begin{cases} nx\,, & 0 \leq x \leq \frac{1}{n} \\ 2 - nx\,, & \frac{1}{n} < x < \frac{2}{n} \\ 0\,, & \frac{2}{n} \leq x \leq 2\,. \end{cases}$$

This function sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the null function in $[0,2]$.

Proof: It must be shown that for all $x \in D$

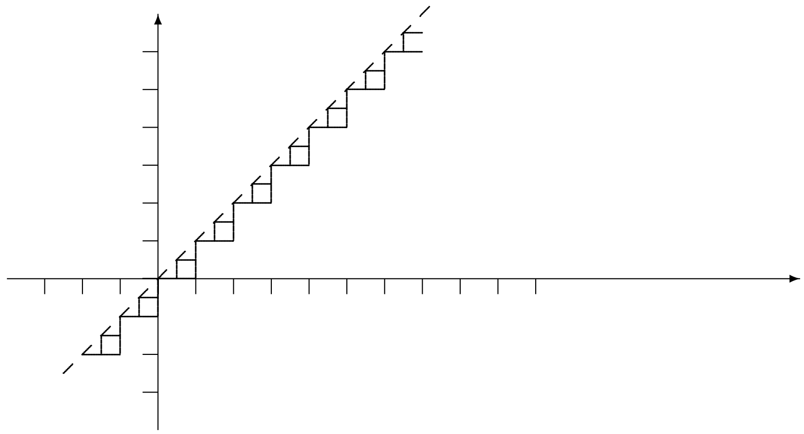
$$\lim_{n\to\infty} f_n(x) = 0.$$

For $x = 0$ we obviously have $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 0 = 0$. Thus, let $x > 0$. Then there is $n_0 \in \mathbb{N}$ such that $\frac{2}{n_0} \leq x$. Since $\frac{2}{n} \leq \frac{2}{n_0} \leq x$ for $n \geq n_0$, the definition of f_n yields $f_n(x) = 0$ for all these n, whence

$$\lim_{n\to\infty} f_n(x) = 0.$$

■

4. Let $D = \mathbb{R}$ and $x \mapsto f_n(x) = \frac{1}{n}[nx]$. Here $[nx]$ denotes the greatest integer less or equal to nx.



$\{f_n\}_{n=1}^{\infty}$ converges pointwise to the identity mapping $x \mapsto f(x) := x$.

Proof: Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then there is $k \in \mathbb{Z}$ with $x \in [\frac{k}{n}, \frac{k+1}{n})$, hence $nx \in [k, k+1)$, and therefore

$$f_n(x) = \frac{1}{n}[nx] = \frac{k}{n}.$$

From $\frac{k}{n} \leq x < \frac{k+1}{n}$ it follows that

$$0 \leq x - \frac{k}{n} < \frac{1}{n},$$

which yields $|x - f_n(x)| = |x - \frac{k}{n}| < \frac{1}{n}$. This implies

$$\lim_{n\to\infty} f_n(x) = x.$$

■

1.2 Uniform convergence, continuity of the limit function

Suppose that $D \subseteq \mathbb{R}$ and that $\{f_n\}_{n=1}^{\infty}$ is a sequence of continuous functions $f_n : D \to \mathbb{R}$, which converges pointwise. It is natural to ask whether the limit function $f : D \to \mathbb{R}$ is continuous. However, the first example considered above shows that this need not be the case, since

$$x \mapsto f_n(x) = x^n : [0,1] \to \mathbb{R}$$

is continuous, but the limit function

$$f(x) = \begin{cases} 0\,, & x \in [0,1) \\ 1\,, & x = 1 \end{cases}$$

is discontinuous. To be able to conclude that the limit function is continuous, a stronger type of convergence must be introduced:

Definition 1.3 *Let D be a set (not necessarily a set of real numbers), and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $f_n : D \to \mathbb{R}$. This sequence is said to be uniformly convergent, if a function $f : D \to \mathbb{R}$ exists such that to every $\varepsilon > 0$ there is a number $n_0 \in \mathbb{N}$ with*

$$|f_n(x) - f(x)| < \varepsilon$$

for all $n \geq n_0$ and all $x \in D$. The function f is called limit function.

With quantifiers, this can be written as

$$\underset{\varepsilon>0}{\forall} \quad \underset{n_0\in\mathbb{N}}{\exists} \quad \underset{x\in D}{\forall} \quad \underset{n\geq n_0}{\forall} \ : \ |f_n(x) - f(x)| < \varepsilon\,.$$

Note that for pointwise convergence the number n_0 may depend on $x \in D$, but for uniform convergence it must be possible to choose the number n_0 independently of $x \in D$. It is obvious that if $\{f_n\}_{n=1}^{\infty}$ converges uniformly, then it also converges pointwise, and the limit functions of uniform convergence and pointwise convergence coincide.

Examples

1. Let $D = [0,1]$ and $x \mapsto f_n(x) := x^n : D \to \mathbb{R}$. We have shown above that the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise. However, this sequence is not uniformly convergent.

Proof: If this sequence would converge uniformly, the limit function had to be

$$f(x) = \begin{cases} 0\,, & x \in [0,1) \\ 1\,, & x = 1\,, \end{cases}$$

since this is the pointwise limit function. We show that for this function the negation of the statement in the definition of uniform convergence is true:

$$\underset{\varepsilon>0}{\exists} \quad \underset{n_0\in\mathbb{N}}{\forall} \quad \underset{x\in D}{\exists} \quad \underset{n\geq n_0}{\exists} \ : \ |f_n(x) - f(x)| \geq \varepsilon\,.$$

Choose $\varepsilon = \frac{1}{2}$ and n_0 arbitrarily. The negation is true if $x \in (0,1)$ can be found with

$$|f_{n_0}(x) - f(x)| = |f_{n_0}(x)| = x^{n_0} = \frac{1}{2} = \varepsilon\,.$$

This is equivalent to

$$x = \Big(\frac{1}{2}\Big)^{\frac{1}{n_0}} = 2^{-\frac{1}{n_0}} = e^{-\frac{\log 2}{n_0}}\,.$$

$\frac{\log 2}{n_0} > 0$ and the strict monotonicity of the exponential function imply $0 < e^{-\frac{\log 2}{n_0}} < e^0 = 1$, whence $0 < \big(\frac{1}{2}\big)^{\frac{1}{n_0}} < 1$, whence $x = \big(\frac{1}{2}\big)^{\frac{1}{n_0}}$ has the sought properties. ■

2. Let $\{f_n\}_{n=1}^\infty$ be the sequence of functions defined in example 3 of section 1.1. This sequence converges pointwise to the function $f = 0$, but it does not converge uniformly. Otherwise it had to converge uniformly to $f = 0$. However, choose $\varepsilon = 1$, let $n_0 \in \mathbb{N}$ be arbitrary and set $x = \frac{1}{n_0}$. Then

$$|f_n\Big(\frac{1}{n_0}\Big) - f\Big(\frac{1}{n_0}\Big)| = |f_n\Big(\frac{1}{n_0}\Big)| = 1 \geq \varepsilon\,,$$

which negates the statement in the definition of uniform convergence.

3. Let $D = \mathbb{R}$ and $x \mapsto f_n(x) = \frac{1}{n}[nx]$. The sequence $\{f_n\}_{n=1}^\infty$ converges uniformly to $x \mapsto f(x) = x$. To verify this, let $\varepsilon > 0$ and remember that in example 4 of section 1.1 we showed that

$$|f_n(x) - f(x)| = |f_n(x) - x| < \frac{1}{n}$$

for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$. Hence, if we choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$, we obtain for all $n \geq n_0$ and all $x \in \mathbb{R}$

$$|f_n(x) - f(x)| < \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon\,.$$

Uniform convergence is important because of the following

Theorem 1.4 *Let $D \subseteq \mathbb{R}$, let $a \in D$ and let all the functions $f_n : D \to \mathbb{R}$ be continuous at a. Suppose that the sequence of functions $\{f_n\}_{n=1}^\infty$ converges uniformly to the limit function $f : D \to \mathbb{R}$. Then f is continuous at a.*

Proof: Let $\varepsilon > 0$. We have to find $\delta > 0$ such that for all $x \in D$ with $|x - a| < \delta$

$$|f(x) - f(a)| < \varepsilon$$

holds. To determine such a number δ, note that for all $x \in D$ and all $n \in \mathbb{N}$

$$\begin{aligned} |f(x) - f(a)| &= |f(x) - f_n(x) + f_n(x) - f_n(a) + f_n(a) - f(a)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)|\,. \end{aligned}$$

Since $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f, there is $n_0 \in \mathbb{N}$ with $|f_n(y) - f(y)| < \frac{\varepsilon}{3}$ for all $n \geq n_0$ and all $y \in D$, whence

$$|f(x) - f(a)| \leq \frac{2}{3}\varepsilon + |f_{n_0}(x) - f_{n_0}(a)| \,.$$

Since f_{n_0} is continuous, there is $\delta > 0$ such that $|f_{n_0}(x) - f_{n_0}(a)| < \frac{\varepsilon}{3}$ for all $x \in D$ with $|x - a| < \delta$. Thus, if $|x - a| < \delta$

$$|f(x) - f(a)| < \frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon \,,$$

which proves that f is continuous at a. ■

This theorem shows that

$$\lim_{x \to a} \lim_{n \to \infty} f_n(x) = \lim_{x \to a} f(x) = f(a) = \lim_{n \to \infty} f_n(a) = \lim_{n \to \infty} \lim_{x \to a} f_n(a) \,.$$

Hence, for a uniformly convergent sequence of functions the limits $\lim_{x \to a}$ and $\lim_{n \to \infty}$ can be interchanged.

Corollary 1.5 *The limit function of a uniformly convergent sequence of continuous functions is continuous.*

Example 2 considered above shows that the limit function can be continuous even if the sequence $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly. However, we have

Theorem 1.6 (of Dini) *Let $D \subseteq \mathbb{R}$ be compact, let $f_n : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$ be continuous, and assume that the sequence of functions $\{f_n\}_{n=1}^{\infty}$ converges pointwise and monotonically to f, i.e. the sequence $\{|f_n(x) - f(x)|\}_{n=1}^{\infty}$ is a decreasing null sequence for every $x \in D$. Then $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f. (Ulisse Dini, 1845-1918).*

Proof: Let $\varepsilon > 0$. To every $x \in D$ a neighborhood $U(x)$ is associated as follows: $\lim_{n \to \infty} f_n(x) = f(x)$ implies that a number $n_0 = n_0(x, \varepsilon)$ exists such that $|f_{n_0}(x) - f(x)| < \varepsilon$. Since f and f_{n_0} are continuous, also $|f_{n_0} - f|$ is continuous, hence there is an open neighborhood $U(x)$ of x such that $|f_{n_0}(y) - f(y)| < \varepsilon$ holds for all $y \in U(x) \cap D$. The system $\mathcal{U} = \{U(x) \mid x \in D\}$ of these neighborhoods is an open covering of the compact set D, hence finitely many of these neighborhoods $U(x_1), \ldots, U(x_m)$ suffice to cover D. Let

$$\tilde{n} = \max \{n_0(x_i, \varepsilon) \mid i = 1, \ldots, m\} \;.$$

To every $x \in D$ there is a number $i \in \{1, \dots, m\}$ with $x \in U(x_i)$. Then, by construction of $U(x_i)$,

$$|f_{n_0(x_i,\varepsilon)}(x) - f(x)| < \varepsilon \,,$$

whence, since $\{f_n(x)\}_{n=1}^{\infty}$ converges monotonically to $f(x)$,

$$|f_n(x) - f(x)| < \varepsilon$$

for all $n \geq n_0(x_i, \varepsilon)$. In particular, this inequality holds for all $n \geq \tilde{n}$. Since $\tilde{n}$ is independent of x, this proves that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f. ■

1.3 Supremum norm

For the definition of convergence and limits of numerical sequences the absolute value, a tool to measure distance for numbers, was of crucial importance. Up to now we have not introduced a tool to measure distance of functions, but we were nevertheless able to define two different types of convergence of sequences of functions, the pointwise convergence and the uniform convergence. Since functions with domain D and target set $\mathbb{R}$ are elements of the algebra $F(D, \mathbb{R})$, it is natural to ask whether a tool can be introduced, which allows to measure the distance of two elements from $F(D, \mathbb{R})$, and which can be used to define convergence on the set $F(D, \mathbb{R})$ just as the absolute value could be used to define convergence on the set $\mathbb{R}$. Here we shall show that this is indeed possible on the smaller algebra $B(D, \mathbb{R})$ of bounded real valued functions. The resulting type of convergence of sequences of functions from $B(D, \mathbb{R})$ is the uniform convergence.

Definition 1.7 *Let D be a set (not necessarily a set of real numbers), and let $f : D \to \mathbb{R}$ be a bounded function. The nonnegative number*

$$\|f\|_\infty := \sup_{x \in D} |f(x)|$$

is called the supremum norm of f.

The norm has properties similar to the properties of the absolute value on $\mathbb{R}$. This is shown by the following

Theorem 1.8 *Let $f, g : D \to \mathbb{R}$ be bounded functions and c be a real number. Then*

(i) $\|f\|_\infty = 0 \iff f = 0$

(ii) $\|cf\|_\infty = |c| \, \|f\|_\infty$

(iii) $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$

(iv) $\|fg\|_\infty \le \|f\|_\infty \, \|g\|_\infty\,.$

Proof: (i) and (ii) are obvious. To prove (iii), note that for $x \in D$

$$\begin{aligned} |(f+g)(x)| &= |f(x)+g(x)| \le |f(x)| + |g(x)| \\ &\le \sup_{y\in D} |f(y)| + \sup_{y\in D} |g(y)| = \|f\|_\infty + \|g\|_\infty\,. \end{aligned}$$

Thus, $\|f\|_\infty + \|g\|_\infty$ is an upper bound for the set $\big\{ |(f+g)(x)| \;\big|\; x \in D \big\}$, whence for the least upper bound

$$\|f+g\|_\infty = \sup_{x\in D} |(f+g)(x)| \le \|f\|_\infty + \|g\|_\infty\,.$$

To prove (iv), we use that for $x \in D$

$$|(fg)(x)| = |f(x)g(x)| = |f(x)|\,|g(x)| \le \|f\|_\infty \, \|g\|_\infty\,,$$

whence

$$\|fg\|_\infty = \sup_{x\in D} |(fg)(x)| \le \|f\|_\infty \, \|g\|_\infty\,.$$

■

Definition 1.9 *Let V be a vector space. A mapping $\|\cdot\| : V \to [0,\infty)$ which has the properties*

(i) $\|v\| = 0 \iff v = 0$

(ii) $\|cv\| = |c|\,\|v\|$ *(positive homogeneity)*

(iii) $\|v+u\| \le \|v\| + \|u\|$ *(triangle inequality)*

is called a norm on V. If V is an algebra, then $\|\cdot\| : V \to [0,\infty)$ is called an algebra norm, provided that (i) - (iii) and

(iv) $\|uv\| \le \|u\|\,\|v\|$

are satisfied. A vector space or an algebra with norm is called a normed vector space or a normed algebra.

Clearly, the absolute value $|\cdot| : \mathbb{R} \to [0,\infty)$ has the properties (i) - (iv) of the preceding definition, hence $|\cdot|$ is an algebra norm on $\mathbb{R}$ and $\mathbb{R}$ is a normed algebra. The preceding theorem shows that the supremum norm $\|\cdot\|_\infty : B(D,\mathbb{R}) \to [0,\infty)$ is an algebra norm on the set $B(D,\mathbb{R})$ of bounded real valued functions, and $B(D,\mathbb{R})$ is a normed algebra.

Definition 1.10 *A sequence of functions* $\{f_n\}_{n=1}^{\infty}$ *from* $B(D,\mathbb{R})$ *is said to converge with respect to the supremum norm to a function* $f \in B(D,\mathbb{R})$ *, if to every* $\varepsilon > 0$ *there is a number* $n_0 \in \mathbb{N}$ *such that*

$$\|f_n - f\|_\infty < \varepsilon$$

for all $n \geq n_0$ *, or, equivalently, if*

$$\lim_{n\to\infty} \|f_n - f\|_\infty = 0\,.$$

Theorem 1.11 *A sequence* $\{f_n\}_{n=1}^{\infty}$ *from* $B(D,\mathbb{R})$ *converges to* $f \in B(D,\mathbb{R})$ *with respect to the supremum norm, if and only if* $\{f_n\}_{n=1}^{\infty}$ *converges uniformly to* f *.*

Proof: $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f , if and only if to every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $x \in D$

$$|f_n(x) - f(x)| \leq \varepsilon\,.$$

This holds if and only if for all $n \geq n_0$

$$\|f_n - f\|_\infty = \sup_{x\in D} |f_n(x) - f(x)| \leq \varepsilon\,,$$

hence if and only if $\{f_n\}_{n=1}^{\infty}$ converges to f with respect to the supremum norm. ■

Definition 1.12 *A sequence* $\{f_n\}_{n=1}^{\infty}$ *of functions from* $B(D,\mathbb{R})$ *is said to be a Cauchy sequence, if to every* $\varepsilon > 0$ *there is* $n_0 \in \mathbb{N}$ *such that*

$$\|f_n - f_m\|_\infty < \varepsilon$$

for all $n, m \geq n_0$ *.*

Theorem 1.13 *A sequence* $\{f_n\}_{n=1}^{\infty}$ *of functions from* $B(D,\mathbb{R})$ *converges uniformly, if and only if it is a Cauchy sequence.*

Proof: If $\{f_n\}_{n=1}^{\infty}$ converges uniformly, then there is a function $f \in B(D,\mathbb{R})$, the limit function, such that $\{\|f_n - f\|_\infty\}_{n=1}^{\infty}$ is a null sequence. Hence to $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n, m \geq n_0$

$$\|f_n - f_m\|_\infty = \|f_n - f + f - f_m\|_\infty \leq \|f_n - f\|_\infty + \|f - f_m\|_\infty < 2\varepsilon\,.$$

This shows that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Conversely, assume that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence. To prove that this sequence converges, we first must identify the limit function. To this end we show that $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence of real numbers for every $x \in D$. For, since $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence, to $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \varepsilon\,,$$

and so $\{f_n(x)\}_{n=1}^{\infty}$ is indeed a Cauchy sequence of real numbers. Since every Cauchy sequence of real numbers converges, we obtain that $\{f_n\}_{n=1}^{\infty}$ converges pointwise with limit function $f : D \to \mathbb{R}$ defined by

$$f(x) = \lim_{n\to\infty} f_n(x)\,.$$

We show that $\{f_n\}_{n=1}^{\infty}$ even converges uniformly to f. For, using again that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence, to $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ with $\|f_n - f_m\|_\infty < \varepsilon$ for $n, m \geq n_0$. Therefore we obtain for $x \in D$ and $n \geq n_0$

$$|f_n(x) - f(x)| = |f_n(x) - \lim_{m\to\infty} f_m(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \varepsilon\,,$$

whence

$$\|f_n - f\|_\infty = \sup_{x\in D} |f_n(x) - f(x)| \leq \varepsilon$$

for $n \geq n_0$, since ε is independent of x. ■

1.4 Uniformly converging series of functions

Let D be a set and let $f_n : D \to \mathbb{R}$ be functions. The series of functions $\sum_{n=1}^{\infty} f_n$ is said to be uniformly convergent, if the sequence $\{\sum_{n=1}^{m} f_n\}_{m=1}^{\infty}$ is uniformly convergent.

Theorem 1.14 (Criterion of Weierstraß) *Let $f_n : D \to \mathbb{R}$ be bounded functions satisfying $\|f_n\|_\infty \leq c_n$, and let $\sum_{n=1}^{\infty} c_n$ be convergent. Then the series of functions $\sum_{n=1}^{\infty} f_n$ converges uniformly.*

Proof: It suffices to show that $\{\sum_{n=1}^{m} f_n\}_{m=1}^{\infty}$ is a Cauchy sequence. Let $\varepsilon > 0$. Since $\sum_{k=1}^{\infty} c_k$ converges, there is $n_0 \in \mathbb{N}$ such that $\left|\sum_{k=n}^{m} c_k\right| = \sum_{k=n}^{m} c_k < \varepsilon$ for all $m \geq n \geq n_0$, whence

$$\|\sum_{k=n}^{m} f_k\|_\infty \leq \sum_{k=n}^{m} \|f_k\|_\infty \leq \sum_{k=n}^{m} c_k < \varepsilon\,,$$

for all $m \geq n \geq n_0$. ■

1.5 Differentiability of the limit function

Let D be a subset of $\mathbb{R}$. We showed that a uniformly convergent sequence $\{f_n\}_{n=1}^{\infty}$ of continuous functions has a continuous limit function $f : D \to \mathbb{R}$. One can ask the question what type of convergence is needed to ensure that a sequence of differentiable functions has a differentiable limit function? Simple examples show that uniform convergence is not sufficient to ensure this. The following is a slightly different question: Assume that $\{f_n\}_{n=1}^{\infty}$ is a uniformly convergent sequence of differentiable functions with limit function f. If f is differentiable, does this imply that the sequence of derivatives $\{f_n'\}_{n=1}^{\infty}$ converges pointwise to f'? Also this need not be true, as is shown by the following example: Let $D = [0,1]$ and let $x \mapsto f_n(x) = \frac{1}{n}x^n : [0,1] \to \mathbb{R}$. The sequence $\{f_n\}_{n=1}^{\infty}$ of differentiable functions converges uniformly to the differentiable limit function $f = 0$. The sequence of derivatives $\{f_n'\}_{n=1}^{\infty} = \{x^{n-1}\}_{n=1}^{\infty}$ does not converge uniformly on $[0,1]$, but it converges pointwise to the limit function

$$g(x) = \begin{cases} 0\,, & 0 \le x < 1 \\ 1\,, & x = 1\,. \end{cases}$$

However, $g \neq f' = 0$.

Our original question is answered by the following

Theorem 1.15 *Let $-\infty < a < b < \infty$ and let $f_n : [a,b] \to \mathbb{R}$ be differentiable functions. If the sequence $\{f_n'\}_{n=1}^{\infty}$ of derivatives converges uniformly and the sequence $\{f_n\}_{n=1}^{\infty}$ converges at least in one point $x_0 \in [a,b]$, then the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to a differentiable limit function $f : [a,b] \to \mathbb{R}$ and*

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

for all $x \in [a,b]$.

This means that under the convergence condition given in this theorem, derivation (which is a limit process) can be interchanged with the limit with respect to n:

$$\big(\lim_{n\to\infty} f_n\big)' = \lim_{n\to\infty} f_n'\,.$$

Proof: First we show that $\{f_n\}_{n=1}^{\infty}$ converges uniformly. Let $\varepsilon > 0$. For $x \in [a,b]$

$$|f_m(x) - f_n(x)| \;\le\; \big|\big(f_m(x) - f_n(x)\big) - \big(f_m(x_0) - f_n(x_0)\big)\big| + |f_m(x_0) - f_n(x_0)|\,. \quad (*)$$

Since $f_m - f_n$ is differentiable, the mean value theorem yields for a suitable z between x_0 and x

$$|\big(f_m(x) - f_n(x)\big) - \big(f_m(x_0) - f_n(x_0)\big)| = |f_m'(z) - f_n'(z)|\,|x - x_0|\,.$$

The sequence of derivatives converges uniformly. Therefore there is $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$

$$|f_m'(z) - f_n'(z)| < \frac{\varepsilon}{2(b-a)}\,,$$

hence

$$|\big(f_m(x) - f_n(x)\big) - \big(f_m(x_0) - f_n(x_0)\big)| \leq \frac{\varepsilon}{2}\,,$$

for all $m, n \geq n_0$ and all $x \in [a, b]$. By assumption the numerical sequence $\{f_n(x_0)\}_{n=1}^{\infty}$ converges, hence there is $n_1 \in \mathbb{N}$ such that for all $m, n \geq n_1$

$$|f_m(x_0) - f_n(x_0)| \leq \frac{\varepsilon}{2}\,.$$

The last two estimates and $(*)$ together yield

$$|f_m(x) - f_n(x)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $m, n \geq n_2 = \max\{n_0, n_1\}$ and all $x \in [a, b]$. This implies that $\{f_n\}_{n=1}^{\infty}$ converges uniformly. The limit function is denoted by f.

Let $c \in [a, b]$ and for $x \in [a, b]$ set

$$F(x) = \begin{cases} \dfrac{f(x) - f(c)}{x - c} - m\,, & x \neq c \\ 0\,, & x = c\,, \end{cases}$$

with $m = \lim_{n\to\infty} f_n'(c)$. The statement of the theorem follows if F is continuous at the point $x = c$, since continuity of F implies that f is differentiable at c with derivative $f'(c) = m = \lim_{n\to\infty} f_n'(c)$. For the proof that F is continuous at c, set

$$F_n(x) = \begin{cases} \dfrac{f_n(x) - f_n(c)}{x - c} - f_n'(c)\,, & x \neq c \\ 0\,, & x = c\,. \end{cases}$$

Obviously $F(x) = \lim_{n\to\infty} F_n(x)$, and since F_n is continuous due to the differentiability of f_n, the continuity of F follows if it can be shown that $\{F_n\}_{n=1}^{\infty}$ converges uniformly. This follows by application of the mean value theorem to the differentiable function $f_m - f_n$:

$$\begin{aligned} F_m(x) - F_n(x) &= \begin{cases} \dfrac{\big(f_m(x) - f_n(x)\big) - \big(f_m(c) - f_n(c)\big)}{x - c} - \big(f_m'(c) - f_n'(c)\big)\,, & x \neq c \\ 0\,, & x = c \end{cases} \\ &= \big(f_m'(z) - f_n'(z)\big) - \big(f_m'(c) - f_n'(c)\big)\,, \end{aligned}$$

for a suitable z between x and c if $x \neq c$, and for $z = c$ if $x = c$. By assumption $\{f_n'\}_{n=1}^{\infty}$ converges uniformly, consequently there is $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$ and all $y \in [a, b]$

$$|f_m'(y) - f_n'(y)| < \varepsilon\,,$$

whence

$$\begin{aligned} |F_m(x) - F_n(x)| &\leq |f_m'(z) - f_n'(z)| + |f_m'(c) - f_n'(c)| \\ &< \varepsilon + \varepsilon = 2\varepsilon\,, \end{aligned}$$

for all $m, n \geq n_0$ and all $x \in [a, b]$. This shows that $\{F_n\}_{n=1}^{\infty}$ converges uniformly and completes the proof. ■

1.6 Power series

Let a numerical sequence $\{a_n\}_{n=1}^{\infty}$ and a real number x_0 be given. For arbitrary $x \in \mathbb{R}$ consider the series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n\,.$$

This series is called a power series. a_n is called the n-th coefficient, x_0 is the center of expansion of the power series. The Taylor series and the series for exp, sin and cos are power series. These examples show that power series are interesting mainly as function series

$$x \mapsto \sum_{n=0}^{\infty} f_n(x)$$

with $f_n(x) = a_n(x - x_0)^n$. First the convergence of power series must be investigated:

Theorem 1.16 *Let*

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

be a power series.

(i) *Suppose first that*

$$a = \overline{\lim_{n \to \infty}} \sqrt[n]{|a_n|} < \infty\,.$$

Then the power series is in case

$$
\begin{aligned}
a = 0 \;:&\qquad \textit{absolutely convergent for all } x \in \mathbb{R}\\
a > 0 \;:&\quad \begin{cases} \textit{absolutely convergent for } |x - x_0| < \frac{1}{a} \\ \textit{convergent or divergent for } |x - x_0| = \frac{1}{a} \\ \textit{divergent for } |x - x_0| > \frac{1}{a}\,. \end{cases}
\end{aligned}
$$

(ii) *If* $\left\{ \sqrt[n]{|a_n|} \right\}_{n=1}^{\infty}$ *is unbounded, then the power series converges only for* $x = x_0$.

Proof: By the root test, the series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges absolutely if

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|\,|x - x_0|^n} = |x - x_0| \,\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = |x - x_0|a < 1\,,$$

and diverges if

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|\,|x - x_0|^n} = |x - x_0|a > 1\,.$$

This proves (i). If $\left\{ \sqrt[n]{|a_n|} \right\}_{n=1}^{\infty}$ is unbounded, then for $x \neq x_0$ also $\left\{ |x - x_0| \sqrt[n]{|a_n|} \right\}_{n=1}^{\infty} = \left\{ \sqrt[n]{|a_n(x - x_0)^n|} \right\}_{n=1}^{\infty}$ is unbounded, hence $\{a_n(x - x_0)^n\}_{n=1}^{\infty}$ is not a null sequence, and consequently $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ diverges. This proves (ii) ■

Definition 1.17 *Let* $a = \overline{\lim}_{n\to\infty} \sqrt[n]{|a_n|}$. *The number*

$$r = \begin{cases} \frac{1}{a}\,, & \textit{if } a \neq 0 \\ \infty\,, & \textit{if } a = 0 \\ 0\,, & \textit{if } \left\{ \sqrt[n]{|a_n|} \right\}_{n=1}^{\infty} \textit{ is unbounded} \end{cases}$$

is called radius of convergence and the open interval

$$(x_0 - r\,,\, x_0 + r) = \left\{ x \in \mathbb{R} \;\middle|\; |x - x_0| < r \right\}$$

is called interval of convergence of the power series

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n\,.$$

Examples

1. The power series

$$\sum_{n=0}^{\infty} x^n\,, \quad \sum_{n=1}^{\infty} \frac{1}{n}\, x^n$$

both have radius of convergence equal to 1. This is evident for the first series. To prove it for the second series, note that

$$\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} e^{\frac{1}{n}\log n} = e^{\lim_{n\to\infty}(\frac{1}{n}\log n)} = e^0 = 1\,,$$

since $\lim_{x\to\infty} \frac{\log x}{x} = 0$, by the rule of de l'Hospital. Thus, the radius of convergence of the second series is given by

$$r = \frac{1}{\overline{\lim}_{n\to\infty} \sqrt[n]{\frac{1}{n}}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{\frac{1}{n}}} = \lim_{n\to\infty} \sqrt[n]{n} = 1 \,.$$

For $x = 1$ both power series diverge, for $x = -1$ the first one diverges, the second one converges.

2. In Analysis I it was proved that the exponential series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges absolutely for all $x \in \mathbb{R}$. (To verify this use the ratio test, for example.) Consequently, the radius of convergence r must be infinite. For, if r would be finite, the exponential series had to diverge for all x with $|x| > r$, which is excluded. (This implies $\frac{1}{r} = \overline{\lim}_{n\to\infty} \sqrt[n]{\frac{1}{n!}} = 0$, by the way.)

Theorem 1.18 *Let* $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ *and* $\sum_{n=0}^{\infty} b_n(x-x_0)^n$ *be power series with radii of convergence* r_1 *and* r_2, *respectively. Then for all* x *with* $|x-x_0| < r = \min(r_1, r_2)$

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n(x-x_0)^n + \sum_{n=0}^{\infty} b_n(x-x_0)^n &= \sum_{n=0}^{\infty} (a_n+b_n)(x-x_0)^n \\
\Big[\sum_{n=0}^{\infty} a_n(x-x_0)^n\Big]\Big[\sum_{n=0}^{\infty} b_n(x-x_0)^n\Big] &= \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} a_k b_{n-k}\Big)(x-x_0)^n \,.
\end{aligned}$$

Proof: The statements follow immediately from the theorems about computing with series and about the Cauchy product of two series. (We note that the radii of convergence of both series on the right are at least equal to r, but can be larger.) ■

Theorem 1.19 *Let* $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ *be a power series with radius of convergence* r. *Then this series converges uniformly in every compact interval* $[x_0 - r_1, x_0 + r_1]$ *with* $0 \le r_1 < r$.

Proof: Let $c_n = |a_n|\, r_1^n$. Then

$$\overline{\lim_{n\to\infty}} \sqrt[n]{c_n} = \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n| r_1^n} = r_1 \frac{1}{r} < 1 \,,$$

whence the root test implies that the series

$$\sum_{n=0}^{\infty} c_n$$

converges. Because of $|a_n(x-x_0)^n| \le |a_n| r_1^n = c_n$ for all x with $|x-x_0| \le r_1$, the Weierstraß criterion (Theorem 1.14) yields that the power series $\sum_{n=0}^\infty a_n(x-x_0)^n$ converges uniformly for $x \in [x_0 - r_1\,,\, x_0 + r_1] = \{y \mid |y - x_0| \le r_1\}$. ■

Corollary 1.20 *Let $\sum_{n=0}^\infty a_n(x-x_0)^n$ be a power series with radius of convergence $r > 0$. Then the function $f : (x_0 - r\,,\, x_0 + r) \to \mathbb{R}$ defined by*

$$f(x) = \sum_{n=0}^\infty a_n(x-x_0)^n$$

is continuous.

Proof: Since $\{x \mapsto \sum_{n=0}^m a_n(x-x_0)^n\}_{m=0}^\infty$ is a sequence of continuous functions, which converges uniformly in every compact interval $[x_0 - r_1\,,\, x_0 + r_1]$ with $r_1 < r$, the limit function f is continuous in each of these intervals. Hence f is continuous in the union

$$(x_0 - r\,,\, x_0 + r) = \bigcup_{0<r_1<r} [x_0 - r_1\,,\, x_0 + r_1]\,.$$ ■

Let

$$f(x) = \sum_{n=0}^\infty a_n(x-x_0)^n$$

be a power series with radius of convergence $r > 0$. Each of the polynomials $f_m(x) = \sum_{n=0}^m a_n(x-x_0)^n$ is differentiable with derivative

$$f_m'(x) = \sum_{n=1}^m n a_n (x-x_0)^{n-1}\,.$$

$\sum_{n=1}^\infty n a_n(x-x_0)^{n-1}$ is a power series, whose radius of convergence r_1 is equal to r. To verify this, note that

$$\sum_{n=1}^\infty n a_n(x-x_0)^{n-1} = \frac{1}{x-x_0}\sum_{n=1}^\infty n a_n (x-x_0)^n\,,$$

and that

$$\overline{\lim_{n\to\infty}}\, \sqrt[n]{|n a_n|} = \lim_{n\to\infty} \sqrt[n]{n}\; \overline{\lim_{n\to\infty}}\, \sqrt[n]{|a_n|} = \overline{\lim_{n\to\infty}}\, \sqrt[n]{|a_n|} = \frac{1}{r}\,,$$

which implies that the series $\sum_{n=1}^\infty n a_n(x-x_0)^{n-1}$ converges for all x with $|x-x_0| < r$ and diverges for all x with $|x-x_0| > r$. By Theorem 1.16 this can only be true if $r_1 = r$.

Thus, Theorem 1.19 implies that the sequence $\{f_m'\}_{m=1}^\infty$ of derivatives converges uniformly in every compact subinterval of the interval of convergence $(x_0 - r, x_0 + r)$.

Consequently, we can use Theorem 1.15 to conclude that the limit function $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ is differentiable with derivative

$$f'(x) = \lim_{m\to\infty} f_m'(x) = \sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1}$$

in all these subintervals. Hence f is differentiable with derivative given by this formula in the interval of convergence $(x_0 - r\,,\, x_0 + r)$, which is the union of these subintervals.

Repeating these arguments we obtain

Theorem 1.21 *Let $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ be a power series with radius of convergence $r > 0$. Then f is infinitely differentiable in the interval of convergence. All the derivatives can be computed termwise:*

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)a_n(x-x_0)^{n-k}\,.$$

Example: In the interval $(0,2]$ the logarithm can be expanded into the power series

$$\log x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n\,.$$

In section 7.4 of the lecture notes to Analysis I we proved that this equation holds true for $\frac{1}{2} \le x \le 2$. To verify that it also holds for $0 < x < \frac{1}{2}$, note that the radius of convergence of the power series on the right is

$$r = \frac{1}{\overline{\lim}_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^{n-1}}{n}\right|}} = \lim_{n\to\infty} \sqrt[n]{n} = 1\,.$$

Hence, this power series converges in the interval of convergence $\{x \mid |x-1| < 1\} = (0,2)$ and represents there an infinitely differentiable function. The derivative of this function is

$$\begin{aligned}\Big[\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n\Big]' &= \sum_{n=1}^{\infty}(-1)^{n-1}(x-1)^{n-1} = \sum_{n=0}^{\infty}(1-x)^n \\ &= \frac{1}{1-(1-x)} = \frac{1}{x} = (\log x)'\,.\end{aligned}$$

Consequently $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$ and $\log x$ both are antiderivatives of $\frac{1}{x}$ in the interval $(0,2)$, and therefore differ at most by a constant:

$$\log x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n + C\,.$$

To determine C, set $x = 1$. From $\log(1) = 0$ we obtain $C = 0$.

Theorem 1.22 (Identity theorem for power series) *Let the radii of convergence r_1 and r_2 of the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ and $\sum_{n=0}^{\infty} b_n(x-x_0)^n$ be greater than zero. Assume that these power series coincide in a neighborhood $U_r(x_0) = \{x \in \mathbb{R} \mid |x - x_0| < r\}$ of x_0 with $r \leq \min(r_1, r_2)$:*

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} b_n(x-x_0)^n$$

for all $x \in U_r(x_0)$. Then $a_n = b_n$ for all $n = 0, 1, 2, \ldots$.

Proof: First choose $x = x_0$, which immediately yields

$$a_0 = b_0 \, .$$

Next let $n \in \mathbb{N} \cup \{0\}$ and assume that $a_k = b_k$ for $0 \leq k \leq n$. It must be shown that $a_{n+1} = b_{n+1}$ holds. From the assumptions of the theorem and from the assumption of the induction it follows that

$$\sum_{k=n+1}^{\infty} a_k(x-x_0)^k = \sum_{k=n+1}^{\infty} b_k(x-x_0)^k \, ,$$

hence

$$(x-x_0)^{n+1} \sum_{k=n+1}^{\infty} a_k(x-x_0)^{k-n-1} = (x-x_0)^{n+1} \sum_{k=n+1}^{\infty} b_k(x-x_0)^{k-n-1}$$

for all $x \in U_r(x_0)$. For x from this neighborhood with $x \neq x_0$ this implies

$$\sum_{k=n+1}^{\infty} a_k(x-x_0)^{k-n-1} = \sum_{k=n+1}^{\infty} b_k(x-x_0)^{k-n-1} \, .$$

The continuity of power series thus implies

$$\begin{aligned} a_{n+1} &= \sum_{k=n+1}^{\infty} a_k(x_0-x_0)^{k-n-1} = \lim_{x \to x_0} \sum_{k=n+1}^{\infty} a_k(x-x_0)^{k-n-1} \\ &= \lim_{x \to x_0} \sum_{k=n+1}^{\infty} b_k(x-x_0)^{k-n-1} = \sum_{k=n+1}^{\infty} b_k(x_0-x_0)^{k-n-1} = b_{n+1} \, . \end{aligned}$$

■

Every power series defines a continuous function in the interval of convergence. Information about continuity of the power series on the boundary of the interval of convergence is provided by the following

Theorem 1.23 *Let $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ be a power series with positive radius of convergence, let $z \in \mathbb{R}$ be a boundary point of the interval of convergence and assume that $\sum_{n=0}^{\infty} a_n(z-x_0)^n$ converges. Then the power series converges uniformly in the interval $[z, x_0]$ (if $z < x_0$), or in the interval $[x_0, z]$ (if $x_0 < z$), respectively.*

A **proof** of this theorem can be found in the book: M. Barner, F. Flohr: Analysis I, p. 317, 318 (in German).

Corollary 1.24 (Abel's limit theorem) *If a power series converges at a point on the boundary of the interval of convergence, then it is continuous at this point. (Niels Hendrick Abel, 1802-1829).*

1.7 Trigonometric functions continued

Since sine is defined by a power series with interval of convergence equal to $\mathbb{R}$,

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

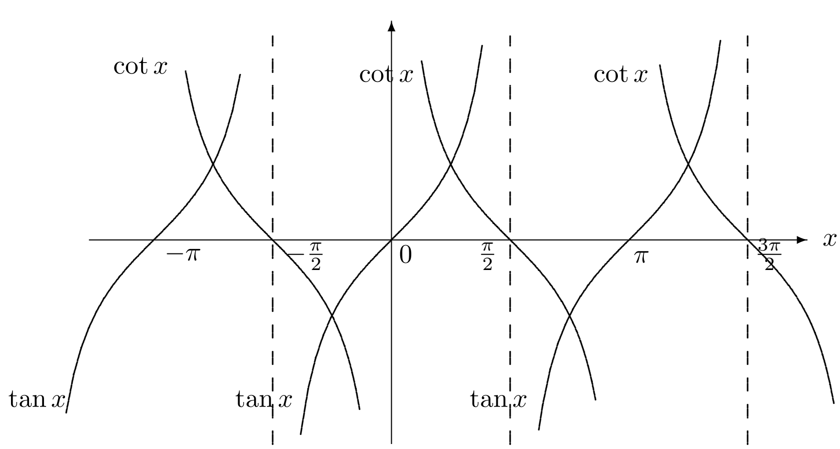
the derivative of sin can be computed by termwise differentiation of the power series, hence

$$\sin' x = \sum_{n=0}^{\infty} (-1)^n (2n+1) \frac{x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x.$$

This result has been proved in Analysis I using the addition theorem for sine.

Tangent and cotangent. One defines

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}.$$



From the addition theorems for sine and cosine addition theorems for tangent and cotangent can be derived:

$$\begin{aligned} \tan(x+y) &= \frac{\tan x + \tan y}{1 - \tan x\ \tan y} \\ \cot(x+y) &= \frac{\cot x\ \cot y - 1}{\cot x + \cot y}\,. \end{aligned}$$

The derivatives are

$$\begin{aligned} \tan' x &= \Big(\frac{\sin x}{\cos x}\Big)' = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \\ \cot' x &= \Big(\frac{\cos x}{\sin x}\Big)' = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x}\,. \end{aligned}$$

Inverse trigonometric functions. sine and cosine are periodic, hence not injective, and consequently do not have inverse functions. However, if sine and cosine are restricted to suitable intervals, inverse functions do exist.

By definition of π, we have $\cos x > 0$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, hence because of $\sin' x = \cos x$, the sine function is strictly increasing in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Consequently, $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ has an inverse function. Moreover, inverse functions also exist to other restrictions of sine:

$$\sin : [\pi(n + \frac{1}{2}), \pi(n + \frac{3}{2})] \to [-1, 1]\,, \quad n \in \mathbb{Z}\,.$$

If one speaks of the inverse function of sine, one has to specify which one of these infinitely many inverses are meant. If no specification is given, the inverse function

$$\arcsin :\ [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$$

of $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ is meant. Because of reasons, which have their origin in the theory of functions of a complex variable, the infinitely many inverse functions

$$x \mapsto (\arcsin x) + 2n\pi\,, \quad n \in \mathbb{Z}$$

and

$$x \mapsto -(\arcsin x) + (2n+1)\pi, \quad n \in \mathbb{Z}$$

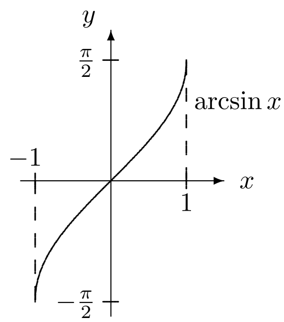
are called *branches of the inverse function of sine* or *branches of arc sine* ("Zweige des Arcussinus"). The function $\arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ is called *principle branch* of the inverse function ("Hauptwert der Umkehrfunktion").

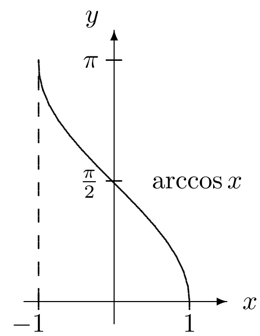
Correspondingly, the inverse function

$$\arccos : [-1, 1] \to [0, \pi]$$

to the function cos : $[0,\pi] \to [-1,1]$ is called principle branch of the inverse function of cosine, but there exist the infinitely many other inverse functions

$$x \to \pm(\arccos x) + 2n\pi, \quad n \in \mathbb{Z}.$$



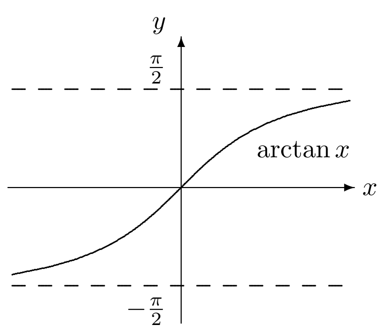


A similar situation arises with tangent and cotangent. The principle branch of the inverse function of tangent is the function

$$\arctan : [-\infty,\infty] \to \Big[-\frac{\pi}{2},\frac{\pi}{2}\Big].$$

One calls this function arc tangent ("Arcustangens"), but there are infinitely many other branches of the inverse function

$$x \mapsto \qquad \arctan x + n\pi, \qquad n \in \mathbb{Z}$$



In the following we consider the principle branches of the inverse functions. For the derivatives one obtains

$$\begin{aligned}
(\arcsin x)' &= \frac{1}{\sin'(\arcsin x)} = \frac{1}{\cos(\arcsin x)} \\
&= \frac{1}{\sqrt{1-(\sin(\arcsin x))^2}} = \frac{1}{\sqrt{1-x^2}} \\
(\arccos x)' &= \frac{1}{\cos'(\arccos x)} = \frac{-1}{\sin(\arccos x)} \\
&= \frac{-1}{\sqrt{1-(\cos(\arccos x))^2}} = \frac{-1}{\sqrt{1-x^2}} \\
(\arctan x)' &= \frac{1}{\tan'(\arctan x)} = \big(\cos(\arctan x)\big)^2 \\
&= \frac{1}{1+(\tan(\arctan x))^2} = \frac{1}{1+x^2}.
\end{aligned}$$

The functions $\arcsin, \arccos$ and $\arctan$ can be expanded into power series. For example,

$$\frac{d}{dt}(\arctan x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n},$$

if $|x| < 1$. Also the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\, x^{2n+1}$$

has radius of convergence equal to 1, and it is an antiderivative of $\sum_{n=0}^{\infty}(-1)^n x^{2n}$, hence

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\, x^{2n+1} + C$$

for $|x| < 1$, with a suitable constant C. From $\arctan 0 = 0$ we obtain $C = 0$, thus

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\, x^{2n+1}$$

for all $x \in \mathbb{R}$ with $|x| < 1$. The convergence criterion of Leibniz shows that the power series on the right converges for $x = 1$, hence Abel's limit theorem implies that the function given by the power series is continuous at 1. Since arctan is continuous, the power series and the function arctan define two continuous extensions of the function arctan from the interval $(-1, 1)$ to $(-1, 1]$. Since the continuous extension is unique, we must have

$$\arctan 1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}\,.$$

Because of

$$\cos(2x) = (\cos x)^2 - (\sin x)^2 = 2(\cos x)^2 - 1,$$

it follows

$$0 = 2\Big(\cos\frac{\pi}{4}\Big)^2 - 1,$$

hence

$$\cos\frac{\pi}{4} = \sqrt{\frac{1}{2}}$$

and

$$\sin\frac{\pi}{4} = \sqrt{1 - (\cos\frac{\pi}{4})^2} = \sqrt{\frac{1}{2}}\,,$$

thus

$$\tan\frac{\pi}{4} = \frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = 1\,.$$

This yields

$$\arctan 1 = \frac{\pi}{4}\,,$$

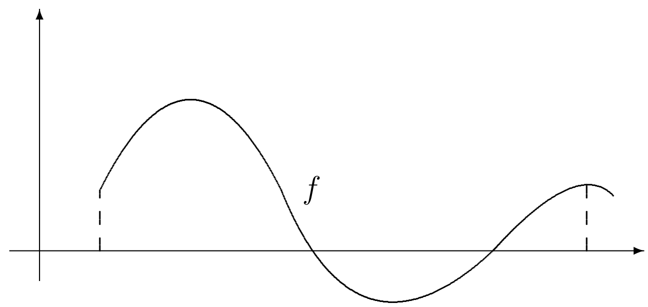
whence

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\,.$$

Theoretically this series allows to compute π, but the convergence is slow.

2 The Riemann integral

For a class of real functions as large as possible one wants to determine the area of the region bounded by the graph of the function and the abscissa. This area is called the integral of the function.



To determine this area might be a difficult task for functions as complicated as the Dirichlet function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \backslash \mathbb{Q}, \end{cases}$$

and in fact, the Riemann integral, which we are going to discuss in this section, is not able to assign a surface area to this function. The Riemann integral was historically the first rigorous notion of an integral. It was introduced by Riemann in his habilitation thesis 1854. Today mathematicians use a more general and advanced integral, the Lebesgue integral, which can assign an integral to the Dirichlet function. The value of the Lebesgue integral of the Dirichlet function is 0. (Bernhard Riemann 1826 – 1866, Henri Lebesgue 1875 – 1941.) Every function, which is Riemann integrable is also Lebesgue integrable and the value of both integrals coincide. However, since the definition of the Riemann integral is simpler and since this integral suffices for many applications, we restrict ourselves here to the study of this integral.

2.1 Definition of the Riemann integral

Let $-\infty < a < b < \infty$ and let $f : [a, b] \to \mathbb{R}$ be a given function. It suggests itself to compute the area of the region below the graph of f by inscribing rectangles into this region. If we refine the subdivision, the total area of these rectangles will converge to the area of the region below the graph of f. It is also possible to cover the region below the graph of f by rectangles. Again, if the subdivision is refined, the total area of these rectangles will converge to the area of the region below the graph of f.

Therefore one expects that in both approximating processes the total areas of the rectangles will converge to the same number. The area of the region below the graph of f is defined to be this number.

Of course, the total areas of the inscribed rectangles and of the covering rectangles will not converge to the same number for all functions f. An example for this is the Dirichlet function.

Those functions f, for which these areas converge to the same number, are called Riemann integrable, and the number is called Riemann integral of f over the interval $[a, b]$.

This program will now be carried through rigorously.

Definition 2.1 *Let* $-\infty < a < b < \infty$ *. A partition* P *of the interval* $[a, b]$ *is a finite set* $\{x_0, \ldots x_n\} \subseteq \mathbb{R}$ *with*

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\,.$$

For brevity we set $\Delta x_i = x_i - x_{x-1}$ $(i = 1, \ldots, n)$ *.*

Let $f : [a, b] \to \mathbb{R}$ be a bounded real function and $P = \{x_0, \ldots, x_n\}$ a partition of $[a, b]$. For $i = 1, \ldots, n$ set

$$\begin{aligned} M_i &= \sup \{f(x) \mid x_{i-1} \le x \le x_i\}\ , \\ m_i &= \inf \{f(x) \mid x_{i-1} \le x \le x_i\}\ , \end{aligned}$$

and define

$$\begin{aligned} U(P, f) &= \sum_{i=1}^{n} M_i \, \Delta x_i \\ L(P, f) &= \sum_{i=1}^{n} m_i \, \Delta x_i\,. \end{aligned}$$

Since f is bounded, there exist numbers m, M such that

$$m \leq f(x) \leq M$$

for all $x \in [a,b]$. This implies $m \leq m_i \leq M_i \leq M$ for all $i = 1, \ldots, n$, hence

$$\begin{aligned} m(b-a) &= \sum_{i=1}^{n} m\,\Delta x_i \leq \sum_{i=1}^{n} m_i\,\Delta x_i = L(P,f) \qquad (*) \\ &\leq \sum_{i=1}^{n} M_i\,\Delta x_i = U(P,f) \leq \sum_{i=1}^{n} M\,\Delta x_i = M(b-a)\,. \end{aligned}$$

Consequently, the infimum and the supremum

$$\begin{aligned} \overline{\int_a^b} f\,dx &= \inf\{U(P,f) \mid P \text{ is a partition of } [a,b]\} \\ \underline{\int_a^b} f\,dx &= \sup\{L(P,f) \mid P \text{ is a partition of } [a,b]\} \end{aligned}$$

exist. The numbers $\overline{\int_a^b} f\,dx$ and $\underline{\int_a^b} f\,dx$ are called upper and lower Darboux integral of f. (Jean Gaston Darboux, 1842 – 1917)

Definition 2.2 *A bounded function $f : [a,b] \to \mathbb{R}$ is called Riemann integrable, if the upper Darboux integral $\overline{\int_a^b} f\,dx$ and the lower Darboux integral $\underline{\int_a^b} f\,dx$ coincide. The common value or the upper and lower Darboux integrals is denoted by*

$$\int_a^b f\,dx \quad \text{or} \quad \int_a^b f(x)\,dx$$

and called the Riemann integral of f. The set of Riemann integrable functions defined on the interval $[a,b]$ is denoted by $\mathcal{R}([a,b])$.

2.2 Criteria for Riemann integrable functions

To work with Riemann integrable functions, one needs simple criteria for a function to be Riemann integrable. In this section we derive such criteria.

Definition 2.3 *Let P, P_1, P_2 and P^* be partitions of $[a,b]$. The partition P^* is called a refinement of P if $P \subseteq P^*$ holds. P^* is called common refinement of P_1 and P_2 if $P^* = P_1 \cup P_2$.*

Theorem 2.4 *Let $f : [a, b] \to \mathbb{R}$ and let P^* be a refinement of the partition P of $[a, b]$. Then*

$$\begin{aligned} L(P, f) &\leq L(P^*, f) \\ U(P^*, f) &\leq U(P, f). \end{aligned}$$

Proof: Let $P = \{x_0, \ldots, x_n\}$ and assume first that P^* contains exactly one point x^* more than P. Then there are $x_{j-1}, x_j \in P$ with $x_{j-1} < x^* < x_j$. Let

$$\begin{aligned} w_1 &= \inf\{f(x) \,\big|\, x_{j-1} \leq x \leq x^*\}, \\ w_2 &= \inf\{f(x) \,\big|\, x^* \leq x \leq x_j\}, \end{aligned}$$

and for $i = 1, \ldots, n$

$$m_i = \inf\{f(x) \,\big|\, x_{i-1} \leq x \leq x_i\}.$$

Then $w_1, w_2 \geq m_j$, hence

$$\begin{aligned} L(P, f) &= \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{j-1} m_i \Delta x_i \\ &\quad + m_j(x^* - x_{j-1} + x_j - x^*) + \sum_{i=j+1}^{n} m_i \Delta x_i \\ &\leq \sum_{i=1}^{j-1} m_i \Delta x_i + w_1(x^* - x_{j-1}) + w_2(x_j - x^*) + \sum_{i=j+1}^{n} m_i \Delta x_i \\ &= L(P^*, f). \end{aligned}$$

By induction we conclude that $L(P, f) \leq L(P^*, f)$ holds if P^* contains k points more than P for any k. The second inequality stated in the theorem is proved analogously. ■

Theorem 2.5 *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then*

$$\underline{\int_a^b} f\,dx \leq \overline{\int_a^b} f\,dx.$$

Proof: Let P_1 and P_2 be partitions and let P^* be the common refinement. Inequality $(*)$ proved above shows that

$$L(P^*, f) \leq U(P^*, f).$$

Combination of this inequality with the preceding theorem yields

$$L(P_1, f) \leq L(P^*, f) \leq U(P^*, f) \leq U(P_2, f),$$

whence

$$L(P_1, f) \leq U(P_2, f)$$

for all partitions P_1 and P_2 of $[a, b]$. Therefore $U(P_2, f)$ is an upper bound of the set

$$\{L(P, f) \mid P \text{ is a partition of } [a, b]\},$$

hence the least upper bound $\underline{\int_a^b} f\,dx$ of this set satisfies

$$\underline{\int_a^b} f\,dx \leq U(P_2, f).$$

Since this inequality holds for every partition P_2 of $[a, b]$, it follows that $\underline{\int_a^b} f\,dx$ is a lower bound of the set

$$\{U(P, f) \mid P \text{ is a partition of } [a, b]\},$$

hence the greatest lower bound of this set satisfies

$$\underline{\int_a^b} f\,dx \leq \overline{\int_a^b} f\,dx.$$

■

Theorem 2.6 *Let $f : [a, b] \to \mathbb{R}$ be bounded. The function f belongs to $\mathcal{R}([a, b])$ if and only if to every $\varepsilon > 0$ there is a partition P of $[a, b]$ such that*

$$U(P, f) - L(P, f) < \varepsilon.$$

Proof: First assume that to every $\varepsilon > 0$ there is a partition P with $U(P, f) - L(P, f) < \varepsilon$. Since

$$L(P, f) \leq \underline{\int_a^b} f\,dx \leq \overline{\int_a^b} f\,dx \leq U(P, f),$$

it follows that

$$0 \leq \overline{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx \leq U(P, f) - L(P, f) < \varepsilon,$$

hence

$$0 \leq \overline{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx < \varepsilon$$

for every $\varepsilon > 0$. This implies

$$\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx,$$

thus $f \in \mathcal{R}([a, b])$.

Conversely, let $f \in \mathcal{R}([a,b])$. By definition of the infimum and the supremum to every $\varepsilon > 0$ there are partitions P_1 and P_2 with

$$\begin{aligned} \int_a^b f\,dx &= \overline{\int_a^b} f\,dx \le U(P_1,f) < \overline{\int_a^b} f\,dx + \frac{\varepsilon}{2} \\ \int_a^b f\,dx &= \underline{\int_a^b} f\,dx \ge L(P_2,f) > \underline{\int_a^b} f\,dx - \frac{\varepsilon}{2}. \end{aligned}$$

Let P be the common refinement of P_1 and P_2. Then

$$\int_a^b f\,dx - \frac{\varepsilon}{2} < L(P,f) \le \int_a^b f\,dx \le U(P,f) < \int_a^b f\,dx + \frac{\varepsilon}{2},$$

hence

$$U(P,f) - L(P,f) < \varepsilon.$$

■

From this theorem we can conclude that $C([a,b]) \subseteq \mathcal{R}([a,b])$:

Theorem 2.7 *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then f is Riemann integrable. Furthermore, to every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\Big| \sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f\,dx \Big| < \varepsilon$$

for every partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ with

$$\max\{\Delta x_1, \ldots, \Delta x_n\} < \delta$$

and for every choice of points $t_1, \ldots, t_n$ with $t_i \in [x_{i-1}, x_i]$.

Note that if $\{P_j\}_{j=1}^{\infty}$ is a sequence of partitions $P_j = \{x_0^{(j)} = a,\ x_1^{(j)}, \ldots, x_{n_j}^{(j)} = b\}$ of $[a,b]$ with

$$\lim_{j\to\infty} \max\{\Delta x_1^{(j)}, \ldots, \Delta x_{n_j}^{(j)}\} = 0$$

and if $t_i^{(j)} \in [x_{i-1}^{(j)}, x_i^{(j)}]$, then this theorem implies

$$\int_a^b f\,dx = \lim_{j\to\infty} \sum_{i=1}^{n_j} f(t_i^{(j)})\Delta x_i^{(j)}.$$

The integral is the limit of the *Riemann sums* $\sum_{i=1}^{n} f(t_i)\Delta x_i$.

Proof: Let $\varepsilon > 0$. We set

$$\eta = \frac{\varepsilon}{b-a}.$$

As a continuous function on the compact interval $[a,b]$, the function f is bounded and uniformly continuous (cf. Theorem 6.43 of the lecture notes to the Analysis I course). Therefore there exists $\delta > 0$ such that for all $x, t \in [a,b]$ with $|x - t| < \delta$

$$|f(x) - f(t)| < \eta. \qquad (*)$$

We choose a partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ with $\max\{\Delta x_1, \ldots, \Delta x_n\} < \delta$. Then $(*)$ implies for all $x, t \in [x_{i-1}, x_i]$

$$f(x) - f(t) < \eta,$$

hence

$$\begin{aligned} M_i - m_i &= \sup_{x_{i-1} \le x \le x_i} f(x) - \inf_{x_{i-1} \le t \le x_i} f(t) \\ &= \max_{x_{i-1} \le x \le x_i} f(x) - \min_{x_{i-1} \le t \le x_i} f(t) \\ &= f(x_0) - f(t_0) < \eta, \end{aligned}$$

for suitable $x_0, t_0 \in [x_{i-1}, x_i]$. This yields

$$\begin{aligned} U(P,f) - L(P,f) &= \sum_{i=1}^{n} (M_i - m_i)\Delta x_i < \eta \sum_{i=1}^{n} \Delta x_i \\ &= \eta(b-a) = \varepsilon. \end{aligned} \qquad (**)$$

Since $\varepsilon > 0$ was arbitrary, the preceding theorem implies $f \in \mathcal{R}([a,b])$. From $(**)$ and from the inequalities

$$\begin{aligned} L(P,f) &= \sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} f(t_i)\Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i \le U(P,f) \\ L(P,f) &\le \int_a^b f\,dx \le U(P,f) \end{aligned}$$

we infer that

$$\Big| \int_a^b f\,dx - \sum_{i=1}^{n} f(t_i)\Delta x_i \Big| < \varepsilon.$$

■

Also the class of monotone functions is a subset of $\mathcal{R}([a,b])$:

Theorem 2.8 *Let $f : [a,b] \to \mathbb{R}$ be monotone. Then f is Riemann integrable.*

Proof: Assume that f is increasing. f is bounded because of $f(a) \le f(x) \le f(b)$ for all $x \in [a,b]$. Let $\varepsilon > 0$. To arbitrary $n \in \mathbb{N}$ set

$$x_i = a + \frac{b-a}{n}\, i,$$

for $i = 0, 1, \ldots, n$. Then $P = \{x_0, \ldots, x_n\}$ is a partition of $[a, b]$, and since f is increasing we obtain

$$\begin{aligned} m_i &= \inf_{x_{i-1} \le x \le x_i} f(x) &= f(x_{i-1}) \\ M_i &= \sup_{x_{i-1} \le x \le x_i} f(x) &= f(x_i), \end{aligned}$$

thence

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i$$
$$= \sum_{i=1}^{n} \Big(f(x_i) - f(x_{i-1})\Big)\frac{b-a}{n} = \Big(f(b) - f(a)\Big)\frac{b-a}{n} < \varepsilon,$$

where the last inequality sign holds if $n \in \mathbb{N}$ is chosen sufficiently large. By Theorem 2.6, this inequality shows that $f \in \mathcal{R}([a, b])$.

For decreasing f the proof is analogous. ■

Example: Let $-\infty < a < b < \infty$. The function $\exp : [a, b] \to \mathbb{R}$ is continuous and therefore Riemann integrable. The value of the integral is

$$\int_a^b e^x dx = e^b - e^a.$$

To verify this equation we use Theorem 2.7. For every $n \in \mathbb{N}$ and all $i = 0, 1, \ldots, n$ we set $x_i^{(n)} = a + \frac{i}{n}(b - a)$. Then $\{P_n\}_{n=1}^{\infty}$ with $P_n = \{x_0^{(n)}, \ldots, x_n^{(n)}\}$ is a sequence of partitions of $[a, b]$ satisfying

$$\lim_{n\to\infty} \max\{\Delta x_1^{(n)}, \ldots, \Delta x_n^{(n)}\} = \lim_{n\to\infty} \frac{b-a}{n} = 0.$$

Thus, with $t_i^{(n)} = x_{i-1}^{(n)}$ we obtain

$$\begin{aligned} \int_a^b e^x dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \exp(t_i^{(n)})\Delta x_i^{(n)} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \exp\Big(a + \frac{i-1}{n}(b-a)\Big)\frac{b-a}{n} \\ &= \lim_{n\to\infty} e^a \frac{b-a}{n} \sum_{i=1}^{n} \Big[e^{(b-a)/n}\Big]^{i-1} \\ &= e^a \lim_{n\to\infty} \frac{b-a}{n} \frac{[e^{(b-a)/n}]^n - 1}{e^{(b-a)/n} - 1} \\ &= \frac{e^a(e^{b-a} - 1)}{\lim_{n\to\infty} \frac{e^{(b-a)/n} - 1}{(b-a)/n}} = e^b - e^a, \end{aligned}$$

since $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$, by the rule of de l'Hospital.

2.3 Simple properties of the integral

Theorem 2.9 (i) *If $f_1, f_2 \in \mathcal{R}([a,b])$, then $f_1 + f_2 \in \mathcal{R}([a,b])$, and*

$$\int_a^b (f_1 + f_2)dx = \int_a^b f_1\,dx + \int_a^b f_2\,dx.$$

If $g \in \mathcal{R}([a,b])$ and $c \in \mathbb{R}$, then $cg \in \mathcal{R}([a,b])$ and

$$\int_a^b cg\,dx = c\int_a^b g\,dx.$$

Hence $\mathcal{R}([a,b])$ is a vector space.
(ii) *If $f_1, f_2 \in \mathcal{R}([a,b])$ and $f_1(x) \leq f_2(x)$ for all $x \in [a,b]$, then*

$$\int_a^b f_1\,dx \leq \int_a^b f_2\,dx.$$

(iii) *If $f \in \mathcal{R}([a,b])$ and if $a < c < b$, then*

$$f\big|_{[a,c]} \in \mathcal{R}([a,b]), \quad f\big|_{[c,b]} \in \mathcal{R}([a,b])$$

and

$$\int_a^c f\,dx + \int_c^b f\,dx = \int_a^b f\,dx.$$

(iv) *If $f \in \mathcal{R}([a,b])$ and $|f(x)| \leq M$ for all $x \in [a,b]$, then*

$$\Big|\int_a^b f\,dx\Big| \leq M(b-a).$$

Proof: (i) Let $f = f_1 + f_2$ and let P be a partition of $[a,b]$. Then

$$\begin{aligned}
\inf_{x_{i-1}\leq x\leq x_i} f(x) &= \inf_{x_{i-1}\leq x\leq x_i} \Big(f_1(x) + f_2(x)\Big) \\
&\geq \inf_{x_{i-1}\leq x\leq x_i} f_1(x) + \inf_{x_{i-1}\leq x\leq x_i} f_2(x) \\
\sup_{x_{-1}\leq x\leq x_i} f(x) &= \sup_{x_{i-1}\leq x\leq x_i} \Big(f_1(x) + f_2(x)\Big) \\
&\leq \sup_{x_{i-1}\leq x\leq x_i} f_1(x) + \sup_{x_{i-1}\leq x\leq x_i} f_2(x),
\end{aligned}$$

hence

$$\begin{aligned}
L(P, f_1) + L(P, f_2) \leq L(P, f) \\
U(P, f) \leq U(P, f_1) + U(P, f_2).
\end{aligned} \qquad (*)$$

Let $\varepsilon > 0$. Since f_1 and f_2 are Riemann integrable, there exist partitions P_1 and P_2 such that for $j = 1, 2$

$$U(P_j, f_j) - L(P_j, f_j) < \varepsilon.$$

For the common refinement P of P_1 and P_2 we have $L(P_j, f_j) \leq L(P, f_j)$ and $U(P, f_j) \leq U(P_j, f_j)$, hence, for $j = 1, 2$,

$$U(P, f_j) - L(P, f_j) < \varepsilon. \tag{$**$}$$

From this inequality and from $(*)$ we obtain

$$U(P, f) - L(P, f) \leq U(P, f_1) + U(P, f_2) - L(P, f_1) - L(P, f_2) < 2\varepsilon.$$

Since $\varepsilon > 0$ was chosen arbitrarily, this inequality and Theorem 2.6 imply $f = f_1 + f_2 \in \mathcal{R}([a, b])$.

From $(**)$ we also obtain

$$U(P, f_j) < L(P, f_j) + \varepsilon \leq \int_a^b f_j \, dx + \varepsilon \,,$$

whence, observing $(*)$

$$\begin{aligned} \int_a^b f \, dx &\leq U(P, f) \leq U(P, f_1) + U(P, f_2) \\ &\leq \int_a^b f_1 \, dx + \int_a^b f_2 \, dx + 2\varepsilon \,. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this yields

$$\int_a^b f \, dx \leq \int_a^b f_1 \, dx + \int_a^b f_2 \, dx \,. \tag{$***$}$$

Similarly, $(**)$ yields

$$L(P, f_j) > U(P, f_j) - \varepsilon \geq \int_a^b f \, dx - \varepsilon \,,$$

which together with $(*)$ results in

$$\begin{aligned} \int_a^b f \, dx &\geq L(P, f) \geq L(P, f_1) + L(P, f_2) \\ &\geq \int_a^b f_1 \, dx + \int_a^b f_2 \, dx - 2\varepsilon \,, \end{aligned}$$

from which we conclude that

$$\int_a^b f \, dx \geq \int_a^b f_1 \, dx + \int_a^b f_2 \, dx \,.$$

This inequality and $(***)$ yield

$$\int_a^b f\,dx = \int_a^b f_1\,dx + \int_a^b f_2\,dx\,.$$

To prove that $cg \in \mathcal{R}([a,b])$ we note that the definition of $L(P,cg)$ immediately yields for every partition P of $[a,b]$

$$L(P,cg) = \begin{cases} cL(P,g)\,, & \text{if } c \geq 0 \\ cU(P,g)\,, & \text{if } c < 0\,. \end{cases}$$

Thus, for $c \geq 0$

$$\begin{aligned} \underline{\int_a^b} cg\,dx &= \sup\left\{cL(P,g)\;\middle|\; P \text{ is a partition of } [a,b]\right\} \\ &= c\sup\left\{L(P,g)\;\middle|\; P \text{ is a partition of } [a,b]\right\} = c\underline{\int_a^b} g\,dx = c\int_a^b g\,dx\,, \end{aligned}$$

and for $c < 0$

$$\begin{aligned} \underline{\int_a^b} cg\,dx &= \sup\left\{cU(P,g)\;\middle|\; P \text{ is a partition of } [a,b]\right\} \\ &= c\inf\left\{U(P,g)\;\middle|\; P \text{ is a partition of } [a,b]\right\} = c\overline{\int_a^b} g\,dx = c\int_a^b g\,dx\,. \end{aligned}$$

In the same manner

$$\overline{\int_a^b} cg\,dx = c\int_a^b g\,dx\,.$$

Therefore

$$\underline{\int_a^b} cg\,dx = c\int_a^b g\,dx = \overline{\int_a^b} cg\,dx\,,$$

which implies $cg \in \mathcal{R}([a,b])$ and $\int_a^b cg\,dx = c\int_a^b g\,dx$.

This completes the proof of (i). The proof of (ii) is left as an exercise. To prove (iii), note first that to any partition P of $[a,b]$ we can define a refinement P^* by

$$P^* = P \cup \{c\}\,.$$

Theorem 2.4 implies

$$L(P,f) \leq L(P^*,f), \quad U(P^*,f) \leq U(P,f)\,. \tag{$*$}$$

From P^* we obtain partitions P_-^* of $[a,c]$ and P_+^* of $[c,b]$ by setting $P_-^* = P^* \cap [a,c]$ and $P_+^* = P^* \cap [c,b]$, and if $P^* = \{x_0, \ldots, x_n\}$ with $x_j = c$, then

$$L(P^*, f) = \sum_{i=1}^{n} m_i \, \Delta x_i = \sum_{i=1}^{j} m_i \, \Delta x_i + \sum_{i=j+1}^{n} m_i \, \Delta x_i = L(P_-^*, f) + L(P_+^*, f) \, .$$

Here for simplicity we wrote $L(P_-^*, f)$ instead of $L(P_-^*, f|_{[a,c]})$ and $U(P_-^*, f)$ instead of $U(P_-^*, f|_{[c,b]})$. Similarly

$$U(P^*, f) = U(P_-^*, f) + U(P_+^*, f) \, .$$

From $(*)$ and from these equations we conclude

$$\begin{aligned} L(P,f) &\le L(P_-^*, f) + L(P_+^*, f) \le \underline{\int_a^c} f \, dx + \underline{\int_c^b} f \, dx \\ U(P,f) &\ge U(P_-^*, f) + U(P_+^*, f) \ge \overline{\int_a^c} f \, dx + \overline{\int_c^b} f \, dx \, . \end{aligned}$$

These estimates hold for any partition P of $[a,b]$, whence

$$\begin{aligned} \int_a^b f \, dx &= \underline{\int_a^b} f \, dx \le \underline{\int_a^c} f \, dx + \underline{\int_c^b} f \, dx \\ \int_a^b f \, dx &= \overline{\int_a^b} f \, dx \ge \overline{\int_a^c} f \, dx + \overline{\int_c^b} f \, dx \, . \end{aligned}$$

Since $\underline{\int_a^c} f \, dx \le \overline{\int_a^c} f \, dx$ and $\underline{\int_c^b} f \, dx \le \overline{\int_c^b} f \, dx$, these inequalities can only hold if

$$\underline{\int_a^c} f \, dx = \overline{\int_a^c} f \, dx \, , \quad \underline{\int_c^b} f \, dx = \overline{\int_c^b} f \, dx \, ,$$

hence $f|_{[a,c]} \in \mathcal{R}([a,c])$, $f|_{[c,b]} \in \mathcal{R}([c,b])$, and

$$\int_a^c f \, dx + \int_c^b f \, dx = \int_a^b f \, dx \, .$$

This proves (iii). The obvious proof of (iv) is left as an exercise. ■

Theorem 2.10 *Let $-\infty < m < M < \infty$ and $f \in \mathcal{R}([a,b])$ with $f : [a,b] \to [m,M]$. Let $\Phi : [m,M] \to \mathbb{R}$ be continuous and let $h = \Phi \circ f$. Then $h \in \mathcal{R}([a,b])$.*

Proof: Let $\varepsilon > 0$. Since Φ is uniformly continuous on $[m,M]$, there is a number $\delta > 0$ such that for all $s,t \in [m,M]$ with $|s-t| \le \delta$

$$|\Phi(s) - \Phi(t)| < \varepsilon \, .$$

Moreover, since $f \in \mathcal{R}([a,b])$ there is a partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ such that

$$U(P,f) - L(P,f) < \varepsilon\delta\,. \tag{$*$}$$

Let

$$\begin{aligned} M_i &= \sup_{x_{i-1}\le x\le x_i} f(x)\,, & m_i &= \inf_{x_{i-1}\le x\le x_i} f(x) \\ M_i^* &= \sup_{x_{i-1}\le x\le x_i} h(x)\,, & m_i^* &= \inf_{x_{i-1}\le x\le x_i} h(x) \end{aligned}$$

and

$$\begin{aligned} A &= \bigl\{ i \bigm| i \in \mathbb{N}\,, \quad 1 \le i \le n\,, \quad M_i - m_i < \delta \bigr\} \\ B &= \{1, \ldots, n\} \setminus A\,. \end{aligned}$$

If $i \in A$, then for all x, y with $x_{i-1} \le x, y \le x_i$

$$|h(x) - h(y)| = |\Phi\bigl(f(x)\bigr) - \Phi\bigl(f(y)\bigr)| < \varepsilon\,,$$

since $|f(x) - f(y)| \le M_i - m_i < \delta$. This yields for $i \in A$

$$M_i^* - m_i^* \le \varepsilon\,.$$

If $i \in B$, then

$$M_i^* - m_i^* \le 2\|\Phi\|\,,$$

with the supremum norm $\|\Phi\| = \sup_{m\le t\le M} |\Phi(t)|$. Furthermore, $(*)$ yields

$$\delta \sum_{i\in B} \Delta x_i \le \sum_{i\in B} (M_i - m_i)\Delta x_i \le \sum_{i=1}^{n} (M_i - m_i)\Delta x_i = U(P,f) - L(P,f) < \varepsilon\delta\,,$$

whence

$$\sum_{i\in B} \Delta x_i \le \varepsilon\,.$$

Together we obtain

$$\begin{aligned} U(P,h) - L(P,h) &= \sum_{i\in A} (M_i^* - m_i^*)\Delta x_i + \sum_{i\in B} (M_i^* - m_i^*)\Delta x_i \\ &\le \varepsilon \sum_{i\in A} \Delta x_i + 2\|\Phi\| \sum_{i\in B} \Delta x_i \\ &\le \varepsilon(b-a) + 2\|\Phi\|\varepsilon = \varepsilon(b - a + 2\|\Phi\|). \end{aligned}$$

Since ε was chosen arbitrarily, we conclude from this inequality that $h \in \mathcal{R}([a,b])$, using Theorem 2.6. ∎

Corollary 2.11 *Let $f, g \in \mathcal{R}([a,b])$. Then*
(i) $fg \in \mathcal{R}([a,b])$,
(ii) $|f| \in \mathcal{R}([a,b])$ *and* $\left|\int_a^b f\,dx\right| \leq \int_a^b |f|\,dx.$

Proof: (i) Setting $\Phi(t) = t^2$ in the preceding theorem yields $f^2 = \Phi \circ f \in \mathcal{R}([a,b])$. From

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$$

we conclude with this result that also $fg \in \mathcal{R}([a,b])$.
(ii) Setting $\Phi(t) = |t|$ in the preceding theorem yields $|f| = \Phi \circ f \in \mathcal{R}([a,b])$. Choose $c = \pm 1$ such that

$$c\int_a^b f\,dx \geq 0.$$

Then

$$\left|\int_a^b f\,dx\right| = c\int_a^b f\,dx = \int_a^b cf\,dx \leq \int_a^b |f|dx,$$

since $cf(x) \leq |f(x)|$ for all $x \in [a,b]$. ■

2.4 Fundamental theorem of calculus

Let $-\infty < a < b < \infty$ and $f \in \mathcal{R}\big([a,b]\big)$. One defines

$$\int_b^a f\,dx = -\int_a^b f\,dx\,.$$

Then

$$\int_u^v f\,dx + \int_v^w f\,dx = \int_u^w f\,dx\,,$$

if u, v, w are arbitrary points of $[a,b]$.

Theorem 2.12 (Mean value theorem of integration) *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then there is a point c with $a \leq c \leq b$ such that*

$$\int_a^b f\,dx = f(c)(b-a)\,.$$

Proof: f is Riemann integrable, since f is continuous. Since the integral is monotone, we obtain

$$\begin{aligned}(b-a)\min_{x\in[a,b]} f(x) &= \int_a^b \min_{y\in[a,b]} f(y)\,dx \leq \int_a^b f(x)\,dx \\ &\leq \int_a^b \max_{y\in[a,b]} f(y)\,dx = \max_{x\in[a,b]} f(x)(b-a)\,.\end{aligned}$$

Since f attains the minimum and the maximum on $[a,b]$, by the intermediate value theorem there exists a number $c \in [a,b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f\,dx\,.$$

■

Theorem 2.13 *Let $f \in \mathcal{R}\big([a,b]\big)$. Then*

$$F(x) = \int_a^x f(t)\,dt$$

defines a continuous function $F : [a,b] \to \mathbb{R}$.

Proof: There is M with $|f(x)| \le M$ for all $x \in [a,b]$. Thus, for $x, x_0 \in [a,b]$ with $x_0 < x$

$$\Big|F(x) - F(x_0)\Big| = \Big|\int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt\Big| = \Big|\int_{x_0}^x f(t)\,dt\Big| \le M(x - x_0)\,.$$

This estimate implies that F is continuous on $[a,b]$. ■

Theorem 2.14 *Let $f \in \mathcal{R}\big([a,b]\big)$ be continuous. Then the function $F : [a,b] \to \mathbb{R}$ defined by*

$$F(x) = \int_a^x f(t)\,dt$$

is continuously differentiable with

$$F' = f\,.$$

Therefore F is an antiderivative of f.

Proof: Let $x_0 \in [a,b]$. The mean value theorem of integration implies

$$\begin{aligned}
\lim_{x\to x_0} \frac{F(x) - F(x_0)}{x - x_0} &= \lim_{x\to x_0} \frac{1}{x - x_0}\Big[\int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt\Big] \\
&= \lim_{x\to x_0} \frac{1}{x - x_0} \int_{x_0}^x f(t)\,dt = \lim_{x\to x_0} \frac{1}{x - x_0}\, f(y)(x - x_0) \\
&= \lim_{x\to x_0} f(y) = f(x_0)\,,
\end{aligned}$$

for suitable y between x_0 and x. Therefore F is differentiable with $F' = f$. Since f is continuous by assumption, F is continuously differentiable. ■

Theorem 2.15 (Fundamental theorem of calculus) *Let F be an antiderivative of the continuous function $f : [a,b] \to \mathbb{R}$. Then*

$$\int_a^b f(t)\,dt = F(b) - F(a) = F(x)\Big|_a^b\,.$$

Proof: The functions $x \mapsto \int_a^x f(t)\,dt$ and F both are antiderivatives of f. Since two antiderivatives differ at most by a constant c, we obtain

$$F(x) = \int_a^x f(t)\,dt + c$$

for all $x \in [a, b]$. This implies $c = F(a)$, whence $F(b) - F(a) = \int_a^b f(t)\,dt$. ■

This theorem is so important because it simplifies the otherwise so tedious computation of integrals.

Examples. 1.) Let $0 < a < b$ and $c \in \mathbb{R}$, $c \neq -1$. Then

$$\int_a^b x^c\,dx = \frac{1}{c+1}\, x^{c+1}\Big|_a^b .$$

For $c < -1$ one obtains

$$\lim_{m\to\infty} \int_a^m x^c\,dx = \lim_{m\to\infty} \frac{1}{c+1}\, m^{c+1} - \frac{1}{c+1}\, a^{c+1} = -\frac{1}{c+1}\, a^{c+1} .$$

Therefore one defines for $a > 0$ and $c < -1$

$$\int_a^\infty x^c\,dx := \lim_{m\to\infty} \int_a^m x^c\,dx = -\frac{1}{c+1}\, a^{c+1} .$$

The integral $\int_a^\infty x^c\,dx$ is called improper Riemann integral and one says that for $c < -1$ the function $x \mapsto x^c$ is improperly Riemann integrable over the interval $[a, \infty)$ with $a > 0$. In particular, one obtains

$$\int_1^\infty x^{-2}\,dx = 1 .$$

For $c < 0$ the function $x \mapsto x^c$ is not defined at $x = 0$ and unbounded on every interval $(0, b]$ with $b > 0$. Therefore the Riemann integral $\int_0^b x^c\,dx$ is not defined. However, for $-1 < c < 0$ one obtains

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_\varepsilon^b x^c\,dx = \frac{1}{c+1}\, b^{c+1} - \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{1}{c+1}\, \varepsilon^{c+1} = \frac{1}{c+1}\, b^{c+1} .$$

Therefore the improper Riemann integral

$$\int_0^b x^c\,dx := \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \int_\varepsilon^b x^c\,dx = \frac{1}{c+1}\, b^{c+1}$$

is defined, x^c is improperly Riemann integrable over $(0, b]$ for $-1 < c < 0$ and $b > 0$. In particular, one obtains

$$\int_0^1 x^{-\frac{1}{2}}\,dx = 2 .$$

2.) For $0 < a < b < \infty$

$$\int_a^b \frac{1}{x}\, dx = \log b - \log a \,.$$

Neither of the limits $\lim_{b\to\infty} \int_a^b \frac{1}{x}\, dx$, $\lim_{a\to 0} \int_a^b \frac{1}{x}\, dx$ exists, so x^{-1} is not improperly Riemann integrable over $[a, \infty)$ or $(0, b]$.

3.) Let $-1 < a < b < 1$. Then

$$\int_a^b \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin b - \arcsin a \,.$$

One defines

$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\, dx = \lim_{\substack{b\to 1\\ b<1}} \lim_{\substack{a\to -1\\ a>-1}} \int_a^b \frac{1}{\sqrt{1-x^2}}\, dx$$

$$= \lim_{\substack{b\to 1\\ b<1}} \arcsin b - \lim_{\substack{a\to -1\\ a>-1}} \arcsin a = \frac{\pi}{2} - \Big(-\frac{\pi}{2}\Big) = \pi \,.$$

$\frac{1}{\sqrt{1-x^2}}$ is improperly Riemann integrable over the interval $(-1, 1)$.

Theorem 2.16 (Substitution) *Let f be continuous, let $g : [a, b] \to \mathbb{R}$ be continuously differentiable and let the composition $f \circ g$ be defined. Then*

$$\int_a^b f\big(g(t)\big)\, g'(t)\, dt = \int_{g(a)}^{g(b)} f(x)\, dx \,.$$

Proof: Since g is a continuous function defined on a compact interval, the range of g is a compact interval $[c, d]$. Therefore we can restrict f to this interval. As a continuous function, $f : [c, d] \to \mathbb{R}$ is Riemann integrable, hence has an antiderivative $F : [c, d] \to \mathbb{R}$. The chain rule implies

$$(F \circ g)' = (F' \circ g) \cdot g' = (f \circ g) \cdot g' \,,$$

whence

$$F\big(g(b)\big) - F\big(g(a)\big) = \int_a^b f\big(g(t)\big)\, g'(t)\, dt \,.$$

Combination of this equation with

$$F\big(g(b)\big) - F\big(g(a)\big) = \int_{g(a)}^{g(b)} f(x)\, dx$$

yields the statement. ∎

Remark: If g^{-1} exists, the rule of substitution can be written in the form

$$\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f\big(g(t)\big)\,g'(t)\,dt\,.$$

Example. We want to compute $\int_0^1 \sqrt{1-x^2}\,dx$. With the substitution $x = x(t) = \cos t$ it follows because of the invertibility of cosine on the interval $[0, \frac{\pi}{2}]$ that

$$\begin{aligned}
\int_0^1 \sqrt{1-x^2}\,dx &= \int_{x^{-1}(0)}^{x^{-1}(1)} \sqrt{1-x(t)^2}\,\frac{dx(t)}{dt}\,dt \\
&= \int_{\frac{\pi}{2}}^{0} \sqrt{1-(\cos t)^2}\,(-\sin t)\,dt = \int_0^{\pi/2} (\sin t)^2\,dt \\
&= \int_0^{\pi/2} \big(\frac{1}{2} - \frac{1}{2}\cos(2t)\big)\,dt = \frac{\pi}{4} - \frac{1}{4}\sin(2t)\Big|_0^{\pi/2} = \frac{\pi}{4}\,,
\end{aligned}$$

where we used the addition theorem for cosine:

$$\cos(2t) = \cos(t+t) = (\cos t)^2 - (\sin t)^2 = 1 - (\sin t)^2 - (\sin t)^2 = 1 - 2(\sin t)^2\,.$$

Theorem 2.17 (Product integration) *Let $f : [a,b] \to \mathbb{R}$ be continuous, let F be an antiderivative of f and let $g : [a,b] \to \mathbb{R}$ be continuously differentiable. Then*

$$\int_a^b f(x)\,g(x)\,dx = F(x)\,g(x)\Big|_a^b - \int_a^b F(x)\,g'(x)\,dx\,.$$

Proof: The product rule gives $(F \cdot g)' = F' \cdot g + F \cdot g' = f \cdot g + F \cdot g'$, thus

$$F(x)\,g(x)\Big|_a^b = \int_a^b f(x)\,g(x)\,dx + \int_a^b F(x)\,g'(x)\,dx\,.$$

■

Example. With $f(x) = g(x) = \sin x$ and $F(x) = -\cos x$ we obtain

$$\begin{aligned}
\int_0^\pi (\sin x)^2\,dx &= -\cos x\ \sin x\Big|_0^\pi + \int_0^\pi (\cos x)^2\,dx \\
&= -\cos x\ \sin x\Big|_0^\pi + \int_0^\pi \big(1 - (\sin x)^2\big)\,dx = \pi - \int_0^\pi (\sin x)^2\,dx\,,
\end{aligned}$$

hence

$$\int_0^\pi (\sin x)^2\,dx = \frac{\pi}{2}\,.$$

3 Continuous mappings on $\mathbb{R}^n$

3.1 Norms on $\mathbb{R}^n$

Let $n \in \mathbb{N}$. On the set of all n-tupels of real numbers

$$\{x = (x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}, \ i = 1, \ldots, n\}$$

the operations of addition and multiplication by real numbers are defined by

$$\begin{aligned} x + y &:= (x_1 + y_1, \ldots, x_n + y_n) \\ cx &:= (cx_1, \ldots, cx_n). \end{aligned}$$

The set of n-tupels together with these operations is a vector space denoted by $\mathbb{R}^n$. A basis of this vector space is for example given by

$$e_1 = (1, 0, \ldots, 0), \ e_2 = (0, 1, 0, \ldots, 0), \ldots, \ e_n = (0, \ldots, 0, 1).$$

On $\mathbb{R}^n$, norms can be defined in different ways. I consider three examples of norms:

1.) **The maximum norm:**

$$\|x\|_\infty := \max\{|x_1|, \ldots, |x_n|\}.$$

To prove that this is a norm, the properties

(i) $\|x\|_\infty = 0 \iff x = 0$

(ii) $\|cx\|_\infty = |c| \, \|x\|_\infty$ (positive homogeneity)

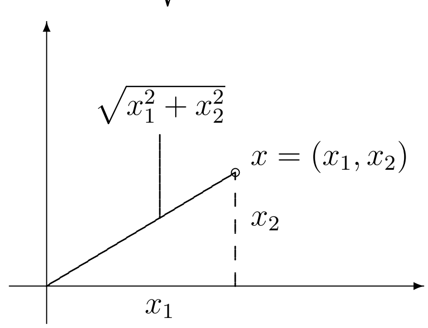
(iii) $\|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$ (triangle inequality)

must be verified. (i) and (ii) are obviously satisfied. To prove (iii) note that there exists $i \in \{1, \ldots, n\}$ such that $\|x + y\|_\infty = |x_i + y_i|$. Then

$$\|x + y\|_\infty = |x_i + y_i| \leq |x_i| + |y_i| \leq \|x\|_\infty + \|y\|_\infty.$$

2.) **The Euclidean norm:**

$$|x| := \sqrt{x_1^2 + \ldots + x_n^2}.$$



Using the **scalar product**

$$x \cdot y := x_1 y_1 + x_2 y_2 + \ldots + x_n y_n \in \mathbb{R}$$

this can also be written as

$$|x| = \sqrt{x \cdot x}\,.$$

It is obvious that $|x| = 0 \Longleftrightarrow x = 0$ and $|cx| = |c|\,|x|$ hold. To verify that $|\cdot|$ is a norm on $\mathbb{R}^n$, it thus remains to verify the triangle inequality. To this end one first proves the

Cauchy-Schwarz inequality

$$|x \cdot y| \le |x|\,|y|\,.$$

Proof: The quadratic polynomial in t

$$|x|^2 t^2 + 2\, x \cdot y\, t + |y|^2 = |tx + y|^2 \ge 0$$

cannot have two different zeros, whence the discriminant must satisfy

$$(x \cdot y)^2 - |x|^2\,|y|^2 \le 0\,.$$

■

Now the triangle inequality is obtained as follows:

$$\begin{aligned} |x+y|^2 &= (x+y)\cdot(x+y) = |x|^2 + 2x\cdot y + |y|^2 \\ &\le |x|^2 + 2|x\cdot y| + |y|^2 \\ &\le |x|^2 + 2|x|\,|y| + |y|^2 = (|x| + |y|)^2\,, \end{aligned}$$

whence

$$|x+y| \le |x| + |y|\,.$$

3.) **The p-norm:**

Let p be a real number with $p \ge 1$. Then the p-norm is defined by

$$\|x\|_p := \big(|x_1|^p + \ldots + |x_n|^p\big)^{\frac{1}{p}}\,.$$

Note that the 2-norm is the Euclidean norm:

$$\|x\|_2 = |x|\,.$$

Here we only verify that $\|\cdot\|_1$ is a norm. Since $\|x\|_1 = 0 \Longleftrightarrow x = 0$ and $\|cx\|_1 = |c|\,\|x\|_1$ are evident, we have to show that the triangle inequality is satisfied:

$$\|x+y\|_1 = \sum_{i=1}^{n} |x_i + y_i| \le \sum_{i=1}^{n} \big(|x_i| + |y_i|\big) = \|x\|_1 + \|y\|_1\,.$$

Definition 3.1 *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. A sequence $\{x_k\}_{k=1}^\infty$ with $x_k \in \mathbb{R}^n$ is said to converge, if $a \in \mathbb{R}^n$ exists such that*

$$\lim_{k\to\infty} \|x_k - a\| = 0\,.$$

a is called limit or limit element of the sequence $\big\{x_k\big\}_{k=1}^\infty$.

Just as in $\mathbb{R} = \mathbb{R}^1$ one proves that a sequence cannot converge to two different limit elements. Hence the limit of a sequence is unique. This limit is denoted by

$$a = \lim_{k\to\infty} x_k\,.$$

In this definition of convergence on $\mathbb{R}^n$ a norm is used. Hence, it seems that convergence of a sequence depends on the norm chosen. The following results show that this is not the case.

Lemma 3.2 *A sequence $\{x_k\}_{k=1}^\infty$ with $x_k = \big(x_k^{(1)}, \ldots, x_k^{(n)}\big) \in \mathbb{R}^n$ converges to $a = (a^{(1)}, \ldots, a^{(n)})$ with respect to the maximum norm, if and only if every sequence of components $\big\{x_k^{(i)}\big\}_{k=1}^\infty$ converges to $a^{(i)}$, $i = 1, \ldots, n$.*

Proof: The statement follows immediately from the inequalities

$$|x_k^{(i)} - a^{(i)}| \leq \|x_k - a\|_\infty \leq |x_k^{(1)} - a^{(1)}| + \ldots + |x^{(n)} - a^{(n)}|\,.$$

■

Theorem 3.3 *Let $\{x_k\}_{k=1}^\infty$ with $x_k \in \mathbb{R}^n$ be a sequence bounded with respect to the maximum norm, i.e. there is a constant $c > 0$ with $\|x_k\|_\infty \leq c$ for all $k \in \mathbb{N}$. Then the sequence $\{x_k\}_{k=1}^\infty$ possesses a subsequence, which converges with respect to the maximum norm.*

Proof: Since $|x_k^{(i)}| \leq \|x_k\|_\infty$ for $i = 1, \ldots, n$, all the component sequences are bounded. Therefore by the Bolzano-Weierstraß Theorem for sequences in $\mathbb{R}$, the sequence $\big\{x_k^{(1)}\big\}_{k=1}^\infty$ possesses a convergent subsequence $\big\{x_{k(j)}^{(1)}\big\}_{j=1}^\infty$. Then $\big\{x_{k(j)}^{(2)}\big\}_{j=1}^\infty$ is a bounded subsequence of $\big\{x_k^{(2)}\big\}_{k=1}^\infty$, hence it has a convergent subsequence $\big\{x_{k(j(\ell))}^{(2)}\big\}_{\ell=1}^\infty$. Also $\big\{x_{k(j(\ell))}^{(1)}\big\}_{\ell=1}^\infty$ converges as a subsequence of the converging sequence $\big\{x_{k(j)}^{(1)}\big\}_{j=1}^\infty$. Thus, for the subsequence $\big\{x_{k(j(\ell))}\big\}_{\ell=1}^\infty$ of $\big\{x_k\big\}_{k=1}^\infty$ the first two component sequences converge. We proceed in the same way and obtain after n steps a subsequence $\big\{x_{k_s}\big\}_{s=1}^\infty$ of $\big\{x_k\big\}_{k=1}^\infty$, for which all component sequences converge. By the preceding lemma this implies that $\big\{x_{k_s}\big\}_{s=1}^\infty$ converges with respect to the maximum norm. ■

Theorem 3.4 *Let $\| \cdot \|$ and $\mathbf{|} \cdot \mathbf{|}$ be norms on $\mathbb{R}^n$. Then there exist constants $a, b > 0$ such that for all $x \in \mathbb{R}^n$*

$$a\|x\| \leq \mathbf{|} x \mathbf{|} \leq b\|x\| \, .$$

Proof: Obviously it suffices to show that for any norm $\| \cdot \|$ on $\mathbb{R}^n$ there exist constants $a, b > 0$ such that for the maximum norm $\| \cdot \|_\infty$

$$\|x\| \leq a\|x\|_\infty \, , \quad \|x\|_\infty \leq b\|x\| \, ,$$

for all $x \in \mathbb{R}^n$. The first one of these estimates is obtained as follows:

$$\begin{aligned} \|x\| &= |x_1e_1 + x_2e_2 + \ldots + x_ne_n| \\ &\leq \|x_1e_1\| + \ldots + \|x_ne_n\| = |x_1| \, \|e_1\| + \ldots + |x_n| \, \|e_n\| \\ &\leq \big(\|e_1\| + \ldots + \|e_n\|\big) \, \|x\|_\infty = a\|x\|_\infty \, , \end{aligned}$$

where $a = \|e_1\| + \ldots + \|e_n\|$.

The second one of these estimates is proved by contradiction: Suppose that such a constant $b > 0$ would not exist. Then for every $k \in \mathbb{N}$ we can choose an element $x_k \in \mathbb{R}^n$ such that

$$\|x_k\|_\infty > k \, \|x_k\| \, .$$

Set $y_k = \frac{x_k}{\|x_k\|_\infty}$. The sequence $\{y_k\}_{k=1}^\infty$ satisfies

$$\|y_k\| = \Big\| \frac{x_k}{\|x_k\|_\infty} \Big\| = \frac{1}{\|x_k\|_\infty} \, \|x_k\| < \frac{1}{k}$$

and

$$\|y_k\|_\infty = \Big\| \frac{x_k}{\|x_k\|_\infty} \Big\|_\infty = \frac{1}{\|x_k\|_\infty} \, \|x_k\|_\infty = 1 \, .$$

Therefore by Theorem 3.3 the sequence $\{y_k\}_{k=1}^\infty$ has a subsequence $\big\{y_{k_j}\big\}_{j=1}^\infty$, which converges with respect to the maximum norm. For brevity we set $z_j = y_{k_j}$. Let z be the limit of $\{z_j\}_{j=1}^\infty$. Then

$$\lim_{j \to \infty} \|z_j - z\|_\infty = 0 \, ,$$

hence, since $\|z_j\|_\infty = \|y_{k_j}\|_\infty = 1$,

$$1 = \lim_{j \to \infty} \|z_j\|_\infty = \lim_{j \to \infty} \|z_j - z + z\|_\infty \leq \|z\|_\infty + \lim_{j \to \infty} \|z_j - z\|_\infty = \|z\|_\infty \, ,$$

whence $z \neq 0$. On the other hand, $\|z_j\| = \|y_{k_j}\| < \frac{1}{k_j} \leq \frac{1}{j}$ together with the estimate $\|x\| \leq a\|x\|_\infty$ proved above implies

$$\begin{aligned} \|z\| &= \|z - z_j + z_j\| = \lim_{j \to \infty} \|z - z_j + z_j\| \\ &\leq \lim_{j \to \infty} \|z - z_j\| + \lim_{j \to \infty} \|z_j\| \leq a \lim_{j \to \infty} \|z - z_j\|_\infty + \lim_{j \to \infty} \frac{1}{j} = 0 \, , \end{aligned}$$

hence $z = 0$. This is a contradiction, hence a constant b must exist such that $\|x\|_\infty \le b\|x\|$ for all $x \in \mathbb{R}$. ■

Definition 3.5 *Let $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ be norms on a vector space V. If constant $a, b > 0$ exist such that*

$$a\|v\| \le |\!|\!|v|\!|\!| \le b\|v\|$$

for all $v \in V$, then these norms are said to be equivalent.

The above theorem thus shows that on $\mathbb{R}^n$ all norms are equivalent. From the definition of convergence it immediately follows that a sequence converging with respect to a norm also converges with respect to an equivalent norm. Therefore **on $\mathbb{R}^n$** the definition of convergence does not depend on the norm.

Moreover, since all norms on $\mathbb{R}^n$ are equivalent to the maximum norm, from Lemma 3.2 and Theorem 3.3 we immediately obtain

Lemma 3.6 *A sequence in $\mathbb{R}^n$ converges to $a \in \mathbb{R}^n$ if and only if the component sequences all converge to the components of a.*

Theorem 3.7 (of Bolzano–Weierstraß for sequences) *Every bounded sequence in $\mathbb{R}^n$ possesses a convergent subsequence.*

Lemma 3.8 (Cauchy convergence criterion) *Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. A sequence $\{x_k\}_{k=1}^\infty$ in $\mathbb{R}^n$ converges if and only if to every $\varepsilon > 0$ there is a $k_0 \in \mathbb{N}$ such that for all $k, \ell \ge k_0$*

$$\|x_k - x_\ell\| < \varepsilon .$$

Proof: $\{x_k\}_{k=1}^\infty$ is a Cauchy sequence on $\mathbb{R}^n$ if and only if every component sequence $\{x_k^{(i)}\}_{k=1}^\infty$ for $i = 1, \ldots, n$ is a Cauchy sequence in $\mathbb{R}$. For, there are constants, $a, b > 0$ such that for all $i = 1, \ldots, n$

$$\begin{aligned} a|x_k^{(i)} - x_\ell^{(i)}| &\le a\|x_k - x_\ell\|_\infty \le \|x_k - x_\ell\| \\ &\le b\|x_k - x_\ell\|_\infty \le b\big(|x_k^{(1)} - x_\ell^{(1)}| + \ldots + |x_k^{(n)} - x_\ell^{(n)}|\big) . \end{aligned}$$

The statement of the lemma follows from this observation, from the fact that the component sequences converge in $\mathbb{R}$ if and only if they are Cauchy sequences, and from the fact that a sequence converges in $\mathbb{R}^n$ if and only if all the component sequences converge. ■

Infinite series: Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^n$. By the infinite series $\sum_{k=1}^{\infty} x_k$ one means the sequence $\{s_\ell\}_{\ell=1}^{\infty}$ of partial sums $s_\ell = \sum_{k=1}^{\ell} x_k$. If $\{s_\ell\}_{\ell=1}^{\infty}$ converges, then $s = \lim_{\ell\to\infty} s_\ell$ is called the sum of the series $\sum_{k=1}^{\infty} x_k$. One writes

$$s = \sum_{k=1}^{\infty} x_k .$$

A series is said to converge absolutely, if

$$\sum_{k=1}^{\infty} \|x_k\|$$

converges, where $\|\cdot\|$ is a norm on $\mathbb{R}^n$. From

$$\|\sum_{k=\ell}^{m} x_k\| \leq \sum_{k=\ell}^{m} \|x_k\|$$

and from the Cauchy convergence criterion it follows that an absolutely convergent series converges. The converse is in general not true.

A series converges absolutely if and only if every component series converges absolutely. This implies that every rearrangement of an absolutely convergent series in $\mathbb{R}^n$ converges to the same sum, since this holds for the component series.

3.2 Topology of $\mathbb{R}^n$

In the following we denote by $\|\cdot\|$ a norm on $\mathbb{R}^n$.

Definition 3.9 *Let $a \in \mathbb{R}^n$ and $\varepsilon > 0$. The set*

$$U_\varepsilon(a) = \{x \in \mathbb{R}^n \mid \|x - a\| < \varepsilon\}$$

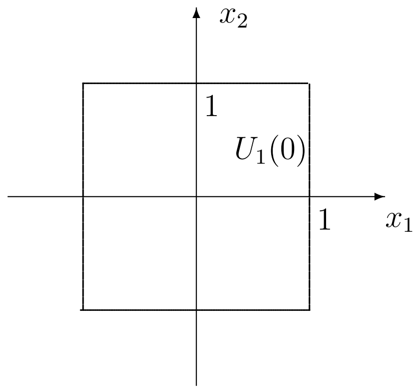
is called open ε-neighborhood of a with respect the the norm $\|\cdot\|$, or ball with center a and radius ε.

A subset U of $\mathbb{R}^n$ is called neighborhood of a if U contains an ε-neighborhood of a.

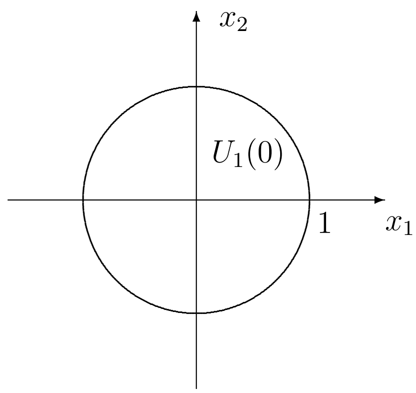
The set $U_1(0) = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ is called open unit ball with respect to $\|\cdot\|$.

In $\mathbb{R}^2$ the unit ball can be pictured for the different norms:

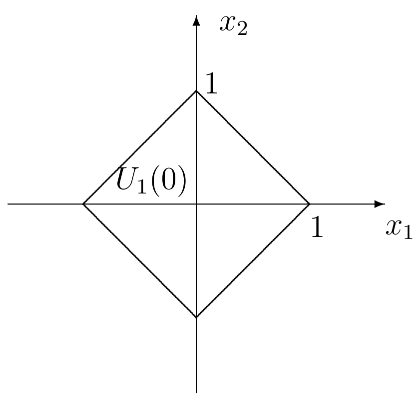
Maximum norm $\|\cdot\|_\infty$:



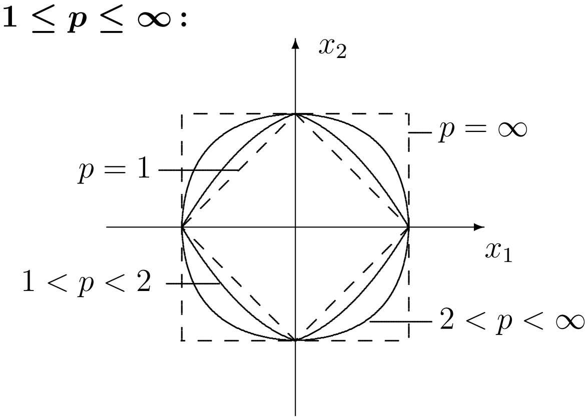
Euclidean norm $|\cdot|$:



1-norm $\|\cdot\|_1$:



p-norm $\|\cdot\|_p$ with $1 \le p \le \infty$:



Whereas the ε-neighborhoods of a point a differ for different norms, the notion of a neighborhood is independent of the norm. For, let $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ be norms on $\mathbb{R}^n$. We show that every ε-neighborhood with respect to $\|\cdot\|$ of $a \in \mathbb{R}^n$ contains a δ-neighborhood with respect to $|\!|\!|\cdot|\!|\!|$.

To this end let

$$\begin{aligned} U_\varepsilon(a) &= \{x \in \mathbb{R}^n \mid \|x-a\| < \varepsilon\}\ , \\ V_\varepsilon(a) &= \{x \in \mathbb{R}^n \mid |\!|\!|x-a|\!|\!| < \varepsilon\}\ . \end{aligned}$$

Since all norms on $\mathbb{R}^n$ are equivalent, there is a constant $c > 0$ such that

$$c\|x-a\| \le |\!|\!|x-a|\!|\!|$$

for all $x \in \mathbb{R}^n$. Therefore, if $x \in V_{c\varepsilon}(a)$ then $|\!|\!|x-a|\!|\!| < c\varepsilon$, which implies $\|x-y\| \le \frac{1}{c}|\!|\!|x-a|\!|\!| < \varepsilon$, and this means $x \in U_\varepsilon(a)$. Consequently, with $\delta = c\varepsilon$,

$$V_\delta(a) \subseteq U_\varepsilon(a)\,.$$

This result implies that if U is a neighborhood of a with respect to $\|\cdot\|$, then it contains a neighborhood $U_\varepsilon(a)$, and then also the neighborhood $V_{c\varepsilon}(a)$, hence U is a neighborhood of a with respect to the norm $|\!|\!|\cdot|\!|\!|$ as well. Consequently, a neighborhood of a with respect to one norm is a neighborhood of a with respect to every other norm on $\mathbb{R}^n$. Therefore the definition of a neighborhood is independent of the norm.

Definition 3.10 *Let M be a subset of $\mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is called interior point of M, if M contains an ε-neighborhood of x, hence if M is a neighborhood of x.*

$x \in \mathbb{R}^n$ is called accumulation point of M, if every neighborhood of x contains a point of M different from x.

$x \in \mathbb{R}$ is called boundary point of M, if every neighborhood of x contains a point of M and a point of the complement $\mathbb{R}^n \backslash M$.

M is called open, if it only consists of its interior points. M is called closed, if it contains all its accumulation points.

The following statements are proved exactly as in $\mathbb{R}^1$:

The complement of an open set is closed, the complement of a closed set is open. The union of an arbitrary system of open sets is open, the intersection of finitely many open sets is open. The intersection of an arbitrary system of closed sets is closed, the union of finitely many closed sets is closed.

A subset M of $\mathbb{R}^n$ is called bounded, if there exists a positive constant C such that

$$\|x\| \leq C$$

for all $x \in M$. The number

$$\operatorname{diam}(M) := \sup_{y,x \in M} \|y - x\|$$

is called diameter of the bounded set M.

Theorem 3.11 *Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of bounded, closed, nonempty subsets A_k of $\mathbb{R}^n$ with $A_{k+1} \subseteq A_k$ and with*

$$\lim_{k\to\infty} \operatorname{diam}(A_k) = 0 .$$

Then there is $x \in \mathbb{R}^n$ such that

$$\bigcap_{k=1}^{\infty} A_k = \{x\} .$$

Proof: For every $k \in \mathbb{N}$ choose $x_k \in A_k$. Then the sequence $\{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence, sind $\lim_{k\to\infty} \operatorname{diam}(A_k) = 0$ implies that to $\varepsilon > 0$ there is k_0 such that diam $A_k < \varepsilon$ for all $k \geq k_0$. Thus, $A_{k+\ell} \subseteq A_k$ implies for all $k \geq k_0$ that

$$\|x_{k+\ell} - x_k\| \leq \text{diam } (A_k) < \varepsilon .$$

The limit x of $\{x_k\}_{k=1}^{\infty}$ satisfies $x \in \bigcap_{k=1}^{\infty} A_k$. For, if $j \in \mathbb{N}$ would exist with $x \notin A_j$, then, since $\mathbb{R}^n \backslash A_j$ is open, a neighborhood $U_\varepsilon(x)$ could be chosen such that $U_\varepsilon(x) \cap A_j = \emptyset$. Thus, $U_\varepsilon(x) \cap A_{j+\ell} = \emptyset$, since $A_{j+\ell} \subseteq A_j$, which implies $\|x - x_{j+\ell}\| \geq \varepsilon$ for all ℓ. This contradicts the property that x is the limit of $\{x_k\}_{k=1}^{\infty}$, and therefore x belongs to the intersection of all sets A_k.

This intersection does not contain any other point. For if $y \in \bigcap_{k=1}^{\infty} A_k$, then $\|x-y\| \leq$ diam (A_k) for all k, whence

$$\|x - y\| = \lim_{k\to\infty} \|x - y\| \leq \lim_{k\to\infty} \text{diam } (A_k) = 0 .$$

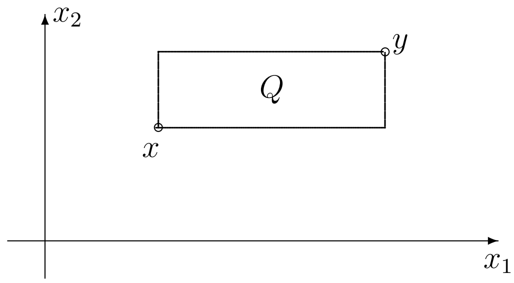
Consequently $y = x$, which proves $\bigcap_{k=1}^{\infty} A_k = \{x\}$. ∎

Definition 3.12 *Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. The set*

$$Q = \{z = (z_1, \ldots, z_n) \in \mathbb{R}^n \mid x_i \leq z_i \leq y_i \, , \, i = 1, \ldots, n\}$$

is called closed interval in $\mathbb{R}^n$. If $y_1 - x_1 = y_2 - x_2 = \ldots = y_n - x_n = a \geq 0$, then this set is called a cube with edge length a.

Let M be a subset of $\mathbb{R}^n$. A system $\mathcal{U}$ of open subsets of $\mathbb{R}^n$ such that $M \subseteq \bigcup_{U\in\mathcal{U}} U$ is called an open covering of M.



Theorem 3.13 *Let $M \subseteq \mathbb{R}^n$. The following three statements are equivalent:*

(i) *M is bounded and closed.*

(ii) *Let $\mathcal{U}$ be an open covering of M. Then there are finitely many $U_1, \ldots, U_m \in \mathcal{U}$ such that $M \subseteq \bigcup_{i=1}^m U_i$.*

(iii) *Every infinite subset of M possesses an accumulation point in M.*

Proof: (i) $\Rightarrow$ (ii): Assume that M is bounded and closed, but that there is an open covering $\mathcal{U}$ of M for which (ii) is not satisfied. As a bounded set M is contained in a sufficiently large closed cube W. Subdivide this cube into 2^n closed cubes with edge length halved. By assumption, there is at least one of the smaller cubes, denoted by W_1, such that $W_1 \cap M$ cannot be covered by finitely many sets from $\mathcal{U}$. Now subdivide W_1 and select W_2 analogously. The sequence $\{M \cap W_k\}_{k=1}^\infty$ of closed sets thus constructed, has the following properties:

1.) $M \cap W \supseteq M \cap W_1 \supseteq M \cap W_2 \supseteq \ldots$

2.) $\lim_{k\to\infty} \text{diam}\,(M \cap W_k) = 0$

3.) $M \cap W_k$ cannot be covered by finitely many sets from $\mathcal{U}$.

3.) implies $M \cap W_k \neq \emptyset$. Therefore, by 1.) and 2.) the sequence $\{M \cap W_k\}_{k=1}^\infty$ satisfies the assumptions of Theorem 3.11, hence there is $x \in \mathbb{R}^n$ such that

$$x \in \bigcap_{k=1}^\infty (M \cap W_k).$$

Since $x \in M$, there is $U \in \mathcal{U}$ with $x \in U$. The set U is open, and therefore contains an ε-neighborhood of x, and then also a δ-neighborhood of x with respect to the maximum norm. Because $\lim_{k\to\infty} \text{diam}\,(W_k) \to 0$ and because $x \in W_k$ for all k, this δ-neighborhood contains the cubes W_k for all sufficiently large k. Hence U contains $M \cap W_k$ for all sufficiently large k. Thus, $M \cap W_k$ can be covered by one set from $\mathcal{U}$, contradicting 3.). We thus conclude that if (i) holds, then also (ii) must be satisfied.

(ii) $\Rightarrow$ (iii): Assume that (ii) holds and let A be a subset of M which does not have accumulation points in M. Then no one of the points of M is an accumulation point of A, consequently to every $x \in M$ there is an open neighborhood, which does not contain a point from A different from x. The system of all these neighborhoods is an open covering of M, hence finitely many of these neighborhoods cover M. Since everyone of these neighborhoods contains at most one point from A, we conclude that A must be finite. An infinite subset of M must thus have an accumulation point in M.

(iii) $\Rightarrow$ (i). Assume that (iii) is satisfied. If M would not be bounded, to every $k \in \mathbb{N}$ there would exist $x_k \in M$ such that

$$\|x_k\| \geq k\,.$$

Let A denote the set of these points. A is an infinite subset of M, but it does not have an accumulation point. For, to an accumulation point y of A there must exist infinitely many $x \in A$ satisfying $\|x - y\| < 1$, which implies

$$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\| < 1 + \|y\|\,.$$

This is not possible, since A only contains finitely many points with norm smaller than $1 + \|y\|$ Thus, the infinite subset A of M does not have an accumulation point. Since this contradicts (iii), M must be bounded.

Let x be an accumulation point of M. For every $k \in \mathbb{N}$ we can select $x_k \in M$ with $0 < \|x_k - x\| < \frac{1}{k}$. The sequence $\{x_k\}_{k=1}^{\infty}$ converges to x, hence x is the only accumulation point of this sequence. Therefore x must belong to M by (iii), thus M contains all its accumulation points, whence M is closed. ■

Definition 3.14 *A subset of $\mathbb{R}^n$ is called compact, if it has one (and therefore all) of the three properties stated in the preceding theorem.*

Theorem 3.15 *A subset M of $\mathbb{R}^n$ is compact, if and only if every sequence in M possesses a convergent subsequence with limit contained in M.*

This theorem is proved as in $\mathbb{R}^1$ (cf. Theorem 6.15 in the classroom notes to Analysis I.)

A set M with the property that every sequence in M has a subsequence converging in M, is called **sequentially compact**. Therefore, in $\mathbb{R}^n$ a set is compact if and only if it is sequentially compact. Finally, just as in $\mathbb{R}^1$, from the Theorem of Bolzano-Weierstraß for sequences (Theorem 3.7) we obtain

Theorem 3.16 (of Bolzano-Weierstraß for sets) *Every bounded infinite subset of* $\mathbb{R}^n$ *has an accumulation point.*

The **proof** is the same as the proof of Theorem 6.11 in the classroom notes to Analysis I.

3.3 Continuous mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$

Let D be a subset of $\mathbb{R}^n$. We consider mappings $f : D \to \mathbb{R}^m$. Such mappings are called *functions of* n *variables*.

For $x \in D$ let $f_1(x), \ldots, f_m(x)$ denote the components of the element $f(x) \in \mathbb{R}^m$. This defines mappings

$$f_i : D \to \mathbb{R}, \quad i = 1, \ldots, m.$$

Conversely, let m mappings $f_1, \ldots, f_m : D \to \mathbb{R}$ be given. Then a mapping

$$f : D \to \mathbb{R}^m$$

is defined by

$$f(x) := \big(f_1(x), \ldots, f_m(x)\big).$$

Thus, every mapping $f : D \to \mathbb{R}^m$ with $D \subseteq \mathbb{R}^n$ is specified by m equations

$$\begin{aligned} y_1 &= f_1(x_1, \ldots, x_n) \\ &\vdots \\ y_m &= f_m(x_1, \ldots, x_n). \end{aligned}$$

Examples

1.) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping, which satisfies for all $x, y \in \mathbb{R}^n$ and all $c \in \mathbb{R}$

$$f(x + y) = f(x) + f(y)$$
$$f(cx) = cf(x)$$

Then f is called a linear mapping. The study of linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$ is the topic of linear algebra. From linear algebra one knows that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping if and only if there exists a matrix

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \\ a_{m1} & \ldots & a_{mn} \end{pmatrix}$$

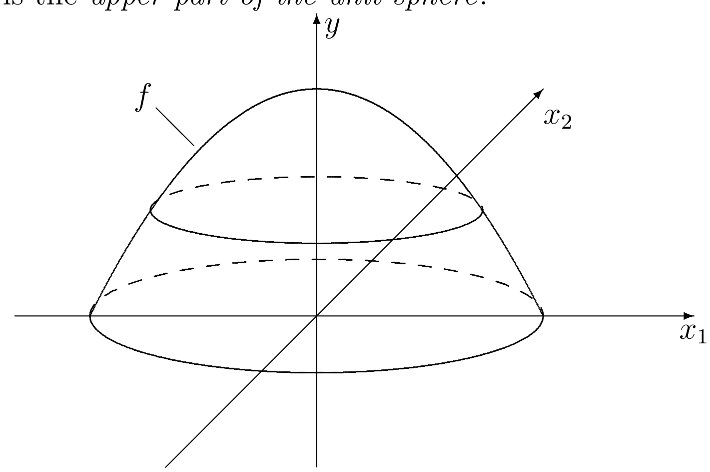
with $a_{ij} \in \mathbb{R}$ such that

$$f(x) = Ax = \begin{pmatrix} a_{11}x_1 & + \ldots + & a_{1n}x_n \\ \vdots & & \\ a_{m1}x_1 & + \ldots + & a_{mn}x_n \end{pmatrix}.$$

2.) Let $n = 2$, $m = 1$ and $D = \left\{x \in \mathbb{R}^2 \;\middle|\; |x| < 1\right\}$. A mapping $f : D \to \mathbb{R}$ is defined by

$$f(x) = f(x_1, x_2) = \sqrt{1 - x_1^2 - x_2^2}\,.$$

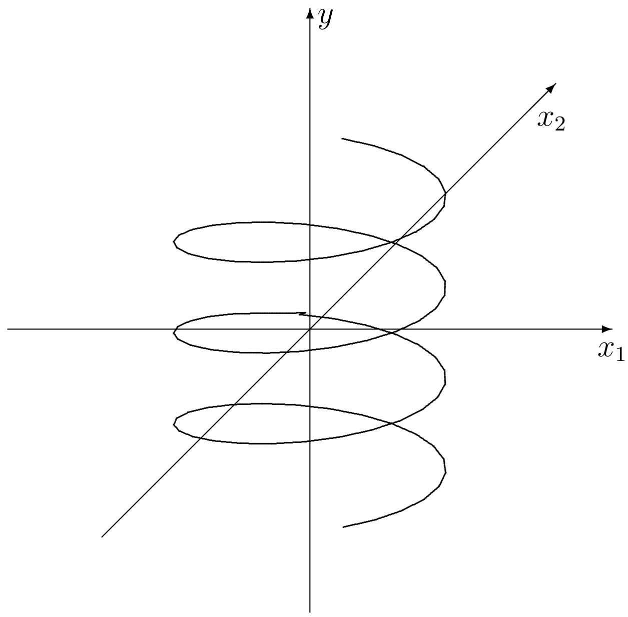
The graph of a mapping from a subset D of $\mathbb{R}^2$ to $\mathbb{R}$ is a surface in $\mathbb{R}^3$. In the present example graph f is the *upper part of the unit sphere*:



3.) Every mapping $f : \mathbb{R} \to \mathbb{R}^m$ is called a path in $\mathbb{R}^m$. For example, let for $t \in \mathbb{R}$

$$f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$$

The range of f is a *helix*.



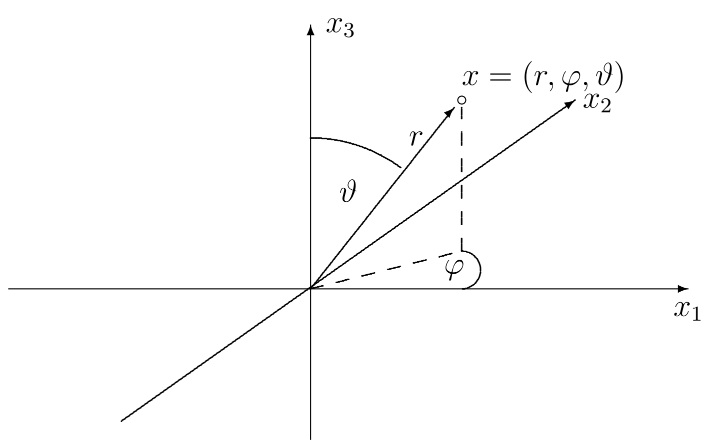
4.) Polar coordinates: Let

$$D = \left\{ (r, \varphi, \psi) \in \mathbb{R}^3 \mid 0 < r,\ 0 \le \varphi < 2\pi,\ 0 < \vartheta < \pi \right\} \subseteq \mathbb{R}^3 ,$$

and let $f : D \to \mathbb{R}^3$,

$$f(r, \varphi, \psi) = \begin{pmatrix} r \cos\varphi \sin\vartheta \\ r \sin\varphi \sin\vartheta \\ r \cos\vartheta \end{pmatrix} .$$

The range of this mapping is $\mathbb{R}^3$ without the x_3-axis:



Definition 3.17 *Let D be a subset of $\mathbb{R}^n$. A mapping $f : D \to \mathbb{R}^m$ is said to be continuous at $a \in D$, if to every neighborhood V of $f(a)$ there is a neighborhood U of a such that $f(U \cap D) \subseteq V$.*

Since every neighborhood of a point contains an ε-neighborhood of this point, irrespective of the norm we use to define ε-neighborhoods, we obtain an equivalent formulation if in this definition we replace V by $V_\varepsilon\big(f(a)\big)$ and U by $U_\delta(a)$. Thus, using the definition of ε-neighborhoods, we immediately get the following

Theorem 3.18 *Let $D \subseteq \mathbb{R}^n$. A mapping $f : D \to \mathbb{R}^m$ is continuous at $a \in D$ if and only if to every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\|f(x) - f(a)\| < \varepsilon$$

for all $x \in D$ with $\|x - a\| < \delta$.

Note that in this theorem we denoted the norms in $\mathbb{R}^n$ and $\mathbb{R}^m$ with the same symbol $\| \cdot \|$.

Almost all results for continuous real functions transfer to continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ with the same proofs. An example is the following

Theorem 3.19 *Let $D \subseteq \mathbb{R}^n$. A function $f : D \to \mathbb{R}^m$ is continuous at $a \in D$, if and only if for every sequence $\{x_k\}_{k=1}^\infty$ with $x_k \in D$ and $\lim_{k\to\infty} x_k = a$*

$$\lim_{k\to\infty} f(x_k) = f(a)$$

holds.

Proof: Cf. the proof of Theorem 6.21 of the classroom notes to Analysis I.

Definition 3.20 *Let $f : D \to \mathbb{R}^m$ and let $a \in \mathbb{R}^n$ be an accumulation point of D. Let $b \in \mathbb{R}^m$. One says that f has the limit b at a and writes*

$$\lim_{x\to a} f(x) = b$$

if to every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\|f(x) - b\| < \varepsilon$$

for all $x \in D\backslash \{a\}$ with $\|x - y\| < \delta$.

Theorem 3.21 *Let $f : D \to \mathbb{R}^m$ and let a be an accumulation point. $\lim_{x\to a} f(x) = b$ holds if and only if for every sequence $\{x_k\}_{k=1}^\infty$ with $x_k \in D\backslash \{a\}$ and $\lim_{k\to\infty} x_k = a$*

$$\lim_{k\to\infty} f(x_k) = b$$

holds.

Proof: Cf. the proof of Theorem 6.39 of the classroom notes to Analysis I.

Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2}\,, & (x,y) \neq 0 \\ 0\,, & (x,y) = 0\,. \end{cases}$$

This function is continuous at every point $(x,y) \in \mathbb{R}^2$ with $(x,y) \neq 0$, but it is not continuous at $(x,y) = 0$. For

$$f(x,0) = f(0,y) = 0\,,$$

whence f vanishes identically on the lines $y = 0$ and $x = 0$. However, on the diagonal $x = y$

$$f(x,y) = f(x,x) = \frac{2x^2}{2x^2} = 1\,.$$

For the two sequences $\{z_k\}_{k=1}^{\infty}$ with $z_k = (\frac{1}{k}, 0)$ and $\{\tilde{z}_k\}_{k=1}^{\infty}$ with $\tilde{z}_\ell = \left(\frac{1}{k}, \frac{1}{k}\right)$ we therefore have $\lim_{k\to\infty} z_k = \lim_{k\to\infty} \tilde{z}_k = 0$, but

$$\lim_{k\to\infty} f(z_k) = 0 = f(0) \neq 1 = \lim_{k\to\infty} f(\tilde{z}_k)\,.$$

Therefore, by Theorem 3.19 f is not continuous at $(0,0)$, and by Theorem 3.21 does not have a limit at $(0,0)$. Hence f cannot be made into a function continuous at $(0,0)$ by modifying the value $f(0,0)$.

Observe however, that the function

$$x \mapsto f(x,y) : \mathbb{R} \to \mathbb{R}$$

is continuous for every $y \in \mathbb{R}$, and

$$y \mapsto f(x,y) : \mathbb{R} \to \mathbb{R}$$

is continuous for every $x \in \mathbb{R}$. Therefore f is continuous in every variable, but as a function $f : \mathbb{R}^2 \to \mathbb{R}$ it is not continuous at $(0,0)$.

Theorem 3.22 *Let $D \subseteq \mathbb{R}^n$ and let $f : D \to \mathbb{R}^m$. The function f is continuous at a point $a \in D$, if and only if all the component functions $f_1, \ldots, f_m : D \to \mathbb{R}$ are continuous at a.*

Proof: f is continuous at a, if and only if for every sequence $\{x_k\}_{k=1}^{\infty}$ with $x_k \in D$ and $\lim_{k\to\infty} x_k = a$ the sequence $\{f(x_k)\}_{k=1}^{\infty}$ converges to $f(a)$. This holds if and only if every component sequence $\{f_i(x_k)\}_{k=1}^{\infty}$ converges to $f_i(a)$ for $i = 1, \ldots, m$, and this is equivalent to the continuity of f_i at a for $i = 1, \ldots, m$. ∎

Definition 3.23 *Let $D \subseteq \mathbb{R}^n$. A function $f : D \to \mathbb{R}^m$ is said to be continuous if it is continuous at every point of D.*

Definition 3.24 *Let D be a subset of $\mathbb{R}^n$. A subset D' of D is said to be relatively open with respect to D, if there exists an open subset O of $\mathbb{R}^n$ such that $D' = O \cap D$.*

Thus, for example, every subset D of $\mathbb{R}^n$ is relatively open with respect to itself, since $D = D \cap \mathbb{R}^n$ and $\mathbb{R}^n$ is open.

Lemma 3.25 *A subset D' of D is relatively open with respect to D, if and only if for every $x \in D$ there is a neighborhood U of x such that $U \cap D \subseteq D'$.*

Proof: If D' is relatively open, there is an open subset O of $\mathbb{R}^n$ such that $D' = O' \cap D$. For every $x \in D'$ the set O is the sought neighborhood.

Conversely, assume that to every $x \in D'$ there is a neighborhood $U(x)$ with $U(x) \cap D \subseteq D'$. Since every neighborhood contains an open neighborhood, we can assume that $U(x)$ is open. Then

$$D' \subseteq D \cap \bigcup_{x \in D'} U(x) = \bigcup_{x \in D'} \big(D \cap U(x)\big) \subseteq D',$$

whence $D' = D \cap O$ with the open set $O = \bigcup_{x \in D'} U(x)$. Consequently D' is relatively open with respect to D. ■

Theorem 3.26 *Let $D \subseteq \mathbb{R}^n$. A function $f : D \to \mathbb{R}^m$ is continuous, if and only if for each open set O of $\mathbb{R}^m$ the inverse image $f^{-1}(O)$ is relatively open with respect to D.*

Proof: Let f be continuous and $x \in f^{-1}(O)$. Then $f(x)$ belongs to the open set O, whence O is a neighborhood of $f(x)$. Therefore, by definition of continuity, there is a neighborhood V of x such that $f(V \cap D) \subseteq O$, which implies $V \cap D \subseteq f^{-1}(O)$. Thus, $f^{-1}(O)$ is relatively open with respect to D.

Assume conversely that the inverse image of every open set is relatively open in D. Let $x \in D$ and let U be an open neighborhood of $f(x)$. Then $f^{-1}(U)$ is relatively open, whence there is an open set $O \subseteq \mathbb{R}^n$ such that $f^{-1}(U) = O \cap D$. This implies $x \in f^{-1}(U) \subseteq O$, whence O is a neighborhood of x. For this neighborhood of x we have

$$f(O \cap D) = f\big(f^{-1}(U)\big) \subseteq U,$$

hence f is continuous. ■

The following theorems and the corollary are proved as the corresponding theorems in $\mathbb{R}$.

Theorem 3.27 (i) *Let $D \subseteq \mathbb{R}^n$ and let $f : D \to \mathbb{R}^m$, $g : D \to \mathbb{R}^m$ be continuous. Then also the mappings $f + g : D \to \mathbb{R}^m$ and $cf : D \to \mathbb{R}^m$ are continuous for every $c \in \mathbb{R}$.*

(ii) *Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ be continuous. Then also $f \cdot g : D \to \mathbb{R}$ and*

$$\frac{f}{g} : \big\{x \in D \mid g(x) \neq 0\big\} \to \mathbb{R}$$

are continuous.

(iii) *Let $f : D \to \mathbb{R}^m$ and $\varphi : D \to \mathbb{R}$ be continuous. Then also φf is continuous.*

Theorem 3.28 *Let $D_1 \subseteq \mathbb{R}^n$ and $D_2 \subseteq \mathbb{R}^p$. Assume that $f : D_1 \to D_2$ and $g : D_2 \to \mathbb{R}^m$ are continuous. Then $g \circ f : D_1 \to \mathbb{R}^m$ is continuous.*

This theorem is proved just as Theorem 6.25 in the classroom notes of Analysis I.

Definition 3.29 *Let D be a subset of $\mathbb{R}^n$. A mapping $f : D \to \mathbb{R}^m$ is said to be uniformly continuous, if to every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\|f(x) - f(y)\| < \varepsilon$$

for all $x, y \in D$ satisfying $\|x - y\| < \delta$.

Theorem 3.30 *Let $D \subseteq \mathbb{R}^n$ be compact and $f : D \to \mathbb{R}^m$ be continuous. Then f is uniformly continuous and $f(D) \subseteq \mathbb{R}^m$ is compact.*

Corollary 3.31 *Let $D \subseteq \mathbb{R}^n$ be compact and $f : D \to \mathbb{R}$ be continuous. Then f attains the maximum and minimum.*

Definition 3.32 *A subset M of $\mathbb{R}^n$ is said to be connected, if it has the following property: Let U_1, U_2 be relatively open subsets of M such that $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = M$. Then $M = U_1$ and $U_2 = \emptyset$ or $M = U_2$ and $U_1 = \emptyset$.*

Example Every interval in $\mathbb{R}$ is connected.

Theorem 3.33 *Let D be a connected subset of $\mathbb{R}^n$ and $f : D \to \mathbb{R}^m$ be continuous. Then $f(D)$ is a connected subset of $\mathbb{R}^m$.*

Proof: Let U_1 and U_2 be relatively open subsets of $f(D)$ with $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = f(D)$. With suitable open subsets O_1, O_2 of $\mathbb{R}^m$ we thus have $U_1 = O_1 \cap f(D)$ and $U_2 = O_2 \cap f(D)$, whence the continuity of f implies that $f^{-1}(U_1) = f^{-1}(O_1)$ and $f^{-1}(U_2) = f^{-1}(O_2)$ are relatively open subsets of D satisfying $f^{-1}(U_1) \cap f^{-1}(U_2) = \emptyset$ and $f^{-1}(U_1) \cup f^{-1}(U_2) = D$. Thus, since D is connected, it follows that $f^{-1}(U_1) = \emptyset$ or $f^{-1}(U_2) = \emptyset$, hence $U_1 = \emptyset$ or $U_2 = \emptyset$. Consequently, $f(D)$ is connected. ■

Definition 3.34 *Let $[a,b]$ be an interval in $\mathbb{R}$ and let $\gamma : [a,b] \to \mathbb{R}^m$ be continuous. Then γ is called a path in $\mathbb{R}^m$.*

Definition 3.35 *A subset M of $\mathbb{R}^n$ is said to be pathwise connected, if any two points in M can be connected by a path in M, i.e. if to $x, y \in M$ there is an interval $[a,b]$ and a continuous mapping $\gamma : [a,b] \to M$ such that $\gamma(a) = x$ and $\gamma(b) = y$.*

$\gamma(a)$ is called starting point, $\gamma(b)$ end point of γ.

Theorem 3.36 *Let $D \subseteq \mathbb{R}^n$ be pathwise connected and let $f : D \to \mathbb{R}^m$ be continuous. Then $f(D)$ is pathwise connected.*

Proof: Let $u, v \in f(D)$ and let $x \in f^{-1}(u)$ and $y \in f^{-1}(v)$. Then there is a path γ, which connects x with y in D. Thus, $f \circ \gamma$ is a path which connects u with v in $f(D)$. ■

Theorem 3.37 *Let $M \subseteq \mathbb{R}^m$ be pathwise connected. Then M is connected.*

Proof: Suppose that M is not connected. Then there are relatively open subsets $U_1 \neq \emptyset$ and $U_2 \neq \emptyset$ such that $U_1 \cap U_2 = \emptyset$ and $U_1 \cup U_2 = M$. Select $x \in U_1$ and $y \in U_2$ and let $\gamma : [a,b] \to M$ be a path connecting x with y. Since M is not connected, it follows that the set $\gamma([a,b])$ is not connected. To see this, set

$$\begin{aligned} V_1 &= \gamma([a,b]) \cap U_1 \,, \\ V_2 &= \gamma([a,b]) \cap U_2 \,. \end{aligned}$$

Then V_1 and V_2 are relatively open subsets of $\gamma([a,b])$ satisfying $V_1 \cap V_2 = \emptyset$ and $V_1 \cap V_2 = \gamma([a,b])$. Therefore, since $x \in V_1$, $y \in V_2$ implies $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, it follows that $\gamma([a,b])$ is not connected.

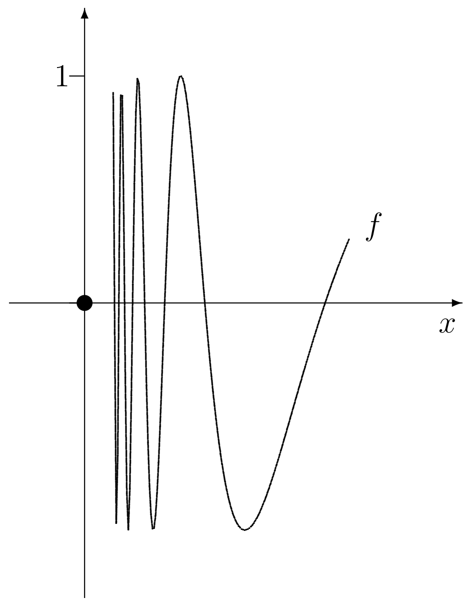
On the other hand, since $[a,b]$ is connected and since γ is continuous, the set $\gamma([a,b])$ must be connected. Our assumption has thus led to a contradiction, hence M is connected.

■

Example Consider the mapping $f : [0,\infty) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin \frac{1}{x} \,, & x > 0 \\ 0 \,, & x = 0 \,. \end{cases}$$

Then $M = \text{graph}(f) = \big\{ \big(x, f(x)\big) \mid x \in [0,\infty) \big\}$ is a subset of $\mathbb{R}^2$, which is connected, but not pathwise connected.



To prove that M is not pathwise connected, assume the contrary. Then, since $(0,0) \in M$ and $(x_0, 1) \in M$ with $x_0 = 1/\frac{\pi}{2}$, a path $\gamma : [a,b] \to M$ exists such that $\gamma(a) = (0,0)$ and $\gamma(b) = (x_0, 1)$. The component functions γ_1 and γ_2 are continuous. Since to every $x \geq 0$ a unique $y \in \mathbb{R}$ exists such that $(x,y) \in M$, namely $y = f(x)$, these component functions satisfy for all $c \in [a,b]$

$$\gamma(c) = \big(\gamma_1(c)\,,\, \gamma_2(c)\big) = \Big(\gamma_1(c)\,,\, f\big(\gamma_1(c)\big)\Big)\,,$$

hence

$$\gamma_2 = f \circ \gamma_1\,.$$

However, this is a contradiction, since $f \circ \gamma_1$ is not continuous.

To see this, set

$$x_n = \frac{1}{\frac{\pi}{2} + 2n\pi}\,.$$

Then $\{x_n\}_{n=1}^{\infty}$ is a null sequence with

$$\gamma_1(a) = 0 < x_n < x_0 = \gamma_1(b)\,.$$

Therefore the intermediate value theorem implies that a sequence $\{c_n\}_{n=1}^{\infty}$ exists with $a \leq c_n \leq b$ such that

$$\gamma_1(c_n) = x_n\,.$$

The bounded sequence $\{c_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{c_{n_j}\}_{j=1}^{\infty}$ with limit

$$c = \lim_{j\to\infty} c_{n_j} \in [a,b]\,.$$

From the continuity of γ_1 it follows that

$$\gamma_1(c) = \lim_{j\to\infty} \gamma_1(c_{n_j}) = \lim_{j\to\infty} x_{n_j} = \lim_{n\to\infty} x_n = 0\,,$$

hence

$$(f \circ \gamma_1)(c) = f\big(\gamma_1(c)\big) = f(0) = 0\,,$$

but

$$\begin{aligned}\lim_{j\to\infty} (f \circ \gamma_1)(c_{n_j}) &= \lim_{j\to\infty} f\big(\gamma_1(c_{n_j})\big) = \lim_{j\to\infty} f(x_{n_j}) \\ &= \lim_{j\to\infty} \sin\big(\frac{\pi}{2} + 2n_j\pi\big) = \lim_{j\to\infty} 1 = 1 \neq (f \circ \gamma_1)(c)\,,\end{aligned}$$

which proves that $f \circ \gamma_1$ is not continuous at c.

To prove that M is connected, assume the contrary. Then there are relatively open subsets U_1, U_2 of M satisfying $U_1 \neq \emptyset$, $U_2 \neq \emptyset$, $U_1 \cap U_2 = \emptyset$, and $U_1 \cup U_2 = M$. The set

$$M' = \big\{\big(x, f(x)\big) \mid x > 0\big\} \subseteq M$$

is connected as the image of the connected set $(0, \infty)$ under the continuous map

$$x \mapsto \big(x, f(x)\big) : (0, \infty) \to \mathbb{R}^2.$$

Consequently, $U_1 \cap M' = \emptyset$ or $U_2 \cap M' = \emptyset$. Without restriction of generality we assume that $U_1 \cap M' = \emptyset$. Then $U_2 = M'$ and $U_1 = \{(0, 0)\}$. However, this is a contradiction, since $\{(0, 0)\}$ is not relatively open with respect to M. Otherwise an open set $O \subseteq \mathbb{R}^2$ would exist such that $\{(0, 0)\} = M \cap O$, hence $(0, 0) \in O$, and therefore O would contain an ε–neighborhood of $(0, 0)$. Since $\sin\big(\frac{1}{x}\big)$ has infinitely many zeros in every neighborhood of $x = 0$, the ε–neighborhood of $(0, 0)$ would contain besides $(0, 0)$ infinitely many points of M on the positive real axis, hence $M \cap O \neq \{(0, 0)\}$. Consequently, M is connected.

This example shows that the statement of Theorem 3.37 cannot be inverted.

Theorem 3.38 *Let D be a compact subset of $\mathbb{R}^n$ and $f : D \to \mathbb{R}^m$ be continuous and injective. Then the inverse $f^{-1} : f(D) \to D$ is continuous.*

The proof of this theorem is obtained by a slight modification of the proof of Theorem 6.28 in the classroom notes of Analysis I.

Definition 3.39 *let $D \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$. A mapping $f : D \to W$ is called homeomorphism, if f is bijective, continuous and has a continuous inverse.*

3.4 The normed spaces of bounded, continuous, and linear functions

The space $\boldsymbol{B(D, \mathbb{R}^m)}$. Let D be a non-empty set, which is not necessarily a subset of $\mathbb{R}^n$. A function $f : D \to \mathbb{R}^m$ is said to be bounded, if the range $f(D)$ is a bounded subset of $\mathbb{R}^m$. The set $B(D, \mathbb{R}^m)$ consists of all bounded functions $f : D \to \mathbb{R}^m$. For bounded functions f and g and for $c \in \mathbb{R}$ also $f + g$ and cf are bounded. Therefore $B(D, \mathbb{R}^m)$ is a vector space. The space $B(D, \mathbb{R})$, which was introduced in Section 1.3, is an algebra with the usual product fg of real valued functions f and g. This is not true for $B(D, \mathbb{R}^m)$ with $m \geq 2$, since the product of two functions with values in $\mathbb{R}^m$ is not defined.

Definition 3.40 *Let $\|\cdot\|$ be a norm on $\mathbb{R}^m$. The supremum norm for $f \in B(D, \mathbb{R}^m)$ is defined by*

$$\|f\|_\infty := \sup_{x \in D} \|f(x)\|.$$

$\|\cdot\|_\infty$ is a norm on the vector space $B(D, \mathbb{R}^m)$, hence $B(D, \mathbb{R}^m)$ is a normed space. This follows as in the proof of Theorem 1.8, where the norm properties for the supremum norm on the space $B(D, \mathbb{R})$ are verified. Of course, the supremum norm on $B(D, \mathbb{R}^m)$ depends on the norm on $\mathbb{R}^m$ used to define the supremum norm. However, from the equivalence of all norms on $\mathbb{R}^m$ it immediately follows that the supremum norms on $B(D, \mathbb{R}^m)$ obtained from different norms on $\mathbb{R}^m$ are equivalent. Therefore the following definition does not depend on the supremum norm chosen:

Definition 3.41 *Let D be a nonempty set and let $\{f_k\}_{k=1}^\infty$ be a sequence of functions $f_k \in B(D, \mathbb{R}^m)$. The sequence $\{f_k\}_{k=1}^\infty$ is said to converge uniformly, if $f \in B(D, \mathbb{R}^m)$ exists such that*

$$\lim_{k \to \infty} \|f_k - f\|_\infty = 0.$$

Theorem 3.42 *A sequence $\{f_k\}_{k=1}^\infty$ with $f_k \in B(D, \mathbb{R}^m)$ converges uniformly if and only if to every $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$ such that for all $k, \ell \geq k_0$*

$$\|f_k - f_\ell\|_\infty < \varepsilon$$

holds, that is, if and only if it is a Cauchy sequence.

The **proof** is the same as the proof of Theorem 1.13.

Definition 3.43 *A normed vector space with the property that every Cauchy sequence converges, is called a complete normed space or a Banach space (Stefan Banach, 1892 – 1945).*

Corollary 3.44 *The space $B(D, \mathbb{R}^m)$ with the supremum norm is a Banach space.*

The space $C(D, \mathbb{R}^m)$. For a subset D of $\mathbb{R}^n$ we denote by $C(D, \mathbb{R}^m)$ the set of all continuous functions from D to $\mathbb{R}^m$. Since for continuous functions f and g and for $c \in \mathbb{R}$ also $f+g$ and cf are continuous, it follows that $C(D, \mathbb{R}^m)$ is a vector space. Also the set of all bounded continuous functions $C(D, \mathbb{R}^m) \cap B(D, \mathbb{R}^m)$ is a vector space. As a subspace of $B(D, \mathbb{R}^m)$ it is a normed space with the supremum norm.

Theorem 3.45 *Let $D \subseteq \mathbb{R}^n$ and let $\{f_k\}_{k=1}^\infty$ be a sequence of functions $f_k \in C(D, \mathbb{R}^m) \cap B(D, \mathbb{R}^m)$, which converges uniformly to $f \in B(D, \mathbb{R}^m)$. Then f is continuous.*

This theorem is proved just as Corollary 1.5. This leads to an important result:

Corollary 3.46 *Let D be a subset of $\mathbb{R}^n$. The space $C(D, \mathbb{R}^m) \cap B(D, \mathbb{R}^m)$ equipped with the supremum norm is complete, hence it is a Banach space.*

Proof: Let $\{f_k\}_{k=1}^\infty$ be a Cauchy sequence in $C(D, \mathbb{R}^m) \cap B(D, \mathbb{R}^m)$. By Theorem 3.42, this sequence converges with respect to the supremum norm to a function $f \in B(D, \mathbb{R}^m)$. Theorem 3.45 implies that $f \in C(D, \mathbb{R}^m)$. Consequently, the Cauchy sequence $\{f_k\}_{k=1}^\infty$ converges with respect to the supremum norm to the limit function $f \in C(D, \mathbb{R}^m) \cap B(D, \mathbb{R}^m)$, from which we see that $C(D, \mathbb{R}^m) \cap B(D, \mathbb{R}^m)$ is a complete space. ∎

The space $L(\mathbb{R}^n, \mathbb{R}^m)$. By $L(\mathbb{R}^n, \mathbb{R}^m)$ we denote the set of all linear mappings $f: \mathbb{R}^n \to \mathbb{R}^m$. Since for linear mappings f, g and for a real number c the mappings $f+g$ and cf are linear, $L(\mathbb{R}^n, \mathbb{R}^m)$ is a vector space.

Theorem 3.47 *Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then f is continuous. If f differs from zero, then f is unbounded.*

Proof: To f there exists a unique $m \times n$–Matrix $A_f = (a_{ij})_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$ such that $f(x) = A_f x$ holds for all $x \in \mathbb{R}^n$. Written in components, this is

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= a_{11}x_1 + \ldots + a_{1n}x_n \\ &\vdots \\ f_m(x_1, \ldots, x_n &= a_{m1}x_1 + \ldots + a_{mn}x_n. \end{aligned}$$

Since everyone of the expressions on the right depends continuously on $x = (x_1, \ldots, x_n)$, it follows that all component functions of f are continuous, hence f is continuous.

If f differs from 0, there is $x \in \mathbb{R}^n$ with $f(x) \neq 0$. From the linearity we then obtain for $\lambda \in \mathbb{R}$

$$\|f(\lambda x)\| = \|\lambda f(x)\| = |\lambda| \, \|f(x)\|,$$

which can be made larger than any constant by choosing $|\lambda|$ sufficiently large. Hence f is not bounded. ■

We want to define a norm on the linear space $L(\mathbb{R}^n, \mathbb{R}^m)$. It is not possible to use the supremum norm, since every linear mapping $f \neq 0$ is unbounded, hence, the supremum of the set $\{\|f(x)\| \mid x \in \mathbb{R}^n\}$ does not exist. However, a norm on $L(\mathbb{R}^n, \mathbb{R}^m)$ can be defined as follows.

Definition 3.48 *For $f \in L(\mathbb{R}^n, \mathbb{R}^m)$ the operator norm is defined by*

$$\|f\| = \sup_{\|x\| \leq 1} \|f(x)\|.$$

The name of this norm stems from the fact that linear mappings are often called operators. Note that $\|x\|$ is a norm on $\mathbb{R}^n$ and $\|f(x)\|$ is a norm on $\mathbb{R}^m$. To see that the supremum in this definition exists, let $B = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ be the closed unit ball in $\mathbb{R}^n$. The set B is bounded and closed, hence compact. Since $f \in L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous, it follows by Theorem 3.30 that the set $f(B)$ is compact, hence bounded. Consequently, the supremum $\sup_{\|x\| \leq 1} \|f(x)\| = \sup_{x \in B} \|f(x)\|$ exists.

The following lemma shows that the operator norm has the properties required from a norm.

Lemma 3.49 *Let $f, g : \mathbb{R}^n \to \mathbb{R}^m$ be linear, let $c \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Then*

(i) $f = 0 \Longleftrightarrow \|f\| = 0$,

(ii) $\|cf\| = |c| \, \|f\|$,

(iii) $\|f + g\| \leq \|f\| + \|g\|$,

(iv) $\|f(x)\| \leq \|f\| \, \|x\|$.

Proof: We first prove (iv). For $x = 0$ the linearity of f implies $f(x) = 0$, whence $\|f(x)\| = 0 \leq \|f\| \, \|x\|$. For $x \neq 0$ we have $\|\frac{x}{\|x\|}\| = 1$, hence the linearity of f yields

$$\|f(x)\| = \Big\|f\Big(\|x\|\frac{x}{\|x\|}\Big)\Big\| = \Big\| \|x\| f\Big(\frac{x}{\|x\|}\Big)\Big\| = \|x\| \, \Big\|f\Big(\frac{x}{\|x\|}\Big)\Big\| \leq \|x\| \sup_{\|y\| \leq 1} \|f(y)\| = \|x\| \, \|f\|.$$

To prove (i), let $f = 0$. Then $\|f\| = \sup_{\|x\|\le 1} \|f(x)\| = 0$. On the other hand, if $\|f\| = 0$, we conclude from (iv) for all $x \in \mathbb{R}^n$ that

$$\|f(x)\| \le \|f\|\,\|x\| = 0,$$

hence $f(x) = 0$, whence $f = 0$. The proof of (ii) is obvious. To verify (iii) consider

$$\begin{aligned}\|f+g\| = \sup_{\|x\|\le 1} \|(f+g)(x)\| = \sup_{\|x\|\le 1} \|(f(x)+g(x)\| &\le \sup_{\|x\|\le 1} (\|f(x)\| + \|g(x)\|)\\ &\le \sup_{\|x\|\le 1} (\|f\| + \|g\|)\|x\| = \|f\| + \|g\|.\end{aligned}$$

■

$L(\mathbb{R}^n, \mathbb{R}^m)$ equipped with the operator norm is a normed vector space. In fact, it is a Banach space.

To show this, let $\mathbb{R}^{m\times n}$ denote the space of $m \times n$–matrices. To $f \in L(\mathbb{R}^n, \mathbb{R}^m)$ let A_f denote the unique matrix in $\mathbb{R}^{m\times n}$ satisfying $f(x) = A_f x$ for all $x \in \mathbb{R}^n$, where $A_f x$ denotes matrix multiplication. The mapping $J_1 : L(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^{m\times n}$ given by $J_1(f) = A_f$ is linear and invertible, hence it is a linear isomorphism. Another linear isomorphism is given by the mapping $J_2 : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\cdot n}$, which associates to every matrix $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ the vector $a_A = (a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{m1}, \ldots, a_{mn}) \in \mathbb{R}^{m\cdot n}$. Also the composition $J = J_2 \circ J_1 : L(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^{m\cdot n}$ is a linear isomporphism. On $\mathbb{R}^{m\times n}$ a norm is defined by

$$\|A\|_\infty = \max_{\substack{i=1,\ldots,m\\ j=1,\ldots,n}} |a_{ij}| = \|a_A\|_\infty\,.$$

Theorem 3.50 *There exist constants $c, C > 0$ such that for every $f \in L(\mathbb{R}^n, \mathbb{R}^m)$*

$$c\|a_{A_f}\|_\infty = c\|A_f\|_\infty \le \|f\| \le C\|A_f\|_\infty = C\|a_{A_f}\|_\infty.$$

Proof: For $a \in \mathbb{R}^{n\cdot m}$ we set

$$|\!|\!|a|\!|\!| = \|J^{-1}(a)\|\,,$$

where the norm on the right hand side is the operator norm. Obviously $|\!|\!|\cdot|\!|\!|$ is a norm on $\mathbb{R}^{n\cdot m}$. Since by Theorem 3.4 all norms on $\mathbb{R}^{n\cdot m}$ are equivalent, there exist constants $c, C > 0$ such that for the maximum norm $\|\cdot\|_\infty$ on $\mathbb{R}^{n\cdot m}$ and for all $a \in \mathbb{R}^{n\cdot m}$

$$c\|a\|_\infty \le |\!|\!|a|\!|\!| \le C\|a\|_\infty$$

holds, which yields for all $f \in L(\mathbb{R}^n, \mathbb{R}^m)$ and for $A_f = J_1(f)$ and $a_{A_f} = J(f)$ that

$$c\|A_f\|_\infty = c\|a_{A_f}\|_\infty \le |\!|\!|a_{A_f}|\!|\!| \le C\|a_{A_f}\|_\infty \; = C\|A_f\|_\infty\,.$$

The inequality stated in the theorem follows from this estimate, since by definition of $|\cdot|$ we have $|a_{A_f}| = \|f\|$. ■

Corollary 3.51 *The space $L(\mathbb{R}^n, \mathbb{R}^m)$ with the operator norm is a Banach space.*

Proof: If $\{f_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $L(\mathbb{R}^n, \mathbb{R}^m)$, we infer from Theorem 3.50 that $\{a_{A_{f_k}}\}_{k=1}^{\infty} = J(f_k)$ is a Cauchy sequence in $\mathbb{R}^{m \cdot n}$. Since $\mathbb{R}^{m \cdot n}$ is complete, $\{a_{A_{f_k}}\}_{k=1}^{\infty}$ converges to an element $a \in \mathbb{R}^{m \cdot n}$. Using Theorem 3.50 again, we see that this implies that $\{f_k\}_{k=1}^{\infty}$ converges to $f := J^{-1}(a)$, hence $L(\mathbb{R}^n, \mathbb{R}^m)$ is complete, so it is a Banach space. ■

4 Differentiable mappings on $\mathbb{R}^n$

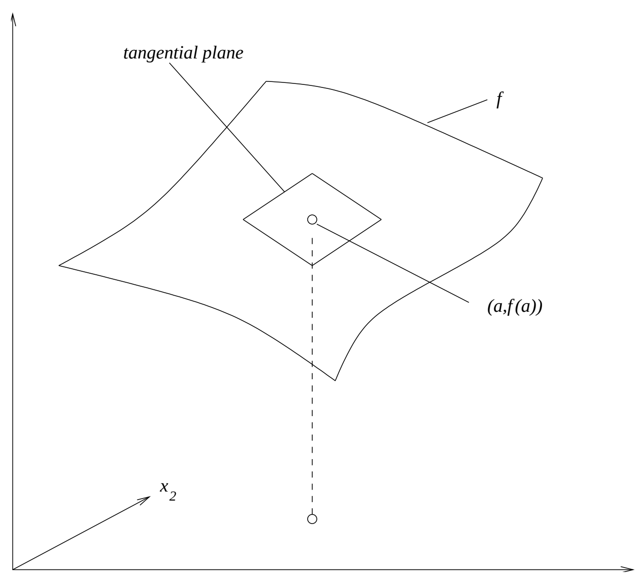
4.1 Definition of the derivative

The derivative of a real function f at a satisfies the equation

$$f(x) = f(a) + f'(a)(x-a) + r(x)(x-a)\,,$$

where the function r is continuous at a and satisfies $r(a) = 0$. Since $x \mapsto f'(a)x$ is a linear map from $\mathbb{R}$ to $\mathbb{R}$, the interpretation of this equation is that under all affine maps $x \mapsto f(a) + T(x-a)$, where $T : \mathbb{R} \to \mathbb{R}$ is linear, the one obtained by choosing $T(x) = f'(a)x$ is the best approximation of the function f in a neighborhood of a.

Viewed in this way, the notion of the derivative can be generalized immediately to mappings $f : D \to \mathbb{R}^m$ with $D \subseteq \mathbb{R}^n$. Thus, the derivative of f at $a \in D$ is the linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ such that under all affine functions the mapping $x \mapsto f(a) + T(x-a)$ approximates f best in a neighborhood of a.



For a mapping $f : \mathbb{R}^2 \to \mathbb{R}$ this means that the linear mapping $T : \mathbb{R}^2 \to \mathbb{R}$, the derivative of f at a, must be chosen such that the graph of the mapping $x \mapsto f(a) + T(x-a)$ is equal to the tangential plane of the graph of f at $\big(a, f(a)\big)$.

This idea leads to the following rigorous definition of a differentiable function:

Definition 4.1 *A function $f : U \to \mathbb{R}^m$ defined on an open set $U \subseteq \mathbb{R}^n$ is said to be differentiable at the point $a \in U$, if there is a linear mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ and a function*

$r : U \to \mathbb{R}^m$, *which is continuous at* a *and satisfies* $r(a) = 0$, *such that for all* $x \in U$

$$f(x) = f(a) + T(x-a) + r(x)\,\|x-a\|\,.$$

Therefore to verify that f is differentiable at $a \in D$ a linear mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ must be found such that the function r defined by

$$r(x) := \frac{f(x) - f(a) - T(x-a)}{\|x-a\|}$$

satisfies

$$\lim_{r \to a} r(x) = 0\,.$$

Later we show how T can be found. However, there is at most one such T:

Lemma 4.2 *The linear mapping* T *is uniquely determined.*

Proof: Let $T_1, T_2 : \mathbb{R}^n \to \mathbb{R}^m$ be linear mappings and $r_1, r_2 : U \to \mathbb{R}^m$ be functions with $\lim_{x\to a} r_1(x) = \lim_{x\to a} r_2(x) = 0$, such that for $x \in U$

$$\begin{aligned} f(x) &= f(a) + T_1(x-a) + r_1(x)\,\|x-a\| \\ f(x) &= f(a) + T_2(x-a) + r_2(x)\,\|x-a\|\,. \end{aligned}$$

Then

$$(T_1 - T_2)(x-a) = \big(r_2(x) - r_1(x)\big)\,\|x-a\|\,.$$

Let $h \in \mathbb{R}^n$. Then, $x = a + th \in U$ for all sufficiently small $t > 0$ since U is open, whence

$$(T_1 - T_2)(th) = t(T_1 - T_2)(h) = \big(r_2(a+th) - r_1(a+th)\big)\,\|th\|\,,$$

thus

$$(T_1 - T_2)(h) = \lim_{t\to 0}(T_1 - T_2)(h) = \lim_{t\to 0}\big(r_2(a+th) - r_1(a+th)\big)\,\|h\| = 0\,.$$

This implies $T_1 = T_2$, since $h \in \mathbb{R}^n$ was chosen arbitrarily. ∎

Definition 4.3 *Let* $U \subseteq \mathbb{R}^n$ *be open and let* $f : U \to \mathbb{R}^m$ *be differentiable at* $a \in U$. *Then the unique linear mapping* $T : \mathbb{R}^n \to \mathbb{R}^m$, *for which a function* $r : U \to \mathbb{R}^m$ *satisfying* $\lim_{x\to a} r(x) = 0$ *exists, such that*

$$f(x) = f(a) + T(x-a) + r(x)\,\|x-a\|$$

holds for all $x \in U$, *is called derivative of* f *at* a. *This linear mapping is denoted by* $f'(a) = T$.

Mostly we drop the brackets around the argument and write $T(h) = Th = f'(a)h$.

For a real valued function f the derivative is a linear mapping $f'(a) : \mathbb{R}^n \to \mathbb{R}$. Such linear mappings are also called linear forms. In this case $f'(a)$ can be represented by a $1\times n$-matrix, and we normally identify $f'(a)$ with this matrix. The transpose $[f'(a)]^T$ of this $1\times n$-matrix is a $n\times 1$-matrix, a column vector. For this transpose one uses the notation

$$\operatorname{grad} f(a) = [f'(a)]^T .$$

$\operatorname{grad} f(a)$ is called the gradient of f at a. With the scalar product on $\mathbb{R}^n$ the gradient can be used to represent the derivative of f: For $h \in \mathbb{R}^n$ we have

$$f'(a)h = \big(\operatorname{grad} f(a)\big) \cdot h .$$

If $h \in \mathbb{R}^n$ is a unit vector and if t runs through $\mathbb{R}$, then the point th moves along the straight line through the origin with direction h. A differentiable real function is defined by

$$t \mapsto \big(\operatorname{grad} f(a)\big) \cdot th = t\big(\operatorname{grad} f(a) \cdot h\big) .$$

The derivative is $\operatorname{grad} f(a) \cdot h$, and this derivative attains the maximum value

$$\operatorname{grad} f(a) \cdot h = |\operatorname{grad} f(a)|$$

if h has the direction of $\operatorname{grad} f(a)$. Since $f(a) + \operatorname{grad} f(a) \cdot (th) = f(a) + f'(a)th$ approximates the value $f(a+th)$, it follows that the vector $\operatorname{grad} f(a)$ points into the direction of steepest ascent of the function f at a, and the length of $\operatorname{grad} f(a)$ determines the slope of f in this direction.

Lemma 4.4 *Let $U \subseteq \mathbb{R}^n$ be an open set. The function $f : U \to \mathbb{R}^m$ is differentiable at $a \in U$, if and only if all component functions $f_1, \ldots, f_m : U \to \mathbb{R}$ are differentiable in a. The derivatives satisfy*

$$(f_j)'(a) = \big(f'(a)\big)_j , \quad j = 1, \ldots, m .$$

Proof: If the derivatives $f'(a)$ exist, then the components satisfy

$$\lim_{h\to 0} \frac{f_j(a+h) - f_j(a) - \big(f'(a)\big)_j h}{\|h\|} = 0 .$$

Since $\big(f'(a)\big)_j : \mathbb{R}^n \to \mathbb{R}$ is linear, it follows that f_j is differentiable at a with derivative $(f_j)'(a) = \big(f'(a)\big)_j$. Conversely, if the derivative $(f_j)'(a)$ of f_j exists at a for all $j =$

$1, \dots, m$, then a linear mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$Th = \begin{pmatrix} (f_1)'(a)h \\ \vdots \\ (f_m)'(a)h \end{pmatrix},$$

for which

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - Th}{\|h\|} = 0 .$$

Thus, f is differentiable at a with derivative $f'(a) = T$. ■

Definition 4.5 *A function $f : U \to \mathbb{R}^m$ defined on an open set $U \subseteq \mathbb{R}^n$ is said to be differentiable, if f is differentiable at every point $a \in U$.*

4.2 Directional derivatives and partial derivatives

Let $U \subseteq \mathbb{R}^n$ be an open set, let $a \in U$ and let $f : U \to \mathbb{R}^m$. Let $v \in \mathbb{R}^n$ be a given vector. Since U is open, there is $\delta > 0$ such that $a + tv \in U$ for all $t \in \mathbb{R}$ with $|t| < \delta$; hence $f(a+tv)$ is defined for all such t. If t runs through the interval $(-\delta, \delta)$, then $a + tv$ runs through a line segment passing through a, which has the direction of the vector v.

Definition 4.6 *We call the limit*

$$D_v f(a) = \lim_{t \to 0} \frac{f(a+tv) - f(a)}{t}$$

derivative of f at a in the direction of the vector v, if this limit exists.

It is possible that the directional derivative $D_v f(a)$ exists, even if f is not differentiable at a. Also, it can happen that the derivative of f at a exists in the direction of some vectors, and does not exist in the direction of other vectors. In any case, the directional derivative contains useful information about the function f. However, if f is differentiable at a, then all directional derivatives of f exist at a:

Lemma 4.7 *Let $U \subseteq \mathbb{R}^n$ be open, let $a \in U$ and let $f : U \to \mathbb{R}^m$ be differentiable at a. Then the directional derivative $D_v f(a)$ exists for every $v \in \mathbb{R}^n$ and satisfies*

$$D_v f(a) = f'(a)v .$$

Proof: Set $x = a + tv$ with $t \in \mathbb{R}$, $t \neq 0$. Then by definition of the derivative $f'(a)$

$$f(a+tv) = f(a) + f'(a)(tv) + r(tv + a)\, |t|\, \|v\| ,$$

hence

$$\frac{f(a+tv)-f(a)}{t} = f'(a)v + r(tv+a)\,\frac{|t|}{t}\,\|v\|\,.$$

Since $\frac{|t|}{t} = \pm 1$ and since $\lim_{t\to 0} r(tv+a) = r(a) = 0$, it follows that $\lim_{t\to 0} r(tv+a)\,\frac{|t|}{t}\,\|v\| = 0$, hence

$$\lim_{t\to 0} \frac{f(a+tv)-f(a)}{t} = f'(a)v\,.$$

■

This result can be used to compute $f'(a)$: If $v_1, \ldots, v_n$ is a basis of $\mathbb{R}^n$, then every vector $v \in \mathbb{R}$ can be represented as a linear combination $v = \sum_{i=1}^n \alpha_i v_i$ of the basis vectors with uniquely determined numbers $\alpha_i \in \mathbb{R}$. The linearity of $f'(a)$ thus yields

$$f'(a)v = f'(a)\Big(\sum_{i=1}^n \alpha_i v_i\Big) = \sum_{i=1}^n \alpha_i f'(a) v_i = \sum_{i=1}^n \alpha_i D_{v_i} f(a)\,.$$

Therefore $f'(a)$ is known if the directional derivatives $D_{v_i} f(a)$ for the basis vectors are known. It suggests itself to use the standard basis $e_1, \ldots, e_n$. The directional derivative $D_{e_i} f(a)$ is called *i-th partial derivative of f at a*. For the i-th partial derivative one uses the notations

$$D_i f,\ \frac{\partial f}{\partial x_i}\,,\ f_{x_i},\ f'_{x_i},\ f_{|i}\,.$$

For $i = 1, \ldots, n$ and $j = 1, \ldots, m$ we have

$$\begin{aligned} \frac{\partial f}{\partial x_i}(a) &= \lim_{t\to 0} \frac{f(a+te_i)-f(a)}{t} = \lim_{x_i\to a_i} \frac{f(a_1,\ldots,x_i,\ldots,a_n) - f(a_1,\ldots,a_i,\ldots,a_n)}{x_i - a_i}\,, \\ \frac{\partial f_j}{\partial x_i}(a) &= \lim_{x_i\to a_i} \frac{f_j(a_1,\ldots,x_i,\ldots,a_n) - f_j(a_1,\ldots,a_i,\ldots,a_n)}{x_i - a_i}\,. \end{aligned}$$

Consequently, to compute partial derivatives the differential calculus for functions of one real variable suffices.

To construct $f'(a)$ from the partial derivatives one proceeds as follows: If $f'(a)$ exists, then all the partial derivatives $D_i f(a) = \frac{\partial f}{\partial x_i}(a)$ exist. For arbitrary $h \in \mathbb{R}^n$ we have $h = \sum_{i=1}^n h_i e_i$, where $h_i \in \mathbb{R}$ are the components of h, hence

$$f'(a)h = f'(a)\Big(\sum_{i=1}^n h_i e_i\Big) = \sum_{i=1}^n \big(f'(a)e_i\big) h_i = \sum_{i=1}^n D_i f(a) h_i\,,$$

or, in matrix notation,

$$f'(a)h = \begin{pmatrix} \big(f'(a)h\big)_1 \\ \vdots \\ \big(f'(a)h\big)_m \end{pmatrix} = \begin{pmatrix} D_1 f_1(a) & \ldots & D_n f_1(a) \\ \vdots & & \\ D_1 f_m(a) & \ldots & D_n f_m(a) \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$$

Thus,

$$f'(a) = \begin{pmatrix} D_1 f_1(a) & \dots & D_n f_1(a) \\ \vdots & & \\ D_1 f_m(a) & \dots & D_n f_m(a) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \dots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1}(a) & \dots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix}$$

is the representation of $f'(a)$ as $m \times n$-matrix belonging to the standard bases $e_1, \dots, e_n$ of $\mathbb{R}^n$ and $e_1, \dots, e_m$ of $\mathbb{R}^m$. This matrix is called Jacobi-matrix of f at a. (Carl Gustav Jacob Jacobi 1804–1851).

It is possible that all partial derivatives exist at a without f being differentiable at a. Then the Jacobi-matrix can be formed, but it does not represent the derivative $f'(a)$, which does not exist.

Therefore, to check whether f is differentiable at a, one first verifies that all partial derivatives exist at a. This is a necessary condition for the existence of $f'(a)$. Then one forms the Jacobi-matrix

$$T = \Big(\frac{\partial f_i}{\partial x_j}(a) \Big)_{\substack{i=1,\dots,m \\ j=1,\dots,n}},$$

and tests whether for this matrix

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - Th}{\|h\|} = 0$$

holds. If this holds, then f is differentiable at a with derivative $f'(a) = T$.

Examples

1.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x_1, x_2) = \begin{pmatrix} f_1(x_2, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_1^2 - x_2^2 \\ 2x_1 x_2 \end{pmatrix}.$$

At $a = (a_1, a_2) \in \mathbb{R}^2$ the Jacobi-matrix is

$$T = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) \end{pmatrix} = \begin{pmatrix} 2a_1 & -2a_2 \\ 2a_2 & 2a_1 \end{pmatrix}.$$

To test the differentiability of f at a, set for $h = (h_1, h_2) \in \mathbb{R}^2$ and $i = 1, 2$

$$r_i(h) = \frac{f_i(a+h) - f_i(a) - T_i(h)}{\|h\|},$$

hence

$$\begin{aligned} r_1(h) &= \frac{(a_1 + h_1)^2 - (a_2 + h_2)^2 - a_1^2 + a_2^2 - 2a_1 h_1 + 2a_2 h_2}{\|h\|} = \frac{h_1^2 - h_2^2}{\|h\|}, \\ r_2(h) &= \frac{2(a_1 + h_1)(a_2 + h_2) - 2a_1 a_2 - 2a_2 h_1 - 2a_1 h_2}{\|h\|} = \frac{2h_1 h_2}{\|h\|}. \end{aligned}$$

Using the maximum norm $\|\cdot\| = \|\cdot\|_\infty$, we obtain

$$\begin{aligned} |r_1(h)| &\le 2\|h\|_\infty \\ |r_2(h)| &\le 2\|h\|_\infty , \end{aligned}$$

thus

$$\lim_{h\to 0} \|r(h)\|_\infty = \lim_{h\to 0} \|\big(r_1(h), r_2(h)\big)\|_\infty \le \lim_{h\to 0} 2\|h\|_\infty = 0 .$$

Therefore f is differentiable at a. Since a was arbitrary, f is everywhere differentiable, i.e. f is differentiable.

2.) Let the affine map $f : \mathbb{R}^n \to \mathbb{R}^m$ be defined by

$$f(x) = Ax + c ,$$

where $c \in \mathbb{R}^m$ and $A : \mathbb{R}^n \to \mathbb{R}^m$ is linear. Then f is differentiable with derivative $f'(a) = A$ for all $a \in \mathbb{R}^n$. For,

$$\frac{f(a+h) - f(a) - Ah}{\|h\|} = \frac{A(a+h) + c - Aa - c - Ah}{\|h\|} = 0 .$$

3.) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x_1, x_2) = \begin{cases} 0 , & \text{for } (x_1, x_2) = 0 , \\ \dfrac{|x_1|\, x_2}{\sqrt{x_1^2 + x_2^2}} , & \text{for } (x_1, x_2) \ne 0 . \end{cases}$$

This function is not differentiable at $a = 0$, but it has all the directional derivatives at 0. To see that all directional derivatives exist, let $v = (v_1, v_2)$ be a vector from $\mathbb{R}^2$ different from zero. Then

$$D_v f(0) = \lim_{t\to 0} \frac{f(tv) - f(0)}{t} = \lim_{t\to 0} \frac{t|t|\, |v_1| v_2}{t|t| \sqrt{v_1^2 + v_2^2}} = \frac{|v_1|\, v_2}{\sqrt{v_1^2 + v_2^2}} .$$

To see that f is not differentiable at 0, note that the partial derivatives satisfy

$$\frac{\partial f}{\partial x_1}(0) = 0 , \quad \frac{\partial f}{\partial x_2}(0) = 0 .$$

Therefore, if f would be differentiable at 0, the derivative had to be

$$f'(0) = \Big(\frac{\partial f}{\partial x_1}(0) \quad \frac{\partial f}{\partial x_2}(0) \Big) = (0 \quad 0) .$$

Consequently, all directional derivatives would satisfy

$$D_v f(0) = f'(0) v = 0 .$$

Yet, the preceding calculation yields for the derivative in the direction of the diagonal vector $v = (1,1)$ that

$$D_v\, f(0) = \frac{1}{\sqrt{2}}\,.$$

Therefore $f'(0)$ cannot exist.

We note that $|f(x_1, x_2)| = \frac{|x_1\, x_2|}{|x|} \leq |x|$, which implies that f is continuous at 0 .

4.3 Elementary properties of differentiable mappings

In the preceding example f was not differentiable at 0, but had all the directional derivatives and was continuous at 0. Here is an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$, which has all the directional derivatives at 0, yet is not continuous at 0: f is defined by

$$f(x_1, x_2) = \begin{cases} 0, & \text{for } (x_1, x_2) = 0 \\ \dfrac{x_1 x_2^2}{x_1^2 + x_2^6}\,, & \text{for } (x_1, x_2) \neq 0\,. \end{cases}$$

To see that all directional derivatives exist at 0, let $v = (v_1, v_2) \in \mathbb{R}^2$ with $v \neq 0$. Then

$$D_v\, f(0) = \lim_{t \to 0} \frac{f(tv) - f(0)}{t} = \begin{cases} \lim\limits_{t \to 0} \dfrac{v_1 v_2^2}{v_1^2 + t^4 v_2^6} = \dfrac{v_2^2}{v_1}\,, & \text{if } v_1 \neq 0 \\ 0\,, & \text{if } v_1 = 0\,. \end{cases}$$

Yet, for $h = (h_1, \sqrt{h_1})$ with $h_1 > 0$ we have

$$\lim_{h_1 \to 0} f(h) = \lim_{h_1 \to 0} \frac{h_1^2}{h_1^2 + h_1^3} = \lim_{h_1 \to 0} \frac{1}{1 + h_1} = 1 \neq f(0)\,.$$

Therefore f is not continuous at 0 . Together with the next result we obtain as a consequence that f is not differentiable at 0 :

Theorem 4.8 *Let U be an open subset of $\mathbb{R}^n$, let $a \in U$ and let $f : U \to \mathbb{R}^m$ be differentiable at a. Then there is a constant $c > 0$ such that for all x from a neighborhood of a*

$$\|f(x) - f(a)\| \leq c\|x - a\|\,.$$

In particular, f is continuous at a .

Proof: We have

$$f(x) = f(a) + f'(a)(x - a) + r(x)\,\|x - a\|\,,$$

whence, with the operator norm $\|f'(a)\|$ of the linear mapping $f'(a) : \mathbb{R}^n \to \mathbb{R}^m$,

$$\|f(x) - f(a)\| \leq \|f'(a)\|\,\|x - a\| + \|r(x)\|\,\|x - a\|\,.$$

Since $\lim_{x\to a} r(x) = 0$, there is $\delta > 0$ such that

$$\|r(x)\| \le 1$$

for all $x \in D$ with $\|x - a\| < \delta$, whence for these x

$$\|f(x) - f(a)\| \le \big(\|f'(a)\| + 1\big)\,\|x - a\| = c\,\|x - a\|\,,$$

with $c = \|f'(a)\| + 1$. In particular, this implies

$$\lim_{x\to a} \|f(x) - f(a)\| \le \lim_{x\to a} c\|x - a\| = 0\,,$$

whence f is continuous at a. ■

Theorem 4.9 *Let $U \subseteq \mathbb{R}^n$ be open and $a \in U$. If $f : U \to \mathbb{R}^m$ and $g : U \to \mathbb{R}^m$ are differentiable at a, then also $f + g$ and cf are differentiable at a for all $c \in \mathbb{R}$, and*

$$\begin{aligned}(f+g)'(a) &= f'(a) + g'(a)\\ (cf)'(a) &= cf'(a)\,.\end{aligned}$$

Proof: We have for $h \in \mathbb{R}^n$ with $a + h \in U$

$$\begin{aligned} f(a+h) &= f(a) + f'(a)h + r_1(a+h)\,\|h\|\,, \quad \lim_{h\to 0} r_1(a+h) = 0\\ g(a+h) &= g(a) + g'(a)h + r_2(a+h)\,\|h\|\,, \quad \lim_{h\to 0} r_2(a+h) = 0\,.\end{aligned}$$

Thus

$$(f+g)(a+h) = (f+g)(a) + \big(f'(a) + g'(a)\big)h + (r_1 + r_2)(a+h)\,\|h\|$$

with $\lim_{h\to 0}(r_1 + r_2)(a + h) = 0$. Consequently $f + g$ is differentiable at a with derivative $(f + g)'(a) = f'(a) + g'(a)$. The statement for cf follows in the same way. ■

Theorem 4.10 (Product rule) *Let $U \subseteq \mathbb{R}^n$ be open and let $f, g : U \to \mathbb{R}$ be differentiable at $a \in U$. Then $f \cdot g : U \to \mathbb{R}$ is differentiable at a with derivative*

$$(f \cdot g)'(a) = f(a)\,g'(a) + g(a)\,f'(a)\,.$$

Proof: We have for $a + h \in U$

$$\begin{aligned}(f\cdot g)(a+h) &= \big(f(a) + f'(a)h + r_1(a+h)\,\|h\|\big)\cdot\big(g(a) + g'(a)h + r_2(a+h)\,\|h\|\big)\\ &= (f\cdot g)(a) + f(a)\,g'(a)h + g(a)\,f'(a)h + r(a+h)\,\|h\|\,,\end{aligned}$$

where

$$\begin{aligned} r(a+h)\,\|h\| &= \big(f'(a)h\,g'(a)\,\frac{h}{\|h\|}\big)\,\|h\| + \big(g(a)+g'(a)h\big)\,r_1(a+h)\,\|h\| \\ &\quad + \big(f(a)+f'(a)h\big)r_2(a+h)\,\|h\| + r_1(a+h)\,r_2(a+h)\,\|h\|^2\,. \end{aligned}$$

The absolute value is a norm on $\mathbb{R}$. Since $r(a+h)\in\mathbb{R}$, we thus obtain with the operator norms $\|f'(a)\|$, $\|g'(a)\|$,

$$\begin{aligned} \lim_{h\to 0}|r(a+h)| &\le \lim_{h\to 0}\Big[\big(\|f'(a)\|\,\|h\|\,\|g'(a)\|\big) \\ &\quad + \big(|g(a)|+\|g'(a)\|\,\|h\|\big)\,|r_1(a+h)| \\ &\quad + \big(|f(a)|+\|f'(a)\|\,\|h\|\big)\,|r_2(a+h)| \\ &\quad + |r_1(a+h)|\,|r_2(a+h)|\,\|h\|\Big] = 0\,. \end{aligned}$$

Since $f(a)\,g'(a)h + g(a)\,f'(a)h = \big(f(a)\,g'(a)+g(a)\,f'(a)\big)h$, it follows that $f\cdot g$ is differentiable at a with derivative given by this linear mapping. ■

Theorem 4.11 (Chain rule) *Let $U\subseteq\mathbb{R}^p$ and $V\subseteq\mathbb{R}^n$ be open, let $f:U\to V$ and $g:V\to\mathbb{R}^m$. Suppose that $a\in U$, that f is differentiable at a and that g is differentiable at $b=f(a)$. Then $g\circ f:U\to\mathbb{R}^n$ is differentiable at a with derivative*

$$(g\circ f)'(a) = g'\big(f(a)\big)\circ f'(a)\,.$$

Remark: Since $g'(b)$ and $f'(a)$ can be represented by matrices, $g'(b)\circ f'(a)$ can also be written as $g'(b)\,f'(a)$, employing matrix multiplication.

Proof: For brevity we set

$$T_2 = g'(b)\,,\quad T_1 = f'(a)\,,$$

and for $h\in\mathbb{R}^p$ with $a+h\in U$

$$R(h) = (g\circ f)(a+h) - (g\circ f)(a) - T_2T_1h\,.$$

The statement of the theorem follows if it can be shown that

$$\lim_{h\to 0}\frac{\|R(h)\|}{\|h\|} = 0.$$

We have for $x\in U$ and $y\in V$

$$\begin{aligned} f(x)-f(a)-T_1(x-a) &= r_1(x-a)\|x-a\|, & \lim_{h\to 0} r_1(h) = 0 \\ g(y)-g(b)-T_2(y-b) &= r_2(y-b)\|y-b\|, & \lim_{k\to 0} r_2(k) = 0. \end{aligned}$$

Since T_2 is linear, we thus obtain for $x = a+h$ and $y = f(a+h)$

$$\begin{aligned} R(h) &= g\big(f(a+h)\big) - g\big(f(a)\big) - T_2\big(f(a+h) - f(a)\big) \\ &\quad + T_2\big(f(a+h) - f(a) - T_1 h\big) \\ &= r_2\big(f(a+h) - f(a)\big)\|f(a+h) - f(a)\| + T_2\big(r_1(h)\|h\|\big), \end{aligned}$$

which yields

$$\begin{aligned} \lim_{h\to 0} \frac{\|R(h)\|}{\|h\|} &\le \lim_{h\to 0} \Big[\frac{1}{\|h\|}\, \|r_2\big(f(a+h) - f(a)\big)\|\, \|f(a+h) - f(a)\| \\ &\qquad + \Big\|T_2\big(r_1(h)\|h\|\big)\Big\|\Big]. \end{aligned}$$

Since f is differentiable at a, for $\|h\|$ sufficiently small the estimate $\|f(a+h) - f(a)\| \le c\|h\|$ holds, cf. Theorem 4.8. Therefore, with the operator norm $\|T_2\|$ we conclude that

$$\lim_{h\to 0} \frac{\|R(h)\|}{\|h\|} \le \lim_{h\to 0} \Big[\|r_2\big(f(a+h) - f(a)\big)\|c + \|T_2\|\,\|r_1(h)\|\Big] = 0.$$

■

For the Jacobi–matrices of $f : U \to V \subseteq \mathbb{R}^n,\;\; g : V \to \mathbb{R}^m$ and $F = g\circ f : U \to \mathbb{R}^m$ we thus obtain

$$\begin{pmatrix} \frac{\partial F_1}{\partial x_1}(a) & \dots & \frac{\partial F_1}{\partial x_p}(a) \\ \vdots & & \\ \frac{\partial F_m}{\partial x_1}(a) & \dots & \frac{\partial F_m}{\partial x_p}(a) \end{pmatrix} = \begin{pmatrix} \frac{\partial g_1}{\partial y_1}(b) & \dots & \frac{\partial g_1}{\partial y_n}(b) \\ \vdots & & \\ \frac{\partial g_m}{\partial y_1}(b) & \dots & \frac{\partial g_m}{\partial y_n}(b) \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \dots & \frac{\partial f_1}{\partial x_p}(a) \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1}(a) & \dots & \frac{\partial f_n}{\partial x_p}(a) \end{pmatrix}.$$

Thus,

$$\frac{\partial F_j}{\partial x_i}(a) = \sum_{k=1}^{n} \frac{\partial g_j}{\partial y_k}(b)\frac{\partial f_k}{\partial x_i}(a), \quad i = 1,\dots,p,\;\; j = 1,\dots,m.$$

Corollary 4.12 *Let U be an open subset of $\mathbb{R}^n$, let $a \in U$ and let $f : U \to \mathbb{R}$ be differentiable at a and satisfy $f(a) \neq 0$. Then $\frac{1}{f}$ is differentiable at a with derivative*

$$\Big(\frac{1}{f}\Big)'(a) = -\frac{1}{f(a)^2}\, f'(a).$$

Proof: Consider the differentiable function $g : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ defined by $g(x) = \frac{1}{x}$. Then

$$\frac{1}{f} = g\circ f : \{x \in U \,\big|\, f(x) \neq 0\} \to \mathbb{R}$$

is differentiable at a with derivative

$$\left(\frac{1}{f}\right)'(a) = g'\big(f(a)\big)f'(a) = -\frac{1}{f(a)^2}\,f'(a). \qquad \blacksquare$$

Assume that U and V are open subsets of $\mathbb{R}^n$ and that $f : U \to V$ is an invertible map with inverse $f^{-1} : V \to U$. If $a \in U$, if f is differentiable at a and if f^{-1} is differentiable at $b = f(a) \in V$, then the derivative $(f^{-1})'(b)$ can be computed from $f'(a)$ using the chain rule. To see this, note that

$$f^{-1} \circ f = \mathrm{id}_U.$$

The identity mapping id_U is obtained as the restriction of the identity mapping $id_{\mathbb{R}^n}$ to U. Since $id_{\mathbb{R}^n}$ is linear, it follows that id_U is differentiable at every $c \in U$ with derivative $(id_U)'(x) = id_{\mathbb{R}^n}$. Consequently

$$id_{\mathbb{R}^n} = (id_U)'(a) = (f^{-1} \circ f)'(a) = (f^{-1})'(b)\, f'(a)\,.$$

From linear algebra we know that this equation implies that $(f^{-1})'(b)$ is the inverse of $f'(a)$. Consequently, $\big(f'(a)\big)^{-1}$ exists and

$$(f^{-1})'(b) = \big(f'(a)\big)^{-1},$$

or

$$(f^{-1})'(b) = \big[f'\big(f^{-1}(b)\big)\big]^{-1}\,.$$

Thus, if one assumes that $f'(a)$ exists and that the inverse mapping is differentiable at $f(a)$, one can conclude that the linear mapping $f'(a)$ is invertible. On the other hand, if one assumes that $f'(a)$ exists and is invertible and that the inverse mapping is continuous at $f(a)$, one can conclude that the inverse mapping is differentiable at $f(a)$. This is shown in the following theorem. We remark that the linear mapping $f'(a)$ is invertible if and only if the determinant $\det f'(a)$ differs from zero, where $f'(a)$ is identified with the $n{\times}n$-matrix representing the linear mapping $f'(a)$.

Theorem 4.13 *Let $U \subseteq \mathbb{R}^n$ be an open subset, let $a \in U$ and let $f : U \to \mathbb{R}^n$ be one-to-one. If f is differentiable at a with invertible derivative $f'(a)$, if the range $f(U)$ contains a neighborhood of $b = f(a)$, and if the inverse mapping $f^{-1} : f(U) \to U$ of f is continuous at b, then f^{-1} is differentiable at b with derivative*

$$(f^{-1})'(b) = \big(f'(a)\big)^{-1} = \Big(f'\big(f^{-1}(b)\big)\Big)^{-1}\,.$$

Proof: For brevity we set $g = f^{-1}$. We must show that

$$\lim_{y \to b} \frac{g(y) - g(b) - \big(f'(a)\big)^{-1}(y - b)}{\|y - b\|} = 0\,. \tag{4.1}$$

Since f is differentiable at a, we have for $x \in U$

$$f(x) - f(a) = f'(a)(x-a) + r(x)\,\|x-a\|\,, \tag{4.2}$$

where r is continuous at a and satisfies $r(a) = 0$. We use $g(b) = a$ to conclude from (4.2), setting $x = g(y)$, that

$$\begin{aligned}
&\frac{g(y) - g(b) - \big(f'(a)\big)^{-1}(y-b)}{\|y-b\|} \\
&\quad = \frac{g(y) - a - \big(f'(a)\big)^{-1}\Big(f\big(g(y)\big) - f(a)\Big)}{\|y-b\|} \\
&\quad = \frac{g(y) - a - \big(f'(a)\big)^{-1}\Big(f'(a)\big(g(y)-a\big) + r\big(g(y)\big)\,\|g(y)-a\|\Big)}{\|y-b\|} \\
&\quad = -\big(f'(a)\big)^{-1}\Big(r\big(g(y)\big)\Big)\,\frac{\|g(y)-a\|}{\|y-b\|}\,.
\end{aligned} \tag{4.3}$$

By assumption, g is continuous at b. Since r is continuous at $a = g(b)$, it follows that $r \circ g$ is continuous at b, hence

$$\lim_{y \to b} r\big(g(y)\big) = r\big(g(b)\big) = r(a) = 0. \tag{4.4}$$

Below we show that there is a neighborhood $V \subseteq f(U)$ of b and a constant $c > 0$ such that

$$\frac{\|g(y)-a\|}{\|y-b\|} \le c, \tag{4.5}$$

for all $y \in V$. Employing this inequality and (4.4), we obtain from (4.3) that

$$\lim_{y \to b} \Big\| \frac{g(y) - g(b) - \big(f'(a)\big)^{-1}(y-b)}{\|y-b\|} \Big\| \le \lim_{y \to b} \|\big(f'(a)\big)^{-1}\|\, \|r\big(g(y)\big)\|\, c = 0\,,$$

where $\|\big(f'(a)\big)^{-1}\|$ denotes the operator norm of $\big(f'(a)\big)^{-1}$. This proves (4.1). To finish the proof it thus remains to verify (4.5).

To this end let $y \in f(U)$. Employing (4.2) with $x = g(y)$ and noting that $b = f(a)$, we obtain from the inverse triangle inequality that

$$\begin{aligned}
\frac{\|g(y)-a\|}{\|y-b\|} &= \frac{\|g(y)-a\|}{\|f\big(g(y)\big) - f(a)\|} = \frac{\|g(y)-a\|}{\big\|f'(a)\big(g(y)-a\big) + r\big(g(y)\big)\,\|g(y)-a\|\,\big\|} \\
&\le \frac{\|\big(f'(a)\big)^{-1} f'(a)\big(g(y)-a\big)\,\|}{\|f'(a)\big(g(y)-a\big)\| - \|r\big(g(y)\big)\|\,\|\big(f'(a)\big)^{-1} f'(a)\big(g(y)-a\big)\|} \\
&\le \frac{\|\big(\big(f'(a)\big)^{-1}\|\,\|f'(a)\big(g(y)-a\big)\|}{\|f'(a)\big(g(y)-a\big)\|\,\Big(1 - \|r\big(g(y)\big)\|\,\|\big(f'(a)\big)^{-1}\|\Big)} = \frac{\|\big(f'(a)\big)^{-1}\|}{1 - \|r\big(g(y)\big)\|\,\|\big(f'(a)\big)^{-1}\|}.
\end{aligned}$$

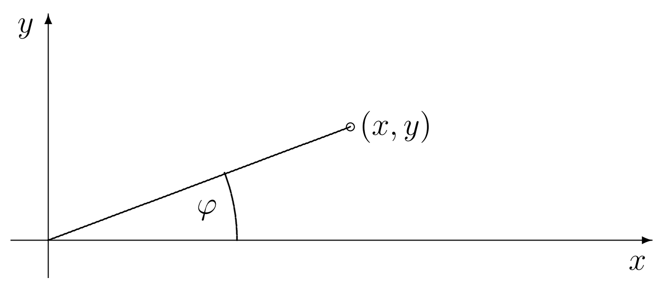
The inequality (4.5) is a consequence of this relation and of (4.4). ■

Example (Polar coordinates). Let

$$U = \{(r, \varphi) \mid r > 0\,, \quad 0 < \varphi < 2\pi\} \subseteq \mathbb{R}^2,$$

and let $f = (f_1, f_2) : U \to \mathbb{R}^2$ be defined by

$$\begin{aligned} x &= f_1(r, \varphi) = r \cos \varphi \\ y &= f_2(r, \varphi) = r \sin \varphi\,. \end{aligned}$$



This mapping is one-to-one with range

$$f(U) = \mathbb{R}^2 \backslash \{(x, 0) \mid x \geq 0\}\,,$$

and has a continuous inverse. From a theorem proved in the next section it follows that f is differentiable. Thus,

$$f'(r, \varphi) = \begin{pmatrix} \frac{\partial f_1}{\partial r}(r, \varphi) & \frac{\partial f_1}{\partial \varphi}(r, \varphi) \\ \frac{\partial f_2}{\partial r}(r, \varphi) & \frac{\partial f_2}{\partial \varphi}(r, \varphi) \end{pmatrix} = \begin{pmatrix} \cos \varphi & -r \sin \varphi \\ \sin \varphi & r \cos \varphi \end{pmatrix}.$$

This matrix is invertible for $(r, \varphi) \in U$, hence the derivative $(f^{-1})'(x, y)$ exists for every $(x, y) = f(r, \varphi) = (r \cos \varphi\,, r \sin \varphi)$ and can be computed without having to determine the inverse function f^{-1}:

$$\begin{aligned} (f^{-1})'(x, y) &= \big(f'(r, \varphi)\big)^{-1} = \begin{pmatrix} \cos \varphi & -r \sin \varphi \\ \sin \varphi & r \cos \varphi \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\frac{1}{r} \sin \varphi & \frac{1}{r} \cos \varphi \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix} \end{aligned}$$

4.4 Mean value theorem

The mean value theorem for real functions can be generalized to *real valued* functions:

Theorem 4.14 (Mean value theorem) *Let U be an open subset of $\mathbb{R}^n$, let $f : U \to \mathbb{R}$ be differentiable, and let $a, b \in U$ be points with $a \neq b$ such that the line segment $\ell = \{a + t(b-a) \mid 0 < t < 1\}$ connecting these points is contained in U. Then there is a point $c \in \ell$ such that*

$$f(b) - f(a) = f'(c)(b-a)\,.$$

Proof: Define a function $\gamma : [0,1] \to U$ by $t \mapsto \gamma(t) := a + t(b-a)$. This function maps the interval $[0,1]$ onto the closed line segment $\overline{\ell}$ connecting a and b. The affine function γ is differentiable with derivative

$$\gamma'(t) = b - a\,.$$

Let $F = f \circ \gamma$ be the composition. Since f and γ are differentiable, $F : [0,1] \to \mathbb{R}$ is differentiable. Thus, the mean value theorem for real functions implies that there is $\vartheta \in (0,1)$ such that

$$f(b) - f(a) = F(1) - F(0) = F'(\vartheta) = f'\big(\gamma(\vartheta)\big)\,\gamma'(\vartheta) = f'(c)(b-a)\,,$$

where we have set $c = \gamma(\vartheta)$. ∎

Of course, the mean value theorem can also be formulated as follows: If U contains together with the points x and $x+h$ also the line segment connecting these points, then there is a number ϑ with $0 < \vartheta < 1$ such that

$$f(x+h) - f(x) = f'(x + \vartheta h)h\,.$$

The mean value theorem does not hold for functions $f : U \to \mathbb{R}^m$ with $m > 1$, but the following weaker result can often be used as a replacement for the mean value theorem:

Corollary 4.15 (Barrier Theorem) *Let $U \subseteq \mathbb{R}^n$ be open, let $f : U \to \mathbb{R}^m$ be differentiable, and let a and b be points from U with $a \neq b$ such that the line segment $\ell = \{a + t(b-a) \mid 0 < t < 1\}$ connecting these points is contained in U. Then there is a point $c \in \ell$ such that*

$$\|f(b) - f(a)\| \leq \|f'(c)\|\,\|b-a\|\,,$$

where $\|f'(c)\|$ *denotes the operator norm of the derivative of* f *at* c*. In particular, if* $\|f'\|$ *is bounded on* ℓ*, i.e.*

$$S = \sup_{0<t<1} \|f'\big(a + t(b-a)\big)\| < \infty\,,$$

then

$$\|f(b) - f(a)\| \le S\|b-a\|\,.$$

To prove this corollary we need the following

Theorem 4.16 *Let* $\|\cdot\|$ *be a norm on* $\mathbb{R}^m$*. Then to every* $u \in \mathbb{R}^m$ *there is a linear mapping* $A_u : \mathbb{R}^m \to \mathbb{R}$ *such that* $\|A_u\| = 1$ *and* $A_u(u) = \|u\|$*.*

This theorem follows from the theorem of Hahn-Banach, which in a more general version also holds for infinite dimensional normed spaces. We do not prove the theorem, but refer for the proof to any book on *functional analysis*.

Example: For the Euclidean norm $\|\cdot\| = |\cdot|$ define A_u by

$$A_u(v) = \frac{u}{|u|}\cdot v\,, \quad v \in \mathbb{R}^m\,.$$

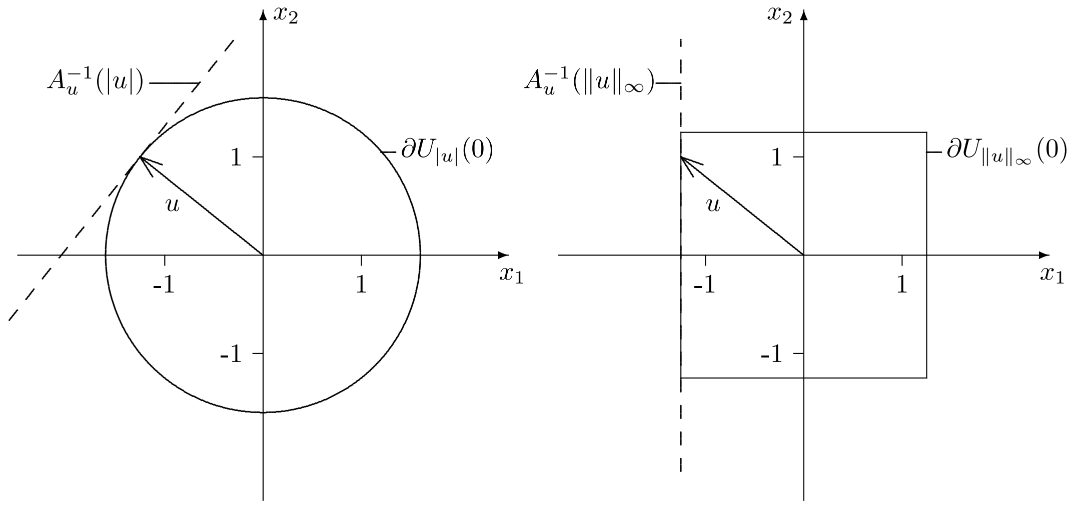
Then $A_u(u) = \frac{u}{|u|}\cdot u = |u|$ and

$$\begin{aligned} 1 = \Big|\frac{u}{|u|}\Big| &= \frac{1}{|u|}A_u(u) \le \frac{1}{|u|}\|A_u\|\,|u| = \|A_u\| \\ &= \sup_{|v|\le 1}|A_u(v)| = \sup_{|v|\le 1}\Big|\frac{u}{|u|}\cdot v\Big| \le \sup_{|v|\le 1}\frac{|u|\cdot|v|}{|u|} = 1\,, \end{aligned}$$

Hence $\|A_u\| = 1$.

Of course, the linear mapping A_u depends on the norm $\|\cdot\|$ on $\mathbb{R}^n$. To illustrate this, we show in the figure the level lines of the different mappings $A_u : \mathbb{R}^2 \to \mathbb{R}$ to the same vector $u \in \mathbb{R}^2$ for the cases when the norm is the Euclidean norm $|\cdot|$ and when the norm is the maximum norm $\|\cdot\|_\infty$.

Proof of the corollary: To $f(b) - f(a) \in \mathbb{R}^m$ choose the linear mapping $A : \mathbb{R}^m \to \mathbb{R}$ such that $\|A\| = 1$ and $A\big(f(b) - f(a)\big) = \|f(b) - f(a)\|$. As a linear mapping, A is differentiable with derivative $A'(y) = A$ for all $y \in \mathbb{R}^m$. Thus, from the mean value theorem applied to the differentiable function $F = A \circ f : U \to \mathbb{R}$ we conclude that a



Level sets $A_u^{-1}(\|u\|)$ of the linear mappings $A_u : \mathbb{R}^2 \to \mathbb{R}$ associated to the vector $u = (-1.25, 1)$. Indicated are also the boundaries $\partial U_{\|u\|}(0)$ of the balls with radius $\|u\|$ and center 0. Left: $\mathbb{R}^2$ with the Euclidean norm $|\cdot|$. Right: $\mathbb{R}^2$ with the norm $\|\cdot\|_\infty$

number ϑ with $0 < \vartheta < 1$ exists such that

$$\begin{aligned}\|f(b) - f(a)\| &= A\big(f(b) - f(a)\big) = A\big(f(b)\big) - A\big(f(a)\big) = F(b) - F(a)\\ &= F'\big(a + \vartheta(b-a)\big)(b-a) = Af'\big(a + \vartheta(b-a)\big)(b-a)\\ &\le \|A\|\,\|f'\big(a + \vartheta(b-a)\big)\|\,\|b-a\| = \|f'\big(a + \vartheta(b-a)\big)\|\,\|b-a\|\,.\end{aligned}$$

■

Theorem 4.17 *Let U be an open and pathwise connected subset of $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^m$ be differentiable. Then f is constant if and only if $f'(x) = 0$ for all $x \in U$.*

To prove this theorem, the following lemma is needed:

Lemma 4.18 *Let $U \subseteq \mathbb{R}^n$ be open and pathwise connected. Then all points $a, b \in U$ can be connected by a polygon in U, i.e. by a curve consisting of finitely many straight line segments.*

A **proof** of this lemma can be found in the book of Barner-Flohr, Analysis II, p. 56.

Proof of the theorem: If f is constant, then evidently $f'(x) = 0$ for all $x \in U$. To prove the converse, assume that $f'(x) = 0$ for all $x \in U$. Let a, b be two arbitrary points in U. These points can be connected in U by a polygon with the corner points

$$a_0 = a\,,\ a_1, \dots, a_{k-1}\,,\ a_k = b\,.$$

We apply Corollary 4.15 to the line segment connecting a_j and a_{j+1} for $j = 0, 1, \dots, k-1$. Since $f'(x) = 0$ for all $x \in U$, the operator norm $\|f'(x)\|$ is bounded on this line segment by 0. Therefore Corollary 4.15 yields $\|f(a_{j+1}) - f(a_j)\| \le 0$, hence $f(a_{j+1}) = f(a_j)$ for all $j = 0, 1, \dots, k-1$, which implies

$$f(b) = f(a)\,.$$

■

From the existence of all the partial derivatives $\frac{\partial f}{\partial x_1}(a), \dots, \frac{\partial f}{\partial x_n}(a)$ at a, one cannot conclude that f is differentiable at a. However, we have the following useful criterion for differentiability of f at a:

Theorem 4.19 *Let U be an open subset of $\mathbb{R}^n$ with $a \in U$ and let $f : U \to \mathbb{R}^m$. If all partial derivatives $\frac{\partial f_j}{\partial x_i}$ exist in U for $i = 1, \dots, n$ and $j = 1, \dots, m$, and if all the functions $x \mapsto \frac{\partial f_j}{\partial x_i}(x) : U \to \mathbb{R}$ are continuous at a, then f is differentiable at a.*

Proof: It suffices to prove that all the component functions $f_1, \dots, f_m$ are differentiable at a. Thus, we can asssume that $f : U \to \mathbb{R}$ is real valued. We have to show that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - Th}{\|h\|_\infty} = 0$$

for the linear mapping T with the matrix representation

$$T = \Big(\frac{\partial f}{\partial x_1}(a), \dots, \frac{\partial f}{\partial x_n}(a)\Big)\,.$$

For $h = (h_1, \dots, h_n) \in \mathbb{R}^n$ define

$$\begin{aligned}
a_0 &:= a \\
a_1 &:= a_0 + h_1 e_1 \\
a_2 &:= a_1 + h_2 e_2 \\
&\vdots \\
a + h = a_n &:= a_{n-1} + h_n e_n\,,
\end{aligned}$$

where $e_1, \dots, e_n$ is the canonical basis of $\mathbb{R}^n$. Then

$$f(a+h) - f(a) = \big(f(a+h) - f(a_{n-1})\big) + \big(f(a_{n-1}) - f(a_{n-2})\big) + \dots + \big(f(a_1) - f(a)\big)\,. \quad (4.6)$$

If x runs through the line segment connecting a_{j-1} to a_j, then only the component x_j of x is varying. Since by assumption the mapping $x_j \to f(x_1, \ldots, x_j, \ldots, x_n)$ is differentiable, the mean value theorem can be applied to every term on the right hand side of (4.6). Let c_j be the intermediate point on the line segment connecting a_{j-1} to a_j. Then

$$f(a+h) - f(a) = \sum_{j=1}^{n} \big(f(a_j) - f(a_{j-1})\big) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(c_j) h_j\,,$$

whence

$$\begin{aligned} |f(a+h) - f(a) - Th| &= \Big|\sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(c_j) h_j - \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a) h_j\Big| \\ &= \Big|\sum_{j=1}^{n} \Big(\frac{\partial f}{\partial x_j}(c_j) - \frac{\partial f}{\partial x_j}(a)\Big) h_j\Big| \\ &\leq \|h\|_\infty \sum_{j=1}^{n} \Big|\frac{\partial f}{\partial x_j}(c_j) - \frac{\partial f}{\partial x_j}(a)\Big|\,. \end{aligned}$$

Because the intermediate points satisfy $\|c_j - a\|_\infty \leq \|h\|_\infty$ for all $j = 1, \ldots, n$, it follows that $\lim_{h\to 0} c_j = a$ for all intermediate points. The continuity of the partial derivatives at a thus implies

$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - Th|}{\|h\|_\infty} \leq \lim_{h\to 0} \sum_{j=1}^{n} \Big|\frac{\partial f}{\partial x_j}(c_j) - \frac{\partial f}{\partial x_j}(a)\Big| = 0\,.$$

■

Example: Let $s \in \mathbb{R}$ and let $f : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ be defined by

$$f(x) = (x_1^2 + \ldots + x_n^2)^s.$$

This mapping is differentiable, since the partial derivatives

$$\frac{\partial f}{\partial x_j}(x) = s(x_1^2 + \ldots + x_n^2)^{s-1} 2x_j$$

are continuous in $\mathbb{R}^n \backslash \{0\}$.

4.5 Continuously differentiable mappings, second derivative

Let $U \subseteq \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}^m$ be differentiable. The derivative of f is a mapping

$$x \mapsto f'(x) : U \to L(\mathbb{R}^n, \mathbb{R}^m).$$

If one applies the linear mapping $f'(x)$ to a vector $h \in \mathbb{R}^n$, a vector of $\mathbb{R}^m$ is obtained. Thus, f' can also be viewed as a mapping

$$(x,h) \mapsto f'(x,h) = f'(x)h : U \times \mathbb{R}^n \to \mathbb{R}^m.$$

The mapping $(x,h) \mapsto f'(x,h)$ is linear with respect to the second argument. What view one takes is a matter of convenience.

We want to define continuity of the mapping f'. Since $L(\mathbb{R}^n, \mathbb{R}^m)$ is a normed space, on which we can measure the distance of two elements $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$ by the operator norm $\|A - B\|$, we can choose the first view and define continuity of f' as follows:

Definition 4.20 *Let $U \subseteq \mathbb{R}^n$ be an open set and suppose that $f : U \to \mathbb{R}^m$ is differentiable.*

(i) *$f' : U \to L(\mathbb{R}^n, \mathbb{R}^m)$ is said to be continuous at $a \in U$ if to every $\varepsilon > 0$ there is $\delta > 0$ such that for all $x \in U$ with $\|x - a\| < \delta$*

$$\|f'(x) - f'(a)\| < \varepsilon\,.$$

f is said to be continuously differentiable if $f' : U \to L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous.

(ii) *Let $U, V \subseteq \mathbb{R}^n$ be open and let $f : U \to V$ be continuously differentiable and invertible. If the inverse $f^{-1} : V \to U$ is also continuously differentiable, then f is called a diffeomorphism.*

The following result makes this definition less abstract:

Theorem 4.21 *Let $U \subseteq \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}^m$. Then the following statements are equivalent:*

(i) *f is continuously differentiable.*

(ii) *All partial derivatives $\frac{\partial}{\partial x_i} f_j$ with $1 \le i \le n$, $1 \le j \le m$ exist in U and are continuous functions*

$$x \mapsto \frac{\partial}{\partial x_i} f_j(x) : U \to \mathbb{R}\,.$$

(iii) *f is differentiable and the mapping $x \mapsto f'(x)h : U \to \mathbb{R}^m$ is continuous for every $h \in \mathbb{R}^n$.*

Proof: First we show that (i) and (ii) are equivalent. If f is differentiable, then all partial derivatives exist in U. Conversely, if all partial derivatives exist in U and are continuous, then by Theorem 4.19 the function f is differentiable. Hence, it remains to show that f' is continuous if and only if all partial derivatives are continuous.

For $a, x \in U$ let

$$\|f'(x) - f'(a)\|_\infty = \max_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}} \Big|\frac{\partial f_j}{\partial x_i}(x) - \frac{\partial f_j}{\partial x_i}(a)\Big| . \tag{4.7}$$

By Theorem 3.50 there exist constants $c, C > 0$, which are independent of x and a, such that $c\|f'(x) - f'(a)\|_\infty \le \|f'(x) - f'(a)\| \le C\|f'(x) - f'(a)\|_\infty$. From this estimate and from (4.7) we see that

$$\lim_{x\to 0} \|f'(x) - f'(a)\| = 0$$

holds, if and only if

$$\lim_{x\to a} \frac{\partial f_j}{\partial x_i}(x) = \frac{\partial f_j}{\partial x_i}(a) ,$$

for all $1 \le i \le n$, $1 \le j \le m$. By Definition 4.20 this means that f' is continuous at a, if and only if all partial derivatives are continuous at a.

To prove that (iii) is equivalent to the first two statements of the theorem it suffices to remark that if f is differentiable, then

$$x \mapsto f'(x)h = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)h_i : U \to \mathbb{R}^m.$$

By choosing for h vectors from the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$, we immediately see from this equation that $x \mapsto f'(x)h$ is continuous for every $h \in \mathbb{R}^n$, if and only if all partial derivatives are continuous. ■

Let $f : U \to \mathbb{R}^m$ be differentiable. To define the second derivative $f''(a)$ of f at a point $a \in U$ we can again view the derivative of f as a function $f' : U \to L(\mathbb{R}^n, \mathbb{R}^m)$ or as a function $f' : U \times \mathbb{R}^n \to \mathbb{R}^m$. In the first view the derivative of f' at a, which is $f''(a)$, is obtained by generalizing Definition 4.1, replacing the norm of $\mathbb{R}^m$ used in this definition by the operator norm of $L(\mathbb{R}^n, \mathbb{R}^m)$. The second derivative $f''(a)$ obtained in this way is a linear function from $\mathbb{R}^n$ to the space $L(\mathbb{R}^n, \mathbb{R}^m)$, hence $f''(a) \in L\big(\mathbb{R}^n, L(\mathbb{R}^n, \mathbb{R}^m)\big)$.

The second view leads to an equivalent result, but the path which one has to follow in the definition of $f''(a)$ is less abstract. In the following definition of $f''(a)$ we therefore follow this path.

Definition 4.22 *Let $U \subseteq \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}^m$ be differentiable. f is said to be two times differentiable at a point $x \in U$, if to every fixed $h \in \mathbb{R}^n$ the mapping $g_h : U \to \mathbb{R}^m$ defined by*

$$g_h(x) = f'(x)h$$

is differentiable at x. The second derivative $f''(x) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ of f at x is defined by

$$f''(x)(h,k) = g_h'(x)(k).$$

If the second derivative $f''(x)$ exists at every $x \in U$, then f is said to be two times differentiable. The second derivative defines then the function

$$(x,h,k) \mapsto f''(x)(h,k) : U \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m.$$

Theorem 4.23 *Let $U \subseteq \mathbb{R}^n$ be an open set, let $f : U \to \mathbb{R}^m$ be differentiable and let x be a point from U.*

(i) *If f is two times differentiable at x, then all second partial derivatives of f at x exist, and for $h = (h_1, \dots h_n) \in \mathbb{R}^n$ and $k = (k_1, \dots, k_n) \in \mathbb{R}^n$*

$$f''(x)(h,k) = \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} f(x) h_i k_j \,. \tag{4.8}$$

(ii) *$f''(x)$ is bilinear, i.e. $(h,k) \to f''(x)(h,k)$ is linear in both arguments.*

(iii) *If all second partial derivatives of f exist in U and are continuous at x, then f is two times differentiable at x.*

Proof: To prove (i), note that if f is two times differentiable at x, then by definition the function

$$y \mapsto g_h(y) = f'(y)h = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(y) h_i$$

is differentiable at $y = x$, hence

$$f''(x)(h,k) = g_h'(x)k = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} g_h(x) k_j = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \Big(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(x) h_i \Big) k_j \,.$$

With $h = e_i$ and $k = e_j$, where e_i and e_j are vectors from the standard basis of $\mathbb{R}^n$, this formula implies that the second partial derivative $\frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} f(x)$ exists. Thus, in this formula the partial derivative and the summation can be interchanged, hence the representation formula (4.8) for $f''(x)(h,k)$ results. This proves (i). The bilinearity of $f''(x)$ stated in (ii) follows immediately from this representation formula.

It remains to verify (iii). If all second partial derivatives of f exist in U, then for every $h \in \mathbb{R}^n$ the first partial derivatives of the function $y \mapsto g_h(y)$ exist and are given by

$$\frac{\partial}{\partial y_j} g_h(y) = \frac{\partial}{\partial y_j}\Big(\sum_{i=1}^{n} \frac{\partial}{\partial y_i} f(y) h_i \Big) = \sum_{i=1}^{n} \frac{\partial}{\partial y_j}\frac{\partial}{\partial y_i} f(y) h_i \,.$$

Since by assumption the second partial derivatives are continuous at x, it follows from this equation that $y \mapsto \frac{\partial}{\partial y_j} g_h(y)$ is continuous at x for $j = 1, \ldots, n$. Theorem 4.19 thus implies that g_h is differentiable at x, which means that f is two times differentiable at x.

■

For the second partial derivatives of f one also uses the notation

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} f, \qquad \frac{\partial^2 f}{\partial x_i^2} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i} f.$$

Note that

$$\frac{\partial^2 f}{\partial x_j \partial x_i}(x) = \begin{pmatrix} \frac{\partial^2}{\partial x_j \partial x_i} f_1(x) \\ \vdots \\ \frac{\partial^2}{\partial x_j \partial x_i} f_m(x) \end{pmatrix} \in \mathbb{R}^m.$$

For $m = 1$, the second partial derivatives $\frac{\partial^2}{\partial x_j \partial x_i} f(x)$ are real numbers. Thus, for $f : U \to \mathbb{R}$ we obtain a matrix representation for $f''(x)$:

$$\begin{aligned} f''(x)(h,k) &= \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial^2}{\partial x_j \partial x_i} f(x) h_i k_j \\ &= (h_1, \ldots, h_n) \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \ldots & \frac{\partial^2 f}{\partial x_n \partial x_1}(x) \\ \vdots & & \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) & \ldots & \frac{\partial^2 f}{\partial x_n^2}(x) \end{pmatrix} \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} = hHk, \end{aligned}$$

with the Hessian matrix

$$H = \Big(\frac{\partial^2 f}{\partial x_j \partial x_i} \Big)_{j,i=1,\ldots,n}.$$

(Ludwig Otto Hesse 1811 – 1874). For $f : U \to \mathbb{R}^m$ with $m > 1$ one obtains

$$\big(f''(x)\big)_\ell (h,k) = hH_\ell k,$$

where H_ℓ is the Hessian matrix for the component function f_ℓ of f. In particular, this yields

$$\big(f''(x)\big)_\ell(h,k) = (f_\ell)''(x)(h,k),$$

i.e. the ℓ–th component of $f''(x)$ is the second derivative of the component function f_ℓ.

It is possible, that all second partial derivatives of f at x exist, even if f is not two times differentiable at x. In this case the Hessian matrices H_ℓ can be formed, but they do not represent the second derivative of f at x, which does not exist.

The following important theorem shows that the second derivative is a symmetric bilinear form:

Theorem 4.24 (of H.A. Schwarz) *Let $U \subseteq \mathbb{R}^n$ be open, let $x \in U$ and let f be two times differentiable at x. Then for all $h, k \in \mathbb{R}^n$*

$$f''(x)(h,k) = f''(x)(k,h).$$

(Hermann Amandus Schwartz, 1843 – 1921.)

This result implies that the Hessian matrices H_ℓ are symmetric if f is two times differentiable at x. To see this, note that Theorem 4.24 and (4.8) yield for the standard basis vectors e_i and e_j that

$$\frac{\partial^2}{\partial x_j \partial x_i} f_\ell(x) = (f''(x))_\ell(e_i, e_j) = (f''(x))_\ell(e_j, e_i) = \frac{\partial^2}{\partial x_i \partial x_j} f_\ell(x), \tag{4.9}$$

for all $1 \le i, j \le n$, whence the order of differentiation does not matter if f is two times differentiable at x.

Proof of Theorem 4.24: Obviously the bilinear mapping $f''(x)$ is symmetric, if and only if every component function $((f''(x))_\ell$ is symmetric. Therefore it suffices to show that every component is symmetric. Since $(f''(x))_\ell = (f_\ell)''(x)$ and since $f_\ell : U \to \mathbb{R}$ is real valued, it is sufficient to prove that for every real valued function $f : U \to \mathbb{R}$ the second derivative $f''(x)$ is symmetric. We thus assume that f is real valued.

To prove symmetry, we show that for all $h, k \in \mathbb{R}^n$

$$\lim_{\substack{s\to 0\\ s>0}} \frac{f(x+sh+sk) - f(x+sh) - f(x+sk) + f(x)}{s^2} = f''(x)(h,k). \tag{4.10}$$

The statement of the theorem ist a consequence of this formula, since the left hand side remains unchanged if h and k are interchanged.

By definition, $f''(x)(h,k)$ is the derivative of the function $x \mapsto f'(x)h$. Thus, for all $h, k \in \mathbb{R}^n$,

$$f'(x+k)h - f'(x)h = f''(x)(h,k) + R_x(h,k)\|k\| \tag{4.11}$$

with

$$\lim_{k \to 0} R_x(h,k) = 0.$$

$R_x(h,k)$ is linear with respect to h, since $f'(x+k)h$, $f'(x)h$ and $f''(x)(h,k)$ are linear with respect to h. We show that a number ϑ with $0 < \vartheta < 1$ exists, which depends on h and k, such that

$$\begin{aligned} &f(x+h+k) - f(x+h) - f(x+k) + f(x) \\ &\qquad = f''(x)(h,k) + R_x(h, \vartheta h + k)\|\vartheta h + k\| - R_x(h, \vartheta h)\|\vartheta h\| \end{aligned} \tag{4.12}$$

holds. For the proof let $F : [0,1] \to \mathbb{R}$ be defined by

$$F(t) = f(x + th + k) - f(x + th).$$

F is differentiable, whence the mean value theorem implies that $0 < \vartheta < 1$ exists with

$$F(1) - F(0) = F'(\vartheta).$$

Therefore, with the definition of F and with (4.11),

$$\begin{aligned} &f(x+h+k) - f(x+h) - f(x+k) + f(x) \\ &\quad = F(1) - F(0) = F'(\vartheta) = f'(x + \vartheta h + k)h - f'(x + \vartheta h)h \\ &\quad = \big(f'(x+\vartheta h + k)h - f'(x)h\big) - \big(f'(x+\vartheta h)h - f'(x)h\big) \\ &\quad = \big(f''(x)(h, \vartheta h + k) + R_x(h, \vartheta h + k)\|\vartheta h + k\|\big) - \big(f''(x)(h, \vartheta h) + R_x(h, \vartheta h)\|\vartheta h\|\big) \\ &\quad = f''(x)(h,k) + R_x(h, \vartheta h + k)\|\vartheta h + k\| - R_x(h, \vartheta h)\|\vartheta h\|, \end{aligned}$$

which is (4.12). In the last step we used the linearity of $f''(x)$ in the second argument.

Let $s > 0$. If one replaces in (4.12) the vector k by sk and the vector h by sh, then on the right hand side the factor s^2 can be extracted, because of the bilinearity or linearity or the positive homogeneity of all the terms. The result is

$$\begin{aligned} &f(x+sh+sk) - f(x+sh) - f(x+sk) + f(x) \\ &\quad = s^2 \Big[f''(x)(h,k) + R_x\big(h, s(\vartheta h + k)\big)\|\vartheta h + k\| - R_x(h, s\vartheta h)\|\vartheta h\| \Big]. \end{aligned}$$

Since

$$\lim_{s \to 0} R_x\big(h, s(\vartheta h + k)\big) = 0, \quad \lim_{s \to 0} R_x(h, s\vartheta h) = 0,$$

this equation yields (4.10). ∎

Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x_1, x_2) = x_1^2 x_2 + x_1 + x_2^3.$$

All first and second partial derivatives of f exist and are continuous, hence f is continuously differentiable and even two times differentiable. We have

$$\operatorname{grad} f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \frac{\partial f}{\partial x_2}(x) \end{pmatrix} = \begin{pmatrix} 2x_1 x_2 + 1 \\ x_1^2 + 3x_2^2 \end{pmatrix},$$

and the Hessian matrix is

$$f''(x) = H(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) \end{pmatrix} = \begin{pmatrix} 2x_2 & 2x_1 \\ 2x_1 & 6x_2 \end{pmatrix}.$$

4.6 Higher derivatives and Taylor formula

Higher derivatives are defined by induction: Let $U \subseteq \mathbb{R}^n$ be open. The p-th derivative of $f : U \to \mathbb{R}^m$ at x is a mapping

$$f^{(p)}(x) : \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{p\text{-factors}} \to \mathbb{R}^m$$

obtained as follows: If f is $(p-1)$–times differentiable in U and if for all $h_1, \ldots, h_{p-1} \in \mathbb{R}^n$ the mapping

$$y \mapsto f^{(p-1)}(y)(h_1, \ldots, h_{p-1}) : U \to \mathbb{R}^m$$

is differentiable at $y = x$, then f is said to be p–times differentiable at x with pth derivative $f^{(p)}(x)$ defined by

$$f^{(p)}(x)(h_1, \ldots, h_p) = \left[f^{(p-1)}(\cdot)(h_1, \ldots, h_{p-1})\right]'(x)h_p,$$

for $h_1, \ldots, h_p \in \mathbb{R}^n$.

The function $(h_1, \ldots, h_p) \to f^{(p)}(x)(h_1, \ldots, h_p)$ is linear in all its arguments, and from Theorem 4.24 one obtains by induction that it is *totally symmetric*, that is, for $1 \leq i < j \leq p$ one has

$$f^{(p)}(x)(h_1, \ldots, h_i, \ldots, h_j, \ldots, h_p) = f^{(p)}(x)(h_1, \ldots, h_j, \ldots, h_i, \ldots, h_p).$$

From the representation formula (4.8) for the second derivatives one immediately obtains by induction with respect to p that with the vectors $h^{(j)} = (h_1^{(j)}, \ldots, h_n^{(j)}) \in \mathbb{R}^n$, $j = 1, \ldots, p$, the representation formula

$$f^{(p)}(x)(h^{(1)}, \ldots, h^{(p)}) = \sum_{i_1=1}^{n} \cdots \sum_{i_p=1}^{n} \frac{\partial^p f}{\partial x_{i_1} \ldots \partial x_{i_p}}(x) h_{i_1}^{(1)} \ldots h_{i_p}^{(p)} \tag{4.13}$$

holds. Also by induction with respect to p we see as in the proof of statement (iii) from Theorem 4.23 that if f is $(p-1)$–times differentiable and if all pth partial derivatives of f exist in U and are continuous at x, then f is p–times differentiable at x.

The spaces $C^p(U,\mathbb{R}^m)$ and $C^\infty(U,\mathbb{R}^m)$. In accordance with Theorem 4.21 one says that f is p–times continuously differentiable, if f is p–times differentiable and the mapping $x \mapsto f^{(p)}(x)(h^{(1)},\ldots,h^{(p)}) : U \to \mathbb{R}^m$ is continuous for all $h^{(1)},\ldots,h^{(p)} \in \mathbb{R}^n$. By choosing in (4.13) for $h^{(1)},\ldots,h^{(p)}$ vectors from the standard basis $e_1,\ldots,e_n$ of $\mathbb{R}^n$, it is immediately seen that f is p–times continuously differentiable, if and only if all partial derivatives of f up to order p exist and are continuous. The set of all p–times continuously differentiable functions $f : U \to \mathbb{R}^m$ is denoted by $C^p(U,\mathbb{R}^m)$.

If $f^{(p)}$ exists for all $p \in \mathbb{N}$, then f is said to be infinitely differentiable. This happens if and only if all partial derivatives of any order exist in U. The set of all infinitely differentiable functions $f : U \to \mathbb{R}^m$ is denoted by $C^\infty(U,\mathbb{R}^m)$.

For $f,g \in C^p(U,\mathbb{R}^m)$ and $c \in \mathbb{R}$ the functions $f+g$ and cf belong to $C^p(U,\mathbb{R}^m)$. This can be seen most easily from the representation formula (4.13). Therefore $C^p(U,\mathbb{R}^m)$ and $C^\infty(U,\mathbb{R}^m)$ are vector spaces. $C^0(U,\mathbb{R}^m)$ is equal to the space $C(U,\mathbb{R}^m)$ of continuous functions studied in Section 3.4.

Theorem 4.25 (Taylor formula) *Let U be an open subset of $\mathbb{R}^n$, let $f : U \to \mathbb{R}$ be $(p+1)$-times differentiable, and assume that the points x and $x+h$ together with the line segment connecting these points belong to U. Then there is a number ϑ with $0 < \vartheta < 1$ such that*

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2!}f''(x)(h,h) + \ldots + \frac{1}{p!}f^{(p)}(x)(\underbrace{h,\ldots,h}_{p\text{-times}}) + R_p(x,h)\,,$$

where

$$R_p(x,h) = \frac{1}{(p+1)!}\, f^{(p+1)}(x+\vartheta h)(\,\underbrace{h,\ldots,h}_{p+1\text{-times}}\,)\,.$$

Proof: Let $\gamma : [0,1] \to U$ be defined by $\gamma(t) = x + th$. To $F = f \circ \gamma : [0,1] \to \mathbb{R}$ apply the Taylor formula for real functions:

$$F(1) = \sum_{j=0}^{p} \frac{F^{(j)}(0)}{j!} + \frac{1}{(p+1)!}\, F^{(p+1)}(\vartheta)\,.$$

Insertion of the derivatives

$$
\begin{aligned}
F'(t) &= f'\big(\gamma(t)\big)\,\gamma'(t) = f'\big(\gamma(t)\big)h\,,\\
F''(t) &= f''\big(\gamma(t)\big)\,\big(h, \gamma'(t)\big) = f''\big(\gamma(t)\big)(h,h)\,,\\
&\ \vdots\\
F^{(p+1)}(t) &= f^{(p+1)}\big(\gamma(t)\big)\big(h,\dots,\gamma'(t)\big) = f^{(p+1)}\big(\gamma(t)\big)(h,\dots,h),
\end{aligned}
$$

into this formula yields the statement. ■

Using the representation (4.13) of $f^{(j)}$ by partial derivatives the Taylor formula can also be written as

$$
\begin{aligned}
f(x+h) &= \sum_{j=0}^{p} \frac{1}{j!}\Big[\sum_{i_1,\dots,i_j=1}^{n} \frac{\partial^j f(x)}{\partial x_{i_1}\dots\partial x_{i_j}}\, h_{i_1}\dots h_{i_j}\Big]\\
&\quad + \frac{1}{(p+1)!}\sum_{i_1,\dots,i_{p+1}=1}^{n} \frac{\partial^{p+1} f(x+\vartheta h)}{\partial x_{i_1}\dots\partial x_{i_{p+1}}}\, h_{i_1}\dots h_{i_{p+1}}\,.
\end{aligned}
$$

In this formula the notation can be simplified using multi-indices. For a multi-index $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}_0^n$ and for $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ set

$$
\begin{aligned}
|\alpha| &:= \alpha_1 + \dots + \alpha_n \qquad \text{(length of } \alpha)\\
\alpha! &:= \alpha_1!\dots\alpha_n!\\
x^\alpha &:= x_1^{\alpha_1}\dots x_n^{\alpha_n}\,,\\
D^\alpha f(x) &:= \frac{\partial^{|\alpha|} f(x)}{\partial^{\alpha_1}x_1\dots\partial^{\alpha_n}x_n}\,.
\end{aligned}
$$

If α is a fixed multi-index with length $|\alpha| = j$, then the sum

$$
\sum_{i_1,\dots,i_j=1}^{n} \frac{\partial^j f(x)}{\partial x_{i_1}\dots\partial x_{i_j}}\, h_{i_1}\dots h_{i_j}
$$

contains $\frac{j!}{\alpha_1!\dots\alpha_n!}$ terms, which are obtained from $D^\alpha f(x)h^\alpha$ by interchanging the order, in which the derivatives are taken. Using this, the Taylor formula can be written in the compact form

$$
\begin{aligned}
f(x+h) &= \sum_{j=0}^{p}\sum_{|\alpha|=j} \frac{1}{\alpha!} D^\alpha f(x)h^\alpha + \sum_{|\alpha|=p+1} \frac{1}{\alpha!} D^\alpha f(x+\vartheta h)h^\alpha\\
&= \sum_{|\alpha|\le p} \frac{1}{\alpha!} D^\alpha f(x)h^\alpha + \sum_{|\alpha|=p+1} \frac{1}{\alpha!} D^\alpha f(x+\vartheta h)h^\alpha\,.
\end{aligned}
$$

5 Local extreme values, inverse function and implicit function

5.1 Local extreme values

Definition 5.1 *Let $U \subseteq \mathbb{R}^n$ be open, let $f : U \to \mathbb{R}$ be differentiable and let $a \in U$. If $f'(a) = 0$, then a is called critical point of f.*

Theorem 5.2 *Let $U \subseteq \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}$ be differentiable. If f has a local extreme value at a, then a is a critical point of f.*

Proof: Without restriction of generality we assume that f has a local maximum at a. Then there is a neighborhood V of a such that $f(x) \leq f(a)$ for all $x \in V$. Let $h \in \mathbb{R}^n$ and choose $\delta > 0$ small enough such that $a + th \in V$ for all $t \in \mathbb{R}$ with $|t| \leq \delta$. Let $F : [-\delta, \delta] \to \mathbb{R}$ be defined by

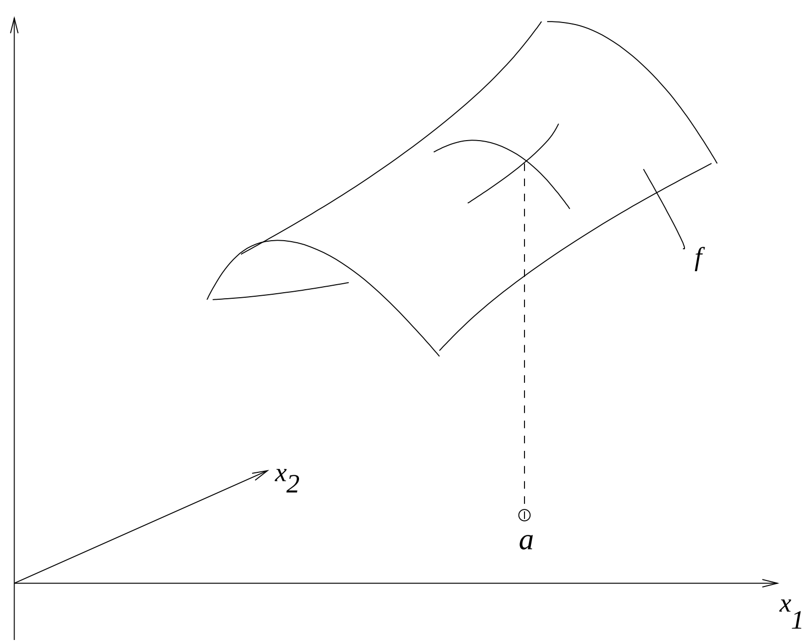
$$F(t) = f(a + th)\,.$$

Then F has a local maximum at $t = 0$, hence

$$0 = F'(0) = f'(a)h\,.$$

Since this holds for every $h \in \mathbb{R}^n$, it follows that $f'(a) = 0$. ∎

Thus, if f has a local extreme value at a, then a is necessarily a critical point. For example, the saddle point a in the following picture is a critical point, but f has not an extreme value there.



This example shows that for functions of several variables the situation is more complicated than for functions of one variable. Still, also for functions of several variables the second derivative can be used to formulate a sufficient criterion for an extreme value. To this end we need the notion of a quadratic form:

Definition 5.3 *Let $Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bilinear mapping. Then the mapping $h \to Q(h,h) : \mathbb{R}^n \to \mathbb{R}$ is called a quadratic form. A quadratic form is called*

(i) *positive definite, if $Q(h,h) > 0$ for all $h \neq 0$,*

(ii) *positive semi-definite, if $Q(h,h) \geq 0$ for all h,*

(iii) *negative definite, if $Q(h,h) < 0$ for all $h \neq 0$,*

(iv) *negative semi definite, if $Q(h,h) \leq 0$ for all h,*

(v) *indefinite, if $Q(h,h)$ has positive and negative values.*

We state some results on quadratic forms from linear algebra without proof: To a quadratic form $Q(h,h)$ one can always find a symmetric coefficient matrix

$$C = \begin{pmatrix} c_{11} & \dots & c_{1n} \\ \vdots & & \\ c_{n1} & \dots & c_{nn} \end{pmatrix}$$

such that

$$Q(h,h) = \sum_{i,j=1}^{n} c_{ij} h_i h_j = h \cdot Ch\,.$$

From this representation it follows that for a quadratic form the mapping $h \mapsto Q(h,h) : \mathbb{R}^n \to \mathbb{R}$ is continuous.

Theorem 5.4 (Criterion of Hurwitz–Sylvester) *The quadratic form $Q(h,h) = h \cdot Ch$ is positive definite, if and only if the leading principal minors of the symmetric coefficient matrix C satisfy*

$$c_{11} > 0\,, \quad \det\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} > 0\,, \quad \det\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & x_{32} & c_{33} \end{pmatrix} > 0\,, \dots, \det(c_{ij})_{i,j=1,\dots,n} > 0\,.$$

(Adolf Hurwitz 1859 - 1919, James Joseph Sylvester 1814 - 1897.)

If $f : U \to \mathbb{R}$ is two times differentiable at $x \in U$, then $(h,k) \mapsto f''(x)(h,k)$ is bilinear, hence $h \mapsto f''(x)(h,h)$ is a quadratic form. Since

$$f''(x)(h,h) = \sum_{i,j=1}^{n} \frac{\partial^2 f(x)}{\partial x_i \partial x_j} h_i h_j \,,$$

the coefficient matrix to this quadratic form is the Hessian matrix

$$H = \Big(\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \Big)_{i,j=1,\ldots,n} .$$

By the theorem of H.A. Schwarz (Theorem 4.24), this matrix is symmetric.

Now we can formulate a sufficient criterion for extreme values:

Theorem 5.5 *Let $U \subseteq \mathbb{R}^n$ be open, let $f : U \to \mathbb{R}$ be two times continuously differentiable, and let $a \in U$ be a critical point of f. If the quadratic form $f''(a)(h,h)$*

(i) *is positive definite, then f has a local minimum at a,*

(ii) *is negative definite, then f has a local maximum at a,*

(iii) *is indefinite, then f does not have an extreme value at a.*

Proof: The Taylor formula yields

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''\big(a + \vartheta(x-a)\big)(x-a, x-a) \,,$$

with a suitable $0 < \vartheta < 1$. Thus, since $f'(a) = 0$,

$$\begin{aligned} f(x) &= f(a) + \frac{1}{2} f''\big(a + \vartheta(x-a)\big)(x-a, x-a) \\ &= f(a) + \frac{1}{2} f''(a)(x-a, x-a) + R(x)(x-a, x-a) \,, \end{aligned} \qquad (5.1)$$

with

$$\begin{aligned} R(x)(h,k) &= \frac{1}{2} f''\big(a + \vartheta(x-a)\big)(h,k) - \frac{1}{2} f''(a)(h,k) \\ &= \frac{1}{2} \sum_{i,j=1}^{n} \Big(\frac{\partial^2 f\big(a + \vartheta(x-a)\big)}{\partial x_i \partial x_j} - \frac{\partial^2 f(a)}{\partial x_i \partial x_j} \Big) h_j k_i . \end{aligned}$$

Since by assumption f is two times continuously differentiable, the second partial derivatives are continuous. Hence to every $\varepsilon > 0$ there is $\delta > 0$ such that for all $x \in U$ with $\|x - a\| < \delta$ and for all $1 \le i,\, j \le n$

$$\Big| \frac{\partial^2 f\big(a + \vartheta(x-a)\big)}{\partial x_i \partial x_j} - \frac{\partial^2 f(a)}{\partial x_i \partial x_j} \Big| < \frac{2}{n^2} \varepsilon \,.$$

Consequently, for $x \in U$ with $\|x-a\| < \delta$

$$|R(x)(h,h)| \le \frac{1}{2}\sum_{i,j=1}^{n} \frac{2}{n^2}\varepsilon\,\|h\|_\infty\,\|h\|_\infty \le \varepsilon c^2\|h\|^2\,, \tag{5.2}$$

where in the last step we used that there is a constant $c > 0$ with $\|h\|_\infty \le c\,\|h\|$ for all $h \in \mathbb{R}^n$.

Assume now that $f''(a)(h,h) > 0$ is a positive definite quadratic form. Then $f''(a)(h,h) > 0$ for all $h \in \mathbb{R}^n$ with $h \neq 0$, and since the continuous mapping $h \mapsto f''(a)(h,h) : \mathbb{R}^n \to \mathbb{R}$ attains the minimum on the closed and bounded, hence compact set $\{h \in \mathbb{R}^n \bigm| \|h\| = 1\}$ at a point h_0 from this set, it follows for all $h \in \mathbb{R}^n$ with $h \neq 0$

$$f''(a)(h,h) = \|h\|^2 f''(a)\Big(\frac{h}{\|h\|}\,,\,\frac{h}{\|h\|}\Big) \ge \|h\|^2 \min_{\|\eta\|=1} f''(a)(\eta,\eta) = \kappa\,\|h\|^2$$

with

$$\kappa = f''(a)(h_0,h_0) > 0\,.$$

Now choose $\varepsilon = \frac{\kappa}{4c^2}$. Then this estimate and (5.1), (5.2) yield that there is $\delta > 0$ such that for all $x \in U$ with $\|x-a\| < \delta$

$$\begin{aligned} f(x)-f(a) &= \frac{1}{2} f''(a)(x-a,x-a) + R(x)(x-a,x-a) \\ &\ge \frac{\kappa}{2}\,\|x-a\|^2 - \frac{\kappa}{4}\,\|x-a\|^2 = \frac{\kappa}{4}\|x-a\|^2 \ge 0. \end{aligned}$$

This means that f attains a local minimum at a.

In the same way one proves that a local maximum is attained at a if $f''(a)(h,h)$ is negative definite. If $f''(a)(h,h)$ is indefinite, there is $h_0 \in \mathbb{R}^n$, $k_0 \in \mathbb{R}^n$ with $\|h_0\| = \|k_0\| = 1$ and with

$$\kappa_1 := f''(a)(h_0,h_0) > 0\,, \quad \kappa_2 := f''(a)(k_0,k_0) < 1\,.$$

From these relations we conclude as above that for all points x on the straight line through a with direction vector h_0 sufficiently close to a the difference $f(x)-f(a)$ is positive, and for x on the straight line through a with direction vector k_0 sufficiently close to a the difference $f(x)-f(a)$ is negative. Thus, f does not attain an extreme value at a. ■

Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = 6xy - 3y^2 - 2x^3$. All partial derivatives of all orders exist, hence f is infinitely differentiable. Therefore the assumptions of the

Theorems 5.2 and 5.5 are satisfied. Thus, if (x, y) is a critical point, then

$$\operatorname{grad} f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) \\ \frac{\partial f}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} 6y - 6x^2 \\ 6x - 6y \end{pmatrix} = 0\,,$$

which yields for the critical points $(x, y) = (0, 0)$ and $(x, y) = (1, 1)$.

To determine, whether these critical points are extremal points, the Hessian matrix must be computed at these points. The Hessian is

$$H(x,y) = \begin{pmatrix} \frac{\partial^2}{\partial x^2} f(x,y) & \frac{\partial^2}{\partial y \partial x} f(x,y) \\ \frac{\partial^2}{\partial x \partial y} f(x,y) & \frac{\partial^2}{\partial y^2} f(x,y) \end{pmatrix} = \begin{pmatrix} -12x & 6 \\ 6 & -6 \end{pmatrix}.$$

The quadratic form $f''(0,0)(h,h)$ defined by the Hessian matrix

$$H(0,0) = \begin{pmatrix} 0 & 6 \\ 6 & -6 \end{pmatrix}$$

is indefinite. For, if $h = (1, 1)$ then

$$f''(0,0)(h,h) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 6 \\ 6 & -6 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 0 \end{pmatrix} = 6\,,$$

and if $h = (0, 1)$ then

$$f''(0,0)(h,h) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 6 \\ 6 & -6 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ -6 \end{pmatrix} = -6\,,$$

Therefore $(0, 0)$ is not an extremal point of f. On the other hand, the quadratic form $f''(1,1)(h,h)$ defined by the matrix

$$H(1,1) = \begin{pmatrix} -12 & 6 \\ 6 & -6 \end{pmatrix}$$

is negative definite. For, by Theorem 5.4 the matrix $-H(1,1)$ is positive definite, since $12 > 0$ and

$$\det \begin{pmatrix} 12 & -6 \\ -6 & 6 \end{pmatrix} = 72 - 36 > 0\,.$$

Consequently $H(1,1)$ is negative definite and $(1, 1)$ is a local maximum of f.

5.2 Banach's fixed point theorem

In this section we state and prove the Banach fixed point theorem, a tool which we need in the later investigations and which has many important applications in mathematics.

Definition 5.6 *Let X be a set and let $d : X \times X \to \mathbb{R}$ be a mapping with the properties*

(i) $d(x, y) \geq 0\,, \quad d(x, y) = 0 \Leftrightarrow x = y$

(ii) $d(x, y) = d(y, x)$ *(symmetry)*

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ *(triangle inequality)*

Then d is called a metric on X, and (X, d) is called a metric space. $d(x, y)$ is called the distance of x and y.

Examples 1.) Let X be a normed vector space with norm $\|\cdot\|$. A metric on X is defined by $d(x, y) := \|x-y\|$. With this definition every normed space becomes a metric space.

2.) Let X be a nonempty set. We define a metric on X by

$$d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y\,. \end{cases}$$

This metric is called discrete.

3.) On $\mathbb{R}$ a metric is defined by

$$d(x, y) = \frac{|x-y|}{1+|x-y|}\,.$$

To see that this is a metric, note that the properties (i) and (ii) of Definition 5.6 are obviously satisfied. It remains to show that the triangle inequality holds. To this end note that $t \mapsto \frac{t}{1+t} : [0, \infty) \to [0, \infty)$ is strictly increasing, since $\frac{d}{dt}\frac{t}{1+t} = \frac{1}{1+t}(1 - \frac{t}{1+t}) > 0$. Thus, for $x, y, z \in \mathbb{R}$

$$\begin{aligned} d(x, y) &= \frac{|x-y|}{1+|x-y|} \leq \frac{|x-z|+|z-y|}{1+|x-z|+|z-y|} \\ &\leq \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|} = d(x, z) + d(z, y)\,. \end{aligned}$$

On a metric space (X, d) a topology can be defined, which is called the topology induced by the metric. In this topology an ε-neighborhood $B_\varepsilon(x)$ of the point $x \in X$ is given by

$$B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}.$$

Starting from this notion, neighborhoods and open and closed sets are defined just as on $\mathbb{R}^n$. This means in particular, that the topology of $\mathbb{R}^n$ is the topology induced by the metric $d(x,y) = \|x-y\|$. Continuous functions between metric spaces are defined just as continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^m$, and convergence of a sequence is defined as usual:

Definition 5.7 *Let (X,d) be a metric space.*
(i) A sequence $\{x_n\}_{n=1}^\infty$ with $x_n \in X$ is said to converge, if $x \in X$ exists such that to every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ with

$$d(x_n, x) < \varepsilon$$

for all $n \geq n_0$. The element x is called the limit of $\{x_n\}_{n=1}^\infty$.
(ii) A sequence $\{x_n\}_{n=1}^\infty$ with $x_n \in X$ is said to be a Cauchy sequence, if to every $\varepsilon > 0$ there is n_0 such that for all $n, k \geq n_0$

$$d(x_n, x_k) < \varepsilon\,.$$

Every converging sequence is a Cauchy sequence, but in a metric space the converse is not necessarily true.

Definition 5.8 *(i) A metric space with the property that every Cauchy sequence converges, is called a complete metric space.*
(ii) A subset of a metric space is called compact, if it has the Heine–Borel covering property.

Note however, that a bounded and closed subset of a metric space does not necessarily have the Heine–Borel property, even if the space is complete, hence on a metric space sets are not necessarily compact, if they are bounded and closed.

Definition 5.9 *Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be a contraction, if there is a number ϑ with $0 \leq \vartheta < 1$, which we call the contraction constant, such that for all $x, y \in X$*

$$d(Tx, Ty) \leq \vartheta d(x,y)\,.$$

Theorem 5.10 (Banach fixed point theorem) *Let (X,d) be a complete metric space and let $T : X \to X$ be a contraction with contraction constant ϑ. Then T possesses exactly one fixed point x , i.e. there is exactly one $x \in X$ such that*

$$Tx = x\,.$$

For arbitrary $x_0 \in X$ define the sequence $\{x_k\}_{k=1}^{\infty}$ by

$$\begin{aligned} x_1 &= Tx_0, \\ x_{k+1} &= Tx_k \,. \end{aligned}$$

Then for $k \geq 0$

$$d(x, x_k) \leq \frac{\vartheta^k}{1-\vartheta}\, d(x_1, x_0)\,, \tag{5.3}$$

hence

$$\lim_{k\to\infty} x_k = x\,.$$

(Stefan Banach, 1892 – 1945.)

Proof: First we show that T can have at most one fixed point. Let $x, y \in X$ be fixed points, hence $Tx = x$, $Ty = y$. Then

$$d(x, y) = d(Tx, Ty) \leq \vartheta d(x, y)\,,$$

which implies $(1-\vartheta)\, d(x, y) = 0$, whence $d(x, y) = 0$, and so $x = y$.

Next we show that a fixed point exists. Let $\{x_k\}_{k=0}^{\infty}$ be the sequence defined in the theorem. This sequence satisfies for $n \geq m \geq 1$

$$d(x_n, x_{n-1}) = d(Tx_{n-1}, Tx_{n-2}) \leq \vartheta d(x_{n-1}, x_{n-2}) \leq \ldots \leq \vartheta^{n-m} d(x_m, x_{m-1}). \tag{5.4}$$

Furthermore, the triangle inequality yields for $k \geq 0$

$$d(x_{k+\ell}, x_k) \leq d(x_{k+\ell}, x_{k+\ell-1}) + d(x_{k+\ell-1}, x_{k+\ell-2}) + \ldots + d(x_{k+1}, x_k)\,.$$

We estimate the terms on the right hand side of this inequality using (5.4) and obtain

$$\begin{aligned} d(x_{k+\ell}, x_k) &\leq (\vartheta^{\ell-1} + \vartheta^{\ell-2} + \ldots + \vartheta + 1)\, d(x_{k+1}, x_k) \\ &\leq \frac{1-\vartheta^\ell}{1-\vartheta}\, \vartheta^k\, d(x_1, x_0) \leq \frac{\vartheta^k}{1-\vartheta}\, d(x_1, x_0)\,. \end{aligned} \tag{5.5}$$

Because of $\lim_{k\to\infty} \vartheta^k = 0$, if follows from this estimate that $\{x_k\}_{k=0}^{\infty}$ is a Cauchy sequence. Since the space X is complete, it has a limit x . For this limit we obtain

$$\begin{aligned} d(Tx, x) &= \lim_{k\to\infty} d(Tx, x) \\ &\leq \lim_{k\to\infty} \big(d(Tx, Tx_k) + d(Tx_k, x)\big) \leq \lim_{k\to\infty} \big(\vartheta d(x, x_k) + d(x_{k+1}, x)\big) = 0\,, \end{aligned}$$

hence $Tx = x$, which shows that x is the uniquely determined fixed point. Moreover, (5.5) yields

$$d(x, x_k) = \lim_{\ell\to\infty} d(x, x_k) \le \lim_{\ell\to\infty} \big[d(x, x_{k+\ell}) + d(x_{k+\ell}, x_k)\big] \le \frac{\vartheta^k}{1-\vartheta}\, d(x_1, x_0)\,,$$

which proves (5.3). ■

Remark. By this theorem we can determine the fixed point approximately by iteration. The estimate (5.3) shows how far we are away from the fixed point after the kth step of iteration. Therefore it is called an error estimate.

5.3 Local invertibility

Since $x \to f(a) + f'(a)x$ is an approximation to f in a neighborhood of a, one can ask whether invertibility of $f'(a)$ (i.e. $\det f'(a) \neq 0$) already suffices to conclude that f is one–to–one in a neighborhood of a. The following example shows that in general this is false:

Example: Let $f : (-1, 1) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x + 3x^2 \sin \dfrac{1}{x}\,, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

f is differentiable for all $|x| < 1$ with derivative

$$f'(x) = \begin{cases} 1 + 6x \sin \dfrac{1}{x} - 3 \cos \dfrac{1}{x}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

In every neighborhood of 0 there are infinitely many intervals, which belong to $(0, \infty)$, and in which f' is continuous and has negative values. Thus, in such an interval one can find $0 < x_1 < x_2$ with $f(x_1) > f(x_2) > 0$. On the other hand, since f is continuous and satisfies $f(0) = 0$, the intermediate value theorem implies that the interval $(0, x_1)$ contains a point x_3 with $f(x_2) = f(x_3)$. Hence in no neighborhood of 0 the function f is one–to–one.

Since $f'(0) = 1$ and since in every neighborhood of 0 there are points x with $f'(x) < 0$, it follows that in this example f' is not continuous at 0. Requiring that f' is continuous, changes the situation:

Theorem 5.11 *Let $U \subseteq \mathbb{R}^n$ be open, let $a \in U$, let $f : U \to \mathbb{R}^n$ be continuously differentiable, and assume that the derivative $f'(a)$ is invertible. Let $b = f(a)$. Then there is a neighborhood V of a and a neighborhood W of b, such that $f|_V : V \to W$ is bijective with a continuously differentiable inverse $g : W \to V$. (Clearly, $g'(y) = \big(f'(g(y))\big)^{-1}$.)*

Proof: We first assume that $a = b = 0$, hence $f(0) = 0$. Morover, we assume that $f'(0) = I$, where $I : \mathbb{R}^n \to \mathbb{R}^n$ is the identity mapping. By $U_r(0)$ we denote the open ball in $\mathbb{R}^n$ with center 0 and radius r. We first show that there is $r > 0$ such that every $y \in U_r(0)$ has a unique inverse image under f in $\overline{U_{2r}(0)}$. To this end we define for $y \in \mathbb{R}^n$ the mapping $\Phi_y : U \to \mathbb{R}^n$ by

$$\Phi_y(x) = x - f(x) + y.$$

Every fixed point x of this mapping is an inverse image of y under f. Therefore it suffices to prove that if $r > 0$ is chosen sufficiently small, then for every $y \in U_r(0)$ the mapping Φ_y has a unique fixed point in $\overline{U_{2r}(0)}$. Since the closed ball $\overline{U_{2r}(0)}$ with the metric $d(x, z) = \|x - z\|$ is a complete metric space, this is guaranteed by the Banach fixed point theorem, if we can show that Φ_y maps $\overline{U_{2r}(0)}$ into itself and is a contraction on $\overline{U_{2r}(0)}$.

To verify this, note that by Definition 4.20 the continuity of the mapping $x \mapsto f'(x) : U \to L(\mathbb{R}^n, \mathbb{R}^n)$ at the point 0 implies that there is $r > 0$ such that for all $x \in \overline{U_{2r}(0)}$

$$\|I - f'(x)\| = \|f'(0) - f'(x)\| \le \frac{1}{2}\,, \tag{5.6}$$

holds, whence

$$\|\Phi_y'(x)\| = \|I - f'(x)\| \le \frac{1}{2}\,, \qquad \text{for } x \in \overline{U_{2r}(0)} \text{ and } y \in \mathbb{R}^n, \tag{5.7}$$

where $\|\cdot\|$ denotes the operator norm on $L(\mathbb{R}^n, \mathbb{R}^n)$. Since for $x, z \in \overline{U_{2r}(0)}$ the line segment connecting these points is contained in $\overline{U_{2r}(0)}$, it follows from (5.7) and from Corollary 4.15 that

$$\|\Phi_y(x) - \Phi_y(z)\| \le \frac{1}{2}\,\|x - z\|, \qquad \text{for } x, z \in \overline{U_{2r}(0)} \text{ and } y \in \mathbb{R}^n. \tag{5.8}$$

From this estimate we obtain for $x \in \overline{U_{2r}(0)}$ and $y \in U_r(0)$ that

$$\|\Phi_y(x)\| = \|(\Phi_y(x) - \Phi_y(0)) + y\| \le \frac{1}{2}\,\|x\| + \|y\| \le 2r,$$

so $\Phi_y(x) \in \overline{U_{2r}(0)}$. This result and the inequality (5.8) imply that $\Phi_y : \overline{U_{2r}(0)} \to \overline{U_{2r}(0)}$ is a contraction for every $y \in U_r(0)$ with contraction constant $\vartheta = \frac{1}{2}$. From Theorem 5.10

we thus conclude that for every $y \in U_r(0)$ the mapping Φ_y has a unique fixed point $x \in \overline{U_{2r}(0)}$, whence x is the unique inverse image of y under f in the set $\overline{U_{2r}(0)}$. The set $V(0) = f^{-1}(U_r(0)) \cap \overline{U_{2r}(0)}$ of all these inverse images is a neighborhood of 0, because by Theorem 3.26 for the continuous function f the inverse image $f^{-1}(U_r(0))$ is open. The function $g : U_r(0) \to V(0)$ defined by

$$g(y) = x.$$

is the local inverse of $f : V(0) \to U_r(0)$.

We must show that g is continuously differentiable. Note first that if x_1 is a fixed point of Φ_{y_1} and x_2 is a fixed point of Φ_{y_2}, then (5.8) yields

$$\begin{aligned} \|x_1 - x_2\| &= \|\Phi_{y_1}(x_1) - \Phi_{y_2}(x_2)\| \leq \|\Phi_0(x_1) - \Phi_0(x_2)\| + \|y_1 - y_2\| \\ &\leq \frac{1}{2}\|x_1 - x_2\| + \|y_1 - y_2\|, \end{aligned}$$

which implies

$$\|g(y_1) - g(y_2)\| = \|x_1 - x_2\| \leq 2\|y_1 - y_2\|.$$

Hence, g is continuous. To show that g is differentiable, observe that (5.6) and Lemma 3.49 (iv) imply for all $x \in U_r(0)$ and $h \in \mathbb{R}^n$ that

$$\|f'(x)h\| = \|h - (I - f'(x))h\| \geq \|h\| - \|(I - f'(x))h\| \geq \|h\| - \|I - f'(x)\|\|h\| \geq \frac{\|h\|}{2},$$

from which we infer that the kernel of the $n \times n$–matrix $f'(x)$ is equal to $\{0\}$, hence, by a result from linear algebra, the matrix $f'(x)$ is invertible for all $x \in V(0)$. Therefore, since the inverse g is continuous, Theorem 4.13 implies that g is differentiable with derivative given by

$$g'(y) = \Big(f'\big(g(y)\big)\Big)^{-1}.$$

This formula implies that g' is continuous. To see this, use that the coefficients of the inverse matrix $(f'(x))^{-1}$ are computed via determinants (Cramer's rule), and that these determinants consist of sums of products of the coefficients of the matrix $f'(x)$, which are the partial derivatives of f. Since by assumption f is continuously differentiable, Theorem 4.21 implies that the partial derivatives are continuous functions of x, hence the determinants are continuous functions of x, whence the composition of the determinants with the continuous function g are continuous, and so are the coefficients of the matrix $g'(y) = \Big(f'\big(g(y)\big)\Big)^{-1}$, which are the partial derivatives of the function g. Again by Theorem 4.21, this means that g is continuously differentiable.

It remains to remove the assumptions $a = b = 0$ and $f'(0) = I$. We assume that f has the properties stated in the theorem and consider the two diffeomorphisms $A, B : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\begin{aligned} Ax &= x + a, \\ By &= \bigl(f'(a)\bigr)^{-1}(y - b). \end{aligned}$$

The function $F = B \circ f \circ A$ is defined in the open set $U - a = \{x - a \,|\, x \in U\}$ containing 0 and satisfies $F(0) = (f'(a))^{-1}(f(a) - b) = 0$ with

$$F'(0) = B'f'(a)A' = \bigl(f'(a)\bigr)^{-1} f'(a) = I.$$

Thus, by what we proved above, there exists a neighborhood $V(0)$ of 0 and $r > 0$ such that $F : V(0) \to U_r(0)$ is invertible with continuously differentiable inverse $G : U_r(0) \to V(0)$. Since $f = B^{-1} \circ F \circ A^{-1}$, it follows that f has the continuously differentiable local inverse

$$g = A \circ G \circ B : W \to V,$$

with the neighborhoods $W = B^{-1}(U_r(0))$ of b and $V = A(V(0))$ of a. ∎

Example. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$\begin{aligned} f_1(x_1, x_2, x_3) &= x_1 + x_2 + x_3\,, \\ f_2(x_1, x_2, x_3) &= x_2x_3 + x_3x_1 + x_1x_2\,, \\ f_3(x_1, x_2, x_3) &= x_1x_2x_3. \end{aligned}$$

Since all partial derivatives exist and are continuous, it follows that f is continuously differentiable with

$$f'(x) = \begin{pmatrix} 1 & 1 & 1 \\ x_3 + x_2 & x_3 + x_1 & x_2 + x_1 \\ x_2x_3 & x_1x_3 & x_1x_2 \end{pmatrix},$$

hence

$$\begin{aligned} \det f'(x) &= \begin{vmatrix} 1 & 0 & 0 \\ x_3 + x_2 & x_1 - x_2 & x_1 - x_3 \\ x_2x_3 & (x_1 - x_2)x_3 & (x_1 - x_3)x_2 \end{vmatrix} \\ &= (x_1 - x_2)(x_1 - x_3)x_2 - (x_1 - x_2)(x_1 - x_3)x_3 \\ &= (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)\,. \end{aligned}$$

Thus, let $b = f(a)$ with $(a_1 - a_2)(a_1 - a_3)(a_2 - a_3) \neq 0$. Then there are neighborhoods V of a and W of b, such that the system of equations

$$\begin{aligned} y_1 &= x_1 + x_2 + x_3 \\ y_2 &= x_2 x_3 + x_3 x_1 + x_1 x_2 \\ y_3 &= x_1 x_2 x_3 \end{aligned}$$

has a unique solution $x \in V$ to every $y \in W$.

We remark that the local invertibility does not imply global invertibility. One can see this at the following example: Let $f : \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \to \mathbb{R}^2$ be defined by

$$\begin{aligned} f_1(x, y) &= y \cos x \\ f_2(x, y) &= y \sin x \,. \end{aligned}$$

f is continuously differentiable with

$$\det f'(x, y) = \begin{vmatrix} -y \sin x & \cos x \\ y \cos x & \sin x \end{vmatrix} = -y \sin^2 x - y \cos^2 x = -y \neq 0$$

for all (x, y) from the domain of definition. Consequently f is locally invertible at every point. Yet, f is not globally invertible, since f is 2π-periodic with respect to the x variable.

5.4 Implicit functions

Let a function $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be given with the components $f_1, \ldots, f_n$, and let $y = (y_1, \ldots, y_m)$ be given. Can one determine $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that the equations

$$\begin{aligned} f_1(x_1, \ldots, x_n, y_1, \ldots, y_m) &= 0 \\ &\vdots \\ f_n(x_1, \ldots, x_n, y_1, \ldots, y_m) &= 0 \end{aligned}$$

hold? These are n equations for n unknowns $x_1, \ldots, x_n$. First we study the situation for a linear function $f = A : \mathbb{R}^{n+m} \to \mathbb{R}^n$,

$$A(x, y) = \begin{pmatrix} A_1(x, y) \\ \vdots \\ A_n(x, a) \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \ldots + a_{1n}x_n + b_{11}y_1 + b_{1m}y_m \\ \vdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n + b_{n1}y_1 + b_{nm}y_m \end{pmatrix}.$$

Suppose that A has the property

$$A(h, 0) = 0 \Rightarrow h = 0 \,.$$

A has this property, if and only if the matrix

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} \frac{\partial A_1}{\partial x_1} & \dots & \frac{\partial A_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial A_n}{\partial x_1} & \dots & \frac{\partial A_n}{\partial x_n} \end{pmatrix}$$

is invertible, hence if and only if

$$\det\left(\frac{\partial A_j}{\partial x_i}\right)_{i,j=1,\dots,n} \neq 0\,.$$

Under this condition the mapping

$$h \mapsto Ch := A(h,0) : \mathbb{R}^n \to \mathbb{R}^n$$

is invertible, consequently the system of equations

$$A(h,k) = A(h,0) + A(0,k) = Ch + A(0,k) = 0$$

has for every $k \in \mathbb{R}^m$ the unique solution

$$h = \varphi(k) := -C^{-1}A(0,k)\,.$$

For $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ one has

$$A\big(\varphi(k), k\big) = 0\,,$$

for all $k \in \mathbb{R}^m$. One says that the function φ is implicitly given by this equation.

The theorem about implicit functions concerns the same situation for continuously differentiable functions f, which are not necessarily linear:

Theorem 5.12 (about implicit functions) *Let $D \subseteq \mathbb{R}^{n+m}$ be open and let $f : D \to \mathbb{R}^n$ be continuously differentiable. Suppose that there are $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ with $(a,b) \in D$, such that $f(a,b) = 0$ and*

$$\det \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a,b) & \dots & \frac{\partial f_1}{\partial x_n}(a,b) \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1}(a,b) & \dots & \frac{\partial f_n}{\partial x_n}(a,b) \end{pmatrix} \neq 0\,. \tag{5.9}$$

Then there is a neighborhood $U \subseteq \mathbb{R}^m$ of the point b and a uniquely determined continuously differentiable function $\varphi : U \to \mathbb{R}^n$ such that $\varphi(b) = a$ and for all $y \in U$

$$f\big(\varphi(y), y\big) = 0\,.$$

Proof: Consider the mapping $F : D \to \mathbb{R}^{n+m}$ defined by

$$F(x,y) = \big(f(x,y), y\big) \in \mathbb{R}^{n+m} .$$

Since f is continuously differentiable, all the partial derivatives of F exist and are continuous in D, hence F is continuously differentiable in D. The derivative $F'(a,b)$ is given by

$$F'(a,b) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial y_1} & \dots & \frac{\partial f_1}{\partial y_m} \\ \vdots & & & & & \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial y_1} & \dots & \frac{\partial f_n}{\partial y_m} \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \vdots & & & \vdots & & \\ 0 & \dots & 0 & 0 & & 1 \end{pmatrix},$$

where the partial derivatives are computed at (a,b). The rules for computation of determinants and the assumption (5.9) yield $\det F'(a,b) = \det\big(\partial_{x_j} f_i(a,b)\big)_{i,j=1,\dots,n} \neq 0$, hence the linear mapping $F'(a,b)$ is invertible. Moreover, we have

$$F(a,b) = \big(f(a,b), b\big) = (0,b) .$$

We thus see that the assumptions of Theorem 5.11 are satisfied for F, and it follows that there are neighborhoods V of (a,b) and W of $(0,b)$ in $\mathbb{R}^{n+m}$ such that

$$F\big|_V : V \to W$$

is invertible. The continuously differentiable inverse $F^{-1} : W \to V$ is of the form

$$F^{-1}(z,w) = \big(\phi(z,w), w\big),$$

with a continuously differentiable function $\phi : W \to \mathbb{R}^n$. Now set

$$U = \{w \in \mathbb{R}^m \mid (0,w) \in W\} \subseteq \mathbb{R}^m$$

and define $\varphi : U \to \mathbb{R}^n$ by

$$\varphi(w) = \phi(0,w) .$$

U is a neighborhood of b, since W is a neighborhood of $(0,b)$, and for all $w \in U$

$$(0,w) = F\big(F^{-1}(0,w)\big) = F\big(\phi(0,w), w\big) = F\big(\varphi(w), w\big) = \big(f\big(\varphi(w), w\big), w\big) ,$$

whence

$$f\big(\varphi(w), w\big) = 0 .$$

■

The derivative of the function φ can be computed using the chain rule: We denote the $m \times m$–unit matrix by $I_{m\times m}$ and define $n \times n$ and $n \times m$–matrices by

$$\begin{aligned} \partial_x f(x,y) &= \Big(\frac{\partial f_j}{\partial x_i}(x,y)\Big)_{i,j=1,\ldots,n}\,, \\ \partial_y f(x,y) &= \Big(\frac{\partial f_j}{\partial y_i}(x,y)\Big)_{j=1,\ldots,n\,,\,i=1,\ldots,m}\,. \end{aligned}$$

With these notations we obtain for the derivative $\frac{d}{dy} f\big(\varphi(y), y\big)$ of the function $y \mapsto f\big(\varphi(y), y\big)$

$$\begin{aligned} 0 = \frac{d}{dy} f\big(\varphi(y), y\big) &= \Big(\big(\partial_x f\big)\big(\varphi(y), y\big)\,,\,\big(\partial_y f\big)\big(\varphi(y), y\big)\Big)\begin{pmatrix}\varphi'(y)\\ I_{m\times m}\end{pmatrix} \\ &= \big(\partial_x f\big)\big(\varphi(y), y\big)\circ\varphi'(y) + \big(\partial_y f\big)\big(\varphi(y), y\big)\,. \end{aligned}$$

Thus,

$$\varphi'(y) = -\Big(\big(\partial_x f\big)\big(\varphi(y), y\big)\Big)^{-1}\circ\big(\partial_y f\big)\big(\varphi(y), y\big)\,.$$

Examples:

1.) Let an equation

$$f(x_1, \ldots, x_n) = 0$$

be given with continuously differentiable $f : \mathbb{R}^n \to \mathbb{R}$. To given $x_1, \ldots, x_{n-1}$ we seek x_n such that this equation is satisfied, i.e. we want to solve this equation for x_n. Assume that $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ is given such that

$$f(a_1, \ldots, a_n) = 0$$

and

$$\frac{\partial f}{\partial x_n}(a_1, \ldots, a_n) \neq 0\,.$$

Then the implicit function theorem implies that there is a neighborhood $U \subseteq \mathbb{R}^{n-1}$ of $(a_1, \ldots, a_{n-1})$, such that to every $(x_1, \ldots, x_{n-1}) \in U$ a unique $x_n = \varphi(x_1, \ldots, x_{n-1})$ can be found, which solves the equation

$$f(x_1, \ldots, x_{n-1}, x_n) = 0\,,$$

and which is a continuously differentiable function of $(x_1, \ldots, x_{n-1})$ and satisfies $x_n = a_n$ for $(x_1, \ldots, x_{n-1}) = (a_1, \ldots, a_{n-1})$. For the derivative of the function φ one obtains

$$\operatorname{grad} \varphi(x_1, \ldots, x_{n-1}) = \frac{-1}{\frac{\partial}{\partial x_n} f(x_1, \ldots, x_n)} \operatorname{grad}_{n-1} f(x_1, \ldots, x_n) = \frac{-1}{\frac{\partial f}{\partial x_n}} \begin{pmatrix} \frac{\partial}{\partial x_1} f \\ \vdots \\ \frac{\partial}{\partial x_{n-1}} f \end{pmatrix},$$

where $x_n = \varphi(x_1, \ldots x_{n-1})$.

2.) Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$\begin{aligned} f_1(x, y, z) &= 3x^2 + xy - z - 3 \\ f_2(x, y, z) &= 2xz + y^3 + xy\,. \end{aligned}$$

We have $f(1, 0, 0) = 0$. To given $z \in \mathbb{R}$ from a neighborhood of 0 we seek $(x, y) \in \mathbb{R}^2$ such that $f(x, y, z) = 0$. To this end we must test, whether the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x}(x, y, z) & \frac{\partial f_1}{\partial y}(x, y, z) \\ \frac{\partial f_2}{\partial x}(x, y, z) & \frac{\partial f_2}{\partial y}(x, y, z) \end{pmatrix} = \begin{pmatrix} 6x + y & x \\ 2z + y & 3y^2 + x \end{pmatrix}$$

is invertible at $(x, y, z) = (1, 0, 0)$. At this point, the determinant of this matrix is

$$\begin{vmatrix} 6 & 1 \\ 0 & 1 \end{vmatrix} = 6 \neq 0\,,$$

hence the matrix is invertible. Consequently, a sufficiently small number $\delta > 0$ and a continuously differentiable function $\varphi : (-\delta, \delta) \to \mathbb{R}^2$ with $\varphi(0) = (1, 0)$ can be found such that $f\big(\varphi_1(z), \varphi_2(z), z\big) = 0$ for all z with $|z| < \delta$. For the derivative of φ we obtain with $(x, y) = \varphi(z)$

$$\begin{aligned} \varphi'(z) &= -\begin{pmatrix} 6x + y & x \\ 2z + y & 3y^2 + x \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial f_1}{\partial z}(x, y, z) \\ \frac{\partial f_2}{\partial z}(x, y, z) \end{pmatrix} \\ &= \frac{-1}{(6x + y)(3y^2 + x) - x(2z + y)} \begin{pmatrix} 3y^2 + x & -x \\ -(2z + y) & 6x + y \end{pmatrix} \begin{pmatrix} -1 \\ 2x \end{pmatrix} \\ &= \frac{-1}{(6x + y)(3y^2 + x) - x(2z + y)} \begin{pmatrix} -3y^2 - x - 2x^2 \\ 2z + y + 12x^2 + 2xy \end{pmatrix}. \end{aligned}$$

Since $\varphi(0) = (1, 0)$, we obtain in particular

$$\varphi'(0) = -\frac{1}{6} \begin{pmatrix} -3 \\ 12 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -2 \end{pmatrix}.$$

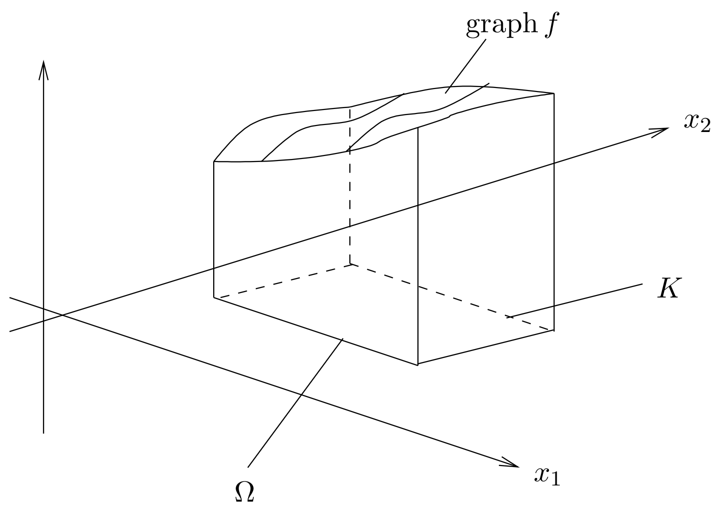
6 Integration of functions of several variables

6.1 Definition of the integral

Let Ω be a bounded subset of $\mathbb{R}^2$ and let $f : \Omega \to \mathbb{R}$ be a real valued function. If f is continuous, then graph f is a surface in $\mathbb{R}^3$. We want to define the integral

$$\int_\Omega f(x)dx$$

such that its value is equal to the volume of the subset K of $\mathbb{R}^3$, which lies between the graph of f and the x_1, x_2-plane. More generally, we want to define integrals for functions defined on $\mathbb{R}^n$, such that for $n = 2$ the integral has this property.



Definition 6.1 *For $a = (a_1, \dots, a_n), b = (b_1, \dots, b_n) \in \mathbb{R}^n$ with $a_i < b_i$ let*

$$Q = \{x \in \mathbb{R}^n \mid a_i \le x_i < b_i,\ i = 1, \dots, n\}$$

be a bounded, half open interval in $\mathbb{R}^n$. A partition P of Q is a cartesian product

$$P = P_1 \times \ldots \times P_n,$$

where $P_i = \{x_0^{(i)}, \dots, x_{k_i}^{(i)}\}$ is a partition of $[a_i, b_i]$, for every $i = 1, \dots, n$.

Q is partitioned into $k = k_1 \cdot k_2 \cdots k_n$ half open subintervals $Q_1, \dots, Q_k$ of the form

$$Q_j = [x_{p_1}^{(1)}, x_{p_1+1}^{(1)}) \times \ldots \times [x_{p_n}^{(n)}, x_{p_n+1}^{(n)}).$$

The number

$$|Q_j| = (x_{p_1+1}^{(1)} - x_{p_1}^{(1)}) \ldots (x_{p_n+1}^{(n)} - x_{p_n}^{(n)})$$

is called measure of Q_j. For a bounded function $f : Q \to \mathbb{R}$ define

$$\begin{aligned} M_j &= \sup f(Q_j), & m_j &= \inf f(Q_j), \\ U(P,f) &= \sum_{j=1}^{k} M_j |Q_j|, & L(P,f) &= \sum_{j=1}^{k} m_j |Q_j|. \end{aligned}$$

The upper and lower Darboux integrals are

$$\begin{aligned} \overline{\int_Q} f\,dx &= \inf\{U(P,f) \mid P \text{ is a partition of Q}\}, \\ \underline{\int_Q} f\,dx &= \sup\{L(P,f) \mid P \text{ is a partition of } Q\}. \end{aligned}$$

Definition 6.2 *A bounded function $f : Q \to \mathbb{R}$ is called Riemann integrable, if the upper and lower Darboux integrals coincide. The common value is denoted by*

$$\int_Q f\,dx \quad \text{or} \quad \int_Q f(x)dx$$

and is called the Riemann integral of f.

To define the integral on more general domains, let $\Omega \subseteq \mathbb{R}^n$ be a bounded subset and let $f : \Omega \to \mathbb{R}$. Choose a bounded interval Q such that $\Omega \subseteq Q$ and extend f to a function $f_Q : Q \to \mathbb{R}$ by

$$f_Q(x) = \begin{cases} f(x), & x \in \Omega, \\ 0, & x \in Q \backslash \Omega. \end{cases}$$

Definition 6.3 *A bounded function $f : \Omega \to \mathbb{R}$ is called Riemann integrable over Ω if the extension f_Q is integrable over Q. We set*

$$\int_\Omega f(x)dx = \int_Q f_Q(x)\,dx.$$

The multi-dimensional integral shares most of the properties with the one dimensional integral. In particular, the integrability criterion of Theorem 2.6, the statements (i), (ii), (iv) of Theorem 2.9, Theorem 2.10 and Corollary 2.11 remain valid when the domain of integration $[a, b]$ is replaced by the multidimensional domain Ω. We do not repeat the proofs, since they are almost the same as in the one dimensional case. Differences arise mainly from the more complicated structure of the domain of integration. Whether a function is integrable over a domain Ω depends not only on the properties of the function but also on the properties of Ω.

Definition 6.4 *A bounded set $\Omega \subseteq \mathbb{R}^n$ is called Jordan measurable, if the characteristic function $\chi_\Omega : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$\chi_\Omega(x) = \begin{cases} 1, & x \in \Omega \\ 0, & x \in \mathbb{R}^n \backslash \Omega \end{cases}$$

is integrable. In this case $|\Omega| = \int_\Omega 1\, dx$ is called the Jordan measure of Ω.

Of course, a bounded interval $Q \subseteq \mathbb{R}^n$ is measurable, and the previously given definition of $|Q|$ coincides with the new definition.

Theorem 6.5 *If the compact domain $\Omega \subseteq \mathbb{R}^n$ is Jordan measurable and if $f : \Omega \to \mathbb{R}$ is continuous, then f is integrable over Ω.*

A **proof** of this theorem can be found in the book "Lehrbuch der Analysis, Teil 2" of H. Heuser, p. 455.

6.2 Limits of integrals, parameter dependent integrals

Theorem 6.6 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded set and let $\{f_k\}_{k=1}^\infty$ be a sequence of Riemann integrable functions $f_k : \Omega \to \mathbb{R}$, which converges uniformly to a Riemann integrable function $f : \Omega \to \mathbb{R}$. Then*

$$\lim_{k\to\infty} \int_\Omega f_k(x)dx = \int_\Omega f(x)dx.$$

Remark It can be shown that the uniform limit f of a sequence of integrable functions is automatically integrable.

Proof Let $\varepsilon > 0$. Then there is $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and all $x \in \Omega$ we have

$$|f_k(x) - f(x)| < \varepsilon,$$

hence

$$\Big|\int_\Omega \big(f_k(x) - f(x)\big)dx\Big| \leq \int_Q |f_k(x) - f(x)|dx \leq \int_Q \varepsilon\, dx \leq \varepsilon|Q|.$$

By definition, this means that $\lim_{k\to\infty} \int_\Omega f_k(x)dx = \int_\Omega f(x)dx$. ■

Corollary 6.7 *Let $D \subseteq \mathbb{R}^k$ and let $Q \subseteq \mathbb{R}^m$ be a bounded interval. If $f : D \times \overline{Q} \to \mathbb{R}$ is continuous, then the function $F : D \to \mathbb{R}$ defined by the parameter dependent integral*

$$F(x) = \int_Q f(x,t)dt$$

is continuous.

Proof Let $x_0 \in D$ and let $\{x_k\}_{k=1}^{\infty}$ be a sequence with $x_k \in D$ and $\lim_{k\to\infty} x_k = x_0$. Then x_0 is the only accumulation point of the set $M = \{x_k \mid k \in \mathbb{N}\} \cup \{x_0\}$, from which it is immediately seen that $M \times \overline{Q}$ is closed and bounded, hence it is a compact subset of $D \times \overline{Q}$. Therefore the continuous function f is uniformly continuous on $M \times \overline{Q}$. This implies that to every $\varepsilon > 0$ there is $\delta > 0$ such that for all $y \in M$ with $|y - x_0| < \delta$ and all $t \in Q$ we have

$$|f(y,t) - f(x_0,t)| < \varepsilon.$$

Choose $k_0 \in \mathbb{N}$ such that $|x_k - x_0| < \delta$ for all $k \geq k_0$. This implies for $k \geq k_0$ and for all $t \in Q$ that

$$|f(x_k,t) - f(x_0,t)| < \varepsilon,$$

which shows that the sequence $\{f_k\}_{k=1}^{\infty}$ of continuous functions $f_k : Q \to \mathbb{R}$ defined by $f_k(t) = f(x_k,t)$ converges uniformly to the continuous function $f_\infty(t) = f(x_0,t)$. Theorem 6.6 implies

$$\lim_{k\to\infty} F(x_k) = \lim_{k\to\infty} \int_Q f(x_k,t)dt = \int_Q f(x_0,t)dx = F(x_0).$$

Therefore F is continuous. ∎

6.3 The theorem of Fubini

The computation of integrals by approximation of the integrand by step functions is impractical. For one-dimensional integrals the computation is strongly simplified by the fundamental theorem of calculus. The next theorem shows that multidimensional integrals can be computed as iterated one-dimensional integrals, which makes also the computation of these integrals practical.

Theorem 6.8 (of Fubini) *For $a = (a_1,\ldots,a_n), b = (b_1,\ldots,b_n) \in \mathbb{R}^n$ with $a_i < b_i$ let*

$$\begin{aligned} Q &= \{x \in \mathbb{R}^n \mid a_i \leq x_i < b_i,\ i = 1,\ldots,n\}, \\ Q' &= \{x' \in \mathbb{R}^{n-1} \mid a_i \leq x_i < b_i,\ i = 1,\ldots,n-1\}, \end{aligned}$$

be half open intervals, and let $f : Q \to \mathbb{R}$ be an integrable function. If for every $x_n \in [a_n,b_n)$ the integral $\int_{Q'} f(x',x_n)dx'$ exists, then the function $F : [a_n,b_n) \to \mathbb{R}$ defined by

$$F(x_n) = \int_{Q'} f(x',x_n)dx'$$

is integrable and

$$\int_Q f(x)dx = \int_{a_n}^{b_n} F(x_n)dx_n = \int_{a_n}^{b_n} \int_{Q'} f(x', x_n)dx'dx_n\,. \qquad (6.1)$$

(Guido Fubini, 1879 – 1943)

Proof First we introduce some notations. If

$$P = P_1 \times P_2 \times \ldots \times P_n$$

is a partition of Q, then $P' = P_1 \times \ldots \times P_{n-1}$ is a partition of Q'. Let $Q'_1, \ldots, Q'_k$ be the subintervals of Q' generated by P' and let $I_1, \ldots, I_{k'} \subseteq [a_n, b_n)$ be the half open subintervals generated by P_n. Then all the subintervals of Q generated by P are given by

$$Q'_j \times I_\ell, \qquad 1 \le j \le k, \quad 1 \le \ell \le k'.$$

For the function $f : Q \to \mathbb{R}$ and for $x_n \in [a_n, b_n)$ we denote by $f(\cdot, x_n)$ the function

$$x' \mapsto f(x', x_n) : Q' \to \mathbb{R}.$$

The upper and lower sums to the partitions P_n, P', P and to the functions F, $f(\cdot, x_n)$ and f are given by

$$U(P_n, F) = \sum_{\ell=1}^{k'} \sup_{x_n \in I_\ell} F(x_n)\,|I_\ell|, \qquad L(P_n, F) = \sum_{\ell=1}^{k'} \inf_{x_n \in I_\ell} F(x_n)\,|I_\ell|,$$

$$U(P', f(\cdot, x_n)) = \sum_{j=1}^{k} \sup_{x' \in Q'_j} f(x', x_n)\,|Q'_j|, \quad L(P', f(\cdot, x_n)) = \sum_{j=1}^{k} \inf_{x' \in Q'_j} f(x', x_n)\,|Q'_j|,$$

$$U(P, f) = \sum_{\substack{j=1\ldots k\\ \ell=1\ldots k'}} \sup_{x \in Q'_j \times I_\ell} f(x)\,|Q'_j \times I_\ell|, \qquad L(P, f) = \sum_{\substack{j=1\ldots k\\ \ell=1\ldots k'}} \inf_{x \in Q'_j \times I_\ell} f(x)\,|Q'_j \times I_\ell|.$$

Now set

$$s = \int_Q f(x)dx.$$

To prove the theorem it suffices to show that for every $\varepsilon > 0$ there is a partition P_n of $[a_n, b_n)$ such that

$$s - \varepsilon < L(P_n, F) \le U(P_n, F) < s + \varepsilon \qquad (6.2)$$

holds. For, by Theorem 2.6 the inequality (6.2) implies that F is integrable. Furthermore, since by definition of the integral we have $L(P_n, F) \le \int_{a_n}^{b_n} F(x_n)dx_n \le U(P_n, F)$ and since ε in the inequality (6.2) is arbitrary, it also follows from this inequality that $\int_{a_n}^{b_n} F(x_n)dx_n = s$. This proves (6.1).

To verify (6.2) let $\varepsilon > 0$ be given. By definition of the integral $\int_Q f(x)dx$, we can choose a partition $P = P' \times P_n$ of Q such that

$$s - \varepsilon < L(P, f) \leq U(P, f) < s + \varepsilon \tag{6.3}$$

holds. For the subintervals Q'_j and I_ℓ generated by this partition we have that

$$\begin{aligned}
\sup_{x_n \in I_\ell} F(x_n) &= \sup_{x_n \in I_\ell} \int_{Q'} f(x', x_n)dx' \leq \sup_{x_n \in I_\ell} U(P', f(\cdot, x_n)) \\
&= \sup_{x_n \in I_\ell} \sum_{j=1}^{k} \sup_{x' \in Q'_j} f(x', x_n)\,|Q'_j| \leq \sum_{j=1}^{k} \sup_{x_n \in I_\ell} \sup_{x' \in Q'_j} f(x', x_n)\,|Q'_j| = \sum_{j=1}^{k} \sup_{x \in Q'_j \times I_\ell} f(x)\,|Q'_j|.
\end{aligned}$$

From this inequality and from $|Q'_j \times I_\ell| = |Q'_j||I_\ell|$ we infer that

$$\begin{aligned}
U(P_n, F) = \sum_{\ell=1}^{k'} \sup_{x_n \in I_\ell} F(x_n)\,|I_\ell| &\leq \sum_{\ell=1}^{k'} \sum_{j=1}^{k} \sup_{x \in Q'_j \times I_\ell} f(x)\,|Q'_j|\,|I_\ell| \\
&= \sum_{\substack{j=1\ldots k \\ \ell=1\ldots k'}} \sup_{x \in Q'_j \times I_\ell} f(x)\,|Q'_j \times I_\ell| = U(P, f) < s + \varepsilon.
\end{aligned}$$

In the last step we used (6.3). This shows that P_n satisfies the last inequality in (6.2). The inequality

$$L(P_n, F) \geq L(P, f) > s - \varepsilon$$

is proved in the same way, hence also the first inequality in (6.2) is satisfied. The second inequality holds always by definition of $L(P_n, F)$ and $U(P_n, F)$, hence the partition P_n satisfies (6.2). The proof of Theorem 6.8 is complete. ■

Remark By repeated application of this theorem we obtain that

$$\int_Q f(x)dx = \int_{a_n}^{b_n} \ldots \int_{a_1}^{b_1} f(x_1, \ldots, x_n)dx_1 \ldots dx_n.$$

In the theorem of Fubini the coordinate x_n can be replaced by any other coordinate, provided of course, that the assumptions of the theorem are satisfied. Therefore the order of integration in the iterated integral can be replaced by any other order.

6.4 The transformation formula

The transformation formula generalizes the rule of substitution for one-dimensional integrals. We start with some preparations.

Definition 6.9 (i) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. The support of f is defined by*

$$\operatorname{supp} f = \overline{\{x \in \mathbb{R}^n \mid f(x) \neq 0\}}.$$

(ii) *Let $D \subseteq \mathbb{R}^n$ and let $\{U_i\}_{i=1}^{\infty}$ be an open covering of D. For every $i \in \mathbb{N}$ let $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ be a continuous function with compact support contained in U_i, such that*

$$\sum_{i=1}^{\infty} \varphi_i(x) = 1, \quad \textit{for all } x \in D.$$

Then $\{\varphi_i\}_{i=1}^{\infty}$ is called partition of unity on D subordinate to the covering $\{U_i\}_{i=1}^{\infty}$.

Theorem 6.10 *Let $D \subseteq \mathbb{R}^n$ be a compact set and let $B_{r_1}(z_1), \ldots, B_{r_m}(z_m)$ be open balls in $\mathbb{R}^n$ with $D \subseteq B_{r_1}(z_1) \cup \ldots \cup B_{r_m}(z_m)$. Then there is a partition of unity $\{\varphi_i\}_{i=1}^{m}$ on D subordinate to the covering $\{B_{r_i}(z_i)\}_{i=1}^{m}$.*

Proof: Let $C = \mathbb{R}^n \setminus \bigcup_{i=1}^{m} B_{r_i}(z_i)$. The distance $\operatorname{dist}(D, C) = \inf\{|x-y| \mid x \in D,\ y \in C\}$ is positive. Otherwise there would be sequences $\{x_j\}_{j=1}^{\infty}$, $\{y_j\}_{j=1}^{\infty}$, $x_j \in D$, $y_j \in C$ such that $\lim_{j\to\infty} |x_j - y_j| = 0$. Since D is compact, $\{x_j\}_{j=1}^{\infty}$ would have an accumulation point $x_0 \in D$. Since x_0 would also be an accumulation point of $\{y_j\}_{j=1}^{\infty}$ and since C is closed, it would follow that $x_0 \in C$, hence $D \cap C \neq \emptyset$, which contradicts the assumptions.

Therefore we can choose balls $B_i' = B_{r_i'}(z_i)$, $i = 1, \ldots m$, with $r_i' < r_i$, such that $D \subseteq \bigcup_{i=1}^{m} B_i'$. For $1 \leq i \leq m$, let β_i be a continuous function on $\mathbb{R}^n$ with support in $B_{r_i}(z_i)$, such that $\beta_i(z) = 1$ for $z \in B_i'$. Put $\varphi_1 = \beta_1$ and set

$$\varphi_j = (1-\beta_1)(1-\beta_2)\cdots(1-\beta_{j-1})\beta_j\,, \quad \text{for } 2 \leq j \leq m.$$

Every φ_j is a continuous function. By induction one obtains that for $1 \leq l \leq m$,

$$\varphi_1 + \cdots + \varphi_l = 1 - (1-\beta_l)(1-\beta_{l-1})\cdots(1-\beta_1).$$

Every $x \in D$ belongs to at least one B_i', hence $1 - \beta_i(x) = 0$. For $l = m$ the product on the right hand side thus vanishes on D, so that $\sum_{i=1}^{m} \varphi_i(x) = 1$ for all $x \in D$. ∎

Theorem 6.11 *Let $U \subseteq \mathbb{R}^n$ be an open set with $0 \in U$ and let $T : U \to \mathbb{R}^n$ be continuously differentiable such that $T(0) = 0$ with invertible derivative $T'(0) : \mathbb{R}^n \to \mathbb{R}^n$. Then there is a number $j \in \{1, \ldots, n\}$ and there are neighborhoods V, W of 0 in $\mathbb{R}^n$, such that the decomposition*

$$T(x) = h\big(g(Bx)\big)$$

is valid for all $x \in V$, where the linear operator $B : \mathbb{R}^n \to \mathbb{R}^n$ merely interchanges the x_j- and x_n-coordinates, and where the functions $h : W \to \mathbb{R}^n$, $g : B(V) \to W$ are of the form

$$g(x) = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ g_n(x) \end{pmatrix}, \quad h(x) = \begin{pmatrix} h_1(x) \\ \vdots \\ h_{n-1}(x) \\ x_n \end{pmatrix}, \tag{6.4}$$

and are continuously differentiable with $\det h' \neq 0$ *in* W*,* $\det g' \neq 0$ *in* $B(V)$*.*

Proof The last row of the Jacobi matrix $T'(0) = \left(\frac{\partial T_i}{\partial x_k}(0)\right)_{i,k=1,\ldots,n}$ contains at least one non-zero element $\frac{\partial T_n}{\partial x_j}(0)$, since otherwise $T'(0)$ would not be invertible. Let B be the mapping, which interchanges the x_j– and x_n–coordinates, where j is the index of this non-zero element, and let $B(U) \subseteq \mathbb{R}^n$ be the image of the set U under B. We have that $B(U) = B^{-1}(U)$, since $B^{-1} = B$. Define

$$\tilde{g}(x) = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ T_n(x_1, \ldots, x_{j-1}, x_n, x_{j+1}, \ldots, x_{n-1}, x_j) \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ T_n(Bx) \end{pmatrix}. \tag{6.5}$$

Then $\tilde{g} : B(U) \to \mathbb{R}^n$ is continuously differentiable with $\tilde{g}(0) = 0$ and

$$\tilde{g}'(x) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ \frac{\partial T_n}{\partial x_1} & & \cdots & \frac{\partial T_n}{\partial x_j} \end{pmatrix},$$

whence $\det \tilde{g}'(0) = \frac{\partial T_n}{\partial x_j}(0) \neq 0$. Consequently the Inverse Function Theorem 5.11 implies that there are neighborhoods $\tilde{V} \subseteq B(U)$ and W of 0 such the restriction $g : \tilde{V} \to W$ of $\tilde{g}$ to $\tilde{V}$ is one-to-one and such that the inverse $g^{-1} : W \to \tilde{V}$ is continuously differentiable with nonvanishing determinants $\det g'$ and $\det(g^{-1})'$. Of course, we have $g^{-1}(0) = 0$. Now set $h = T \circ B \circ g^{-1}$. Since $B \circ g^{-1}(W) = B(\tilde{V}) \subseteq B(B(U)) = U$, it follows that h is defined on W. Moreover, h is continuously differentiable and satisfies $h(0) = 0$. Since $\tilde{g}|_{\tilde{V}} = g$, the definition of h and the definition of $\tilde{g}$ in (6.5) yield for $y \in W$ that

$$h_n(y) = (T_n \circ B)(g^{-1}(y)) = \tilde{g}_n(g^{-1}(y)) = g_n(g^{-1}(y)) = \big((g \circ g^{-1})(y)\big)_n = y_n.$$

This equation and (6.5) show that h and g have the form required in (6.4).

Set $V = B(\tilde{V})$. Then $h \circ g \circ B : V \subseteq U \to \mathbb{R}^n$, and we have

$$h \circ g \circ B = T \circ B \circ g^{-1} \circ g \circ B = T \circ B \circ B = T,$$

which is the decomposition of T required in the theorem. Since $B' = B$, the chain rule yields

$$h' = (T \circ B \circ g^{-1})' = (T' \circ B \circ g^{-1})(B \circ g^{-1})(g^{-1})',$$

whence $\det h' = \big(\det T'(B \circ g^{-1})\big) \det B \, \det(g^{-1})'$. We have $\det B = \pm 1$. Moreover, $\det(g^{-1})'$ does not vanish by construction. Thus, because $\det T'(0) \neq 0$ and because $\det T'$ is continuous, we can reduce the sizes of V and W, if necessary, such that $\det h'(x) \neq 0$ for all $x \in W$. ■

With this theorem we can prove the transformation rule, which generalizes the rule of substitution:

Theorem 6.12 (Transformation rule) *Let $U \subseteq \mathbb{R}^n$ be open and let $T : U \to \mathbb{R}^n$ be an injective, continuously differentiable transformation such that $\det T'(x) \neq 0$ for all $x \in U$. Suppose that Ω is a compact Jordan-measurable subset of U and that $f : T(\Omega) \to \mathbb{R}$ is continuous. Then $T(\Omega)$ is a Jordan measurable subset of $\mathbb{R}^n$, the function f is integrable over $T(\Omega)$ and*

$$\int_{T(\Omega)} f(y)\, dy = \int_\Omega f(T(x)) |\det T'(x)|\, dx. \tag{6.6}$$

Proof For simplicity we prove this theorem only under the assumptions that $\det T'(x)$ has the same sign for all $x \in U$ and that $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function with $\operatorname{supp} f \subseteq T(\Omega)$. In this case $f \circ T$ is defined on all of U and is continuous. The injectivity of T implies for $x \in U \backslash \Omega$ that $T(x) \notin T(\Omega)$. Since f vanishes outside of $T(\Omega)$, it therefore follows that $f \circ T$ vanishes in $U \setminus \Omega$, whence we can extend $f \circ T$ by zero to a continuous function on all of $\mathbb{R}^n$. For the extended function, which we also denote by $f \circ T$, we have $\operatorname{supp}(f \circ T) \subseteq \Omega$. We also extend the function $f \circ T \,|\det T'|$ from U by zero to all of $\mathbb{R}^n$ and for simplicity denote the extended function by $f \circ T \,|\det T'|$, though the functions T and $\det T'$ themselfes are not defined outside of U. With these definitions we can extend the integration on both sides of (6.6) to all of $\mathbb{R}^n$, since f vanishes outside of $T(\Omega)$ and $f \circ T \,|\det T'|$ vanishes outside of Ω.

We prove (6.6) by induction with respect to the dimension n. Consider first the case $n = 1$. By assumption $\Omega \subseteq \mathbb{R}$ is compact, hence we can choose a bounded interval $[a, b]$ containing Ω. Since we assume that $\det T'(x) = T'(x)$ has the same sign for all $x \in \Omega$, it follows either that $T'(x) > 0$ for all $x \in \Omega$ or that $T'(x) < 0$ for all $x \in \Omega$. In the first case we have $T(a) < T(b)$, in the second case $T(b) < T(a)$. If we take the plus sign in the first case and the minus sign in the second case we obtain from the rule of substitution

$$\int_{\mathbb{R}} f(y)dy = \int_{T([a,b])} f(y)\,dy = \pm \int_{T(a)}^{T(b)} f(y)\,dy = \pm \int_a^b f\big(T(x)\big)T'(x)\,dx$$
$$= \int_a^b f\big(T(x)\big)|T'(x)|\,dx = \int_a^b f\big(T(x)\big)|\det T'(x)|\,dx = \int_{\mathbb{R}} f\big(T(x)\big)|\det T'(x)|\,dx.$$

Therefore (6.6) holds for one-dimensional integrals. Assume next that $n \geq 2$ and that (6.6) holds for $(n-1)$–dimensional integrals. We shall prove that this implies that (6.6) holds for n–dimensional integrals.

To this end note first that if T has the properties stated in the theorem and if $y \in U$, then the transformation $\hat{T}(z) := T(z+y) - T(y)$ satisfies all assumptions of Theorem 6.11, since $\hat{T}(0) = 0$. It follows by this theorem that there is a neighborhood $\hat{V}$ of 0, in which $\hat{T}$ has the decomposition $\hat{T} = h \circ g \circ B$ with transformations h, g and B of the form described in Theorem 6.11. For x from the neighborhood $V(y) = y + \hat{V}$ of y we have that $z = x - y \in \hat{V}$, which implies that

$$T(x) = T(y) + \hat{T}(x-y) = T(y) + (h \circ g \circ B)(x-y).$$

With the transformation $T_c(x) = x + c$, where $c \in \mathbb{R}^n$ is a fixed vector, it follows from this equation that the restriction $T\big|_{V(y)}$ of the transformation T to $V(y)$ can be decomposed in the form

$$T\big|_{V(y)} = T_{T(y)} \circ h \circ g \circ B \circ T_{-y}. \tag{6.7}$$

We call the transformations on the right hand side of this equation elementary transformations. We show next that (6.6) holds in n dimensions for every one of these elementary transformations.

Consider first transformations of the form of g, that is $T(x) = T(x', x_n) = \big(x', T_n(x', x_n)\big)$. This implies that $\det T'(x) = \frac{\partial}{\partial x_n} T_n(x)$, from which we see that by our assumpton $\frac{\partial}{\partial x_n} T_n(x)$ has the same sign for all $x \in \Omega$. Consequently, the Theorem of Fubini and the transformation formula (6.6) for the case $n = 1$ together yield

$$\int_{\mathbb{R}^n} f(y)\,dy = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} f(y', y_n)\,dy_n\,dy'$$
$$= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} f(x', T_n(x', x_n)) \Big| \frac{\partial}{\partial x_n} T_n(x', x_n) \Big|\,dx_n\,dx'$$
$$= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} f(T(x))|\det T'(x)|\,dx_n\,dx' = \int_{\mathbb{R}^n} f(T(x))|\det T'(x)|\,dx.$$

For such elementary transformations (6.6) thus holds in n dimensions. Next, assume that the transformation is of the form of the function h, that is $T(x) = \big(\tilde{T}(x', x_n), x_n\big)$ with $\tilde{T}(x', x_n) \in \mathbb{R}^{n-1}$. With the Jacobi matrix $\partial_{x'}\tilde{T}(x) = \Big(\frac{\partial \tilde{T}_i}{\partial x_j}(x)\Big)_{i,j=1,\ldots,n-1}$ we have

$$\det T'(x) = \det \begin{pmatrix} \partial_{x'}\tilde{T}(x) & \partial_{x_n}\tilde{T}(x) \\ 0 & 1 \end{pmatrix} = \det\big(\partial_{x'}\tilde{T}(x)\big).$$

This equation implies that $\det\big(\partial_{x'}\tilde{T}(x)\big)$ has the same sign for all $x \in \Omega$. Therefore $x' \mapsto \tilde{T}(x', x_n)$ is a transformation in $\mathbb{R}^{n-1}$, for which by the induction hypotheses the transformation formula (6.6) holds. The Theorem of Fubini thus yields

$$\begin{aligned}\int_{\mathbb{R}^n} f(y)\,dy &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} f(y', y_n)\,dy'\,dy_n \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} f\big(\tilde{T}(x', x_n), x_n\big) |\det\big(\partial_{x'}\tilde{T}(x', x_n)\big)| dx'\,dx_n = \int_{\mathbb{R}^n} f(T(x)) |\det T'(x)|\,dx.\end{aligned}$$

The transformation formula (6.6) therefore holds also for such elementary transformations in n dimensions. It holds in every dimension when the transformation T is a linear operator B, which merely interchanges coordinates, since this amounts to a change of the order of integration when the integral is computed iteratively, and by the Theorem of Fubini the order of integration does not matter. By definition of the integral the transformation rule also holds in every dimension for transformations T_c, since such transformations simply translate the domain of integration.

If (6.6) holds for the transformations R and S, then it also holds for the transformation $T = R \circ S$. For,

$$\begin{aligned}\int_{\mathbb{R}^n} f(z)\,dz &= \int_{\mathbb{R}^n} f(R(y)) |\det R'(y)|\,dy \\ &= \int_{\mathbb{R}^n} f\big(R(S(x))\big) |\det R'(S(x))|\,|\det S'(x)|\,dx \\ &= \int_{\mathbb{R}^n} f(T(x)) |\det\big(R'(S(x))S'(x)\big)|\,dx = \int_{\mathbb{R}^n} f(T(x)) |\det T'(x)|\,dx,\end{aligned}$$

since by the determinant multiplication theorem for $n \times n$–matrices M_1 and M_2 we have $\det M_1 \det M_2 = \det(M_1 M_2)$.

From these results and from the decomposition (6.7) we conclude that the transformation rule in n dimensions holds for $T|_{V(y)}$. We thus proved that every $y \in U$ has a neighborhood $V(y)$ such that (6.6) holds for functions f with $\operatorname{supp} f \subseteq T\big(V(y)\big)$. In the last step of the proof we must extend (6.6) now to functions f with $\operatorname{supp} f \subseteq T(\Omega)$.

To this end note that since $\det T'(y) \neq 0$, the inverse function theorem implies that T is locally a diffeomorphismus. Therefore $T\big(V(y)\big)$ is a neighborhood of $z = T(y)$ and contains an open ball $W(z)$ with center z, such that (6.6) holds for all continuous f with $\operatorname{supp} f \subseteq W(z)$. Since $\{W(z)\}_{z\in T(\Omega)}$ is an open covering of the compact set $T(\Omega)$, there are points $z_1, \dots, z_p$ in $T(\Omega)$ such that the union of the open balls $W(z_i)$ covers $T(\Omega)$. By Theorem 6.10 there is a partition of unity $\{\varphi_i\}_{i=1}^p$ on $T(\Omega)$ subordinate to the covering $\{W(z_i)\}_{i=1}^p$. If f is a continuous function with $\operatorname{supp} f \subseteq T(\Omega)$, we thus have for every $x \in T(\Omega)$

$$f(x) = f(x)\sum_{i=1}^{p} \varphi_i(x) = \sum_{i=1}^{p} \big(\varphi_i(x) f(x)\big).$$

Since $\operatorname{supp}(\varphi_i f) \subseteq \operatorname{supp}\varphi_i \subseteq W(z_i)$, the transformation rule (6.6) holds for every function $\varphi_i f$, whence it holds for the sum of these functions, which is f. We thus proved that the transformation rule (6.6) holds in n dimensions. By induction, it follows that the transformation rule holds in any space dimension. ∎

7 Submanifolds, curve and surface integrals, theorems of Gauß and Stokes

The integral theorems of Gauß and Stokes, which we discuss at the end of this section, involve integrals of functions defined on surfaces. There are many other occasions, where such surface integrals occur. In this section we first introduce surfaces and study surface integrals before we come to the integral theorems of Gauß and Stokes.

A typical example for a surface is a sphere in $\mathbb{R}^3$, which is a two dimensional subset of the three dimensional space $\mathbb{R}^3$. We study general p–dimensional subsets of $\mathbb{R}^n$ with $p < n$. Such lower dimensional subsets in $\mathbb{R}^n$ are called parametrized surfaces or, more generally, submanifolds in $\mathbb{R}^n$.

7.1 Parametrized surfaces, coordinate mappings

In Section 3.4 we introduced the normed space $L(\mathbb{R}^n, \mathbb{R}^m)$ of all linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$. For $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ the range $A(\mathbb{R}^n)$ is a linear subspace of $\mathbb{R}^m$.

Definition 7.1 *Let $A \in L(\mathbb{R}^n, \mathbb{R}^m)$. The dimension of the subspace $A(\mathbb{R}^n)$ is called rank of A.*

From linear algebra we know that the rank of the linear mapping $A : \mathbb{R}^n \to \mathbb{R}^m$ is equal to the rank of the $m \times n$–matrix representing it and that A is injective if the rank is n.

Definition 7.2 *Let $1 \le p < n$ and let $U \subseteq \mathbb{R}^p$ be an open set. The mapping $\gamma : U \to \mathbb{R}^n$ is called a parametrized p–dimensional surface in $\mathbb{R}^n$, if γ is continuously differentiable and the derivative $\gamma'(u) \in L(\mathbb{R}^p, \mathbb{R}^n)$ has rank p for all $u \in U$. The range $\gamma(U) \subseteq \mathbb{R}^n$ is called trace of γ. If $p = 1$, then γ is called a parametrized curve in $\mathbb{R}^n$*

Note that γ need not be injective. The surface may have double points.

Example 1: Let $U = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1\}$ and let $\gamma : U \to \mathbb{R}^3$ be defined by

$$\gamma(u,v) = \begin{pmatrix} \gamma_1(u,v) \\ \gamma_2(u,v) \\ \gamma_3(u,v) \end{pmatrix} = \begin{pmatrix} u \\ v \\ \sqrt{1-(u^2+v^2)} \end{pmatrix}.$$

γ is a parametrized two dimensional surface in $\mathbb{R}^3$, since the two columns of the Jacobi matrix

$$\gamma'(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{u}{\sqrt{1-(u^2+v^2)}} & -\frac{v}{\sqrt{1-(u^2+v^2)}} \end{pmatrix}.$$

are linearly independent for all $(u,v) \in U$, whence the rank of the linear mapping $\gamma'(u,v)$ is 2 for all $(u,v) \in U$. The trace of γ is the upper half of the unit sphere in $\mathbb{R}^3$.

Example 2: In the preceding example the trace of the parametrized surface is given by the graph of the function $(u,v) \mapsto \sqrt{1-(u^2+v^2)}$. More generally, let $U \subseteq \mathbb{R}^p$ be an open set and $f : U \to \mathbb{R}^{n-p}$ be continuously differentiable. Then the mapping $\gamma : U \to \mathbb{R}^n$ defined by

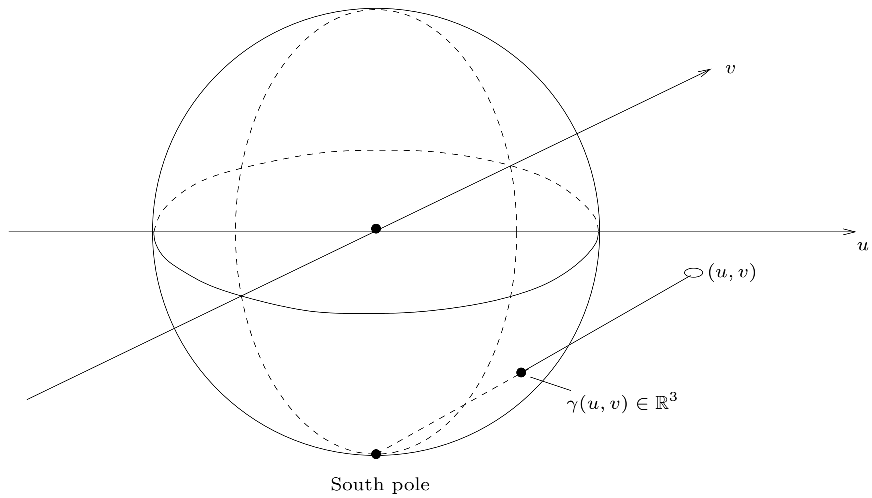
$$\begin{aligned} \gamma_1(u) &:= u_1 \\ \gamma_2(u) &:= u_2 \\ &\vdots \\ \gamma_p(u) &:= u_p \\ \gamma_{p+1}(u) &:= f_1(u_1 \ldots, u_p) \\ &\vdots \\ \gamma_n(u) &:= f_{n-p}(u_1 \ldots, u_p) \end{aligned}$$

is a parametrized p–dimensional surface in $\mathbb{R}^n$, since the p column vectors of the matrix

$$\gamma'(u) = \begin{pmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \\ 0 & \ldots & 1 \\ \partial_{x_1} f_1(u) & \ldots & \partial_{x_p} f_1(u) \\ \vdots & & \vdots \\ \partial_{x_1} f_{n-p}(u) & \ldots & \partial_{x_p} f_{n-p}(u) \end{pmatrix},$$

are linearly independent. Therefore the rank is p. The trace of γ is the graph of f.

Example 3: By stereographic projection, the sphere with center in the origin, which is punched at the south pole, can be mapped one-to-one onto the plane. The inverse γ of this projection maps the plane onto the punched sphere:



From the figure we see that the components $\gamma_1, \ldots, \gamma_3$ of the mapping $\gamma : \mathbb{R}^2 \to \mathbb{R}^3$ satisfy

$$\frac{\gamma_1}{\gamma_2} = \frac{u}{v}, \quad \frac{\sqrt{u^2+v^2} - \sqrt{\gamma_1^2+\gamma_2^2}}{\gamma_3} = \frac{\sqrt{u^2+v^2}}{-1}, \quad \gamma_1^2+\gamma_2^2+\gamma_3^2 = 1.$$

Solution of these equations for $\gamma_1, \ldots, \gamma_3$ yields

$$\begin{aligned}
\gamma_1(u,v) &= \frac{2u}{1+u^2+v^2} \\
\gamma_2(u,v) &= \frac{2v}{1+u^2+v^2} \\
\gamma_3(u,v) &= \frac{1-u^2-v^2}{1+u^2+v^2}\,.
\end{aligned}$$

To see that γ is a parametrized surface consider the derivative

$$\gamma'(u,v) = \frac{2}{(1+u^2+v^2)^2}\begin{pmatrix} 1-u^2+v^2 & -2uv \\ -2uv & 1+u^2-v^2 \\ -2u & -2v \end{pmatrix}.$$

For $u^2+v^2 \neq 1$ we have

$$\begin{aligned}
\begin{vmatrix} \partial_u\gamma_1(u,v) & \partial_v\gamma_1(u,v) \\ \partial_u\gamma_2(u,v) & \partial_v\gamma_2(u,v) \end{vmatrix} &= (1+(v^2-u^2))(1-(v^2-u^2)) - 4u^2v^2 \\
&= 1-(v^2-u^2)^2 - 4u^2v^2 = 1-(v^2+u^2)^2 \neq 0\,,
\end{aligned}$$

and for $u \neq 0$

$$\begin{vmatrix} \partial_u\gamma_2(u,v) & \partial_v\gamma_2(u,v) \\ \partial_u\gamma_3(u,v) & \partial_v\gamma_3(u,v) \end{vmatrix} = 4uv^2 + 2u(1+u^2-v^2) = 2u(1+u^2+v^2) \neq 0\,.$$

Correspondingly, for $v \neq 0$ we get

$$\begin{vmatrix} \partial_u\gamma_1(u,v) & \partial_v\gamma_1(u,v) \\ \partial_u\gamma_3(u,v) & \partial_v\gamma_3(u,v) \end{vmatrix} = -2v(1+u^2+v^2) \neq 0\,.$$

These relations show that $\gamma'(u,v)$ has rank 2 for all $(u,v) \in \mathbb{R}^2$, which means that γ is a parametrized two dimensional surface in $\mathbb{R}^3$. The trace of γ is the unit sphere with the south pole removed.

Example 4: Let $\tilde{\gamma} : \tilde{U} \to \mathbb{R}^3$ be the parametrized surface obtained by restricting the parametrized surface γ from Example 3 to the unit disk $\tilde{U} = \{(u,v) \in \mathbb{R}^2 \mid u^2+v^2 < 1\}$. The trace of $\tilde{\gamma}$ is the upper half of the unit sphere. $\tilde{\gamma}$ differs from the parametrized surface given in Example 1, but both have the same trace.

Example 5: Let $R > 0$ and set $a = \frac{\pi}{180}$. Let $U = \{(\varphi,\vartheta) \in \mathbb{R}^2 \mid -180 < \varphi < 180,\ -90 < \vartheta < 90\}$ and let $\gamma : U \to \mathbb{R}^3$ be given by

$$\gamma(\varphi,\vartheta) := R \begin{pmatrix} \cos(a\varphi)\cos(a\vartheta) \\ \sin(a\varphi)\cos(a\vartheta) \\ \sin(a\vartheta) \end{pmatrix}.$$

γ is a parametrized two dimensional surface in $\mathbb{R}^3$, since the two columns of the Jacobi matrix

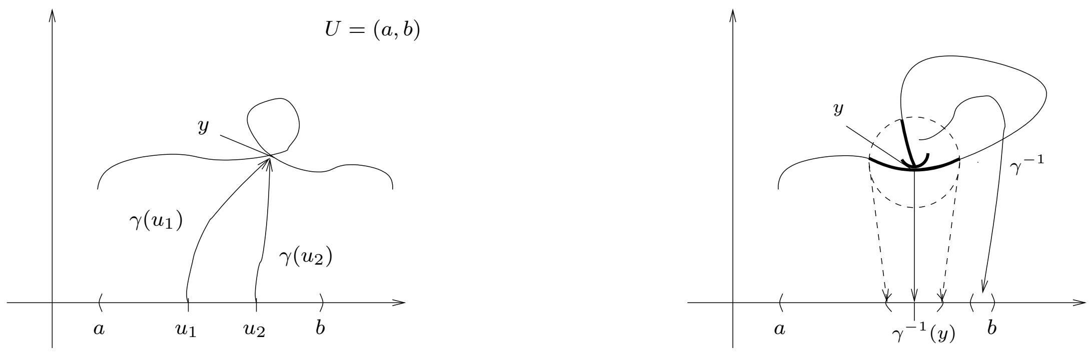
$$\gamma'(\varphi,\vartheta) = aR \begin{pmatrix} -\sin(a\varphi)\cos(a\vartheta) & -\cos(a\varphi)\sin(a\vartheta) \\ \cos(a\varphi)\cos(a\vartheta) & -\sin(a\varphi)\sin(a\vartheta) \\ 0 & \cos(a\vartheta) \end{pmatrix}$$

are linearly independent for all $(\varphi,\vartheta) \in U$, hence $\operatorname{rank}(\gamma'(\varphi,\vartheta)) = 2$ for all $(\varphi,\vartheta) \in U$. The trace of γ is the sphere with radius R, with the north and south poles and a half circle connecting the poles removed. If we choose for R the radius of the earth, then (φ,ϑ) are the usual geographical coordinates of the globe. φ is the longitude, ϑ is the latitude.

This example shows that for a parametrized surface $u \mapsto \gamma(u)$ the parameter values $u = (u_1,\ldots,u_p)$ can be used to associate coordinates to any point of the trace $\gamma(U)$. This is however only possible in a unique way, if the mapping γ is injective. This gives rise to the following

Definition 7.3 *Let $U \subseteq \mathbb{R}^p$ be an open set. The parametrized surface $\gamma : U \to \mathbb{R}^n$ is called simple, if γ is injective with continuous inverse $\kappa = \gamma^{-1} : \gamma(U) \to U$. The mapping κ is called coordinate mapping on the set $\gamma(U)$.*

This definition is explained in the figure at the example of parametrized curves in $\mathbb{R}^2$, which are not simple.



γ is not injective: The two different parameter values u_1 and u_2 are mapped to the same double point of the curve.

γ^{-1} is not continuous: The image of every sphere around y contains points, whose distance to $\gamma^{-1}(y)$ is greater than $\varepsilon = \frac{1}{2}\big(b - \gamma^{-1}(u)\big)$.

Examples of parametrized curves $\gamma : (a, b) \to \mathbb{R}^2$, which are not simple.

Tangent space. The derivative $\gamma'(u)$ of a p–dimensional parametrized surface $\gamma : U \to \mathbb{R}^n$ can be interpreted geometrically. To explain this interpretation, note that for given $u \in U$ the column vectors of the Jacobi matrix $\gamma'(u)$ are the partial derivatives $\frac{\partial\gamma}{\partial u_1}(u), \ldots, \frac{\partial\gamma}{\partial u_p}(u)$. These vectors are tangential to the trace $\gamma(U)$ at the point $x = \gamma(u)$. To see this, observe that the mapping $\xi \mapsto \gamma(u_1 \ldots, u_{i-1}, \xi, u_{i+1}, \ldots, u_p)$ is a parametrized curve, whose trace lies in the trace $\gamma(U)$. The domain of definition of this mapping is an open set in $\mathbb{R}$ containing the point $\xi = u_i$, and the trace passes through the point x. The derivative $\frac{\partial\gamma}{\partial u_i}(u)$ is a tangential vector to the trace of the curve at x, hence it is also a tangential vector to the trace $\gamma(U)$.

The vectors $\frac{\partial\gamma}{\partial u_1}(u), \ldots, \frac{\partial\gamma}{\partial u_p}(u)$ span the range of the linear mapping $\gamma'(u) \in L(\mathbb{R}^p, \mathbb{R}^n)$. By Definition 7.2 the dimension of this range space is p, which means that the set of tangent vectors $\frac{\partial\gamma}{\partial u_1}(u), \ldots, \frac{\partial\gamma}{\partial u_p}(u)$ is linearly independent. Since every linear combination of tangential vectors is again a tangential vector, we see that every vector from the range of the linear mapping $\gamma'(u)$ is a tangential vector to the trace $\gamma(U)$ at the point $x = \gamma(u)$.

This motivates the following

Definition 7.4 *Let $\gamma : U \to \mathbb{R}^n$ be a simple p–dimensional parametrized surface in $\mathbb{R}^n$ and let $x = \gamma(u)$ be a point of the trace $\gamma(U)$. The p–dimensional range of the linear mapping $\gamma'(u) \in L(\mathbb{R}^p, \mathbb{R}^n)$ is called the tangent space to the parametrized surface γ at x. We denote this tangent space by $T_x(\gamma)$.*

7.2 Integration on parametrized surfaces

By Definition 6.4 the measure of a bounded subset Ω of $\mathbb{R}^n$ is computed by integration of the constant function 1 over the set Ω. For $n = 2$ this measure is equal to the area of Ω. Accordingly, if we want to compute the area of a sphere in $\mathbb{R}^3$ we must integrate the function 1 over the sphere. Similarly, if we want to compute the weight of a thin metallic shell of variable thickness, we model this shell as a set K in $\mathbb{R}^3$ with zero thickness, that is, as a surface, and integrate the function $f(x)$, whose value is the thickness of the shell at the point $x \in K$ multiplied by the specific weight, over the surface K. These are two examples, where one needs to compute surface integrals. In this section we give the precise definition of such integrals.

The idea behind the definition is to choose a parametrized surface γ such that the trace of γ is equal to the surface K, over which we want to integrate, and to use this function γ to define the integral over K as an integral over the domain of definition of γ, which is a subset of $\mathbb{R}^p$. However, Examples 3 and 4 show that there are different parametrized surfaces with the same trace; there is no natural way how to choose one of these parametrized surfaces to define the integral. We must therefore define the surface integral in such a way that it is independent of the parametrized surface chosen to compute the integral. To give this definition we first need to introduce Gram's determinant.

Definition 7.5 *Let U be an open subset of $\mathbb{R}^p$ with $p < n$ and let $\gamma : U \to \mathbb{R}^n$ be a parametrized surface. Define $G : U \to \mathbb{R}^{p\times p}$ by*

$$G(u) = \begin{pmatrix} g_{11}(u) & \dots & g_{1p}(u) \\ \vdots & & \\ g_{p1}(u) & \dots & g_{pp}(u) \end{pmatrix},$$

where for $1 \le i, j \le p$ the continuous functions $g_{ij} : U \to \mathbb{R}$ are given by

$$g_{ij}(u) = \frac{\partial\gamma}{\partial u_i}(u) \cdot \frac{\partial\gamma}{\partial u_j}(u) = \begin{pmatrix} \frac{\partial\gamma_1}{\partial u_i}(u) \\ \vdots \\ \frac{\partial\gamma_n}{\partial u_i}(u) \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial\gamma_1}{\partial u_j}(u) \\ \vdots \\ \frac{\partial\gamma_n}{\partial u_j}(u) \end{pmatrix} = \sum_{k=1}^{n} \frac{\partial\gamma_k}{\partial u_i}(u) \frac{\partial\gamma_k(u)}{\partial u_j}\,.$$

The function $g : U \to \mathbb{R}$ defined by $g(u) := \det(G(u))$ is called Gram's determinant to γ. (Jørgen Pedersen Gram, 1850 – 1916.)

Remark Gram's determinant g is continuous, since $\det(G(u))$ is computed as a sum of products of the continuous functions g_{ij}.

Definition 7.6 *Let U be an open subset of $\mathbb{R}^p$ with $p < n$ and let $\gamma : U \to \mathbb{R}^n$ be a simple p–dimensional parametrized surface with coordinate mapping $\kappa = \gamma^{-1} : \gamma(U) \to U$.*

(i) A compact subset K of the trace $\gamma(U) \subseteq \mathbb{R}^n$ is called measurable if the image $\kappa(K) \subseteq U$ is Jordan measurable. The measure of K is given by

$$|K| = \int_{\kappa(K)} \sqrt{g(u)}du.$$

(ii) If K is a compact measurable subset of $\gamma(U)$ and if $f : K \to \mathbb{R}$ is a continuous function, then the surface integral is defined by

$$\int_K f(x)dS(x) := \int_{\kappa(K)} f(\gamma(u))\sqrt{g(u)}du.$$

Remarks 1) $dS(x)$ is called the p–dimensional surface element of K. Symbolically one writes

$$dS(x) = \sqrt{g(u)}du\,, \quad x = \gamma(u)\,.$$

2) The set $\kappa(K)$ is compact. This follows from Theorem 3.30, since by Definition 7.3 of simple parametrized surfaces the inverse κ of γ is continuous and since K is compact.
3) By Theorem 6.5 the integrals $\int_{\kappa(K)} \sqrt{g(u)}du$ and $\int_{\kappa(K)} f(\gamma(u))\sqrt{g(u)}du$ exist as integrals of the continuous functions $\sqrt{g(u)}$ and $f(\gamma(u))\sqrt{g(u)}$ over the measurable subset $\kappa(K)$ of $\mathbb{R}^p$.
4) If $p = 1$ and if $U = (a, b)$ is an interval, then $\gamma : (a, b) \to \mathbb{R}^n$ is a curve, Gram's determinant is equal to $g(u) = \left|\frac{d\gamma(u)}{du}\right|^2$, and if $K = \gamma([c, d])$, where $[c, d]$ is a compact interval contained in (a, b), the integral

$$\int_K f(x)ds(x) = \int_c^d f(\gamma(u))\sqrt{g(u)}du = \int_c^d f(\gamma(u))\Big|\frac{d\gamma(u)}{du}\Big|du$$

is called a curve integral. The length of the curve between the points $\gamma(c)$ and $\gamma(d)$ is $\ell(K) = \int_c^d \left|\frac{d\gamma(u)}{du}\right|du$, and $ds(x) = \left|\frac{d\gamma(u)}{du}\right|du$ is called line element.

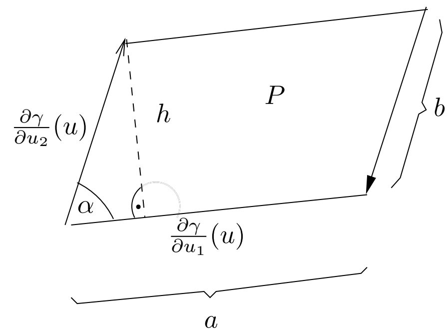
To motivate Definition 7.6 we must determine the meaning of the expression $\sqrt{g(u)}$. To this end fix $u \in U$ and consider the set

$$P = \Big\{\sum_{i=1}^p r_i \,\frac{\partial\gamma}{\partial u_i}\,(u) \;\Big|\; r_i \in \mathbb{R}\,,\, 0 \le r_i \le 1\Big\}.$$

P is a parallelotope. Every element from P is a linear combination of the tangent vectors $\frac{\partial\gamma}{\partial u_1}(u), \ldots, \frac{\partial\gamma}{\partial u_p}(u)$ to $\gamma(U)$ at $x = \gamma(u)$, hence by Definition 7.4 P is a subset of the tangent space $T_x(\gamma)$.

Theorem 7.7 *We have $g(u) > 0$ and $\sqrt{g(u)}$ is equal to the measure $|P|$ of the p–dimensional parallelotope P.*

This result is known from linear algebra. We give here the simple **proof** in the case $p = 2$. In this case P is the parallelogram shown in the figure.



With $a = \left|\frac{\partial\gamma}{\partial u_1}(u)\right|$ and $b = \left|\frac{\partial\gamma}{\partial u_2}(u)\right|$ it follows that

$$\sqrt{g(u)} = \sqrt{\det(G(u))}$$

$$= \sqrt{\begin{vmatrix} \frac{\partial\gamma}{\partial u_1}(u)\cdot\frac{\partial\gamma}{\partial u_1}(u) & \frac{\partial\gamma}{\partial u_1}(u)\cdot\frac{\partial\gamma}{\partial u_2}(u) \\ \frac{\partial\gamma}{\partial u_2}(u)\cdot\frac{\partial\gamma}{\partial u_1}(u) & \frac{\partial\gamma}{\partial u_2}(u)\cdot\frac{\partial\gamma}{\partial u_2}(u) \end{vmatrix}} = \sqrt{\begin{vmatrix} a^2 & ab\cos\alpha \\ ab\cos\alpha & b^2 \end{vmatrix}}$$

$$= \sqrt{a^2b^2 - a^2b^2\cos^2\alpha} = ab\sqrt{1-\cos^2\alpha} = ab\sin\alpha = a\cdot h.$$

$a \cdot h$ is the area of P. ∎

The interpretation of the formula $dS = \sqrt{g(u)}du$ is therefore that locally the mapping γ distorts a small p–dimensional volume element du, which is a p–dimensional cube with measure $|du|$, into the small surface element $dS(x)$, a parallelotope with measure $|dS(x)| = \sqrt{g(u)}|du|$. The factor $\sqrt{g(u)}$ in the surface integral corrects the change in measure caused by this distortion.

Before we can show that the measure and the integral on parametrized surfaces are well defined, we need another definition:

Definition 7.8 *Let $U, V \subseteq \mathbb{R}^p$ be open sets. Parametrized surfaces $\gamma : U \to \mathbb{R}^n$ and $\tilde{\gamma} : V \to \mathbb{R}^n$ are called equivalent, if there exists a diffeomorphism $\varphi : V \to U$ with*

$$\tilde{\gamma} = \gamma \circ \varphi\,.$$

This is an equivalence relation on the set of parametrized surfaces. Of course, this definition implies that equivalent parametrized surfaces γ and $\tilde{\gamma}$ have the same trace. The next theorem shows that simple parametrized surfaces are equivalent if and only if they have the same trace.

Theorem 7.9 *Let $U, V \subseteq \mathbb{R}^p$ be open sets and let $\gamma : U \to \mathbb{R}^n$ and $\tilde{\gamma} : V \to \mathbb{R}^n$ be simple parametrized surfaces with $\gamma(U) = \tilde{\gamma}(V)$. Then*

$$\gamma^{-1} \circ \tilde{\gamma} : V \to U$$

is a diffeomorphism.

A **proof** of this theorem can be found in Barner-Flohr, Analysis II, pp. 384 (in German). By this theorem $\varphi = \gamma^{-1} \circ \tilde{\gamma}$ is a diffeomorphism which satisfies $\tilde{\gamma} = \gamma \circ \varphi$, hence γ and $\tilde{\gamma}$ are indeed equivalent.

Example 6: Let U be the open unit ball in $\mathbb{R}^2$ and let $\gamma : U \to \mathbb{R}^3$ and $\tilde{\gamma} : U \to \mathbb{R}^3$, respectively, be the parametrized surfaces from Example 1 and Example 4, respectively. Both parametrized surfaces are simple and the trace is the upper half of the unit sphere in $\mathbb{R}^3$. Therefore these parametrized surfaces are equivalent. A diffeomorphism $\varphi : U \to U$ is given by

$$\varphi(u, v) = \begin{pmatrix} \frac{2u}{1+u^2+v^2} \\ \frac{2v}{1+u^2+v^2} \end{pmatrix}.$$

We have

$$(\gamma \circ \varphi)(u, v) = \begin{pmatrix} \frac{2u}{1+u^2+v^2} \\ \frac{2v}{1+u^2+v^2} \\ \sqrt{1 - \frac{4u^2+4v^2}{(1+u^2+v^2)^2}} \end{pmatrix} = \frac{1}{1+u^2+v^2} \begin{pmatrix} 2u \\ 2v \\ 1-u^2-v^2 \end{pmatrix} = \tilde{\gamma}(u, v).$$

We are now in a position to show that the content of Definition 7.6 does not change if we replace the simple parametrized surface γ by an equivalent simple parametrized surface $\tilde{\gamma}$.

Theorem 7.10 *Let $U, V \subseteq \mathbb{R}^p$ be open sets, let $\gamma : U \to \mathbb{R}^n$ and $\tilde{\gamma} : V \to \mathbb{R}^n$ be simple parametrized surfaces with $\gamma(U) = \tilde{\gamma}(V) = F$ and with coordinate mappings $\kappa = \gamma^{-1}$ and $\tilde{\kappa} = \tilde{\gamma}^{-1}$. Set $\varphi = \kappa \circ \tilde{\gamma} : V \to U$. The Gram determinants to γ and $\tilde{\gamma}$ are denoted by $g : U \to \mathbb{R}$ and $\tilde{g} : V \to \mathbb{R}$, respectively. Then we have:*

(i) For all $v \in V$

$$\tilde{g}(v) = g(\varphi(v))\, |\det \varphi'(v)|^2.$$

(ii) If $K \subseteq F$ is a compact set such that $\kappa(K) \subseteq U$ is measurable, and if $f : K \to \mathbb{R}$ is continuous, then $\tilde{\kappa}(K) \subseteq V$ is measurable and

$$\int_{\kappa(K)} f(\gamma(u))\sqrt{g(u)}du = \int_{\tilde{\kappa}(K)} f(\tilde{\gamma}(v))\sqrt{\tilde{g}(v)}dv\,.$$

Remark. We have that $\tilde{\gamma} = \gamma \circ \varphi$. By Theorem 7.9 the mapping φ is a diffeomorphism.
Proof of Theorem 7.10: (i) From

$$g_{ij}(u) = \sum_{k=1}^{n} \frac{\partial \gamma_k(u)}{\partial u_i} \frac{\partial \gamma_k(u)}{\partial u_j}\,,$$

we obtain that

$$G(u) = \gamma'(u)^T \gamma'(u)\,,$$

where $\gamma'(u)^T$ denotes the transpose matrix to the matrix $\gamma'(u)$. The chain rule and the rule of multiplication of determinants thus imply

$$\begin{aligned}\tilde{g} = \det \tilde{G} = \det\big((\tilde{\gamma}')^T \tilde{\gamma}'\big) &= \det\big([(\gamma' \circ \varphi)\varphi']^T (\gamma' \circ \varphi)\varphi'\big) = \det\big((\varphi')^T [\gamma' \circ \varphi]^T [\gamma' \circ \varphi]\varphi'\big) \\ &= (\det \varphi') \det\big([\gamma' \circ \varphi]^T [\gamma' \circ \varphi]\big)(\det \varphi'\big) = (\det \varphi')^2 \det(G \circ \varphi) = (\det \varphi')^2 (g \circ \varphi)\,.\end{aligned}$$

(ii) Since K is compact and since κ is continuous, it follows that $\kappa(K)$ is a compact, measurable subset of the open set $U \subseteq \mathbb{R}^p$. Since φ is a diffeomorphism, also the invers $\psi = \varphi^{-1} : U \to V$ is diffeomorphic. The transformation rule Theorem 6.12 thus implies that the set $\tilde{\kappa}(K) = \psi(\kappa(K)) \subseteq V$ is measurable. Moreover, since the function $\tilde{f} = (f \circ \tilde{\gamma})\sqrt{\tilde{g}} : \tilde{\kappa}(K) \to \mathbb{R}$ is continuous, the transformation rule together with statement (i) of the theorem imply

$$\begin{aligned}&\int_{\kappa(K)} f(\gamma(u))\sqrt{g(u)}du \\ &\qquad = \int_{\tilde{\kappa}(K)} f((\gamma \circ \varphi)(v))\sqrt{g(\varphi(v))}\,|\det \varphi'(v)|dv = \int_{\tilde{\kappa}(K)} f(\tilde{\gamma}(v))\sqrt{\tilde{g}(v)}dv\,.\end{aligned}$$

■

7.3 Submanifolds, tangent and normal space

The trace of the parametrized surface in Example 3 is the punched sphere, the trace of the parametrized surface in Example 5 is the sphere with a half circle connecting the poles removed. However, for topological reasons there exists no parametrized surface, whose trace is the entire sphere. To parametrize the entire sphere we have to split it into several parts, of which everyone is a trace of a parametrized surfaces. Therefore we define:

Definition 7.11 *A subset $M \subseteq \mathbb{R}^n$ is called p–dimensional submanifold of $\mathbb{R}^n$ if there exists for each $x \in M$ an open n–dimensional neighborhood V of x such that $V \cap M$ is the trace of a simple p–dimensional parametrized surface $\gamma : U \to V \cap M$.*

The set $V \cap M$ is called a coordinate neighborhood of x, the continuous inverse mapping $\kappa = \gamma^{-1} : V \cap M \to U \subseteq \mathbb{R}^p$ is called a local coordinate mapping or a chart. From now on we also say that the parametrized surface γ is a simple parametrization of the coordinate neighborhood $V \cap M$.

If V and $\tilde{V}$ are coordinate neighborhoods in M, whose intersection is not empty, if $\kappa : V \to U$, $\tilde{\kappa} : \tilde{V} \to \tilde{U}$ are coordinate mappings, and if $\gamma : U \to V$, $\tilde{\gamma} : \tilde{U} \to \tilde{V}$ are parametrizations, then the intersection $W = V \cap \tilde{V}$ is a coordinate neighborhood,

$$\kappa\big|_W : W \to \kappa(W) = U_1 \subseteq U \quad \text{and} \quad \tilde{\kappa}\big|_W : W \to \tilde{\kappa}(W) = U_2 \subseteq \tilde{U}$$

are local coordinate mappings on W, and

$$\gamma\big|_{U_1} : U_1 \to W, \qquad \tilde{\gamma}\big|_{U_2} : U_2 \to W$$

are parametrizations of W. Since the parametrizations $\gamma\big|_{U_1}$ and $\tilde{\gamma}\big|_{U_2}$ are simple and the traces of the parametrizations coincide, Theorem 7.9 implies that these parametrizations of W are equivalent. That is,

$$\varphi = \tilde{\kappa} \circ \gamma\big|_{U_1} : U_1 \to U_2 \quad \text{and} \quad \varphi^{-1} = \kappa \circ \tilde{\gamma}\big|_{U_2} : U_2 \to U_1$$

are diffeomorphisms.

The tangent space $T_x(\gamma)$ to a parametrized surface γ was introduced in Definition 7.4. Based on the above, we can extend this definition to submanifolds:

Definition 7.12 *Let x be a point of the submanifold M with coordinate neighborhood V and simple parametrization $\gamma : U \to V$. The p–dimensional tangent space T_xM to M at the point x is defined by $T_xM = T_x(\gamma)$. The $(n-p)$–dimensional orthogonal space*

$$T_x^{\perp}M = \{\nu \in \mathbb{R}^n \mid \nu \cdot \tau = 0 \textit{ for all } \tau \in T_xM\}$$

is called the normal space to M at x.

This definition is independent of the coordinate neighborhood V and parametrization $\gamma : U \to V$ chosen. For, if $\tilde{V}$ is another coordinate neighborhood of x with simple parametrization $\tilde{\gamma} : \tilde{U} \to \tilde{V}$ and if $u \in U$, $\tilde{u} \in \tilde{U}$ are points such that $x = \gamma(u) = \tilde{\gamma}(\tilde{u})$, then by the above there are open subsets $U_1 \subseteq U$, $U_2 \subseteq \tilde{U}$ and a diffeomorphism $\varphi : U_1 \to U_2$, such that $\gamma|_{U_1} = \tilde{\gamma}|_{U_2} \circ \varphi$. From this equation we obtain by the chain rule

$$\gamma'(u) = \tilde{\gamma}'(\tilde{u})\varphi'(u).$$

Since φ is a diffeomorphism, $\varphi'(u) : \mathbb{R}^p \to \mathbb{R}^p$ is an invertible linear mapping, hence this equation implies that the ranges $T_x(\gamma)$ and $T_x(\tilde{\gamma})$ of the linear mappings $\gamma'(u)$ and $\tilde{\gamma}'(\tilde{u})$ coincide. This means that T_xM and $T_x^{\perp}M$ are independent of the choice of the parametrization.

Example 7: Let $S = \{x \in \mathbb{R}^3 \mid |x| = 1\}$ be the unit sphere in $\mathbb{R}^3$. The simple parametrized surface $\gamma : \mathbb{R}^2 \to \mathbb{R}^3$ of Example 3 has the trace $\gamma(\mathbb{R}^2) = S \setminus \{(0,0,-1)\}$. This trace is a coordinate neighborhood for everyone of its points, the local coordinate mapping $\kappa = \gamma^{-1} : S \setminus \{(0,0,-1)\} \to \mathbb{R}^2$ is the stereographic projection from the south pole to the plane $\mathbb{R}^2$. Similarly, the set $S \setminus \{(0,0,1)\}$ is a coordinate neighborhood for everyone of its points, and in particular for the point $(0,0,-1)$, which does not belong to the first coordinate neighborhood. A local coordinate mapping on $S \setminus \{(0,0,1)\}$ is given by the stereographic projection from the north pole onto $\mathbb{R}^2$. Therefore S is a two dimensional submanifold of $\mathbb{R}^3$. Two charts suffice to cover this submanifold.

7.4 Integration on submanifolds

To define the integral on submanifolds we need some preparations. Let M be a p–dimensional submanifold of $\mathbb{R}^n$ and let $V \subseteq M$ be a coordinate neighborhood in M. We define the support $\mathrm{supp} f$ of a continuous function $f : M \to \mathbb{R}$ by

$$\mathrm{supp} f = \overline{\{x \in M \mid f(x) \neq 0\}}$$

and say that f has compact support in V, if $\mathrm{supp} f$ is a compact subset of V. For continuous functions with compact support in V we define the surface integral $\int_V f(x)dS(x)$ as follows:

Let $\gamma : U \to V$ be a parametrization, $\kappa = \gamma^{-1} : V \to U$ the coordinate mapping and $g : U \to \mathbb{R}$ the Gram determinant to γ. Here U denotes an open subset of $\mathbb{R}^p$. Purely to avoid technical details we assume that U is bounded. In this case we can choose a compact interval Q such that $U \subseteq Q$. If f has compact support in V, then $\kappa(\mathrm{supp} f)$ is a compact subset of U, since κ is continuous. Because U is open, this implies that the distance of $\kappa(\mathrm{supp} f)$ to the boundary ∂U of U is positive: $\mathrm{dist}(\kappa(\mathrm{supp} f), \partial U) > 0$. Therefore the continuous function $f \circ \gamma : U \to \mathbb{R}$ and consequently also the function $(f \circ \gamma)\sqrt{g} : U \to \mathbb{R}$

vanish in a neighborhood of ∂U. If we extend the latter function by zero from U to all of Q, we thus obtain a continuous function on Q. By Theorem 6.5 this function is integrable over the measurable set Q because it is continuous, hence by Definition 6.3 it is integrable over U. For continuous functions $f : M \to \mathbb{R}$ with compact support in the coordinate neighborhood V we therefore define the surface integral by

$$\int_V f(x)dS(x) = \int_U (f \circ \gamma)(u)\sqrt{g(u)}\, du. \tag{7.1}$$

To show that this definition is independent of the choice of the parametrization, let K be a compact, measurable subset of V such that

$$\text{supp} f \subseteq K \subseteq V. \tag{7.2}$$

Such a set K exists. For, it is not difficult to construct a set Ω, which consists of finitely many closed intervals in $\mathbb{R}^p$, such that $\kappa(\text{supp} f) \subseteq \Omega \subseteq U$ holds. Every closed interval is compact and Jordan measurable, whence also the finite union Ω. This implies that the set $K = \gamma(\Omega)$ is compact, satisfies (7.2) and, by Definition 7.6, is measurable as a subset of V.

Now if $\tilde{\gamma} : \tilde{U} \to V$ is another simple parametrization of V with the coordinate mapping $\tilde{\kappa} = \tilde{\gamma}^{-1} : V \to \tilde{U}$ and the Gram determinant $\tilde{g} : \tilde{U} \to \mathbb{R}$, then for every such K we have

$$\begin{aligned}\int_U (f \circ \gamma)(u)\sqrt{g(u)}\, du &= \int_{\kappa(K)} (f \circ \gamma)(u)\sqrt{g(u)}\, du \\ &= \int_{\tilde{\kappa}(K)} (f \circ \tilde{\gamma})(v)\sqrt{\tilde{g}(v)}\, dv = \int_{\tilde{U}} (f \circ \tilde{\gamma})(v)\sqrt{\tilde{g}(v)}\, dv. \end{aligned}\tag{7.3}$$

The first and last equality sign hold, because (7.2) implies the relations $\kappa(\text{supp} f) \subseteq \kappa(K) \subseteq U$ and $\tilde{\kappa}(\text{supp} f) \subseteq \tilde{\kappa}(K) \subseteq \tilde{U}$, and the second equality sign holds by statement (ii) of Theorem 7.10, which theorem can be applied, since K is compact and measurable. Equation (7.3) shows, that the definition of the surface integral in (7.1) is independent of the choice of the parametrization.

If V and V' are coordinate neighborhoods in M with $V' \subseteq V$ and if $f : M \to \mathbb{R}$ is a continuous function with compact support contained in V', then the definition of the surface integral implies

$$\int_V f(x)dS(x) = \int_{V'} f(x)dS(x). \tag{7.4}$$

We leave the obvious proof to the reader.

Definition 7.13 *Let M be a p–dimensional submanifold of $\mathbb{R}^n$ and let $\{V_i\}_{i=1}^{\infty}$ be a family of coordinate neighborhoods in M such that $M = \bigcup_{i=1}^{\infty} V_i$. For each $i \in \mathbb{N}$ let $\alpha_i : M \to \mathbb{R}$*

be a continuous function with compact support contained in V_i. *We say that* $\{\alpha_i\}_{i=1}^{\infty}$ *is a partition of unity on* M *subordinate to the covering* $\{V_i\}_{i=1}^{\infty}$, *if*

$$\sum_{i=1}^{\infty} \alpha_i(x) = 1, \quad \textit{for all } x \in M.$$

Now we can define the surface integral on submanifolds. For simplicity we restrict ourselves to submanifolds, for which a partition of unity subordinate to a finite covering exists.

Definition 7.14 *Let* M *be a* p*–dimensional submanifold of* $\mathbb{R}^n$ *and assume that there is a partition of unity* $\{\alpha_i\}_{i=1}^{m}$ *of continuous functions subordinate to the covering* $\{V_i\}_{i=1}^{m}$ *of* M *by coordinate neighborhoods* V_i. *Let* $f : M \to \mathbb{R}$ *be continuous. We define the surface integral of* f *over* M *by*

$$\int_M f(x)\, dS(x) = \sum_{i=1}^{m} \int_{V_i} \alpha_i(x) f(x)\, dS(x).$$

Remark. Since $\alpha_i f$ is a continuous function on M with compact support in the coordinate neighborhood V_i, the integral $\int_{V_i} \alpha_i(x) f(x)\, dS(x)$ is defined as in equation (7.1).

The partition of unity $\{\alpha_i\}_{i=1}^{m}$ is not uniquely determined. Therefore this definition is meaningful only if the surface integral is independent of the choice of the partition. We show this in the next theorem.

Theorem 7.15 *Let* M *be a submanifold and let* $f : M \to \mathbb{R}$ *be a continuous function. Assume that* $\{\alpha_i\}_{i=1}^{m}$ *and* $\{\beta_j\}_{j=1}^{\ell}$ *are partitions of unity of continuous functions subordinate to the coverings* $\{V_i\}_{i=1}^{m}$ *and* $\{\tilde{V}_j\}_{j=1}^{\ell}$ *of* M *by coordinate neighborhoods, respectively. Then we have*

$$\sum_{i=1}^{m} \int_{V_i} \alpha_i(x) f(x) dS(x) = \sum_{j=1}^{\ell} \int_{\tilde{V}_j} \beta_j(x) f(x) dS(x)\,.$$

Proof: Note first that $V_i \cap V_j'$ is a coordinate neighborhood on M satisfying $V_i \cap V_j' \subseteq V_i$ and $V_i \cap V_j' \subseteq V_j'$. Moreover, $\alpha_i \beta_j f$ is a continuous function on M with compact support contained in $V_i \cap V_j'$. By (7.4) we thus have

$$\int_{V_i} \alpha_i(x)\beta_j(x) f(x) dS(x) = \int_{V_i \cap V_j'} \alpha_i(x)\beta_j(x) f(x) dS(x) = \int_{V_j'} \alpha_i(x)\beta_j(x) f(x) dS(x).$$

This equation yields

$$\sum_{i=1}^{m}\int_{V_i}\alpha_i(x)f(x)dS(x)=\sum_{i=1}^{m}\int_{V_i}\alpha_i(x)\sum_{j=1}^{\ell}\beta_j(x)f(x)dS(x)$$

$$=\sum_{i=1}^{m}\sum_{j=1}^{\ell}\int_{V_i}\alpha_i(x)\beta_j(x)f(x)dS(x)=\sum_{i=1}^{m}\sum_{j=1}^{\ell}\int_{V_j'}\alpha_i(x)\beta_j(x)f(x)dS(x)$$

$$=\sum_{j=1}^{\ell}\int_{V_j'}\sum_{i=1}^{m}\alpha_i(x)\beta_j(x)f(x)dS(x)=\sum_{j=1}^{\ell}\int_{V_j'}\beta_j(x)f(x)dS(x).$$

■

7.5 The integral theorem of Gauß

In the remainder of Section 7 we state without proofs the integral theorems of Gauß and Stokes and Green's identities. These theorems belong to the central results of the theory of multidimensional integration. They are of fundamental importance in the theory of partial differential equations and in all applications of science and engineering. To demonstrate the application of the theorems we discuss simple examples from potential theory and physics.

We start with the theorem of Gauß. To formulate it, we need some preparations.

Normal vector fields. Let M be an $(n-1)$–dimensional submanifold of $\mathbb{R}^n$. The normal space $T_x^{\perp}M$ at $x \in M$ is one-dimensional. Every element $\nu \neq 0$ from $T_x^{\perp}M$ is called a normal vector to M at x. If $|\nu| = 1$ holds, then ν is called a unit normal vector. There are two unit normal vectors ν and $-\nu$ at the point x. If $W \subseteq M$ and if $\nu(x)$ is a unit normal vector for every $x \in W$, then the mapping $x \mapsto \nu(x) : W \to \mathbb{R}^n$ is called a unit normal vector field on W.

Theorem 7.16 *Let W be coordinate neighborhood in the $(n-1)$–dimensional submanifold M of $\mathbb{R}^n$. Then there is a continuous unit normal vector field ν on W.*

Proof: Let $\gamma : U \to W$ be a simple parametrization of W and let $u \in U$. With the Jacobi matrix $\gamma'(u) = (\partial_{u_k}\gamma_j(u))_{j=1,\ldots,n;k=1,\ldots,n-1}$ define the vector $N(\gamma(u)) = (N_i(\gamma(u))_{i=1,\ldots,n}$ by

$$N_i(\gamma(u)) = (-1)^{1+i}\det(\partial_{u_k}\gamma_j(u))_{j\neq i}\,. \tag{7.5}$$

For $k = 1,\ldots,n-1$ we have

$$N(\gamma(u))\cdot\partial_{u_k}\gamma(u) = \det\big(\partial_{u_k}\gamma(u),\partial_{u_1}\gamma(u),\ldots,\partial_{u_{(n-1)}}\gamma(u)\big) = 0. \tag{7.6}$$

The first equality in (7.6) is seen to hold by expansion of the determinant with respect to the first column $\partial_{u_k}\gamma(u)$, the second equality holds since the determinant contains two times the same column $\partial_{u_k}\gamma(u)$. We conclude from (7.6) that $N(\gamma(u))$ is a normal vector to M at $\gamma(u)$. The determinant in (7.5) is a sum of products of the continuous functions $\partial_{u_k}\gamma_j$, hence the component functions $N_i \circ \gamma : U \to \mathbb{R}$ of the function $N \circ \gamma : U \to \mathbb{R}^n$ are continuous. By Theorem 3.22 this means that $N \circ \gamma$ is continuous, hence also the function $N = N \circ \gamma \circ \kappa : W \to \mathbb{R}^n$ is continuous, where $\kappa = \gamma^{-1}$ is the continuous coordinate mapping on W. From this we conclude that $|N| = \sqrt{\sum_{i=1}^n N_i^2}$ is a continuous function, so also the unit normal vector field $\nu = N/|N| : W \to \mathbb{R}^3$ is continuous. ∎

Remark. For $n = 3$ equation (7.5) yields $N(\gamma(u)) = \partial_{u_1}\gamma(u) \times \partial_{u_2}\gamma(u)$, with the vector product $\times$ on $\mathbb{R}^3$.

From this local result one cannot conclude that on every $(n-1)$–dimensional submanifold M in $\mathbb{R}^n$ there exists a unit normal vector field, which is continuous on all of M. A counterexample will be given later in this section.

Definition 7.17 (Exterior unit normal) *Let $A \subseteq \mathbb{R}^n$ be a nonempty set. We say that A has a smooth boundary, if ∂A is an $(n-1)$–dimensional submanifold of $\mathbb{R}^n$.*

A normal vector $N \in \mathbb{R}^n$ to the $(n-1)$–dimensional submanifold ∂A at $x \in \partial A$ is called exterior normal vector, if there exists $\delta_0 > 0$ such that $x + \delta N$ belongs to the complement $\mathbb{R}^n \setminus A$ of A for all $0 < \delta < \delta_0$. A mapping $\nu : \partial A \to \mathbb{R}^n$ is called exterior unit normal vector field, if $\nu(x)$ is an exterior unit normal vector for every $x \in \partial A$.

Definition 7.18 (Divergence) *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $f : \Omega \to \mathbb{R}^n$ be differentiable. The function $\operatorname{div} f : \Omega \to \mathbb{R}$ is defined by*

$$\operatorname{div} f(x) := \sum_{i=1}^n \frac{\partial}{\partial x_i} f_i(x)\,.$$

$\operatorname{div} f(x)$ is called the divergence of f at x.

We can now state the Gauß theorem:

Theorem 7.19 (Integral theorem of Gauß) *Let $A \subseteq \mathbb{R}^n$ be a bounded open set with smooth boundary ∂A, for which a continuous exterior unit normal vector field $\nu : \partial A \to \mathbb{R}^n$ exists. Then for every continuously differentiable vector field $f : \Omega \to \mathbb{R}^n$, which is defined on an open set $\Omega \subseteq \mathbb{R}^n$ with $\overline{A} \subseteq \Omega$, the equation*

$$\int_{\partial A} \nu(x) \cdot f(x) dS(x) = \int_A \operatorname{div} f(x) dx$$

holds. (Carl Friedrich Gauß, 1777 – 1855.)

For $n = 1$ the theorem says: Let $a, b \in \mathbb{R}$, $a < b$. Then

$$f(b) - f(a) = \int_a^b \frac{d}{dx} f(x) dx \, ,$$

and we see that the theorem of Gauß is the generalization of the fundamental theorem of calculus to $\mathbb{R}^n$ with $n > 1$.

Example: Application in hydrostatics. A body A is submerged in a liquid with specific weight c. The surface of the liquid is given by the plane $x_3 = 0$. Then the pressure at a point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ with $x_3 < 0$ is

$$p(x) = -cx_3 \, .$$

This pressure exerts a force on the body A. If $\nu(x)$ is the exterior unit normal vector to ∂A at $x \in \partial A$, then the force per unit area at x is

$$-\nu(x)p(x) = -cx_3(-\nu(x)) = cx_3\nu(x).$$

The total force on the body is thus equal to

$$K = \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \int_{\partial A} cx_3 \nu(x) dS(x) \, .$$

Application of the Gauß Theorem to the functions $f_1, f_2, f_3 : A \to \mathbb{R}^3$ defined by

$$f_1(x_1, x_2, x_3) = (x_3, 0, 0), \quad f_2(x_1, x_2, x_3) = (0, x_3, 0), \quad f_3(x_1, x_2, x_3) = (0, 0, x_3)$$

yields for $i = 1, 2$

$$K_i = \int_{\partial A} cx_3 \nu_i(x) dS(x) = c \int_{\partial A} \nu(x) \cdot f_i(x) dS(x) = c \int_A \frac{\partial}{\partial x_i} x_3 \, dx = 0,$$

and for $i = 3$

$$K_3 = \int_{\partial A} cx_3 \nu_3(x) dS(x) = c \int_{\partial A} \nu(x) \cdot f_3(x) dS(x) = c \int_A \frac{\partial}{\partial x_3} x_3 dx = c \int_A dx = c \, |A|.$$

K has the direction of the positive x_3-axis. Therefore K is a buoyant force acting on A with the value $c \, |A| = c \, \mathrm{Vol}(A)$. This is equal to the weight of the liquid displaced by the body A.

7.6 Green's identities

Assume that $A \subseteq \mathbb{R}^n$ is a bounded open set with smooth boundary, for which a continuous exterior unit normal vector field $\nu : \partial A \to \mathbb{R}^n$ exists, and let Ω be an open set with $\overline{A} \subseteq \Omega$. In the following we write $\nabla f(x)$ to denote the gradient of differentiable $f : \Omega \to \mathbb{R}$.

Definition 7.20 *Let the function $f : \Omega \to \mathbb{R}$ be continuously differentiable. Then the normal derivative of f at the point $x \in \partial A$ is defined by*

$$\frac{\partial f}{\partial \nu}(x) := f'(x)\nu(x) = \nu(x) \cdot \nabla f(x) = \sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} \nu_i(x) .$$

For twice differentiable $f : \Omega \to \mathbb{R}$ set

$$\Delta f(x) := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} f(x) .$$

Remarks. 1.) The normal derivative of f at the point $x \in \partial A$ is the derivative of f in the direction of the normal vector $\nu(x)$.
2.) The differential operator Δ is called Laplace operator, after Pierre-Simon de Laplace (1749 – 1827).

Theorem 7.21 *For $f, g \in C^2(\Omega, \mathbb{R})$ we have* Green's first identity

$$\int_{\partial A} f(x) \frac{\partial g}{\partial \nu}(x) dS(x) = \int_A \big(\nabla f(x) \cdot \nabla g(x) + f(x) \Delta g(x)\big) dx,$$

and Green's second identity

$$\int_{\partial A} \big(f(x) \frac{\partial g}{\partial \nu}(x) - g(x) \frac{\partial f}{\partial \nu}(x)\big) dS(x) = \int_A \big(f(x) \Delta g(x) - g(x) \Delta f(x)\big) dx.$$

(George Green, 1793 – 1841.)

Proof: To prove Green's first identity apply the Gauß Theorem to the continuously differentiable function

$$f \,\nabla g : \Omega \to \mathbb{R}^n.$$

Because of $\operatorname{div}(f \,\nabla g) = \nabla f \cdot \nabla g + f \Delta g$ we obtain

$$\begin{aligned} \int_{\partial A} f(x) \frac{\partial g}{\partial \nu}(x) dS(x) &= \int_{\partial A} \nu(x) \cdot (f \,\nabla g)(x) dS(x) \\ &= \int_A \operatorname{div}(f \,\nabla g)(x) dx = \int_A \big(\nabla f(x) \cdot \nabla g(x) + f(x) \Delta g(x)\big) dx . \end{aligned}$$

To prove Green's second identity use Green's first identity. We obtain

$$\begin{aligned}
&\int_{\partial A}\Big(f(x)\frac{\partial g}{\partial \nu}(x) - g(x)\frac{\partial f}{\partial \nu}(x)\Big)dS(x)\\
&= \int_A (\nabla f(x)\cdot\nabla g(x) + f(x)\Delta g(x))dx - \int_A (\nabla f(x)\cdot\nabla g(x) + g(x)\Delta f(x))dx\\
&= \int_A (f(x)\Delta g(x) - g(x)\Delta f(x))dx\,.
\end{aligned}$$

■

Example: Application in potential theory. A twice differentiable function $u : A \to \mathbb{R}$, which satisfies the *partial differential equation* $\Delta u(x) = 0$ for all $x \in A$, is called a harmonic function or a potential function in A, since for $n = 3$ the gravitational potential and the electric potential satisfy this equation. An important problem is to determine a continuous function $u : \overline{A} \to \mathbb{R}$, which is a potential function in A and which is equal to a given function f at the boundary ∂A. We must therefore find a solution u of the *boundary value problem*

$$\begin{aligned}
\Delta u(x) &= 0, &&\text{for all } x \in A,\\
u(x) &= f(x), &&\text{for all } x \in \partial A.
\end{aligned}$$

Theorem 7.22 *If the set A is pathwise connected, then there exists at most one solution u of this boundary value problem, which is two times continuously differentiable in an open set Ω with $\overline{A} \subseteq \Omega$.*

Proof: If u and v are solutions then the difference $w = u - v$ solves the new boundary value problem

$$\Delta w(x) = 0,\ \text{for } x \in A, \qquad w(x) = u(x) - v(x) = 0,\ \text{for } x \in \partial A. \tag{7.7}$$

The first Green identity thus yields

$$\int_A |\nabla w(x)|^2 dx = \int_A \nabla w(x)\cdot\nabla w(x)dx = \int_{\partial A} w(x)\frac{\partial w}{\partial \nu}(x)dS(x) - \int_A w(x)\Delta w(x)dx = 0. \tag{7.8}$$

This implies $\nabla w(x) = 0$ for all $x \in A$. For, otherwise there would exist $x_0 \in A$ with $|\nabla w(x_0)| > 0$. Since ∇w is continuous, $|\nabla w(x)| > 0$ would hold for all x from a neighborhood of x_0. From the definition of the integral one sees immediately that this would imply $\int_A |\nabla w(x)|^2 dx > 0$, contradicting (7.8). By Theorem 4.17, $w'(x) = \big(\nabla w(x)\big)^T = 0$ for all $x \in A$ implies that w is constant in the pathwise connected set A. Since by (7.7) the function w is equal to zero at the boundary ∂A, we conclude that $u(x) - v(x) = w(x) = 0$ for all $x \in \overline{A}$, hence $u = v$. ■

7.7 The integral theorem of Stokes

In a general version, which is formulated best using differential forms, the Stokes integral theorem holds in every space dimension. We avoid to introduce differential forms and discuss only the versions for $n = 2$ and $n = 3$, which are the relevant versions for most applications.

We start from the integral theorem of Gauß in $\mathbb{R}^2$. Let $A \subseteq \mathbb{R}^2$ be a bounded open set with smooth boundary ∂A. The boundary ∂A is a curve. If $g = (g_1, g_2) : \Omega \to \mathbb{R}^2$ is a continuously differentiable function defined on an open set $\Omega \subseteq \mathbb{R}^2$ with $\overline{A} \subseteq \Omega$, then the Gauß theorem states that

$$\int_A \Big(\frac{\partial g_1}{\partial x_1}(x) + \frac{\partial g_2}{\partial x_2}(x)\Big) dx = \int_{\partial A} (\nu_1(x) g_1(x) + \nu_2(x) g_2(x)) ds(x) \tag{7.9}$$

holds, where $\nu(x) = (\nu_1(x), \nu_2(x))$ is the exterior unit normal vector at $x \in \partial A$. If $f = (f_1, f_2) : \Omega \to \mathbb{R}^2$ is another continuously differentiable function and if we choose for g in (7.9) the function

$$g(x) = \begin{pmatrix} f_2(x) \\ -f_1(x) \end{pmatrix},$$

then we obtain

$$\int_A \Big(\frac{\partial f_2}{\partial x_1}(x) - \frac{\partial f_1}{\partial x_2}(x)\Big) dx = \int_{\partial A} \big(\nu_1(x) f_2(x) - \nu_2(x) f_1(x)\big) ds(x) = \int_{\partial A} \tau(x) \cdot f(x) ds(x)\,, \tag{7.10}$$

where

$$\tau(x) = \begin{pmatrix} -\nu_2(x) \\ \nu_1(x) \end{pmatrix}.$$

$\tau(x)$ is a unit vector perpendicular to the normal vector $\nu(x)$ and is obtained by rotating $\nu(x)$ by 90^o in the mathematically positive sense (counterclockwise). Therefore $\tau(x)$ is a unit tangent vector to ∂A in $x \in \partial A$. If we define for differentiable $f : \Omega \to \mathbb{R}^2$ the *rotation* of f by

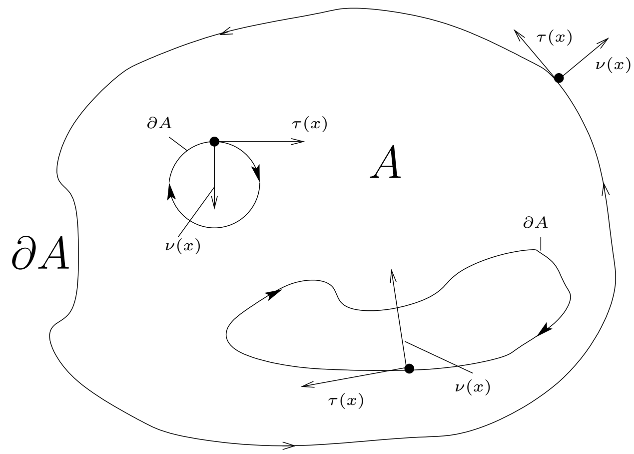
$$\operatorname{rot} f(x) = \frac{\partial f_2}{\partial x_1}(x) - \frac{\partial f_1}{\partial x_2}(x)\,,$$

then (7.10) can be written in the form

$$\int_A \operatorname{rot} f(x) dx = \int_{\partial A} \tau(x) \cdot f(x) ds(x)\,.$$

This formula is the *Stokes integral theorem* in the plane. In the Anglo-Saxon literature this formula, usually written in the form of (7.10), is called Green's theorem.

Note that A is not assumed to be *simply connected*. This means that A can have *holes*, as in the following picture, which displays a domain A in $\mathbb{R}^2$ with two holes. The boundary ∂A is not connected, but consists of three components. For every component the direction of the tangential vector τ is indicated.

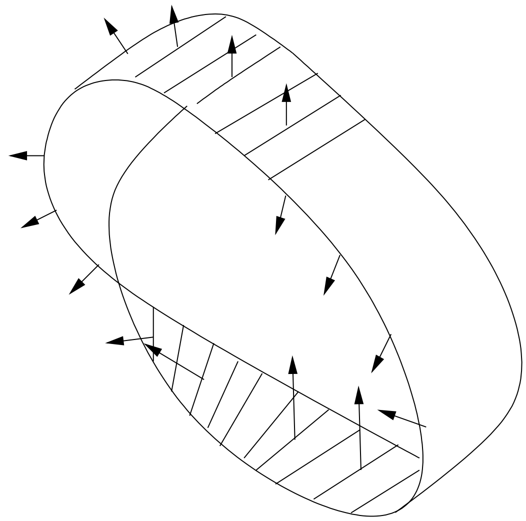


We can identify the subset $A \subseteq \mathbb{R}^2$ with the planar submanifold $A \times \{0\}$ of $\mathbb{R}^3$ and the integral over A in the Stokes formula with the surface integral over this submanifold. This interpretation suggests the idea that this formula can be generalized and that the Stokes formula is not only valid for planar submanifolds but also for more general two dimensional submanifolds of $\mathbb{R}^3$. This idea leads to the general Stokes integral theorem in $\mathbb{R}^3$. To formulate the theorem some preparations are necessary.

Orientable submanifolds. By Theorem 7.16 locally on every $(n-1)$–dimensional submanifold of $\mathbb{R}^n$ a continuous normal vector field exists. The Stokes theorem holds on submanifolds, on which a continuous normal vector field exists globally.

Definition 7.23 *An $(n-1)$–dimensional submanifold M in $\mathbb{R}^n$ is called orientable, if there exists a continuous unit normal vector field on M.*

Example: 1.) Let $R > 0$ and $n \geq 2$. The sphere $M = \{x \in \mathbb{R}^n \mid |x| = R\}$ is an orientable $(n-1)$–dimensional submanifold of $\mathbb{R}^n$. A continuous unit normal vector field is $\nu(x) = \frac{x}{|x|}$.
2.) The Möbius strip is a two dimensional submanifold of $\mathbb{R}^3$, which is not orientable:



Möbius strip

A unit normal vector field on the Möbius strip has necessarily a jump. (August Ferdinand Möbius, 1790 – 1868.)

Submanifolds with boundary. So far we only considered parametrized surfaces and submanifolds without boundary, that is, surfaces and submanifolds, which do not include boundary points. To introduce parametrized surfaces and submanifolds with boundary, set

$$H = \mathbb{R}^{n-2} \times [0, \infty), \qquad \partial H = \mathbb{R}^{n-2} \times \{0\}.$$

H ist the upper closed half space of $\mathbb{R}^{n-1}$.

Definition 7.24 *Let $U \subseteq \mathbb{R}^{n-1}$ be an open set such that $U \cap \partial H \neq \emptyset$. A mapping $\gamma : U \cap H \to \mathbb{R}^n$ is called simple $(n-1)$–dimensional parametrized surface with boundary, if γ can be extended to a simple parametrized surface $\tilde{\gamma} : U \to \mathbb{R}^n$. For the trace $F = \gamma(U \cap H)$ we set $\partial F = \gamma(U \cap \partial H)$.*

For the trace $F = \gamma(U)$ of a parametrized surface $\gamma : U \to \mathbb{R}^n$ without boundary we set $\partial F = \emptyset$.

To simplify the notation we denote in the following also the extended mapping by γ.

Theorem 7.25 *Assume that $\gamma : U \cap H \to \mathbb{R}^n$ is an $(n-1)$–dimensional parametrized surface with boundary. If we identify the set $U \cap \partial H$ with an open set in the space $\mathbb{R}^{n-2}$, then the restriction $\hat{\gamma} = \gamma|_{U \cap \partial H} : U \cap \partial H \to \mathbb{R}^n$ is a simple $(n-2)$–dimensional parametrized surface with trace $\partial F = \hat{\gamma}(U \cap \partial H)$.*

Proof: Let $u \in U \cap \partial H$. Since $\gamma : U \to \mathbb{R}^n$ is an $(n-1)$–dimensional parametrized surface, the Jacobi matrix $\gamma'(u)$ has rank $n-1$, and therefore the column vectors $\partial_{u_1}\gamma(u), \ldots, \partial_{u_{n-1}}\gamma(u)$, which compose $\gamma'(u)$, are linearly independent. This means that the first $n-2$ of these column vectors of which the matrix $\hat{\gamma}'(u)$ consists, has rank $n-2$. Moreover, the inverse $\hat{\kappa} = \hat{\gamma}^{-1} : \partial F \to U \cap \partial H$ is continuous as restriction of the continuous mapping γ^{-1} to the domain ∂F. ∎

Definition 7.26 *A set $M \subset \mathbb{R}^n$ is called an $(n-1)$–dimensional submanifold of $\mathbb{R}^n$ with boundary, if the following two conditions are satisfied:*

(i) *M is closed,*

(ii) *There exists for each $x \in M$ an open n–dimensional neighborhood $V(x)$ of x such that $V(x) \cap M$ is the trace of a simple $(n-1)$–dimensional parametrized surface with or without boundary.*

$V(x) \cap M$ is called a coordinate neighborhood of x in M. The boundary of M is defined by

$$\partial M = \bigcup_{x \in M} \partial\big(V(x) \cap M\big).$$

Theorem 7.27 *The set ∂M is an $(n-2)$–dimensional submanifold of $\mathbb{R}^n$.*

Proof: It must be shown that to every $x \in \partial M$ there exists a coordinate neighborhood in ∂M. To construct such a coordinate neighborhood of x, choose an open set V in $\mathbb{R}^n$ such that $V \cap M$ is a coordinate neighborhood of x in M. By Theorem 7.25, the set $V \cap \partial M = \partial(V \cap M)$ is the sought coordinate neighborhood of x in ∂M. ∎

Orientation of ∂M. Suppose that M is a two-dimensional orientable submanifold of $\mathbb{R}^3$ with boundary ∂M. Since every $x \in \partial M$ belongs to the two-dimensional submanifold M as well as to the one-dimensional submanifold ∂M, there exists in x the two-dimensional tangent space $T_x M$ to M, which is defined as for submanifolds without boundary, and the one-dimensional tangent space $T_x(\partial M)$ to ∂M. The space $T_x(\partial M)$ is a linear subspace of $T_x M$, since every tangent vector to ∂M at x is also a tangent vector to M. The orthogonal space of $T_x(\partial M)$ in $T_x M$ is one-dimensional, hence this orthogonal space contains two unit vectors. Intuitively, one of these vectors points into the submanifold M, the other one out of M. We denote the second one by $\mu(x)$.

We can define $\mu(x)$ rigorously as follows: To the boundary point $x \in \partial M$ choose an $(n-1)$–dimensional coordinate neighborhood W in M with parametrization $\gamma : U \cap H \to W$ and coordinate mapping $\kappa = \gamma^{-1} : W \to U \cap H$. For $u \in U \cap \partial H$ set

$$\hat{\mu}(\gamma(u)) = \partial_{u_1}\gamma(u) \times \big(\partial_{u_1}\gamma(u) \times \partial_{u_2}\gamma(u)\big), \qquad \mu(\gamma(u)) = \frac{\hat{\mu}(\gamma(u))}{|\hat{\mu}(\gamma(u))|}. \tag{7.11}$$

The properties of the vector product $\times$ imply that the vector $\hat{\mu}(\gamma(u))$ and therefore also the unit vector $\mu(\gamma(u))$ belong to the tangent space $T_{\gamma(u)}M$ and are orthogonal to the space $T_{\gamma(u)}(\partial M)$. Since for $\omega_i = \partial_{u_i}\gamma(u)$ we have

$$\partial_{u_2}\gamma(u) \cdot \hat{\mu}(\gamma(u)) = \omega_2 \cdot (\omega_1 \times (\omega_1 \times \omega_2)) = -(\omega_1 \times \omega_2) \cdot (\omega_1 \times \omega_2) < 0,$$

it follws that $\mu(\gamma(u))$ is that one of the two unit tangent vectors sharing these properties, which points outside of M. We leave it to the reader to prove that $\mu(\gamma(u))$ is independent of the choice of the parametrization γ. To this end it suffices to show that $\partial_{u_2}\gamma(u) \cdot \hat{\mu}^*(\gamma^*(u^*)) < 0$ holds for the vector $\hat{\mu}^*(\gamma^*(u^*))$, which is constructed using another parametrization γ^*.

Since $\partial_{u_1}\gamma$ and $\partial_{u_2}\gamma$ are continuous, it follows from the equations (7.11), that $\mu \circ \gamma$ is a continuous mapping on $U \cap \partial H$, whence also $\mu = \mu \circ \gamma \circ \kappa : W \cap \partial M \to \mathbb{R}^3$ is continuous. In particular, μ is continuous at the point $x \in W \cap \partial M$. Because $x \in \partial M$ was chosen arbitrarily, (7.11) defines a continuous vector field $\mu : \partial M \to \mathbb{R}^3$.

Let $\nu : M \to \mathbb{R}^3$ be a continuous unit normal vector field on M. With the continuous vector field μ we define a continuous unit tangent vector field $\tau : \partial M \to \mathbb{R}^3$ by

$$\tau(x) = \nu(x) \times \mu(x), \quad x \in \partial M,$$

and say that *the vector field τ orients ∂M positively with respect to ν.*

Definition 7.28 *Let $\Omega \subseteq \mathbb{R}^3$ be an open set and $f : \Omega \to \mathbb{R}^3$ be a differentiable function. The rotation* $\operatorname{rot} f : \Omega \to \mathbb{R}^3$ *of f is defined by*

$$\operatorname{rot} f(x) := \begin{pmatrix} \partial_{x_2} f_3(x) - \partial_{x_3} f_2(x) \\ \partial_{x_3} f_1(x) - \partial_{x_1} f_3(x) \\ \partial_{x_1} f_2(x) - \partial_{x_2} f_1(x) \end{pmatrix}.$$

In the Anglo-Saxon literature the notation $\operatorname{curl} f$ is common instead of $\operatorname{rot} f$.

Theorem 7.29 (Integral theorem of Stokes) *Let M be a two-dimensional orientable submanifold of $\mathbb{R}^3$ with boundary ∂M, let $\nu : M \to \mathbb{R}^3$ be a continuous unit normal* *vector field and let* $\tau : \partial M \to \mathbb{R}^3$ *be the continuous unit tangent vector field, which orients* ∂M *positively with respect to* ν*. Then for every continuously differentiable function* $f : \Omega \to \mathbb{R}^3$*, which is defined on an open set* $\Omega \subseteq \mathbb{R}^3$ *with* $M \subseteq \Omega$*, the equation*

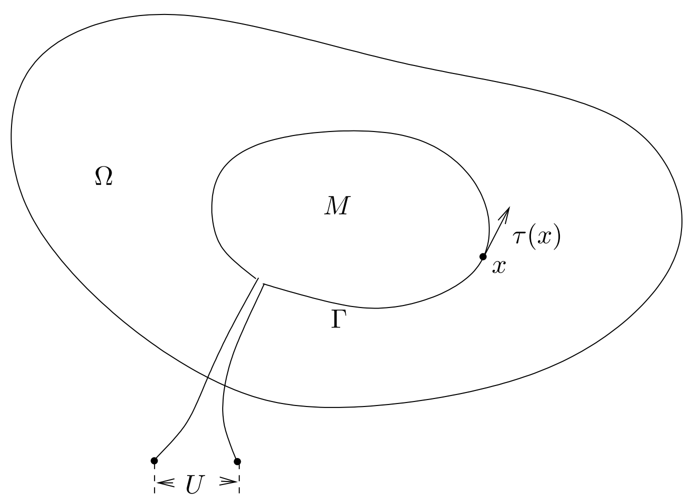
$$\int_M \nu(x) \cdot \mathrm{rot} f(x)\, dS(x) = \int_{\partial M} \tau(x) \cdot f(x)\, ds(x)$$

holds. (George Gabriel Stokes, 1819 – 1903.)

Example: Application in electro-magnetism. Let Ω be an open set in $\mathbb{R}^3$. Assume that in Ω there exist an electric field E and a field of magnetic induction B, which depend on the location $x \in \Omega$ and the time $t \in \mathbb{R}$. Thus, E and B are vector fields

$$E : \Omega \times \mathbb{R} \to \mathbb{R}^3 , \qquad B : \Omega \times \mathbb{R} \to \mathbb{R}^3 .$$

We place a metallic wire loop Γ into Ω.



If B varies in time, then after Faraday's law of induction an electric voltage $U(t)$ is induced in Γ. We can calculate this voltage as follows: The vector fields E and B satisfy the equation

$$\mathrm{rot}_x E(x,t) = -\frac{\partial}{\partial t} B(x,t)$$

for all $(x,t) \in \Omega \times \mathbb{R}$. This is one of Maxwell's equations. As a vector equation, it consists of three coupled partial differential equations. The operator rot_x on the left hand side of this equation acts with respect to the variable $x = (x_1, x_2, x_3)$. To determine $U(t)$ from this equation choose an orientable two-dimensional submanifold $M \subseteq \Omega$ with boundary $\partial M = \Gamma$, and choose a continuous unit normal vector field $\nu : M \to \mathbb{R}^3$. This normal vector field determines a unit tangent vector field $\tau : \Gamma \to \mathbb{R}^3$, which orients Γ positively with respect to ν. Then Stokes' theorem yields

$$U(t) = \int_\Gamma \tau(x) \cdot E(x,t) ds(x) = \int_M \nu(x) \cdot \mathrm{rot}_x E(x,t) dS(x)$$

$$= -\int_M \nu(x) \cdot \frac{\partial}{\partial t} B(x,t) dS(x) = -\frac{\partial}{\partial t} \int_M \nu(x) \cdot B(x,t) dS(x) \, .$$

The integral $\int_M \nu(x) \cdot B(x,t) \, dS(x)$ is called *flux of the magnetic induction through* M. Therefore $U(t)$ is equal to the negative time variation of the flux of B through M. This is Faraday's law of induction. (Michael Faraday, 1791 – 1876, James Clerk Maxwell, 1831 – 1879)

Appendix

German translation of Section 7

A Untermannigfaltigkeiten, Kurven- und Flächenintegrale, Integralsätze von Gauß und Stokes

Flächenintegrale spielen eine große Rolle in vielen Bereichen der Mathematik, unter anderem werden sie auch zur Formulierung der Integralsätze von Gauß und Stokes benötigt. In diesem Kapitel führen wir Flächen ein und studieren wir Flächenintegrale bevor wir die Integralsätze von Gauß und Stokes behandeln.

Ein typische Beispiel für eine Fläche ist eine Sphäre im $\mathbb{R}^3$. Dies ist eine zweidimensionale Punktmenge im dreidimensionalen Raum. Wie behandeln allgemeine p–dimensionale Teilmengen im $\mathbb{R}^n$ mit $p < n$. Derartige niederdimensionale Teilmengen m $\mathbb{R}^n$ heißen parametrisierte Flächen oder, allgemeiner, Untermannigfaltigkeiten im $\mathbb{R}^n$.

A.1 Parametrisierte Flächen, Koordinatenabbildungen

In Section 3.4 haben wir den normierten Raum $L(\mathbb{R}^n, \mathbb{R}^m)$ aller linearen Abbildungen von $\mathbb{R}^n$ nach $\mathbb{R}^m$ eingeführt. Für $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ ist die Bildmenge $A(\mathbb{R}^n)$ ein linearer Unterraum von $\mathbb{R}^m$.

Definition A.1 *Sei $A \in L(\mathbb{R}^n, \mathbb{R}^m)$. Als Rang von A bezeichnet man die Dimension des Unterraumes $A(\mathbb{R}^n)$.*

Aus der Theorie der linearen Abbildungen ist bekannt, dass der Rang der linearen Abbildung $A : \mathbb{R}^n \to \mathbb{R}^m$ gleich dem Rang der $m \times n$–Matrix ist, die diese Abbildung repräsentiert, und dass A injektiv ist wenn der Rang gleich n ist.

Definition A.2 *Sei $1 \leq p < n$ und sei $U \subseteq \mathbb{R}^p$ eine offene Menge. Die Abbildung $\gamma : U \to \mathbb{R}^n$ heißt parametrisierte p–dimensionale Fläche im $\mathbb{R}^n$, wenn γ stetig differenzierbar ist und die Ableitung $\gamma'(u) \in L(\mathbb{R}^p, \mathbb{R}^n)$ für alle $u \in U$ den Rang p hat. Der Wertebereich $\gamma(U) \subseteq \mathbb{R}^n$ heißt Spur von γ. Ist $p = 1$, dann heißt γ parametrisierte Kurve im $\mathbb{R}^n$.*

Man beachte, daß γ nicht injektiv zu sein braucht. Die Fläche kann Doppelpunkte haben.

Beispiel 1: Sei $U = \{(u,v) \in \mathbb{R}^2 \,|\, u^2 + v^2 < 1\}$ und sei $\gamma : U \to \mathbb{R}^3$ definiert durch

$$\gamma(u,v) = \begin{pmatrix} \gamma_1(u,v) \\ \gamma_2(u,v) \\ \gamma_3(u,v) \end{pmatrix} = \begin{pmatrix} u \\ v \\ \sqrt{1-(u^2+v^2)} \end{pmatrix}.$$

γ ist eine zweidimensionale parametrisierte Fläche im $\mathbb{R}^3$, da die beiden Spalten der Jacobi Matrix

$$\gamma'(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{u}{\sqrt{1-(u^2+v^2)}} & -\frac{v}{\sqrt{1-(u^2+v^2)}} \end{pmatrix}$$

für alle $(u,v) \in U$ linear unabhängig sind, also die lineare Abbildug $\gamma'(u,v)$ den Rang 2 hat. Die Spur von γ ist die obere Hälfte der Einheitssphäre im $\mathbb{R}^3$.

Beispiel 2: Im vorangehenden Beispiel ist die Spur der parametrisierten Fläche durch den Graphen der Funktion $(u,v) \mapsto \sqrt{1-(u^2+v^2)}$ gegeben. Allgemeiner sei $U \subseteq \mathbb{R}^p$ eine offene Menge, $f : U \to \mathbb{R}^{n-p}$ sei stetig differenzierbar und die Abbildung $\gamma : U \to \mathbb{R}^n$ sei definiert durch

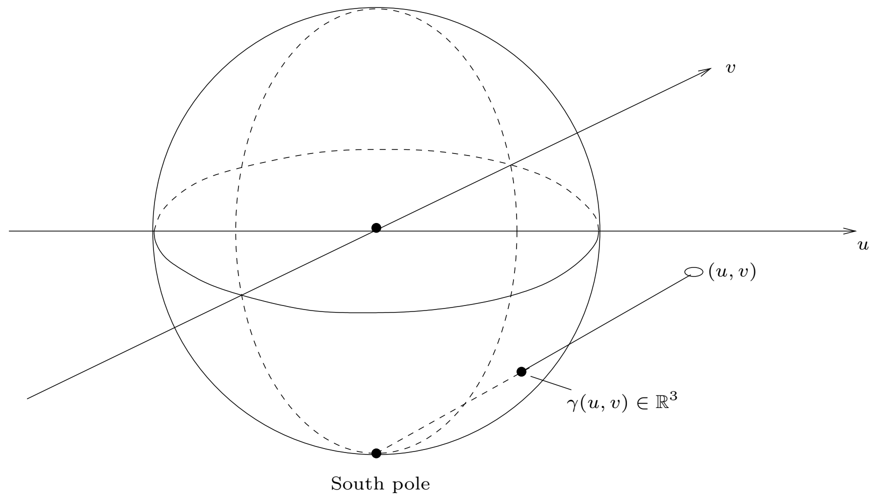
$$\begin{aligned} \gamma_1(u) &:= u_1 \\ \gamma_2(u) &:= u_2 \\ &\vdots \\ \gamma_p(u) &:= u_p \\ \gamma_{p+1}(u) &:= f_1(u_1 \ldots, u_p) \\ &\vdots \\ \gamma_n(u) &:= f_{n-p}(u_1 \ldots, u_p). \end{aligned}$$

Dann ist γ eine parametrisierte p–dimensionale Fläche im $\mathbb{R}^n$, weil die p Spaltenvektoren der Matrix

$$\gamma'(u) = \begin{pmatrix} 1 & \ldots & 0 \\ \vdots & \ddots & \\ 0 & \ldots & 1 \\ \partial_{x_1} f_1(u) & \ldots & \partial_{x_p} f_1(u) \\ \vdots & & \vdots \\ \partial_{x_1} f_{n-p}(u) & \ldots & \partial_{x_p} f_{n-p}(u) \end{pmatrix},$$

linear unabhängig sind, also der Rang p ist. Die Spur von γ ist der Graph von f.

Beispiel 3: Durch stereographische Projektion kann die am Südpol gelochte Sphäre mit Mittelpunkt im Ursprung eineindeutig auf die Ebene abgebildet werden. Die Inverse γ dieser Projektion bildet die Ebene auf die gelochte Sphäre ab:



Aus der Abbildung ergibt sich, dass für die Komponenten $\gamma_1, \ldots, \gamma_3$ der Abbildung $\gamma : \mathbb{R}^2 \to \mathbb{R}^3$ die Gleichungen

$$\frac{\gamma_1}{\gamma_2} = \frac{u}{v}, \quad \frac{\sqrt{u^2+v^2} - \sqrt{\gamma_1^2+\gamma_2^2}}{\gamma_3} = \frac{\sqrt{u^2+v^2}}{-1}, \quad \gamma_1^2+\gamma_2^2+\gamma_3^2 = 1.$$

gelten. Auflösen dieser Gleichungen nach $\gamma_1, \ldots, \gamma_3$ ergibt

$$\begin{aligned} \gamma_1(u,v) &= \frac{2u}{1+u^2+v^2} \\ \gamma_2(u,v) &= \frac{2v}{1+u^2+v^2} \\ \gamma_3(u,v) &= \frac{1-u^2-v^2}{1+u^2+v^2}\,. \end{aligned}$$

γ ist eine parametriserte Fläche im $\mathbb{R}^3$. Zum Beweis muss der Rang der Ableitung

$$\gamma'(u,v) = \frac{2}{(1+u^2+v^2)^2} \begin{pmatrix} 1-u^2+v^2 & -2uv \\ -2uv & 1+u^2-v^2 \\ -2u & -2v \end{pmatrix}$$

bestimmt werden. Für $u^2+v^2 \neq 1$ ist

$$\begin{aligned} \begin{vmatrix} \partial_u\gamma_1(u,v) & \partial_v\gamma_1(u,v) \\ \partial_u\gamma_2(u,v) & \partial_v\gamma_2(u,v) \end{vmatrix} &= (1+(v^2-u^2))(1-(v^2-u^2)) - 4u^2v^2 \\ &= 1-(v^2-u^2)^2 - 4u^2v^2 = 1-(v^2+u^2)^2 \neq 0\,. \end{aligned}$$

Für $u \neq 0$ gilt

$$\begin{vmatrix} \partial_u \gamma_2(u,v) & \partial_v \gamma_2(u,v) \\ \partial_u \gamma_3(u,v) & \partial_v \gamma_3(u,v) \end{vmatrix} = 4uv^2 + 2u(1+u^2-v^2) = 2u(1+u^2+v^2) \neq 0\,,$$

und für $v \neq 0$ entsprechend

$$\begin{vmatrix} \partial_u \gamma_1(u,v) & \partial_v \gamma_1(u,v) \\ \partial_u \gamma_3(u,v) & \partial_v \gamma_3(u,v) \end{vmatrix} = -2v(1+u^2+v^2) \neq 0\,,$$

also hat $\gamma'(u,v)$ den Rang 2 für alle $(u,v) \in \mathbb{R}^2$, folglich ist γ eine zweidimensionale parametrisierte Fläche. Die Spur von γ ist die Einheitssphäre bei herausgenommenem Südpol.

Beispiel 4: Es sei $\tilde{\gamma} : \tilde{U} \to \mathbb{R}^3$ die Einschränkung der parametrisierten Fläche γ aus Beispiel 3 auf die Einheitskreisscheibe $\tilde{U} = \{(u,v) \in \mathbb{R}^2 \mid u^2+v^2 < 1\}$. Die Spur von $\tilde{\gamma}$ ist die obere Hälfte der Einheitsphäre. $\tilde{\gamma}$ unterscheidet sich von der parametrisierten Fläche aus Beispiel 1, aber beide haben dieselbe Spur.

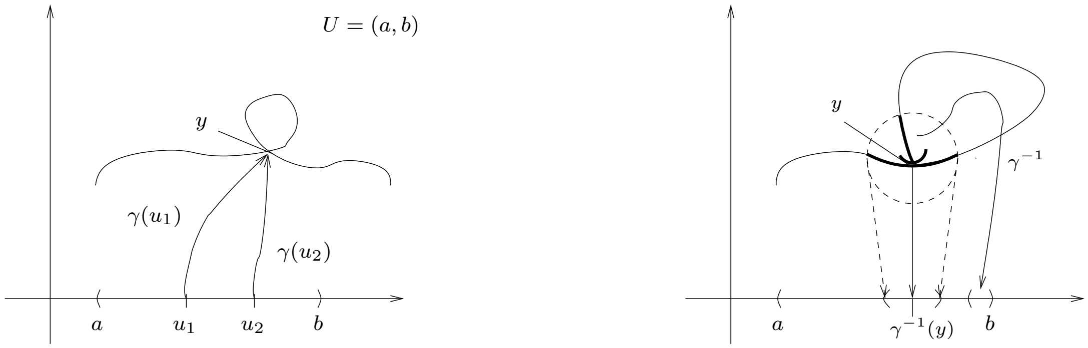
Beispiel 5: Seien $R > 0$ und $a = \frac{\pi}{180}$. Für $U = \{(\varphi, \vartheta) \in \mathbb{R}^2 \mid -180 < \varphi < 180,\ -90 < \vartheta < 90\}$ sei $\gamma : U \to \mathbb{R}^3$ gegeben durch

$$\gamma(\varphi, \vartheta) := R \begin{pmatrix} \cos(a\varphi)\cos(a\vartheta) \\ \sin(a\varphi)\cos(a\vartheta) \\ \sin(a\vartheta) \end{pmatrix}.$$

γ ist eine parametrisierte zweidimensionale Fläche in $\mathbb{R}^3$, weil die beiden Spalten der Jacobi Matrix

$$\gamma'(\varphi, \vartheta) = aR \begin{pmatrix} -\sin(a\varphi)\cos(a\vartheta) & -\cos(a\varphi)\sin(a\vartheta) \\ \cos(a\varphi)\cos(a\vartheta) & -\sin(a\varphi)\sin(a\vartheta) \\ 0 & \cos(a\vartheta) \end{pmatrix}$$

linear unabhängig sind für alle $(\varphi, \vartheta) \in U$, also gilt $\text{Rang}(\gamma'(\varphi, \vartheta)) = 2$ für alle $(\varphi, \vartheta) \in U$. Die Spur von γ ist die Sphäre mit Radius R, wenn man den Nord- und Südpol und einen die beiden Pole verbindenden Halbkreis entfernt. Wenn für R der Erdradius gewählt wird, dann sind (φ, ϑ) die üblichen geographischen Koordinaten des Globuses. φ ist die geographische Länge, ϑ ist die geographische Breite.



γ ist nicht injektiv: Die beiden verschiedenen Parameterwerte u_1 und u_2 werden auf denselben Doppelpunkt y der Kurve abgebildet.

γ^{-1} ist nicht stetig: Das Bild jeder Kugel um y enthält Punkte, deren Abstand von $\gamma^{-1}(y)$ größer als $\varepsilon = \frac{1}{2}(b - \gamma^{-1}(y))$ ist.

Figure 1: Beispiele für nicht einfache Parametrisierungen $\gamma : (a, b) \to \mathbb{R}^2$.

Wie dieses Beispiel zeigt, können für eine parametrisierte Fläche $u \mapsto \gamma(u)$ die Parameterwerte $u = (u_1, \ldots, u_p)$ als Koordinaten auf der Fläche verwendet werden. Weil jedem Punkt der Fläche eindeutige Koordinaten zugeordnet werden müssen, ist dies allerdings nur dann möglich, wenn die Abbildung γ injektiv ist. Dies motiviert die folgende Definition.

Definition A.3 *Sei $U \subseteq \mathbb{R}^p$ eine offene Menge. Eine parametrisierte Fläche $\gamma : U \to \mathbb{R}^n$ heißt einfach, wenn γ injektiv ist mit stetiger Inverser $\kappa = \gamma^{-1} : \gamma(U) \to U$. Die Abbildung κ heißt Koordinatenabbildung auf der Menge $\gamma(U)$.*

In Figure 1 wird diese Definition am Beispiel von parametrisierten Kurven im $\mathbb{R}^2$ erläutert, die nicht einfach sind.

Tangentialraum. Die Ableitung $\gamma'(u)$ einer p–dimensionalen parametrisierten Fläche $\gamma : U \to \mathbb{R}^n$ kann geometrisch interpretiert werden. Zur Erklärung dieser Interpretation erinnern wir daran, dass zu gegebenem $u \in U$ die Spaltenvektoren der Jacobi Matrix $\gamma'(u)$ durch die partiellen Ableitungen $\frac{\partial \gamma}{\partial u_1}(u), \ldots, \frac{\partial \gamma}{\partial u_p}(u)$ gegeben sind. Diese Vektoren sind tangential an die Spur $\gamma(U)$ im Punkt $x = \gamma(u)$. Um dies einzusehen, beachte man, dass die Abbildung $\xi \mapsto \gamma(u_1 \ldots, u_{i-1}, \xi, u_{i+1}, \ldots, u_p)$ eine parametrisierte Kurve ist, deren Spur in der Spur $\gamma(U)$ liegt. Der Definitionsbereich dieser Abbildung ist eine offene Menge in $\mathbb{R}$, die den Punkt $\xi = u_i$ enthält, und ihre Spur verläuft durch den Punkt x. Die Ableitung $\frac{\partial \gamma}{\partial u_i}(u)$ stellt einen Tangentenvektor an die Spur der Kurve dar und ist folglich auch ein Tangentenvektor an die Spur $\gamma(U)$.

Die Vektoren $\frac{\partial\gamma}{\partial u_1}(u), \ldots, \frac{\partial\gamma}{\partial u_p}(u)$ spannen den Wertebereich der linearen Abbildung $\gamma'(u) \in L(\mathbb{R}^p, \mathbb{R}^n)$ auf. Da nach Definition A.2 der Wertebereich ein linearer Raum mit Dimension p ist, muss die Menge der Tangentenvektoren $\frac{\partial\gamma}{\partial u_1}(u), \ldots, \frac{\partial\gamma}{\partial u_p}(u)$ linear unabhängig sein. Weil jede Linearkombination von Tangentenvektoren wieder ein Tangentenvektor ist, folgt, dass jeder Vektor aus dem Wertebereich der linearen Abbildung $\gamma'(u)$ ein Tangentenvektor an die Spur $\gamma(U)$ im Punkt $x = \gamma(u)$ ist. Dies führt auf die

Definition A.4 *Es sei $\gamma : U \to \mathbb{R}^n$ eine einfache p–dimensionale parametrisierte Fläche im $\mathbb{R}^n$ und $x = \gamma(u)$ sei ein Punkt der Spur $\gamma(U)$. Der p–dimensionale Wertebereich der linearen Abbildung $\gamma'(u) \in L(\mathbb{R}^p, \mathbb{R}^n)$ heißt Tangentialraum an die parametrisierte Fläche γ im Punkt x. Wir bezeichnen diesen Tangentialraum mit $T_x(\gamma)$.*

A.2 Integration auf parametrisierten Flächen

Nach Definition 6.4 ergibt sich das Maß einer beschränkten Teilmenge Ω des $\mathbb{R}^n$ durch Integration der konstanten Funktion 1 über die Menge Ω. Für $n = 2$ ist dieses Maß gleich der Fläche von Ω. Wenn wir entsprechend die Größe der Oberfläche einer Sphäre im $\mathbb{R}^3$ berechnen wollen, müssen wir die Funktion 1 über die Sphäre integrieren, und wenn wir das Gewicht einer dünnen metallischen Schale mit variabler Dicke bestimmen wollen, dann modellieren wir die Schale als Menge $K \subseteq \mathbb{R}^3$ mit verschwindender Dicke, also als Fläche, und integrierten die Funktion $f(x)$, deren Wert die Dicke der Schale an der Stelle $x \in K$ multipliziert mit dem spezifischen Gewicht ist, über die Fläche K. Dies sind zwei Beispiele, in denen Flächenintegrale berechnet werden müssen. In diesem Kapitel besprechen wir die genaue Definition solcher Integrale.

Die Idee hinter der Definition ist, eine parametrisierte Fläche γ zu wählen, deren Spur gleich der Fläche K ist, über die integriert werden soll, und diese Funktion γ zu benutzen, um das Integral über K auf ein Integral über den Definitionsbereich von γ, eine Teilmenge von $\mathbb{R}^p$, zurück zu führen. Die Beispiele 3 und 4 zeigen aber, dass verschiedene parametrisierte Flächen mit derselben Spur existieren können; keine dieser parametrisierten Flächen ist in natürlicher Weise ausgezeichnet, um damit das Integral zu definieren. Daher muss das Flächenintegral so definiert werden, dass es unabhängig ist von der parametrisierten Fläche, die zur Berechnung des Integrals gewählt wird. Um die Definition anzugeben, müssen wir die Gramsche Determinante einführen.

Definition A.5 *Es sei U eine offene Teilmenge von $\mathbb{R}^p$ mit $p < n$ und sei $\gamma : U \to \mathbb{R}^n$*

eine parametrisierte Fläche. Die Funktion $G : U \to \mathbb{R}^{p\times p}$ *sei definiert durch*

$$G(u) = \begin{pmatrix} g_{11}(u) & \dots & g_{1p}(u) \\ \vdots & & \\ g_{p1}(u) & \dots & g_{pp}(u) \end{pmatrix},$$

wobei für $i \leq i, j \leq p$ *die stetigen Funktionen* $g_{ij} : U \to \mathbb{R}$ *gegeben seien durch*

$$g_{ij}(u) = \frac{\partial\gamma}{\partial u_i}(u) \cdot \frac{\partial\gamma}{\partial u_j}(u) = \begin{pmatrix} \frac{\partial\gamma_1}{\partial u_i}(u) \\ \vdots \\ \frac{\partial\gamma_n}{\partial u_i}(u) \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial\gamma_1}{\partial u_j}(u) \\ \vdots \\ \frac{\partial\gamma_n}{\partial u_j}(u) \end{pmatrix} = \sum_{k=1}^{n} \frac{\partial\gamma_k}{\partial u_i}(u)\frac{\partial\gamma_k(u)}{\partial u_j} \quad .$$

Die durch $g(u) := \det(G(u))$ *definierte Funktion* $g : U \to \mathbb{R}$ *heißt Gramsche Determinante zu* γ*. (Jørgen Pedersen Gram, 1850 – 1916.)*

Bemerkung. Die Gramsche Determinante g ist stetig, weil $\det(G(u))$ sich als Summe von Produkten der stetigen Funktionen g_{ij} ergibt.

Definition A.6 *Sei* U *eine offene Teilmenge von* $\mathbb{R}^p$ *mit* $p < n$ *und sei* $\gamma : U \to \mathbb{R}^n$ *eine einfache* p*–dimensionale parametrisierte Fläche mit Spur* $\gamma(U)$ *und Koordinatenabbildung* $\kappa = \gamma^{-1} : \gamma(U) \to U$*.*

(i) Eine kompakte Teilmenge $K \subseteq \gamma(U)$ *heißt messbar, wenn das Bild* $\kappa(K) \subseteq U$ *Jordan messbar ist. Das Maß von* K *ist gegeben durch*

$$|K| = \int_{\kappa(K)} \sqrt{g(u)}du.$$

(ii) Für eine eine kompakte messbare Teilmenge K *von* $\gamma(U)$ *und für eine stetige Funktion* $f : K \to \mathbb{R}$ *ist das Flächenintegral definiert durch*

$$\int_K f(x)dS(x) = \int_{\kappa(K)} f(\gamma(u))\sqrt{g(u)}du.$$

Bemerkungen 1) In anschaulicher Weise spricht man von $dS(x)$ als dem p–dimensionalen Flächenelement von K und schreibt symbolisch

$$dS(x) = \sqrt{g(u)}du\,, \quad x = \gamma(u)\,.$$

2) Die Menge $\kappa(K)$ ist kompakt. Dies folgt aus Theorem 3.30, weil bei einer einfachen parametrisierten Fläche nach Definition A.3 die Inverse κ von γ stetig ist und weil K nach

Voraussetzung kompakt ist.
3) Nach Theorem 6.5 existieren die Integrale $\int_{\kappa(K)} \sqrt{g(u)}du$ und $\int_{\kappa(K)} f(\gamma(u))\sqrt{g(u)}du$ als Integrale der stetigen Funktionen $\sqrt{g(u)}$ und $f(\gamma(u))\sqrt{g(u)}$ über die messbare Teilmenge $\kappa(K)$ von $\mathbb{R}^p$.
4) Ist $p = 1$ und $U = (a,b)$ ein Intervall, dann ist $\gamma : (a,b) \to \mathbb{R}^n$ eine Kurve und für die Gramsche Determinante ergibt sich $g(u) = \left|\frac{d\gamma(u)}{du}\right|^2$. Ist $[c,d]$ ein in (a,b) enthaltenes kompaktes Intervall und gilt $K = \gamma([c,d])$, dann nennt man

$$\int_K f(x)ds(x) = \int_c^d f(\gamma(u))\sqrt{g(u)}du = \int_c^d f(\gamma(u))\left|\frac{d\gamma(u)}{du}\right|du$$

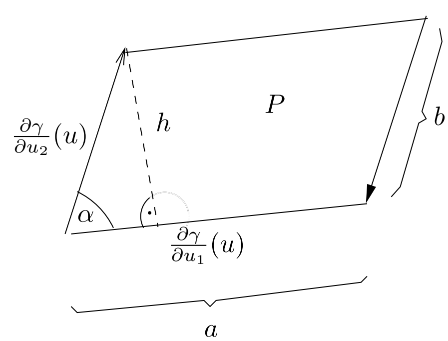
Kurvenintegral. Die Länge der Kurve zwischen den Punkten $\gamma(c)$ und $\gamma(d)$ ist $\ell(K) = \int_c^d \left|\frac{d\gamma(u)}{du}\right|du$, und man spricht vom Linienelement $ds(x) = \left|\frac{d\gamma(u)}{du}\right|du$.
Um Definition A.6 zu motivieren, muss die Bedeutung von $\sqrt{g(u)}$ bestimmt werden. Dazu betrachte man für fest gewähltes $u \in U$ die Menge

$$P = \Big\{\sum_{i=1}^{p} r_i \frac{\partial\gamma}{\partial u_i}(u) \;\Big|\; r_i \in \mathbb{R}\,,\ 0 \le r_i \le 1\Big\}.$$

P ist ein Parallelotop. Jedes Element von P ist eine Linearkombination der Tangentenvektoren $\frac{\partial\gamma}{\partial u_1}(u), \dots, \frac{\partial\gamma}{\partial u_p}(u)$ an $\gamma(U)$ im Punkt $x = \gamma(u)$. Nach Definition A.4 ist P also eine Teilmenge des Tangentialraumes $T_x(\gamma)$.

Satz A.7 *Es gilt $g(u) > 0$ und $\sqrt{g(u)}$ ist gleich dem p-dimensionalen Maß des Parallelotops P.*

Der Einfachheit halber beweisen wir diesen Satz nur für $n = 2$. Im diesem Fall ist P das im Bild dargestellte Parallelogramm.



Mit $a = \left|\frac{\partial\gamma}{\partial u_1}(u)\right|$ und $b = \left|\frac{\partial\gamma}{\partial u_2}(u)\right|$ gilt

$$\begin{aligned}
\sqrt{g(u)} &= \sqrt{\det(G(u))} \\
&= \sqrt{\begin{vmatrix} \frac{\partial\gamma}{\partial u_1}(u)\cdot\frac{\partial\gamma}{\partial u_1}(u) & \frac{\partial\gamma}{\partial u_1}(u)\cdot\frac{\partial\gamma}{\partial u_2}(u) \\ \frac{\partial\gamma}{\partial u_2}(u)\cdot\frac{\partial\gamma}{\partial u_1}(u) & \frac{\partial\gamma}{\partial u_2}(u)\cdot\frac{\partial\gamma}{\partial u_2}(u) \end{vmatrix}} = \sqrt{\begin{vmatrix} a^2 & ab\cos\alpha \\ ab\cos\alpha & b^2 \end{vmatrix}} \\
&= \sqrt{a^2b^2 - a^2b^2\cos^2\alpha} = ab\sqrt{1-\cos^2\alpha} = ab\sin\alpha = a\cdot h.
\end{aligned}$$

$a \cdot h$ ist der Flächeninhalt von P. ■

Aus diesem Satz ergibt sich folgende Interpretation der Formel $dS = \sqrt{g(u)}du$: Die lokale Wirkung der Abbildung γ ist, das kleine p–dimensionale Volumenelement du, ein p–dimensionaler Würfel mit dem Maß $|du|$, in ein kleines verzerrtes Flächenelement $dS(x)$, ein Parallelotop mit dem Maß $|dS(x)| = \sqrt{g(u)}|du|$, abzubilden. Der Faktor $\sqrt{g(u)}$ im Flächenintegral korrigiert die mit dieser Verzerrung verbundene Veränderung des Maßes.

Um zu zeigen, dass das Maß und das Integral auf parametrisierten Flächen wohldefiniert sind, benötigen wir eine weitere Definition:

Definition A.8 *Seien $U, V \subseteq \mathbb{R}^p$ offene Mengen. Zwei parametrisierte Flächen $\gamma : U \to \mathbb{R}^n$ und $\tilde{\gamma} : V \to \mathbb{R}^n$ heißen äquivalent, wenn ein Diffeomorphismus $\varphi : V \to U$ existiert mit*

$$\tilde{\gamma} = \gamma \circ \varphi\,.$$

Dies ist eine Äquivalenzrelation auf der Menge der parametrisierten Flächen. Aus dieser Definition folgt natürlich, dass zwei äquivalente parametrisierte Flächen γ und $\tilde{\gamma}$ dieselbe Spur haben. Der nächste Satz zeigt, daß für einfache parametrisierte Flächen auch die Umkehrung dieser Aussage gilt.

Satz A.9 *Seien $U, V \subseteq \mathbb{R}^p$ offene Mengen und seien $\gamma : U \to \mathbb{R}^n$ und $\tilde{\gamma} : V \to \mathbb{R}^n$ einfache parametriserte Flächen mit $\gamma(U) = \tilde{\gamma}(V)$. Dann ist*

$$\gamma^{-1} \circ \tilde{\gamma} : V \to U$$

ein Diffeomorphismus.

Einen **Beweis** dieses Satzes findet man in Barner-Flohr, Analysis II, 384 ff. Nach diesem Satz ist $\varphi = \gamma^{-1} \circ \tilde{\gamma}$ ein Diffeomorphismus, für den $\tilde{\gamma} = \gamma \circ \varphi$ gilt, also sind γ und $\tilde{\gamma}$ äquivalent.

Beispiel 6: Es sei U die offene Einheitskreisscheibe im $\mathbb{R}^2$ und es seien $\gamma : U \to \mathbb{R}^3$ und $\tilde{\gamma} : U \to \mathbb{R}^3$ die parametrisierten Flächen aus den Beispielen 1 und 4. Beide Parametrisierungen sind einfach, und die Spur ist die obere Hälfte der Einheitssphäre im $\mathbb{R}^3$. Daher sind diese Parametrisierungen äquivalent. Ein Diffeomorphismus $\varphi : U \to U$ ist gegeben durch

$$\varphi(u,v) = \begin{pmatrix} \frac{2u}{1+u^2+v^2} \\ \frac{2v}{1+u^2+v^2} \end{pmatrix}.$$

Es gilt

$$(\gamma \circ \varphi)(u,v) = \begin{pmatrix} \frac{2u}{1+u^2+v^2} \\ \frac{2v}{1+u^2+v^2} \\ \sqrt{1 - \frac{4u^2+4v^2}{(1+u^2+v^2)^2}} \end{pmatrix} = \frac{1}{1+u^2+v^2} \begin{pmatrix} 2u \\ 2v \\ 1-u^2-v^2 \end{pmatrix} = \tilde{\gamma}(u,v).$$

Nun können wir zeigen, dass Definition A.6 sinnvoll ist, das heißt, dass sich das Maß $|K|$ und das Flächenintegral $\int_K f(x)\, dS(x)$ nicht ändern, wenn wir die einfache parametriserte Fläche γ durch eine äquivalente parametriserte Fläche $\tilde{\gamma}$ ersetzen.

Satz A.10 *Seien $U, V \subseteq \mathbb{R}^p$ offene Mengen, seien $\gamma : U \to \mathbb{R}^n$ und $\tilde{\gamma} : V \to \mathbb{R}^n$ einfache parametrisierte Flächen mit $\gamma(U) = \tilde{\gamma}(V) = F$ und mit Koordinatenabbildungen $\kappa = \gamma^{-1}$ und $\tilde{\kappa} = \tilde{\gamma}^{-1}$. Wir setzen $\varphi = \kappa \circ \tilde{\gamma} : V \to U$. Die Gramschen Determinanten zu γ und $\tilde{\gamma}$ seien mit $g : U \to \mathbb{R}$ und $\tilde{g} : V \to \mathbb{R}$ bezeichnet. Dann gilt:*

(i) Für alle $v \in V$ ist

$$\tilde{g}(v) = g(\varphi(v))\, |\det \varphi'(v)|^2.$$

(ii) Wenn $K \subseteq F$ eine kompakte Menge ist mit messbarer Bildmenge $\kappa(K) \subseteq U$ und wenn $f : K \to \mathbb{R}$ stetig ist, dann ist die Menge $\tilde{\kappa}(K) \subseteq V$ auch messbar und es gilt Gleichheit der Integrale:

$$\int_{\kappa(K)} f(\gamma(u)) \sqrt{g(u)} du = \int_{\tilde{\kappa}(K)} f(\tilde{\gamma}(v)) \sqrt{\tilde{g}(v)} dv\,.$$

Bemerkung. Es gilt $\tilde{\gamma} = \gamma \circ \varphi$. Nach Satz A.9 ist φ ein Diffeomorphismus.

Beweis: (i) Nach Definition ist

$$g_{ij}(u) = \sum_{k=1}^{n} \frac{\partial \gamma_k(u)}{\partial u_i} \frac{\partial \gamma_k(u)}{\partial u_j}\,,$$

woraus

$$G(u) = \gamma'(u)^T \gamma'(u)$$

folgt, wobei $\gamma'(u)^T$ die transponierte Matrix zur Matrix $\gamma'(u)$ bedeutet. Nach der Kettenregel und dem Determinantenmultiplikationssatz gilt also

$$\begin{aligned}
\tilde{g} &= \det \tilde{G} = \det((\tilde{\gamma}')^T \tilde{\gamma}') \\
&= \det([(\gamma' \circ \varphi)\varphi']^T (\gamma' \circ \varphi)\varphi') = \det((\varphi')^T [\gamma' \circ \varphi]^T [\gamma' \circ \varphi]\varphi') \\
&= (\det \varphi') \det([\gamma' \circ \varphi]^T [\gamma' \circ \varphi])(\det \varphi') = (\det \varphi')^2 \det(G \circ \varphi) = (\det \varphi')^2 (g \circ \varphi) \,.
\end{aligned}$$

(ii) Weil K kompakt ist und weil κ stetig ist, folgt dass $\kappa(K)$ eine kompakte, messbare Teilmenge der offenen Menge $U \subseteq \mathbb{R}^p$ ist. Da φ ein Diffeormorphismus ist, ist auch die Umkehrabbildung $\psi = \varphi^{-1} : U \to V$ diffeomorph mit $\psi(\kappa(K)) = \tilde{\kappa}(K) \subseteq V$. Der Transformationssatz Theorem 6.12 impliziert daher, dass auch die Menge $\tilde{\kappa}(K) = \psi(\kappa(K)) \subseteq V$ messbar ist. Weil die Funktion $\tilde{f} = (f \circ \tilde{\gamma})\sqrt{\tilde{g}} : \tilde{\kappa}(K) \to \mathbb{R}$ stetig ist, liefert eine erneute Anwendung des Transformationssatzes zusammen mit der Behauptung (i), dass

$$\begin{aligned}
&\int_{\kappa(K)} f(\gamma(u))\sqrt{g(u)}du \\
&\qquad = \int_{\tilde{\kappa}(K)} f((\gamma \circ \varphi)(v))\sqrt{g(\varphi(v))}\,|\det \varphi'(v)|dv = \int_{\tilde{\kappa}(K)} f(\tilde{\gamma}(v))\sqrt{\tilde{g}(v)}dv \,.
\end{aligned}$$

■

A.3 Untermannigfaltigkeiten, Tangential- und Normalraum

Die Spur der parametrisierten Fläche aus Beispiel 3 ist die gelochte Sphäre, die Spur der parametrisierten Fläche aus Beispiel 5 ist eine Sphäre, aus der ein Halbkreis entfernt ist, der die beiden Pole miteinander verbindet. Aus topologischen Gründen kann es keine parametrisierte Fläche geben, deren Spur die gesamte Sphäre ist. Um die gesamte Sphäre zu parametrisieren, muss sie daher in mehrere Teile aufgeteilt werden, die jeweils als Spur einer parametrisierten Fläche dargestellt werden können. Daher definiert man:

Definition A.11 *Eine Teilmenge $M \subseteq \mathbb{R}^n$ heißt p–dimensionale Untermannigfaltigkeit des $\mathbb{R}^n$, wenn zu jedem $x \in M$ eine offene n–dimensionale Umgebung V von x existiert, so, dass $V \cap M$ die Spur einer einfachen p–dimensionalen parametrisierten Fläche $\gamma : U \to V \cap M$ ist.*

Die Menge $V \cap M$ heißt Koordinatenumgebung von x, die stetige Inverse $\kappa = \gamma^{-1} : V \cap M \to U \subseteq \mathbb{R}^p$ heißt lokale Koordinatenabbildung. Man bezeichnet sie auch als Karte. Ab jetzt verwenden wir auch die bequeme Sprechweise und sagen, die parametrisierte Fläche γ sei eine einfache Parametrisierung der Koordinatenumgebung $V \cap M$.

Wenn V und $\tilde{V}$ Koordinatenumgebungen in M sind, deren Durchschnitt nicht leer ist, wenn $\kappa : V \to U$, $\tilde{\kappa} : \tilde{V} \to \tilde{U}$ Koordinatenabbildungen sind und wenn $\gamma : U \to V$, $\tilde{\gamma} : \tilde{U} \to \tilde{V}$ Parametrisierungen sind, dann ist der Durchschnitt $W = V \cap \tilde{V}$ eine Koordinatenumgebung,

$$\kappa|_W : W \to \kappa(W) = U_1 \subseteq U \quad \text{and} \quad \tilde{\kappa}|_W : W \to \tilde{\kappa}(W) = U_2 \subseteq \tilde{U}$$

sind lokale Koordinatenabbildungen auf W, und

$$\gamma|_{U_1} : U_1 \to W, \qquad \tilde{\gamma}|_{U_2} : U_2 \to W$$

sind Parametrisierungen von W. Da die Parametrisierungen $\gamma|_{U_1}$ und $\tilde{\gamma}|_{U_2}$ einfach sind und die Spuren der beiden Parametrisierungen übereinstimmen, folgt aus Satz A.9, dass diese beiden Parametrisierungen äquivalent sind. Das bedeutet, dass

$$\varphi = \tilde{\kappa} \circ \gamma|_{U_1} : U_1 \to U_2 \quad \text{und} \quad \varphi^{-1} = \kappa \circ \tilde{\gamma}|_{U_2} : U_2 \to U_1$$

Diffeomorphismen sind.

In Definition A.4 haben wir den Tangentialraum $T_x(\gamma)$ zu einer parametrisierten Fläche γ eingeführt. Wir können diese Definition nun auf Untermannigfaltigkeiten verallgemeinern.

Definition A.12 *Sei x ein Punkt der Untermannigfaltigkeit M mit Koordinatenumgebung V und einfacher Parametrisierung $\gamma : U \to V$. Wir definieren den p–dimensionalen Tangentialraum T_xM an M im Punkt x durch $T_xM = T_x(\gamma)$. Der $(n{-}p)$–dimensionale Orthogonalraum*

$$T_x^{\perp}M = \{\nu \in \mathbb{R}^n \mid \nu \cdot \tau = 0 \textit{ für alle } \tau \in T_xM\}$$

heißt Normalraum zu M im Punkt x.

Diese Definition ist unabhängig von der gewählten Koordinatenumgebung V und der gewählten Parametrisierung $\gamma : U \to V$. Denn wenn $\tilde{V}$ eine zweite Koordinatenumgebung von x ist mit einfacher Parametrisierung $\tilde{\gamma} : \tilde{U} \to \tilde{V}$ und wenn $u \in U$, $\tilde{u} \in \tilde{U}$ Punkte sind mit $x = \gamma(u) = \tilde{\gamma}(\tilde{u})$, dann gibt es nach obenstehendem Ergebnis offene Mengen $U_1 \subseteq U$, $U_2 \subseteq \tilde{U}$ und einen Diffeomorphismus $\varphi : U_1 \to U_2$, so dass $\gamma|_{U_1} = \tilde{\gamma}|_{U_2} \circ \varphi$. Nach der Kettenregel folgt aus dieser Gleichung

$$\gamma'(u) = \tilde{\gamma}'(\tilde{u})\varphi'(u).$$

Da φ ein Diffeomorphismus ist, ist $\varphi'(u) : \mathbb{R}^p \to \mathbb{R}^p$ eine invertierbare lineare Abbildung, also impliziert die Gleichung, dass die Wertebereiche $T_x(\gamma)$ und $T_x(\tilde{\gamma})$ der linearen Abbildungen $\gamma'(u)$ und $\tilde{\gamma}'(\tilde{u})$ übereinstimmen. Dies bedeutet, dass $T_x M$ und $T_x^{\perp} M$ unabhängig sind von der Wahl der Parametrisierung.

Beispiel 7: Es sei $S = \{x \in \mathbb{R}^3 \mid |x| = 1\}$ die Einheitssphäre im $\mathbb{R}^3$. Die einfache parametrisierte Fläche $\gamma : \mathbb{R}^2 \to \mathbb{R}^3$ aus Beispiel 3 hat die Spur $\gamma(\mathbb{R}^2) = S \setminus \{(0, 0, -1)\}$. Diese Spur ist eine Koordinatenumgebung für jeden ihrer Punkte, die lokale Koordinatenabbildung $\kappa = \gamma^{-1} : S \setminus \{(0, 0, -1)\} \to \mathbb{R}^2$ ist die stereographische Projektion vom Südpol auf die Ebene $\mathbb{R}^2$. Ebenso ist die Menge $S \setminus \{(0, 0, 1)\}$ eine Koordinatenumgebung für jeden ihrer Punkte und insbesondere für den Punkt $(0, 0, -1)$, der nicht zur ersten Koordinatenumgebung gehört. Eine lokale Koordinatenabbildung auf $S \setminus \{(0, 0, 1)\}$ erhält man durch stereographische Projektion vom Nordpol auf $\mathbb{R}^2$. Daher ist S eine zweidimensionale Untermannigfaltigkeit von $\mathbb{R}^3$. Zwei Karten genügen, um diese Untermannigfaltigkeit zu überdecken.

A.4 Integration auf Untermannigfaltigkeiten

Zur Definition des Integrals auf Untermannigfaltigkeiten benötigen wir Vorbereitungen. Sei M eine p-dimensionale Untermannigfaltigkeit von $\mathbb{R}^n$ und sei $V \subseteq M$ eine Koordinatenumgebung von M. Wir definieren den Träger $\mathrm{supp} f$ einer stetigen Funktion $f : M \to \mathbb{R}$ durch

$$\mathrm{supp} f = \overline{\{x \in M \mid f(x) \neq 0\}}$$

und sagen, dass f kompakten Träger in V habe, wenn $\mathrm{supp} f$ eine kompakte Teilmenge von V ist. Für stetige Funktionen mit kompaktem Träger in V definieren wir das Flächenintegral $\int_V f(x) dS(x)$ folgendermaßen:

Es sei $\gamma : U \to V$ eine Parametrisierung, $\kappa = \gamma^{-1} : V \to U$ die Koordinatenabbildung und $g : U \to \mathbb{R}$ die Gramsche Determinante zu γ. Hierbei sei U eine offene Teilmenge von $\mathbb{R}^p$. Nur um technische Einzelheiten zu vermeiden, nehmen wir an, dass U beschränkt ist. In diesem Fall können wir U in einen kompakten Quader Q einschliessen. Wenn f kompakten Träger in V hat, ist $\kappa(\mathrm{supp} f)$ eine kompakte Teilmenge von U, weil κ stetig ist. Weil U offen ist, folgt hieraus, dass der Abstand von $\kappa(\mathrm{supp} f)$ zum Rand ∂U von U positiv ist: $\mathrm{dist}(\kappa(\mathrm{supp} f), \partial U) > 0$. Daher verschwindet die stetige Funktion $f \circ \gamma : U \to \mathbb{R}$ und also auch die Funktion $(f \circ \gamma)\sqrt{g} : U \to \mathbb{R}$ in einer Umgebung von ∂U. Setzen wir die zweite dieser Funktionen durch Null von U auf ganz Q fort, erhalten wir also eine stetige Funktion auf Q. Nach Satz 6.5 ist diese Funktion über der messbaren

Menge Q integrierbar, weil sie stetig ist, folglich ist sie nach Definition 6.3 integrierbar über U. Für Funktionen $f : M \to \mathbb{R}$ mit kompaktem Träger in der Koordinatenumgebung V definieren wir daher das Flächenintegral

$$\int_V f(x)dS(x) = \int_U (f \circ \gamma)(u)\sqrt{g(u)}\, du. \tag{A.1}$$

Um zu zeigen, dass diese Definition von der Wahl der Parametrisierung unabhängig ist, sei K eine kompakte, messbare Teilmenge von M mit

$$\text{supp} f \subseteq K \subseteq V. \tag{A.2}$$

Ein solches K existiert. Denn es ist nicht schwer, eine Vereinigungsmenge Ω von endlich vielen abgeschlossenen Quadern zu konstruieren, so dass $\kappa(\text{supp} f) \subseteq \Omega \subseteq U$ gilt. Jeder abgeschlossene Quader ist kompakt und Jordan messbar, als auch die endliche Vereinigung Ω. Folglich ist die Menge $K = \gamma(\Omega)$ kompakt, erfüllt (A.2) und ist nach Definition A.6 auch messbar in M.

Ist nun $\tilde{\gamma} : \tilde{U} \to V$ eine andere einfache Parametrisierung von V mit der Koordinatenabbildung $\tilde{\kappa} = \tilde{\gamma}^{-1} : V \to \tilde{U}$ und Gramscher Determinante $\tilde{g} : \tilde{U} \to \mathbb{R}$, dann folgt für jedes solche K, dass

$$\begin{aligned}\int_U (f \circ \gamma)(u)\sqrt{g(u)}\, du &= \int_{\kappa(K)} (f \circ \gamma)(u)\sqrt{g(u)}\, du \\ &= \int_{\tilde{\kappa}(K)} (f \circ \tilde{\gamma})(v)\sqrt{\tilde{g}(v)}\, dv = \int_{\tilde{U}} (f \circ \tilde{\gamma})(v)\sqrt{\tilde{g}(v)}\, dv \end{aligned} \tag{A.3}$$

gilt. Das erste und dritte Gleichheitszeichen gelten, weil aus (A.2) die Relationen $\kappa(\text{supp} f) \subseteq \kappa(K) \subseteq U$ und $\tilde{\kappa}(\text{supp} f) \subseteq \tilde{\kappa}(K) \subseteq \tilde{U}$ folgen, und das zweite Gleichheitszeichen gilt nach Aussage (ii) in Satz A.10, der auf die messbare und kompakte Menge K anwendbar ist. Gleichung (A.3) zeigt, dass die Definition des Flächenintegrals in (A.1) unabhängig von der Wahl der Parametrisierung ist.

Sind V und V' Koordinatenumgebungen in M mit $V' \subseteq V$ und ist $f : M \to \mathbb{R}$ eine stetige Funktion mit kompaktem Träger in V', dann folgt aus der Definition des Flächenintegrals, dass

$$\int_V f(x)dS(x) = \int_{V'} f(x)dS(x). \tag{A.4}$$

gilt. Wir überlassen den offensichtlichen Beweis dem Leser.

Definition A.13 *Sei M eine p–dimensionale Untermannigfaltigkeit von $\mathbb{R}^n$ und sei $\{V_i\}_{i=1}^{\infty}$ eine Familie von Koordinatenumgebungen in M, so dass $M = \bigcup_{i=1}^{\infty} V_i$ gilt. Für* *jedes $i \in \mathbb{N}$ sei $\alpha_i : M \to \mathbb{R}$ eine stetige Funktion mit kompaktem Träger in V_i. Wir sagen, dass $\{\alpha_i\}_{i=1}^{\infty}$ eine der Überdeckung $\{V_i\}_{i=1}^{\infty}$ von M untergeordnete Zerlegung der Eins sei, wenn*

$$\sum_{i=1}^{\infty} \alpha_i(x) = 1, \quad \textit{für alle } x \in M,$$

gilt.

Nun können wir das Flächenintegral auf Untermannigfaltigkeiten definieren. Der Einfachheit halber beschränken wir uns dabei auf Untermannigfaltigkeiten M, für die eine Zerlegung der Eins existiert, die einer endlichen Überdeckung von M untergeordnet ist.

Definition A.14 *Sei M eine p–dimensionale Untermannigfaltigkeit von $\mathbb{R}^n$ und sei $\{\alpha_i\}_{i=1}^{m}$ eine Zerlegung der Eins aus stetigen Funktionen, die einer Überdeckung $\{V_i\}_{i=1}^{m}$ von M durch Koordinatenumgebungen V_i untergeordnet ist. $f : M \to \mathbb{R}$ sei eine stetige Funktion. Wir definieren das Flächenintegral von f über M durch*

$$\int_M f(x)\, dS(x) = \sum_{i=1}^{m} \int_{V_i} \alpha_i(x) f(x)\, dS(x).$$

Bemerkung. Da $\alpha_i f$ eine stetige Funktion auf M mit kompaktem Träger in der Koordinatenumgebung V_i ist, ist das Integral $\int_{V_i} \alpha_i(x) f(x)\, dS(x)$ wie in Gleichung (A.1) definiert.

Die Zerlegung der Eins $\{\alpha_i\}_{i=1}^{m}$ ist nicht eindeutig bestimmt. Daher ist diese Definition nur sinnvoll, wenn das Flächenintegral unabhängig von der Wahl der Zerlegung ist. Im nächsten Satz wird dies gezeigt.

Satz A.15 *Sei M eine Untermannigfaltigkeit und sei $f : M \to \mathbb{R}$ eine stetige Funktion. $\{\alpha_i\}_{i=1}^{m}$ und $\{\beta_j\}_{j=1}^{\ell}$ seien Zerlegungen der Eins aus stetigen Funktionen, die der Überdeckung $\{V_i\}_{i=1}^{m}$ beziehungsweise der Überdeckung $\{\tilde{V}_j\}_{j=1}^{\ell}$ von M durch Koordinatenumgebungen untergeordnet sind. Dann gilt*

$$\sum_{i=1}^{m} \int_{V_i} \alpha_i(x) f(x) dS(x) = \sum_{j=1}^{\ell} \int_{\tilde{V}_j} \beta_j(x) f(x) dS(x)\,.$$

Beweis: Für alle $i = 1, \ldots, m$ und $j = 1, \ldots, \ell$ ist $V_i \cap V_j'$ eine Koordinatenumgebung auf M mit $V_i \cap V_j' \subseteq V_i$ und $V_i \cap V_j' \subseteq V_j'$. Der kompakte Träger der stetigen Funktion $\alpha_i \beta_j f$ ist in der Koordinatenumgebung $V_i \cap V_j'$ enthalten. Nach (A.4) gilt also

$$\int_{V_i} \alpha_i(x)\beta_j(x) f(x) dS(x) = \int_{V_i \cap V_j'} \alpha_i(x)\beta_j(x) f(x) dS(x) = \int_{V_j'} \alpha_i(x)\beta_j(x) f(x) dS(x).$$

Aus dieser Gleichung folgt

$$\sum_{i=1}^{m} \int_{V_i} \alpha_i(x) f(x) dS(x) = \sum_{i=1}^{m} \int_{V_i} \alpha_i(x) \sum_{j=1}^{\ell} \beta_j(x) f(x) dS(x)$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{\ell} \int_{V_i} \alpha_i(x)\beta_j(x) f(x) dS(x) = \sum_{i=1}^{m} \sum_{j=1}^{\ell} \int_{V_j'} \alpha_i(x)\beta_j(x) f(x) dS(x)$$
$$= \sum_{j=1}^{\ell} \int_{V_j'} \sum_{i=1}^{m} \alpha_i(x)\beta_j(x) f(x) dS(x) = \sum_{j=1}^{\ell} \int_{V_j'} \beta_j(x) f(x) dS(x).$$

■

A.5 Der Gaußsche Integralsatz

Im letzten Teil von Kapitel A formulieren wir die Integralsätze von Gauß und Stokes und die Greenschen Formeln ohne Beweis. Diese Sätze gehören zu den zentralen Ergebnissen der Theorie der mehrdimensionalen Integration und sind von fundamentaler Bedeutung für die Theorie partieller Differentialgleichungen und für alle Anwendungen in Wissenschaft und Technik. Um ihre Anwendung zu demonstrieren, behandeln wir einfache Beispiele aus der Potentialtheorie und der Physik.

Wir beginnen mit dem Gaußschen Satz. Zur Formulierung des Satzes benötigen wir einige Vorbereitungen.

Normalenfelder. Sei M eine $(n-1)$–dimensionale Untermanigfaltigkeit von $\mathbb{R}^n$. Der Normalraum $T_x^{\perp}M$ an der Stelle $x \in M$ ist eindimensional. Die Elemente $\nu \neq 0$ von $T_x^{\perp}M$ heißen Normalenvektoren an M im Punkt x. Gilt $|\nu| = 1$, dann heißt ν Einheitsnormalenvektor. Im Punkt x gibt es zwei Einheitsnormalenvektoren ν und $-\nu$. Ist W eine Teilmenge von M und ist $\nu(x)$ ein Einheitsnormalenvektor für jedes $x \in W$, dann nennt man die Abbildung $x \mapsto \nu(x) : W \to \mathbb{R}^n$ ein Einheitsnormalenfeld auf W.

Satz A.16 *Es sei W eine Koordinatenumgebung in der $(n-1)$–dimensionalen Untermannigfaltigkeit M von $\mathbb{R}^n$. Dann gibt es ein stetiges Einheitsnormalenfeld ν auf W.*

Proof: Sei $\gamma : U \to W$ eine einfache Parametrisierung von W und sei u ein Punkt in U. Mit der Jacobi Matrix $\gamma'(u) = (\partial_{u_k}\gamma_j(u))_{j=1,\dots,n;k=1,\dots,n-1}$ definieren wir das Vektorfeld $N(\gamma(u)) = (N_i(\gamma(u))_{i=1,\dots,n}$ durch

$$N_i(\gamma(u)) = (-1)^{1+i} \det(\partial_{u_k}\gamma_j(u))_{j\neq i}\,. \tag{A.5}$$

Für $k = 1, \ldots, n-1$ gilt

$$N(\gamma(u)) \cdot \partial_{u_k}\gamma(u) = \det\big(\partial_{u_k}\gamma(u), \partial_{u_1}\gamma(u), \ldots, \partial_{u_{(n-1)}}\gamma(u)\big) = 0. \tag{A.6}$$

Dass das erste Gleichheitszeichen (A.6) gilt, sieht man, wenn man die Determinante nach der ersten Spalte $\partial_{u_k}\gamma(u)$ entwickelt, das zweite Gleichheitszeichen gilt, weil die Determinante zweimal dieselbe Spalte $\partial_{u_k}\gamma(u)$ enthält. Aus (A.6) folgt, dass $N(\gamma(u))$ ein Normalenvektor an M im Punkt $\gamma(u)$ ist. Weil die Determinante in (A.5) eine Summe von Produkten der stetigen Funktionen $\partial_{u_k}\gamma_j$ ist, sind die Komponentenfunktionen $N_i \circ \gamma : U \to \mathbb{R}$ der Funktion $N \circ \gamma : U \to \mathbb{R}^n$ stetig. Nach Theorem 3.22 bedeutet dies, dass $N \circ \gamma$ stetig ist, also ist auch die Hintereinanderausführung $N = N \circ \gamma \circ \kappa : W \to \mathbb{R}^n$ stetig, wobei $\kappa = \gamma^{-1}$ die stetige Koordinantenabbildung auf W ist. Hieraus schließen wir, dass $|N| = \sqrt{\sum_{i=1}^n N_i^2}$ eine stetige Funktion ist, also ist auch das Einheitsnormalenfeld $\nu = N/|N| : W \to \mathbb{R}^3$ stetig. ∎

Bemerkung. Für $n = 3$ liefert (A.5), dass $N(\gamma(u)) = \partial_{u_1}\gamma(u) \times \partial_{u_2}\gamma(u)$, mit dem Vektorprodukt $\times$ im $\mathbb{R}^3$.

Aus diesem lokalen Resultat kann man nicht schließen, dass auf jeder $(n-1)$–dimensionalen Untermannigfaltigkeit M von $\mathbb{R}^n$ ein stetiges Einheitsnormalenfeld existiert, das auf ganz M stetig ist. Ein Gegenbeispiel werden wir später in diesem Kapitel angeben.

Definition A.17 (Äußere Einheitsnormale) *Sei $A \subseteq \mathbb{R}^n$ eine nichtleere Menge. Man sagt, A habe glatten Rand, wenn ∂A eine $(n-1)$–dimensionale Untermannigfaltigkeit von $\mathbb{R}^n$ ist.*

Ein Normalenvektor $N \in \mathbb{R}^n$ an die $(n-1)$–dimensionale Untermannigfaltigkeit ∂A im Punkt $x \in \partial A$ heißt äußerer Normalenvektor, wenn ein Zahl $\delta_0 > 0$ existiert, so dass $x + \delta N$ für alle $0 < \delta < \delta_0$ zum Komplement $\mathbb{R}^n \setminus A$ von A gehört. Eine Abbildung $\nu : \partial A \to \mathbb{R}^n$ heißt äußeres Einheitsnormalenfeld, wenn $\nu(x)$ ein äußerer Einheitsnormalenvektor ist für alle $x \in \partial A$.

Definition A.18 (Divergenz) *Sei $\Omega \subseteq \mathbb{R}^n$ eine offene Menge. Zu differenzierbarem $f : \Omega \to \mathbb{R}^n$ definiert man die Divergenz* $\operatorname{div} f : \Omega \to \mathbb{R}$ *von f durch*

$$\operatorname{div} f(x) := \sum_{i=1}^n \frac{\partial}{\partial x_i} f_i(x)\,.$$

Nun können wir den Gaußschen Satz formulieren:

Satz A.19 (Gaußscher Integralsatz) *Sei $A \subseteq \mathbb{R}^n$ eine beschränkte offene Menge mit glattem Rand ∂A, zu der ein stetiges äußeres Einheitsnormalenfeld $\nu : \partial A \to \mathbb{R}^n$ existiert. Dann gilt für jedes stetig differenzierbare Vektorfeld $f : \Omega \to \mathbb{R}^n$, das auf einer offenen Menge $\Omega \subseteq \mathbb{R}^n$ mit $\overline{A} \subseteq \Omega$ definiert ist, die Gleichung*

$$\int_{\partial A} \nu(x) \cdot f(x) dS(x) = \int_A \operatorname{div} f(x) dx\,.$$

(Carl Friedrich Gauß, 1777 – 1855.)

Für $n = 1$ lautet der Satz: Seien $a, b \in \mathbb{R}$, $a < b$. Dann ist

$$f(b) - f(a) = \int_a^b \frac{d}{dx} f(x)\, dx\,,$$

und man sieht, daß der Gaußsche Satz die Verallgemeinerung des Hauptsatzes der Differential- und Integralrechnung auf den $\mathbb{R}^n$ mit $n > 1$ ist.

Beispiel: Anwendung in der Hydrostatik. Ein Körper A sei in eine Flüssigkeit mit dem spezifischen Gewicht c eingetaucht. Die Oberfläche der Flüssigkeit falle mit der Ebene $x_3 = 0$ zusammen. Der Druck im Punkt $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ mit $x_3 < 0$ ist dann

$$p(x) = -cx_3\,.$$

Dieser Druck übt eine Kraft auf den Körper A aus. Ist $\nu(x)$ der äußere Einheitsnormalenvektor an ∂A in $x \in \partial A$, dann ist die Kraft pro Flächeneinheit im Punkt x

$$-\nu(x)p(x) = -cx_3(-\nu(x)) = cx_3\nu(x).$$

Für die gesamte Oberflächenkraft ergibt sich also

$$K = \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \int_{\partial A} cx_3\nu(x)\, dS(x)\,.$$

Anwendung des Gaußschen Satzes auf die Funktionen $f_1, f_2, f_3 : A \to \mathbb{R}^3$ mit

$$f_1(x_1, x_2, x_3) = (x_3, 0, 0), \quad f_2(x_1, x_2, x_3) = (0, x_3, 0), \quad f_3(x_1, x_2, x_3) = (0, 0, x_3)$$

liefert für $i = 1, 2$

$$K_i = \int_{\partial A} cx_3\nu_i(x)dS(x) = c\int_{\partial A} \nu(x) \cdot f_i(x)dS(x) = c\int_A \frac{\partial}{\partial x_i}\, x_3\, dx = 0,$$

und für $i = 3$

$$K_3 = \int_{\partial A} cx_3\nu_3(x)dS(x) = c\int_{\partial A} \nu(x) \cdot f_3(x)dS(x) = c\int_A \frac{\partial}{\partial x_3}\, x_3 dx = c\int_A dx = c\,|A|.$$

K ist somit in Richtung der positiven x_3-Achse gerichtet, also erfährt A einen Auftrieb der Größe $c\,|A| = c\,\mathrm{Vol}(A)$. Dieser Wert ist gleich dem Gewicht der verdrängten Flüssigkeit.

A.6 Greensche Formeln

Es sei $A \subseteq \mathbb{R}^n$ eine beschränkte offene Menge mit glattem Rand, zu der ein stetiges äußeres Einheitsnormalenfeld $\nu : \partial A \to \mathbb{R}^n$ existiert. Ω sei eine offene Menge mit $\overline{A} \subseteq \Omega$. Im Folgenden bezeichnen wir den Gradienten einer differenzierbaren Abbildung $f : \Omega \to \mathbb{R}$ mit $\nabla f(x)$.

Definition A.20 *Für stetig differenzierbares $f : \Omega \to \mathbb{R}$ definiert man die Normalableitung von f im Punkt $x \in \partial A$ durch*

$$\frac{\partial f}{\partial \nu}(x) := f'(x)\nu(x) = \nu(x) \cdot \nabla f(x) = \sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i}\nu_i(x)\,.$$

Für zweimal stetig differenzierbares $f : \Omega \to \mathbb{R}$ setzt man

$$\Delta f(x) := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} f(x)\,.$$

Bemerkungen. 1.) Die Normalableitung von f im Punkt $x \in \partial A$ ist die Ableitung von f in Richtung des Normalenvektors $\nu(x)$.
2.) Der Differentialoperator Δ heißt Laplaceoperator, nach Pierre-Simon de Laplace (1749 – 1827).

Satz A.21 *Für $f, g \in C^2(\Omega, \mathbb{R})$ gilt die* erste Greensche Formel

$$\int_{\partial A} f(x)\frac{\partial g}{\partial \nu}(x)dS(x) = \int_A \big(\nabla f(x) \cdot \nabla g(x) + f(x)\Delta g(x)\big)dx$$

und die zweite Greensche Formel

$$\int_{\partial A} \Big(f(x)\frac{\partial g}{\partial \nu}(x) - g(x)\frac{\partial f}{\partial \nu}(x)\Big)dS(x) = \int_A \big(f(x)\Delta g(x) - g(x)\Delta f(x)\big)dx\,.$$

(George Green, 1793 – 1841.)

Beweis: Zum Beweis der ersten Greenschen Formel wende den Gaußschen Integralsatz auf die stetig differenzierbare Funktion

$$f\,\nabla g : \Omega \to \mathbb{R}^n$$

an. Mit $\operatorname{div}(f\,\nabla g) = \nabla f \cdot \nabla g + f\Delta g$ folgt

$$\begin{aligned}\int_{\partial A} f(x)\frac{\partial g}{\partial \nu}(x)dS(x) &= \int_{\partial A} \nu(x) \cdot (f\,\nabla g)(x)dS(x)\\ &= \int_A \operatorname{div}(f\,\nabla g)(x)dx = \int_A \big(\nabla f(x) \cdot \nabla g(x) + f(x)\Delta g(x)\big)dx\,.\end{aligned}$$

Für den Beweis der zweiten Greenschen Formel benutzt man die erste Greensche Formel. Danach gilt

$$\int_{\partial A} \Big(f(x)\frac{\partial g}{\partial \nu}(x) - g(x)\frac{\partial f}{\partial \nu}(x)\Big) dS(x)$$
$$= \int_A (\nabla f(x) \cdot \nabla g(x) + f(x)\Delta g(x))dx - \int_A (\nabla f(x) \cdot \nabla g(x) + g(x)\Delta f(x))dx$$
$$= \int_A (f(x)\Delta g(x) - g(x)\Delta f(x))dx\,. \qquad \blacksquare$$

Beispiel: Anwendung in der Potentialtheorie. Eine zweimal stetig differenzierbare Funktion $u : A \to \mathbb{R}$, die die *partielle Differentialgleichung* $\Delta u(x) = 0$ für alle $x \in A$ erfüllt, heißt harmonische Funktion oder Potentialfunktion in A, weil das Gravitationspotential und das elektrische Potential diese Gleichung für $n = 3$ erfüllen. Eines der zentralen Probleme der Theorie der partiellen Differentialgleichungen besteht in der Bestimmung einer stetigen Funktion $u : \overline{A} \to \mathbb{R}$, die in A eine Potentialfunktion ist und die am Rand ∂A mit einer gegebenen Funktion f übereinstimmt. Gesucht ist also eine Lösung u des *Randwertproblems*

$$\begin{aligned} \Delta u(x) &= 0, && \text{für all } x \in A, \\ u(x) &= f(x), && \text{für all } x \in \partial A. \end{aligned}$$

Satz A.22 *Wenn die Menge A wegzusammenhängend ist, dann gibt es höchstens eine Lösung dieses Randwertptoblems, die in einer offenen Menge Ω with $\overline{A} \subseteq \Omega$ zweimal stetig differenzierbar ist.*

Beweis: Wenn u und v Lösungen sind, dann löst $w = u - v$ das neue Randwertproblem

$$\Delta w(x) = 0, \text{ für } x \in A, \qquad w(x) = u(x) - v(x) = 0, \text{ für } x \in \partial A. \tag{A.7}$$

Die erste Greensche Formel liefert daher

$$\int_A |\nabla w(x)|^2 dx = \int_A \nabla w(x) \cdot \nabla w(x) dx = \int_{\partial A} w(x)\frac{\partial w}{\partial \nu}(x) dS(x) - \int_A w(x)\Delta w(x) dx = 0. \tag{A.8}$$

Hieraus folgt $\nabla w(x) = 0$ für alle $x \in A$. Denn andernfalls würde ein Punkt $x_0 \in A$ existieren mit $|\nabla w(x_0)| > 0$. Weil ∇w stetig ist, würde sogar $|\nabla w(x)| > 0$ gelten für alle x in einer Umgebung von x_0. Aus der Definition des Integrals würde sich hieraus unmittelbar $\int_A |\nabla w(x)|^2 dx > 0$ ergeben, im Widerspruch zu (A.8). Nach Theorem 4.17 folgt aus $w'(x) = \big(\nabla w(x)\big)^T = 0$ für alle $x \in A$, dass w konstant ist in der wegzusammenhängenden Menge A. Weil w nach (A.7) am Rand ∂A verschwindet, folgt $u(x) - v(x) = w(x) = 0$ für alle $x \in \overline{A}$, also $u = v$. $\blacksquare$

A.7 Der Integralsatz von Stokes

In einer allgemeinen Version, zu deren Formulierung man am besten Differentialformen verwendet, gilt der Stokessche Integralsatz in jeder Raumdimension. Wir verzichten auf die Einführung von Differentialformen und diskutieren nur die Versionen für $n = 2$ und $n = 3$, die für die meisten Anwendungen relevant sind.

Wir gehen vom Gaußschen Integralsatz in $\mathbb{R}^2$ aus. Sei $A \subseteq \mathbb{R}^2$ eine beschränkte offene Menge mit glattem Rand ∂A. Der Rand ∂A ist eine Kurve. Wenn $g = (g_1, g_2) : \Omega \to \mathbb{R}^2$ eine stetig differenzierbare Funktion auf einer offenen Menge $\Omega \subseteq \mathbb{R}^2$ mit $\overline{A} \subseteq \Omega$ ist, dann gilt nach dem Gaußschen Satz

$$\int_A \Big(\frac{\partial g_1}{\partial x_1}(x) + \frac{\partial g_2}{\partial x_2}(x)\Big)dx = \int_{\partial A} (\nu_1(x)g_1(x) + \nu_2(x)g_2(x))ds(x), \tag{A.9}$$

mit dem äußeren Einheitsnormalenvektor $\nu(x) = (\nu_1(x), \nu_2(x))$ im Punkt $x \in \partial A$. Ist $f : U \to \mathbb{R}^2$ eine andere stetig differenzierbare Funktion und wählt man für g in (A.9) die Funktion

$$g(x) = \begin{pmatrix} f_2(x) \\ -f_1(x) \end{pmatrix},$$

dann erhält man

$$\int_A \Big(\frac{\partial f_2}{\partial x_1}(x) - \frac{\partial f_1}{\partial x_2}(x)\Big)dx = \int_{\partial A} \big(\nu_1(x)f_2(x) - \nu_2(x)f_1(x)\big)ds(x) = \int_{\partial A} \tau(x) \cdot f(x)ds(x)\,, \tag{A.10}$$

mit

$$\tau(x) = \begin{pmatrix} -\nu_2(x) \\ \nu_1(x) \end{pmatrix}.$$

$\tau(x)$ ist ein Einheitsvektor, der senkrecht auf dem Normalenvektor $\nu(x)$ steht, also ist $\tau(x)$ ein Einheitstangentenvektor an ∂A im Punkt $x \in \partial A$, und zwar derjenige, den man aus $\nu(x)$ durch Drehung um 90^o im mathematisch positiven Sinn erhält. Definiert man für differenzierbares $f : U \to \mathbb{R}^2$ die *Rotation* von f durch

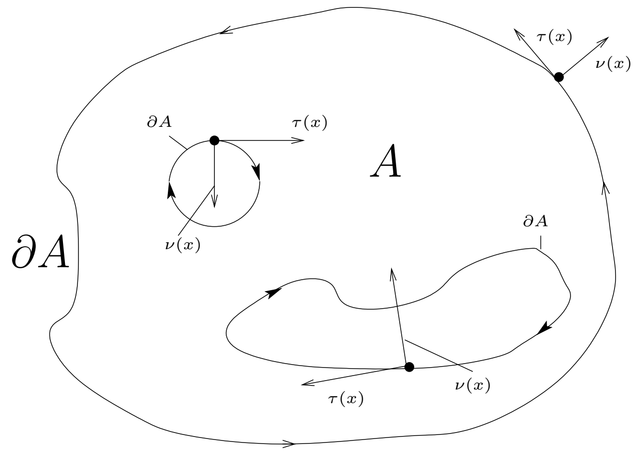
$$\operatorname{rot} f(x) := \frac{\partial f_2}{\partial x_1}(x) - \frac{\partial f_1}{\partial x_2}(x)\,,$$

dann kann (A.10) in der Form

$$\int_A \operatorname{rot} f(x)dx = \int_{\partial A} \tau(x) \cdot f(x)ds(x)$$

geschrieben werden. Diese Formel heißt *Stokesscher Satz* in der Ebene. In der angelsächsischen Literatur wird diese Formel auch als Satz von Green bezeichnet und dabei oft in der Form (A.10) geschrieben.

Man beachte, dass A nicht als *einfach zusammenhängend* vorausgesetzt wurde. Das heißt, dass A wie in der folgenden Abildung *Löcher* haben kann. Dargestellt ist eine Menge mit zwei Löchern. Der Rand ∂A ist nicht zusammenhängend, sondern besteht aus drei Zusammenhangskomponenten. Für jede Komponente ist die Richtung des Tangentenvektors τ eingezeichnet.



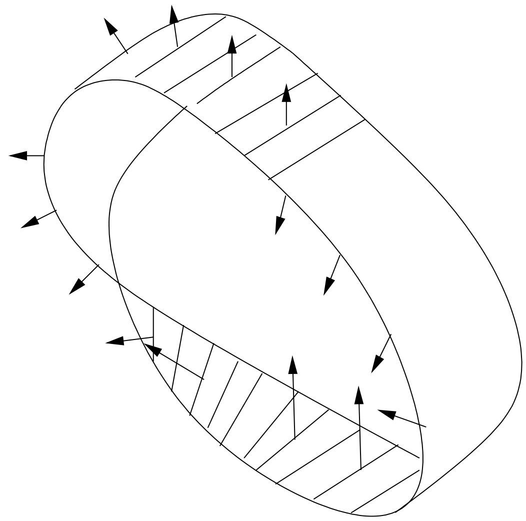
Man kann die Teilmenge $A \subseteq \mathbb{R}^2$ mit der ebenen Untermannigfaltigkeit $A \times \{0\}$ von $\mathbb{R}^3$ identifizieren und das Integral über A im Stokesschen Satz mit dem Flächenintegral über diese Untermannigfaltigkeit. Diese Interpretation legt die Vermutung nahe, dass diese Formel verallgemeinert werden kann und der Stokessche Satz nicht nur für ebene Untermannigfaltigkeiten, sondern für allgemeinere zweidimensionale Untermannigfaltigkeiten des $\mathbb{R}^3$ gilt. Damit gelangt man zum allgemeinen Stokesschen Integralsatz im $\mathbb{R}^3$. Um den Satz zu formulieren, sind Vorbereitungen nötig.

Orientierbare Untermannigfaltigkeiten. Nach Satz A.16 existiert lokal auf jeder $(n-1)$–dimensionalen Untermannigfaltigkeit von $\mathbb{R}^n$ ein stetiges Normalenfeld. Der Stokessche Satz gilt auf Untermannigfaltigkeiten, die ein globales stetiges Normalenfeld zulassen.

Definition A.23 *Eine $(n-1)$–dimensionale Untermannigfaltigkeit M des $\mathbb{R}^n$ heißt orientierbar, wenn ein stetiges Einheitsnormalenfeld auf M existiert.*

Beispiel: 1.) Sei $R > 0$ und $n \geq 2$. Die Sphäre $M = \{x \in \mathbb{R}^n \mid |x| = R\}$ ist eine $(n-1)$–dimensionale orientierbare Untermannigfaltigkeit von $\mathbb{R}^n$. Ein Einheitsnormalenfeld ist $\nu(x) = \frac{x}{|x|}$.

2.) Das Möbiusband ist eine zweidimensionale nicht orientierbare Untermannigfaltigkeit des $\mathbb{R}^3$:



Möbius strip

Ein Einheitsnormalenfeld auf dem Möbiusband hat notwendigerweise einen Sprung. (August Ferdinand Möbius, 1790 – 1868.)

Untermannigfaltigkeiten mit Rand. Bis jetzt haben wir nur parametrisierte Flächen und Untermannigfaltigkeiten ohne Rand betrachtet, also parametrisierte Flächen und Untermannigfaltigkeiten, die keine Randpunkte enthalten. Um parametrisierte Flächen und Untermannigfaltigkeiten mit Rand einzuführen, sei

$$H = \mathbb{R}^{n-2} \times [0, \infty), \qquad \partial H = \mathbb{R}^{n-2} \times \{0\}.$$

H ist der obere abgeschlossene Halbraum von $\mathbb{R}^{n-1}$.

Definition A.24 *Sei $U \subseteq \mathbb{R}^{n-1}$ eine offene Menge mit $U \cap \partial H \neq \emptyset$. Eine Abbildung $\gamma : U \cap H \to \mathbb{R}^n$ heißt einfache (n−1)–dimensionale parametrisierte Fläche mit Rand, wenn γ zu einer einfachen parametriserten Fläche $\tilde{\gamma} : U \to \mathbb{R}^n$ fortgesetzt werden kann. Für die Spur $F = \gamma(U \cap H)$ setzen wir $\partial F = \gamma(U \cap \partial H)$.*

Für die Spur $F = \gamma(U)$ einer parametrisierten Fläche $\gamma : U \to \mathbb{R}^n$ ohne Rand definieren wir $\partial F = \emptyset$.

Zur Vereinfachung der Bezeichnung werden wir im Folgenden auch die Fortsetzung $\tilde{\gamma}$ mit γ bezeichnen.

Satz A.25 *Es sei $\gamma : U \cap H \to \mathbb{R}^n$ eine (n−1)–dimensionale parametriserte Fläche mit Rand. Identifiziert man die Menge $U \cap \partial H$ mit einer offenen Menge im Raum $\mathbb{R}^{n-2}$, dann* *ist die Einschränkung* $\hat\gamma = \gamma\big|_{U\cap\partial H} : U \cap \partial H \to \mathbb{R}^n$ *eine einfache* $(n-2)$*–dimensionale parametrisierte Fläche mit Spur* $\partial F = \hat\gamma(U \cap \partial H)$.

Beweis: Sei $u \in U \cap \partial H$. Da $\gamma : U \to \mathbb{R}^n$ eine $(n-1)$–dimensionale parametrisiert Fläche ist, hat die Jacobimatrix $\gamma'(u)$ den Rang $n-1$, folglich sind die Spaltenvektoren $\partial_{u_1}\gamma(u), \dots, \partial_{u_{n-1}}\gamma(u)$, aus denen sich $\gamma'(u)$ zusammensetzt, linear unabhängig. Dies bedeutet, dass die aus den ersten $n-2$ dieser Spaltenvektoren bestehende Matrix $\hat\gamma'(u)$ den Rang $n-2$ hat. Außerdem ist die Inverse $\hat\kappa = \hat\gamma^{-1} : \partial F \to U \cap \partial H$ stetig als Einschränkung der stetigen Abbildung γ^{-1} auf ∂F. ■

Definition A.26 *Eine Menge* $M \subset \mathbb{R}^n$ *heißt* $(n-1)$*–dimensionale Untermannigfaltigkeit von* $\mathbb{R}^n$ *mit Rand, wenn die folgenden zwei Bedingungen erfüllt sind:*

(i) *M ist abgeschlossen,*

(ii) *Zu jedem Punkt $x \in M$ gibt es eine offene n–dimensionale Umgebung $V(x)$ von x, so dass $V(x) \cap M$ die Spur einer einfachen $(n-1)$–dimensionalen parametrisierten Fläche mit oder ohne Rand ist.*

$V(x) \cap M$ wird als Koordinatenumgebug von x in M bezeichnet. Unter dem Rand von M versteht man die Menge

$$\partial M = \bigcup_{x\in M} \partial\big(V(x) \cap M\big).$$

Satz A.27 *Die Menge ∂M ist eine $(n-2)$–dimensionale Untermannigfaltigkeit von $\mathbb{R}^n$.*

Beweis: Es muss gezeigt werden, dass es zu jedem $x \in \partial M$ eine Koordinatenumgebung in ∂M gibt. Um eine solche Koordinatenumgebung von x zu konstruieren, wähle eine offene Menge V in $\mathbb{R}^n$, so dass $V \cap M$ eine Koordinatenumgebung von x in M ist. Nach Satz A.25 ist $V \cap \partial M = \partial(V \cap M)$ die gesuchte Koordinatenumgebung von x in ∂M. ■

Orientierung von ∂M. Es sei M eine zweidimensionale orientierbare Untermannigfaltigkeit von $\mathbb{R}^3$ mit Rand ∂M. Weil jedes $x \in \partial M$ sowohl zur zweidimensionalen Untermannigfaltigkeit M als auch zur eindimensionalen Untermannigfaltigkeit ∂M gehört, gibt es im Punkt x den zweidimensionalen Tangentialraum T_xM an M, der wie bei Mannigfaltigkeiten ohne Rand definiert ist, und den eindimensionalen Tangentialraum $T_x(\partial M)$ an ∂M. Der Raum $T_x(\partial M)$ ist ein linearer Unterraum von T_xM, weil jeder Tangentenvektor an ∂M in x auch Tangentenvektor an M ist. Der Orthogonalraum von $T_x(\partial M)$ in T_xM ist eindimensional, also gibt es in diesem Orthogonalraum zwei Einheitsvektoren. Nach intuitiver Vorstellung zeigt einer dieser Vektoren in die Untermanigfaltigkeit M hinein, der andere aus M heraus. Wir bezeichnen den Zweiten mit $\mu(x)$.

Rigoros kann $\mu(x)$ folgendermaßen definiert werden: Zum Randpunkt $x \in \partial M$ wähle man eine $(n-1)$–dimensionale Koordinatenumgebung W in M mit Parametrisierung $\gamma : U \cap H \to W$ und Koordinatenabbildung $\kappa = \gamma^{-1} : W \to U \cap H$. Für $u \in U \cap \partial H$ setze

$$\hat{\mu}(\gamma(u)) = \partial_{u_1}\gamma(u) \times \big(\partial_{u_1}\gamma(u) \times \partial_{u_2}\gamma(u)\big), \qquad \mu(\gamma(u)) = \frac{\hat{\mu}(\gamma(u))}{|\hat{\mu}(\gamma(u))|}. \tag{A.11}$$

Aus den Eigenschaften des Vektorproduktes $\times$ folgt, dass der Vektor $\hat{\mu}(\gamma(u))$ und also auch der Einheitsvektor $\mu(\gamma(u))$ zum Tangentialraum $T_{\gamma(u)}M$ gehören und orthogonal zum Raum $T_{\gamma(u)}(\partial M)$ sind. Weil mit $\omega_i = \partial_{u_i}\gamma(u)$

$$\partial_{u_2}\gamma(u) \cdot \hat{\mu}(\gamma(u)) = \omega_2 \cdot (\omega_1 \times (\omega_1 \times \omega_2)) = -(\omega_1 \times \omega_2) \cdot (\omega_1 \times \omega_2) < 0$$

gilt, ist $\mu(\gamma(u))$ derjenige der beiden Einheitsvektoren mit diesen Eigenschaften, der aus M hinauszeigt. Wir überlassen dem Leser den Beweis, dass $\mu(\gamma(u))$ unabhängig von der Wahl der Parametrisierung γ ist. Es genügt dazu zu zeigen, dass $\partial_{u_2}\gamma(u) \cdot \hat{\mu}^*(\gamma^*(u^*)) < 0$ ist für den mit einer anderen Parametrisierung γ^* definierten Vektor $\hat{\mu}^*(\gamma^*(u^*))$.

Weil $\partial_{u_1}\gamma$ und $\partial_{u_2}\gamma$ stetig sind, folgt aus den Gleichungen (A.11), dass $\mu \circ \gamma$ eine stetige Abbildung auf $U \cap \partial H$ ist, folglich ist auch $\mu = \mu \circ \gamma \circ \kappa : W \cap \partial M \to \mathbb{R}^3$ stetig. Insbesondere ist μ im Punkt $x \in W \cap \partial M$ stetig. Weil $x \in \partial M$ beliebig gewählt war, wird durch (A.11) ein stetiges Vektorfeld $\mu : \partial M \to \mathbb{R}^3$ definiert.

Es sei $\nu : M \to \mathbb{R}^3$ ein stetiges Einheitsnormalenfeld an M. Mit dem stetigen Vektorfeld μ definieren wir ein stetiges Einheitstangentenfeld $\tau : \partial M \to \mathbb{R}^3$ durch

$$\tau(x) = \nu(x) \times \mu(x), \quad x \in \partial M,$$

und sagen, *das Vektorfeld τ orientiere den Rand ∂M positiv bezüglich ν.*

Definition A.28 *Seien $\Omega \subseteq \mathbb{R}^3$ eine offene Menge und $f : \Omega \to \mathbb{R}^3$ eine differenzierbare Funktion. Die Rotation* $\operatorname{rot} f : \Omega \to \mathbb{R}^3$ *von f ist definiert durch*

$$\operatorname{rot} f(x) := \begin{pmatrix} \partial_{x_2} f_3(x) - \partial_{x_3} f_2(x) \\ \partial_{x_3} f_1(x) - \partial_{x_1} f_3(x) \\ \partial_{x_1} f_2(x) - \partial_{x_2} f_1(x) \end{pmatrix}.$$

In der angelsächsischen Literatur wird die Bezeichung $\operatorname{curl} f$ anstelle von $\operatorname{rot} f$ verwendet.

Satz A.29 (Stokesscher Integralsatz) *Sei M eine zweidimensionale orientierbare Untermannigfaltigkeit von $\mathbb{R}^3$ mit Rand ∂M, sei $\nu : M \to \mathbb{R}^3$ ein stetiges Einheitsnormalenfeld und sei $\tau : \partial M \to \mathbb{R}^3$ das stetige Einheitstangentenfeld, das ∂M positiv orientiert bezüglich ν. Für jede stetig differenzierbare Funktion $f : \Omega \to \mathbb{R}^3$, die auf einer offenen Menge $\Omega \subseteq \mathbb{R}^3$ mit $M \subseteq \Omega$ definiert ist, gilt dann die Gleichung*

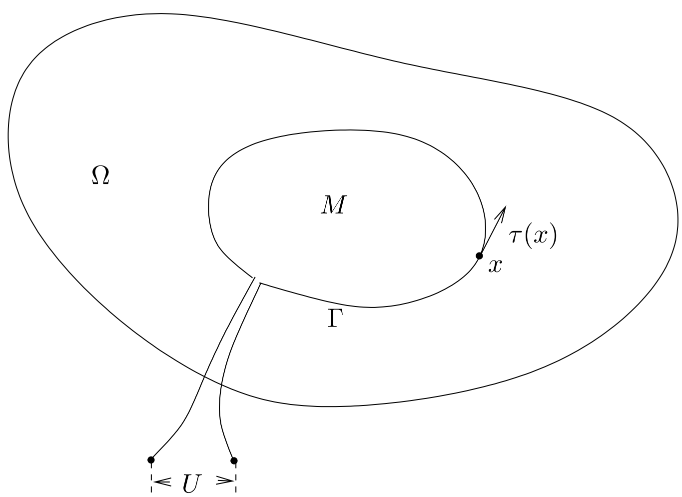
$$\int_M \nu(x) \cdot \mathrm{rot} f(x)\, dS(x) = \int_{\partial M} \tau(x) \cdot f(x)\, ds(x).$$

(George Gabriel Stokes, 1819 – 1903.)

Beispiel: Anwendung im Elektromagnetismus. Sei Ω eine offene Menge im $\mathbb{R}^3$. In Ω seien ein elektrisches Feld E und ein magnetisches Induktionsfeld B vorhanden, die vom Ort $x \in \Omega$ und der Zeit $t \in \mathbb{R}$ abhängen. E und B sind Vektorfelder

$$E : \Omega \times \mathbb{R} \to \mathbb{R}^3\,, \qquad B : \Omega \times \mathbb{R} \to \mathbb{R}^3\,.$$

Wir plazieren eine Drahtschleife Γ in Ω.



Wenn B zeitlich veränderlich ist, wird nach dem Faradayschen Induktionsgesetz in Γ eine elektrische Spannung induziert, die folgendermaßen berechnet werden kann: Die Vektorfelder E und B erfüllen die Gleichung

$$\mathrm{rot}_x E(x,t) = -\frac{\partial}{\partial t} B(x,t)$$

für alle $(x,t) \in \Omega \times \mathbb{R}$. Dies ist eine der Maxwellschen Gleichungen. Diese vektorielle Gleichung besteht aus drei untereinander gekoppelten partiellen Differentialgleichungen. Der Operator rot_x auf der linken Seite der Gleichung wirkt nur bezüglich der Variablen

$x = (x_1, x_2, x_3)$. Um $U(t)$ aus dieser Gleichung zu bestimmen, wähle man eine orientierbare zweidimensionale Untermannigfaltigkeit $M \subseteq \Omega$ mit Rand $\partial M = \Gamma$ und ein stetiges Einheitsnormalenfeld $\nu : M \to \mathbb{R}^3$. Dieses Normalenfeld bestimmt ein Einheitstangentenfeld $\tau : \Gamma \to \mathbb{R}^3$, durch das Γ positiv orientiert wird bezüglich ν. Der Stokessche Satz liefert nun

$$U(t) = \int_\Gamma \tau(x) \cdot E(x,t)\, ds(x) = \int_M \nu(x) \cdot \mathrm{rot}_x E(x,t)\, dS(x)$$
$$= -\int_M \nu(x) \cdot \frac{\partial}{\partial t} B(x,t)\, dS(x) = -\frac{\partial}{\partial t} \int_M \nu(x) \cdot B(x,t)\, dS(x)\,.$$

Das Integral $\int\limits_M \nu(x) \cdot B(x,t)\, dS(x)$ heißt *Fluß der magnetischen Induktion durch* M. Somit ist $U(t)$ gleich der negativen zeitlichen Änderung des Flusses von B durch M. Dies ist das Faradaysche Induktionsgesetz. (Michael Faraday, 1791 – 1876, James Clerk Maxwell, 1831 – 1879)

Postface

The lecture notes Analysis I and II originated from my handwritten notes in German for the course Infinitesimalrechnung I – IV, which I gave at the Universität Bonn during the years 1982 – 1984. In writing the notes I drew from several textbooks and reference works. Those books, which I mainly used, are listed below. As a guideline for the course I used the two volume work of Barner and Flohr. Therefore these lecture notes owe much to this excellent work.

The notes have been revised and typed in latex when I gave the analysis course several times during the years 1990 – 2008 at the Technische Universität Darmstadt. The English version, which covers the material of a two semester introductory course, was originally prepared in 2001.

Hans-Dieter Alber

Bibliography

Martin Barner, Friedrich Flohr: Analysis I, II. Walter de Gruyter, Berlin 1974.

Heinrich Behnke, Reinhold Remmert (Herausgeber): Das Fischer Lexikon, Band 29,1: Mathematik 1. Fischer-Taschenbuch-Verlag 1964.

Friedhelm Erwe: Differential- und Integralrechnung I. Hochschultaschenbücher, Bibliographisches Institut, Mannheim 1962.

Harro Heuser: Lehrbuch der Analysis Teil 1, Teil 2. Teubner Verlag, Stuttgart 1980/1981.

Walter Rudin: Principles of Mathematical Analysis, McGraw-Hill Book Company 1964.