Nonlinear elliptic and parabolic partial differential equations
(PDE 2)

Lecture Notes

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2007 – 2013

last modified June 10, 2018

Contents

1 Nonlinear partial differential equations modeling diffusion and elastic deformation

1.1 Nonlinear diffusion processes

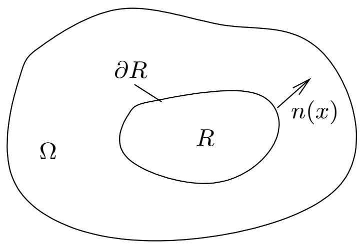
Heat conduction and transport of electric charge are examples for diffusion processes. Let $\Omega \subseteq \mathbb{R}^3$ be a body, which conducts heat or electric charge. $u(t,x)$ is the temperature or the charge density at the point x at time t. The amount of heat or charge contained in a subvolume R of Ω is given by

$$\int_R u(t,x)dx.$$

Let $q(t,x) \in \mathbb{R}^3$ be the flux at the position x at time t and let $b(t,x) \in \mathbb{R}^3$ be the amount of the quantity u generated by a source at time t and position x in unit time. The temporal variation of $\int_R u(t,x)dx$ is given by

$$-\int_R q(t,x) \cdot n(x)dx + \int_R b(t,x)dx,$$

where $n(x) \in \mathbb{R}^3$ is the outer unit normal vector of ∂R at the point $x \in \partial R$. The first term gives the amount of heat or charge entering R over the boundary ∂R in unit time. Therefore the equation



$$\frac{d}{dt}\int_R u(t,x)dx = -\int_{\partial R} q(t,x) \cdot n(x)dx + \int_R b(t,x)dx$$

must hold. Application of Gauss' theorem yields

$$\begin{aligned} 0 &= \int_R \frac{\partial}{\partial t} u(t,x)dx + \int_{\partial R} q(t,x)\, n(x)dx - \int_R b(t,x)dx \\ &= \int_R \Big(\frac{\partial}{\partial t} u(t,x) + \operatorname{div}_x q(t,x) - b(t,x)\Big)dx. \end{aligned}$$

This equation must hold for every subset R of Ω, which implies that

$$\frac{\partial}{\partial t} u(t,x) + \operatorname{div}_x q(t,x) = b(t,x), \quad x \in \Omega,\ t \geq 0. \tag{1.1}$$

This is one equation for four unknowns

$$u(t,x), \quad q(t,x) = \big(q_1(t,x),\ q_2(t,x),\ q_3(t,x)\big).$$

To obtain three more equations it must be known how the flux q depends on u. Equations describing this dependence are called constitutive equations. The form of these equations depends on the material properties of the conductor Ω. Because of the atomistic structure of Ω, these material properties are very complicated and therefore in general not very well known. To formulate constitutive equations one thus makes reasonable assumptions about these material properties and verifies the resulting equations by comparison with experiments.

The simplest assumption is Fourier's law of heat conduction:

$$q(t,x) = -c\nabla_x u(t,x), \tag{1.2}$$

with a constant $c > 0$. This law assumes that heat is flowing from regions of higher temperatures to regions of lower temperature. The strength $|q(t,x)|$ of the flow is proportional to the absolute value of $|\nabla_x u(t,x)|$ of the temperature gradient. Insertion of (1.2) into (1.1) yields the heat equation

$$\frac{\partial}{\partial t} u(t,x) = c\Delta_x u(t,x) + b(t,x), \quad x \in \Omega,\ t \geq 0. \tag{1.3}$$

A more general assumption is that the flow $q(t,x)$ is a linear or nonlinear function of $\nabla_x u(t,x)$:

$$q(t,x) = -F\big(\nabla_x u(t,x)\big), \tag{1.4}$$

with a given function $F : \mathbb{R}^3 \to \mathbb{R}^3$. Such a relation often holds in electric semi-conductors, where the conductivity is a function of the flow. Insertion of (1.4) into (1.1) gives

$$\frac{\partial}{\partial t} u(t,x) = \operatorname{div}_x F\big(\nabla_x u(t,x)\big) + b(t,x), \quad x \in \Omega,\ t \geq 0. \tag{1.5}$$

Equations (1.3) and (1.5) both are partial differential equations for the unknown function u, which must hold in the domain $Z = [0,\infty) \times \Omega$. One wants to find solutions of these equations which also satisfy boundary and initial conditions. In the Dirichlet initial-boundary value problem the solution u of (1.3) or (1.5) must also satisfy the equations

$$u(t,x) = \gamma_D(t,x), \qquad t \geq 0,\ x \in \partial\Omega, \tag{1.6}$$

$$u(0,x) = u_0(x), \qquad x \in \Omega \tag{1.7}$$

with given functions $\gamma_D : [0,\infty) \times \partial\Omega \to \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}$. In the Neumann initial-boundary value problem a solution u of (1.3) or (1.5) is sought which satisfies

$$F\big(\nabla_x u(t,x)\big) \cdot n(x) = \gamma_N(t,x), \qquad t \geq 0,\ x \in \partial\Omega, \tag{1.8}$$

$$u(0,x) = u_0(x), \qquad x \in \Omega, \tag{1.9}$$

with given functions $\gamma_N : [0,\infty) \times \partial\Omega \to \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}$.

1.2 Uniqueness of solutions for the initial-boundary value problem and monotonicity

One cannot expect that initial-boundary value problems to the diffusion equation (1.5) have a unique solution for every function F. After all, F describes material properties of the conductor Ω, and one expects that only those initial-boundary value problems with properties in accordance with real materials have unique solutions.

In order to get an idea what conditions must be imposed on F, we consider two solutions u, v of the initial-boundary value problem

$$\begin{aligned} u_t(t,x) &= \mathrm{div} F\big(\nabla u(t,x)\big) + b(t,x), \\ u(0,x) &= u_0(x), \quad x \in \Omega, \end{aligned}$$

with the Dirichlet boundary condition

$$u(t,x) = \gamma_D(t,x), \quad (t,x) \in [0,\infty) \times \partial\Omega$$

or the Neumann boundary condition

$$F\big(\nabla_x u(x,t)\big) \cdot n(x) = \gamma_N(t,x), \quad (t,x) \in [0,\infty) \times \Omega.$$

In a diffusion problem one expects that the solutions tend to a unique equilibrium distributions for large times. Therefore the distance of two solutions, measured in a suitable norm, should tend to zero for $t \to \infty$. We use the square of the L^2-norm to measure the distance of u and v at time t:

$$\|u(t) - v(t)\|_\Omega^2 = \int_\Omega \big(u(t,x) - v(t,x)\big)^2 \, dx,$$

where for a function $w : [0,\infty) \times \Omega \to \mathbb{R}$ the function $w(t) : \Omega \to \mathbb{R}$ is defined by

$$w(t)(x) = w(t,x), \quad x \in \Omega.$$

Using (1.5), we obtain

$$\begin{aligned} \frac{d}{dt} \|u(t) - v(t)\|_\Omega^2 &= \int_\Omega \frac{\partial}{\partial t}\big(u(t,x) - v(t,x)\big)^2 \, dx \\ &= 2 \int_\Omega \big(u(t,x) - v(t,x)\big)\big(u_t(t,x) - v_t(t,x)\big) \, dx \\ &= 2 \int_\Omega \big(u(t,x) - v(t,x)\big)\Big(\mathrm{div}_x F\big(\nabla_x u(t,x)\big) - \mathrm{div}_x F\big(\nabla_x v(t,x)\big)\Big) dx. \end{aligned}$$

We now employ Gauss' theorem and (1.7) or (1.8):

$$\begin{aligned}
&\frac{d}{dt}\,\|u(t)-v(t)\|_\Omega^2 \qquad (1.10)\\
&\quad= -2\int_\Omega \big(\nabla_x u(t,x)-\nabla_x v(t,x)\big)\cdot\big(F\big(\nabla_x u(t,x)\big)-F\big(\nabla_x v(t,x)\big)\big)\,dx\\
&\qquad+2\int_{\partial\Omega}\big(u(t,x)-v(t,x)\big)\big(F\big(\nabla_x u(t,x)\big)-F\big(\nabla_x v(t,x)\big)\big)\cdot n(x)dS_x\\
&\quad= -2\int_\Omega \big(\nabla_x u(t,x)-\nabla_x v(t,x)\big)\cdot\big(F\big(\nabla_x u(t,x)\big)-F\big(\nabla_x v(t,x)\big)\big)dx.
\end{aligned}$$

Now assume that the vector field $F:\mathbb{R}^3\to\mathbb{R}^3$ satisfies

$$\big(F(\xi)-F(\eta)\big)\cdot(\xi-\eta)\ge 0 \quad \text{for all } \xi,\eta\in\mathbb{R}^3. \tag{1.11}$$

A vector field satisfying this condition is called **monotone**. This condition implies that the right hand side of (1.10) is nonpositive, whence we have $\frac{d}{dt}\,\|u(t)-v(t)\|_\Omega^2\le 0$. Integration yields for $s\le t$

$$\begin{aligned}
\|u(t)-v(t)\|_\Omega^2 &= \int_s^t \frac{d}{d\tau}\,\|u(\tau)-v(\tau)\|_\Omega^2\,d\tau+\|u(s)-v(s)\|_\Omega^2\\
&\le \|u(s)-v(s)\|_\Omega^2.
\end{aligned}$$

In particular, for $s=0$ we obtain

$$\|u(t)-v(t)\|_\Omega^2\le\|u(t)-v(0)\|_\Omega^2=\|u_0-u_0\|_\Omega^2=0,$$

whence $\|u(t)-v(t)\|_\Omega$ for all $t\ge 0$, and therefore $u=v$ in $[0,\infty)\times\Omega$.

We thus have proved the following result:

Theorem 1.1 *Assume that F satisfies* (1.11). *Then the Dirichlet and Neumann initial-boundary value problems* (1.5), (1.6), (1.7) *and* (1.5), (1.8), (1.9) *have at most one solution.*

Examples for monotone vector fields. 1. For $\xi\in\mathbb{R}^3$ let $F(\xi)=|\xi|^p\xi$ with $p\ge 0$. Young's inequality yields

$$\begin{aligned}
&\big(F(\xi)-F(\eta)\big)\cdot(\xi-\eta)=(|\xi|^p\xi-|\eta|^p\eta)\cdot(\xi-\eta)\\
&\quad=|\xi|^{p+2}-|\xi|^p\xi\cdot\eta-|\eta|^p\eta\cdot\xi+|\eta|^{p+2}\\
&\quad\ge|\xi|^{p+2}-|\xi|^{p+1}|\eta|-|\eta|^{p+1}|\xi|+|\eta|^{p+2}\\
&\quad\ge|\xi|^{p+2}-\frac{1}{r}\,|\xi|^{(p+1)r}-\frac{1}{s}\,|\eta|^s-\frac{1}{r}\,|\eta|^{(p+1)r}-\frac{1}{s}\,|\xi|^s+|\eta|^{p+2},
\end{aligned}$$

where $r, s > 1$ with $\frac{1}{r} + \frac{1}{s} = 1$. We choose $r = \frac{p+2}{p+1}$ and $s = p + 2$. Then we obtain

$$\begin{aligned}\big(F(\xi) - F(\eta)\big) \cdot (\xi - \eta) \geq |\xi|^{p+2} - \frac{p+1}{p+2} |\xi|^{p+2} - \frac{1}{p+2} |\eta|^{p+2} \\ - \frac{p+1}{p+2} |\eta|^{p+2} - \frac{1}{p+2} |\xi|^{p+2} + |\eta|^{p+2} = 0.\end{aligned}$$

Therefore F is monotone. The differential equation to this example is

$$u_t = \operatorname{div}(|\nabla_x u|^p \nabla_x u) + b.$$

2. Clearly, the vector field $F(\xi) = c\xi$ with $c > 0$ used in Fourier's law satisfies $\big(F(\xi) - F(\eta)\big) \cdot (\xi - \eta) = c|\xi - \eta|^2$, whence F is strongly monotone.

1.3 Elastic deformation of a nonlinear membrane

We want to find the static (equilibrium) states of an elastic membrane. To this end assume that the membrane has a deformation state, in which it is flattened in the plane $\mathbb{R}^2$. This state is not necessarily an equilibrium state. Let $\Omega \subseteq \mathbb{R}^2$ be the set of material points of the membrane in this deformation state. We call Ω the reference configuration.

For a given other deformation state let $\varphi(x) \in \mathbb{R}^3$ be the position in space of the material point, which in the reference configuration is at the point $x \in \Omega$. This defines a function $\varphi : \Omega \to \mathbb{R}^3$ describing the actual configuration.

With every deformation state of the membrane there is associated an amount of potential energy stored in the membrane. In a static deformation state this potential energy is minimal compared to all other deformation states possible for the membrane under the given kinematic restrictions. To find these deformation states with minimal stored energy we must know an expression for the potential energy. We assume first that no exterior forces are present. In this case the potential energy only depends on the material properties of the membrane and is a local function, i.e. it depends on the relative positions of atoms, which are neighboring in the reference configuration, but it does not depend on the relative positions of atoms, which in the reference configuration have a larger distance.

It thus only depends on the local deformation state in the neighborhood of every point. Since by Taylor's formula φ is given by

$$\varphi(x) = \sum_{\substack{|\alpha| \leq m \\ \alpha \in \mathbb{N}_0^2}} \frac{\partial^\alpha \varphi(x_0)}{\alpha!} (x - x_0)^\alpha + R_m(x_0, x)$$

with the remainder R_m, which is small for x close to $x_0 \in \Omega$, it follows that the potential energy $\Psi(\varphi)$ of the membrane in the actual configuration must have the form

$$\Psi(\varphi) = \int_\Omega \psi\big(\varphi(x), \nabla\varphi(x), \ldots, \nabla^m \varphi(x)\big)dx$$

with a suitable function $\psi : \mathbb{R}^3 \times \ldots \times \mathbb{R}^{3^m} \to [0, \infty)$. This expression can be simplified. First, under the assumption that no exterior forces are present, the potential energy does not depend on the absolute position $\varphi(x)$ of the material point x. Moreover, derivatives $\partial^\alpha \varphi(x)$ with a multi-index $\alpha \in \mathbb{N}_0^2$ satisfying $|\alpha| \geq 2$ describe bending of the membrane. For a thin shell the potential energy depends on such bending terms, but a membrane can be bent without resistance. Therefore, the energy density ψ does not contain such terms, and hence we obtain

$$\Psi(\varphi) = \int_\Omega \psi\big(\nabla\varphi(x)\big)dx,$$

with a function $\psi : \mathbb{R}^{3\times3} \to [0, \infty)$.

We consider next the case where exterior forces are present, but for simplicity we assume that these forces do not depend on the deformation state of the membrane. In this case the total potential energy of the membrane is obtained by adding to the energy given above an amount, which is equal to the work done against the exterior force during the movement of the membrane from the reference configuration to the actual configuration. If $b(x) \in \mathbb{R}^3$ is the exterior force per surface area, measured in the reference configuration, which acts at the material point x, we thus obtain for the total potential energy

$$\Psi(\varphi) = \int_\Omega \psi\big(\nabla\varphi(x)\big)dx + \int_\Omega \varphi(x) \cdot \big(-b(x)\big)dx.$$

With this potential energy we can determine the equilibrium state $\varphi : \Omega \to \mathbb{R}^3$ of a membrane. We assume that the membrane is fixed at the boundary, i.e. we assume that the position of the material points belonging to the boundary is determined by a given function $\gamma_D : \partial\Omega \to \mathbb{R}^3$. We thus have to find a deformation state $\varphi : \overline{\Omega} \to \mathbb{R}^3$ satisfying

$$\varphi\big|_{\partial\Omega} = \gamma_D,$$

and

$$\begin{aligned} &\int_\Omega \psi\big(\nabla\varphi(x)\big) - \varphi(x) \cdot b(x)dx \qquad (1.12) \\ &\quad = \min\Big\{ \int_\Omega \psi\big(\nabla\hat{\varphi}(x)\big) - \hat{\varphi}(x) \cdot b(x)dx \;\Big|\; \hat{\varphi} : \Omega \to \mathbb{R}^3,\ \hat{\varphi}\big|_{\partial\Omega} = \gamma_D \Big\}. \end{aligned}$$

Of course, φ and $\hat{\varphi}$ must satisfy differentiability properties, which we discuss later.

The minimization problem is associated to a Dirichlet problem for a partial differential equation of second order. We determine this Dirichlet problem under the simplifying assumption that the material points are displaced from the reference configuration only in the direction orthogonal to the plane of the reference configuration. Though this Dirichlet problem can be determined in the same way for the general case, the simplification will be essential for the succeeding considerations.

If the position of the material point $x \in \Omega$ is

$$\varphi(x) = \big(x_1, x_2, \varphi_3(x_1, x_2)\big) \in \mathbb{R}^3,$$

then

$$\nabla\varphi(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \multicolumn{2}{c}{\big(\nabla\varphi_3(x)\big)^T} \end{pmatrix}$$

and

$$\varphi(x) \cdot b(x) = x_1 b_1(x) + x_2 b_2(x) + \varphi_3(x) b_3(x).$$

To simplify the notation we denote φ_3 by φ and b_3 by b, whence $b : \Omega \to \mathbb{R}$, $\varphi : \Omega \to \mathbb{R}$. In this case the energy density in (1.12) is a function $\psi : \mathbb{R}^2 \to [0, \infty)$.

Now assume that φ is a two times differentiable solution of (1.12) and that $h \in C_0^\infty(\Omega, \mathbb{R})$, $s \in \mathbb{R}$. Since $\frac{d}{ds}\psi(\varphi + sh)\big|_{s=0} = 0$ is necessary for φ being a minimum, we obtain

$$\begin{aligned} 0 &= \frac{d}{ds}\Psi(\varphi + sh)\big|_{s=0} \\ &= \int_\Omega \frac{d}{ds}\Big(\psi\big(\nabla\varphi(x) + s\nabla h(x)\big) - \big(\varphi(x) + sh(x)\big)b(x)\Big)\big|_{s=0} dx \\ &= \int_\Omega (\nabla\psi)\big(\nabla\varphi(x)\big) \cdot \nabla h(x) - h(x)b(x)dx \\ &= \int_\Omega \Big(-\operatorname{div}_x\big[\nabla\psi\big(\nabla_x\varphi(x)\big)\big] - b(x)\Big)h(x)dx, \end{aligned}$$

where in the last step we used Gauss' theorem, noting that $h\big|_{\partial\Omega} = 0$. This equation must hold for every $h \in C_0^\infty(\Omega, \mathbb{R}^3)$, which implies

$$-\operatorname{div}_x\big[F\big(\nabla_x\varphi(x)\big)\big] = b(x), \qquad x \in \Omega \tag{1.13}$$

$$\varphi(x) = \gamma_D(x), \qquad x \in \partial\Omega, \tag{1.14}$$

with $F : \mathbb{R}^3 \to \mathbb{R}^3$ given by $F(\xi) = \nabla\psi(\xi)$. We obtain that φ must solve the Dirichlet problem (1.13), (1.14). The partial differential equation (1.13) contains the same nonlinear differential operator as (1.5). However, in the present case F is a gradient and thus is more special than in (1.5).

1.4 Equation for minimal surfaces

Consider a soap film, which at the boundary is attached to a wire forming a closed loop. The shape of the soap film is determined by the surface tension: the surface tension tends to bring the soap film into a shape with minimal surface area. To compute this shape, assume that the projection of the soap film to the x_1, x_2–axis is given by an open set $\Omega \subseteq \mathbb{R}^2$, and that the position of the wire loop is given by the graph of a function

$$\gamma : \partial\Omega \to \mathbb{R}.$$

The position of the soap film is given by the graph of a function $u : \overline{\Omega} \to \mathbb{R}$. This function must satisfy the Dirichlet boundary condition

$$u(x) = \gamma(x), \quad x \in \partial\Omega.$$

To determine a partial differential equation for the soap film we use that the potential energy of the soap film is proportional to the surface of the soap film:

$$\Psi(u) = \int_\Omega \sqrt{1 + |\nabla u(x)|^2}\, dx.$$

If we assume that the shape of the soap film can be described by the graph of a continuously differentiable function, then u must be determined such that

$$\Psi(u) = \min\{\Psi(v) \mid v \in C_1(\Omega \cap C(\overline{\Omega})\}.$$

A soap film is therefore a minimal surface. We are thus in the same situation as in Section 1.3 with the special energy density $\psi(\xi) = \sqrt{1 + |\xi|^2}$ and the volume force $b = 0$. Consequently u must satisfy the Dirichlet problem (1.13), (1.14) with $F(\xi) = \nabla\psi(\xi) = \frac{\xi}{\sqrt{1+|\xi|^2}}$. For this function the boundary value problem becomes

$$-\mathrm{div}\Big(\frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}}\Big) = 0, \qquad x \in \Omega, \tag{1.15}$$

$$u(x) = \gamma(x), \qquad x \in \partial\Omega. \tag{1.16}$$

This is the partial differential equation for minimal surfaces. If we assume that $u \in C_2(\Omega)$, we obtain from the chain rule

$$\begin{aligned}
\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u(x)|^2}}\Big) &= \frac{1}{\sqrt{1+|\nabla u(x)|^2}}\,\Delta u \\
&\quad -\frac{1}{\sqrt{1+|\nabla u(x)|^2}^3}\,\nabla u(x)\cdot\big(\nabla^2 u(x)\big)\nabla u(x) \\
&= \frac{(1+|\nabla u(x)|^2)\Delta u(x) - \nabla u(x)\cdot\big(\nabla^2 u(x)\big)\nabla u(x)}{\sqrt{1+|\nabla u(x)|^2}^3} = 0.
\end{aligned}$$

This differential operator yields the mean curvature of the graph of the function u at the point $\big(x, u(x)\big)$. In the one-dimensional case the equation reduces to

$$\frac{u_{xx}}{\sqrt{1+u_x^2}^3} = 0.$$

Therefore the mean curvature of a minimal surface vanishes everywhere.

1.5 Uniqueness of solutions of the boundary value problem and convexity

We showed that a two times differentiable solution φ of the minimization problem (1.12) is a solution of the Dirichlet problem (1.13), (1.14). Here we discuss under what conditions a solution of (1.13), (1.14) is also a solution of (1.12) and under what condition solutions of (1.13), (1.14) are unique.

Definition 1.2 A function $\psi \in C_1(\mathbb{R}^n, \mathbb{R})$ is convex, if for all $\eta, \xi \in \mathbb{R}^n$

$$\psi(\eta) \geq \nabla\psi(\eta)\cdot(\eta - \xi) + \psi(\xi).$$

ψ is strictly convex, if strict inequality holds for $\eta \neq \xi$.

Theorem 1.3 *If ψ is convex, then a two times differentiable function φ is a solution of the Dirichlet problem* (1.13), (1.14) *if and only if φ is a solution of the minimization problem* (1.12).

If ψ is strictly convex, then solutions of the Dirichlet problem (1.13), (1.14) *and of the minimization problem* (1.12) *are unique.*

Proof. Let ψ be convex and assume that φ is a two times differentiable solution of (1.13), (1.14). Let $\hat{\varphi} : \Omega \to \mathbb{R}$ be an arbitrary differentiable function satisfying $\hat{\varphi}|_{\partial\Omega} = \gamma_D$. Then

$(\hat{\varphi} - \varphi)|_{\partial\Omega} = 0$ and the convexity condition yield

$$\begin{aligned}
\Psi(\hat{\varphi}) &= \int_\Omega \psi\big(\nabla_x \hat{\varphi}(x)\big)dx - \int_\Omega \hat{\varphi}(x) \cdot b(x)dx \\
&\geq \int_\Omega \nabla\psi\big(\nabla_x \varphi(x)\big) \cdot \big(\nabla_x \hat{\varphi}(x) - \nabla_x \varphi(x)\big) + \psi\big(\nabla\varphi(x)\big)dx \qquad (1.17)\\
&\quad - \int_\Omega \big(\hat{\varphi}(x) - \varphi(x)\big) \cdot b(x) + \varphi(x) \cdot b(x)dx \\
&= \int_\Omega \Big(-\text{div}\big(\nabla_x \varphi(x)\big) - b(x)\Big)\big(\hat{\varphi}(x) - \varphi(x)\big)dx + \Psi(\varphi) = \Psi(\varphi).
\end{aligned}$$

This inequality implies that φ is the global minimum of Ψ and therefore solves the minimization problem (1.12).

If ψ is strictly convex and if $\hat{\varphi} \neq \varphi$, then there are points $x \in \Omega$ with $\nabla\hat{\varphi}(x) \neq \nabla\varphi(x)$. For these x we have

$$\nabla\psi\big(\nabla_x \varphi(x)\big) \cdot \big(\nabla\hat{\varphi}(x) - \nabla_x \varphi(x)\big) + \psi\big(\nabla\varphi(x)\big) > 0.$$

This implies that in (1.17) the strict inequality sign holds. Consequently, φ is the only minimum of Ψ, and therefore the minimization problem and the Dirichlet problem have at most one solution. ∎

In the next section we show that the function ψ is convex if and only if the vector field $F = \nabla\psi$ appearing in (1.13) is monotone. Therefore the uniqueness condition (1.11) for the diffusion problems (1.5) – (1.9) and the strict convexity condition guaranteeing uniqueness of the solution of the boundary value problem (1.13), (1.14) are closely related. We shall show in Sections 3 and 4 that monotonicity conditions are also needed in the existence theory to these problems. Because of this important role of monotonicity, we investigate now the relations between various monotonicity and convexity conditions more closely.

1.6 Convexity and monotonicity

Definition 1.4 Let $G : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field.

(i) G is called monotone, if for all $\xi, \eta \in \mathbb{R}^n$

$$\big(G(\xi) - G(\eta)\big) \cdot (\xi - \eta) \geq 0.$$

(ii) G is called strictly monotone, if for all $\xi, \eta \in \mathbb{R}^n$, $\xi \neq \eta$,

$$\big(G(\xi) - G(\eta)\big) \cdot (\xi - \eta) > 0.$$

(iii) G is called strongly monotone, if there is $c > 0$ such that for all $\xi, \eta \in \mathbb{R}^n$

$$\big(G(\xi) - G(\eta)\big) \cdot (\xi - \eta) \geq c|\xi - \eta|^2.$$

Definition 1.5 Let $\psi \in C^2(\mathbb{R}^2, \mathbb{R})$.

(i) $\nabla^2\psi(\xi)$ is uniformly positive definite if there is $c > 0$ such that

$$\eta \cdot \big[\nabla^2\psi(\xi)\big]\eta \geq c|\eta|^2$$

for all $\xi, \eta \in \mathbb{R}^2$.

(ii) ψ is uniformly convex if there is $c > 0$ such that

$$\psi(\xi) \geq \psi(\eta) + \big[\nabla\psi(\eta)\big](\xi - \eta) + \frac{c}{2}|\xi - \eta|^2$$

for all $\xi, \eta \in \mathbb{R}^2$.

Theorem 1.6 *Let $\psi \in C^2(\mathbb{R}^2, \mathbb{R})$. Then the following conditions are equivalent:*

(i) $\nabla^2\psi(\xi)$ *is uniformly positive definite.*

(ii) *The vector field $\xi \mapsto \nabla\psi(\xi) : \mathbb{R}^2 \to \mathbb{R}^2$ is strongly monotone.*

(iii) ψ *is uniformly convex.*

Proof. Let $\xi, \eta \in \mathbb{R}^2$. Then

$$\begin{aligned}\big(\nabla\psi(\xi) - \nabla\psi(\eta)\big) \cdot (\xi - \eta) &= \int_0^1 \frac{d}{ds}\nabla\psi\big(s\xi + (1-s)\eta\big) \cdot (\xi - \eta)ds \\ &= \int_0^1 (\xi - \eta) \cdot \nabla^2\psi\big(s\xi + (1-s)\eta\big)(\xi - \eta)ds.\end{aligned}$$

If $\nabla^2\psi$ is uniformly positive definite, it follows from this formula that

$$\big(\nabla\psi(\xi) - \nabla\psi(\eta)\big) \cdot (\xi - \eta) \geq \int_0^1 c|\xi - \eta|^2 ds = c|\xi - \eta|^2,$$

whence $\nabla\psi$ is a strongly monotone vector field.

We also have

$$\begin{aligned}\psi(\xi) - \psi(\eta) &= \int_0^s \frac{d}{ds}\psi\big(s(\xi - \eta) + \eta\big)ds \\ &= \int_0^1 \nabla\psi\big(s(\xi - \eta) + \eta\big) \cdot (\xi - \eta)ds.\end{aligned}$$

If $\nabla\psi$ is a strongly monotone vector field we thus obtain

$$\begin{aligned}
&\psi(\xi) - \psi(\eta) - \nabla\psi(\eta)\cdot(\xi-\eta) \\
&\quad = \int_0^1 \Big(\nabla\psi\big(s(\xi-\eta)+\eta\big) - \nabla\psi(\eta)\Big)\cdot(\xi-\eta)ds \\
&\quad \geq \int_0^1 sc|(\xi-\eta)|^2\, ds = \frac{c}{2}|\xi-\eta|^2,
\end{aligned}$$

and therefore ψ is uniformly convex. We next use Taylor's formula

$$\begin{aligned}
\psi(\xi) &= \psi(\eta) + \frac{d}{ds}\,\psi\big(s(\xi-\eta)+\eta\big)_{|_{s=0}} + \int_0^1 (1-s)\,\frac{d^2}{ds^2}\,\psi\big(s(\xi-\eta)+\eta\big)ds \\
&= \psi(\eta) + \nabla\psi(\eta)\cdot(\xi-\eta) + \int_0^1 (1-s)(\xi-\eta)\cdot\nabla^2\psi\big(s(\xi-\eta)+\eta\big)(\xi-\eta)ds.
\end{aligned}$$

If ψ is uniformly convex, we thus obtain from this equation that

$$\frac{c}{2}\,|\xi-\eta|^2 \leq \int_0^1 (1-s)(\xi-\eta)\cdot\Big[\nabla^2\psi\big(s(\xi-\eta)+\eta\big)\Big](\xi-\eta)ds. \tag{1.18}$$

Let $\zeta \in \mathbb{R}^2$ be an arbitrary vector different from zero and let $t > 0$. Then

$$\lim_{t\to 0}(\nabla^2\psi)(st\zeta+\eta) = (\nabla^2\psi)(\eta),$$

uniformly with respect to $s \in [0,1]$. Thus,

$$\begin{aligned}
\zeta\cdot\big[\nabla^2\psi(\eta)\big]\zeta &= 2\int_0^1 (1-s)\zeta\cdot\big[\nabla^2\psi(\eta)\big]\zeta\, ds \\
&= 2\int_0^1 (1-s)\lim_{t\to 0}\zeta\cdot\big[\nabla^2\psi(st\zeta+\eta)\big]\zeta\, ds \\
&= \lim_{t\to 0}\frac{2}{t^2}\int_0^1 (1-s)\, t\zeta\cdot\big[\nabla^2\psi(st\zeta+\eta)\big]\, t\zeta ds \\
&\geq \lim_{t\to 0}\frac{1}{t^2}\, c|t\zeta|^2 = c|\zeta|^2,
\end{aligned}$$

where we used (1.18) with $\xi = \eta + t\zeta$. This shows that $\nabla^2\psi(\eta)$ is uniformly positive definite.

The foregoing proof is valid for $c = 0$. This yields the following

Corollary 1.7 *For $\psi \in C^2(\mathbb{R}^2,\mathbb{R})$ the following statements are equivalent*

(i) $\nabla^2\psi(\xi)$ *is positive semi-definite for all* $\xi \in \mathbb{R}^2$.

(ii) *The vector field* $\xi \mapsto \nabla\psi(\xi) : \mathbb{R}^3 \to \mathbb{R}^3$ *is monotone.*

(iii) ψ *is convex.*

Examples. 1. $\psi(\xi) = \frac{1}{p+2}\,|\xi|^{p+2}$ satisfies

$$\begin{aligned}
\nabla\psi(\xi) &= |\xi|^{p+1}\,\frac{\xi}{|\xi|} = |\xi|^p\,\xi, \\
\nabla^2\psi(\xi) &= \nabla(|\xi|^p\xi) = \xi \otimes \big(p|\xi|^{p-1}\,\frac{\xi}{|\xi|}\big) + |\xi|^p\ I \\
&= p|\xi|^{p-2}(\xi\otimes\xi) + |\xi|^p I,
\end{aligned}$$

with the 3×3-identity matrix I. For $\eta \in \mathbb{R}^3$ with $\eta \neq 0$ we thus obtain

$$\eta \cdot \big[\nabla^2\psi(\xi)\big]\eta = p|\xi|^{p-2}(\xi\cdot\eta)^2 + |\xi|^p\,|\eta|^2 > 0.$$

Consequently, $\nabla^2\psi(\xi)$ is positive definite for all $\xi \neq 0$ and positive semi-definite for all $\xi \in \mathbb{R}^3$.

2. For $\psi(\xi) = \frac{c}{2}\,|\xi|^2$ we obtain

$$\begin{aligned}
\nabla\psi(\xi) &= c|\xi|\,\frac{\xi}{|\xi|} = c\xi, \\
\nabla^2\psi(\xi) &= \nabla(c\xi) = cI.
\end{aligned}$$

The partial differential equation to this example is

$$-\mathrm{div}(c\nabla\varphi) = -c\Delta\varphi = b.$$

This is Poisson's equation.

2 Sobolev spaces

2.1 The Banach space $L^p(\Omega)$. Fundamental lemma of the calculus of variations.

Let $1 \le p < \infty$ and let $\Omega \subseteq \mathbb{R}^n$ be a nonempty measurable set. $L^p(\Omega) = L^p(\Omega, \mathbb{R})$ is the set of all functions, whose p-th power is integrable:

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \text{ is measurable, } \int_{\mathbb{R}^n} |f(x)|^p \, dx < \infty\}.$$

We show that $L^p(\Omega)$ is a vector space:

Theorem 2.1 (Hölder's inequality) *Let $1 < p,\ q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and let $f \in L^p(\Omega)$, $g \in L^q(\Omega)$. Then the product $f \cdot g$ is integrable and*

$$\Big| \int_\Omega f(x)g(x)\, dx \Big| \le \Big(\int_\Omega |f(x)|^p \, dx \Big)^{1/p} \Big(\int_\Omega |g(x)|^q \, dx \Big)^{1/q}.$$

Proof. For $a, b \ge 0$ Young's inequality states

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}\,.$$

With

$$a = \frac{|f(x)|}{\big(\int_\Omega |f(x)|^p \, dx \big)^{1/p}}\,, \quad b = \frac{|g(x)|}{\big(\int_\Omega |g(x)|^q \, dx \big)^{1/q}}$$

we obtain from this inequality that

$$\frac{|f(x)g(x)|}{\big(\int_\Omega |f(x)|^p \, dx \big)^{1/p} \big(\int_\Omega |g(x)|^q \, dx \big)^{1/q}} \le \frac{|f(x)|^p}{p \int_\Omega |f(x)|^p \, dx} + \frac{|g(x)|^q}{q \int_\Omega |g(x)|^q \, dx}\,.$$

Since the right hand side is integrable, we conclude that fg is integrable and that

$$\frac{\int_\Omega |f(x)g(x)|\, dx}{\big(\int_\Omega |f(x)|^p \, dx \big)^{1/p} \int_\Omega |g(x)|^q \, dx \big)^{1/q}} \le \frac{\int_\Omega |f(x)|^p \, dx}{p \int_\Omega |f(x)|^p \, dx} + \frac{\int_\Omega |g(x)|^q \, dx}{q \int_\Omega |g(x)|^q \, dx} = 1\,.$$

This shows that Hölder's inequality holds. ■

Corollary 2.2 (Minkowski's inequality) *Let $f, g \in L^p(\Omega)$. Then $f + g \in L^p(\Omega)$ and*

$$\Big(\int_\Omega |f(x) + g(x)|^p \, dx \Big)^{1/p} \le \Big(\int_\Omega |f(x)|^p \, dx \Big)^{1/p} + \Big(\int_\Omega |g(x)|^p \, dx \Big)^{1/p}.$$

Proof. The function $\xi \mapsto |\xi|^p : \mathbb{R} \to [0, \infty)$ is convex. This yields for $a, b \in \mathbb{R}$ that

$$|a+b|^p = 2^p \, |\frac{1}{2}a + \frac{1}{2}b|^p \le 2^p\big(\frac{1}{2}|a|^p + \frac{1}{2}|b|^p\big) = 2^{p-1}\big(|a|^p + |b|^p\big).$$

We use this inequality and $q = \frac{p}{p-1}$ to conclude

$$\big(|f(x)+g(x)|^{p-1}\big)^q = |f(x)+g(x)|^p \le 2^p\big(|f(x)|^p + |g(x)|^p\big).$$

This implies that $f+g \in L^p(\Omega)$ and $|f+g|^{p-1} \in L^q(\Omega)$. Hölder's inequality thus yields

$$\begin{aligned}
\int_\Omega |f(x)+g(x)|^p \, dx &= \int_\Omega |f(x)+g(x)|^{p-1} \, |f(x)+g(x)| \, dx \\
&\le \int_\Omega |f(x)+g(x)|^{p-1} \big(|f(x)| + |g(x)|\big) \, dx \\
&\le \Big(\int_\Omega |f(x)+g(x)|^{q(p-1)} \, dx\Big)^{1/q} \bigg(\Big(\int_\Omega |f(x)|^p \, dx\Big)^{1/p} + \Big(\int_\Omega |g(x)|^p \, dx\Big)^{1/p}\bigg) \\
&= \Big(\int_\Omega |f(x)+g(x)|^p \, dx\Big)^{1-\frac{1}{p}} \bigg(\Big(\int_\Omega |f(x)|^p \, dx\Big)^{1/p} + \Big(\int_\Omega |g(x)|^p \, dx\Big)^{1/p}\bigg).
\end{aligned}$$

We divide by $\big(\int_\Omega |f(x)+g(x)|^p \, dx\big)^{1-\frac{1}{p}}$ to obtain Minkowski's inequality. ■

For $1 \le p < \infty$, $f \in L^p(\Omega)$ and $\lambda \in \mathbb{R}$ we have that $\lambda f \in L^p(\Omega)$. Moreover, if $f, g \in L^p(\Omega)$ then $f+g \in L^p(\Omega)$. This is obvious for $p = 1$ and follows from Corollary 2.2 for $1 < p < \infty$. Consequently, $L^p(\Omega)$ is a vector space. Also, for

$$\|f\|_p = \|f\|_{p,\Omega} = \Big(\int_\Omega |f(x)|^p \, dx\Big)^{1/p}$$

the triangle inequality $\|f+g\|_p \le \|f\|_p + \|g\|_p$ holds. Again, this is obvious for $p = 1$ and follows otherwise from Corollary 2.2.

Corollary 2.3 *$L^p(\Omega)$ is a normed vector space for $1 \le p < \infty$ with the norm $\|f\|_{p,\Omega}$.*

Theorem 2.4 (Fischer-Riesz) *$L^p(\Omega)$ is a Banach space for $1 \le p < \infty$, i.e. the vector space $L^p(\Omega)$ is complete with the norm $\|f\|_{p,\Omega}$.*

The proof can be found in the book "Lineare Funktionalanalysis" of H.W. Alt, Springer Verlag Berlin, 1999, p. 49, 50, and also in my lecture notes "Variationsrechnung und Sobolevräume", which are online available.

Our next goal is to study the approximation of functions from $L^p(\Omega)$ by continuous functions and even by infinitely differentiable functions.

Definition 2.5 Let $\Omega \subseteq \mathbb{R}^n$ be open. For $m \in \mathbb{N}_0 \cup \{\infty\}$ we define the vector spaces

$$\begin{aligned} C_m(\Omega) = C_m(\Omega, \mathbb{R}) &= \{f : \Omega \to \mathbb{C} \mid D^\alpha f \text{ exists and is continuous} \\ &\qquad\qquad \text{for all } \alpha \in \mathbb{N}_0^n \text{ such that } |\alpha| \le m\}, \\ C_m(\overline{\Omega}) &= \{f \in C_m(\Omega) \mid D^\alpha f \text{ can be extended continuously} \\ &\qquad\qquad \text{up to the boundary}\}, \\ \overset{\circ}{C}_m(\Omega) &= \{\varphi \in C_\infty(\mathbb{R}^n) \mid \operatorname{supp}\varphi \text{ is a compact subset of } \Omega\}. \end{aligned}$$

For $m = 0$ we also write $C(\Omega) = C_0(\Omega)$ and $\overset{\circ}{C}(\Omega) = \overset{\circ}{C}_0(\Omega)$.

Definition 2.6 A family of functions $\{\varphi_\varepsilon\}_{\varepsilon>0} \subseteq L^1(\mathbb{R}^n)$ satisfying $\varphi_\varepsilon \ge 0$, $\int_{\mathbb{R}^n} \varphi_\varepsilon(x)\, dx = 1$ and

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\delta(0)} \varphi_\varepsilon(x) dx = 0, \quad \text{for all } \delta > 0,$$

is called Dirac family.

Example. Choose $\varphi \in L^1(\mathbb{R}^n)$ with $\varphi \ge 0$ and $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$. Then $\{\varphi_\varepsilon\}_{\varepsilon>0}$ defined by

$$\varphi_\varepsilon(x) = \varepsilon^{-n}\, \varphi\Big(\frac{x}{\varepsilon}\Big) \tag{2.1}$$

is a Dirac family. In particular, if we define

$$\psi(x) = \begin{cases} \exp\big(-\frac{1}{1-|x|^2}\big), & |x| < 1 \\ 0, & |x| \ge 1 \end{cases}$$

and set

$$\varphi(x) = \frac{1}{\|\psi\|_1}\, \psi(x),$$

then (2.1) yields a Dirac family $\{\varphi_\varepsilon\}_{\varepsilon>0}$ with $\varphi_\varepsilon \in \overset{\circ}{C}_\infty(\mathbb{R}^n)$, $\operatorname{supp}\varphi_\varepsilon = \overline{B_\varepsilon(0)}$.

Roughly speaking, functions in a Dirac family approximate the Dirac distribution, in the sense that under weak assumptions for f the function f_ε defined by

$$f_\varepsilon(x) = \int_{\mathbb{R}^n} \varphi_\varepsilon(x-y)\, f(y) dy$$

converges to f for $\varepsilon \to 0$. Since for $\varphi_\varepsilon \in \overset{\circ}{C}_\infty(\Omega)$ the function f_ε is infinitely differentiable, this opens the possibility to approximate f by infinitely differentiable functions. In the following lemmas and theorems this idea is carried through rigorously.

Theorem 2.7 *Let $1 \le p \le \infty$, $\varphi \in L^1(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n)$. Then for almost all $x \in \mathbb{R}^n$ the integral*

$$F(x) = \int_{\mathbb{R}^n} \varphi(x-y)\, f(y)\, dy = \int_{\mathbb{R}^n} \varphi(y)\, f(x-y)\, dy$$

exists and satisfies

$$\|F\|_{L^p} \le \|\varphi\|_{L^1} \, \|f\|_{L^p} \,. \tag{2.2}$$

A **proof** of this theorem can be found for example in my lecture notes "Introduction to the theory of linear partial differential equations", which are online available.

One uses the notation

$$(\varphi * f)(x) = \int_{\mathbb{R}^n} \varphi(x-y)\, f(y)\, dy.$$

The operator $*$ is called convolution.

Lemma 2.8 *For $f \in L^p(\mathbb{R}^n)$ and $\varphi \in \overset{\circ}{C}_\infty(\mathbb{R}^n)$ the function $F = \varphi * f$ is infinitely differentiable and the partial derivatives are obtained by differentiation under the integral sign:*

$$D^\alpha F(x) = \int_{\mathbb{R}^n} D_x^\alpha \varphi(x-y)\, f(y)\, dy, \quad \alpha \in \mathbb{N}_0^n.$$

The **proof** follows from standard theorems on differentiation of integrals with respect to parameters and is left to the reader.

Lemma 2.9 *Let $1 \le p < \infty$ and let $\{\varphi_\varepsilon\}_{\varepsilon>0}$ be a Dirac family. For $f \in \overset{\circ}{C}(\mathbb{R}^n)$ we have*

$$\lim_{\varepsilon \to 0} \|\varphi_\varepsilon * f - f\|_{p,\mathbb{R}^n} = 0.$$

Proof. For $\delta > 0$ set

$$\begin{aligned} \varphi_{\varepsilon\delta}(x) &= \begin{cases} \varphi_\varepsilon(x), & |x| < \delta, \\ 0, & |x| > \delta, \end{cases} \\ \psi_{\varepsilon\delta}(x) &= \begin{cases} 0, & |x| \le \delta, \\ \varphi_\varepsilon(x), & |x| > \delta. \end{cases} \end{aligned}$$

Since $\int_{\mathbb{R}^n} \varphi_\varepsilon(x) dx = 1$ we have

$$\begin{aligned} (\varphi_\varepsilon * f)(x) - f(x) &= \int_{\mathbb{R}^n} \varphi_\varepsilon(y) \big(f(x-y) - f(x)\big)\, dy \\ &= \int_{\mathbb{R}^n} \varphi_{\varepsilon\delta}(y) \big(f(x-y) - f(x)\big)\, dy + \int_{\mathbb{R}^n} \psi_{\varepsilon\delta}(y) f(x-y) dy \\ &\quad - f(x) \int_{\mathbb{R}^n} \psi_{\varepsilon\delta}(y) dy. \end{aligned} \tag{2.3}$$

Note that

$$\Big|\int_{\mathbb{R}^n}\varphi_{\varepsilon\delta}(y)\big(f(x-y)-f(x)\big)dy\Big|$$
$$\leq \sup_{|y|\leq\delta}|f(x-y)-f(x)|\int_{\mathbb{R}^n}\varphi_{\varepsilon\delta}(y)dy \leq \sup_{|y|\leq\delta}|f(x-y)-f(x)|, \qquad (2.4)$$

where the right hand side vanishes for x outside of the bounded set

$$\Gamma_\delta = \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, \operatorname{supp} f) < \delta\}.$$

Thus, (2.2), (2.3) and (2.4) imply

$$\|\varphi_\varepsilon * f - f\|_p$$
$$\leq \|\sup_{y\in\Gamma_\delta}|f(x-y)-f(x)|\,\|_\infty \Big(\int_{\Gamma_\delta} dx\Big)^{1/p} + 2\|\psi_{\varepsilon\delta}\|_1\,\|f\|_p. \qquad (2.5)$$

Let $\eta > 0$ be given. Since $f \in \overset{\circ}{C}(\mathbb{R}^n)$ is uniformly continuous, there is $\delta > 0$ such that $|f(x-y)-f(x)| < \eta$ for all $x \in \mathbb{R}^n$ and all $|y| < \delta$. Since $\{\varphi_\varepsilon\}_{\varepsilon>0}$ is a Dirac family, for fixed δ we can choose $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$

$$\|\psi_{\varepsilon\delta}\|_1 = \int_{|x|>\delta}\varphi_\varepsilon(x)dx < \eta.$$

Together we obtain form (2.5) that

$$\|\varphi_\varepsilon * f - f\|_p \leq C\eta, \quad 0 < \varepsilon \leq \varepsilon_0,$$

with the constant $C = \big(\int_{\Gamma_1} dx\big)^{1/p} + \|f\|_p$, which is independent of η. Since $\eta > 0$ was chosen arbitrarily, the statement of the lemma follows from this inequality. ■

Theorem 2.10 *Let $1 \leq p < \infty$, let $\{\varphi_\varepsilon\}_{\varepsilon>0}$ be a Dirac family and let $f \in L^p(\mathbb{R}^n)$. Then we have*

$$\lim_{\varepsilon\to 0}\|\varphi_\varepsilon * f - f\|_{p,\mathbb{R}^n} = 0.$$

Proof. It follows from Lebesgue integration theory that there is a sequence $\{f_k\}_{k=1}^\infty \subseteq \overset{\circ}{C}(\mathbb{R}^n)$ such that $\|f - f_k\|_{p,\mathbb{R}^n} \to 0$ for $k \to \infty$. We thus obtain

$$\begin{aligned}
\|\varphi_\varepsilon * f - f\|_{p,\mathbb{R}^n} &\leq \|\varphi_\varepsilon * f - \varphi_\varepsilon * f_k\|_{p,\mathbb{R}^n} + \|\varphi_\varepsilon * f_k - f_k\|_{p,\mathbb{R}^n} + \|f_k - f\|_{p,\mathbb{R}^n} \\
&= \|\varphi_\varepsilon * (f - f_k)\|_{p,\mathbb{R}^n} + \|\varphi_\varepsilon * f_k - f_k\|_{p,\mathbb{R}^n} + \|f_k - f\|_{p,\mathbb{R}^n} \\
&\leq \big(\|\varphi_\varepsilon\|_{1,\mathbb{R}^n} + 1\big)\,\|f - f_k\|_{p,\mathbb{R}^n} + \|\varphi_\varepsilon * f_k - f_k\|_{p,\mathbb{R}^n} \\
&= 2\|f - f_k\|_{p,\mathbb{R}^n} + \|\varphi_\varepsilon * f_k - f_k\|_{p,\mathbb{R}^n}.
\end{aligned}$$

For $\vartheta > 0$ choose $k \in \mathbb{N}$ with

$$\|f - f_k\|_{p,\mathbb{R}^n} < \frac{1}{4}\vartheta.$$

To this k choose ε such that

$$\|\varphi_\varepsilon * f_k - f_k\|_{p,\mathbb{R}^n} < \frac{1}{2}\vartheta.$$

It follows that

$$\|\varphi_\varepsilon * f - f\|_{p,\mathbb{R}^n} \leq \frac{1}{2}\vartheta + \frac{1}{2}\vartheta = \vartheta.$$

■

Theorem 2.11 *Let $\Omega \subseteq \mathbb{R}^n$ be open and let $1 \leq p < \infty$. Then $\overset{\circ}{C}_\infty(\Omega)$ is dense in $L^p(\mathbb{R}^n)$.*

Proof. Choose a Dirac family $\{\varphi_\varepsilon\}_{\varepsilon>0} \subseteq \overset{\circ}{C}_\infty(\mathbb{R}^n)$ with $\operatorname{supp}\varphi_\varepsilon \subseteq \overline{B_\varepsilon}(0)$. Let $f \in L^p(\Omega)$ and $\vartheta > 0$. The assertion follows if we can show that there is $g \in \overset{\circ}{C}_\infty(\Omega)$ with $\|f - g\|_{p,\Omega} \leq \vartheta$. To prove this let $\delta > 0$ and set

$$\begin{aligned} \Omega_\delta &= \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \delta,\ |x| < \frac{1}{\delta}\}, \\ f_\delta(x) &= \begin{cases} f(x), & x \in \Omega_\delta \\ 0, & x \in \mathbb{R}^n \backslash \Omega_\delta. \end{cases} \end{aligned}$$

The dominated convergence theorem implies

$$\lim_{\delta\to 0} \|f - f_\delta\|_{p,\Omega}^p = \lim_{\delta \to 0} \int_\Omega |f(x) - f_\delta(x)|^p dx = 0,$$

since $\lim_{\delta\to 0}\big(f(x) - f_\delta(x)\big) = 0$ for all $x \in \Omega$ and since $|f(x) - f_\delta(x)| \leq |f(x)|$, hence $|f|$ is an integrable dominating function for the family $\{f - f_\delta\}_{\delta>0}$. We can thus choose $\delta > 0$ such that

$$\|f - f_\delta\|_{p,\Omega} < \frac{\vartheta}{2}.$$

Define $f_{\delta,\varepsilon} = \varphi_\varepsilon * f_\delta$. Since $f_\delta \in L^p(\mathbb{R}^n)$, there is $\varepsilon > 0$ such that

$$\|f_\delta - f_{\delta,\varepsilon}\|_{p,\Omega} < \frac{\vartheta}{2},$$

for all $\varepsilon \leq \varepsilon_0$, by the preceding theorem. $f_{\delta,\varepsilon}$ belongs to $C_\infty(\mathbb{R}^n)$ and for $\varepsilon < \min(\varepsilon_0, \delta)$ and $x \in \mathbb{R}^n$ with $\operatorname{dist}(x, \Omega_\delta) > \varepsilon$

$$f_{\delta,\varepsilon}(x) = \int_{\mathbb{R}^n} \varphi_\varepsilon(x-y) f_\delta(y) dy = \int_{\Omega_\delta} \varphi_\varepsilon(x-y) f_\delta(y) dy = 0,$$

since $y \in \Omega_\delta$ implies $|x-y| > \text{dist}(x, \Omega_\delta) > \varepsilon$, which yields $\varphi_\varepsilon(x-y) = 0$. It thus follows that

$$\begin{aligned} \text{supp} f_{\delta,\varepsilon} &\subseteq \{x \in \mathbb{R}^n \mid \text{dist}(x, \Omega_\delta) \leq \varepsilon\} \\ &= \{x \in \Omega \mid \text{dist}(x, \partial\Omega) \geq \delta - \varepsilon, \ |x| \leq \frac{1}{\delta} + \varepsilon\}. \end{aligned}$$

The last set is a compact subset of Ω, so $f_{\delta,\varepsilon} \in \overset{\circ}{C}_\infty(\Omega)$. Together we conclude

$$\|f - f_{\delta,\varepsilon}\|_{p,\Omega} \leq \|f - f_\delta\|_{p,\Omega} + \|f_\delta - f_{\delta,\varepsilon}\|_{p,\Omega} < \vartheta.$$

■

Lemma 2.12 (Fundamental lemma of the calculus of variations) *Let Ω be an open subset of $\mathbb{R}^n$ and assume that $g \in L^{1,\text{loc}}(\Omega)$ satisfies*

$$\int_\Omega g(x)\,\varphi(x)\,dx = 0$$

for all $\varphi \in C^0_\infty(\Omega)$. Then $g(x) = 0$ for almost all $x \in \Omega$.

Proof. Let $E \subseteq \Omega$ be bounded and measurable such that $E \subseteq \Omega$, whence $\text{dist}(E, \partial\Omega) = \delta > 0$. Let $\{\varphi_\varepsilon\}_{\varepsilon>0}$ be a Dirac family with $\varphi_\varepsilon \in \overset{\circ}{C}_\infty\big(B_\varepsilon(0)\big)$, where $B_\varepsilon(0) = \{x \in \mathbb{R}^n \mid |x| < \varepsilon\}$. Set

$$\Phi_\varepsilon(x) = \int_E \varphi_\varepsilon(x-y)dy = \varphi_\varepsilon * \chi_E.$$

By Theorem 2.10 we have $\lim_{\varepsilon\to 0} \|\Phi_\varepsilon - \chi_E\|_{1,\Omega} = 0$. A well known result of Lebesgue integration theory implies that there is a sequence $\{\varepsilon_k\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} \Phi_{\varepsilon_k}(x) - \chi_E(x) = 0 \tag{2.6}$$

for almost all $x \in \mathbb{R}^n$. Also

$$|\Phi_\varepsilon(x)| = \Big|\int_E \varphi_\varepsilon(x-y)dy\Big| \leq \int_{\mathbb{R}^n} \varphi_\varepsilon(x-y)dy = 1,$$

which yields for $g \in L^{1,\text{loc}}(\Omega)$ and

$$x \in E_{\delta/2} = \{x \in \mathbb{R}^n \mid \text{dist}(x, E) < \frac{\delta}{2}\} \subseteq \Omega$$

that

$$|g(x)\Phi_{\varepsilon_k}(x)| \leq |g(x)|. \tag{2.7}$$

Since $g \in L^1(E_{\delta/2})$, we see from (2.6), (2.7) that we can apply Lebesgue's dominated convergence theorem to conclude

$$\begin{aligned}\lim_{k\to\infty}\int_{E_{\delta/2}} g(x)\Phi_{\varepsilon_k}(x)dx &= \int_{E_{\delta/2}} \lim_{k\to\infty} g(x)\,\Phi_{\varepsilon_k}(x)\,dx \\ &= \int_{E_{\delta/2}} g(x)\chi_E(x)\,dx = \int_E g(x)\,dx.\end{aligned} \tag{2.8}$$

On the other hand, for $|\varepsilon_k| < \frac{\delta}{2}$ we have $\Phi_{\varepsilon_k} \in \overset{\circ}{C}_\infty(E_{\delta/2})$, whence by assumption

$$\int_{E_{\delta/2}} g(x)\Phi_{\varepsilon_k}(x)\,dx = \int_\Omega g(x)\Phi_{\varepsilon_k}(x)\,dx = 0.$$

(2.8) thus yields

$$\int_E g(x)dx = 0.$$

For $\eta > 0$ let

$$\Omega_\eta = \{x \in \Omega \;\big|\; \operatorname{dist}(x, \partial\Omega) > \eta,\ |x| < \frac{1}{\eta}\}.$$

Choose

$$E_\pm = \{x \in \Omega_\eta \;\big|\; \pm g(x) > 0\}.$$

Then $E_\pm$ is bounded, measurable with $\overline{E_\pm} \subseteq \Omega$. Therefore we can replace E in the preceding considerations by $E_\pm$, and obtain

$$\int_{\Omega_\eta} |g(x)|\,dx = \int_{E_+} g(x)dx - \int_{E_-} g(x)\,dx = 0.$$

This implies $g = 0$ in Ω_η. Since $\Omega = \bigcup_{\eta>0}\Omega_\eta$, we conclude $g = 0$ on Ω. ■

2.2 Weak derivatives and Sobolev spaces

Definition 2.13 Assume that $\Omega \subseteq \mathbb{R}^n$ is an open set, let $u \in L^{1,\text{loc}}(\Omega)$ and $\alpha \in \mathbb{N}_0^n$. If there is a function $v \in L^{1,\text{loc}}(\Omega)$ such that

$$(-1)^{|\alpha|}(u, D^\alpha\varphi)_\Omega = (v, \varphi)_\Omega$$

for all $\varphi \in \overset{\circ}{C}_\infty(\Omega)$, then v is called α-th weak derivative of v.

The α-th weak derivative is uniquely determined. To see this, let $v_1, v_2 \in L^{1,\text{loc}}(\Omega)$ be weak derivatives of $u \in L^{1,\text{loc}}(\Omega)$. Then, for all $\varphi \in \overset{\circ}{C}_\infty(\Omega)$

$$(v_1, \varphi)_\Omega = (-1)^{|\alpha|}\,(u, D^\alpha\varphi)_\Omega = (v_2, \varphi)_\Omega\,,$$

hence

$$\int_\Omega \big(v_1(x) - v_2(x)\big)\varphi(x)\,dx = 0.$$

By the fundamental lemma of the calculus of variations this implies $v_1 = v_2$.

It is immediately seen by partial integration that for $u \in C_m(\Omega)$ and $|\alpha| \le m$ the α-th weak derivative coincides with the classical derivative $D^\alpha u$. Because of these results one uses the notation $D^\alpha u$ also for the weak derivative. Confusion is not possible, since the weak derivative is equal to the classical derivative, if the latter exists. We note that $L^{p,\mathrm{loc}}(\Omega) \subseteq L^{1,\mathrm{loc}}(\Omega)$ for $1 \le p < \infty$, since Hölder's inequality yields for $u \in L^{p,\mathrm{loc}}(\Omega)$ and every compact subset K of Ω that

$$\int_K |u(x)|\,dx \le \Big(\int_K 1\,dx\Big)^{1/q}\Big(\int_K |u(x)|^p\,dx\Big)^{1/p} < \infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Therefore Definition 2.13 applies to functions $u \in L^{p,\mathrm{loc}}(\Omega)$ and $u \in L^p(\Omega)$.

Definition 2.14 Assume that $1 \le p < \infty$, let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $m \in \mathbb{N}_0$. The vector space of functions

$$H^p_m(\Omega) = \{u \in L^p(\Omega) \mid \text{ the weak derivative } D^\alpha u \in L^p(\Omega) \text{ exists for } |\alpha| \le m\}$$

is called Sobolev space. For $u \in H^p_m(\Omega)$ set

$$\|u\|_{m,p} = \|u\|_{m,p,\Omega} = \Big(\sum_{|\alpha|\le m} \|D^\alpha u\|^p_{p,\Omega}\Big)^{1/p}.$$

Theorem 2.15 *$H^p_m(\Omega)$ is a Banach space with the norm $\|u\|_{m,p}$.*

Proof. It is immediately seen that $H^p_m(\Omega)$ is a vector space. The triangle inequality $\|u + v\|_{m,p} \le \|u\|_{m,p} + \|v\|_{m,p}$ follows from the triangle inequality for $\|u\|_p$ and from the discrete Minkowski inequality $\big(\sum_{i=1}^k (a_i + b_i)^p\big)^{1/p} \le \big(\sum_{i=1}^k a_i^p\big)^{1/p} + \big(\sum_{i=1}^k b_i^p\big)^{1/p}$, which can be proved as the Minkowski inequality. We leave the details to the reader. It remains to show that $H^p_m(\Omega)$ is complete. Thus, let $\{u_k\}_k$ be a Cauchy sequence in $H^p_m(\Omega)$. Since

$$\|u_\ell - u_k\|^p_{m,p} = \sum_{|\alpha|\le m} \|D^\alpha u_\ell - D^\alpha u_k\|^p_{m,p},$$

it follows that $\{D^\alpha u_k\}_k$ is a Cauchy sequence in $L^p(\Omega)$ for $|\alpha| \le m$. Because $L^p(\Omega)$ is complete, $\{D^\alpha u_k\}_k$ has a limit function $u^{(\alpha)} \in L^p(\Omega)$. We write $u = u^{(0)}$ and show that $u^{(\alpha)} = D^\alpha u$ for $0 < |\alpha| \le m$. To this end let $\varphi \in \overset{\circ}{C}_\infty(\Omega)$. Then

$$(-1)^{|\alpha|}(u, D^\alpha\varphi)_\Omega = \lim_{k\to\infty} (-1)^{|\alpha|}(u_k, D^\alpha\varphi)_\Omega = \lim_{k\to\infty} (D^\alpha u_k, \varphi)_\Omega = (u^{(\alpha)}, \varphi)_\Omega.$$

This implies $u^{(\alpha)} = D^\alpha u$. Consequently $\|u - u_k\|_m \to 0$ for $k \to \infty$, whence $H^p_m(\Omega)$ is complete. ∎

Lemma 2.16 (Leibniz rule for weak derivatives) *For $1 \leq p \leq \infty$ let $f \in H^p_m(\Omega)$ and let $\eta \in \overset{\circ}{C}_\infty(\Omega)$. Then $\eta f \in H^p_m(\Omega)$ and*

$$D^\alpha(\eta f) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \eta D^{\alpha-\beta} f.$$

We write $\beta \leq \alpha$ if and only if $\beta_i \leq \alpha_i$ for all i and use the notation

$$\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n}.$$

Proof. Let $\varphi \in \overset{\circ}{C}_\infty(\Omega)$. Then

$$\begin{aligned} \int_\Omega \eta f \frac{\partial}{\partial x_i} \varphi \, dx &= \int_\Omega f \frac{\partial}{\partial x_i} (\eta\varphi) dx - \int_\Omega f \big(\frac{\partial}{\partial x_i} \eta\big) \varphi \, dx \\ &= \int_\Omega \Big[\big(\frac{\partial}{\partial x_i} f\big)\eta + f \frac{\partial}{\partial x_i} \eta \Big] \varphi \, dx. \end{aligned}$$

Therefore ηf has the weak derivative

$$\big(\frac{\partial}{\partial x_i} f\big)\eta + \big(\frac{\partial}{\partial x_i} \eta\big) f \in L^p(\Omega).$$

For the second derivatives we obtain the statement by application of these arguments to $\big(\frac{\partial}{\partial x_i} f\big)\eta$ and to $f\big(\frac{\partial}{\partial x_i} \eta\big)$. The general statement follows by induction. ∎

Definition 2.17 Let $\Omega \subseteq \mathbb{R}^n$ be an open set. The closure of the linear subspace $\overset{\circ}{C}_\infty(\Omega)$ in $H^p_m(\Omega)$ is denoted by $\overset{\circ}{H}{}^p_m(\Omega)$.

$\overset{\circ}{H}{}^p_m(\Omega)$ is complete as a closed subspace of the complete space $H^p_m(\Omega)$. Therefore $\overset{\circ}{H}{}^p_m(\Omega)$ is a Banach space.

Theorem 2.18 (Poincaré inequality) *Let $1 \leq p < \infty$ and let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded subset. Let*

$$d = \operatorname{diam} \Omega = \sup_{x,y \in \Omega} |x - y|.$$

Then for every $u \in \overset{\circ}{H}{}^p_1(\Omega)$ we have

$$\|u\|_{p,\Omega} \leq p^{-1/p} \, d |u|_{p,1,\Omega},$$

where

$$|u|_{p,1,\Omega} = \Big(\sum_{|\alpha|=1} \|D^\alpha u\|^p_{p,\Omega} \Big)^{1/p}.$$

Proof. Let $\varphi \in \overset{\circ}{C}_\infty(\Omega)$, let $x = (x_1, \ldots, x_n) \in \Omega$ and let $y_1 \in \mathbb{R}$ be the smallest number with $y = (y_1, x_2, \ldots, x_n) \in \partial\Omega$. Then

$$\varphi(x) = \int_{y_1}^{x_1} \frac{\partial}{\partial x_1} \varphi(\xi, x_2, \ldots, x_n) d\xi.$$

Hölder's inequality yields

$$\begin{aligned} |\varphi(x)|^p &\leq \Big(\int_{y_1}^{x_1} \Big| \frac{\partial}{\partial x_1} \varphi(\xi, x_2, \ldots, x_n) \Big| \, d\xi \Big)^p \\ &\leq |x_1 - y_1|^{\frac{p}{q}} \int_{y_1}^{x_1} \Big| \frac{\partial}{\partial x_1} \varphi \Big|^p \, d\xi \\ &\leq |x_1 - y_1|^{\frac{p}{q}} \sum_{|\alpha|=1} \int_{-\infty}^{\infty} |D^\alpha \varphi(\xi, x_2, \ldots, x_n)|^p \, d\xi. \end{aligned}$$

Integration with respect to x_1 yields

$$\int_{y_1}^{y_1+d} |\varphi(x)|^p \, dx_1 \leq \frac{1}{1 + \frac{p}{q}} \, d^{1+\frac{p}{q}} \sum_{|\alpha|=1} \int_{-\infty}^{\infty} |D^\alpha \varphi(\xi, x_2, \ldots, x_n)^p \, dx.$$

We integrate with respect to the other variables and obtain with $1 + \frac{p}{q} = 1 + p(1 - \frac{1}{p}) = p$ that

$$\int_\Omega |\varphi(x)|^p \, dx \leq \frac{1}{p} \, d^p \sum_{|\alpha|=1} \int_\Omega |D^\alpha \, \varphi(x)|^p \, dx,$$

whence

$$\|\varphi\|_{p,\Omega} \leq p^{-\frac{1}{p}} \, d|\varphi|_{p,1,\Omega} \, .$$

The inequality stated in the lemma therefore holds for $\varphi \in \overset{\circ}{C}_\infty(\Omega)$. Now let $u \in \overset{\circ}{H}{}_1^p(\Omega)$. Choose a sequence $\{\varphi_k\}_{k=1}^\infty \subseteq \overset{\circ}{H}{}_1^p(\Omega)$ such that

$$\lim_{k\to\infty} \|u - \varphi_k\|_{p,1,\Omega} = 0.$$

Then

$$\begin{aligned} \|u\|_{p,\Omega} &\leq \lim_{k\to\infty} (\|u - \varphi_k\|_{p,\Omega} + \|\varphi_k\|_{p,\Omega}) \\ &= \lim_{k\to\infty} \|\varphi_k\|_{p,\Omega} \leq p^{-\frac{1}{p}} \, d \lim_{k\to\infty} |\varphi_k|_{p,1,\Omega} \\ &\leq p^{-\frac{1}{p}} \, d \lim_{k\to\infty} \big(|\varphi_k - u|_{p,1,\Omega} + |u|_{p,1,\Omega} \big) \\ &= p^{-\frac{1}{p}} \, d|u|_{p,1,\Omega}. \end{aligned}$$

■

Definition 2.19 Let $\Omega \subseteq \mathbb{R}^n$ be open, $1 \leq p \leq \infty$ and $k \in \mathbb{N}$. We define

$$H_m^p(\Omega, \mathbb{R}^k) = \left(H_m^p(\Omega, \mathbb{R})\right)^k.$$

and for $u \in H_1^p(\Omega, \mathbb{R}^n)$, $v \in H_1^p(\Omega, \mathbb{R})$ set

$$\text{div} u = \sum_{i=1}^n \frac{\partial}{\partial x_i} u_i, \quad \nabla v = \begin{pmatrix} \frac{\partial}{\partial x_1} v \\ \vdots \\ \frac{\partial}{\partial x_n} v \end{pmatrix}.$$

Lemma 2.20 *Let $\Omega \subseteq \mathbb{R}^n$ be open and let $1 < p < \infty$, $1 < q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. For $u \in H_1^p(\Omega, \mathbb{R}^n)$ and $v \in \overset{\circ}{H}{}_1^q(\Omega, \mathbb{R})$ we have*

$$(\text{div} u, v)_\Omega = -(u, \nabla v)_\Omega\,.$$

Proof. Let $\{\varphi_k\}_{k=1}^\infty \subseteq \overset{\circ}{C}_\infty(\Omega)$ be a sequence with $\lim_{k\to\infty} \|v - \varphi_k\|_{q,1,\Omega} = 0$. By definition of weak derivatives we have

$$(\text{div} u, \varphi_k)_\Omega = -(u, \nabla \varphi_k)_\Omega\,,$$

whence

$$(\text{div} u, v)_\Omega = \lim_{k\to\infty} (\text{div} u, \varphi_k)_\Omega = -\lim_{k\to\infty} (u, \nabla \varphi_k)_\Omega = -\lim_{k\to\infty} (u, \nabla \varphi)_\Omega\,.$$

■

2.3 Density of $C_\infty(\Omega)$ in $H_m^p(\Omega)$

By definition, $\overset{\circ}{C}_\infty(\Omega)$ is dense in $\overset{\circ}{H}{}_m^p(\Omega)$, but it is not dense in $H_m^p(\Omega)$, since $\overset{\circ}{H}{}_m^p(\Omega)$ is a proper closed subspace of $H_m^p(\Omega)$. Instead, in this section we show that $C_\infty(\Omega) \cap H_m^p(\Omega)$ is dense in $H_m^p(\Omega)$. To this end we first show that convolution of $u \in H_m^p(\Omega)$ with infinitely differentiable functions from a Dirac family can be used to approximate u by infinitely differentiable functions in the interior of Ω. Thus, let $1 \leq p < \infty$ and assume that $\Omega \subseteq \mathbb{R}^n$ is an open set. Choose a Dirac family $\{\varphi_\varepsilon\}_{\varepsilon>0} \subseteq \overset{\circ}{C}_\infty(\mathbb{R}^n)$ with $\text{supp}\, \varphi_\varepsilon \subseteq \overline{B_\varepsilon(0)}$. For $u \in H_m^p(\Omega)$ set

$$u_\varepsilon(x) = (\varphi_\varepsilon * u)(x) = \int_\Omega \varphi_\varepsilon(x-y) u(y) dy.$$

We then have $u_\varepsilon \in C_\infty(\mathbb{R}^n)$ and $\lim_{\varepsilon\to 0} \|u - u_\varepsilon\|_{p,\Omega} = 0$. Moreover, for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$ we have

$$D^\alpha u_\varepsilon(x) = \int_\Omega D_x^\alpha \varphi_\varepsilon(x-y) u(y) dy = (-1)^{|\alpha|} \int_\Omega \left[D_y^\alpha \varphi_\varepsilon(x-y)\right] u(y) dy.$$

For $\delta > 0$ set

$$\Omega_\delta = \{x \in \Omega \;\big|\; \operatorname{dist}(x, \partial\Omega) > \delta\}.$$

For $x \in \Omega_\delta$ and $\varepsilon < \delta$ we have $\operatorname{supp}\big(y \mapsto \varphi_\varepsilon(x-y)\big) \subseteq \overline{B_\varepsilon(x)} \subseteq \Omega$, whence

$$\big(y \mapsto \varphi_\varepsilon(x-y)\big) \in \overset{\circ}{C}_\infty(\Omega).$$

By definition of weak derivatives we can shift the derivative D^α from φ_ε to u in the integral on the right hand side, whence, for $x \in \Omega_\delta$ and $\varepsilon < \delta$,

$$D^\alpha u_\varepsilon(x) = \int_\Omega \varphi_\varepsilon(x-y)\, D^\alpha u(y) dy,$$

so

$$(D^\alpha u_\varepsilon)\big|_{\Omega_\delta} = (D^\alpha u)_\varepsilon\big|_{\Omega_\delta}.$$

Since by Theorem 2.10 $\lim_{\varepsilon\to 0} \|D^\alpha u - (D^\alpha u)_\varepsilon\|_{p,\Omega} = 0$, we conclude

$$\begin{aligned}
\lim_{\varepsilon\to 0} \|u - u_\varepsilon\|^p_{p,m,\Omega_\delta} &= \lim_{\varepsilon\to 0}\Big(\sum_{|\alpha|\le m} \|D^\alpha u - D^\alpha u_\varepsilon\|^p_{p,m,\Omega_\delta}\Big) \\
&= \lim_{\varepsilon\to 0}\Big(\sum_{|\alpha|\le m} \|D^\alpha u - (D^\alpha u)_\varepsilon\|^p_{p,m,\Omega_\delta}\Big) = 0.
\end{aligned}$$

We thus proved:

Corollary 2.21 *Let $\Omega \subseteq \mathbb{R}^n$ be open and let $1 \le p < \infty$. Let $u \in H^p_m(\Omega)$. Then there is a family $\{u_\varepsilon\}_{\varepsilon>0} \subseteq C^p_\infty(\mathbb{R}^n)$ such that for every open set $D \subseteq \Omega$ with $\operatorname{dist}(D, \partial\Omega) > 0$ we have*

$$\lim_{\varepsilon\to 0} \|u - u_\varepsilon\|_{p,m,D} = 0.$$

Corollary 2.22 *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, let $1 \le p < \infty$, and let $u \in H^p_m(\Omega)$ with $u(x) = 0$ for all $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) < \delta$. Then there is a family $\{u_\varepsilon\}_{\varepsilon>0} \subseteq \overset{\circ}{C}_\infty(\Omega)$ such that*

$$\lim_{\varepsilon\to 0} \|u - u_\varepsilon\|_{p,m,\Omega} = 0.$$

Proof. We define $u_\varepsilon = \varphi_\varepsilon * u$ as in the preceding proof. For $\varepsilon < \delta$ we have $\operatorname{supp} u_\varepsilon \subseteq D = \{x \in \Omega \;\big|\; \operatorname{dist}(x, \partial\Omega) < \delta - \varepsilon\}$, hence $u_\varepsilon \in \overset{\circ}{C}_\infty(\Omega)$. Corollary 2.21 thus yields

$$\lim_{\varepsilon\to 0} \|u - u_\varepsilon\|_{p,m,\Omega} = \lim_{\varepsilon\to 0} \|u - u_\varepsilon\|_{p,m,D} = 0.$$

■

Definition 2.23 Let $\Omega \subseteq \mathbb{R}^n$. A family $\{U_i\}_{i=1}^\infty$ of subsets U_i of $\mathbb{R}^n$ is called a covering of Ω if

$$\Omega \subseteq \bigcup_{i=1}^\infty U_i.$$

$\{U_i\}_{i=1}^\infty$ is an open covering, if every U_i is open. $\{U_i\}_{i=1}^\infty$ is a locally finite covering, if to every $x \in \Omega$ there is a neighborhood V of x such that $U_i \cap \overline{V} \neq \emptyset$ for at most finitely many $i \in \mathbb{N}$.

Lemma 2.24 *Let $\Omega \subseteq \mathbb{R}^n$ be open. Then there is an open, locally finite covering $\{U_i\}_{i=1}^\infty$ of Ω such that U_i is bounded and $\overline{U_i} \subseteq \Omega$ for all $i \in \mathbb{N}$.*

Proof. Let $\Omega_0 = \emptyset$ and for $i \in \mathbb{N}$ set

$$\Omega_i = \{x \in \Omega \bigm| |x| < i,\ \mathrm{dist}(x, \partial\Omega) > \frac{1}{i}\}.$$

Ω_i is open, bounded, and

$$\overline{\Omega_i} \subseteq \Omega_{i+1} \subseteq \overline{\Omega_{i+1}} \subseteq \Omega, \quad \bigcup_{i=1}^\infty \Omega_i = \Omega.$$

Let $U_i = \Omega_{i+1} \backslash \overline{\Omega_{i-1}}$. Then U_i is open, bounded, and

$$\overline{U_i} \subseteq \overline{\Omega_{i+1}} \subseteq \Omega.$$

Moreover,

$$\bigcup_{i=2}^{j} (\Omega_{i+1} \backslash \Omega_i) \cup \Omega_2 = (\Omega_{j+1} \backslash \Omega_{j-1}) \cup \bigcup_{i=2}^{j-2} (\Omega_{i+1} - \Omega_i) \cup \Omega_2 = \Omega_{j+1}$$

implies

$$\Omega \supseteq \bigcup_{i=1}^\infty U_i = \bigcup_{i=1}^\infty (\Omega_{i+1} \backslash \overline{\Omega_{i-1}}) \supseteq \bigcup_{i=2}^\infty (\Omega_{i+1} \backslash \Omega_i) \cup \Omega_2 = \Omega.$$

Finally, $\{U_i\}_{i=1}^\infty$ is locally finite, since for $x \in \Omega$ and every open ball V centered at x and satisfying $\overline{V} \subseteq \Omega$ there is a $\delta > 0$ with $\mathrm{dist}(V, \partial\Omega) > 2\delta$. Choose $i \in \mathbb{N}$ such that $\frac{1}{i} < \delta$. Then $\overline{V} \subseteq \Omega_i$, whence, for $m \geq i+1$

$$\begin{aligned} U_m \cap \overline{V} &\subseteq U_m \cap \Omega_i = (\Omega_{m+1} \backslash \overline{\Omega_{m-1}}) \cap \Omega_i \\ &\subseteq (\Omega_{m+1} \backslash \overline{\Omega_{m-1}}) \cap \Omega_{m-1} = \emptyset. \end{aligned}$$

This implies $U_i \cap \overline{V} \neq \emptyset$ for at most finitely many $i \in \mathbb{N}$. ∎

Definition 2.25 Let $\Omega \subseteq \mathbb{R}^n$ be open, let $\{U_i\}_{i=1}^{\infty}$ be a locally finite, open covering of Ω. A sequence $\{\eta_i\}_{i=1}^{\infty} \subseteq \overset{\circ}{C}_{\infty}(\mathbb{R}^n)$ is called a partition of unity subordinate to $\{U_i\}_{i=1}^{\infty}$, if

$$\eta_i \in \overset{\circ}{C}_{\infty}(U_i), \quad \eta_i \geq 0, \quad \sum_{i=1}^{\infty} \eta_i(x) = 1 \quad \text{for all } x \in \Omega.$$

We note that for every x the sum $\sum_{i=1}^{\infty} \eta_i(x)$ has at most finitely many nonvanishing terms.

Lemma 2.26 *Let $\Omega \subseteq \mathbb{R}^n$ be open and let $\{U_i\}_{i=1}^{\infty}$ be an open, locally finite covering of Ω. If every U_i is bounded and satisfies $\overline{U}_i \subseteq \Omega$, then there is a partition of unity $\{\eta_i\}_{i=1}^{\infty}$ subordinate to this covering.*

Proof. We first construct by induction a sequence of open sets $\{V_i\}_{i=1}^{\infty}$ such that $\overline{V}_i \subseteq U_i$ and $\bigcup_{i=1}^{\infty} V_i = \Omega$. This implies that also $\{V_i\}_{i=1}^{\infty}$ is an open, locally finite covering of Ω consisting of bounded sets.

To construct this sequence, set $V_0 = \emptyset$ and assume for $m \geq 0$ that $V_0, \ldots, V_m$ are chosen such that

$$\Omega = \Big(\bigcup_{i=0}^{m} V_i\Big) \cup \Big(\bigcup_{i=m+1}^{\infty} U_i\Big) = \Big(\bigcup_{i=1}^{m} V_i\Big) \cup \Big(\bigcup_{i=m+1}^{\infty} U_i\Big). \tag{2.9}$$

The closed and bounded, therefore compact set ∂U_{m+1} thus satisfies

$$\begin{aligned} \partial U_{m+1} \cap U_{m+1} &= \emptyset, \\ \partial U_{m+1} \subseteq \overline{U_{m+1}} \subseteq \Omega &= \Big(\bigcup_{i=1}^{m} V_i\Big) \cup \Big(\bigcup_{i=m+1}^{\infty} U_i\Big), \end{aligned}$$

hence

$$\partial U_{m+1} \subseteq \Big(\bigcup_{i=1}^{m} V_i\Big) \cup \Big(\bigcup_{i=m+2}^{\infty} U_i\Big) = M.$$

This implies

$$\operatorname{dist}(\Omega \backslash M,\, \partial U_{m+1}\Big) \geq \operatorname{dist}(\mathbb{R}^n \backslash M,\, \partial U_{m+1}) = \delta > 0.$$

Set $V_{m+1} = \{x \in U_{m+1} \mid \operatorname{dist}(x, \partial U_{m+1}) > \frac{\delta}{2}\}$. Then

$$\Omega = \Big(\bigcup_{i=1}^{m+1} V_i\Big) \cup \Big(\bigcup_{i=m+2}^{\infty} U_i\Big).$$

The sequence $\{V_i\}_{i=1}^{\infty}$ thus constructed satisfies

$$\Omega = \bigcup_{i=1}^{\infty} V_i. \tag{2.10}$$

For, to $x \in \Omega$ there are at most finitely many $i_1 < \ldots < i_k$ such that $x \in U_{i_\ell}$, $\ell = 1, \ldots, k$. Therefore $x \notin \bigcup_{j=i_k+1}^{\infty} U_j$, hence by (2.9)

$$x \in \bigcup_{j=1}^{i_k} V_j \subseteq \bigcup_{j=1}^{\infty} V_j,$$

which proves (2.10). Because of $\overline{V}_i \subseteq U_i$ we have $\mathrm{dist}(V_i, \partial U_i) = \delta_i > 0$. Consequently the open intermediate set $E_i = \bigcup_{x \in V_i} B_{\delta_i/2}(x)$ satisfies

$$\overline{V}_i \subseteq E_i \subseteq \overline{E}_i \subseteq U_i,$$

thus

$$\mathrm{dist}\left(\partial E_i,\, V_i \cup (\mathbb{R}^n \backslash U_i)\right) = \varepsilon_i > 0. \tag{2.11}$$

With a Dirac family $\{\varphi_\varepsilon\}_{\varepsilon>0} \subseteq \overset{\circ}{C}_\infty(\mathbb{R}^n)$ satisfying $\mathrm{supp}\, \varphi_{\varepsilon_i} \subseteq B_{\varepsilon_i}(0)$ define

$$\tilde{\eta}_i(x) = (\varphi_{\varepsilon_i} * \chi_{E_i})(x) = \int_{E_i} \varphi_{\varepsilon_i}(x-y)dy,$$

where χ_{E_i} is the characteristic function of E_i. Since $\int_{|z|<\varepsilon_i} \varphi_{\varepsilon_i}(z)dz = 1$, we have $\tilde{\eta}_i \leq 1$. Moreover, (2.11) yields $\tilde{\eta}_i \in \overset{\circ}{C}_\infty(U_i)$ and

$$\chi_{V_i} \leq \tilde{\eta}_i,$$

whence, because of (2.10),

$$\sum_{i=1}^{\infty} \tilde{\eta}_i(x) \geq \sum_{i=1}^{\infty} \chi_{V_i}(x) \geq 1, \quad x \in \Omega.$$

This series converges and defines an infinitely differentiable function on Ω, since the covering $\{V_i\}_{i=1}^{\infty}$ is locally finite, which implies that to every $x \in \Omega$ there is a neighborhood where at most finitely many of th infinitely differentiable functions $\tilde{\eta}_i$ are different from zero. Now define

$$\eta_i(x) = \frac{\tilde{\eta}_i(x)}{\sum_{j=1}^{\infty} \tilde{\eta}_j(x)}\,, \quad x \in \Omega.$$

Then $\eta_i \in \overset{\circ}{C}_\infty(U_i)$ and $\sum_{i=1}^{\infty} \eta_i(x) = 1$ on Ω. This proves the lemma. ■

Theorem 2.27 *Let* $1 \leq p < \infty$ *and let* $\Omega \subseteq \mathbb{R}^n$ *be open. Then*

$$\overline{C_\infty(\Omega) \cap H_m^p(\Omega} = H_m^p(\Omega).$$

Proof. Let $f \in H^p_m(\Omega)$ and choose $\varepsilon > 0$. To prove the theorem it must be shown that there is a function $g \in C_\infty(\Omega) \cap H^p_m(\Omega)$ such that

$$\|f - g\|_{p,m,\Omega} < \varepsilon.$$

To construct g, let $\{U_i\}_{i=1}^\infty$ be a locally finite open covering of Ω such that U_i is bounded and $\overline{U}_i \subseteq \Omega$ for every $i \in \mathbb{N}$. Let $\{\eta_i\}_{i=1}^\infty$ be a partition of unity subordinate to this covering. Since $\eta_i \in \overset{\circ}{C}_\infty(U_i)$ it follows from the Leibniz rule that $\eta_i f \in H^p_m(U_i)$. Also, $\eta_i f$ vanishes outside the set $\operatorname{supp} \eta_i$, which is a compact subset of U_i. Thus, by Corollary 2.22, there is a function $f_i \in \overset{\circ}{C}_\infty(U_i)$ such that

$$\|\eta_i f - f_i\|_{p,m,\Omega} \le \frac{\varepsilon}{2^i}\,.$$

Define $g : \Omega \to \mathbb{R}$ by $g(x) = \sum_{i=1}^\infty f_i(x)$. Note that for every $x \in \Omega$ at most finitely many f_i are different from zero. We thus have $g \in C_\infty(\Omega)$, and

$$\begin{aligned} \|f - g\|_{p,m,\Omega} &= \Big\|\sum_{i=1}^\infty \eta_i f - \sum_{i=1}^\infty f_i\Big\|_{p,m,\Omega} \\ &= \Big\|\sum_{i=1}^\infty (\eta_i f - f_i)\Big\|_{p,m,\Omega} \le \sum_{i=1}^\infty \|\eta_i f - f_i\|_{p,m,\Omega} \\ &\le \sum_{i=1}^\infty \frac{\varepsilon}{2^i} = \varepsilon\,. \end{aligned}$$

This proves the theorem. ■

3 Nonlinear elliptic boundary value problems

3.1 Preliminary results from functional analysis

In this section we study existence and uniqueness of solutions of the boundary value problem (1.13), (1.14). We first collect some results from functional analysis needed in the existence proof.

Lemma 3.1 *There is a countable, linearly independent set* $\{\nu_k\}_{k=1}^{\infty} \subseteq \overset{\circ}{C}_{\infty}(\Omega)$ *such that the linear span*

$$\operatorname{span}\{\nu_k\} = \Big\{ \sum_{k=1}^{\ell} a_k \nu_k \;\Big|\; \ell \in \mathbb{N},\, a_1, a_2, \ldots \in \mathbb{R} \Big\}$$

is dense in $\overset{\circ}{H}{}_1^p(\Omega)$.

Proof. The space $\overset{\circ}{H}{}_1^p(\Omega)$ is separable for $1 \leq p < \infty$, cf. H.W. Alt, Lineare Funktionalanalysis, p. 101. Thus, there is a countable dense subset $\{w_k\}_{k=1}^{\infty}$ of $\overset{\circ}{H}{}_1^p(\Omega)$. To every w_k choose a sequence $\{\varphi_{k\ell}\}_{\ell=1}^{\infty} \subseteq \overset{\circ}{C}_{\infty}(\Omega)$ with $\lim_{\ell\to\infty} \|w_k - \varphi_{k\ell}\|_{p,m,\Omega} = 0$. Then $\{\varphi_{k\ell} \,|\, k, \ell = 1, 2, \ldots\}$ is a countable, dense subset of $\overset{\circ}{H}{}_1^p(\Omega)$. We write this set as a sequence $\{v_k\}_{k=1}^{\infty}$. We select a linearly independent subset $A = \{\nu_k\}_{k=1}^{\infty}$ of $\{v_k\}_{k=1}^{\infty}$ as follows:

$$v_k \in A \Leftrightarrow v_k \notin \operatorname{span}\{v_1, \ldots, v_{k-1}\}.$$

It follows by induction that $\{\nu_k\}_{k=1}^{\infty} \subseteq \operatorname{span}(A)$. Hence, $\operatorname{span}(A)$ is dense in $\overset{\circ}{H}{}_1^p(\Omega)$. ∎

For the set $\{\nu_k\}_{k=1}^{\infty}$ constructed in this lemma define

$$V_m = \operatorname{span}\{\nu_k \mid k = 1, \ldots, m\}.$$

Then to every $v \in \overset{\circ}{H}{}_1^p(\Omega)$ there is $u \in V_m$ such that

$$\|v - u\|_{p,1,\Omega} = \operatorname{dist}(v, V_m).$$

To see this, let $\{u_\ell\}_{\ell=1}^{\infty} \subseteq V_m$ be a sequence with

$$\lim_{\ell\to\infty} \|v - u_\ell\|_{p,1,\Omega} = \operatorname{dist}(v, V_m).$$

$\{u_\ell\}_{\ell=1}^{\infty}$ is bounded, since

$$\|u_\ell\|_{p,1,\Omega} \leq \|v - u_\ell\|_{p,1,\Omega} + \|v\|_{p,1,\Omega}\,.$$

Thus, since the m-dimensional space V_m is isomorphic to $\mathbb{R}^m$, there is a converging subsequence $\{u_{\ell_k}\}_{k=1}^{\infty}$ with limit u. Because V_m is closed, we have $u \in V_m$ and

$$\|v - u\|_{p,1,\Omega} = \lim_{k\to\infty} \|v - u_{\ell_k}\|_{p,1,\Omega} = \text{dist}(v, V_m).$$

Without proof we remark that u is unique. We can thus define a mapping $P_m : \overset{\circ}{H}{}_1^p(\Omega) \to V_m$ by setting

$$P_m v = u.$$

Lemma 3.2 *Let* $v \in \overset{\circ}{H}{}_1^p(\Omega)$ *and set* $v_m = P_m v$. *Then*

$$\lim_{m\to\infty} \|v - v_m\|_{p,1,\Omega} = 0.$$

Proof. Since $V_m \subseteq V_\ell$ for $\ell \geq m$, we have

$$\|v - v_\ell\|_{p,1,\Omega} \leq \|u - u_m\|_{p,1,\Omega}, \quad \ell \geq m.$$

Let $\varepsilon > 0$. Since $\text{span}\{\nu_k\}_{k=1}^{\infty}$ is dense in $\overset{\circ}{H}{}_1^p(\Omega)$, there is $m \in \mathbb{N}$ and $u = \sum_{k=1}^{m} a_k \nu_k$ with $\|v - u\|_{p,1,\Omega} < \varepsilon$. Therefore

$$\|v - v_\ell\|_{p,1,\Omega} \leq \|v - v_m\|_{p,1,\Omega} \leq \|v - u\|_{p,1,\Omega} < \varepsilon,$$

for $\ell \geq m$. ■

In the infinite dimensional space $\overset{\circ}{H}{}_1^p(\Omega)$ bounded closed sets are not necessarily compact. This introduces difficulties in existence proofs to partial differential equations, since compactness properties are essentially used in these proofs. To overcome this difficulty we need the weak topology on Banach spaces, where a similar compactness result holds even in the infinite dimensional case. We briefly discuss weak convergence.

Definition 3.3 Let X be a Banach space and let X' be the dual space. A sequence $\{x_k\}_{k=1}^{\infty} \subseteq X$ is said to be weakly convergent if there is $x \in X$ such that for all $y' \in X'$ we have

$$\lim_{k\to\infty} \langle x_k, y' \rangle = \langle x, y' \rangle.$$

x is called the weak limit of $\{x_k\}_{k=1}^{\infty}$. If $\{x_k\}_{k=1}^{\infty}$ converges weakly to x, one writes $x_k \rightharpoonup x$. Weak limits are unique. If $\{x_k\}_{k=1}^{\infty}$ converges in the norm of X to x, then $\{x_k\}_{k=1}^{\infty}$ converges also weakly to x.

Lemma 3.4 *If* $\{x_k\}_{k=1}^{\infty}$ *converges weakly, then there is a constant* C *such that*

$$\|x_k\| \le C$$

for all k*, i.e. weakly converging sequences are bounded.*

Proof. Every element $z \in X$ defines a bounded linear mapping $\hat{z}$ on X' with $\|\hat{z}\|_{x'} = \|z\|_x$ by $\hat{z}(y) = \langle z, y\rangle$ for all $y \in X'$. If $x_k \rightharpoonup x$ we have $\hat{x}_k(y) = \langle x_k, y\rangle \to \langle x, y\rangle = \hat{x}(y)$, hence the family $\{\hat{x}_k\}_{k=1}^{\infty}$ of bounded operators on X' is pointwise bounded. By the Banach-Steinhaus theorem it thus follows that $C > 0$ exists such that

$$\|x_k\|_X = \|\hat{x}_k\|_{X'} \le C,$$

for all k. ■

Lemma 3.5 *Let* X *and* Y *be Banach spaces and assume that* $T : X \to Y$ *is a bounded linear mapping. Then* T *is also weakly sequentially continuous, i.e. if* $\{x_k\}_{k=1}^{\infty}$ *converges weakly in* X *to* x*, then* $\{Tx_k\}_{k=1}^{\infty}$ *converges weakly in* Y *to* Tx*.*

Proof. Let $y' \in Y'$. Then $x' = y' \circ T$ is a bounded linear mapping on X, whence $x' \in X'$. Consequently

$$\lim_{k\to\infty} \langle Tx_k, y'\rangle = \lim_{k\to\infty} (y' \circ T)(x_k) = (y' \circ T)(x) = \langle Tx, y'\rangle,$$

whence $Tx_k \rightharpoonup Tx$. ■

Corollary 3.6 *Assume that the sequence* $\{u_k\}_{k=1}^{\infty}$ *converges weakly in* $H_1^p(\Omega)$ *or* $\overset{\circ}{H}{}_1^p(\Omega)$ *to* u*. Then the sequences* $\{u_k\}_{k=1}^{\infty}$, $\left\{\frac{\partial}{\partial x_i} u_k\right\}_{k=1}^{\infty}$, $i = 1, \ldots, n$*, converge weakly in* $L^p(\Omega)$ *to* u *or* $\frac{\partial}{\partial x_i} u$*.*

Proof. For $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le 1$ let $T_\alpha : H_1^p(\Omega) \to L^p(\Omega)$ be defined by

$$T_\alpha u = D^\alpha u.$$

Since $\|T_\alpha u\|_{p,\Omega} = \|D^\alpha u\|_{p,\Omega} \le \|u\|_{p,1,\Omega}$, the operator T_α is bounded. Therefore T_α is weakly continuous. Thus, if $u_k \rightharpoonup u$ in $H_1^p(\Omega)$, we obtain

$$D^\alpha u_k = T_\alpha u_k \rightharpoonup T_\alpha u = D^\alpha u.$$ ■

Theorem 3.7 *Assume that* X *is a reflexive Banach space. Then, the closed unit ball* $B = \{x \in X \mid \|x\| \le 1\}$ *is weakly sequentially compact, i.e. every sequence* $\{x_k\}_{k=1}^{\infty} \subseteq \overline{B}$ *has a subsequence, which converges weakly in* $\overline{B}$*.*

For a proof of this theorem cf. the book H.W. Alt: Lineare Funktionalanalysis, Springer 1999, p. 218.

For $1 < p < \infty$ the Banach spaces $L^p(\Omega)$, $H_1^p(\Omega)$, and $\overset{\circ}{H}{}_1^p(\Omega)$ are reflexive with dual spaces $L^q(\Omega), H_1^q(\Omega)$ and $\overset{\circ}{H}{}_1^q(\Omega)$, respectively. The compactness result from Theorem 3.7 thus holds for these spaces.

3.2 Existence of solutions for nonlinear boundary value problems, Minty-Browder method

Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field; let $b \in L^{1,\mathrm{loc}}(\Omega, \mathbb{R})$ be a given function. Consider the homogeneous Dirichlet boundary value problem

$$-\mathrm{div} F\big(\nabla u(x)\big) = b(x), \qquad x \in \Omega \tag{3.1}$$

$$u(x) = 0, \qquad x \in \partial\Omega. \tag{3.2}$$

Definition 3.8 A function $u \in \overset{\circ}{H}{}_1^1(\Omega)$ is a weak solution of (3.1), (3.2), if $F(\nabla u) \in L^{1,\mathrm{loc}}(\Omega)$ and if for all $\varphi \in \overset{\circ}{C}_\infty(\Omega)$

$$\big(F(\nabla u), \nabla\varphi\big)_\Omega = (b, \varphi)_\Omega.$$

In this subsection we prove that the boundary value problem (3.1), (3.2) has a unique weak solution in $\overset{\circ}{H}{}_1^p(\Omega)$ with $1 < p < \infty$, if F belongs to $C_1(\mathbb{R}^n, \mathbb{R}^n)$ and satisfies the following conditions:

(i) There is a constant $c > 0$ such that

$$\big(F(\xi) - F(\eta)\big) \cdot (\xi - \eta) \geq c|\xi - \eta|^2. \tag{3.3}$$

(F is strongly monotone.)

(ii) There are constants $c_1 > 0$, $c_2 > 0$ such that

$$F(\xi) \cdot \xi \geq c_1|\xi|^p - c_2. \tag{3.4}$$

(F is coercive.)

(iii) There is a constant $c_3 > 0$ such that

$$|F(\xi)| \leq c_3(|\xi| + 1)^{p-1}. \tag{3.5}$$

(F is bounded.)

We first prove two technical lemmas, which we need in the existence proof.

Lemma 3.9 *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. If F satisfies* (3.4), *then there is $C > 0$ such that for all $u \in \overset{\circ}{C}_\infty(\Omega)$*

$$\int_\Omega F\big(\nabla u(x)\big) \cdot \nabla u(x) dx \geq c_1 C |u|^p_{p,1,\Omega} - c_2|\Omega|, \tag{3.6}$$

where $|\Omega| = \int_\Omega dx$.

Proof. $F(\nabla u)$ is a continuous function with compact support, hence the integral $\int_\Omega F(\nabla u) \cdot \nabla u \, dx$ exists. (3.4) implies

$$\begin{aligned}\int_\Omega F\big(\nabla u(x)\big) \cdot \nabla u(x)\, dx &\geq \int_\Omega c_1 |\nabla u(x)|^p \, dx - c_2|\Omega| \\ &\geq c_1 \int_\Omega C \sum_{i=1}^n \Big| \frac{\partial}{\partial x_i} u(x)\Big|^p \, dx - c_2|\Omega| = c_1 C |u|^p_{p,1,\Omega} - c_2|\Omega|.\end{aligned}$$

■

Lemma 3.10 *Assume that F satisfies* (3.5) *and let $1 < q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then there is a constant C such that $F(\nabla v) \in L^q(\Omega, \mathbb{R}^n)$ and*

$$\|F(\nabla v)\|_{q,\Omega} \leq C\big(|v|_{p,1,\Omega} + 1\big)^{\frac{p}{q}} \tag{3.7}$$

for all $v \in H^p_1(\Omega, \mathbb{R})$.

Proof. Since $\frac{p}{q} = p\big(1 - \frac{1}{p}\big) = p - 1$, it follows from (3.5) and from Minkowski's inequality

$$\begin{aligned}\int_\Omega |F\big(\nabla v(x)\big)|^q \, dx &\leq \int_\Omega c_3^q \big(|\nabla v(x)| + 1\big)^{(p-1)q} \, dx \\ &= c_3^q \int_\Omega \big(|\nabla v(x)| + 1\big)^p \, dx \\ &\leq c_3^q \bigg(\Big(\int_\Omega |\nabla v(x)|^p \, dx\Big)^{\frac{1}{p}} + \Big(\int_\Omega dx\Big)^{\frac{1}{p}}\bigg)^p \\ &\leq c_3^q \bigg(\Big(\int_\Omega C_1 \sum_{i=1}^n \big(\frac{\partial}{\partial x_i} v(x)\big)^p \, dx\Big)^{\frac{1}{p}} + |\Omega|^{\frac{1}{p}}\bigg)^p \\ &= c_3^q \big(C_1^{\frac{1}{p}} |v|_{p,1,\Omega} + |\Omega|^{\frac{1}{p}}\big)^p.\end{aligned}$$

Taking q-th roots yields (3.7). ■

In the first step of the existence proof we use the Galerkin procedure to construct a sequence $\{u_m\}_{m=1}^\infty$ of approximate solutions of (3.1), (3.2), which consist of linear combinations of $\{\nu_k\}_{k=1}^\infty$:

Theorem 3.11 *Let* $F \in C_1(\mathbb{R}^n, \mathbb{R}^n)$ *satisfy* (3.3) *and* (3.4), *and let* $b \in L^q(\Omega)$. *Then to every* $m \in \mathbb{N}$ *there is* $u_m \in \overset{\circ}{C}_\infty(\Omega)$ *given by*

$$u_m(x) = \sum_{k=1}^{m} a_{mk}\, \nu_k(x)$$

with $a_{m1}, \ldots, a_{mm} \in \mathbb{R}$, *such that*

$$\big(F(\nabla u_m), \nabla \nu_\ell\big)_\Omega = (b, \nu_\ell)_\Omega, \quad \ell = 1, \ldots, m. \tag{3.8}$$

Proof. I) For $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$ define $E(a) = \big(E_1(a), \ldots, E_m(a)\big) \in \mathbb{R}^m$ by

$$E_\ell(a) = \Big(F\Big(\sum_{i=1}^{m} a_i \nabla \nu_i\Big), \nabla \nu_\ell\Big)_\Omega - (b, \nu_\ell) \in \mathbb{R}. \tag{3.9}$$

This defines a differentiable mapping $E : \mathbb{R}^m \to \mathbb{R}^m$ with derivative

$$\frac{\partial}{\partial a_k} E_\ell(a) = \Big(\nabla F\Big(\sum_{i=1}^{m} a_i \nabla \nu_i\Big) \nabla \nu_k, \nabla \nu_\ell\Big)_\Omega.$$

The Jacobi-matrix

$$\nabla E(a) = \Big(\frac{\partial}{\partial a_k} E_\ell(a)\Big)_{\ell,k=1,\ldots,m}$$

is positive definite. To see this, we use that (3.3) and Theorem 1.6 imply that ∇F is uniformly positive definite, and hence for $a = (a_1, \ldots, a_m)$, $\eta = (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m$

$$\begin{aligned}
\eta \cdot \big[\nabla E(a)\big]\eta &= \sum_{k,\ell=1}^{m} \Big(\nabla F\Big(\sum_{i=1}^{m} a_i \nabla \nu_i\Big)\eta_k \nabla \nu_k, \eta_\ell \nabla \nu_\ell\Big)_\Omega \\
&= \int_\Omega \Big(\nabla F\Big(\sum_{i=1}^{m} a_i \nabla \nu_i\Big) \sum_{k=1}^{m} \eta_k \nabla \nu_k\Big) \cdot \Big(\sum_{\ell=1}^{m} \eta_\ell \nu_\ell\Big) dx \\
&\geq \int_\Omega c \Big|\sum_{k=1}^{m} \eta_k \nabla \nu_k\Big|^2 dx = c \Big\|\sum_{k=1}^{m} \eta_k \nabla \nu_k\Big\|_\Omega^2 \\
&\geq cC \sum_{k=1}^{m} \eta_k^2 \;=\; cC|\eta|^2,
\end{aligned}$$

with a constant $C > 0$, independent of a and η, since the set $\{\nabla \nu_1, \ldots \nabla \nu_m\}$ is linearly independent.

II) By definition of E it follows that

$$u = \sum_{k=1}^{m} a_k \nu_k$$

satisfies (3.8) if and only if $a = (a_1, \ldots, a_m)$ satisfies $E(a) = 0$.

Now consider the function

$$\eta \mapsto |E(\eta)|^2 : \mathbb{R}^m \to [0, \infty).$$

a is a stationary point of this function if and only if

$$\nabla |E(a)|^2 = 2E(a)^T \, \nabla E(a) = 0.$$

Since $\nabla E(a)$ is positive definite, the inverse $\big(\nabla E(a)\big)^{-1}$ exists, whence a is a stationary point if and only if $E(a) = 0$, and therefore if and only if $u = \sum_{k=1}^m a_k \nu_k$ solves (3.8). Thus, to prove the theorem it suffices to show that $|E(\eta)|^2$ assumes the minimum, since the minimum is a stationary point.

III) To prove that a minimum exists we use (3.9) and Lemma 3.9, which yields for $a \in \mathbb{R}^m$ and $v = \sum_{k=1}^m a_k \nu_k$

$$\begin{aligned} |E(a)|\,|a| \geq E(a) \cdot a \;&=\; \big(F(\nabla v), \nabla v\big)_\Omega - (b, v)_\Omega \;\geq\; c_1 C \|\nabla v\|_{p,\Omega}^p - c_2 |\Omega| \\ &=\; c_1 C \int_\Omega \Big| \sum_{k=1}^m a_k \nabla \nu_k \Big|^p dx - c_2 |\Omega| \;\geq\; C_1 |a|^p - c_2 |\Omega|, \end{aligned}$$

with a constant $C_1 > 0$, since all norms on $\mathbb{R}^m$ are equivalent. We divide by $|a|$ to obtain

$$|E(a)| \geq C_1 |a|^{p-1} - \frac{c_2 |\Omega|}{|a|}\,. \tag{3.10}$$

Choose $r > 0$ such that

$$C_1 r^{p-1} - \frac{c_2 |\Omega|}{r} > |E(0)|\,. \tag{3.11}$$

The continuous function $\eta \mapsto |E(\eta)|^2$ assumes the minimum on the compact set $K_r = \{x \in \mathbb{R}^m \;\big|\; |x| \leq r\}$, and from (3.10), (3.11) we conclude

$$\min_{K_r} |E(\eta)|^2 \leq |E(0)|^2 < \min_{\partial K_r} |E(\eta)|^2,$$

whence the minimum is assumed at an interior point of K_r, and therefore this minimum is a stationary point. ∎

Our goal is to show that from the sequence $\{u_m\}_{m=1}^\infty$ we can select a subsequence, which converges weakly to a solution of the boundary value problem (3.1), (3.2). To this end we show that the sequence $\{u_m\}_{m=1}^\infty$ is bounded.

Lemma 3.12 *For the sequence* $\{u_m\}_{m=1}^\infty \subseteq \overset{\circ}{C}_\infty(\Omega)$ *satisfying* (3.8) *there is a constant* $K = K(\|b\|_{q,\Omega})$ *such that*

$$\|u_m\|_{p,1,\Omega} \leq K(\|b\|_{q,\Omega}). \tag{3.12}$$

Proof. From (3.8) we obtain

$$\begin{aligned}\big(F(\nabla u_m), \nabla u_m\big)_\Omega &= \sum_{k=1}^{m} a_{mk}\big(F(\nabla u_m), \nabla \nu_k\big)_\Omega \\ &= \sum_{k=1}^{\infty} a_{mk}(b, \nu_k)_\Omega \;=\; (b, u_m)_\Omega.\end{aligned}$$

Therefore Lemma 3.9 and Hölder's inequality imply

$$\begin{aligned}c_1 C |u_m|_{p,1,\Omega}^p - c_2|\Omega| &\le \big(F(\nabla u_m), \nabla u_m\big)_\Omega \\ = (b, u_m)_\Omega &\le \|b\|_{q,\Omega}\,\|u_m\|_{p,\Omega}\,.\end{aligned}$$

Using Poincaré's inequality we obtain with $d = \mathrm{diam}(\Omega)$ that

$$c_1 C |u_m|_{p,1,\Omega}^p \le p^{-\frac{1}{p}}\, d\|b\|_{q,\Omega}\,|u_m|_{p,1,\Omega} + c_2|\Omega|.$$

For $|u_m|_{p,1,\Omega} \ge 1$ we thus have

$$|u_m|_{p,1,\Omega}^{p-1} \le p^{-\frac{1}{p}}\,\frac{d}{c_1 C}\,\|b\|_{q,\Omega} + \frac{c_2}{c_1 C}\,|\Omega|\,,$$

whence

$$|u_m|_{p,1,\Omega} \le \max\Big\{1, \Big(p^{-\frac{1}{p}}\, d\frac{d}{c_1 C}\,\|b\|_{q,\Omega} + \frac{c_2}{c_1 C}\,|\Omega|\Big)^{\frac{1}{p-1}}\Big\}\,.$$

(3.12) is obtained from this inequality using again Poincaré's inequality. ∎

We complete now the existence proof. The main difficulty of the proof arises from the fact that we can only work with weakly converging subsequences and that the nonlinear function F is not continuous with respect to weak convergence. Because of this, it is not obvious that the limit function of a weakly converging subsequence of approximate solutions is a solution of the partial differential equation (3.1). The idea used in the proof to overcome this difficulty is called Minty-Browder method.

Theorem 3.13 *Assume that $F \in C_1(\mathbb{R}^n, \mathbb{R}^n)$ satisfies the conditions* (3.3) – (3.5). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Then to every $b \in L^q(\Omega)$ the boundary value problem*

$$\begin{aligned}-\mathrm{div} F\big(\nabla u(x)\big) &= b(x), \qquad x \in \Omega, \\ u(x) &= 0, \qquad\;\;\; x \in \partial\Omega,\end{aligned}$$

has a weak solution $u \in \overset{\circ}{H}{}_1^p(\Omega)$.

Proof. Let $\{u_m\}_{m=1}^{\infty} \subseteq \overset{\circ}{C}_{\infty}(\Omega)$ be the sequence of approximate solutions satisfying (3.8). I can select a subsequence $\{u_{m_k}\}_{k=1}^{\infty}$, which converges weakly in $\overset{\circ}{H}{}_1^p(\Omega)$. To simplify the notation, I denote the subsequence again by $\{u_m\}_{m=1}^{\infty}$. Thus

$$u_m \rightharpoonup u \in \overset{\circ}{H}{}_1^p(\Omega).$$

For $v \in \overset{\circ}{H}{}_1^p(\Omega)$ let $v_m = P_m v \in V_m$. Lemma 3.2 yields that

$$||v - v_m||_{p,1,\Omega} \to 0, \quad m \to \infty, \tag{3.13}$$

and Lemma 3.10 implies $F(\nabla v), F(\nabla u_m) \in L^q(\Omega;\mathbb{R}^n)$, hence the following integrals exist. Using that F is monotone, by (3.3), we conclude

$$\begin{aligned} 0 \;&\le\; \int_\Omega \big(F(\nabla v) - F(\nabla u_m)\big)\cdot(\nabla v - \nabla u_m)dx \qquad (3.14)\\ &=\; \int_\Omega F(\nabla v)\cdot(\nabla v - \nabla u_m) - F(\nabla u_m)\cdot(\nabla v_m - \nabla u_m)\\ &\qquad\qquad - F(\nabla u_m)\cdot(\nabla v - \nabla v_m)dx\\ &=\; \int_\Omega F(\nabla v)\cdot(\nabla v - \nabla u_m) - b\cdot(v_m - u_m) - F(\nabla u_m)\cdot(\nabla v - \nabla v_m)dx. \end{aligned}$$

Here we used (3.8), which yields

$$\begin{aligned} \big(F(\nabla u_m),(\nabla v_m - \nabla u_m)\big)_\Omega \;&=\; \Big(F(\nabla u_m),\, \nabla\Big(\sum_{k=1}^{m}(b_k - a_{mk})\nu_k\Big)\Big)_\Omega\\ &=\; \sum_{k=1}^{m}(b_k - a_{mk})\big(F(\nabla u_m),\nabla\nu_k\big)_\Omega\\ &=\; \sum_{k=1}^{m}(b_k - a_{mk})(b,\nu_k)_\Omega = (b, v_m - u_m)_\Omega. \end{aligned}$$

(3.12), (3.7) and Hölder's inequality imply

$$\Big|\int_\Omega F(\nabla u_m)\cdot(\nabla v - \nabla v_m)dx\Big| \le \|F(\nabla u_m)\|_{q,\Omega}\,\|\nabla v - \nabla v_m\|_{p,\Omega} \le C\|v - v_m\|_{p,1,\Omega},$$

with a suitable constant C independent of m. From (3.13) we thus conclude

$$\lim_{m\to\infty}\big(F(\nabla u_m),(\nabla v - \nabla v_m)\big)_\Omega = 0.$$

We use this relation, the relation $v_m \to v$ in $L^p(\Omega)$, and note that Corollary 3.6 yields $u_m \rightharpoonup u$, $\nabla u_m \rightharpoonup \nabla u$ weakly in $L^p(\Omega)$, to infer from (3.14) that

$$\begin{aligned} 0 \;&\le\; \lim_{m\to\infty}\big[\big(F(\nabla v),\,\nabla(v - u_m)\big)_\Omega - (b, v_m - u_m)_\Omega\big]\\ &\quad - \lim_{m\to\infty}\big(F(\nabla u_m),\nabla v - \nabla v_m\big)_\Omega\\ &=\; \big(F(\nabla v),\nabla(v-u)\big)_\Omega - (b, v-u)_\Omega. \end{aligned}$$

We insert $v = u + \lambda w$ with $\lambda > 0$ and arbitrary $w \in \overset{\circ}{H}{}^p_1(\Omega)$ to obtain

$$\lambda\big[\big(F\big(\nabla(u+\lambda w)\big), \nabla w\big)_\Omega - (b,w)_\Omega\big] \geq 0.$$

Thus

$$\int_\Omega F\big(\nabla u(x) + \lambda \nabla w(x)\big) \cdot \nabla w(x) - b(x)w(x)dx \geq 0.$$

Using Lemma 3.14 we obtain for $\lambda \to 0$

$$\int_\Omega F\big(\nabla u(x)\big) \cdot \nabla w(x) - b(x)w(x)dx \geq 0.$$

Replacing w by $-w$ in this computation yields

$$-\int_\Omega F\big(\nabla u(x)\big) \cdot \nabla w(x) - b(x)w(x)dx \geq 0,$$

whence

$$\big(F(\nabla u), \nabla w\big)_\Omega = (b,w)_\Omega$$

for all $w \in \overset{\circ}{H}{}^p_1(\Omega)$. Therefore u is a weak solution of (3.1), (3.2). ∎

Lemma 3.14 *We have*

$$\lim_{\substack{\lambda\to 0\\ \lambda>0}} \int_\Omega F\big(\nabla u(x) + \lambda\nabla w(x)\big) \cdot \nabla w(x)dx = \int_\Omega F\big(\nabla u(x)\big) \cdot \nabla w(x)dx\,.$$

Proof. The convexity of $\xi \mapsto |\xi|^{p-1}$ and (3.5) yield for $0 < \lambda < 1$ and for $C = 2^{p-2}c_3$ that

$$\begin{aligned}
\big|F\big(\nabla u(x) + \lambda\nabla w(x)\big)\big| &\leq c_3\big(\big|\nabla u(x) + \lambda \nabla w(x)\big| + 1\big)^{p-1}\\
&\leq C|\nabla u(x) + \lambda\nabla w(x)|^{p-1} + C\\
&= C\big|(1-\lambda)\nabla u(x) + \lambda\big(\nabla u(x) + \nabla w(x)\big)\big|^{p-1} + C\\
&\leq C\big((1-\lambda)\big|\nabla u(x)\big|^{p-1} + \lambda\big|\nabla u(x) + \nabla w(x)\big|^{p-1} + 1\big)\\
&\leq C\big(\big|\nabla u(x)\big|^{p-1} + \big|\nabla u(x) + \nabla w(x)\big|^{p-1} + 1\big).
\end{aligned}$$

Thus, for $0 < \lambda < 1$,

$$\begin{aligned}
&\big|F\big(\nabla u(x) + \lambda\nabla w(x)\big) \cdot \nabla w(x)\big| \\
&\quad \leq C\big(\big|\nabla u(x)\big|^{p-1} + \big|\nabla u(x) + \nabla w(x)\big|^{p-1} + 1\big)\,|\nabla w| = g(x)\,.
\end{aligned} \tag{3.15}$$

Therefore g defined in this equation is a dominating function for $|F(\nabla u + \lambda\nabla w)\cdot\nabla w|$. The statement of the lemma follows from Lebesgue's dominated convergence theorem if we show that $g \in L^1(\Omega)$, since the continuity of F implies

$$\lim_{\lambda\to 0} F\big(\nabla u(x) + \lambda \nabla w(x)\big) \cdot \nabla w(x) = F\big(\nabla u(x)\big) \cdot \nabla w(x).$$

To show that $g \in L^1(\Omega)$ we use Hölder's inequality, which for $q = \frac{p}{p-1}$ yields

$$\begin{aligned}\int_\Omega |\nabla u(x)|^{p-1} |\nabla w| dx &\leq \Big(\int_\Omega |\nabla u(x)|^{(p-1)q} \, dx \Big)^{1/q} \Big(\int_\Omega |\nabla w(x)|^p \, dx \Big)^{1/p} \\ &= \|\nabla u\|_{L^p(\Omega)}^{\frac{p}{q}} \|\nabla w\|_{L^p(\Omega)} < \infty,\end{aligned}$$

and similarly

$$\int_\Omega \big|\nabla u(x) + \nabla w(x)\big|^{p-1} |\nabla w(x)| \, dx \leq \|\nabla u + \nabla w\|_{L^p(\Omega)}^{\frac{p}{q}} \|\nabla w\|_{L^p(\Omega)} < \infty.$$

These estimates and $\int_\Omega |\nabla w(x)| dx \leq \big(\int_\Omega dx \big)^{\frac{1}{q}} \|\nabla w\|_{L^p(\Omega)}$ yield that $g \in L^1(\Omega)$. ■

Lemma 3.15 *The solution of the boundary value problem* (3.1), (3.2) *is unique.*

Proof. Let $u, v \in \overset{\circ}{H}{}^p_1(\Omega)$ be two solutions. Then we have $F(\nabla u), F(\nabla v) \in L^q(\Omega)$. The monotonicity condition (3.3) then yields

$$\begin{aligned}&\int_\Omega \Big(F\big(\nabla u(x)\big) - F\big(\nabla v(x)\big)\Big) \cdot \big(\nabla u(x) - \nabla v(x)\big) dx \\ &\quad \geq \int_\Omega c |\nabla u(x) - \nabla v(x)|^2 \, dx.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\int_\Omega \Big(F\big(\nabla u(x)\big) - F\big(\nabla v(x)\big)\Big) \cdot \big(\nabla u(x) - \nabla v(x)\big) dx \\ &= \int_\Omega F\big(\nabla u(x)\big) \cdot \nabla\big(u(x) - v(x)\big) dx \\ &\quad - \int_\Omega F\big(\nabla v(x)\big) \cdot \nabla\big(u(x) - v(x)\big) dx \\ &= \int_\Omega b(x)\big(u(x) - v(x)\big) dx - \int_\Omega b(x)\big(u(x) - v(x)\big) dx = 0.\end{aligned}$$

Together it follows that

$$|u - v|_{2,1,\Omega} = 0,$$

whence $\|u - v\|_{2,1,\Omega} = 0$, by Poincaré's inequality. This implies $u = v$. ■

Example. Let $p \geq 2$, $c > 0$ and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$F(\xi) = c\big(|\xi|^{p-2} + 1\big)\xi, \quad \xi \in \mathbb{R}^n.$$

Then we have that $F \in C_1(\mathbb{R}^n, \mathbb{R}^n)$ and that the conditions (3.3) – (3.5) are satisfied. To see this, note that for $\xi, \eta \in \mathbb{R}^n$

$$\begin{aligned}\big(F(\xi) - F(\eta)\big) \cdot (\xi - \eta) &= c\big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big) \cdot (\xi - \eta) + c(\xi - \eta) \cdot (\xi - \eta) \\ &\geq c|\xi - \eta|^2,\end{aligned}$$

where we used that by the example in Section 1.2 we have

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) \geq 0.$$

Therefore condition (3.3) holds. Moreover, we have

$$F(\xi) \cdot \xi = c\big(|\xi|^{p-2} + 1\big)|\xi|^2 \geq c|\xi|^p,$$

which is (3.4).

Finally we have

$$|F(\xi)| \leq c(|\xi|^{p-2} + 1)|\xi| \leq c \cdot 2(|\xi| + 1)^{p-2}|\xi| \leq 2c(|\xi| + 1)^{p-1},$$

which shows that (3.5) holds. We thus have that the homogeneous boundary value problem

$$-c\,\mathrm{div}\Big(\big(|\nabla u(x)|^{p-2} + 1\big)\nabla u\Big) = b(x), \quad x \in \Omega,$$
$$u(x) = 0, \quad x \in \partial\Omega,$$

has a unique weak solution in $\overset{\circ}{H}{}^p_1(\Omega)$ for $p \geq 2$ and $\Omega \subseteq \mathbb{R}^n$ open and bounded.

For two times differentiable u we can use the chain rule and compute

$$\begin{aligned}
&\mathrm{div}\big((|\nabla u|^{p-2} + 1)\nabla u\big) \\
&\quad = (|\nabla u|^{p-2} + 1)\Delta u + (p-2)|\nabla u|^{p-2} \frac{\nabla u}{|\nabla u|} \cdot (\nabla^2 u) \frac{\nabla u}{|\nabla u|}, \\
&\quad = |\nabla u|^{p-2}\big(\Delta u + (p-2) \frac{\nabla u}{|\nabla u|} \cdot (\nabla^2 u) \frac{\nabla u}{|\nabla u|}\big) + \Delta u,
\end{aligned}$$

with the Hessian matrix

$$\nabla^2 u = \Big(\frac{\partial^2}{\partial x_i \partial x_j}\, u\Big)_{i,j=1,\ldots,n}.$$

3.3 Variable coefficients and inhomogeneous boundary conditions

The results from the preceding chapter can be generalized to boundary value problems of the form

$$-\mathrm{div} F\big(x, \nabla u(x)\big) = b(x), \qquad x \in \Omega \tag{3.16}$$
$$u(x) = 0, \qquad x \in \partial\Omega. \tag{3.17}$$

Assume that $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following assumptions:

For every $x \in \Omega$ the function $\xi \mapsto F(x, \xi)$ belongs to $C_1(\mathbb{R}^n, \mathbb{R}^n)$ and for every $\xi \in \mathbb{R}^n$ the function $x \mapsto F(x, \xi)$ is measurable. Moreover, for $1 < p < \infty$ we have:

(i) There is $c > 0$ such that for all $x \in \Omega$

$$\big(F(x, \xi) - F(x, \eta)\big) \cdot (\xi - \eta) \geq c|\xi - \eta|^2.$$

(ii) There are $c_1 > 0$, $c_2 \geq 0$ such that for all $x \in \Omega$

$$F(x, \xi) \cdot \xi \geq c_1|\xi|^p - c_2.$$

(iii) There is $c_3 > 0$ such that for all $x \in \Omega$

$$|F(x, \xi)| \leq c_3(|\xi| + 1)^{p-1}.$$

(iv) To every $k > 0$ there is K such that for all $x \in \Omega$ and all $\xi \in \mathbb{R}^n$ with $|\xi| \leq k$

$$|\nabla_\xi F(x, \xi)| \leq K.$$

If F does not depend on x, then condition (iv) is a consequence of the assumption that F is continuously differentiable with respect to x.

The proofs in the preceding section can be repeated without modification under these assumptions. We thus have the following result:

Theorem 3.16 *Assume that F satisfies the assumptions above. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Then to every $b \in L^q(\Omega)$ the boundary value problem* (3.16), (3.17) *has a unique weak solution in $\overset{\circ}{H}{}^p_1(\Omega)$.*

We can apply this result to solve the inhomogeneous boundary value problem

$$-\text{div} F\big(x, \nabla u(x)\big) = b(x), \qquad x \in \Omega, \tag{3.18}$$

$$u(x) = \gamma(x), \qquad x \in \partial\Omega. \tag{3.19}$$

Let $\gamma \in H^p_1(\Omega) \cap C(\overline{\Omega})$. Then for every $v \in \overset{\circ}{H}{}^p_1(\Omega)$ the function

$$u = \gamma + v : \Omega \to \mathbb{R}$$

satisfies the boundary condition

$$u|_{\partial\Omega} = \gamma|_{\partial\Omega},$$

in a generalized sense. We extend this definition to $u \in H^p_1(\Omega)$:

Definition 3.17 Let $\gamma \in H^p_1(\Omega)$. A function $u \in H^p_1(\Omega)$ satisfies the boundary condition

$$u|_{\partial\Omega} = \gamma|_{\partial\Omega}$$

in the weak sense if $u - \gamma \in \overset{\circ}{H}{}^p_1(\Omega)$.

Definition 3.18 Let $\gamma \in H_1^p(\Omega)$ and $b \in L^q(\Omega)$. A function $u \in H_1^p(\Omega)$ is a weak solution of (3.18), (3.19), if $u - \gamma \in \overset{\circ}{H}{}_1^p(\Omega)$ and if for all $v \in \overset{\circ}{C}_\infty(\Omega)$

$$\big(F\big(x, \nabla u(x)\big), \nabla v(x)\big)_\Omega = (b, v)_\Omega.$$

Theorem 3.19 *Assume that the function $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the assumptions given above. Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Then to every $\gamma \in H_1^\infty(\Omega)$ and to all $b \in L^q(\Omega)$ there is a unique weak solution of* (3.18), (3.19).

Proof. Consider the problem

$$\begin{aligned} -\mathrm{div} F\big(x, \nabla w(x) + \nabla\gamma(x)\big) &= b(x), \qquad && x \in \Omega, \\ w(x) &= 0, && x \in \partial\Omega. \end{aligned}$$

$w \in \overset{\circ}{H}{}_1^p(\Omega)$ is a weak solution of this problem if and only if $u = \gamma + w$ is a weak solution of (3.18), (3.19). To prove the theorem it therefore suffices to show that this homogeneous boundary value problem has a unique solution, and by the preceding theorem this follows if we show that the function

$$(x, \xi) \mapsto G(x, \xi) = F\big(x, \xi + \nabla\gamma(x)\big) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

satisfies the assumptions formulated above.

Note first that $x \mapsto F\big(x, \xi + \nabla\gamma(x)\big)$ is measurable, since $\nabla\gamma(x) \in L^\infty(\Omega)$ is measurable, and since F is measurable with respect to the first argument and continuous with respect to the second. (F is a Caratheodory function.) Also, $\xi \mapsto F\big(x, \xi + \nabla\gamma(x)\big)$ is continuously differentiable. Since F is strongly monotone, we have for $\xi, \eta \in \mathbb{R}^n$ and $x \in \Omega$

$$\begin{aligned} &\big(G(x,\xi) - G(x,\eta)\big) \cdot (\xi - \eta) \\ &\quad \Big(F\big(x, \xi + \nabla\gamma(x)\big) - F\big(x, \eta + \nabla\gamma(x)\big)\Big) \cdot \Big(\big(\xi + \nabla\gamma(x)\big) - \big(\eta + \nabla\gamma(x)\big)\Big) \\ &\quad \geq c\big|\big(\xi + \nabla\gamma(x)\big) - \big(\eta + \nabla\gamma(x)\big)\big|^2 = c|\xi - \eta|^2. \end{aligned}$$

Therefore (i) is satisfied. Also,

$$\begin{aligned} G(x,\xi) \cdot \xi &= F\big(x, \xi + \nabla\gamma(x)\big) \cdot \big(\xi + \nabla\gamma(x) - \nabla\gamma(x)\big) \\ &\geq c_1\big|\xi + \nabla\gamma(x)\big|^p - c_2 - \big|F\big(x, \xi + \nabla\gamma(x)\big)\big|\,|\nabla\gamma(x)| \\ &\geq c_1\big|\xi + \nabla\gamma(x)\big|^p - c_2 - c_3\big(|\xi + \nabla\gamma(x)| + 1\big)^{p-1}\,\|\nabla\gamma\|_{\Omega,\alpha} \\ &\geq \frac{c_1}{2}\,\big|\xi + \nabla\gamma(x)\big|^p - c_2 - C_3 \\ &\geq \frac{c_1}{2}\,|\xi|^p - C_4\,, \end{aligned}$$

where

$$C_3 = \max_{r\geq 0} c_3\|\nabla\gamma\|_{\Omega,\infty}(r+1)^{p-1} - \frac{c_1}{2}\, r^p.$$

Therefore (ii) holds. Next, (iii) holds since

$$\begin{aligned}
|G(x,\xi)| &= \big|F\big(x,\xi+\nabla\gamma(x)\big)\big| \\
&\leq c_3\big(|\xi+\nabla\gamma(x)|+1\big)^{p-1} \\
&\leq c_3\big(|\xi|+\|\nabla\gamma\|_{\Omega,\infty}+1\big)^{p-1} \leq C_5\big(|\xi|+1\big)^{p-1}.
\end{aligned}$$

Also (iv) is satisfied, because for $|\xi|\leq k$ we have $|\xi+\nabla\gamma(x)|\leq k+\|\nabla\gamma\|_{\Omega,\infty}$, whence there is K such that for all $x\in\Omega$

$$|\nabla_\xi G(x,\xi)| = \big|\nabla_\xi F\big(x,\xi+\nabla\gamma(x)\big)\big| \leq K. \qquad \blacksquare$$

3.4 Existence of solutions for the boundary value problem to minimal surfaces

To see whether Theorem 3.19 can be used to prove existence of solutions for the boundary value problem (1.15), (1.16) of minimal surfaces, we must study whether the function $F:\mathbb{R}^2\to\mathbb{R}^2$ given by

$$F(\xi) = \frac{\xi}{\sqrt{1+|\xi|^2}}$$

satisfies the assumptions of this theorem. Note first that $F\in C_\infty(\mathbb{R}^n,\mathbb{R}^n)$. We note that F is strongly monotone in a convex set $M\subseteq\mathbb{R}^2$ if and only if

$$\eta\cdot[\nabla F(\xi)]\eta \geq c|\eta|^2$$

for all $\xi\in M$. Now

$$\nabla F(\xi) = \nabla\,\frac{\xi}{\sqrt{1+|\xi|^2}} = \frac{1}{\sqrt{1+|\xi|^2}}\, I - \frac{1}{\sqrt{1+|\xi|^2}^{\,3}}\,\xi\otimes\xi,$$

thus

$$\begin{aligned}
\eta\cdot\nabla F(\xi)\eta &= \frac{1}{\sqrt{1+|\xi|^2}}\left(|\eta|^2 - \frac{(\xi\cdot\eta)^2}{1+|\xi|^2}\right) \\
&\geq \frac{1}{\sqrt{1+|\xi|^2}}\left(|\eta|^2 - \frac{|\xi|^2}{1+|\xi|^2}\,|\eta|^2\right) \\
&= \frac{1}{\sqrt{1+|\xi|^2}^{\,3}}\,|\eta|^2.
\end{aligned}$$

For $R > 0$ and $c = \frac{1}{\sqrt{1+R^2}^3}$ we therefore have

$$\eta \cdot F(\xi)\eta \geq c|\eta|^2,$$

for all $|\xi| \leq R$. We thus see that F is strongly monotone in every ball $B_R(0) \subseteq \mathbb{R}^2$, yet if is not strongly monotone on all of $\mathbb{R}^2$. Also, for $\xi \in \mathbb{R}^2$ we have

$$F(\xi) \cdot \xi = \frac{|\xi|^2}{\sqrt{1+|\xi|^2}} \leq |\xi|.$$

Therefore F does not satisfy a coercivity condition on $\mathbb{R}^2$ with $p > 1$. So, two of the three conditions for F are not satisfied, and the existence theory from the preceding sections cannot be applied.

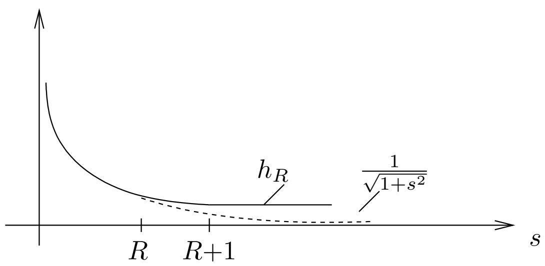
Still, at least partly, the difficulties can be circumvented as follows:

We can modify F for large values of $|\xi|$ such that the modified function satisfies the conditions needed in the existence theorem. For $R > 0$ choose a function $h_R \in C_1\big([0,\infty)\big)$ with

$$h_R(s) = \begin{cases} \frac{1}{\sqrt{1+s^2}}, & 0 \leq s \leq R \\ \text{const}, & R+1 \leq s, \end{cases}$$

and with

$$\frac{d}{ds}\frac{1}{\sqrt{1+s^2}} \leq h_R'(s) \leq 0.$$



This implies $h_R(s) = \text{const} > 0$ for $s \geq R+1$ and $h_R(s) \geq \frac{1}{\sqrt{1+s^2}}$. Set

$$G_R(\xi) = h_R(|\xi|)\xi.$$

$G_R \in C_1(\mathbb{R}^2, \mathbb{R}^2)$ satisfies

$$\nabla G_R(\xi) = h_R(|\xi|)I + h_R'(|\xi|)\frac{1}{|\xi|}\,\xi \otimes \xi,$$

hence, for $|\xi| \le R+1$

$$\begin{aligned}
\eta \cdot \nabla G_R(\xi)\eta &= h_R(|\xi|)\,|\eta|^2 + h_R'(|\xi|)\,\frac{1}{|\xi|}\,(\xi\cdot\eta)^2 \\
&\ge \big(h_R(|\xi|) + h_R'(|\xi|)\,|\xi|\big)\,|\eta|^2 \\
&\ge \Big(\frac{1}{\sqrt{1+|\xi|^2}} - \frac{|\xi|^2}{\sqrt{1+|\xi|^2}^3}\Big)\,|\eta|^2 \\
&= \frac{1}{\sqrt{1+|\xi|^2}^3}\,|\eta|^2 \ \ge\ \frac{1}{\sqrt{1+(R+1)^2}^3}\,|\eta|^2.
\end{aligned}$$

For $|x| > R+1$ we have

$$\eta \cdot \nabla G_R(\xi)\eta = \text{const}\,|\eta|^2.$$

Together it follows that G_R is a strongly monotone function on $\mathbb{R}^2$. We also have

$$\begin{aligned}
G_R(\xi)\cdot\xi &= h_R(|\xi|)\,|\xi|^2 \ \ge\ \text{const}\,|\xi|^2, \\
|G_R(\xi)| &= h_R(|\xi|)\,|\xi| \ \le\ |\xi|,
\end{aligned}$$

hence G_R satisfies the coercivity and boundedness condition with $p = 2$. We therefore conclude that for every open and bounded set $\Omega \subseteq \mathbb{R}^2$ and for every $\gamma \in \overset{\circ}{H}{}_1^{\infty}(\Omega)$ the boundary value problem

$$\operatorname{div} G_R\big(\nabla u(x)\big) = 0, \qquad x \in \Omega, \tag{3.20}$$

$$u(x) = \gamma(x), \qquad x \in \partial\Omega, \tag{3.21}$$

has a unique weak solution.

The following theorem will be proved in Section 5:

Theorem 3.20 *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with two times continuously differentiable boundary $\partial\Omega$, such that the curvature of $\partial\Omega$ is everywhere positive. Let $s \mapsto y(s) : [0,\ell) \to \partial\Omega$ be a two times continuously differentiable parametrization of $\partial\Omega$.*

Then there is a constant L, which only depends on $\frac{d^m}{ds^m}\,y$, $m = 0,1,2$, such that for every $\gamma \in C_2(\overline{\Omega})$ the solution $u \in H_1^2(\Omega)$ of (3.20), (3.21) satisfies

$$\|\nabla u\|_{\infty,\Omega} \le L \sum_{m=0}^{2} \max_{0\le s<\ell} \Big|\frac{d^m}{ds^m}\,\gamma\big(y(s)\big)\Big|.$$

With this theorem we can construct minimal surfaces. We have the following theorem:

Theorem 3.21 *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with two times continuously differentiable boundary $\partial\Omega$, such that the curvature of $\partial\Omega$ is everywhere positive. Let $\gamma \in C_2(\overline{\Omega})$. Then there is a unique weak solution $u \in H_1^2(\Omega)$ of*

$$\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = 0, \qquad x \in \Omega,$$
$$u(x) = \gamma(x), \qquad x \in \partial\Omega.$$

The solution satisfies $\|\nabla u\|_{\infty,\Omega} \le C$.

Proof. Let $F(\xi) = \frac{\xi}{\sqrt{1+\xi^2}}$ and let L be the constant from the previous theorem. Choose

$$R > L \sum_{m=0}^{2} \max_{0\le s\le \ell} \Big|\frac{d^m}{ds^m}\gamma\big(y(s)\big)\Big|.$$

With this R construct G_R as above. With this function G_R the boundary value problem (3.20), (3.21) has a unique solution $u \in H_1^2(\Omega)$, which satisfies

$$\|\nabla u\|_{\infty,\Omega} \le L \sum_{m=0}^{2} \max_{0\le s<\ell} \Big|\frac{d^m}{ds^m}\gamma\big(y(s)\big)\Big| < R.$$

Since $G_R(\xi) = F(\xi)$ for all $\xi \le R$, it follows that $G_R\big(\nabla u(x)\big) = F\big(\nabla u(x)\big)$ for all $x \in \Omega$, whence u solves

$$\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+\nabla u|^2}}\Big) = \operatorname{div} G_R(\nabla u) = 0,$$
$$u(x) = \gamma(x), \quad x \in \partial\Omega.$$

To prove uniqueness, assume that $u, v \in H_1^2(\Omega)$ are two solutions. Then, with $F(\xi) = \frac{1}{\sqrt{1+|\xi|^2}}\,\xi$,

$$\begin{aligned}\big(F(\nabla u), \nabla u - \nabla v\big)_\Omega &= 0,\\ \big(F(\nabla v), \nabla u - \nabla v\big)_\Omega &= 0.\end{aligned}$$

We have

$$\big(F(\xi) - F(\eta)\big)\cdot(\xi - \eta) \ge \min\Big(\frac{1}{\sqrt{1+|\xi|^2}}, \frac{1}{\sqrt{1+|\eta|^2}}\Big)\,|\xi-\eta|^2 > 0,$$

for $\xi \ne \eta$. Thus, if $\nabla u(x) \ne \nabla v(x)$ on a subset of Ω of positive measure,

$$\begin{aligned}0 &= \big(F(\nabla u) - F(\nabla v), \nabla u - \nabla v\big)_\Omega\\ &= \int_\Omega \big(F(\nabla u(x)) - F\big(\nabla v(x)\big)\cdot\big(\nabla u(x) - \nabla v(x)\big)\,dx > 0.\end{aligned}$$

This is a contradiction, and therefore we must have $\nabla u(x) = \nabla v(x)$ almost everywhere, whence by Poincaré's inequality,

$$\|u - v\|_{2,1,\Omega} \leq C|u - v|_{2,1,\Omega} = 0.$$

Thus, $u = v$. ■

4 Nonlinear parabolic initial-boundary value problems

4.1 The orthonormal system of eigenfunctions. The space $H_{-1}(\Omega)$

In Section 4 I study nonlinear initial-boundary value problems for the diffusion equation derived in Section 1.1 and prove that solutions exist. As in Section 3, the proof is based on the construction of approximate solutions using the Galerkin method. To simplify the computations we restrict ourselves in this section to $p = 2$ and choose for the countable dense function system $\{\nu_k\}_{k=1}^{\infty}$ needed in this method a special orthonormal system in the Hilbert space $\overset{\circ}{H}{}_1^2(\Omega)$. We introduce this system here and show that it can be conveniently used to write the elements of $\overset{\circ}{H}{}_1^2(\Omega)$ and of the dual space $H_{-1}^2(\Omega)$ in terms of this orthonormal system.

For simplicity we use in Section 4 the notations

$$\overset{\circ}{H}_1(\Omega) = \overset{\circ}{H}{}_1^2(\Omega), \quad \|u\|_\Gamma = \|u\|_{L^2(\Gamma)}, \quad \|u\|_{1,\Gamma} = \|u\|_{1,2,\Gamma} = \|u\|_{H_1(\Gamma)}.$$

We always assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded open set. For $T > 0$ we define $Q = (0, T) \times \Omega$.

Theorem 4.1 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. There is a countable complete orthonormal system $\{\nu_k\}_{k=1}^{\infty}$ of $L^2(\Omega)$, which consists of functions $\nu_k \in \overset{\circ}{H}_1(\Omega)$, and for which numbers*

$$0 < \alpha_1 \le \alpha_2 \le \alpha_3 \le \ldots \to \infty$$

exist such that

$$(\nabla u, \nabla \nu_k)_\Omega = \alpha_k^2 (u, \nu_k)_\Omega \tag{4.1}$$

for all $u \in \overset{\circ}{H}_1(\Omega)$.

(4.1) means that ν_k is a weak solution of the Dirichlet problem

$$\begin{aligned} -\Delta \nu_k(x) &= \alpha_k^2 \nu_k(x), && x \in \Omega \\ \nu_k(x) &= 0, && x \in \partial\Omega, \end{aligned}$$

hence ν_k is an eigenfunction to the Laplace operator to the eigenvalue $\lambda_k = \alpha_k^2$. The proof of this theorem is based on the solution theory to this linear boundary value problem. It can be found for example in my lecture notes "Partial differential equations I", which are online available.

Theorem 4.2 (i) *The orthonormal system satisfies*

$$(\nabla \nu_k, \nabla \nu_\ell)_\Omega = \alpha_k^2\, \delta_{k\ell}.$$

(ii) *Let $u \in L^2(\Omega)$ and let*

$$u = \sum_{k=1}^{\infty} a_k \nu_k$$

with $a_k = (u, \nu_k)_\Omega$ be the Fourier series. We have $u \in \overset{\circ}{H}_1(\Omega)$ if and only if

$$\sum_{k=1}^{\infty} \alpha_k^2 \, a_k^2 < \infty.$$

If this holds we have

$$\lim_{m\to\infty} \|\nabla u - \sum_{k=1}^{m} a_k \, \nabla \nu_k\|_\Omega = 0, \tag{4.2}$$

and

$$\|\nabla u\|^2 = \sum_{k=1}^{\infty} \alpha_k^2 \, a_k^2 \, .$$

Proof. (i) follows immediately from (4.1). To prove (ii) assume that $\sum_{k=1}^{\infty} \alpha_k^2 a_k^2 < \infty$. This implies that to $\varepsilon > 0$ we can find m_0 such that

$$\begin{aligned}
\Big\| \sum_{k=\ell}^{m} a_k \, \nabla \nu_k \Big\|^2 &= \Big(\sum_{k=\ell}^{m} a_k \, \nabla \nu_k, \, \sum_{k=\ell}^{m} a_k \, \nabla \nu_k \Big)_\Omega \\
&= \sum_{k=\ell}^{m} \sum_{j=\ell}^{m} a_k a_j (\nabla_k, \nabla \nu_j)_\Omega = \sum_{k=\ell}^{m} \alpha_k^2 a_k^2 < \varepsilon,
\end{aligned}$$

for $m \geq \ell \geq m_0$. Consequently, $\sum_{k=1}^{\infty} a_k \, \nabla \nu_k$ is a Cauchy sequence and thus converges in $L^2(\Omega)$. This yields that $\sum_{k=1}^{\infty} a_k \nu_k$ converges in $\overset{\circ}{H}_1(\Omega)$, whence u belongs to $\overset{\circ}{H}_1(\Omega)$ with weak derivative ∇u given by the function

$$\sum_{k=1}^{\infty} a_k \, \nabla \nu_k = \lim_{m\to\infty} \sum_{k=1}^{m} a_k \, \nabla \nu_k = \lim_{m\to\infty} \nabla \Big(\sum_{k=1}^{m} a_k \nu_k \Big).$$

Obviously, this implies (4.2). To prove the converse we assume that $u \in \overset{\circ}{H}_1(\Omega)$. Then (4.1) yields

$$\begin{aligned}
0 &\leq \Big(\nabla u - \sum_{k=1}^{m} a_k \, \nabla \nu_k, \, \nabla u - \sum_{k=1}^{m} a_k \nabla \nu_k \Big)_\Omega \\
&= (\nabla u, \nabla u)_\Omega - 2 \sum_{k=1}^{m} a_k (\nabla u, \nabla \nu_k)_\Omega + \sum_{k=1}^{m} a_k^2 (\nabla \nu_k, \nabla \nu_k)_\Omega \\
&= \|\nabla u\|^2 - 2 \sum_{k=1}^{m} \alpha_k^2 a_k^2 + \sum_{k=1}^{m} \alpha_k^2 a_k^2 = \|\nabla u\|^2 - \sum_{k=1}^{m} \alpha_k^2 u_k^2,
\end{aligned}$$

from which we see that $\sum_{k=1}^{\infty} \alpha_k^2 a_k^2$ converges. As we showed above, this implies that (4.2) holds. Therefore the last estimate yields

$$\sum_{k=1}^{\infty} \alpha_k^2 a_k^2 = \|\nabla u\|^2 - \lim_{m\to\infty} \|\nabla u - \sum_{k=1}^{m} a_k \nabla \nu_k\|^2 = \|\nabla u\|^2. \qquad \blacksquare$$

The Poincaré inequality implies that to the bounded, open set $\Omega \subseteq \mathbb{R}^n$ there is a constant $C > 0$ such that

$$\|\nabla u\|_\Omega \leq \|u\|_{1,\Omega} \leq C\|\nabla u\|_\Omega, \quad u \in \overset{\circ}{H}_1(\Omega).$$

Therefore $\overset{\circ}{H}_1(\Omega)$ is a Hilbert space with scalar product $(\nabla u, \nabla v)_\Omega$ and norm $\|\nabla u\|_\Omega$, which is equivalent to $\|u\|_{1,\Omega}$. We always assume that $\overset{\circ}{H}_1(\Omega)$ is equipped with this scalar product and norm. Theorem 4.2 shows that $L^2(\Omega)$ and $\overset{\circ}{H}_1(\Omega)$ can be written in terms of the orthonormal system as follows:

Corollary 4.3 *Let $\overset{\circ}{H}_0(\Omega) = L^2(\Omega)$. Then, for $i = 0, 1$,*

$$\overset{\circ}{H}_i(\Omega) = \Big\{ u = \sum_{k=1}^{\infty} a_k \nu_k \;\Big|\; \sum_{k=1}^{\infty} (\alpha_k^i a_k)^2 < \infty \Big\}.$$

For $u = \sum_{k=1}^{\infty} a_k \nu_k$, $v = \sum_{k=1}^{\infty} d_k \nu_k \in \overset{\circ}{H}_i(\Omega)$, the scalar products and norms are

$$(\nabla^i u, \nabla^i v)_\Omega = \sum_{k=1}^{\infty} \alpha_k^{2i} a_k d_k\,, \quad \|\nabla^i u\|_\Omega^2 = \sum_{i=1}^{\infty} (\alpha_k^i a_k)^2.$$

This result suggests to define a space $H_{-1}(\Omega)$ as follows:

Definition 4.4 Let

$$H_{-1}(\Omega) = \Big\{ \sum_{k=1}^{\infty} a_k \nu_k \;\Big|\; \sum_{k=1}^{\infty} (\alpha_k^{-1} a_k)^2 < \infty \Big\},$$

and for $u = \sum_{k=1}^{\infty} d_k \nu_k$, $v = \sum_{k=1}^{\infty} \hat{d}_k \nu_k \in H_{-1}(\Omega)$ define

$$(u, v)_{-1} = \sum_{k=1}^{\infty} \alpha_k^{-2} d_k \hat{d}_k, \quad \|u\|_{-1} = \sqrt{\sum_{k=1}^{\infty} \alpha_k^{-2} d_k^2}. \tag{4.3}$$

It is not difficult to prove that $H_{-1}(\Omega)$ is a Hilbert space with scalar product and norm defined in (4.3).

Note that we do not require that $\sum_{k=1}^{\infty} a_k^2 < \infty$. Therefore the series in the space $H_{-1}(\Omega)$ do not necessarily converge in $L^2(\Omega)$. We consider the elements of $H_{-1}(\Omega)$ to be formal series. Yet, the following Lemma gives another interpretation of the elements of the space $H_{-1}(\Omega)$.

Lemma 4.5 *To every bounded linear form v on $\overset{\circ}{H}_1(\Omega)$ there is a unique series $\sum_{k=1}^{\infty} d_k\nu_k \in H_{-1}(\Omega)$ such that for all $u = \sum_{k=1}^{\infty} a_k\nu_k \in \overset{\circ}{H}_1(\Omega)$*

$$\langle v, u\rangle = \sum_{k=1}^{\infty} d_k a_k. \tag{4.4}$$

On the other hand, every series $\sum_{k=1}^{\infty} d_k\nu_k \in H_{-1}(\Omega)$ defines a bounded linear form v on $\overset{\circ}{H}_1(\Omega)$ by (4.4) with

$$\|v\| = \Big\| \sum_{k=1}^{\infty} d_k\nu_k \Big\|_{-1}, \tag{4.5}$$

hence $H_{-1}(\Omega)$ is isometrically isomorphic to the dual space $\overset{\circ}{H}_1(\Omega)'$.

Proof. Let v be a bounded linear form on $\overset{\circ}{H}_1(\Omega)$. Since $\nu_k \in \overset{\circ}{H}_1(\Omega)$, we can apply v to ν_k and define

$$d_k = \langle v, \nu_k\rangle, \quad k \in \mathbb{N}.$$

The continuity of v implies for $u = \sum_{k=1}^{\infty} a_k\nu_k \in \overset{\circ}{H}_1(\Omega)$

$$\langle v, u\rangle = \Big\langle v, \sum_{k=1}^{\infty} a_k\nu_k \Big\rangle = \sum_{k=1}^{\infty} a_k \langle v, \nu_k\rangle = \sum_{k=1}^{\infty} a_k d_k.$$

To finish the first part of the proof we show that $\sum_{k=1}^{\infty} d_k\nu_k \in H_{-1}(\Omega)$. To this end we verify that $\sum_{k=1}^{\infty} \alpha_k^{-2} d_k^2 < \infty$. Note that the boundedness of v implies for $\sum_{k=1}^{m} \alpha_k^{-2} d_k\nu_k \in \overset{\circ}{H}_1(\Omega)$ that

$$\begin{aligned}
\sum_{k=1}^{m} \alpha_k^{-2} d_k^2 &= \sum_{k=1}^{m} \alpha_k^{-2} d_k \langle v, \nu_k\rangle = \Big\langle v, \sum_{k=1}^{m} \alpha_k^{-2} d_k \nu_k \Big\rangle \\
&\leq \|v\| \Big\| \sum_{k=1}^{m} \alpha_k^{-2} d_k\nu_k \Big\|_1 \\
&= \|v\| \Big(\sum_{k=1}^{m} \alpha_k^2 (\alpha_k^{-2} d_k)^2 \Big)^{1/2} = \|v\| \sqrt{\sum_{k=1}^{m} \alpha_k^{-2} d_k^2}.
\end{aligned}$$

This inequality yields

$$\sqrt{\sum_{k=1}^{m} \alpha_k^{-2} d_k^2} \leq \|v\| \tag{4.6}$$

for all $m \in \mathbb{N}$, whence $\sum_{k=1}^{\infty} \alpha_k^{-2} d_k^2 < \infty$. Consequently, $\sum_{k=1}^{\infty} d_k\nu_k$ is the unique element in H_{-1} satisfying (4.4).

Conversely, for all $\sum_{k=1}^{\infty} d_k\nu_k \in H_{-1}(\Omega)$ and $u = \sum_{k=1}^{\infty} a_k\nu_k \in \overset{\circ}{H}_1(\Omega)$ we have

$$\sum_{k=1}^{\infty} |d_k a_k| = \sum_{k=1}^{\infty} |\alpha_k^{-1} d_k \alpha_k a_k| \le \sqrt{\sum_{k=1}^{\infty} \alpha_k^{-2} d_k^2}\sqrt{\sum_{k=1}^{\infty} \alpha_k^2 a_k^2} = \Big\| \sum_{k=1}^{\infty} d_k\nu_k \Big\|_{-1} |u|_{1,\Omega}.$$

Therefore $\sum_{k=1}^{\infty} d_k\nu_k$ defines a bounded linear form v on $\overset{\circ}{H}_1(\Omega)$ via (4.4) with

$$\|v\| \le \Big\| \sum_{k=1}^{\infty} d_k\nu_k \Big\|_{-1} = \sqrt{\sum_{k=1}^{\infty} \alpha_k^{-2} d_k^2}.$$

Equation (4.5) follows from this estimate and from (4.6). ■

4.2 The Bochner spaces $L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$, $L^2\big(0,T;H_{-1}(\Omega)\big)$ and V

To study the initial-boundary value problem we need Sobolev spaces of functions, which depend on space and time, and which have weak derivatives with respect to the space and time variables of different order. These spaces are called Bochner spaces. Here we introduce and study several such spaces and in particular the space V, in which the solutions of the initial-boundary value problem lie.

We use the following notation: For $u : \overline{Q} \to \mathbb{R}$ and $0 \le t \le T$ let the function $u(t) : \Omega \to \mathbb{R}$ be defined by

$$u(t)(x) = u(t,x), \quad x \in \Omega.$$

Let X be a separable Banach space with dual space X'. A function $u : [0,T] \to X$ is called weakly measurable, if $t \mapsto \langle f, u(t)\rangle$ is measurable for every $f \in X'$. Since X is separable, it follows by Petti's theorem (cf. K. Yoshida, Functional Analysis, p. 131) that $t \mapsto \|u(t)\|$ is measurable. This is used in the definition of the Bochner space $L^2(0,T;X)$:

$$L^2(0,T;X) = \Big\{ u : [0,T] \to X \mid u \text{ is weakly measurable, } \int_0^T \|u(t)\|^2\, dt < \infty \Big\}. \quad (4.7)$$

Theorem 4.6 $L^2(0,T;X)$ *is a Banach space with the norm*

$$\|u\|_{L^2(X)} = \sqrt{\int_0^T \|u(t)\|^2 dt}\,.$$

For a proof cf. the book “Linear Operators I” by N. Dunford and J.T. Schwartz.

This definition applies to $X = \overset{\circ}{H}_1(\Omega)$ and $X = H_{-1}(\Omega)$, since the set $\{\nu_k\}_{k=1}^{\infty}$ is dense in both spaces, which means that $\overset{\circ}{H}_1(\Omega)$ and $H_{-1}(\Omega)$ are separable. We note that the norm in $L^2(0,T;\overset{\circ}{H}_1(\Omega))$ is

$$\|u\|^2_{L^2(H_1)} = \int_0^T \|\nabla_x u(t)\|^2_\Omega\,dt = \|\nabla_x u\|^2_Q,$$

and the norm in $L^2(0,T;H_{-1}(\Omega))$ is

$$\|u\|^2_{L^2(H_{-1})} = \int_0^T \Big\|\sum_{k=1}^\infty a_k\nu_k\Big\|^2_{-1}\,dt = \int_0^T \sum_{k=1}^\infty \alpha_k^{-2} a_k(t)^2\,dt.$$

To give other characterizations of the Bochner spaces $L^2(0,T;\overset{\circ}{H}_1(\Omega))$ and $L^2(0,T;H_{-1}(\Omega))$, we need the following lemma.

Lemma 4.7 (i) *Assume that $a_k \in L^2((0,T))$ for $k \in \mathbb{N}$. The series $\sum_{k=1}^\infty a_k\nu_k$ converges in $L^2(0,T;\overset{\circ}{H}_1(\Omega))$ or in $L^2(0,T;H_{-1}(\Omega))$ respectively, if and only if $\sum_{k=1}^\infty \alpha_k^2\|a_k\|^2_{(0,T)} < \infty$ or $\sum_{k=1}^\infty \alpha_k^{-2}\|a_k\|^2_{(0,T)} < \infty$, respectively.*

(ii) *If $u \in L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$ then the function $t \mapsto a_k(t) = \big(u(t),\nu_k\big)_\Omega$ belongs to $L^2\big((0,T)\big)$ and the series $\sum_{k=1}^\infty a_k\nu_k$ converges to u in $L^2(0,T;\overset{\circ}{H}_1(\Omega))$.*

(iii) *If $u \in L^2(0,T;H_{-1}(\Omega))$, then the function $t \mapsto a_k(t) = \langle u(t),\nu_k\rangle$ belongs to $L^2\big((0,T)\big)$ and the series $\sum_{k=1}^\infty a_k\nu_k$ converges to u in $L^2(0,T;H_{-1}(\Omega))$.*

Proof. To prove (i) note that

$$\begin{aligned}\Big\|\nabla_x\Big(\sum_{k=l}^m a_k\nu_k\Big)\Big\|^2_Q &= \int_0^T \Big(\sum_{k=l}^m a_k(t)\nu_k, \sum_{k=l}^m a_k(t)\nu_k\Big)_\Omega\,dt\\ &= \int_0^T \sum_{k=l}^m a_k(t)^2\alpha_k^2\,dt = \sum_{k=l}^m \alpha_k^2\|a_k\|^2_{(0,T)}.\end{aligned}$$

¿From this equation it follows that $\sum_{k=1}^\infty a_k\nu_k$ is a Cauchy sequence in the space $L^2(0,T;\overset{\circ}{H}_1(\Omega))$ if and only if $\sum_{k=1}^\infty \alpha_k^2\|a_k\|^2_{(0,T)}$ is a Cauchy sequence. For $L^2(0,T;H_{-1}(\Omega))$ the statement is proved in the same way.

For the proof of statement (ii) we use that if $u \in L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$, then

$$t \mapsto a_k(t) = (u(t),\nu_k)_\Omega : [0,T] \to \mathbb{R}$$

is weakly measurable, since $w \mapsto (w,\nu_k)_\Omega : \overset{\circ}{H}_1(\Omega) \to \mathbb{R}$ is a bounded linear form, and we have

$$\int_0^T a_k(t)^2 dt \le C\int_0^T \|\nabla u(t)\|^2_\Omega\,dt = C\|u\|_{L^2(H_1)} < \infty,$$

hence u belongs to $L^2((0,T))$. Moreover, the series $\sum_{k=1}^\infty a_k(t)\nu_k$ converges for all t in $\overset{\circ}{H}_1(\Omega)$ to $u(t)$ with $\|\nabla u(t)\|_\Omega^2 = \sum_{k=1}^\infty \alpha_k^2 a_k(t)^2$. Therefore

$$\sum_{k=1}^\infty \alpha_k^2 \|a_k\|_{(0,T)}^2 = \int_0^T \sum_{k=1}^\infty \alpha_k^2 a_k(t)^2 \, dt = \|u\|_{L^2(H_1)}^2 < \infty.$$

The first part of the proof thus implies that $\sum_{k=1}^\infty a_k\nu_k$ converges in the space $L^2(0,T;\overset{\circ}{H}_1(\Omega))$ to a function v. Since the series converges for all t to $u(t)$, it follows that $u = v$.

Assertion (iii) follows by a slight modification of the arguments in (ii). ■

Corollary 4.8 *The Bochner spaces* $L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$ *and* $L^2\big(0,T;H_{-1}(\Omega)\big)$ *satisfy*

$$\begin{aligned} L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big) &= \Big\{u = \sum_{k=1}^\infty a_k\nu_k \;\Big|\; t \mapsto a_k(t) \text{ is measurable,} \\ &\qquad a_k \in L^2\big((0,T)\big), \sum_{k=1}^\infty \alpha_k^2 \|a_k\|_{(0,T)}^2 < \infty\Big\} \qquad (4.8) \\ &= \Big\{u \in L^2(Q) \;\Big|\; \nabla_x u \in L^2(Q), u(t) \in \overset{\circ}{H}_1(\Omega) \text{ for a.e. } t\Big\}, \\ L^2\big(0,T;H_{-1}(\Omega)\big) &= \Big\{v = \sum_{k=1}^\infty d_k\nu_k \;\Big|\; t \mapsto d_k(t) \text{ is measurable,} \\ &\qquad d_k \in L^2\big((0,T)\big), \sum_{k=1}^\infty \alpha_k^{-2} \|d_k\|_{(0,T)}^2 < \infty\Big\}, \qquad (4.9) \end{aligned}$$

$$\|u\|_{L^2(H_1)}^2 = \|\nabla_x u\|_Q^2 = \sum_{k=1}^\infty \alpha_k^2 \|a_k\|_{(0,T)}^2, \quad \|v\|_{L^2(H_{-1})}^2 = \sum_{k=1}^\infty \alpha_k^{-2} \|d_k\|_{(0,T)}^2.$$

Lemma 4.9 *To every bounded linear form* $v \in L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$ *there is a unique element* $\sum_{k=1}^\infty d_k\nu_k \in L^2\big(0,T;H_{-1}(\Omega)\big)$ *such that for all* $u = \sum_{k=1}^\infty a_k\nu_k \in L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$

$$\langle v, u\rangle = \sum_{k=1}^\infty (d_k, a_k)_{(0,T)}. \qquad (4.10)$$

On the other hand, every series $\sum_{k=1}^\infty d_k\nu_k \in L^2\big(0,T;H_{-1}(\Omega)\big)$ *defines a bounded linear form* v *on* $L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$ *by (4.10) with*

$$\|v\|^2 = \sum_{k=1}^\infty \alpha_k^{-2}\, \|d_k\|_{(0,T)}^2 = \Big\|\sum_{k=1}^\infty d_k\nu_k\Big\|_{L^2(H_{-1})}^2, \qquad (4.11)$$

hence the space $L^2\big(0,T;H_{-1}(\Omega)\big)$ *is isometrically isomorphic to the dual space* $L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)'$.

The proof is very similar to the proof of Lemma 4.5, but we give it for completeness.

Proof. Let v be a bounded linear form on $L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$. If $a_k \in L^2\big((0,T)\big)$ then $a_k\nu_k \in L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$, and

$$a_k \mapsto \langle v, a_k\nu_k\rangle : \; L^2\big((0,T)\big) \to \mathbb{R}$$

defines a bounded linear functional. Consequently, there is $d_k \in L^2\big((0,T)\big)$ such that

$$\langle v, a_k\nu_k\rangle = (d_k, a_k)_{(0,T)}\,. \tag{4.12}$$

The continuity of v implies for $u = \sum_{k=1}^{\infty} a_k\nu_k \in L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$ that

$$\langle v, u\rangle = \langle v, \sum_{k=1}^{\infty} a_k\nu_k\rangle = \sum_{k=1}^{\infty}\langle v, a_k\nu_k\rangle = \sum_{k=1}^{\infty}(d_k, a_k)_{(0,T)}.$$

We show that

$$\sum_{k=1}^{\infty} d_k\nu_k \in L^2\big(0,T;H_{-1}(\Omega)\big). \tag{4.13}$$

To this end note that (4.12) yields

$$\begin{aligned}
\sum_{k=1}^{m}\alpha_k^{-2}\|d_k\|^2_{(0,T)} &= \sum_{k=1}^{m}\alpha_k^{-2}(d_k, d_k)_{(0,T)} \\
&= \sum_{k=1}^{m}\alpha_k^{-2}\langle v, d_k\nu_k\rangle = \langle v, \sum_{k=1}^{m}\alpha_k^{-2}d_k\nu_k\rangle \\
&\le \|v\|\,\Big|\sum_{k=1}^{m}\alpha_k^{-2}d_k\nu_k\Big|_{L^2(\overset{\circ}{H}_1)} = \|v\|\Big(\sum_{k=1}^{m}\alpha_k^{-2}\|d_k\|^2_{(0,T)}\Big)^{1/2}.
\end{aligned}$$

This inequality yields

$$\sqrt{\sum_{k=1}^{m}\alpha_k^{-2}\|d_k\|^2_{(0,T)}} \le \|v\| \tag{4.14}$$

for all $m \in \mathbb{N}$, whence $\sum_{k=1}^{\infty}\alpha_k^{-2}\|d_k\|^2_{(0,T)} < \infty$, which proves (4.13). Consequently, $\sum_{k=1}^{\infty} d_k\nu_k$ is the unique element in $L^2\big(0,T;H_{-1}(\Omega)\big)$ satisfying (4.10).

Conversely, for all $\sum_{k=1}^{\infty} d_k\nu_k \in L^2\big(0,T;H_{-1}(\Omega)\big)$ and $u = \sum_{k=1}^{\infty} a_k\nu_k \in L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$ we have

$$\begin{aligned}
\sum_{k=1}^{\infty}|(d_k, a_k)_{(0,T)}| &\le \sum_{k=1}^{\infty}\alpha_k^{-1}\|d_k\|_{(0,T)}\alpha_k\|a_k\|_{(0,T)} \\
&\le \sqrt{\sum_{k=1}^{\infty}\alpha_k^{-2}\|d_k\|^2_{(0,T)}}\,\sqrt{\sum_{k=1}^{\infty}\alpha_k^{2}\|a_k\|^2_{(0,T)}} \\
&= \sqrt{\sum_{k=1}^{\infty}\alpha_k^{-2}\|d_k\|^2_{(0,T)}}\;\|u\|_{L^2(H_1)}\,.
\end{aligned}$$

Therefore $\sum_{k=1}^{\infty} d_k \nu_k$ defines a bounded linear form v on $L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$ via (4.10) with

$$\|v\| \le \sqrt{\sum_{k=1}^{\infty} \alpha_k^{-2} \|d_k\|_{(0,T)}^2}\,.$$

Equation (4.11) follows from this estimate and from (4.14). ■

We also need several spaces of functions with time derivatives. The first such space can be defined using the ordinary weak derivative u_t: Let

$$H_1\big(0,T;\overset{\circ}{H}_1(\Omega)\big) = \{u \in L^2(Q) \mid u_t, \nabla_x u, \nabla_x u_t \in L^2(Q)\}.$$

This is a Hilbert space with norm

$$\|u\|_{H_1(H_1)}^2 = \|\nabla_x u\|_Q^2 + \|\nabla_x u_t\|_Q^2.$$

It is immediately seen from Corollary 4.8 that $u \in H_1\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$ implies $u, u_t \in L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$. To define the space $H_1\big(0,T;H_{-1}(\Omega)\big)$ we need

Lemma 4.10 *The space $\overset{\circ}{C}_\infty\big((0,T)\times\Omega\big)$ is dense in the space $L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$.*

The proof is left to the reader.

Definition 4.11 $v \in L^2(0,T;H_{-1}(\Omega))$ *is called weak time derivative of* $u \in L^2(0,T;H_{-1}(\Omega))$*, if for all* $w \in \overset{\circ}{C}_\infty\big((0,T)\times\Omega\big)$

$$\langle v, w\rangle = -\langle u, w_t\rangle.$$

Weak time derivatives are uniquely defined, since for two weak time derivatives $v_1, v_2 \in L^2(0,T;H_{-1}(\Omega))$ of u we have

$$\langle v_1, w\rangle = -\langle u, w_t\rangle = \langle v_2, w\rangle \tag{4.15}$$

for all $w \in \overset{\circ}{C}_\infty\big((0,T)\times\Omega\big)$. The dual space $L^2(0,T;\overset{\circ}{H}_1(\Omega))$ of $L^2(0,T;H_{-1}(\Omega))$ separates points in $L^2(0,T;\overset{\circ}{H}_1(\Omega))$. Since $\overset{\circ}{C}_\infty\big((0,T)\times\Omega\big)$ is dense in the space $L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$, also $\overset{\circ}{C}_\infty\big((0,T)\times\Omega\big)$ separates points, which implies that (4.15) can only hold if $v_1 = v_2$.

We denote the uniquely defined weak time derivative of u by u_t, and the space of all $u \in L^2(0,T;H_{-1}(\Omega))$ having a weak time derivative $u_t \in L^2(0,T;H_{-1}(\Omega))$ is denoted by $H_1(0,T;H_{-1}(\Omega))$.

Lemma 4.12 *We have*

$$H_1\big(0,T;H_{-1}(\Omega)\big) \;=\; \Big\{\sum_{k=1}^{\infty} a_k\nu_k \mid a_k \in H_1\big((0,T)\big),$$
$$\sum_{k=1}^{\infty} \alpha_k^{-2}\big(\|a_k\|^2_{(0,T)} + \|a_k'\|^2_{(0,T)}\big) < \infty\Big\}.$$

Proof. (4.9) implies that u belongs to $H_1(0,T;H_{-1}(\Omega))$, if and only if u and u_t have the representations $u = \sum_{k=1}^{\infty} a_k\nu_k$, $u_t = \sum_{k=1}^{\infty} d_k\nu_k$ with $a_k, d_k \in L^2((0,T))$ and $\sum_{k=1}^{\infty} \alpha_k^{-2}(\|a_k\|^2_{(0,T)}+\|d_k\|^2_{(0,T)}) < \infty$, such that for every $w = \sum_{k=1}^{\infty} w_k\nu_k \in \overset{\circ}{C}_\infty((0,T)\times\Omega)$

$$\sum_{k=1}^{\infty}(d_k, w_k)_{(0,T)} = \langle u_t, w\rangle = -\langle u, w_t\rangle = -\sum_{k=1}^{\infty}(a_k, w_k')_{(0,T)}.$$

Here we used (4.10). This equation holds for all $\sum_{k=1}^{\infty} w_k\nu_k$ if and only if

$$(d_k, w)_{(0,T)} = -(a_k, w')_{(0,T)}$$

for all k and all $w \in \overset{\circ}{C}_\infty((0,T))$. This last equation is equivalent to $d_k = a_k'$. ∎

Finally we introduce the space V which we need in the proof of existence of solutions to the initial-boundary value problem and which contains the solutions. Let

$$\begin{aligned} V &= L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big) \cap H_1\big(0,T;H_{-1}(\Omega)\big) \\ &= \Big\{u = \sum_{k=1}^{\infty} a_k\nu_k \mid a_k \in H_1\big((0,T)\big), \sum_{k=1}^{\infty}\big(\alpha_k^2\|a_k\|^2_{(0,T)} + \alpha_k^{-2}\|a_k'\|^2_{(0,T)}\big) < \infty\Big\}. \end{aligned}$$

This is a Hilbert space with scalar product

$$[u,v] = \Big[\sum_{k=1}^{\infty} a_k\nu_k, \sum_{k=1}^{\infty} d_k\nu_k\Big] = \sum_{k=1}^{\infty}\big(\alpha_k^2(a_k,d_k)_{(0,T)} + \alpha_k^{-2}(a_k',d_k')_{(0,T)}\big)$$

and norm

$$\|u\|_V^2 = \sqrt{\sum_{k=1}^{\infty}\big(\alpha_k^2\|a_k\|^2_{(0,T)} + \alpha_k^{-2}\|a_k'\|^2_{(0,T)}\big)}.$$

For $u, v \in V$ we have $v \in L^2(0,T;\overset{\circ}{H}_1(\Omega))$ and $u_t \in L^2(0,T;H_{-1}(\Omega)) = L^2(0,T;\overset{\circ}{H}_1(\Omega))'$. Therefore $\langle u_t, v\rangle$ is defined and Lemma 4.9 immediately yields

Corollary 4.13 *Let* $u = \sum_{k=1}^{\infty} a_k\nu_k \in V$, $v = \sum_{k=1}^{\infty} d_k\nu_k \in V$. *Then*

$$\langle u_t, v\rangle = \sum_{k=1}^{\infty}(a_k', d_k)_{(0,T)}.$$

4.3 The trace theorem for the space V

We now show that for functions $u \in V$ one can define boundary values $(x \mapsto u(0,x)) \in L^2(\Omega)$ and $(x \mapsto u(T,x)) \in L^2(\Omega)$. For preparation we prove the trace theorem for functions in $H_1((a,b))$.

Lemma 4.14 *Let (a,b) be a bounded or unbounded interval. $u \in C_1\big((a,b)\big) \cap H_1\big((a,b)\big)$ satisfies*

$$\begin{aligned} |u(x) - u(y)| &\leq \|u'\|_{(a,b)} \, |x-y|^{1/2}, \\ |u(x) &\leq r^{1/2} \, \|u'\|_{(a,b)} + r^{-1/2} \, \|u\|_{(a,b)} \, , \end{aligned} \tag{4.16}$$

for all $x, y \in (a,b)$ and for all $0 < r < b-a$.

Proof. The fundamental theorem of calculus yields

$$\begin{aligned} |u(x) - u(y)| &= \Big| \int_y^x u'(z) dz \Big| \leq \Big(\int_y^x dz \Big)^{1/2} \Big(\int_y^x |u'(z)|^2 \, dz \Big)^{1/2} \\ &\leq |x-y|^{1/2} \, \|u'\|_{(a,b)} \, . \end{aligned}$$

Using this estimate we have for all $y \in (a,b)$

$$|u(x)| \leq |u(x) - u(y)| + |u(y)| \leq |x-y|^{1/2} \, \|u'\|_{(a,b)} + |u(y)|.$$

Let (c,d) with $x \in (c,d) \subseteq (a,b)$ be an interval of finite length. We integrate with respect to y from c to d and obtain

$$\begin{aligned} |u(x)|(d-c) &\leq \|u'\|_{(a,b)} \int_c^d |x-y|^{1/2} \, dx + \int_c^d |u(y)| dy \\ &\leq \|u'\|_{(a,b)} \, (d-c)^{3/2} + (d-c)^{1/2} \Big(\int_c^d |u(y)|^2 dy \Big) u^{1/2}. \end{aligned}$$

Division by $(d-c)$ yields

$$|u(x)| \leq (d-c)^{1/2} \, \|u'\|_{(a,b)} + (d-c)^{-1/2} \|u\|_{(a,b)} \, .$$

■

Corollary 4.15 (Trace theorem for $H_1\big((a,b)\big)$) *Let (a,b) be a bounded interval. Then for every $c \in [a,b]$ there is a unique bounded linear operator $P_c : H_1\big((a,b)\big) \to \mathbb{R}$, which for $u \in H_1\big((a,b)\big) \cap C_1([a,b])$ satisfies*

$$P_c u = u(c).$$

One calls $P_c u$ the trace of $u \in H_1\big((a,b)\big)$ and writes $P_c u = u(c)$.

Proof. By the first estimate of the preceding lemma, $u \in C_1\big((a,b)\big) \cap H_1\big((a,b)\big)$ is uniformly continuous. Therefore u can be extended in a unique way to a continuous function on $[a,b]$, which we also denote by u. It is immediately seen that u satisfies both estimates of the preceding lemma for all $x, y \in [a,b]$. Choose $c \in [a,b]$. The mapping $P_c : C_1\big([a,b]\big) \cap H_1\big((a,b)\big) \to \mathbb{R}$ defined by

$$P_c u = u(c)$$

is linear, and the second estimate yields

$$|P_c u| \le C\|u\|_{1,(a,b)} ,$$

hence P_c is continuous with respect to the norm of $H_1\big((a,b)\big)$. Since $C_1([a,b]) \cap H_1\big((a,b)\big)$ is dense in $H_1\big((a,b)\big)$, the mapping P_c can be extended in a unique way to a continuous linear mapping on $H_1\big((a,b)\big)$. ■

Corollary 4.16 *For all $u, v \in H_1\big((a,b)\big)$ one has*

$$(u', v)_{(a,b)} + (u, v')_{(a,b)} = u(b)v(b) - u(a)v(a). \tag{4.17}$$

Let $\tilde{u}$ be a representative of the equivalence class u. Then

$$|\tilde{u}(x) - \tilde{u}(y)| \le \|u'\|_{(a,b)}\, |x-y|^{1/2} \tag{4.18}$$

for almost all x, y. For $u \in \overset{\circ}{H}_1\big((a,b)\big)$ one has

$$u(a) = u(b) = 0. \tag{4.19}$$

Proof. We choose sequences $\{u_\ell\}_{\ell=1}^\infty$, $\{v_\ell\}_{\ell=1}^\infty \subseteq C_1\big((a,b)\big) \cap H_1\big((a,b)\big)$ such that

$$\|u - u_\ell\|_{1,\Omega} \to 0, \quad \|v - v_\ell\|_{1,\Omega} \to 0$$

for $\ell \to \infty$. By the fundamental theorem of calculus we have

$$(u_\ell', v_\ell)_{(a,b)} + (u_\ell, v_\ell')_{(a,b)} = u_\ell(b)v_\ell(b) - u_\ell(a)v_\ell(a).$$

¿From the continuity of the trace operator it follows that the right hand side converges to $u(b)v(b) - u(a)v(a)$ for $\ell \to \infty$. The left hand side converges to $(u', v)_{(a,b)} + (u, v')_{(a,b)}$, since the scalar products are continuous on $L^2\big((a,b)\big)^2$. This proves (4.17). To prove (4.18), we choose a subsequence $\{u_{\ell_k}\}_{k=1}^\infty$ of $\{u_\ell\}_{\ell=1}^\infty$ such that

$$\lim_{k\to\infty} u_{\ell_k}(x) = \tilde{u}(x)$$

for all x from a subset $M \subseteq (a,b)$ with $\mathrm{meas}([a,b]\backslash M) = 0$. This is possible by a well known theorem from integration theory. Using (4.16) we obtain for $x, y \in M$

$$\begin{aligned} |\tilde{u}(x) - \tilde{u}(y)| &\leq \lim_{k\to\infty} \big(|\tilde{u}(x) - u_{\ell_k}(x)| + |u_{\ell_k}(x) - u_{\ell_k}(y)| + |u_{\ell_k}(y) - \tilde{u}(y)|\big) \\ &\leq \lim_{k\to\infty} |x-y|^{1/2}\, \|u'_{\ell_k}\|_{(a,b)} = |x-y|^{1/2}\, \|u'\|_{(a,b)}. \end{aligned}$$

To prove (4.19) we choose a sequence $\{\varphi_\ell\}_{\ell=1}^\infty \subseteq \overset{\circ}{C}_\infty\big((a,b)\big)$ such that $\|u - \varphi_\ell\|_{1,(a,b)} \to 0$. The continuity of the trace operator implies $u(a) = \lim_{\ell\to\infty} \varphi_\ell(a) = 0$, $u(b) = \lim_{\ell\to\infty} \varphi_\ell(b) = 0$. ■

Remark. In the sense of traces, the estimate

$$|u(x) - u(y)| \leq \|u'\|_{(a,b)}\, |x-y|^{1/2}$$

holds for all $u \in H_1\big((a,b)\big)$ and all $x, y \in (a,b)$. Thus $x \mapsto P_x u$ is a uniformly Hölder continuous mapping. Every representative coincides almost everywhere with this mapping, whence $x \mapsto P_x u$ is itself a representative of u. Therefore $u \in H_1\big((a,b)\big)$ has exactly one Hölder continuous representative.

Theorem 4.17 (Trace theorem for the space V) *There are continuous linear mappings $P_0, P_T : V \to L^2(\Omega)$ such that for all $u, v \in V$ we have*

$$\begin{aligned} \langle u_t, v\rangle + \langle u, v_t\rangle &= (P_T u, P_T v)_\Omega - (P_0 u, P_0 v)_\Omega, \\ \langle u_t, u\rangle &= \frac{1}{2}\,\|P_T u\|_\Omega^2 - \frac{1}{2}\,\|P_0 u\|_\Omega^2. \end{aligned}$$

One writes $P_T u = u(T)$, $P_0 u = u(0)$ and calls $u(T), u(0)$ the traces of u on the boundary parts $\{T\}\times\Omega$, $\{0\}\times\Omega$ of Q.

Proof. Let $u = \sum_{k=1}^\infty a_k \nu_k$, $v = \sum_{k=1}^\infty b_k \nu_k \in V$. Since $a_k, b_k \in H_1\big((0,T)\big)$, the traces $a_k(T), b_k(T), a_k(0), b_k(0)$ exist and satisfy

$$(a'_k, b_k)_{(0,T)} + (a_k, b'_k)_{(0,T)} = a_k(T)b_k(T) - a_k(0)b_k(0).$$

This implies

$$\begin{aligned} (a'_k, a_k)_{(0,T)} &= \frac{1}{2}(a'_k, a_k)_{(0,T)} + \frac{1}{2}(a'_k, a_k)_{(0,T)} \\ &= \frac{1}{2}(a'_k, a_k)_{(0,T)} - \frac{1}{2}(a'_k, a_k)_{(0,T)} + \frac{1}{2}\,a_k(T)^2 - \frac{1}{2}\,a_k(0)^2 \\ &= \frac{1}{2}\,a_k(T)^2 - \frac{1}{2}\,a_k(0)^2. \end{aligned}$$

Thus,

$$\langle u_t, u\rangle = \sum_{k=1}^{\infty} (a_k', a_k)_{(0,T)} = \sum_{k=1}^{\infty} \frac{1}{2}\big(a_k(T)^2 - a_k(0)^2\big). \tag{4.20}$$

Let $\varphi \in C_\infty(\mathbb{R})$ with $0 \le \varphi \le 1$ and

$$\varphi(t) = \begin{cases} 0, & t \le \frac{1}{4} \\ 1, & t \ge \frac{3}{4}\,. \end{cases}$$

Then the function $(\varphi u)(t,x) = \varphi(t)u(t,x)$ belongs to $L^2(Q)$ and

$$\varphi u = \sum_{k=1}^{\infty} (\varphi a_k)\nu_k$$

with

$$\begin{aligned} \|(\varphi a_k)'\|_{(0,T)}^2 &= \|\varphi' a_k + \varphi a_k'\|_{(0,T)}^2 \\ &\le 2\|\varphi'\|_{L^\infty((0,T))}^2 \|a_k\|_{(0,T)}^2 + 2\|a_k'\|_{(0,T)}^2\,, \end{aligned}$$

whence

$$\begin{aligned} &\alpha_k^2 \|\varphi a_k\|_{(0,T)}^2 + \frac{1}{\alpha_k^2} \|(\varphi a_k)'\|_{(0,T)}^2 \\ &\quad \le \Big(\alpha_k^2 + \frac{2}{\alpha_k^2}\|\varphi'\|_{L^\infty((0,T))}^2\Big)\|a_k\|_{(0,T)}^2 + \frac{2}{\alpha_k^2}\|a_k'\|_{(0,T)}^2 \\ &\quad \le C\Big(\alpha_k^2\|a_k\|_{(0,T)}^2 + \frac{1}{\alpha_k^2}\|a_k'\|_{(0,T)}^2\Big), \end{aligned}$$

with $C = 2\big(1 + \frac{1}{\alpha^4}\|\varphi'\|_{L^\infty((0,T))}^2\big)$. This implies $\varphi u \in V$ and

$$\|\varphi u\|_V \le \sqrt{C}\|u\|_V\,. \tag{4.21}$$

Therefore we can insert φu into (4.20) and obtain

$$\langle(\varphi u)_t, \varphi u\rangle = \frac{1}{2}\sum_{k=1}^{\infty} a_k(T)^2, \tag{4.22}$$

which shows that the sum on the right hand side converges. Since $\{\nu_k\}_{k=1}^{\infty}$ is an orthonormal system, we obtain that

$$P_T u = \sum_{k=1}^{\infty} a_k(T)\nu_k$$

belongs to $L^2(\Omega)$ and

$$\|P_T u\|_\Omega^2 = \sum_{k=1}^{\infty} a_k(T)^2.$$

This defines a mapping $P_T : V \to L^2(\Omega)$. Since $a_k \to a_k(T)$ is linear, it is immediately seen that P_T is linear. The continuity of P_T follows from (4.22), which together with (4.21) implies

$$\begin{aligned}\|P_T u\|_\Omega^2 &\le 2\langle(\varphi u)_t, \varphi u\rangle \\ &\le 2\|\varphi u\|_V \, \|\nabla(\varphi u)\|_Q \le 2\|\varphi u\|_V^2 \le 2C\|u\|_V^2 .\end{aligned}$$

$P_0 : V \to L^2(\Omega)$ is defined by $P_0 u = \sum_{k=1}^\infty a_k(0)\nu_k$. The linearity and continuity of P_0 is seen in an analogous way. From (4.20) we infer that

$$\begin{aligned}\langle u_t, u\rangle &= \frac{1}{2}\sum_{k=1}^\infty a_k(T)^2 - \frac{1}{2}\sum_{k=1}^\infty a_k(0)^2 \\ &= \frac{1}{2}\|P_T u\|_\Omega^2 - \frac{1}{2}\|P_0 u\|_\Omega^2 .\end{aligned}$$

Moreover, for $v = \sum_{k=1}^\infty b_k\nu_k \in V$ we obtain

$$\begin{aligned}\langle u_t, v\rangle + \langle u, v_t\rangle &= \sum_{k=1}^\infty \big((a_k', b_k)_{(0,T)} + (a_k, b_k')_{(0,T)}\big) \\ &= \sum_{k=1}^\infty \big(a_k(T)b_k(T) - a_k(0)b_k(0)\big) \\ &= \sum_{k=1}^\infty a_k(T)b_k(T) - \sum_{k=1}^\infty a_k(0)b_k(0) \\ &= (P_T u, P_T v)_\Omega - (P_0 u, P_0 v)_\Omega .\end{aligned}$$

■

4.4 Existence of solutions for nonlinear initial-boundary value problems

Assume that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field, and let $b \in L^1(Q, \mathbb{R})$, $u_0 \in L^{1,\mathrm{loc}}(\Omega)$ be given. We consider the initial-boundary value problem

$$u_t(x) = \mathrm{div}_x F\big(\nabla_x u(t,x)\big) + b(t,x), \quad (t,x) \in Q, \tag{4.23}$$

$$u(t,x) = 0, \quad (t,x) \in (0,T) \times \partial\Omega \tag{4.24}$$

$$u(0,x) = u_0(x), \quad x \in \Omega. \tag{4.25}$$

Definition 4.18 A function $u \in L^2\big(0,T; \overset{\circ}{H}_1(\Omega)\big)$ is a weak solution of (4.23) – (4.25), if $F(\nabla_x u) \in L^1(Q)$ and if for all $\varphi \in \overset{\circ}{C}_\infty\big((-\infty, T) \times \Omega\big)$

$$-(u, \varphi_t)_Q + \big(F(\nabla_x u), \nabla_x \varphi\big)_Q = (b, \varphi)_Q + \big(u_0, \varphi(0)\big)_\Omega .$$

In this subsection we prove that the initial-boundary value problem (4.23) – (4.25) has a unique weak solution $u \in V$. For weak solutions in V, the condition in Definition 4.18 can be reformulated:

Lemma 4.19 *Assume that $u \in V$, $F(\nabla u) \in L^2(Q)$, $b \in L^2(Q)$, $u_0 \in L^2(\Omega)$. Then u is a weak solution of (4.23) – (4.25) if and only if the equations*

$$u(0) = u_0 \tag{4.26}$$

$$\langle u_t, v\rangle + \big(F(\nabla u), \nabla v\big)_Q - (b, v)_Q = 0, \quad \textit{for all } v \in V, \tag{4.27}$$

hold.

Proof. Let $\varphi \in \overset{\circ}{C}_\infty\big((-\infty, T)\times\Omega\big)$. The function φ belongs to V, since $\overset{\circ}{C}_\infty\big((-\infty, T)\times\Omega\big)$ is a subspace of V. Therefore we obtain from Theorem 4.17 for $u \in V$ that

$$\langle u_t, \varphi\rangle + \langle u, \varphi_t\rangle = \big(u(T), \varphi(T)\big)_\Omega - \big(u(0), \varphi(0)\big)_\Omega = -\big(u(0), \varphi(0)\big)_\Omega.$$

Using this equation and $\langle u, \varphi_t\rangle = (u, \varphi_t)_Q$ we see that the two equations

$$-(u, \varphi_t)_Q + \big(F(\nabla u), \nabla\varphi\big)_Q = (b, \varphi)_Q + \big(u_0, \varphi(0)\big)_\Omega \tag{4.28}$$

and

$$\langle u_t, \varphi\rangle + \big(F(\nabla u), \nabla\varphi\big)_Q = (b, \varphi)_Q + \big(u_0 - u(0), \varphi(0)\big)_\Omega \tag{4.29}$$

are equivalent.

Assume now that (4.26) and (4.27) hold. We conclude that (4.29) and also (4.28) are satisfied. By Definition 4.18, this means that u is a weak solution of (4.23) – (4.25).

On the other hand, assume that $u \in V$ is a weak solution, which means that (4.28) is satisfied for all $\varphi \in \overset{\circ}{C}_\infty\big((-\infty, T)\times\Omega\big)$. Thus, also the equivalent equation (4.29) holds. Let $v \in L^2\big(0, T; \overset{\circ}{H}_1(\Omega)\big)$. Since $\overset{\circ}{C}_\infty\big((0, T)\times\Omega\big)$ is dense in $L^2\big(0, T; \overset{\circ}{H}_1(\Omega)\big)$, by Lemma 4.10, we can choose a sequence $\{\varphi_k\}_{k=1}^\infty \subseteq \overset{\circ}{C}_\infty\big((0, T)\times\Omega\big)$ which converges in $L^2\big(0, T; \overset{\circ}{H}_1(\Omega)\big)$ to v, whence

$$\|v - \varphi_k\|_Q + \|\nabla(v - \varphi_k)\|_Q \to 0 \quad \text{for } k \to \infty.$$

Since $v \mapsto \langle u_t, v\rangle$ is a continuous linear form on $L^2\big(0, T; \overset{\circ}{H}_1(\Omega)\big)$, we thus obtain

$$\begin{aligned} \lim_{k\to\infty} \langle u_t, \varphi_k\rangle &= (u_t, v)_Q \\ \lim_{k\to\infty} \big(F(\nabla u), \nabla\varphi_k\big)_Q &= \big(F(\nabla u), \nabla v\big)_Q \\ \lim_{k\to\infty} (b, \varphi_k)_Q &= (b, v)_Q\,. \end{aligned}$$

Observing that $\varphi_k(0) = 0$, we obtain from (4.29)

$$\begin{aligned} &\langle u_t, v\rangle + \big(F(\nabla u), \nabla v\big)_Q - (b, v)_Q \\ &\quad = \lim_{k\to\infty} \Big[\langle u_t, \varphi_k\rangle + \big(F(\nabla u), \nabla\varphi_k\big)_Q - (b, \varphi_k)_Q\Big] = 0. \end{aligned}$$

Since $V \subseteq L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$, this shows that (4.27) holds for all $v \in V$. Furthermore, noting that $\overset{\circ}{C}_\infty\big((-\infty,T)\times\Omega\big) \subseteq L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$, we infer from (4.27) and (4.29) that

$$\big(u_0 - u(0), \varphi(0)\big)_\Omega = 0,$$

for all $\varphi \in \overset{\circ}{C}_\infty\big((-\infty,T)\times\Omega\big)$. Since the set of traces $\big\{\varphi(0) \mid \varphi \in \overset{\circ}{C}_\infty\big((-\infty,T)\times\Omega\big)\big\}$ is dense in $L^2(\Omega)$, this implies $u(0) = u_0$, which is (4.26). ■

Corollary 4.20 *For $b \in L^2(Q)$, $u_0 \in L^2(\Omega)$ there is at most one weak solution u of* (4.23) – (4.25), *which belongs to the space V and satisfies $F(\nabla u) \in L^2(Q)$.*

Proof. Assume that $u, v \in V$ are two weak solutions. Lemma 4.19 then yields

$$\langle (u-v)_t, u-v\rangle + \big(F(\nabla u) - F(\nabla v), \nabla u - \nabla v\big)_Q = 0.$$

Using Theorem 4.17, which implies

$$\langle (u-v)_t, u-v\rangle = \frac{1}{2}\,\|(u-v)(T)\|_\Omega^2 - \frac{1}{2}\,\|(u-v)(0)\|_\Omega^2\,,$$

we conclude from $(u-v)(0) = u(0) - v(0) = u_0 - u_0 = 0$ and from the monotonicity of F that

$$\frac{1}{2}\,\|(u-v)(T)\|_\Omega^2 \le -\big(F(\nabla u) - F(\nabla v), \nabla u - \nabla v)_Q \le 0,$$

whence $u(T) = v(T)$. Since $T > 0$ can be chosen arbitrarily, we find that $u = v$. ■

Next we prove that solutions exist. In the proof we need that $F \in C_1(\mathbb{R}^n, \mathbb{R}^n)$ satisfies the following conditions:

(i) $\big(F(\xi) - F(\eta)\big)\cdot(\xi - \eta) \ge 0$ for all $\xi, \eta \in \mathbb{R}^n$.

(ii) $F(0) = 0$.

(iii) There are constants $c_1 > 0$ and $c_2 \ge 0$ such that

$$F(\xi)\cdot\xi \ge c_1|\xi|^2 - c_2, \quad \xi \in \mathbb{R}^n.$$

(iv) There is $c_3 > 0$ such that

$$|F(\xi)| \le c_3|\xi|, \quad \xi \in \mathbb{R}^n.$$

(v) $\|\nabla F\|_{L^\infty(\mathbb{R}^n)} < \infty$.

Theorem 4.21 *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded and assume that $F \in C_1(\mathbb{R}^n, \mathbb{R}^n)$ satisfies the conditions (i), (ii) and (iv), (v). Let $u_0 \in L^2(\Omega)$ and $b \in L^2(Q) \cap C(\overline{Q})$. Then to every $m \in \mathbb{N}$ there is*

$$u_m(t,x) = \sum_{k=1}^{m} a_{mk}(t)\nu_k(x) \tag{4.30}$$

with $a_{mk} \in C_1([0,T])$ such that

$$\frac{d}{dt}\big(u_m(t), \nu_\ell\big)_\Omega + \big(F\big(\nabla u_m(t)\big), \nabla \nu_\ell\big)_\Omega = \big(b(t), \nu_\ell\big)_\Omega, \tag{4.31}$$

$$\big(u_m(0), \nu_\ell\big)_\Omega = (u_0, \nu_\ell)_\Omega, \tag{4.32}$$

for $\ell = 1, \ldots, m$ and $t \in [0,T]$. Moreover, u_m satisfies

$$\|u_m(t)\|_\Omega \le \|u_0\|_\Omega + \int_0^t \|b(s)\|_\Omega \, ds \tag{4.33}$$

for all $0 \le t \le T$ and

$$\begin{aligned}\|u_m(T)\|_\Omega^2 &+ \big(F(\nabla u_m), \nabla u_m\big)_Q \\ &\le T\|b\|_Q^2 + \|u_0\|_\Omega \big(\frac{1}{2} + T^{1/2}\,\|b\|_Q\big).\end{aligned} \tag{4.34}$$

Proof. The following equations are equivalent to (4.31), (4.32):

$$\frac{d}{dt}\, a_{m\ell}(t) + \Big(F\Big(\sum_{k=1}^{m} a_{mk}(t)\nabla\nu_k\Big), \nabla\nu_\ell\Big)_\Omega = \big(b(t), \nu_\ell\big)_\Omega, \tag{4.35}$$

$$a_{m\ell}(0) = (u_0, \nu_\ell)_\Omega, \tag{4.36}$$

for $\ell = 1, \ldots, m$. These equations are obtained by insertion of (4.30) into (4.31), (4.32). The equations (4.35), (4.36) form an initial value problem to a system of nonlinear ordinary differential equations for the vector function $t \mapsto a(t) = \big(a_{m1}(t), \ldots, a_{mm}(t)\big) : [0,T] \to \mathbb{R}^m$.

This initial value problem can be written in the form

$$\begin{aligned}\frac{d}{dt}\, a(t) &= -G\big(a(t)\big) + f(t), \\ a(0) &= \big((u_0, \nu_1)_\Omega, \ldots, (u_0, \nu_m)_\Omega\big),\end{aligned}$$

with the continuous function

$$t \mapsto f(t) = \Big(\big(b(t), \nu_1\big)_\Omega, \ldots, \big(b(t), \nu_m\big)_\Omega\Big) \in \mathbb{R}^m$$

and with the mapping $G : \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$G(a) = \left(\Big(F\Big(\sum_{k=1}^{m} a_k \nabla \nu_k \Big), \nabla \nu_1 \Big)_\Omega, \ldots, \Big(F\Big(\sum_{k=1}^{m} a_k \nabla \nu_k \Big), \nabla \nu_m \Big)_\Omega \right).$$

G is Lipschitz continuous on $\mathbb{R}^m$. To prove this, note that G is continuously differentiable with

$$\begin{aligned} \frac{\partial}{\partial a_i} G_\ell(a) &= \frac{\partial}{\partial a_i} \int_\Omega F\Big(\sum_{k=1}^{m} a_k \nabla \nu_k(x) \Big) \cdot \nabla \nu_\ell(x) dx \\ &= \int_\Omega \nabla \nu_\ell(x) \cdot \left(\Big[\nabla F\Big(\sum_{k=1}^{m} a_k | \nabla \nu_k(x) \Big) \Big] \nabla \nu_i \right) dx \\ &\leq \| \nabla F \|_{L^\infty(\mathbb{R}^n)} \, \| \nabla \nu_\ell \|_\Omega \, \| \nabla \nu_i \|_\Omega =: L_{i\ell}. \end{aligned}$$

Here we used that F satisfies condition (v). Consequently, the mean value theorem yields

$$|G(\eta) - G(\xi)| \leq \sqrt{\sum_{i,\ell=1}^{m} L_{i\ell}^2} \, |\eta - \xi|,$$

for all $\eta, \xi \in \mathbb{R}^m$, which means that G is Lipschiz continuous. By the theorem of Picard-Lindelöf it thus follows that there is $\tau > 0$ such that the initial value problem (4.35), (4.36) has a unique solution $a \in C_1\big([0,\tau)\big)$, and that a can be extended to a solution $a \in C_1([0,T])$ provided there is $g \in C([0,T])$ such that

$$|a(t)| \leq g(t) \tag{4.37}$$

for all $0 \leq t \leq T$. To construct such a function g, remember that $u_m = \sum_{k=1}^{m} a_{mk} \nu_k$ satisfies (4.31), (4.32). We multiply (4.31) and (4.32) by $a_{m\ell}(t)$ and sum with respect to ℓ from 1 to m. The result is

$$\frac{1}{2} \frac{d}{dt} \big(u_m(t), u_m(t) \big)_\Omega + \Big(F\big(\nabla u_m(t) \big), \, \nabla u_m(t) \Big)_\Omega = \big(b(t), u_m(t) \big)_\Omega, \tag{4.38}$$

$$\| u_m(0) \|_\Omega^2 = \big(u_0, u_m(0) \big)_\Omega. \tag{4.39}$$

Since $F(0) = 0$, the monotonicity of F yields

$$F(\xi) \cdot \xi = \big(F(\xi) - F(0) \big) \cdot (\xi - 0) \geq 0,$$

whence

$$\Big(F\big(\nabla u_m(t) \big), \, \nabla u_m(t) \Big)_\Omega \geq 0.$$

¿From (4.38) we thus obtain

$$\begin{aligned} \|u_m(t)\|_\Omega \frac{d}{dt} \|u_m(t)\|_\Omega &= \frac{1}{2} \frac{d}{dt} \|u_m(t)\|_\Omega^2 \\ &= \frac{1}{2} \frac{d}{dt} \big(u_m(t), u_m(t)\big)_\Omega \\ &\leq \big(b(t), u_m(t)\big)_\Omega \leq \|b(t)\|_\Omega \|u_m(t)\|_\Omega. \end{aligned}$$

We divide by $\|u_m(t)\|_\Omega$ and obtain

$$\frac{d}{dt} \|u_m(t)\|_\Omega \leq \|b(t)\|_\Omega.$$

Integration yields

$$\|u_m(t)\|_\Omega \leq \int_0^t \|b(s)\|_\Omega \, ds + \|u_m(0)\|_\Omega.$$

To estimate $\|u_m(0)\|_\Omega$ we use (4.39), which yields

$$\|u_m(0)\|_\Omega^2 = \big(u_0, u_m(0)\big)_\Omega \leq \|u_0\|_\Omega \|u_m(0)\|_\Omega,$$

whence

$$\|u_m(0)\|_\Omega \leq \|u_0\|_\Omega. \tag{4.40}$$

Together we obtain

$$\|u_m(t)\|_\Omega \leq \int_0^t \|b(s)\|_\Omega \, ds + \|u_0\|_\Omega, \tag{4.41}$$

for all $0 \leq t \leq T$. Since

$$|a(t)| = \sqrt{\sum_{k=1}^m a_{mk}(t)^2} = \|u_m(t)\|_\Omega,$$

we see from (4.41) that (4.37) holds with g defined by

$$g(t) = \int_0^t \|b(s)\|_\Omega \, ds + \|u_0\|_\Omega \leq T^{1/2} \sqrt{\int_0^t \|b(s)\|_\Omega^2 \, ds} + \|u_0\|_\Omega = T^{1/2} \|b\|_Q + \|u_0\|_\Omega.$$

To finish the proof note that (4.33) follows from (4.41). To verify (4.34) we use the estimates (4.40), (4.41) and obtain by integration of (4.38) that

$$\begin{aligned} &\frac{1}{2} \|u_m(t)\|_\Omega^2 + \int_0^t \Big(F\big(\nabla u_m(s)\big) \cdot \nabla u_m(s)\Big)_\Omega \, ds \\ &\quad \leq \frac{1}{2} \|u_m(0)\|_\Omega^2 + \int_0^t \|b(s)\|_\Omega \|u_m(s)\|_\Omega \, ds \\ &\quad \leq \frac{1}{2} \|u_m(0)\|_\Omega^2 + \int_0^t \|b(s)\|_\Omega \Big(\int_0^s \|b(r)\|_\Omega \, dr + \|u_0\|_\Omega \Big) ds \\ &\quad \leq \Big(\int_0^t \|b(s)\|_\Omega \, ds \Big)^2 + \|u_0\|_\Omega \Big(\int_0^t \|b(s)\|_\Omega \, ds + \frac{1}{2} \Big). \\ &\quad \leq T \|b\|_Q^2 + \|u_0\|_\Omega \big(\frac{1}{2} + T^{1/2} \|b\|_Q \big). \end{aligned}$$

This is (4.34). ■

Lemma 4.22 *Assume that* $F \in C_1(\mathbb{R}^n, \mathbb{R}^n)$ *fulfills the conditions (i) – (v). Then there are constants* $K_1, K_2 > 0$ *such that the sequence* $\{u_m\}_{m=1}^{\infty} \subseteq C\big([0,T],\, \overset{\circ}{H}{}_1^2(\Omega)\big)$ *of approximate solutions with* $u_m(t,x) = \sum_{k=1}^{m} a_{mk}(t)\nu_k(x)$ *satisfies*

$$\|\nabla u_m\|_Q \;\leq\; K_1 \tag{4.42}$$

$$\|F(\nabla u_m)\|_Q \;\leq\; c_3^{1/2}\|\nabla u_m\|_Q \tag{4.43}$$

$$\sqrt{\sum_{k=1}^{m} \frac{1}{\alpha_k^2}\|a'_{mk}\|^2_{(0,T)}} \;\leq\; \|F(\nabla u_m)\|_Q + K_2\|b\|_Q \tag{4.44}$$

for all $m \in \mathbb{N}$.

Proof. Condition (iii) and the estimate (4.34) imply

$$\begin{aligned} c_1\|\nabla u_m\|_Q^2 - c_2|Q| \;&\leq\; \big(F(\nabla u_m), \nabla u_m\big)_Q \\ &\leq\; T\|b\|_Q^2 + \|u_0\|_\Omega\Big(\frac{1}{2} + T^{1/2}\|b\|_Q\Big). \end{aligned}$$

This yields (4.42) with

$$K_1 = \frac{1}{c_1}\Big(T\|b\|_Q^2 + \|u_0\|_\Omega\left(\frac{1}{2} + T^{1/2}\|b\|_Q\right) + c_2|Q|\Big).$$

(4.43) is obtained from condition (iv), which yields

$$\begin{aligned} \|F(\nabla u_m)\|_Q^2 \;&=\; \int_Q |F\big(\nabla u_m(t,x)\big)|^2\, d(t,x) \\ &\leq\; \int_Q c_3^2|\nabla u_m(t,x)|^2\, d(t,x) \;=\; c_3\|\nabla u_m\|_Q^2. \end{aligned}$$

To prove (4.44) we need that u_m satisfies (4.31). Let $v = \sum_{k=1}^{m} d_k(t)\nu_k$ with $d_k \in C([0,T])$. We multiply (4.31) by d_ℓ, sum with respect to ℓ from 1 to m and integrate with respect to t. This yields

$$\big(\frac{\partial}{\partial t}\, u_m, v\big)_Q + \big(F(\nabla u_m), \nabla v\big)_Q = (b,v)_Q. \tag{4.45}$$

Since the Poincaré inequality implies

$$\begin{aligned} |(b,v)_Q|^2 \;&\leq\; \|b\|_Q^2\,\|v\|_Q^2 \;=\; \|b\|_Q^2 \int_0^T \|v(t)\|_\Omega^2\, dt \\ &\leq\; \|b\|_Q^2 \int_0^T K_2^2\|\nabla v(t)\|_\Omega^2 = \; K_2^2\|b\|_Q^2\,\|\nabla v\|_Q^2\,, \end{aligned}$$

we obtain from (4.45) and from (4.42), (4.43) that

$$\begin{aligned}\sum_{k=1}^{m}(a'_{mk},d_k)_{(0,T)} &= \Big(\sum_{k=1}^{m}a'_{mk}\nu_k,\sum_{\ell=1}^{m}d_\ell\nu_\ell\Big)_Q\\ &= \big(\frac{\partial}{\partial t}u_m,v\big)_Q \le \big(\|F(\nabla u_m)\|_Q+K_2\|b\|_Q\big)\,\|\nabla v\|_Q\\ &= M\,\|\nabla v\|_Q\,.\end{aligned}$$

Using $\|\nabla v\|_Q^2=\sum_{k=1}^m \alpha_k^2\,\|d_k\|^2_{(0,T)}$, we thus get

$$\sum_{k=1}^{m}\big(a'_{mk},d_k\big)_{(0,T)}\le M\sqrt{\sum_{k=1}^{m}\alpha_k^2\|d_k\|^2_{(0,T)}}\,.$$

Now choose $d_k=\frac{1}{\alpha_k^2}\,a'_{mk}$. Then

$$\sum_{k=1}^{m}\frac{1}{\alpha_k^2}\|a'_{mk}\|^2_{(0,T)}\le M\sqrt{\sum_{k=1}^{m}\frac{1}{\alpha_k^2}\|a'_{mk}\|^2_{(0,T)}}\,.$$

We divide both sides by the square root and obtain (4.44). ∎

Corollary 4.23 *There are constants* $K,\hat K>0$ *such that*

$$\begin{aligned}\|F(\nabla u_m)\|_Q &\le K,\\ \|u_m\|_V &\le \hat K.\end{aligned}$$

Proof. Combination of (4.42), (4.43) results in $\|F(\nabla u_m)\|_Q\le c_3^{1/2}K_1=K$. Insertion of this inequality into (4.44) yields for $u_m(t,x)=\sum_{k=1}^m a_{mk}(t)\nu_k(x)$ that

$$\|u_m\|_V^2=\|\nabla u_m\|_Q^2+\sum_{k=1}^{m}\alpha_k^{-2}\|a'_{km}\|^2_{(0,T)}\le K_1^2+(K+K_2\|b\|_Q)^2=\hat K^2. \qquad ∎$$

Now we are in a position to complete the existence proof for the initial-boundary value problem (4.23) – (4.25).

Theorem 4.24 *Let* $\Omega\subseteq\mathbb{R}^n$ *be open and bounded and assume that* $F\in C_1(\mathbb{R}^n,\mathbb{R}^n)$ *has the properties (i) – (v). Then to every* $u_0\in L^2(\Omega)$ *and* $f\in L^2(Q)\cap C(\overline{Q})$ *there is a weak solution* $u\in V$ *of the initial-boundary value problem* (4.23) – (4.25).

Proof. By Corollary 4.23 the sequence $\{u_m\}_{m=1}^\infty$ is bounded in V. Since V is a Hilbert space, it follows that this sequence has a subsequence, which converges weakly in V. For simplicity we denote the subsequence again by $\{u_m\}_{m=1}^\infty$. We thus have that there is $u \in V$ with

$$u_m \rightharpoonup u \quad \text{in } V.$$

Since $\|\nabla v\|_Q \le \|v\|_V$, the injection mapping $J : V \to L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$ is continuous and therefore weakly continuous. It follows that $u_m \rightharpoonup u$ in $L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$ and that $u_m \rightharpoonup u$, $\nabla u_m \rightharpoonup \nabla u$ in $L^2(Q)$. Since the trace-map $P_0 : V \to L^2(\Omega)$ is continuous, it follows that P_0 is weakly continuous, and so

$$u_m(0) = P_0 u_m \rightharpoonup P_0 u = u(0) \quad \text{in } L^2(\Omega). \tag{4.46}$$

On the other hand, (4.32) implies

$$u_m(0) = \sum_{\ell=1}^m \big(u_m(0), \nu_\ell\big)_\Omega \, \nu_\ell = \sum_{\ell=1}^m (u_0, \nu_\ell)_\Omega \, \nu_\ell \, .$$

Since the orthonormal system $\{\nu_\ell\}_{\ell=1}^\infty$ is complete, we have

$$u_0 = \sum_{\ell=1}^\infty (u_0, \nu_\ell)_\Omega \, \nu_\ell,$$

whence

$$\|u_0 - u_m(0)\|_\Omega^2 = \Big\| \sum_{\ell=m+1}^\infty (u_0, \nu_\ell)_\Omega \, \nu_\ell \Big\|_\Omega^2 = \sum_{\ell=m+1}^\infty (u_0, \nu_\ell)_\Omega^2 \to 0$$

for $m \to \infty$, and therefore $u_m(0) \to u_0$ in $L^2(\Omega)$. This implies $u_m(0) \rightharpoonup u_0$, consequently (4.46) yields $u(0) = u_0$. Summing up, we have

$$u_m(0) \to u_0 = u(0) \quad \text{in } L^2(\Omega). \tag{4.47}$$

Now choose $v \in V$ arbitrary. Then Theorem 4.17 yields

$$\begin{aligned} &\frac{1}{2} \|v(0) - u_m(0)\|_\Omega^2 + \langle v_t - u_{m,t}, v - u_m \rangle + \big(F(\nabla v) - F(\nabla u_m), \nabla v - \nabla u_m\big)_Q \\ &\quad = \frac{1}{2} \|v(T) - u_m(T)\|_\Omega^2 + \big(F(\nabla v) - F(\nabla u_m), \nabla v - \nabla u_m\big)_Q \ge 0. \end{aligned} \tag{4.48}$$

Here we used that F is monotone, by condition (i). Since $V \subseteq L^2(0,T;\overset{\circ}{H}_1(\Omega))$, Lemma 4.7 shows that $v = \sum_{k=1}^\infty d_k \nu_k$, and the sum converges in the norm of $L^2(0,T;\overset{\circ}{H}_1(\Omega))$. We define $v_m = \sum_{k=1}^m d_k \nu_k$. Corollary 4.13 yields

$$\langle u_{m,t}, v \rangle = \sum_{k=1}^m (a'_{mk}, d_k)_{(0,T)} = (u_{m,t}, v_m)_Q$$

whence (4.48) implies

$$\begin{aligned} 0 &\leq \frac{1}{2}\,\|v(0)-u_m(0)\|_\Omega^2 + \langle v_t, v-u_m\rangle + \big(F(\nabla v), \nabla v - \nabla u_m\big)_Q \\ &\quad -(u_{m,t}, v_m - u_m)_Q - \big(F(\nabla u_m), \nabla(v_m-u_m)\big)_Q - \big(F(\nabla u_m), \nabla(v-v_m)\big)_Q \\ &= \frac{1}{2}\,\|v(0)-u_m(0)\|_\Omega^2 + \langle v_t, v-u_m\rangle + \big(F(\nabla v), \nabla v - \nabla u_m\big)_Q \\ &\quad -(b, v_m-u_m)_Q - \big(F(\nabla u_m), \nabla(v-v_m)\big)_Q, \end{aligned} \tag{4.49}$$

where we employed (4.31). Since $u_m(0) \to u(0)$ strongly in $L^2(\Omega)$, we see that

$$\|v(0)-u_m(0)\|_\Omega \to \|v(0)-u(0)\|_\Omega, \quad m \to \infty.$$

v_t is a continuous linear form on $L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$ for $v \in V$. From $u_m \rightharpoonup u$ in $L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$ we consequently obtain

$$\langle v_t, v-u_m\rangle \to \langle v_t, v-u\rangle\,.$$

Similarly, $\nabla u_m \rightharpoonup \nabla u$ in $L^2(Q)$ yields

$$\big(F(\nabla v), \nabla v - \nabla u_m\big)_Q \to \big(F(\nabla v), \nabla(v-u)\big)_Q\,.$$

Since $v_m \to v$ strongly in $L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$, we obtain together with $u_m \rightharpoonup u$ in $L^2(Q)$ that

$$(b, v_m-u_m)_Q \to (b, v-u)_Q,$$

and together with Corollary 4.23 we find

$$\begin{aligned} \Big|\big(F(\nabla u_m), \nabla(v-v_m)\big)_Q\Big| &\leq \|F(\nabla u_m)\|_Q\,\|\nabla(v-v_m)\|_Q \\ &\leq K\|\nabla(v-v_m)\|_Q \to 0. \end{aligned}$$

We employ these limit relations to conclude from (4.49) that

$$0 \leq \frac{1}{2}\,\|v(0)-u(0)\|_\Omega^2 + \langle v_t, v-u\rangle + \big(F(\nabla v), \nabla(v-u)\big)_Q - (b, v-u)_Q\,. \tag{4.50}$$

We now choose $v = u + \lambda w$ with $\lambda > 0$ and with arbitrary $w \in V$. Then (4.50) becomes

$$0 \leq \frac{1}{2}\,\lambda^2\,\|w(0)\|_\Omega^2 + \lambda\langle u_t + \lambda w_t, w\rangle + \lambda\big(F(\nabla u + \lambda\nabla w), \nabla w\big)_Q - \lambda(b, w)_Q\,.$$

Division by λ yields

$$0 \leq \frac{1}{2}\,\lambda\,\|w(0)\|_\Omega^2 + \langle u_t, w\rangle + \lambda\langle w_t, w\rangle + \big(F(\nabla u + \lambda\nabla w), \nabla w\big)_Q - (b, w)_Q\,. \tag{4.51}$$

For $\lambda \to 0$ we have $F(\nabla u + \lambda \nabla w) \to F(\nabla u)$, strongly in $L^2(Q)$. The proof is the same as in Lemma 3.14. Letting $\lambda \to 0$ in (4.51) therefore results in

$$0 \le \langle u_t, w \rangle + \big(F(\nabla u), \nabla w)_Q - (b, w)_Q \,.$$

Replacing w by $-w$ shows that this inequality can only hold for all $w \in V$ if

$$\langle u_t, w \rangle + \big(F(\nabla u), \nabla w\big)_Q - (b, w)_Q = 0.$$

By Lemma 4.19 this equation and (4.47) imply that u is a weak solution of (4.23) – (4.25). ■

Examples. 1. Obviously, the function $F(\xi) = \xi$ satisfies the conditions (i) - (v). In this case the initial boundary value problem is

$$\begin{aligned} u_t &= \Delta u + b, \\ u(t,x) &= 0, \qquad (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) &= u_0(x), \qquad x \in \Omega, \end{aligned}$$

the Dirichlet initial-boundary value problem for the heat equation. We conclude that this problem has a unique solution.

2. $F(\xi) = c(1 - e^{-|\xi|})\xi$ with $c > 0$ satisfies conditions (i) - (v). We only verify the monotonicity of F. We have

$$\nabla F(\xi) = c(1 - e^{-|\xi|})I + ce^{-|\xi|}\, \frac{1}{|\xi|}\, \xi \otimes \xi,$$

whence, for $\xi, \eta \in \mathbb{R}^n$

$$\eta \cdot \nabla F(\xi)\eta = c(1 - e^{-|\xi|})|\eta|^2 + ce^{-|\xi|}\, \frac{(\xi - \eta)^2}{|\xi|} \ge 0.$$

The initial-boundary value problem becomes

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= c\,\mathrm{div}\Big[\big(1 - e^{-|\nabla_x u(t,x)|}\big)\nabla_x u(t,x)\Big] + b(t,x) \\ u(t,x) &= 0, \qquad (t,x) \in (0,T) \times \partial\Omega \\ u(0,x) &= u_0(x), \qquad x \in \Omega. \end{aligned}$$

Also this problem has a unique solution in V.

4.5 Higher regularity in the case that the nonlinearity is a gradient

For the weak solution $u \in V$ of the partial differential equation

$$u_t = \operatorname{div}_x F(\nabla_x u) + b$$

the time derivative u_t belongs to the space $L^2\big(0,T;H_{-1}(\Omega)\big)$. This low regularity of u_t cannot be improved in general. In this section we show however, that if F is the gradient of a convex function ψ, then the time derivative u_t is contained in $L^2\big(0,T;L^2(\Omega)\big) = L^2(Q)$, hence the solution u belongs to the space $V \cap H_1\big(0,T;L^2(\Omega)\big) \subseteq H_1(Q)$. To prove this, we need some preparations.

Definition 4.25 Let X be a Banach space. A function $f : X \to \mathbb{R}$ is lower semicontinuous or weakly lower semicontinuous, respectively, if for every $\vartheta \in \mathbb{R}$ the set

$$\{z \in X \mid f(z) \le \vartheta\}$$

is closed or weakly closed, respectively.

Lemma 4.26 *Assume that $f : X \to \mathbb{R}$ is weakly lower semicontinuous. If the sequence $\{z_n\}_{n=1}^\infty \subseteq X$ converges weakly to $z_0 \in X$, then we have*

$$f(z_0) \le \liminf_{n\to\infty} f(z_n).$$

Proof. We set $c = \liminf_{n\to\infty} f(z_n)$. Choose a subsequence $\{z_{n_k}\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} f(z_{n_k}) = c.$$

This implies that to every $\varepsilon > 0$ there is k_0 such that $f(z_{n_k}) \le c + \varepsilon$ for all $k \ge k_0$, whence $z_{n_k} \in A_\varepsilon = \{z \in X \mid f(z) \le c + \varepsilon\}$. Since A_ε is weakly closed and since $\{z_{n_k}\}_{k=1}^\infty$ converges weakly to z_0, it follows that $z_0 \in A_\varepsilon$, which implies $f(z_0) \le c + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the statement follows. ∎

Lemma 4.27 *Let $f : X \to \mathbb{R}$ be convex. Then f is lower semicontinuous if and only if it is weakly lower semicontinuous.*

Proof. Since f is convex, the set $\{z \in X \mid f(z) \le \vartheta\}$ is convex for every $\vartheta \in \mathbb{R}$. The statement follows immediately from the fact that a convex set is closed if and only if it is weakly closed. ∎

Lemma 4.28 *Assume that $\Gamma \subseteq \mathbb{R}^n$ is an open, bounded set and that $\psi \in C_1(\mathbb{R}^n, \mathbb{R})$ satisfies $|\nabla\psi(\xi)| \le C|\xi|$, where C is a constant independent of $\xi \in \mathbb{R}^n$. Let $\varphi \in L^\infty(\Gamma)$. Then the integral $\int_\Gamma \psi\big(w(x)\big)\varphi(x)\,\mathrm{d}x$ exists for all $w \in L^2(\Gamma, \mathbb{R}^n)$ and*

$$w \mapsto f(w) = \int_\Gamma \psi\big(w(x)\big)\varphi(x)\,\mathrm{d}x : L^2(\Gamma, \mathbb{R}^n) \to \mathbb{R}$$

is a continuous function.

Proof. From the mean value theorem it follows that to $\xi, \eta \in \mathbb{R}^n$ there is a number $0 < \vartheta < 1$ such that

$$\begin{aligned} |\psi(\xi) - \psi(\eta)| &= |\nabla\psi\big(\vartheta\xi + (1-\vartheta)\eta\big) \cdot (\xi - \eta)| \\ &\le C|\vartheta\xi + (1-\vartheta)\eta|\ |\xi - \eta| \le C\big(|\xi - \eta| + |\eta|\big)\ |\xi - \eta|. \end{aligned} \tag{4.52}$$

For $w \in L^2(\Gamma, \mathbb{R}^n)$ and $x \in \Gamma$ it follows from this estimate by setting $\xi = w(x)$ and $\eta = 0$ that

$$|\psi\big(w(x)\big)|\ |\varphi(x)| \le \|\varphi\|_\infty \ \big(|\psi(0)| + C|w(x)|^2\big),$$

which shows that the integral $\int_\Gamma \psi\big(w(x)\big)\varphi(x)\,\mathrm{d}x$ exists, since Γ is bounded. To see that f is continuous at $w_0 \in L^2(\Gamma, \mathbb{R}^n)$, set $\xi = w(x)$, $\eta = w_0(x)$ in (4.52) to obtain

$$\begin{aligned} |f(w) - f(w_0)| &\le \int_\Gamma |\psi\big(w(x)\big) - \psi\big(w_0(x)\big)|\ |\varphi(x)\,\mathrm{d}x| \\ &\le C \int_\Gamma \big(|w(x) - w_0(x)| + |w_0(x)|\big)\ |w(x) - w_0(x)|\ \|\varphi\|_\infty\,\mathrm{d}x \\ &\le C\|\varphi\|_\infty\big(\|w - w_0\|_\Gamma + \|w_0\|_\Gamma\big)\ \|w - w_0\|_\Gamma, \end{aligned}$$

where we applied the Cauchy-Schwarz inequality. This estimate yields

$$\lim_{w \to w_0} f(w) = f(w_0),$$

hence f is continuous at w_0. ∎

Now we can study the regularity of the solution of the initial boundary value problem (4.23) – (4.25) if the function F is a gradient and satisfies the condition (i) – (v). We shall therefore assume that there is a function $\psi \in C_2(\mathbb{R}^n, \mathbb{R})$ such that

$$F(\xi) = \nabla\psi(\xi), \text{ for all } \xi \in \mathbb{R}^n. \tag{4.53}$$

By condition (i) the function $F = \nabla\psi$ is monotone and by Corollary 1.7 this condition is equivalent to the convexity of ψ. By condition (ii) and Definition 1.2 we have $\psi(\eta) \geq \psi(0)$ for all $\eta \in \mathbb{R}^n$. For simplicity we shall assume that

$$\psi(\eta) \geq 0, \qquad \eta \in \mathbb{R}^n. \tag{4.54}$$

We first show that if (4.53) holds then the approximate solutions constructed in Theorem 4.21 satisfy an additional estimate. In the following we use the notation

$$Q_t = (0,t) \times \Omega$$

and write as usual $Q = (0,T) \times \Omega$.

Theorem 4.29 *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded and assume that $F = \nabla\psi$ satisfies the conditions (i), (ii) and (iv), (v). Let $u_0 \in L^2(\Omega)$ and $b \in L^2(Q) \cap C(\bar{Q})$. Then the functions $u_m = \sum_{k=1}^{m} a_{mk}\nu_k \in C_1\big([0,T], \overset{\circ}{H}_1(\Omega)\big)$ constructed in Theorem 4.21 satisfy for all $0 \leq t \leq T$ the estimate*

$$\frac{1}{2}\|\partial_t u_m\|^2_{Q_t} + \int\limits_\Omega \psi\big(\nabla u_m(t,x)\big)\,\mathrm{d}x \leq \int\limits_\Omega \psi\big(\nabla u_m(0,x)\big)\,\mathrm{d}x + \frac{1}{2}\|b\|^2_{Q_t}. \tag{4.55}$$

Proof. Multiply (4.31) by $\partial_t a_{m\ell}$ and sum with respect to ℓ from 1 to m to obtain

$$\big(\partial_t u_m(t), \partial_t u_m(t)\big)_\Omega + \Big(F\big(\nabla u_m(t)\big), \partial_t \nabla u_m(t)\Big)_\Omega = \big(b(t), \partial_t u_m(t)\big)_\Omega. \tag{4.56}$$

Since

$$\begin{aligned}\Big(F\big(\nabla u_m(t)\big), \partial_t \nabla u_m(t)\Big)_\Omega &= \int\limits_\Omega \nabla_\xi \psi\big(\nabla u_m(t,x)\big) \cdot \partial_t \nabla u_m(t,x)\,\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_\Omega \psi\big(\nabla u_m(t,x)\big)\,\mathrm{d}x,\end{aligned}$$

it follows by integration of (4.56) and application of the Cauchy-Schwarz inequality that

$$\begin{aligned}\|\partial_t u_m\|^2_{Q_t} + &\int\limits_\Omega \psi\big(\nabla u_m(t,x)\big)\,\mathrm{d}x - \int\limits_\Omega \psi\big(\nabla u_m(0,x)\big)\,\mathrm{d}x \\ &= (b, \partial_t u_m)_{Q_t} \leq \|b\|_{Q_t}\,\|\partial_t u_m\|_{Q_t} \leq \frac{1}{2}\|b\|^2_{Q_t} + \frac{1}{2}\|\partial_t u_m\|^2_{Q_t}.\end{aligned}$$

(4.55) follows from this estimate. ■

Lemma 4.30 *Assume that u_0 belongs to $\overset{\circ}{H}_1(\Omega)$. Then we have*

$$\lim_{m\to\infty} \int\limits_\Omega \psi\big(\nabla u_m(0,x)\big)\,\mathrm{d}x = \int\limits_\Omega \psi\big(\nabla u_0(x)\big)\,\mathrm{d}x. \tag{4.57}$$

Proof. Since $F = \nabla\psi$ satisfies condition (iv), it follows from Lemma 4.28 that $w \mapsto \int_\Omega \psi\big(w(x)\big)\,\mathrm{d}x$ is a continuous function on $L^2(\Omega,\mathbb{R}^n)$. To verify (4.57) it therefore suffices to show that $\nabla u_m(0) \to \nabla u_0$ in $L^2(\Omega)$. To verify this note that (4.32) implies

$$u_m(0) = \sum_{\ell=1}^{m} \big(u_m(0),\nu_\ell\big)_\Omega\,\nu_\ell = \sum_{\ell=1}^{m}(u_0,\nu_\ell)_\Omega\,\nu_\ell,$$

whence Theorem 4.2(ii) yields

$$\|\nabla\big(u_0 - u_m(0)\big)\|_\Omega^2 = \|\nabla\Big(\sum_{\ell=m+1}^{\infty}(u_0,\nu_\ell)_\Omega\,\nu_\ell\Big)\|_\Omega^2 = \sum_{\ell=m+1}^{\infty}\alpha_\ell^2(u_0,\nu_\ell)_\Omega^2.$$

The term on the right hand side tends to zero for $m\to\infty$ since $u_0 \in \overset{\circ}{H}_1(\Omega)$, which by Theorem 4.2(ii) is equivalent to $\sum_{\ell=1}^{\infty}\alpha_\ell^2(u_0,\nu_\ell)_\Omega^2 < \infty$. ■

Theorem 4.31 *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Assume that $F = \nabla\psi$ satisfies the conditions (i) – (v), that $u_0 \in \overset{\circ}{H}_1(\Omega)$ and that $b \in L^2(Q)\cap C(\bar{Q})$. Then there is a unique weak solution $u \in H_1\big(0,T;L^2(\Omega)\big)\cap L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big) \subseteq H_1(Q)$ of the initial boundary value problem* (4.23) – (4.25). *For almost all $t\in[0,T]$ this solution satisfies the estimate*

$$\frac{1}{2}\|\partial_t u\|_{Q_t}^2 + \int_\Omega \psi\big(\nabla u(t,x)\big)\,\mathrm{d}x \le \int_\Omega \psi\big(\nabla u_0(x)\big)\,\mathrm{d}x + \frac{1}{2}\|b\|_{Q_t}^2. \tag{4.58}$$

Proof. By our assumption (4.54), both sides of the inequality (4.55) are nonnegative. Moreover, from Lemma 4.30 it follows that the right hand side is bounded by a constant C, which is independent of $m\in\mathbb{N}$ and $t\in[0,T]$. From (4.55) it thus follows that

$$\|\partial_t u_m\|_Q^2 \le 2C. \tag{4.59}$$

In the proof of Theorem 4.24 we constructed the solution u as weak limit of a subsequence of $\{u_m\}_{m=1}^\infty$, which is still denoted by $\{u_m\}_{m=1}^\infty$ and which has the property that $u_m \rightharpoonup u$ in $L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$ and $u_m \rightharpoonup u$, $\nabla u_m \rightharpoonup \nabla u$ in $L^2(Q)$. Since by (4.59) the sequence $\{\partial_t u_m\}_{m=1}^\infty$ is bounded in $L^2(Q)$, we can select another subsequence, again denoted by $\{u_m\}_{m=1}^\infty$, such that we also have $\partial_t u_m \rightharpoonup \partial_t u$ in $L^2(Q)$. This implies that $u \in L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)\cap H_1\big(0,T;L^2(\Omega)\big) \subseteq H_1(Q)$. It remains to prove the inequality (4.58).

Since the left hand side of (4.55) is nonnegative and the right hand side of (4.55) is bounded by C, it follows that the function $h_m : [0,T] \to \mathbb{R}$ defined by

$$h_m(t) = \frac{1}{2}\|\partial_t u_m\|_{Q_t}^2 + \int_\Omega \psi\big(\nabla u_m(t,x)\big)\,\mathrm{d}x \\ - \int_\Omega \psi\big(\nabla u_m(0,x)\big)\,\mathrm{d}x - \frac{1}{2}\|b\|_{Q_t}^2$$

satisfies $-C \leq h_m(t) \leq 0$ for $0 \leq t \leq T$. For every function $\varphi \in L^\infty\big([0,T],\mathbb{R}\big)$ with $\varphi(t) \geq 0$ for $0 \leq t \leq T$ we thus have that

$$\begin{aligned} 0 &\geq \int_0^T h_m(t)\varphi(t)\,\mathrm{d}t \\ &= \int_0^T \frac{1}{2}\|\partial_t u_m\|_{Q_t}^2\,\varphi(t)\,\mathrm{d}t + \int_Q \psi\big(\nabla u_m(t,x)\big)\varphi(t)\,\mathrm{d}(t,x) \\ &\quad - \int_\Omega \psi\big(\nabla u_m(0,x)\big)\,\mathrm{d}x \int_0^T \varphi(t)\,\mathrm{d}t - \int_0^T \frac{1}{2}\|b\|_{Q_t}^2\,\varphi(t)\,\mathrm{d}t. \end{aligned} \tag{4.60}$$

Since the mapping $v \mapsto v\big|_{Q_t} : L^2(Q) \to L^2(Q_t)$ is linear and bounded and since $\partial_t u_m \rightharpoonup \partial_t u$ in $L^2(Q)$, it follows that $\partial_t u_m\big|_{Q_t} \rightharpoonup \partial_t u\big|_{Q_t}$ in $L^2(Q_t)$. This implies that

$$\|\partial_t u\|_{Q_t} \leq \liminf_{m\to\infty} \|\partial_t u_m\|_{Q_t},$$

hence

$$\begin{aligned} \int_0^T \frac{1}{2}\|\partial_t u\|_{Q_t}^2\,\varphi(t)\,\mathrm{d}(t) &\leq \int_0^T \frac{1}{2}\liminf_{m\to\infty}\|\partial_t u_m\|_{Q_t}^2\,\varphi(t)\,\mathrm{d}(t) \\ &\leq \liminf_{m\to\infty}\int_0^T \frac{1}{2}\|\partial_t u_m\|_{Q_t}^2\,\varphi(t)\,\mathrm{d}(t). \end{aligned} \tag{4.61}$$

Since $F = \nabla\psi$ satisfies condition (iv) and since $\varphi \in L^\infty((0,T))$, it follows from Lemma 4.28 that

$$w \mapsto f(w) = \int_Q \psi\big(w(t,x)\big)\,\varphi(t)\,\mathrm{d}(t,x) : L^2(Q,\mathbb{R}^n) \to \mathbb{R}$$

is continuous, hence it is lower semicontinuous. Since ψ is convex, it follows for $w, v \in L^2(Q, \mathbb{R}^n)$ and $0 < \vartheta < 1$ that

$$f\big(\vartheta w + (1-\vartheta)v\big) = \int\limits_Q \psi\big(\vartheta w(t,x) + (1-\vartheta)v(t,x)\big)\, \varphi(t)\, \mathrm{d}(t,x)$$
$$\leq \int\limits_Q \Big(\vartheta\psi\big(w(t,x)\big) + (1-\vartheta)\psi\big(v(t,x)\big)\Big)\, \varphi(t)\, \mathrm{d}(t,x)$$
$$= \vartheta f(w) + (1-\vartheta) f(v),$$

hence f is convex. Here we used that $\varphi \geq 0$. ¿From Lemma 4.27 we thus conclude that f is weakly lower semicontinuous. Since $\nabla u_m \rightharpoonup \nabla u$ in $L^2(Q)$, we therefore obtain from Lemma 4.26 that

$$\int\limits_Q \psi\big(\nabla u(t,x)\big)\, \varphi(t)\, \mathrm{d}(t,x) = f(\nabla u) \leq \liminf_{m\to\infty} f(\nabla u_m)$$
$$= \liminf_{m\to\infty} \int\limits_Q \psi\big(\nabla u_m(t,x)\big)\, \varphi(t)\, \mathrm{d}(t,x). \tag{4.62}$$

We combine (4.57), (4.61) and (4.62) with (4.60) to obtain

$$\int\limits_0^T \Big(\frac{1}{2}\|\partial_t u\|_{Q_t}^2 + \int\limits_\Omega \psi\big(\nabla u(t,x)\big)\, \mathrm{d}x - \int\limits_\Omega \psi\big(\nabla u_0(x)\big)\, \mathrm{d}x - \frac{1}{2}\|b\|_{Q_t}^2\Big)\, \varphi(t)\, \mathrm{d}t$$
$$\leq \liminf_{m\to\infty} \int\limits_0^T \frac{1}{2}\|\partial_t u_m\|_{Q_t}^2\, \varphi(t)\, \mathrm{d}t + \liminf_{m\to\infty} \int\limits_Q \psi\big(\nabla u_m(t,x)\big)\, \varphi(t)\, \mathrm{d}(t,x)$$
$$- \lim_{m\to\infty} \int\limits_\Omega \psi\big(\nabla u_m(0,x)\big)\, \mathrm{d}x \int\limits_0^T \varphi(t)\, \mathrm{d}t - \int\limits_0^T \frac{1}{2}\|b\|_{Q_t}^2\, \varphi(t)\, \mathrm{d}t$$
$$\leq \liminf_{m\to\infty} \int\limits_0^T \Big(\frac{1}{2}\|\partial_t u_m\|_{Q_t}^2 + \int\limits_\Omega \psi\big(\nabla u_m(t,x)\big)\, \mathrm{d}x$$
$$- \int\limits_\Omega \psi\big(\nabla u_m(0,x)\big) - \frac{1}{2}\|b\|_{Q_t}^2\Big)\, \varphi(t)\, \mathrm{d}t = \liminf_{m\to\infty} \int\limits_0^T h_m(t)\, \varphi(t)\, \mathrm{d}t \leq 0. \tag{4.63}$$

Let

$$h(t) = \frac{1}{2}\|\partial_t u\|_{Q_t}^2 + \int\limits_\Omega \psi\big(\nabla u(t,x)\big)\, \mathrm{d}x - \int\limits_\Omega \psi\big(\nabla u_0(x)\big)\, \mathrm{d}x - \frac{1}{2}\|b\|_{Q_t}^2, \tag{4.64}$$

let $E = \{t \in [0,T] \mid h(t) > 0\}$ and choose

$$\varphi(t) = \begin{cases} 1, & t \in E \\ 0, & t \in [0,T] \setminus E. \end{cases}$$

Then $h\varphi$ is nonnegative, hence, together with (4.63),

$$0 \leq \int_0^T h\varphi \, \mathrm{d}t \leq 0,$$

and so $\int_0^T h\varphi \, \mathrm{d}t = \int_E h \, \mathrm{d}t = 0$. Since $h(t) > 0$ for all $t \in E$, this can only hold if E is a null set. Since $h(t) \leq 0$ for all $t \in [0,T] \setminus E$, it follows that $h(t) \leq 0$ for almost all t. By definition of h in (4.64), this means that (4.58) holds for almost all t. ■

Lemma 4.32 *With the constants $c_1 > 0$, $c_2 \geq 0$ from condition (iii) the function ψ satisfies for all $\xi \in \mathbb{R}^n$*

$$\psi(\xi) \geq \frac{c_1}{4}|\xi|^2 - \frac{c_2}{2}. \tag{4.65}$$

Proof. Let $R = \sqrt{2\frac{c_2}{c_1}}$. For $|\xi| \geq R$ we then have

$$c_1|\xi|^2 - c_2 = \big(c_1 - \frac{c_2}{|\xi|^2}\big)|\xi|^2 \geq \big(c_1 - \frac{c_2}{R^2}\big)|\xi|^2 = \frac{c_1}{2}|\xi|^2.$$

Since $F(\xi) = \nabla\psi(\xi)$, we obtain from this inequality and from condition (iii) for $\xi \in \mathbb{R}^n$ with $|\xi| \geq R$ that

$$\begin{aligned} \psi(\xi) - \psi\left(R\frac{\xi}{|\xi|}\right) &= \int_R^{|\xi|} \frac{\mathrm{d}}{\mathrm{d}s}\psi\left(\frac{\xi}{|\xi|}s\right) \mathrm{d}s \\ &= \int_R^{|\xi|} \frac{\xi}{|\xi|} \cdot \nabla\psi\left(\frac{\xi}{|\xi|}s\right) \mathrm{d}s = \int_R^{|\xi|} \frac{1}{s}\left(\frac{\xi}{|\xi|}s\right) \cdot F\left(\frac{\xi}{|\xi|}s\right) \mathrm{d}s \\ &\geq \int_R^{|\xi|} \frac{1}{s}\frac{c_1}{2}\left|\frac{\xi}{|\xi|}s\right|^2 \mathrm{d}s = \int_R^{|\xi|} \frac{c_1}{2} s \, \mathrm{d}s = \frac{c_1}{4}|\xi|^2 - \frac{c_1}{2}, \end{aligned}$$

hence

$$\psi(\xi) \geq \frac{c_1}{4}|\xi|^2 - \frac{c_2}{2} + \psi\left(R\frac{\xi}{|\xi|}\right).$$

The estimate (4.65) follows from this inequality for $|\xi| \geq R$, since $\psi\big(R\frac{\xi}{|\xi|}\big) \geq 0$, by (4.54). For $|\xi| < R$ the right hand side of (4.65) is negative, whence because of (4.54) the estimate (4.65) holds also in this case. ■

Corollary 4.33 *Under the assumptions of Theorem 4.31 the solution* u *belongs to the space* $L^\infty\big(0,T;\overset{\circ}{H}_1(\Omega)\big)\cap H_1(Q)$.

Proof. From (4.65) we obtain

$$\frac{c_1}{4}\,|\nabla u(t,x)|^2-\frac{c_2}{2}\le\psi\big(\nabla u(t,x)\big),$$

thus

$$\|\nabla u(t)\|_\Omega^2\le\frac{4}{c_1}\int\limits_\Omega\psi\big(\nabla u(t,x)\big)\,\mathrm{d}x+2\frac{c_2}{c_1}|\Omega|.$$

The statement of the corollary follows from this estimate and from (4.58). ∎

4.6 Discussion and examples

Following the definition of the Friedrich's extension of linear differential operators (cf. Section 9 in the script PDE I), for $u\in H_1(\Omega)$ we say that the divergence of the function $F(\nabla u)$ exists in $L^2(\Omega)$ if there is a function $g\in L^2(\Omega)$ such that for all $v\in\overset{\circ}{H}_1(\Omega)$

$$\big(F(\nabla u),\nabla v\big)_\Omega=-(g,v)_\Omega.$$

In this case we define $\mathrm{div}F(\nabla u)$ by

$$\mathrm{div}F(\nabla u)=g$$

and say that u belongs to the domain of definition of the differential operator

$$u\mapsto\mathrm{div}F(\nabla u).$$

If F is a gradient and if the initial data u_0 belong to $\overset{\circ}{H}_1(\Omega)$, then for the solution u of the initial-boundary value problem (4.23) – (4.25) the function $u(t):\Omega\to\mathbb{R}$ belongs to the domain of definition of the differential operator $\mathrm{div}F(\nabla u)$ for almost all $t\in[0,T]$. To see this we use that by Theorem 4.31 the time derivative u_t belongs to $L^2(Q)$, which means that for the first term in (4.27) we have $\langle u_t,v\rangle=(u_t,v)_Q$. Thus, (4.27) can be written as

$$\int\limits_0^T\big(F(\nabla u(t)),\nabla v(t)\big)_\Omega-\big(b(t)-u_t(t),v(t)\big)_\Omega\,\mathrm{d}t=0,\qquad(4.66)$$

for all $v\in V$. In particular, we can choose $v(t,x)=\varphi(t)w(x)$ with $\varphi\in C_0^\infty\big((0,T),\mathbb{R}\big)$ and $w\in\overset{\circ}{H}_1(\Omega)$. Then (4.66) becomes

$$\int\limits_0^T\Big(\big(F(\nabla u(t)),\nabla w\big)_\Omega-\big(b(t)-u_t(t),w\big)_\Omega\Big)\,\varphi(t)\,\mathrm{d}t=0.$$

Thus, Lemma 2.12 implies

$$\Big(F\big(\nabla u(t)\big), \nabla w\Big)_\Omega = \big(b(t) - u_t(t), w\big)_\Omega, \tag{4.67}$$

for almost all t and all $w \in \overset{\circ}{H}_1(\Omega)$. Since $b(t) - u_t(t) \in L^2(\Omega)$, this means that $u(t)$ belongs to the domain of definition of $\mathrm{div} F(\nabla \cdot)$ and that

$$\mathrm{div} F\big(\nabla u(t)\big) = u_t(t) - b(t),$$

for almost all t.

There is a slight difficulty in this argument, since the set of all t for which (4.67) holds might depend on $w \in \overset{\circ}{H}_1(\Omega)$. However, it suffices to show that (4.67) holds for all w from a dense subset of $\overset{\circ}{H}_1(\Omega)$. Since $\overset{\circ}{H}_1(\Omega)$ is separable, we can choose a countable dense subset. Since a countable union of null sets is a null set, we obtain that there is a null set E such that for all $t \in [0, T] \setminus E$ the equation (4.67) holds for all w from this countable dense set. We leave the details to the reader.

To obtain $u \in H_1(Q)$ and $\mathrm{div} F(\nabla u) \in L^2(Q)$ we only need to assume in Theorem 4.31 that the right hand side b of the differential equation belongs to $L^2(Q)$. If F is a gradient and if the initial data belong to $\overset{\circ}{H}_1(\Omega)$ the regularity of the solution u with respect to the time variable is therefore one order higher than the regularity of the right hand side, and the regularity with respect to the x-variable is increased by more than one order, since $\nabla u \in L^2(Q)$ and since $\mathrm{div} F(\nabla u)$, which is a certain second x-derivative of u, belongs to $L^2(Q)$.

In fact, since $\mathrm{div} F(\nabla \cdot)$ is an elliptic differential operator, one can conclude from $\nabla u \in L^2(Q)$ and $\mathrm{div} F(\nabla u) \in L^2(Q)$ that $u \in L^2\big(0, T; H_2(\Omega)\big)$, provided the boundary $\partial\Omega$ is sufficiently regular. This follows from the regularity theory of such differential operators, cf. [H.-D. Alber, Elliptische partielle Differentialgleichungen, Vorlesungsskript]. In this case the x-regularity of u is two orders higher than the regularity of b.

It is not possible to replace the assumption $u_0 \in \overset{\circ}{H}_1(\Omega)$ in Theorem 4.31 by $u_0 \in H_1(Q)$. To get a solution with higher regularity it is necessary that the initial data satisfy the same boundary conditions as the solution. This is called compatibility condition for the initial data.

In Theorem 4.24, where we studied the case of general monotone F, we also assumed that $b \in L^2(Q)$, but we only obtained $u \in V$ and could not prove that the time derivative u_t belongs to $L^2(Q)$. Instead, the time derivative belongs to $L^2\big(0, T; H_{-1}(\Omega)\big)$, hence the t-regularity of the solution is not higher than the regularity of the right hand side b. The term $\mathrm{div}F(\nabla u)$ does not belong to $L^2(Q)$ but has the same regularity as u_t because of

$$\mathrm{div}F(\nabla u) = u_t - b \in L^2\big(0,T;H_{-1}(\Omega)\big).$$

We have $u \in L^2\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$, but the weak regularity of $\mathrm{div}F(\nabla u)$ is not sufficient to show that $u \in L^2\big(0,T;H_2(\Omega)\big)$. Therefore the x-regularity of u is higher just by one order than the regularity of b. For the initial data we only assumed $u_0 \in L^2(\Omega)$, but even if we assumed that $u_0 \in \overset{\circ}{H}_1(\Omega)$ it would not help to prove higher regularity of u.

Examples. Note that the vector field $F \in C_1(\mathbb{R}^n,\mathbb{R}^n)$ is a gradient if and only if $\mathrm{rot}F(\xi) = 0$. For $n \neq 3$ this means that

$$\frac{\partial}{\partial \xi_i}F_j(\xi) = \frac{\partial}{\partial \xi_j}F_i(\xi), \quad i,j = 1,\dots,n.$$

3. The vector field $F(\xi) = \xi$ from Example 1 satisfies $\frac{\partial}{\partial \xi_i}F_j(\xi) = \frac{\partial}{\partial \xi_i}\xi_j = 0$ for $i \neq j$, hence it is a gradient field. A potential for F is $\psi(\xi) = \frac{1}{2}|\xi|^2$. The initial boundary value problem to the heat equation

$$\begin{aligned} &u_t = \Delta u + b, \\ &u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0 \\ &u(x,0) = u_0(x), \quad x \in \Omega, \end{aligned}$$

has therefore a unique solution

$$u \in L^\infty\big(0,T;\overset{\circ}{H}_1(\Omega)\big) \cap H_1(Q)$$

for all $u_0 \in \overset{\circ}{H}_1(\Omega)$, $b \in L^2(Q)$.

4. The vector field $F(\xi) = c(1 - e^{-|\xi|})\xi$ with $c > 0$ from Example 2 satisfies for $i \neq j$

$$\frac{\partial}{\partial \xi_i}F_j(\xi) = \frac{\partial}{\partial \xi_i}c\,(1 + e^{-|\xi|})\xi_j = -c\,e^{-|\xi|}\,\frac{\xi_i\xi_j}{|\xi|} = \frac{\partial}{\partial \xi_j}F_i(\xi),$$

hence F is a gradient field. The solutions of the initial-boundary value problem with homogeneous Dirichlet boundary condition to the differential equation

$$u_t = c\,\mathrm{div}\big((1 - e^{-|\nabla_x u|})\nabla_x u\big) + b$$

thus belong to $L^\infty\big(0,T;\overset{\circ}{H}_1(\Omega)\big) \cap H_1(Q)$.

5. If $F \in C_1(\mathbb{R}^n,\mathbb{R}^n)$ is a linear mapping, then there is an $n \times n$–matrix $A = (a_{ij})_{i,j=1,\dots,n}$ such that $F(\xi) = A\xi$, and the differential equation (4.23) becomes

$$u_t = \mathrm{div}(A\nabla u) + b. \tag{4.68}$$

This F satisfies the conditions (i) – (v) if A is positive definite. In this case we obtain from Theorem 4.31 that there is a unique solution $u \in V$ of the initial-boundary value problem to the differential equation (4.68). In fact, u even belongs to the space $H_1\big(0,T;L^2(\Omega)\big) \cap L^\infty\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$. To see this note that the condition for $F(\xi) = A\xi$ to be a gradient is

$$a_{ij} = \frac{\partial}{\partial \xi_j} F_i(\xi) = \frac{\partial}{\partial \xi_i} F_j(\xi) = a_{ji},$$

which means that the matrix A must be symmetric. Now since $\partial_i \partial_j u = \partial_j \partial_i u$, we have $\operatorname{div}(A\nabla u) = \operatorname{div}(A^T \nabla u) = \operatorname{div}(A_s \nabla u)$ with the symmetric part $A_s = \frac{1}{2}(A + A^T)$ of A. Therefore (4.68) can be written in the form

$$u_t = \operatorname{div}(A_s \nabla u) + b. \tag{4.69}$$

Moreover, any $\xi \in \mathbb{R}^n$ satisfies $\xi \cdot A\xi = (A^T\xi) \cdot \xi = \xi \cdot (A^T)\xi$, hence $\xi \cdot A\xi = \xi \cdot A_s\xi$. From this we see that if A is positive definite then also A_s. Consequently, Theorem 4.31 and Corollary 4.33 guarantee that if A is positive definite, then the initial-boundary value problem to (4.69) has a unique solution $u \in H_1\big(0,T;L^2(\Omega)\big) \cap L^\infty\big(0,T;\overset{\circ}{H}_1(\Omega)\big)$. This function u is also the unique solution of the initial-boundary value problem to (4.68).

The proof of Theorem 4.31 shows that if $b = 0$ then in the inequality (4.58) we can drop the factor $\frac{1}{2}$ in front of $\|\partial_t u\|_{Q_t}^2$. The resulting inequality satisfied by the solution u of the initial boundary value problem can thus be written in the form

$$\int_\Omega \psi\big(\nabla u(t)\big)\,\mathrm{d}x \le \int_\Omega \psi\big(\nabla u(0)\big)\,\mathrm{d}x - \|\partial_t u\|_{Q_t}^2. \tag{4.70}$$

To give a physical interpretation of this inequality note that $\int_\Omega \psi\big(\nabla u(t)\big)\,\mathrm{d}x$ can be considered to be a measure for the variability of the quantity u, which for example can be the temperature or the electric charge density. This measure is bounded by the right hand side of (4.70), which decreases in time. This means that u becomes more equally distributed in the course of time.

5 Maximum principles and L^∞-estimates for weak solutions of elliptic equations

5.1 The absolute value and the positive part of Sobolev functions

To derive maximum principles for elliptic equations we need to consider functions of the form $|u|$ and $\max(u,0)$ with $u \in H_1^p(\Omega)$. Here we study these functions.

Lemma 5.1 *Let $\theta \in C_1(\mathbb{R},\mathbb{R})$ with*

$$M = \sup_{\xi\in\mathbb{R}} |\theta'(\xi)| < \infty.$$

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and let $u \in H_1^p(\Omega)$ with $1 \le p < \infty$. Then the function $\theta \circ u$ belongs to $H_1^p(\Omega)$ and the chain rule holds: For almost all $x \in \Omega$

$$\frac{\partial}{\partial x_i}\theta\big(u(x)\big) = \theta'\big(u(x)\big)\frac{\partial u}{\partial x_i}(x).$$

Proof. By Theorem 2.27 there is a sequence $\{u_m\}_{m=1}^\infty \subseteq C_\infty(\Omega)\cap H_1^p(\Omega)$, which converges to u in $H_1^p(\Omega)$. We can select a subsequence, again denoted by $\{u_m\}_{m=1}^\infty$, which converges to u pointwise for almost all $x \in \Omega$. From the mean value theorem we have

$$\begin{aligned}\big|\theta\big(u_m(x)\big) - \theta\big(u(x)\big)\big| &\le \sup_{\xi\in\mathbb{R}} |\theta'(\xi)|\,|u_m(x) - u(x)|,\\ &= M|u_m(x) - u(x)|,\end{aligned}$$

whence

$$\|\theta\circ u_m - \theta\circ u\|_p \le M\|u_m - u\|_p \to 0, \quad m\to\infty. \tag{5.1}$$

Moreover,

$$\begin{aligned}&\|(\theta'\circ u_m)\partial_{x_i}u_m - (\theta'\circ u)\partial_{x_i}u\|_p\\ &\qquad \le \|(\theta'\circ u_m)(\partial_{x_i}u_m - \partial_{x_i}u)\|_p + \|(\theta'\circ u_m - \theta'\circ u)\partial_{x_i}u\|_p.\end{aligned} \tag{5.2}$$

Both terms on the right hand side tend to zero for $m\to\infty$. This is obvious for the first term, since $\|\theta'\circ u_m\|_\infty \le \sup_{\xi\in\mathbb{R}^n} |\theta'(\xi)| = M$ implies

$$\|(\theta'\circ u_m)(\partial_{x_i}u_m - \partial_{x_i}u)\|_p \le M\|\partial_{x_i}u_m - \partial_{x_i}u\|_p \to 0.$$

To see that also the second term tends to zero, note that

$$\|(\theta'\circ u_m - \theta'\circ u)\partial_{x_i}u\|_p^p = \int_\Omega \big|\theta'\big(u_m(x)\big) - \theta'\big(u(x)\big)\big|^p\,|\partial_{x_i}u(x)|^p\,\mathrm{d}x. \tag{5.3}$$

The integrand on the right hand side tends to zero pointwise almost everywhere in Ω for $m \to \infty$, since θ' is continuous. Moreover, the integrand is bounded uniformly with respect to m by the integrable function

$$(2M)^p \, |\partial_{x_i} u(x)|^p,$$

hence the Lebesgue theorem on dominated convergence implies that the integral in (5.3) tends to zero.

By (5.1) we see that $\{\theta \circ u_m\}_{m=1}^{\infty}$ tends to $\theta \circ u$ in $L^p(\Omega)$ and (5.2) implies that $\{\partial_{x_i}(\theta \circ u_m)\}_{m=1}^{\infty}$ tends to $(\theta' \circ u)\partial_{x_i} u$ in $L^p(\Omega)$. Together this implies that $\theta \circ u \in L^p(\Omega)$ and that $\theta \circ u$ has a weak derivative in $L^p(\Omega)$ given by $(\theta' \circ u)\partial_{x_i} u$, hence $\theta \circ u \in H_1^p(\Omega)$.

■

We use the notation

$$\operatorname{sgn}(\xi) = \begin{cases} -1, & \xi < 0, \\ 0, & \xi = 0, \\ 1, & \xi > 0. \end{cases}$$

and write for the composition of the function sgn with a function u

$$\operatorname{sgn}(u) = \operatorname{sgn} \circ u.$$

Lemma 5.2 *Let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded set, let $1 \le p < \infty$ and assume that $u \in H_1^p(\Omega)$. Then $|u| \in H_1^p(\Omega)$ with $\partial_{x_i}|u| = \operatorname{sgn}(u)\partial_{x_i} u$ and*

$$\partial_{x_i} u(x) = \partial_{x_i}|u|(x) = 0,$$

for almost all $x \in N = \{x \in \Omega \,|\, u(x) = 0\}$.

Proof: For $\gamma \in [-1, 1]$ and $\varepsilon > 0$ define

$$\sigma_{\gamma\varepsilon}(t) = \begin{cases} -1, & -\infty < t \le -\varepsilon(1+\gamma) \\ \gamma + \frac{t}{\varepsilon} & -\varepsilon(1+\gamma) < t < \varepsilon(1-\gamma) \\ 1, & \varepsilon(1-\gamma) \le t < \infty. \end{cases}$$

$\sigma_{\gamma\varepsilon}$ is a continuous function with $\sigma_{\gamma\varepsilon}(0) = \gamma$ and satisfies

$$\lim_{\varepsilon \to 0} \sigma_{\gamma\varepsilon}(t) = \operatorname{sgn}_{\gamma}(t) = \begin{cases} -1, & t < 0 \\ \gamma, & t = 0 \\ 1, & t > 0. \end{cases} \tag{5.4}$$

Set

$$\theta_{\gamma\varepsilon}(t) = \int\limits_0^t \sigma_{\gamma\varepsilon}(s)\,\mathrm{d}s. \tag{5.5}$$

Then $\theta_{\gamma\varepsilon} \in C_1(\mathbb{R})$ satisfies $|\theta'_{\gamma\varepsilon}(t)| = |\sigma_{\gamma\varepsilon}(t)| \leq 1$, hence Lemma 5.1 implies $\theta_{\gamma\varepsilon} \circ u \in H_1^p(\Omega)$ and

$$\partial_{x_i}\theta_{\gamma\varepsilon}\big(u(x)\big) = \theta'_{\gamma\varepsilon}\big(u(x)\big)\partial_{x_i}u(x) = \sigma_{\gamma\varepsilon}\big(u(x)\big)\partial_{x_i}u(x). \tag{5.6}$$

From (5.4) and (5.5) it follows that

$$\lim_{\varepsilon\to 0}\theta_{\gamma\varepsilon}(x) = \int\limits_0^t \operatorname{sgn}_\gamma(s)\,\mathrm{d}s = |t|,$$

which yields

$$\lim_{\varepsilon\to 0}\theta_{\varepsilon\gamma}\big(u(x)\big) = |u(x)|, \tag{5.7}$$

whereas (5.4) and (5.6) show that

$$\lim_{\varepsilon\to 0}\partial_{x_i}\theta_{\gamma\varepsilon}\big(u(x)\big) = \operatorname{sgn}_\gamma\big(u(x)\big)\partial_{x_i}u(x). \tag{5.8}$$

From $|\sigma_{\gamma\varepsilon}(t)| \leq 1$ and from (5.5) we obtain

$$|\theta_{\gamma\varepsilon} \circ u| \leq |u| \in L^p(\Omega),$$

from which we conclude together with (5.7) by Lebesgue's convergence theorem that

$$\theta_{\gamma\varepsilon} \circ u \to |u| \text{ in } L^p(\Omega). \tag{5.9}$$

Similarly, from (5.8) and from

$$|\partial_{x_i}(\theta_{\gamma\varepsilon} \circ u)| = |\sigma_{\gamma\varepsilon} \circ u|\,|\partial_{x_i}u| \leq |\partial_{x_i}u| \in L^p(\Omega)$$

we infer by Lebesgue's theorem that

$$\partial_{x_i}(\theta_{\gamma\varepsilon} \circ u) \to (\operatorname{sgn}_\gamma \circ u)\partial_{x_i}u \text{ in } L^p(\Omega).$$

This relation and (5.9) together imply that $|u| \in H_1^p(\Omega)$ with

$$\partial_{x_i}|u| = (\operatorname{sgn}_\gamma \circ u)\partial_{x_i}u.$$

In particular, for almost all $x \in N$ we obtain

$$\partial_{x_i}|u|(x) = \gamma\partial_{x_i}u(x).$$

Since $\gamma \in [-1, 1]$ was chosen arbitrary, this equation can only hold if

$$\partial_{x_i}|u|(x) = \partial_{x_i}u(x) = 0,$$

for almost all $x \in N$. ■

Theorem 5.3 *Let $\Omega \subseteq \mathbb{R}^n$ be an open, bounded set and let $1 \leq p < \infty$. If $u \in H_1^p(\Omega)$, then* $\max(u, 0)$ *belongs to* $H_1^p(\Omega)$ *and the weak derivative satisfies*

$$\big(\partial_{x_i} \max(u,0)\big)(x) = \begin{cases} \partial_{x_i} u(x), & \text{if } u(x) > 0, \\ 0, & \text{if } u(x) \leq 0, \end{cases}$$

for almost all $x \in \Omega$.

Proof: By Lemma 5.2 we have

$$\max(u,0) = \frac{1}{2}(u + |u|) \in H_1^p(\Omega)$$

and $\partial_{x_i} \max(u,0)(x) = \frac{1}{2}\big(\partial_{x_i} u(x) + \partial_{x_i} u(x)\big) = \partial_{x_i} u(x)$ for $u(x) > 0$ and $\partial_{x_i} \max(u,0)(x) = 0$ for almost all x from the set $\{x \in \Omega \,|\, \max(u,0)(x) = 0\} = \{x \in \Omega \,|\, u(x) \leq 0\}$. ■

Definition 5.4 Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. For a measurable function $u : \Omega \to \mathbb{R}$ consider the set B_1 of all $r \in \mathbb{R}$ such that for every neighborhood V of $\partial\Omega$ in $\bar{\Omega}$ we have

$$\operatorname*{ess\,sup}_{x \in V} u(x) \geq r.$$

We define

$$\limsup_{V \to \partial\Omega} u = \begin{cases} -\infty, & \text{if } B_1 = \emptyset \\ \sup B_1, & \text{if } B_1 \neq \emptyset, \mathbb{R}, \\ \infty, & \text{if } B_1 = \mathbb{R}. \end{cases}$$

Similarly, let B_2 be the set of all $r \in \mathbb{R}$ such that for every neighborhood V of $\partial\Omega$ in $\bar{\Omega}$ we have

$$\operatorname*{ess\,inf}_{x \in V} u(x) \leq r.$$

We define

$$\liminf_{V \to \partial\Omega} u = \begin{cases} -\infty, & \text{if } B_2 = \mathbb{R} \\ \inf B_2, & \text{if } B_2 \neq \emptyset, \mathbb{R} \\ \infty, & \text{if } B_2 = \emptyset. \end{cases}$$

Note that if $B_1 \neq \emptyset$ and $B_1 \neq \mathbb{R}$, then $\sup B_1 < \infty$ because B_1 is bounded above. For, if $r_0 \notin B_1$, then obviously $[r_0, \infty) \cap B_1 = \emptyset$. Similarly, if $B_2 \neq \emptyset$ and $B_2 \neq \mathbb{R}$, then B_2 is bounded below and therefore $\inf B_2 > -\infty$.
These notions have the following properties:

Lemma 5.5 *Let $r \in \mathbb{R}$.*

(i) *We have* $r = \limsup_{V\to\partial\Omega} u$ *if and only if for every* $\varepsilon > 0$ *there is a neighborhood* V_0 *of* $\partial\Omega$ *in* $\bar{\Omega}$ *such that for every neighborhood* $V \subseteq V_0$ *of* $\partial\Omega$

$$r - \varepsilon \leq \operatorname*{ess\,sup}_{x\in V} u(x) \leq r + \varepsilon.$$

(ii) *We have* $r = \liminf_{V\to\partial\Omega} u$ *if and only if for every* $\varepsilon > 0$ *there is a neighborhood* V_0 *of* $\partial\Omega$ *in* $\bar{\Omega}$ *such that for every neighborhood* $V \subseteq V_0$ *of* $\partial\Omega$

$$r - \varepsilon \leq \operatorname*{ess\,inf}_{x\in V} u(x) \leq r + \varepsilon.$$

(iii) *For* $u \in C(\bar{\Omega})$ *we have*

$$\limsup_{V\to\partial\Omega} u = \sup_{x\in\partial\Omega} u(x), \quad \liminf_{V\to\partial\Omega} u = \inf_{x\in\partial\Omega} u(x).$$

Proof: We have $r = \sup B_1$ if and only if for every $\varepsilon > 0$ the two relations $r - \varepsilon \in B_1$ and $r + \varepsilon \notin B_1$ hold. Now $r - \varepsilon \in B_1$ if and only if for every neighborhood V of $\partial\Omega$ in $\bar{\Omega}$ we have $\operatorname*{ess\,sup}_{x\in V} u(x) \geq r - \varepsilon$. Also, $r + \varepsilon \notin B_1$ if and only if there is a neighborhood V_0 of $\partial\Omega$ in $\bar{\Omega}$ such that $\operatorname*{ess\,sup}_{x\in V_0} u(x) < r + \varepsilon$. Statement (i) follows from these relations. (ii) is proved in the same way.

To prove (iii), let $u \in C(\bar{\Omega})$ and let $\varepsilon > 0$. For every $y \in \partial\Omega$ there is an open ball $B(y)$ such that $\sup_{x\in B(y)\cap\Omega} u(x) < u(y) + \varepsilon$, hence $V_0 = \bar{\Omega} \cap \bigcup_{y\in\partial\Omega} B(y)$ is a neighborhood of $\partial\Omega$ in $\bar{\Omega}$ such that for every neighborhood $V \subseteq V_0$ of $\partial\Omega$

$$\sup_{y\in\partial\Omega} u(y) - \varepsilon < \sup_{y\in\partial\Omega} u(y) \leq \sup_{x\in V} u(x) \leq \sup_{x\in V_0} u(x) \leq \sup_{y\in\partial\Omega} u(y) + \varepsilon.$$

From statement (i) of the lemma we thus obtain $\sup_{y\in\partial\Omega} u(y) = \limsup_{V\to\partial\Omega} u$. The assertion for $\liminf_{V\to\partial\Omega} u$ is proved analogously. ∎

Lemma 5.6 *Let* $\Omega \subseteq \mathbb{R}^n$ *be a bounded open set and let* $1 \leq p < \infty$. *Assume that* $\omega \in H_1^p(\Omega)$ *satisfies* $\liminf_{V\to\partial\Omega} \omega \geq 0$. *Then for every* $u \in \overset{\circ}{H}{}_1^p(\Omega)$ *we have*

$$\max(u - \omega, 0) \in \overset{\circ}{H}{}_1^p(\Omega).$$

Proof: Since $u - \omega \in H_1^p(\Omega)$, it follows from Theorem 5.3 that $\max(u - \omega, 0) \in H_1^p(\Omega)$. To prove that this function belongs to $\overset{\circ}{H}{}_1^p(\Omega)$ it suffices to show that there is a sequence $\{u_m\}_{m=1}^\infty \subseteq \overset{\circ}{H}{}_1^p(\Omega)$ with $u_m \to \max(u - \omega, 0)$ in $H_1^p(\Omega)$.

To this end let $\{\varphi_m\}_{m=1}^\infty \subseteq C_0^\infty(\Omega)$ be a sequence with $\varphi_m \to u$ in $H_1^p(\Omega)$ and with $\varphi_m(x) \to u(x)$ for almost all $x \in \Omega$. We set

$$u_m = \min\left(\varphi_m - \omega - \frac{1}{m}, 0\right).$$

Again by Theorem 5.3 we have $u_m \in H_1^p(\Omega)$. The assumption $\liminf\limits_{V\to\partial\Omega} \omega \geq 0$ implies that for every $m \in \mathbb{N}$ there is a neighborhood V of $\partial\Omega$ in $\bar\Omega$ such that $\omega + \frac{1}{m} \geq 0$ on V. Since $\partial\Omega$ is compact, there is $\delta > 0$ such that all $x \in \Omega$ with $\mathrm{dist}(x, \partial\Omega) < \delta$ belong to V. Moreover, there is $\delta_m > 0$ such that $\varphi_m(x) = 0$ if $\mathrm{dist}(x, \partial\Omega) < \delta_m$, hence we have $\varphi_m(x) - \omega(x) - \frac{1}{m} \leq 0$ for all x with $\mathrm{dist}(x, \partial\Omega) < \min(\delta, \delta_m)$, hence $u_m(x) = 0$ for all such x. Corollary 2.22 thus implies that $u_m \in \overset{\circ}{H}{}_1^p(\Omega)$. To finish the proof it therefore suffices to show that $u_m \mapsto \max(u, 0)$ in $H_1^p(\Omega)$, and to prove this relation it is sufficient to show that

$$\left|\varphi_m - \omega - \frac{1}{m}\right| \to |u - \omega|, \quad \text{in } H_1^p(\Omega), \tag{5.10}$$

because (5.10) and $\varphi_m \to u$ in $H_1^p(\Omega)$ imply

$$\begin{aligned} u_m &= \max\left(\varphi_m - \omega - \frac{1}{m}, 0\right) = \frac{1}{2}\left(\varphi_m - \omega - \frac{1}{m} + \left|\varphi_m - \omega - \frac{1}{m}\right|\right) \\ &\to \frac{1}{2}\big(u - \omega + |u - \omega|\big) = \max(u - \omega, 0), \end{aligned}$$

in $H_1^p(\Omega)$.

To verify (5.10) note first that the inverse triangle inequality yields

$$\begin{aligned} &\int_\Omega \left| |u - \omega| - \left|\varphi_m - \omega - \frac{1}{m}\right| \right|^p \mathrm{d}x \\ &\qquad \leq \int_\Omega \left| (u - \omega) - \left(\varphi_m - \omega - \frac{1}{m}\right) \right|^p \mathrm{d}x \\ &\qquad = \left\| u - \varphi_m + \frac{1}{m} \right\|_p^p \leq \left(\|u - \varphi_m\|_p + \frac{1}{m}|\Omega|^{1/p} \right)^p \to 0, \end{aligned}$$

for $m \to \infty$, whence

$$\left|\varphi_m - \omega - \frac{1}{m}\right| \to |u - \omega| \quad \text{in } L^p(\Omega). \tag{5.11}$$

Lemma 5.2 implies

$$\begin{aligned} &\left\| \partial_{x_i}|u - \omega| - \partial_{x_i}\left|\varphi_m - \omega - \frac{1}{m}\right| \right\|_p \\ &= \left\| \mathrm{sgn}(u - \omega)\partial_{x_i}(u - \omega) - \mathrm{sgn}\left(\varphi_m - \omega - \frac{1}{m}\right)\partial_{x_i}(\varphi_m - \omega) \right\|_p \\ &\leq \left\| \left(\mathrm{sgn}(u - \omega) - \mathrm{sgn}\left(\varphi_m - \omega - \frac{1}{m}\right) \right) \partial_{x_i}(u - \omega) \right\|_p \\ &\quad + \left\| \mathrm{sgn}\left(\varphi_m - \omega - \frac{1}{m}\right)(\partial_{x_i}u - \partial_{x_i}\varphi_m) \right\|_p. \end{aligned} \tag{5.12}$$

The second term on the right hand side satisfies

$$\Big\|\operatorname{sgn}\Big(\varphi_m - \omega - \frac{1}{m}\Big)(\partial_{x_i} u - \partial_{x_i}\varphi_m)\Big\|_p \le \|\partial_{x_i} u - \partial_{x_i}\varphi_m\|_p \to 0$$

for $m \to \infty$. For the first term we have

$$\begin{aligned} &\Big\|\Big(\operatorname{sgn}(u-\omega) - \operatorname{sgn}\Big(\varphi_m - \omega - \frac{1}{m}\Big)\Big)\partial_{x_i}(u-\omega)\Big\|_p^p \\ &\quad = \int\limits_{\Omega\setminus N} \Big|\operatorname{sgn}\big((u-\omega)(x)\big) - \operatorname{sgn}\Big(\Big(\varphi_m - \omega - \frac{1}{m}\Big)(x)\Big)\Big|^p \, \big|\partial_{x_i}(u-\omega)(x)\big|^p \,\mathrm{d}x, \end{aligned} \tag{5.13}$$

where $N = \{x \in \Omega \,|\, (u-\omega)(x) = 0\}$. To obtain this equation we used Lemma 5.2, which implies that $\partial_{x_i}(u-\omega)(x) = 0$ for almost all $x \in N$. For $(u-\omega)(x) \neq 0$ the function sgn is constant in a neighborhood of the number $(u-\omega)(x) \in \mathbb{R}$. By our choice of φ_m we have $(\varphi_m - \omega)(x) - \frac{1}{m} \to (u-\omega)(x)$ for almost all $x \in \Omega$, whence

$$\operatorname{sgn}\big((u-\omega)(x)\big) - \operatorname{sgn}\Big(\Big(\varphi_m - \omega - \frac{1}{m}\Big)(x)\Big) \to 0, \quad m \to \infty.$$

Therefore the integrand in (5.13) tends to zero pointwise for almost all $x \in \Omega \setminus N$. Moreover, we have

$$\begin{aligned} &\Big|\operatorname{sgn}(u-\omega) - \operatorname{sgn}\Big(\varphi_m - \omega - \frac{1}{m}\Big)\Big|^p \, |\partial_{x_i}(u-\omega)|^p \\ &\quad \le 2^p \, |\partial_{x_i}(u-\omega)|^p \in L^1(\Omega \setminus N). \end{aligned}$$

From these properties we conclude by the Lebesgue convergence theorem that the integral on the right hand side of (5.13) tends to zero for $m \to \infty$, whence both terms on the right hand side of (5.12) tend to zero for $m \to \infty$. The relation (5.10) is a consequence of (5.11) and (5.12). ■

5.2 Maximum principles for weak solutions

In this section we derive maximum principles for weak solutions of linear and nonlinear equations of the form

$$-\operatorname{div} F\big(x, \nabla u(x)\big) = 0$$

where F is a monotone vector function. Such equations are called elliptic equations in divergence form. As before, in this section we use the notation

$$H_m(\Omega) = H_m^2(\Omega), \quad \overset{\circ}{H}_m(\Omega) = \overset{\circ}{H}{}_m^2(\Omega).$$

We start with a result for linear equations. In this case we have

$$F(x,\xi) = A(x)\xi, \quad (x,\xi) \in \Omega \times \mathbb{R}^n,$$

with a matrix $A(x) = \big(a_{ij}(x)\big)_{i,j=1,\ldots,n}$. We assume that the functions $a_{ij} : \Omega \to \mathbb{R}$ are bounded and measurable. The vector field $\xi \mapsto F(x,\xi) = A(x)\xi$ is uniformly strongly monotone if $A(x)$ is uniformly positive definite: There is a constant $c > 0$ such that

$$\xi \cdot A(x)\xi \ge c|\xi|^2, \quad (x,\xi) \in \Omega \times \mathbb{R}^n. \tag{5.14}$$

In this case the linear differential operator

$$(Lu)(x) = -\mathrm{div}\big(A(x)\nabla u(x)\big) \tag{5.15}$$

is called uniformly strongly elliptic on Ω, cf. [H.-D. Alber: Partial Differential Equations I, (lecture notes), section 8]. Since by assumption $A \in L^\infty(\Omega, \mathbb{R}^{n\times n})$, for $u \in H_1(\Omega)$ we have that $x \mapsto A(x)\nabla u(x) \in L^2(\Omega)$. In this case the Definition 3.18 of weak solutions $u \in H_1(\Omega)$ of the Dirichlet problem

$$Lu = 0, \qquad \text{in } \Omega \tag{5.16}$$

$$u\big|_{\partial\Omega} = \gamma\big|_{\partial\Omega}, \tag{5.17}$$

with $\gamma \in H_1(\Omega)$ is equivalent to $u - \gamma \in \overset{\circ}{H}_1(\Omega)$ and

$$\big(A(\cdot)\nabla u, \nabla v\big)_\Omega = 0, \tag{5.18}$$

for all $v \in \overset{\circ}{H}_1(\Omega)$.

Theorem 5.7 (Maximum principle for linear equations) *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Assume that L defined in* (5.15) *is uniformly strongly elliptic on Ω. For $\gamma \in H_1(\Omega)$ let $u \in H_1(\Omega)$ be a weak solution of the boundary value problem* (5.16), (5.17). *Then*

$$\|u\|_{\infty,\Omega} \le \limsup_{V \to \partial\Omega} |\gamma|, \tag{5.19}$$

with the norm $\|w\|_{\infty,\Gamma} = \operatorname*{ess\,sup}_{x\in\Gamma} |w(x)|$.

Proof: Set $M = \limsup_{V\to\partial\Omega} |\gamma|$ and

$$w = \max(u - M, 0).$$

Since $u - M = (u - \gamma) - (M - \gamma)$, where $u - \gamma \in \overset{\circ}{H}_1(\Omega)$ and where $\omega = M - \gamma \in H_1(\Omega)$ satisfies

$$\liminf_{V \to \partial\Omega} \omega \geq M - \limsup_{V \to \partial\Omega} |\gamma| = 0,$$

Lemma 5.6 implies $w \in \overset{\circ}{H}_1(\Omega)$. Moreover, Theorem 5.3 yields

$$\partial_{x_i} w(x) = \begin{cases} \partial_{x_i} u(x), & \text{if } w(x) > 0, \\ 0, & \text{if } w(x) = 0, \end{cases}$$

for almost all $x \in \Omega$, hence

$$\partial_{x_j} u(x) \partial_{x_i} w(x) = \partial_{x_j} w(x) \partial_{x_i} w(x), \tag{5.20}$$

for almost all $x \in \Omega$. We insert w for v in (5.18) and use (5.20) and (5.14) to compute

$$\begin{aligned} 0 = (A\nabla u, \nabla w)_\Omega &= \int_\Omega \sum_{i,j=1}^n a_{ij}(x) \partial_{x_j} u(x) \partial_{x_i} w(x) \, \mathrm{d}x \\ &= \int_\Omega \sum_{i,j=1}^n a_{ij}(x) \partial_{x_j} w(x) \partial_{x_i} w(x) \, \mathrm{d}x \geq \int_\Omega c |\nabla w(x)|^2 \, \mathrm{d}x = c \|\nabla w\|_\Omega^2. \end{aligned}$$

The Poincaré inequality yields

$$\|w\|_\Omega \leq K \|\nabla w\|_\Omega = 0,$$

whence $w(x) = 0$ for almost all $x \in \Omega$, which implies

$$u(x) \leq M = \limsup_{V \to \partial\Omega} |\gamma|, \tag{5.21}$$

for almost all $x \in \Omega$. The function $-u$ is a weak solution of the problem (5.16), (5.17) to the boundary data $-\gamma$. From the foregoing we thus obtain

$$-u(x) \leq \limsup_{V \to \partial\Omega} |-\gamma| = \limsup_{V \to \partial\Omega} |\gamma|,$$

which together with (5.21) proves (5.19). ■

Remarks. We cannot deduce Theorem 5.7 from the classical maximum principle for solutions of elliptic equations, since the coefficient matrix A is only assumed to be bounded and measurable. Therefore weak solutions of (5.16) need not have two classical derivatives. Note also that we did not assume any regularity for the boundary $\partial\Omega$.

In the next two theorems we deduce a general maximum principle for weak solutions of nonlinear equations.

Theorem 5.8 (Maximum principle for nonlinear equations) *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open set. Assume that $F \in C_1(\mathbb{R}^n, \mathbb{R}^n)$ is strictly monotone and satisfies*

$$|F(\xi)| \leq c_1 |\xi|, \tag{5.22}$$

for all $\xi \in \mathbb{R}^n$ with a constant $c_1 > 0$. Let $\gamma \in H_1(\Omega)$ and suppose that the affine function $\omega(x) = \omega_0 + a \cdot (x - x_0)$ with $\omega_0 \in \mathbb{R}$ and $x_0, a \in \mathbb{R}^n$ satisfies

$$\liminf_{V \to \partial\Omega} (\omega - \gamma) \geq 0. \tag{5.23}$$

Then for every weak solution $u \in H_1(\Omega)$ of the boundary value problem

$$-\mathrm{div} F\big(\nabla u(x)\big) = 0, \qquad x \in \Omega, \tag{5.24}$$

$$u(x) = \gamma(x), \qquad x \in \partial\Omega \tag{5.25}$$

the inequality

$$u(x) \leq \omega(x),$$

holds for almost all $x \in \Omega$.

Remarks If $\gamma \in C_1(\bar{\Omega})$, then by Lemma 5.5 condition (5.23) is equivalent to $\inf_{\partial\Omega}(\omega - \gamma) \geq 0$, which holds if and only if

$$\gamma(y) \leq \omega(y),$$

for all $y \in \partial\Omega$.

By Definition 1.4 F is strictly monotone if

$$\big(F(\xi) - F(\eta)\big) \cdot (\xi - \eta) > 0$$

for all $\xi, \eta \in \mathbb{R}^n$ with $\xi \neq \eta$. From (5.22) it follows that $F(\nabla u) \in L^2(\Omega)$ for $u \in H_1(\Omega)$. In this case Definition 3.18 of weak solutions $u \in H_1(\Omega)$ of (5.24), (5.25) is equivalent to $u - \gamma \in \overset{\circ}{H}_1(\Omega)$ and

$$\big(F(\nabla u), \nabla w\big)_\Omega = 0, \tag{5.26}$$

for all $w \in \overset{\circ}{H}_1(\Omega)$.

Proof of Theorem 5.8: Define

$$v(x) = \min\big(u(x), \omega(x)\big), \quad x \in \Omega.$$

To prove the theorem it suffices to show that $v(x) = u(x)$ for almost all $x \in \Omega$. To verify this observe that

$$\begin{aligned} v - u &= \min(u, \omega) - u = \min(0, \omega - u) \\ &= -\max(u - \omega, 0) = -\max\big((u - \gamma) - (\omega - \gamma), 0\big). \end{aligned} \tag{5.27}$$

Since $u - \gamma \in \overset{\circ}{H}_1(\Omega)$ and since $\omega - \gamma \in H_1(\Omega)$ satisfies (5.23), it follows from Lemma 5.6 that $v - u \in \overset{\circ}{H}_1(\Omega)$, hence $v = u + (v - u) \in H_1(\Omega)$ and

$$\nabla v(x) = \nabla u(x) + \nabla(v - u)(x), \tag{5.28}$$

for almost all $x \in \Omega$. With

$$\Omega_+ = \{x \in \Omega \mid u(x) > \omega(x)\}$$

we obtain by application of Theorem 5.3 to (5.27) that

$$\nabla(v - u)(x) = \begin{cases} 0, & \text{for almost all } x \in \Omega \setminus \Omega_+, \\ \nabla\omega(x) - \nabla u(x), & \text{for almost all } x \in \Omega_+. \end{cases} \tag{5.29}$$

(5.28) and (5.29) yield that

$$\nabla v(x) = \nabla\omega(x), \tag{5.30}$$

for almost all $x \in \Omega_+$. Next, by definition of the affine function ω we have $\nabla\omega(x) = a \in \mathbb{R}^n$, hence $F\big(\nabla\omega(x)\big)$ is constant, which yields $\operatorname{div} F\big(\nabla\omega(x)\big) = 0$. Thus,

$$\big(F(\nabla\omega), \nabla(v - u)\big)_\Omega = -\big(\operatorname{div} F(\nabla\omega), v - u\big)_\Omega = 0. \tag{5.31}$$

Since u is a weak solution, we obtain by insertion of $v - u$ for w into (5.26) that

$$\big(F(\nabla v), \nabla(v - u)\big)_\Omega = 0,$$

thence, by (5.29), (5.30) and (5.31),

$$\begin{aligned} &\int_\Omega \big(F(\nabla v) - F(\nabla u)\big) \cdot (\nabla v - \nabla u)\,\mathrm{d}x \\ &\qquad = \int_\Omega F(\nabla v) \cdot \nabla(v - u)\,\mathrm{d}x = \int_{\Omega_+} F(\nabla v) \cdot \nabla(v - u)\,\mathrm{d}x, \\ &\qquad = \int_{\Omega_+} F(\nabla\omega) \cdot \nabla(v - u)\,\mathrm{d}x = 0. \end{aligned} \tag{5.32}$$

Since $\big(F(\nabla v) - F(\nabla u)\big) \cdot (\nabla v - \nabla u) \geq 0$ by the strict monotonicity of F, equation (5.32) can only hold if

$$\Big(F\big(\nabla v(x)\big) - F\big(\nabla u(x)\big)\Big) \cdot \big(\nabla v(x) - \nabla u(x)\big) = 0,$$

for almost all $x \in \Omega$, which again by the strict monotonicity implies

$$\nabla v(x) = \nabla u(x),$$

for almost all $x \in \Omega$. Because of $v - u \in \overset{\circ}{H}_1(\Omega)$, Poincaré's inequality yields

$$\|v - u\|_\Omega \le K\|\nabla(v - u)\|_\Omega = 0,$$

whence $v(x) = u(x)$ for almost all $x \in \Omega$. ■

Theorem 5.9 (Minimum principle for nonlinear equations) *Let Ω and F satisfy the same assumptions as in Theorem 5.8. Let $\gamma \in H_1(\Omega)$ and suppose that the affine function $\omega(x) = \omega_0 + a \cdot (x - x_0)$ satisfies*

$$\liminf_{V \to \partial\Omega} (\gamma - \omega) \ge 0.$$

Then for every weak solution $u \in H_1(\Omega)$ of the boundary value problem (5.24), (5.25) *the inequality*

$$\omega(x) \le u(x) \tag{5.33}$$

holds for almost all $x \in \Omega$.

Proof: Define

$$v(x) = \max\big(u(x), \omega(x)\big), \quad x \in \Omega.$$

Then

$$\begin{aligned} v - u = \max(u, \omega) - u &= \max(\omega - u, 0) \\ &= \max\big(-(u - \gamma) - (\gamma - \omega), 0\big), \end{aligned}$$

with $-(u - \gamma) \in \overset{\circ}{H}_1(\Omega)$, $\gamma - \omega \in H_1(\Omega)$ and $\liminf_{V \to \partial\Omega} (\gamma - \omega) \ge 0$. Thus, Lemma 5.6 yields $v - u \in \overset{\circ}{H}_1(\Omega)$, whence $v = u + (v - u) \in H_1(\Omega)$. Proceeding as in the proof of Theorem 5.8 we obtain equation (5.32), which holds with Ω_+ redefined as

$$\Omega_+ = \{x \in \Omega \mid u(x) < \omega(x)\}.$$

Continuing as in that proof, we obtain $v(x) = u(x)$ almost everywhere, which implies (5.33). ■

Corollary 5.10 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Assume that $F \in C_1(\mathbb{R}^n, \mathbb{R}^n)$ is strictly monotone and satisfies the estimate* (5.22). *Let $\gamma \in H_1(\Omega)$ and let $u \in H_1(\Omega)$ be a weak solution of* (5.24), (5.25). *Then*

$$\|u\|_{\infty,\Omega} \le \limsup_{V \to \partial\Omega} |\gamma|.$$

Proof: Choose for ω in Theorem 5.8 the constant function $\omega(x) = \limsup_{V\to\partial\Omega} |\gamma|$ and in Theorem 5.9 the function $\omega(x) = -\limsup_{V\to\partial\Omega} |\gamma|$. Then these two theorems yield

$$-\limsup_{V\to\partial\Omega} |\gamma| \leq u(x) \leq \limsup_{V\to\partial\Omega} |\gamma|,$$

for almost all $x \in \Omega$. ■

Remark Of course, Theorems 5.8, 5.9 and Corollary 5.10 hold for linear functions $F(\xi) = A\xi$ with a positive definite $n \times n$-matrix A. Yet, A must be a constant matrix whereas $A(x)$ in Theorem 5.7 can be a measurable and bounded function of x.

We conclude this section by deriving a maximum principle for the first derivatives of weak solutions to nonlinear equations. Also in this result we pose no restrictions on $\partial\Omega$, but we assume that the weak solution, which we study, belongs to $H_2(\Omega)$. If $\partial\Omega$ is of class C_1, then it can in fact be proved that weak solutions belong to $H_2(\Omega)$. We state this result from the regularity theory of nonlinear elliptic equations at the end of this section.

Theorem 5.11 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, open set. Assume that there are constants c, $c_1 > 0$ such that $F \in C_1(\mathbb{R}^n, \mathbb{R}^n)$ satisfies*

$$\big(F(\xi) - F(\eta)\big) \cdot (\xi - \eta) \geq c|\xi - \eta|^2, \quad \xi, \eta \in \mathbb{R}^n \tag{5.34}$$

$$|\nabla F(\xi)| \leq c_1, \quad \xi \in \mathbb{R}^n. \tag{5.35}$$

Let $\gamma_i \in \overset{\circ}{H}_1(\Omega), i = 1, \ldots, n$, and suppose that $u \in H_2(\Omega)$ is a weak solution of

$$-\mathrm{div} F\big(\nabla u(x)\big) = 0, \quad x \in \Omega. \tag{5.36}$$

$$\partial_{x_i} u\big|_{\partial\Omega} = \gamma_i\big|_{\partial\Omega}. \tag{5.37}$$

Then we have for $i = 1, \ldots, n$

$$\|\partial_{x_i} u\|_{\infty,\Omega} \leq \limsup_{V\to\partial\Omega} |\gamma_i|. \tag{5.38}$$

Proof: The weak solution u of (5.36) satisfies

$$\big(F(\nabla u), \nabla\varphi\big)_\Omega = 0 \tag{5.39}$$

for all $\varphi \in C_0^\infty(\Omega)$. For $x \in \Omega$ we define

$$A(x) = (\nabla_\xi F)\big(\nabla u(x)\big) \in \mathbb{R}^{n\times n}, \tag{5.40}$$

where $\mathbb{R}^{n\times n}$ denotes the set of $n \times n$-matrices. From (5.35) we have $|A(x)| \leq c_1$, hence

$$A \in L^\infty(\Omega, \mathbb{R}^{n\times n}). \tag{5.41}$$

By (5.34) the vector field F is strongly monotone. Theorem 1.6 thus yields that the matrix $\nabla F(\eta)$ is uniformly positive definite for all $\eta \in \mathbb{R}^n$, hence we have for all $(x,\xi) \in \Omega \times \mathbb{R}^n$

$$\xi \cdot A(x)\xi = \xi \cdot \Big((\nabla_\xi F)\big(\nabla u(x)\big)\xi \Big) \geq c|\xi|^2. \tag{5.42}$$

Since $u \in H_2(\Omega)$ we have $\nabla u \in H_1(\Omega)$. Noting (5.35), we can thus apply Lemma 5.1 to conclude that $F \circ \nabla u \in H_1(\Omega)$ and that by the chain rule

$$\partial_{x_i} F\big(\nabla_x u(x)\big) = \nabla_\xi F\big(\nabla_x u(x)\big)\partial_{x_i} \nabla_x u(x) = A(x)\nabla_x u_{x_i}(x),$$

with the notation $u_{x_i} = \partial_{x_i} u$. Using that $\big(x \mapsto A(x)\nabla_x u_{x_i}(x)\big) \in L^2(\Omega)$, which results from (5.41), we obtain from this equation for $\varphi \in C_0^\infty(\Omega)$ that

$$\begin{aligned} \int_\Omega \big(A(x)\nabla_x u_{x_i}(x)\big) \cdot \nabla_x \varphi(x)\,\mathrm{d}x &= \int_\Omega \Big(\partial_{x_i} F\big(\nabla_x u(x)\big)\Big) \cdot \nabla_x \varphi(x)\,\mathrm{d}x \\ &= \int_\Omega F\big(\nabla_x u(x)\big) \cdot \nabla_x \partial_{x_i} \varphi(x)\,\mathrm{d}x = 0. \end{aligned} \tag{5.43}$$

To obtain the last equality we used that $\partial_{x_i}\varphi \in C_0^\infty(\Omega)$ and applied (5.39). Equation (5.43) together with (5.37) means that the function $u_{x_i} \in H_1(\Omega)$ is a weak solution of the boundary value problem

$$\begin{aligned} -\mathrm{div}\big(A(x)\nabla u_{x_i}(x)\big) &= 0, \qquad x \in \Omega, \\ u_{x_i}(x) &= \gamma_i(x), \qquad x \in \partial\Omega. \end{aligned}$$

From (5.41) and (5.42) we see that the assumptions of Theorem 5.7 are satisfied for this boundary value problem. This theorem yields (5.38). ■

5.3 L^∞-estimates for derivatives of solutions of nonlinear Dirichlet problems

In this section we derive L^∞-estimates for derivatives of weak solutions of Dirichlet boundary value problems to nonlinear elliptic equations. For simplicity we restrict our considerations to $\mathbb{R}^2$, but the results can be generalized to $\mathbb{R}^n$ with arbitrary n.

We need some preparations. Let c be a C_2-curve in $\mathbb{R}^2$ with a two times continuously differentiable parametrization $s \mapsto y(s) : [a,b] \to \mathsf{c}$, where $[a,b] \subseteq \mathbb{R}$ is a bounded interval.

$$\tau(x) = \frac{y'(s)}{|y'(s)|}$$

is a unit tangential vector to c at $x = y(s)$. If $|y'(s)| = 1$ for all $s \in [a,b]$, then $s \mapsto y(s)$ is called arc length parametrization. If $f : \mathsf{c} \to \mathbb{R}^m$ is such that $f \circ y : [a,b] \to \mathbb{R}^m$ is measurable, we define as usual

$$\int_{\mathsf{c}} f(x)\,\mathrm{d}\sigma(x) = \int_a^b f\big(y(s)\big)\,|y'(s)|\,\mathrm{d}s, \tag{5.44}$$

$$\partial_\tau f(x) = \frac{\mathrm{d}}{\mathrm{d}s} f\big(y(s)\big)/|y'(s)|, \quad x = y(s).$$

The space $L^2(\mathsf{c})$ consists of all $f : \mathsf{c} \to \mathbb{R}^m$ such that

$$\|f\|_{\mathsf{c}} = \Big(\int_{\mathsf{c}} |f(x)|^2\,\mathrm{d}\sigma(x)\Big)^{1/2} < \infty.$$

For $u, v : \mathsf{c} \to \mathbb{R}^m$ with $u \circ y :\in C_1\big([a,b]\big)$ and $v \circ y \in C_0^1\big((a,b)\big)$ we have

$$\begin{aligned}\int_{\mathsf{c}} \partial_\tau u(x) \cdot v(x)\,\mathrm{d}\sigma(x) &= \int_a^b (\partial_\tau u)\big(y(s)\big) \cdot v\big(y(s)\big)\,|y'(s)|\,\mathrm{d}s \\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}s} u\big(y(s)\big) \cdot v\big(y(s)\big)\,\mathrm{d}s = -\int_a^b u\big(y(s)\big) \frac{\mathrm{d}}{\mathrm{d}s} v\big(y(s)\big)\,\mathrm{d}s \\ &= -\int_{\mathsf{c}} u(x) \cdot \partial_\tau v(x)\,\mathrm{d}\sigma(x).\end{aligned}$$

Based on this formula, $g \in L^2(\mathsf{c})$ is called weak derivative of $f \in L^2(\mathsf{c})$, if for all $\varphi : \mathsf{c} \to \mathbb{R}^m$ with $\varphi \circ y \in C_0^\infty\big((a,b)\big)$ we have

$$\int_{\mathsf{c}} g(x) \cdot \varphi(x)\,\mathrm{d}\sigma(x) = -\int_{\mathsf{c}} f(x)\partial_\tau \varphi(x)\,\mathrm{d}\sigma(x).$$

The Sobolev space $H_1(\mathsf{c})$ is defined in the usual way.

In the following we always assume that Ω is a bounded open set in $\mathbb{R}^2$ with boundary $\partial\Omega \in C_2$ and that $s \mapsto y(s) : [a,b] \to \partial\Omega$ is an arc length parametrization of $\partial\Omega$. For $x = y(s) \in \partial\Omega$ let $n(x) = n(s)$ be the unit normal vector to $\partial\Omega$ at x pointing into the interior of Ω.

We assume that the orientation of $\partial\Omega$ defined by the parametrization is chosen such that $y'(s)^\perp = n(s)$, where $y'(s)^\perp$ is the vector obtained by rotation of the unit tangent vector $y'(s)$ in the mathematical positive sense by an angle of $\frac{\pi}{2}$. Then there is $\delta > 0$ such that

$$(s,\xi) \mapsto x(s,\xi) = y(s) + \xi n(s) : [a,b) \times [0,\delta] \to \mathbb{R}^2$$

is an invertible, continuously differentiable mapping of $[a,b) \times [0,\delta)$ to an open neighborhood V of $\partial\Omega$ in $\bar{\Omega}$. For every fixed number $\xi \in [0,\delta)$ the mapping

$$s \mapsto y_\xi(s) = y(s) + \xi n(s) : [a,b] \to V \tag{5.45}$$

is a C_1-parametrization of a curve in Ω parallel to $\partial\Omega$. We denote this curve by $\partial\Omega_\xi$. With the curvature $\varkappa(s)$ of $\partial\Omega$ at $y(s)$ we have $n'(s) = -\varkappa(s)y'(s)$, hence

$$\frac{\mathrm{d}}{\mathrm{d}s} y_\xi(s) = y'(s) + \xi n'(s) = \big(1 - \xi\varkappa(s)\big)y'(s) \tag{5.46}$$

is a tangential vector to $\partial\Omega_\xi$ at $y_\xi(s)$ with length

$$|y'_\xi(s)| = \big(1-\xi\varkappa(s)\big)\,|y'(s)| = 1-\xi\varkappa(s). \tag{5.47}$$

To every function $u : \partial\Omega_\xi \to \mathbb{R}^m$ we define a corresponding function $u^{(\xi)} : \partial\Omega \to \mathbb{R}^m$ by

$$u^{(\xi)}(x) = u\big(x + \xi n(x)\big), \quad x \in \partial\Omega.$$

We need the following result, a version of the Sobolev embedding theorem:

Theorem 5.12 *Let $\Omega \subseteq \mathbb{R}^2$ be open and bounded with $\partial\Omega \in C_2$.*

(i) *Then the space $C_1(\bar{\Omega})$ is dense in $H_1(\Omega)$. For every $\xi \in [0, \delta)$ there is a uniquely determined linear, continuous mapping $P_\xi : H_1(\Omega) \to L^2(\partial\Omega_\xi)$ such that for all $u \in C_1(\bar{\Omega})$*

$$P_\xi u = u\big|_{\partial\Omega_\xi}. \tag{5.48}$$

(ii) *There is a constant $C > 0$ such that for every $\xi_1, \xi_2 \in [0, \delta)$ and $u \in H_1(\Omega)$ we have*

$$\|(P_{\xi_1}u)^{(\xi_1)} - (P_{\xi_2}u)^{(\xi_2)}\|_{\partial\Omega} \le C\,|\xi_1 - \xi_2|^{1/2}\,\|u\|_{1,\Omega}. \tag{5.49}$$

(iii) *$u \in H_1(\Omega)$ belongs to $\overset{\circ}{H}_1(\Omega)$ if and only if $P_0 u = 0$*

Because of (5.48), we use $u\big|_{\partial\Omega_\xi}$ to denote $P_\xi u$ for any $u \in H_1(\Omega)$ and call $u\big|_{\partial\Omega_\xi}$ the trace of u on $\partial\Omega_\xi$.

Consider a weak solution u of the Dirichlet boundary value problem

$$-\mathrm{div}F\big(\nabla u(x)\big) = 0, \quad x \in \Omega, \tag{5.50}$$

$$u(x) = \gamma(x), \quad x \in \partial\Omega, \tag{5.51}$$

with a strongly monotone vector field $F \in C_1(\mathbb{R}^2, \mathbb{R}^2)$. We assume that both the data γ and the solution u belong to the space $H_2(\Omega) \cap C_1(\bar{\Omega})$. We want to estimate $\|\nabla u\|_{\infty,\Omega}$ by the values of the function $\gamma\big|_{\partial\Omega}$ and the derivatives of this function. To this end we first construct a function $g = (g_1, g_2) \in H_1(\Omega, \mathbb{R}^2) \cap C(\bar{\Omega}, \mathbb{R}^2)$, with $\partial_{x_i} u\big|_{\partial\Omega} = g_i\big|_{\partial\Omega}$,

$i = 1, 2$. For this construction we extend the vector fields τ and n from $\partial\Omega$ to V. For $x = x(s, \xi) = y(s) + \xi n(s) \in V$ set

$$\tau(x) = \frac{\partial_s x(s,\xi)}{|\partial_s x(s,\xi)|} = \frac{y'_\xi(s)}{1 - \xi\varkappa}, \quad n(x) = n(s).$$

$\tau(x)$ and $n(x)$ are orthogonal unit vectors and we have $\tau, n \in C_1(V)$. We can therefore decompose any vector field $w : V \to \mathbb{R}^2$ as

$$w(x) = \big(\tau(x) \cdot w(x)\big)\tau(x) + \big(n(x) \cdot w(x)\big)n(x).$$

In particular, for $u \in H_2(\Omega)$ we have that

$$\nabla u = (\tau \cdot \nabla u)\tau + (n \cdot \nabla u)n. \tag{5.52}$$

Since $\nabla u \in H_1(\Omega)$ and $\tau, n \in C_1(V)$, it follows that $\tau \cdot \nabla u$, $n \cdot \nabla u \in H_1(V)$.
We also need a function $\psi \in C_\infty(\bar\Omega, \mathbb{R})$ with

$$\psi(x) = \begin{cases} 0, & \text{if } x \in \Omega \setminus V, \\ 1, & \text{if } x \in \bar\Omega,\ \operatorname{dist}(x, \partial V) \le \delta/2. \end{cases}$$

Lemma 5.13 *Assume that* $\gamma, u \in H_2(\Omega) \cap C_1(\bar\Omega)$ *with* $u - \gamma \in \overset{\circ}{H}_1(\Omega)$. *Then the function* $g : \bar\Omega \to \mathbb{R}^2$ *defined by*

$$g(x) = \Big(\big(\tau(x) \cdot \nabla\gamma(x)\big)\tau(x) + \big(n(x) \cdot \nabla u(x)\big)n(x)\Big)\psi(x) \tag{5.53}$$

belongs to the space $H_1(\Omega) \cap C(\bar\Omega)$ *and satisfies*

$$g\big|_{\partial\Omega} = \nabla u\big|_{\partial\Omega}. \tag{5.54}$$

Proof: Since $\gamma, u \in H_2(\Omega) \cap C_1(\bar\Omega)$ we have that $\tau \cdot \nabla\gamma \in H_1(V) \cap C(\bar V)$ and $n \cdot \nabla u \in H_1(V) \cap C(\bar V)$. By definition the function ψ vanishes outside of V. Therefore $\psi\tau \cdot \nabla\gamma$ and $\psi n \cdot \nabla u$ belong to $H_1(\Omega) \cap C(\bar\Omega)$, hence g defined by (5.53) belongs to $H_1(\Omega) \cap C(\bar\Omega)$. From the assumption $u - \gamma \in \overset{\circ}{H}_1(\Omega)$ it follows by Theorem 5.12(iii) that $(u - \gamma)\big|_{\partial\Omega} = 0$. Since $u - \gamma$ belongs to $C_1(\bar\Omega)$, we thus conclude

$$\tau \cdot \nabla(u - \gamma)\big|_{\partial\Omega} = \partial_\tau(u - \gamma)\big|_{\partial\Omega} = 0.$$

Since ψ is equal to 1 in a neighborhood of $\partial\Omega$, we obtain from this equation and from (5.53) for $x \in \partial\Omega$

$$\begin{aligned} \nabla u(x) - g(x) &= \big(\tau(x) \cdot \nabla u(x)\big)\tau(x) + \big(n(x) \cdot \nabla u(x)\big)n(x) \\ &\qquad - \big(\tau(x) \cdot \nabla\gamma(x)\big)\tau(x) - \big(n(x) \cdot \nabla u(x)\big)n(x) \\ &= \tau(x) \cdot \nabla(u - \gamma)(x) = 0. \end{aligned}$$

This is (5.54). ∎

Since $g \in C(\bar{\Omega})$, we obtain from Lemma 5.5(iii) for the i-th component g_i of g and from (5.53) that

$$\begin{aligned}\limsup_{V\to\partial\Omega} |g_i| = \sup_{\partial\Omega} |g_i| = \sup_{\partial\Omega} |(\tau\cdot\nabla\gamma)\tau_i + (n\cdot\nabla u)n_i| \\ \le \sup_{\partial\Omega} |\tau\cdot\nabla\gamma| + \sup_{\partial\Omega} |n\cdot\nabla u| = \sup_{\partial\Omega} |\partial_\tau\gamma| + \sup_{\partial\Omega} |n\cdot\nabla u|. \end{aligned} \tag{5.55}$$

Corollary 5.14 *Let $F \in C_1(\mathbb{R}^2, \mathbb{R}^2)$ satisfy the conditions* (5.34) *and* (5.35). *Assume that the weak solution u of the boundary value problem* (5.50), (5.51) *and the data γ belong to the space $H_2(\Omega) \cap C_1(\bar{\Omega})$. Then we have for $i = 1, 2$*

$$\|\partial_{x_i} u\|_{\infty,\Omega} \le \sup_{\partial\Omega} |\partial_\tau\gamma| + \sup_{\partial\Omega} |n\cdot\nabla u|. \tag{5.56}$$

Proof: From (5.37) we obtain $g_i\big|_{\partial\Omega} = \partial_{x_i} u\big|_{\partial\Omega}$, which by Theorem 5.12(iii) means that $\partial_{x_i} u - g \in \overset{\circ}{H}_1(\Omega)$. Consequently $u \in H_2(\Omega)$ is a weak solution of the boundary value problem

$$\begin{aligned} -\mathrm{div} F\big(\nabla u(x)\big) = 0, \quad x \in \Omega \\ \partial_{x_i} u\big|_{\partial\Omega} = g_i\big|_{\partial\Omega}. \end{aligned}$$

Theorem 5.11 yields $\|\partial_{x_i} u\|_{\infty,\Omega} \le \limsup_{V\to\partial\Omega} |g_i|$, which together with (5.55) implies (5.56). ∎

To obtain an estimate for $\|\partial_{x_i} u\|_{\infty,\Omega}$ we must estimate the normal derivative $n\cdot\nabla u$ on the right hand side of (5.56). We have the following result.

Theorem 5.15 *Let $F \in C_1(\mathbb{R}^2, \mathbb{R}^2)$ satisfy the conditions* (5.34) *and* (5.35). *Suppose that $\gamma \in H_2(\Omega) \cap C_1(\bar{\Omega})$ and that for every $x \in \partial\Omega$ there are affine functions $\omega_x^\pm$ such that*

$$\omega_x^-(y) \le \gamma(y) \le \omega_x^+(y), \quad y \in \partial\Omega, \tag{5.57}$$

$$\omega_x^-(x) = \gamma(x) = \omega_x^+(x). \tag{5.58}$$

Assume that the weak solution u of the Dirichlet boundary value problem (5.50), (5.51) *belongs to the space $H_2(\Omega) \cap C_1(\bar{\Omega})$. Then we have*

$$\|\partial_{x_i} u\|_{\infty,\Omega} \le \sup_{\partial\Omega} |\partial_\tau\gamma| + \sup_{x\in\partial\Omega} \max\{|\nabla\omega_x^+|, |\nabla\omega_x^-|\}. \tag{5.59}$$

Proof: Let $x \in \partial\Omega$. Under the conditions stated in this theorem the assumptions of Theorem 5.8 are satisfied with the affine function $\omega = \omega_x^+$, and the assumptions of Theorem 5.9 are satisfied with the affine function $\omega = \omega_x^-$. These two theorems yield for all $y \in \bar{\Omega}$ that

$$\omega_x^-(y) \leq u(y) \leq \omega_x^+(y).$$

From these inequalities and from (5.58) we obtain for $0 < \xi < \delta$ that

$$\frac{1}{\xi}\Big(\omega_x^-\big(x+\xi n(x)\big) - \omega_x^-(x)\Big) \leq \frac{1}{\xi}\Big(u\big(x+\xi n(x)\big) - u(x)\Big) \leq \frac{1}{\xi}\Big(\omega_x^+\big(x+\xi n(x)\big) - \omega_x^+(x)\Big). \tag{5.60}$$

Since the affine functions $\omega_x^\pm(y) = a_x^\pm \cdot y + \omega_0^\pm$ satisfy

$$\omega_x^\pm\big(x+\xi n(x)\big) - \omega_x^\pm(x) = a_x^\pm \cdot n(x)\xi \leq |a_x^\pm|\xi = |\nabla\omega_x^\pm|\xi,$$

we infer from (5.60) that

$$\begin{aligned} |n(x)\cdot\nabla u(x)| &= \Big|\frac{\mathrm{d}}{\mathrm{d}\xi}u\big(x+\xi n(x)\big)_{|_{\xi=0}}\Big| \\ &= \Big|\lim_{\xi\searrow 0}\frac{u\big(x+\xi n(x)\big)-u(x)}{\xi}\Big| \leq \max\{|\nabla\omega_x^+|, |\nabla\omega_x^-|\}. \end{aligned}$$

Combination of this inequality with (5.56) yields (5.59). ■

In the next lemma we study for what domains Ω and boundary data γ affine functions $\omega_x^\pm$ with the stated properties can be found.

Lemma 5.16 *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with $\partial\Omega \in C_2$. Assume that the curvature $\varkappa$ of $\partial\Omega$ is everywhere positive. Then to every $\gamma \in C_2(\partial\Omega)$ and every $x \in \partial\Omega$ there are affine functions $\omega_x^\pm : \mathbb{R}^2 \to \mathbb{R}$, which satisfy* (5.57) *and* (5.58). *Moreover, there is a constant K, which is independent of γ, such that*

$$\sup_{x\in\partial\Omega}\max\{|\nabla\omega_x^+|, |\nabla\omega_x^-|\} \leq K\sum_{m=0}^{2}\|\partial_\tau^m\gamma\|_{\infty,\partial\Omega}. \tag{5.61}$$

Proof: Fix $x \in \Omega$ and choose the coordinate system such that $x = (0,0)$ and such that the x_1-axis is tangential to $\partial\Omega$ at x and points into the direction of the vector $\tau(x)$. By our choice of the normal vector $n(x)$ it follows that the x_2-axis points into the direction of $n(x)$. We can assume that the arclength parametrization $s : [a,b] \to \partial\Omega$ is such that $a < 0 < b$ and $y(0) = x$.

Let $\gamma \in C_2(\partial\Omega)$ be given. We define a function $h : \partial\Omega \to \mathbb{R}$ by

$$h(z) = \gamma(z) - \big(\gamma(x) + \partial_\tau\gamma(x)z_1\big), \quad z = (z_1, z_2) \in \partial\Omega. \tag{5.62}$$

Since the curvature of $\partial\Omega$ is everywhere positive, the domain $\bar{\Omega}$ is strictly convex and we have for all $z = (z_1, z_2) \in \bar{\Omega}$ with $z \neq x$ that $z_2 > 0$. Therefore we obviously have for $z \in \partial\Omega$ with $z \neq x$ that

$$z_2 \inf_{\substack{y\in\partial\Omega \\ y\neq x}} \frac{h(y)}{y_2} \leq z_2 \frac{h(z)}{z_2} = h(z) \leq z_2 \sup_{\substack{y\in\partial\Omega \\ y\neq x}} \frac{h(y)}{y_2}. \tag{5.63}$$

If we therefore define the affine functions $\omega_x^\pm : \mathbb{R}^2 \to \mathbb{R}$ by

$$\omega_x^+(z) = \gamma(x) + \partial_\tau\gamma(x) z_1 + z_2 \sup_{\substack{y\in\partial\Omega \\ y\neq x}} \frac{h(y)}{y_2}, \tag{5.64}$$

$$\omega_x^-(z) = \gamma(x) + \partial_\tau\gamma(x) z_1 + z_2 \inf_{\substack{y\in\partial\Omega \\ y\neq x}} \frac{h(y)}{y_2}, \tag{5.65}$$

we obtain from (5.62) and (5.63) for $z \in \partial\Omega$ that

$$\gamma(z) = \gamma(x) + \partial_\tau\gamma(x) z_1 + h(z) \leq \omega_x^+(z),$$
$$\gamma(z) = \gamma(x) + \partial_\tau\gamma(x) z_1 + h(z) \geq \omega_x^-(z).$$

We clearly have $\omega_x^+(x) = \omega_x^-(x) = \gamma(x)$. Therefore the relations (5.57) and (5.58) hold. To complete the proof it remains to verify (5.61).

We have that

$$\partial_{z_1}\omega_x^+(z) = \partial_\tau\gamma(x), \quad \partial_{z_2}\omega_x^+(z) = \sup_{\substack{y\in\partial\Omega \\ y\neq x}} \frac{h(y)}{y_2}, \quad \partial_{z_2}\omega_x^-(z) = \inf_{\substack{y\in\partial\Omega \\ y\neq x}} \frac{h(y)}{y_2},$$

hence, because of $|\inf h(y)/y_2| \leq \sup |h(y)/y_2|$,

$$|\nabla\omega_x^\pm|^2 = \big(\partial_\tau\gamma(x)\big)^2 + \Big(\sup_{\substack{y\in\partial\Omega \\ y\neq x}} \Big|\frac{h(y)}{y_2}\Big|\Big)^2. \tag{5.66}$$

To estimate the last term on the right note that by our assumptions on the parametrization $s \mapsto y(s)$ of $\partial\Omega$ and by the differential geometric properties of curves we have

$$y(0) = x = (0,0), \quad y'(0) = \tau(x) = (1,0), \quad y''(0) = \varkappa n(x) = (0, \varkappa),$$

hence $y_1'(0) = 1$, $y_2(0) = y_2'(0) = 0$, $y_2''(0) = \varkappa(0) > 0$. Moreover, from (5.62),

$$h \circ y(0) = h(x) = 0, \quad \partial_s(h \circ y)(0) = \partial_\tau\gamma(x) - \partial_\tau\gamma(x) y_1'(0) = 0.$$

Therefore Taylor's formula yields with a suitable s^* between 0 and s that

$$\Big|\frac{h\big(y(s)\big)}{y_2(s)}\Big| = \frac{|\frac{1}{2}(h\circ y)''(s^*)s^2|}{\frac{1}{2}\big(y_2''(0)+o(1)\big)s^2} \le \frac{\max\limits_{[a,b]}|(h\circ y)''|}{\varkappa(0)\big(1+\frac{o(1)}{\varkappa(0)}\big)}$$
$$\le 2\max_{[a,b]}|(h\circ y)''|/\min_{z\in\partial\Omega}\varkappa(z), \tag{5.67}$$

for all s with $|s|\le\varepsilon$, where $\varepsilon>0$ is chosen small enough such that $|o(1)/\varkappa(0)|\le 1/2$ for all $|s|\le\varepsilon$.

For $s\in[a,b]$ with $|s|>\varepsilon$ we have because of the convexity of $\bar\Omega$ that

$$y_2(s)\ge\min\{y_2(-\varepsilon),y_2(\varepsilon)\}=\frac{1}{2}\big(y_2''(0)+o(1)\big)\varepsilon^2$$
$$=\frac{\varepsilon^2}{2}\varkappa(0)\big(1+\frac{o(1)}{\varkappa(0)}\big)\ge\frac{\varepsilon^2}{4}\varkappa(0),$$

whence

$$\Big|\frac{h\big(y(s)\big)}{y_2(s)}\Big|\le\frac{4}{\varepsilon^2}\max_{[a,b]}|h\circ y|/\min_{z\in\partial\Omega}\varkappa(z).$$

From this inequality and from (5.66), (5.67) we find that

$$\Big|\frac{h\big(y(s)\big)}{y_2(s)}\Big|\le K_1\sum_{m=0}^{2}\|\partial_\tau^m h\|_{\infty,\partial\Omega}\le K_2\sum_{m=0}^{2}\|\partial_\tau^m\gamma\|_{\infty,\partial\Omega} \tag{5.68}$$

with constants K_1 and K_2, which only depend on the parametrization $s\mapsto y(s)$; this means that the constants only depend on $\partial\Omega$. To get the last inequality sign in (5.68) we used the definition (5.62) of h. We combine (5.68) with (5.66) and obtain the estimate (5.61). ■

From Theorem 5.15 and Lemma 5.16 we obtain the following

Corollary 5.17 *Suppose that* $F\in C_1(\mathbb{R}^2,\mathbb{R}^2)$ *satisfies the conditions* (5.34) *and* (5.35). *Let* $\Omega\subseteq\mathbb{R}^2$ *be a bounded open set with* $\partial\Omega\in C_2$. *Assume that the curvature* $\varkappa$ *of* $\partial\Omega$ *is everywhere positive. Then there exists a constant* K' *such that for all* $\gamma\in H_2(\Omega)\cap C_1(\bar\Omega)$ *with* $\gamma\big|_{\partial\Omega}\in C_2(\partial\Omega)$ *and for every weak solution* u *of the boundary value problem* (5.50), (5.51), *which belongs to* $H_2(\Omega)\cap C_1(\bar\Omega)$, *the estimate*

$$\|\partial_{x_i}u\|_{\infty,\Omega}\le K'\sum_{m=0}^{2}\|\partial_\tau^m\gamma\|_{\infty,\partial\Omega}$$

holds for $i=1,2$.