Linear Partial Differential Equations
(PDE 1)

Lecture Notes

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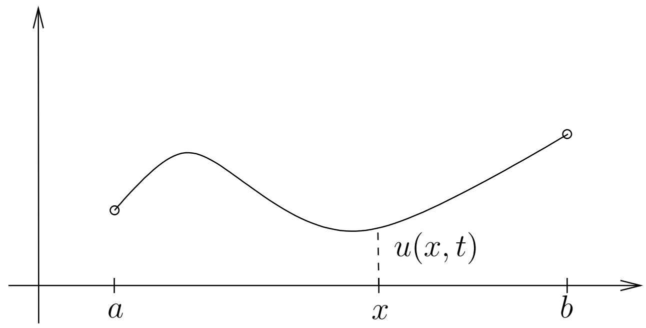
1 The wave equation as mathematical model for the vibrating string and the vibrating membrane.

1.1 Potential energy of the linear elastic string

We want to formulate mathematical equations, which allow to compute the vibrations in time of an elastic string, which is fixed at both ends. To this end we first compute the potential energy stored in the string at time t. This requires to make an assumption for the elastic material properties of the string. Let $-\infty < a < b < \infty$ be given numbers.



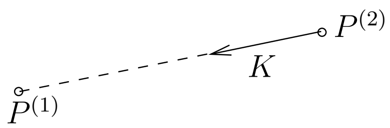
Actual configuration of the string

Imagine first that the string is linearly stretched along the x-axis with ends fixed at $(a, 0)$ and $(b, 0)$ and is at rest. We call this the reference configuration of the string. Now consider an actual configuration, where the string is displaced from this reference configuration.

Hypothesis: Let $P^{(1)}$, $P^{(2)}$ be material points of the string, whose positions are $(x_1, 0)$, $(x_2, 0)$ in the reference configuration and $(x^{(1)}, y^{(1)})$, $(x^{(2)}, y^{(2)})$ in the actual configuration. Assume that in the actual configuration the string is linearly and uniformly stretched between $P^{(1)}$ and $P^{(2)}$. Then the force K acting on $P^{(2)}$ is

$$K = \kappa \frac{|P^{(1)} - P^{(2)}|}{|x_1 - x_2|} \frac{P^{(1)} - P^{(2)}}{|P^{(1)} - P^{(2)}|} = \kappa \frac{P^{(1)} - P^{(2)}}{|x_1 - x_2|},$$

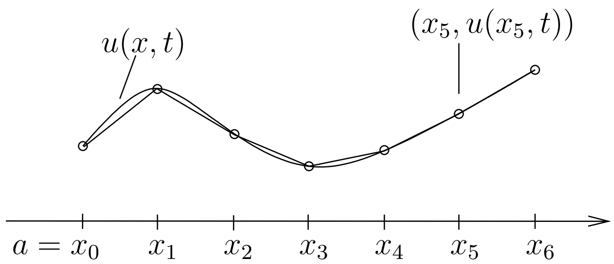
with a material constant $\kappa > 0$. Here we identified $P^{(1)}$, $P^{(2)}$ with the positions in the actual configuration. (Linearly elastic material behavior.)



Note that by this law of force a variation of the actual positions of the points $P^{(1)}, P^{(2)}$ in direction orthogonal to the x-axis does not alter the component of the force K parallel of the x-axis, and a variation of the positions parallel to this axis does not alter the force component orthogonal to this axis. Therefore the movements of the material points of the string in directions parallel and orthogonal to the x-axis are not coupled. We thus can and shall assume in the following that the material points of the string move only in the direction orthogonal to the x-axis. This implies that at time $t \geq 0$ the string can be represented by the graph of a function

$$x \mapsto u(x,t) : [a,b] \to \mathbb{R}.$$

To compute the potential energy we approximate the graph of this function by a piecewise linear function:



With $h = \frac{b-a}{n}$ let

$$x_i = ih + a\,, \quad i = 0, 1, \ldots, n$$

be the x-coordinates of the node points of the polygonal arc. We first determine the potential energy stored in the polygonial arc. To this end we successively deform the string from the reference configuration to the polygon and compute the work done in every step. In the first step we move all material points of the string vertically and in parallel from the x-axis to the horizontal line passing through the point $(x_0, u(x_0,t))$. No work is done in this step. We then fix the endpoint of the string at $(x_0, u(x_0,t))$ and move all points on the line segment $\{(x, u(x_0,t)) \mid x_1 \leq x \leq b\}$ vertically and in parallel to the line segment $\{(x, u(x_1,t)) \mid x_1 \leq x \leq b\}$. During this movement an amount of energy V_1 is stored in the straight line segment above the interval $[x_0, x_1]$, which is equal to the work done in moving the material point at the position $(x_1, u(x_0,t))$ along a vertical path to the position $(x_1, u(x_1,t))$ against the vertical component K_2 of the elastic force K in this straight line segment. A parametrization of this path is

$$s \mapsto P(s) = (x_1, s) : [u(x_0,t),\, u(x_1,t)] \to \mathbb{R}^2.$$

Since in the reference configuration the position of the point $P(s) = (x_1, s)$ is $(x_1, 0)$ and the position of $(x_0, u(x_0, t))$ is $(x_0, 0)$, our hypothesis implies that the elastic force $K(s)$ at the point $P(s)$ is given by

$$K(s) = (K_1(s), K_2(s)) = \kappa \frac{(x_0, u(x_0, t)) - (x_1, s)}{x_1 - x_0},$$

whence

$$K_2(s) = -\kappa \frac{s - u(x_0, t)}{x_1 - x_0}.$$

We thus have

$$\begin{aligned}
V_1 &= -\int_{u(x_0,t)}^{u(x_1,t)} K_2(s) ds = \frac{\kappa}{x_1 - x_0} \int_{u(x_0,t)}^{u(x_1,t)} (s - u(x_0, t))\, ds \\
&= \frac{\kappa}{x_1 - x_0} \frac{1}{2} (s - u(x_0, t))^2 \Big|_{s=u(x_0,t)}^{s=u(x_1,t)} = \frac{\kappa}{2} \frac{(u(x_1, t) - u(x_0, t))^2}{x_1 - x_0} \\
&= \frac{\kappa}{2} \left(\frac{u(x_1, t) - u(x_0, t)}{x_1 - x_0} \right)^2 (x_1 - x_0) = \frac{\kappa}{2} \left(\frac{\partial}{\partial x} u(x_1^*, t) \right)^2 h,
\end{aligned}$$

where x_1^* is a point between x_0 and x_1. Here we used the mean value theorem. We proceed in the same way and obtain for the elastic energy V_i stored in the straight line segment above $[x_{i-1}, x_i]$ that

$$V_i = \frac{\kappa}{2} \left(\frac{\partial}{\partial x} u(x_i^*, t) \right)^2 h.$$

For the total energy $V^{(h)}(t)$ of the polygonal arc we thus have

$$V^{(h)}(t) = \sum_{i=1}^{n} V_i(t) = \frac{\kappa}{2} \sum_{i=1}^{n} \big(\frac{\partial}{\partial x} u(x_i^*, t) \big)^2 h.$$

For $h \to 0$ the polygon converges to the string. Therefore one defines the potential energy $V(t)$ of the string at time t by

$$V(t) = \lim_{h \to 0} V^{(h)}(t).$$

On the other hand, $\sum_{i=1}^{n} (\frac{\partial}{\partial x} u(x_i^*, t))^2 h$ is a Riemann sum. If $x \mapsto \frac{\partial}{\partial x} u(x, t)$ is continuous we thus obtain by Riemann integration theory that

$$\lim_{n \to 0} \frac{\kappa}{2} \sum_{i=1}^{n} \big(\frac{\partial}{\partial x} u(x_i^*, t) \big)^2 h = \frac{\kappa}{2} \int_a^b \big(\frac{\partial}{\partial x} u(x, t) \big)^2 dx.$$

Therefore we conclude that the **stored energy of the string at time t** is

$$V(t) = \frac{\kappa}{2} \int_a^b \big(u_x(x, t) \big)^2 dx.$$

1.2 The Hamiltonian principle

The velocity of the material point $\big(x, u(x,t)\big)$ of the string at time t in the direction orthogonal to the x-axis is $\frac{d}{dt}\,u(x,t)$. Therefore the kinetic energy $E(t)$ of the string at time t is

$$E(t) = \int_a^b \frac{1}{2}\,\rho(x)\big(u_t(x,t)\big)^2 dx\,,$$

where $\rho(x)$ is the mass of the string per unit length.

To formulate Hamilton's principle I use the following notations: For a continuously differentiable function $v : [a,b] \times [0,T] \to \mathbb{R}$ let

$$\begin{aligned} V_v(t) &= \int_a^b \frac{\kappa}{2}\big(v_x(x,t)\big)^2 dx \\ E_v(t) &= \int_a^b \frac{\rho(x)}{2}\big(v_t(x,t)\big)^2 dx\,. \end{aligned}$$

Hamilton's principle: Let $T > 0$, let the movement of the string be given by the continuously differentiable function

$$u : [a,b] \times [0,T] \to \mathbb{R},$$

and let $w : [a,b] \times [0,T] \to \mathbb{R}$ be a continuously differentiable function satisfying

$$w(x,0) = w(x,T) = w(a,t) = w(b,t) = 0 \tag{1.1}$$

for all $a \le x \le b$ and all $0 \le t \le T$. Let s denote real numbers. Hamilton's principle states that the movement is such that

$$\frac{d}{ds}\int_0^T E_{u+sw}(t) - V_{u+sw}(t)dt\big|_{s=0} = 0\,. \tag{1.2}$$

Remark. If $|s|$ is a small number, then

$$v(x,t) = u(x,t) + sw(x,t)$$

is a small perturbation of the movement of the string, which because of (1.1) does not change the boundary, initial and final values. Therefore Hamilton's principle states that the material points of the string move such that the integral

$$\int_0^T E(t) - V(t)dt$$

is stationary when the movement of the string is perturbed such that the initial, final and boundary values are not changed.

The equation (1.2) can be used to derive an equation for the movement of the string. For, (1.2) yields

$$\begin{aligned} 0 &= \frac{d}{ds}\int_0^T\int_a^b \frac{\rho(x)}{2}\big(u_t(x,t)+sw_t(x,t)\big)^2 - \frac{\kappa}{2}\big(u_x(x,t)+sw_x(x,t)\big)^2 dxdt\big|_{s=0} \\ &= \int_0^T\int_a^b \rho(x)\big(u_t(x,t)+sw_t(x,t)\big)w_t(x,t) \\ &\qquad\qquad - \kappa\big(u_x(x,t)+sw_x(x,t)\big)w_x(x,t)dxdt\big|_{s=0} \\ &= \int_0^T\int_a^b \big(\rho(x)u_t(x,t)w_t(x,t) - \kappa u_x(x,t)w_x(x,t)\big)dxdt =: I\,. \end{aligned}$$

If u is two times continuously differentiable, then the last integral can be transformed using partial integration. Since w vanishes at the boundary of the rectangle $[a,b]\times[0,T]$ we obtain

$$0 = I = -\int_0^T\int_a^b \big(\rho(x)u_{tt}(x,t) - \kappa u_{xx}(x,t)\big)w(x,t)dxdt\,.$$

This must hold for all continuously differentiable functions w vanishing at the boundary. If $\rho(x)u_{tt}(x,t) - \kappa u_{xx}(x,t)$ is continuous, this can only hold if

$$\rho(x)u_{tt}(x,t) - \kappa u_{xx}(x,t) = 0 \tag{1.3}$$

for all $(x,t)\in[a,b]\times[0,T]$.

1.3 Initial-boundary value problems for the one-dimensional wave equation

Since T is an arbitrary chosen positive number, we conclude that the vibrating string must satisfy the equation (1.3) in the whole domain $[a,b]\times[0,\infty)$. We thus have

$$\rho(x)u_{tt}(x,t) = \kappa u_{xx}(x,t), \quad (x,t)\in[a,b]\times[0,\infty).$$

This is a linear partial differential equation of second order for u, the wave equation in one space dimension. Since the ends of the string at $x=a$ or $x=b$ can be fixed or can be subjected to arbitrarily given motions, and since at time $t=0$ the material points of the string can be displaced arbitrarily and can be submitted to arbitrarily given velocities, one wants to solve the following initial-boundary value problem to determine the motion of the string:

$$
\begin{aligned}
&\rho(x)u_{tt}(x,t) = \kappa u_{xx}(x,t), && (x,t) \in [a,b] \times [0,\infty), \\
\text{(BD)} \quad & u(a,t) = u^{(a)}(t), \quad u(b,t) = u^{(b)}(t), && t \in [0,\infty), \\
\text{(IC)} \quad & u(x,0) = u^{(0)}(x), \quad u_t(x,0) = u^{(1)}(x), && x \in [a,b],
\end{aligned}
$$

with given functions $u^{(a)}, u^{(b)} : [0,\infty) \to \mathbb{R}$, $u^{(0)}, u^{(1)} : [a,b] \to \mathbb{R}$. This is the **Dirichlet initial-boundary value problem** for the wave equation. The **Neumann initial-boundary value problem** is obtained if instead of the values $u(a,t)$ and $u(b,t)$ the values $u_x(a,t)$ and $u_x(b,t)$ for the x derivatives are prescribed:

$$
\begin{aligned}
&\rho(x)u_{tt}(x,t) = \kappa u_{xx}(x,t), && (x,t) \in [a,b] \times [0,\infty), \\
\text{(BC)} \quad & u_x(a,t) = v^{(a)}(t), \quad u_x(b,t) = v^{(b)}(t), && t \in [0,\infty), \\
\text{(IC)} \quad & u(x,0) = u^{(0)}(x), \quad u_t(x,0) = u^{(1)}(x), && x \in [a,b].
\end{aligned}
$$

If $a = -\infty$ and $b = \infty$ and no boundary conditions are posed, then one speaks of the **initial value problem** or **Cauchy problem**:

$$
\begin{aligned}
&\rho(x)u_{tt}(x,t) = \kappa u_{xx}(x,t), && (x,t) \in (-\infty,\infty) \times [0,\infty), \\
\text{(IC)} \quad & u(x,0) = u^{(0)}(x), \quad u_t(x,0) = u^{(1)}(x), && x \in (-\infty,\infty).
\end{aligned}
$$

1.4 Initial-boundary value problems for the wave equation in higher space dimensions

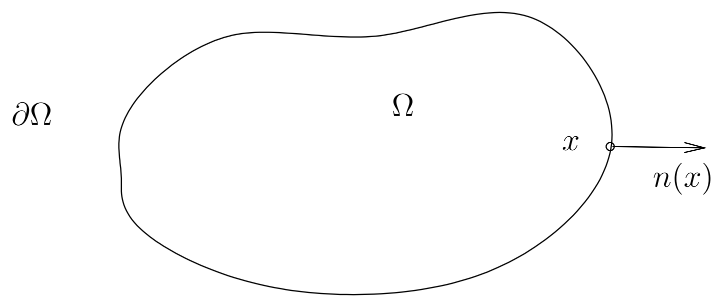
Consider an elastic membrane, which at the boundary is fixed to a wire forming a closed loop. The projection $\overline{\Omega}$ of the membrane to the plane $\mathbb{R}^2$ is a closed bounded set, the interior of which is Ω. We assume that the boundary $\partial\Omega$ is continuously differentiable, and that the wire is given by the graph of a continuously differentiable function $\mu : \partial\Omega \to \mathbb{R}$. Let $u(x,t) \in \mathbb{R}$ be the height of the membrane above the point $x \in \overline{\Omega}$ at time $t \geq 0$. Thus, at time t the membrane is represented by the graph of the function

$$x \mapsto u(x,t) : \overline{\Omega} \to \mathbb{R}.$$

Since the membrane is attached at the boundary to the wire, we have the Dirichlet boundary condition

$$u(x,t) = \mu(x), \quad x \in \partial\Omega, \ t \geq 0.$$

To determine a partial differential equation for the function u we again apply Hamilton's principle. We first need to make assumptions for the elastic properties of the membrane,



The exterior unit normal vector $n(x)$

or equivalently for the form of the potential energy stored in the membrane. Generalizing the one-dimensional potential energy we assume here that the potential energy $V_u(t)$ of the membrane at time t is given by

$$V_u(t) = \frac{\kappa}{2} \int_\Omega |\nabla_x u(x,t)|^2 dx,$$

with the gradient

$$\nabla_x u(x,t) = \begin{pmatrix} \frac{\partial}{\partial x_1} u(x,t) \\ \frac{\partial}{\partial x_2} u(x,t) \end{pmatrix}.$$

The kinetic energy of the membrane is

$$E_u(t) = \int_\Omega \frac{\rho(x)}{2} \big(u_t(x,t)\big)^2 dx.$$

If $T > 0$ and if

$$w : \overline{\Omega} \times [0,T] \to \mathbb{R}$$

is continuously differentiable with

$$\begin{aligned} &w(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T] \\ &w(x,0) = w(x,T) = 0, \quad x \in \Omega, \end{aligned}$$

then Hamilton's principle yields

$$\begin{aligned} 0 &= \frac{d}{ds} \int_0^T E_{u+sw}(t) - V_{u+sw}(t) dt\Big|_{s=0} \\ &= \frac{d}{ds} \int_0^T \int_\Omega \frac{\rho(x)}{2} \big(u_t(x,t) + s w_t(x,t)\big)^2 - \frac{\kappa}{2} |\nabla_x u(x,t) + s \nabla_x w(x,t)|^2 dx\, dt\Big|_{s=0} \\ &= \int_0^T \int_\Omega \big(\rho(x) u_t(x,t) w_t(x,t) - \kappa \nabla_x u(x,t) \cdot \nabla_x w(x,t)\big) dx\, dt\,. \end{aligned}$$

If u is two times continuously differentiable then the first Green's formula yields

$$\begin{aligned}0 = -\int_0^T \int_\Omega \big(\rho(x) u_{tt}(x,t) - \kappa \Delta_x u(x,t)\big) w(x,t) dx\, dt \\ + \int_\Omega \rho(x) u_t(x,T) w(x,T) - \rho(x) u_t(x,0) w(x,0) dx \\ - \int_0^T \int_{\partial\Omega} \kappa \big(\frac{\partial}{\partial n_x} u(x,t)\big) w(x,t) d\sigma_x\, dt \\ = -\int_0^T \int_\Omega \big(\rho(x) u_{tt}(x,t) - \kappa \Delta_x u(x,t)\big) w(x,t) dx\, dt\,, \quad (1.4)\end{aligned}$$

with the Laplace operator

$$\Delta_x u(x,t) = \sum_{i=1}^2 \frac{\partial^2}{\partial x_i^2} u(x,t)$$

and the normal derivative

$$\frac{\partial}{\partial n} u(x,t) = n(x) \cdot \nabla_x u(x,t),$$

where $n(x) \in \mathbb{R}^2$ is the unit normal vector to the boundary $\partial\Omega$ at $x \in \partial\Omega$ pointing into the exterior $\mathbb{R}^2 \backslash \overline{\Omega}$ of $\overline{\Omega}$.

(1.4) must be satisfied for all w with the stated properties. This is only possible if the bracketed expression in the integrand on the right hand side vanishes identically, whence u must satisfy the equation

$$\rho(x) u_{tt}(x,t) = \kappa \Delta_x u(x,t), \quad (x,t) \in \overline{\Omega} \times [0,T]. \tag{1.5}$$

Since T is arbitrary, it follows that u must satisfy this equation for all $(x,t) \in \overline{\Omega} \times [0,\infty)$.

(1.5) is is the wave equation in two space dimensions. In the derivation of this equation and in the derivation of the wave equation in one space dimension we assumed that no forcing term is acting on the membrane or on the string. Such forces are often present, for example the gravitational force. If such a force is present and if $f(x,t)$ denotes the force per unit area of the membrane, then instead of (1.5) the equation

$$\rho(x) u_{tt}(x,t) = \kappa \Delta_x u(x,t) + f(x,t) \tag{1.6}$$

is obtained. (1.6) is the inhomogeneous wave equation.

We already noted that u must satisfy the Dirichlet boundary condition. Therefore u must be a solution of the Dirichlet initial-boundary value problem, which we immediately formulate for the n-dimensional inhomogeneous wave equation. Thus, for $n \in \mathbb{N}$ let

$$\Delta_x u(x,t) = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} u(x,t), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n$$

be the n-dimensional Laplace operator. With this operator the inhomogeneous Dirichlet initial-boundary value problem in a domain $\Omega \subseteq \mathbb{R}^n$ is

$$\rho(x)u_{tt}(x,t) = \kappa\Delta_x u(x,t) + f(x,t), \qquad (x,t) \in \overline{\Omega} \times [0,\infty), \tag{1.7}$$

$$\text{(BC)} \qquad u(x,t) = \mu(x,t), \qquad (x,t) \in \partial\Omega \times [0,\infty), \tag{1.8}$$

$$\text{(IC)} \qquad u(x,0) = u^{(0)}(x), \quad u_t(x,0) = u^{(1)}(x), \qquad x \in \overline{\Omega}, \tag{1.9}$$

with given functions $f : \overline{\Omega} \times [0,\infty) \to \mathbb{R}, \quad \mu : \partial\Omega \times [0,\infty) \to \mathbb{R}, \quad u^{(0)}, u^{(1)} : \overline{\Omega} \to \mathbb{R}.$

The Neumann initial-boundary value problem for the wave equation in $\Omega \subseteq \mathbb{R}^n$ is

$$\rho(x)u_{tt}(x,t) = \kappa\Delta_x u(x,t) + f(x,t), \qquad (x,t) \in \overline{\Omega} \times [0,\infty), \tag{1.10}$$

$$\text{(BC)} \qquad \frac{\partial}{\partial n} u(x,t) = \nu(x,t), \qquad (x,t) \in \partial\Omega \times [0,\infty), \tag{1.11}$$

$$\text{(IC)} \qquad u(x,0) = u^{(0)}(x), \quad u_t(x,0) = u^{(1)}(x), \qquad x \in \overline{\Omega}, \tag{1.12}$$

and the Cauchy problem is

$$\rho(x)u_{tt}(x,t) = \kappa\Delta_x u(x,t) + f(x,t), \qquad (x,t) \in \mathbb{R}^n \times [0,\infty), \tag{1.13}$$

$$\text{(IC)} \qquad u(x,0) = u^{(0)}(x), \quad u_t(x,0) = u^{(1)}(x), \qquad x \in \mathbb{R}^n. \tag{1.14}$$

In the investigation of the wave equation it is often convenient to merge the coefficients $\rho(x)$ and κ into one coefficient

$$c(x) = \sqrt{\frac{\kappa}{\rho(x)}} > 0.$$

With this coefficient and with $g(x,t) = \frac{1}{\rho(x)} f(x)$ the equations (1.5) and (1.6) become

$$u_{tt}(x,t) = c(x)^2 \Delta_x u(x,t), \tag{1.15}$$

$$u_{tt}(x,t) = c(x)^2 \Delta_x u(x,t) + g(x,t). \tag{1.16}$$

In a strict sense, (1.15) is called wave equation only if $c(x) = \text{const}$. We shall use this name for (1.15) also if the coefficient $c(x)$ is variable.

2 The Helmholtz equation obtained by reduction of the wave equation

2.1 Complex valued solution

Up to now we considered solutions of the equations (1.15) and (1.16) with values in the real numbers. We shall often consider complex valued solution. To define such solutions, set for a complex valued function $u : \Omega\times[0,\infty) \to \mathbb{C}$ with $u = u_1+iu_2$, $u_1, u_2 : \Omega\times[0,\infty) \to \mathbb{R}$,

$$\frac{\partial}{\partial x_j}u(x,t) = \frac{\partial}{\partial x_j}u_1(x,t) + i\frac{\partial}{\partial x_j}u_2(x,t), \qquad \frac{\partial}{\partial t}u(x,t) = \frac{\partial}{\partial t}u_1(x,t) + i\frac{\partial}{\partial t}u_2(x,t),$$

hence

$$\Delta_x u(x,t) = \Delta_x u_1(x,t) + i\Delta_x u_2(x,t), \qquad u_{tt}(x,t) = \frac{\partial^2}{\partial t^2}u_1(x,t) + i\frac{\partial^2}{\partial t^2}u_2(x,t). \tag{2.1}$$

$u : \Omega \times [0,\infty) \to \mathbb{C}$ is a complex valued solution of the wave equation (1.15) or of the inhomogeneous wave equation (1.16) with a complex valued right hand side $f : \Omega \times [0,\infty) \to \mathbb{C}$, if u satisfies (1.15) or (1.16) with $\Delta_x u$ and u_{tt} defined by (2.1).

Clearly, since $c(x)^2$ is real, a complex valued function u is a solution of (1.15) or (1.16), if both the real part u_1 and the imaginary part u_2 solve the respective equation.

Though complex valued solutions seem to be more complicated than real valued solutions, it turns out that allowing complex valued solutions elucidates the situation considerably.

2.2 Separation of variables and boundary value problems for the Helmholtz equation

To find a solution $u : \Omega \times [0,\infty) \to \infty$ of the wave equation (1.15) we try the product ansatz

$$u(x,t) = w(t)\,v(x).$$

Insertion into the wave equation yields

$$w_{tt}(t)\,v(x) = c(x)^2 w(t)\Delta v(x),$$

hence

$$\frac{w_{tt}(t)}{w(t)} = \frac{c(x)^2\Delta v(x)}{v(x)}\,.$$

This equation must hold for all $x \in \Omega$ and all $t \in [0,\infty)$. Since the left hand side only depends on t and the right hand side on x, this is only possible if the fractions on both sides have a constant value $-\lambda \in \mathbb{R}$. Thus,

$$w_{tt}(t) + \lambda w(t) = 0, \qquad t \in [0, \infty) \tag{2.2}$$

$$c(x)^2 \Delta v(x) + \lambda v(x) = 0, \qquad x \in \overline{\Omega}. \tag{2.3}$$

The first equation is a linear, homogeneous ordinary differential equation of second order. The second equation is called Helmholtz equation or reduced wave equation, a linear second order partial differential equation. In a strict sense, these names are reserved for the equation (2.3) with $c(x) = 1$, but we use them also when the coefficient $c(x) > 0$ is variable. For $\lambda = 0$ one obtains the potential equation

$$\Delta v(x) = 0, \quad x \in \overline{\Omega}.$$

For $\lambda \neq 0$ the general solution of (2.2) is

$$\begin{aligned} w(t) = C_1 e^{\sqrt{-\lambda}\, t} + C_2 e^{-\sqrt{-\lambda}\, t} \\ &= C_1 e^{\operatorname{Re}\sqrt{-\lambda}\, t}\big(\cos(\operatorname{Im}\sqrt{-\lambda}\, t) + i \sin(\operatorname{Im}\sqrt{-\lambda}\, t)\big) \\ &\qquad + C_2 e^{-\operatorname{Re}\sqrt{-\lambda}\, t}\big(\cos(\operatorname{Im}\sqrt{-\lambda}\, t) - i \sin(\operatorname{Im}\sqrt{-\lambda}\, t)\big), \end{aligned}$$

whereas for $\lambda = 0$ the general solution of (2.2) is

$$w(t) = C_1 t + C_2.$$

By choosing the constant λ suitably we can thus construct solutions of the wave equation with special behavior in time. For example, if $\lambda > 0$ and if v is a solution of the Helmholtz equation to this λ, then

$$u(x,t) = \big(C_1 \cos(\sqrt{\lambda}\, t) + C_2 \sin(\sqrt{\lambda}\, t)\big) v(x)$$

is a solution representing an undamped oscillation. If we choose the coefficients such that $C_1^2 + C_2^2 = 1$, then $|v(x)|$ is the amplitude of this oscillation. If $\lambda < 0$ then

$$u(x,t) = e^{\sqrt{-\lambda}\, t}\, v(x)$$

is a solution, which increases exponentially in time, and

$$u(x,t) = e^{-\sqrt{-\lambda}\, t}\, v(x)$$

is an exponentially decreasing solution.

The method to solve the wave equation (1.15) with the product ansatz $u(x,t) = w(t)v(x)$ is called separation of variables. Of course, with this ansatz the Dirichlet boundary condition

$$w(t)\,v(x) = u(x,t) = \mu(x,t), \quad (x,t) \in \partial\Omega \times [0,\infty)$$

can only be solved if also the given boundary data μ are of the form

$$\mu(x,t) = w(t)\,\gamma(x)$$

with a function $\gamma : \partial\Omega \to \mathbb{R}$. In this case $u(x,t) = w(t)v(x)$ solves the wave equation and the Dirichlet boundary condition if w solves (2.2) and v solves the Dirichlet boundary value problem

$$\begin{aligned} c(x)^2 \Delta v(x) + \lambda v(x) &= 0, \qquad x \in \overline{\Omega}, \\ v(x) &= \gamma(x), \qquad x \in \partial\Omega, \end{aligned}$$

for the Helmholtz equation. Also, $u(x,t) = w(t)v(x)$ solves the wave equation and the Neumann boundary condition

$$\frac{\partial}{\partial n} u(x,t) = \nu(x,t) = w(t)\,\hat{\gamma}(t), \quad (x,t) \in \partial\Omega \times [0,\infty),$$

if v satisfies the Neumann boundary value problem

$$\begin{aligned} c(x)^2 \Delta v(x) + \lambda v(x) &= 0, \qquad x \in \overline{\Omega}, \\ \frac{\partial}{\partial n} v(x) &= \hat{\gamma}(x), \qquad x \in \partial\Omega, \end{aligned}$$

for the Helmholtz equation.

2.3 Linear partial differential equations of order m, well posed problem

More general solutions of the wave equation (1.15) can be obtained by adding two solutions $u_1(x,t) = w_1(t)v_1(x)$ and $u_2(x,t) = w_2(t)v_2(x)$ of the wave equation constructed with the method of separation of variables, for example by choosing different constants λ_1 and λ_2. More precisely, any linear combination

$$\sum_{j=1}^{m} a_j u_j(x,t), \quad a_j \in \mathbb{C},$$

of solutions $u_j(x,t)$ of the wave equation is itself a solution of the wave equation. Even infinite series of solutions of the wave equation can yield new solutions. This is shown by the following

Theorem 2.1 *Let* $\{u_m\}_{m=1}^{\infty}$ *be a sequence of two times continuously differentiable solutions of the wave equation* (1.15) *in the domain* $\overline{\Omega} \times [0,\infty)$. *If the function series*

$$\sum_{m=1}^{\infty} \frac{\partial^{j_1+\ldots+j_n+k}}{\partial x_1^{j_1} \ldots \partial x_n^{j_n} \partial t^k} u_m(x_1,\ldots,x_n,t), \quad k + j_1 + \ldots + j_n \le 2,$$

converge uniformly in every compact subset of $\overline{\Omega} \times [0,\infty)$, *then*

$$u(x,t) = \sum_{m=1}^{\infty} u_m(x,t)$$

is a two times differentiable solution of (1.15) *in* $\overline{\Omega} \times [0,\infty)$.

The **proof** follows from the well known result of calculus, that under the assumptions of the theorem the function u is two times continuously differentiable. We leave the proof to the reader.

Every linear combination of solutions of the wave equation is a solution since the unknown function u and its derivatives appear only linearly in the wave equation. Partial differential equations with this property are called linear. To define precisely the notion of a linear partial differential equation I introduce the following notations:

For a multi-index $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}_0^n$ and for $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ let

$$\begin{aligned} |\alpha| &= \alpha_1 + \ldots + \alpha_n \quad \text{(length of the multi-index)}, \\ D^\alpha v(x) &= \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_n} x_n} v(x_1,\ldots,x_n), \\ \alpha! &= \alpha_1!\,\alpha_2! \ldots \alpha_n!\,. \end{aligned}$$

Definition 2.2 *Let* $m \in \mathbb{N}$ *be a given number and let* $x = (x_1,\ldots,x_n)$ *denote points in* $\mathbb{R}^n$. *The expression*

$$\sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| \le m}} a_\alpha(x)\, D^\alpha v(x) = f(x)$$

is called linear partial differential equation for the function v *with given (real or complex valued) coefficient functions* a_α *and given right hand side* f. *This equation is called of order* m *at the point* x *if at least one of the coefficient functions* a_α *with* $|\alpha| = m$ *does not vanish at* x. *The partial differential equation is called homogeneous if* $f \equiv 0$.

Partial differential equations are grouped into various classes comprising equations with similar properties. Most important are the classes of elliptic, parabolic and hyperbolic equations. The Helmholtz equation is the prototype of a linear elliptic equation, the wave equation is the prototype of a linear hyperbolic equation, the heat equation

$$u_t(x,t) = c(x)\Delta_x u(x,t), \quad (x,t) \in \overline{\Omega} \times [0,\infty), \quad c(x) > 0,$$

is the prototype of a linear parabolic equation. The precise definitions of elliptic and hyperbolic equations are given in Sections 8.1 and 10.1.

Definition 2.3 (Well posed problem) *A boundary value problem, initial value problem or initial-boundary value problem is called well posed, if it has the following three properties:*

1. *a solution of the problem exists,*
2. *the solution is unique,*
3. *the solution depends continuously on the right hand side, on the boundary data and on the initial data.*

The meaning of continuos dependence has to be made precise in the context of the particular problem studied. These lecture notes are mostly devoted to the study of questions of well posedness of linear elliptic partial differential equations. Only in Section 10 we return to the wave equation, where we show how the solution theory for the Helmholtz equation, and more generally, for elliptic equations developed in Sections 8 and 9 can be used to solve initial-boundary value problems for the wave equation and for more general hyperbolic equations.

3 Tools from functional analysis. Weak solutions of one-dimensional boundary value problems to the Helmholtz equation

3.1 The Hilbert space $L^2(\Omega, \mathbb{C})$

Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty, open or closed set. $L^2(\Omega) = L^2(\Omega, \mathbb{C})$ is the space of all quadratically integrable functions:

$$L^2(\Omega) = \{f : \Omega \to \mathbb{C} \,\Big|\, \int_\Omega |f(x)|^2 \, dx < \infty\}.$$

We show that $L^2(\Omega)$ is a vector space:

Theorem 3.1 (Cauchy-Schwarz inequality) *Let $f, g \in L^2(\Omega)$. Then the product $f \cdot \overline{g}$ is integrable and satisfies*

$$|\int_\Omega f(x)\overline{g(x)} \, dx| \leq \left(\int_\Omega |f(x)|^2 \, dx\right)^{1/2} \left(\int_\Omega |g(x)|^2 \, dx\right)^{1/2}.$$

Proof: Let $a, b \geq 0$. From $0 \leq (a-b)^2 = a^2 - 2ab + b^2$ we infer that $ab \leq \frac{1}{2}(a^2 + b^2)$. Setting

$$a = \frac{|f(x)|}{\left(\int_\Omega |f(x)|^2 \, dx\right)^{1/2}}, \quad b = \frac{|g(x)|}{\left(\int_\Omega |g(x)|^2 \, dx\right)^{1/2}},$$

we conclude that

$$\frac{|f(x)\overline{g}(x)|}{\left(\int_\Omega |f(x)|^2 \, dx\right)^{1/2} \left(\int_\Omega |g(x)|^2 \, dx\right)^{1/2}} \leq \frac{|f(x)|^2}{2\int_\Omega |f(x)|^2 \, dx} + \frac{|g(x)|^2}{2\int_\Omega |g(x)|^2 \, dx}.$$

Since the right hand side is integrable we see from this inequality that $f \cdot \overline{g}$ is integrable and that

$$\frac{\int_\Omega |f(x)\overline{g}(x)| \, dx}{(\int_\Omega |f(x)|^2 \, dx)^{1/2} (\int_\Omega |g(x)|^2 \, dx)^{1/2}} \leq \frac{\int_\Omega |f(x)|^2 \, dx}{2\int_\Omega |f(x)|^2 \, dx} + \frac{\int_\Omega |g(x)|^2 \, dx}{2\int_\Omega |g(x)|^2 \, dx} = 1.$$

This shows that the Cauchy-Schwarz inequality holds. ∎

Corollary 3.2 (Minkowski inequality) *Let $f, g \in L^2(\Omega)$. Then $f + g \in L^2(\Omega)$ and*

$$\left(\int_\Omega |f(x) + g(x)|^2 \, dx\right)^{1/2} \leq \left(\int_\Omega |f(x)|^2 \, dx\right)^{1/2} + \left(\int_\Omega |g(x)|^2 \, dx\right)^{1/2}.$$

Proof: The Cauchy-Schwarz inequality implies

$$\begin{aligned}
&\int_\Omega |f(x)+g(x)|^2\,dx = \int_\Omega (f(x)+g(x))(\overline{f(x)}+\overline{g(x)})dx \\
&= \int_\Omega |f(x)|^2 + g(x)\overline{f(x)} + f(x)\overline{g(x)} + |g(x)|^2\,dx \\
&= \int_\Omega |f(x)|^2\,dx + 2\,\mathrm{Re}\int_\Omega f(x)\overline{g(x)}\,dx + \int_\Omega |g(x)|^2\,dx, \\
&\le \int_\Omega |f(x)|^2 dx + 2\Big(\int_\Omega |f(x)|^2 dx\Big)^{1/2}\Big(\int_\Omega |g(x)|^2 dx\Big)^{1/2} + \int_\Omega |g(x)|^2 dx \\
&= \bigg(\Big(\int_\Omega |f(x)|^2 dx\Big)^{1/2} + \Big(\int_\Omega |g(x)|^2 dx\Big)^{1/2}\bigg)^2 .
\end{aligned}$$

This implies Minkowski's inequality. ■

For $f, g \in L^2(\Omega)$ let

$$\begin{aligned}
\|f\| = \|f\|_\Omega &= \Big(\int_\Omega |f(x)|^2 dx\Big)^{1/2}, \\
(f,g) = (f,g)_\Omega &= \int_\Omega f(x)\,\overline{g(x)}\,dx\,.
\end{aligned}$$

Corollary 3.3 *$L^2(\Omega)$ is a vector space, $\|f\|$ is a norm and (f,g) is a scalar product on this vector space with*

$$\|f\| = (f,f)^{1/2}.$$

Therefore $L^2(\Omega)$ is a pre-Hilbert space.

Theorem 3.4 (of Fischer-Riesz.) *$L^2(\Omega)$ is a Hilbert space, i.e. the pre-Hilbert space $L^2(\Omega)$ is complete with respect to the norm $\|f\|$.*

The **proof** can be found in the book "Lineare Funktionalanalysis“ of H.W. Alt, Springer Verlag Berlin, 1999, pp. 49, 50.

3.2 The Riesz representation theorem and the projection theorem

Let X be an abstract Hilbert space over $\mathbb{C}$ with the scalar product (u,v) and the norm $\|u\| = (u,u)^{1/2}$, let $F : X \to \mathbb{C}$ be a linear functional (linear mapping), and define the mapping $J_F : X \to \mathbb{R}$ by

$$J_F(u) = \frac{1}{2}\,\|u\|^2 - \mathrm{Re}\,F(u), \qquad \text{for all } u \in X.$$

Theorem 3.5 *Let Y be a closed subspace of X. Then $u \in Y$ satisfies*

$$J_F(u) = \min_{v \in Y} J_F(v)$$

if and only if for all $v \in Y$

$$(v, u) = F(v).$$

Proof: Let $J_F(u) = \min_{v \in Y} J_F(v)$. Then for all $v \in Y$ the function

$$\lambda \mapsto J_F(u + \lambda v) : \mathbb{R} \to \mathbb{R}$$

has the minimum at $\lambda = 0$, hence

$$\begin{aligned}
0 &= \frac{d}{d\lambda} J_F(u + \lambda v)|_{\lambda=0} = \frac{d}{d\lambda} \Big(\frac{1}{2} (u + \lambda v, u + \lambda v) - \mathrm{Re}\, F(u + \lambda v) \Big) \\
&= \frac{d}{d\lambda} \Big(\frac{1}{2} (u, u) + \lambda \, \mathrm{Re}\, (v, u) + \lambda^2 \frac{1}{2} (v, v) - \mathrm{Re}\, F(u) - \lambda \mathrm{Re}\, F(v) \Big) \\
&= \big(\mathrm{Re}\, (v, u) + \lambda (v, v) - \mathrm{Re}\, F(v) \big)|_{\lambda=0} \\
&= \mathrm{Re}\, (v, u) - \mathrm{Re}\, F(v).
\end{aligned}$$

Therefore we have

$$\mathrm{Re}\, (v, u) = \mathrm{Re}\, F(v)$$

for all $v \in Y$. Thus, we also have for $v \in Y$

$$\mathrm{Im}\, (v, u) = -\mathrm{Re}\, i(v, u) = -\mathrm{Re}\, (iv, u) = -\mathrm{Re}\, F(iv) = -\mathrm{Re}\, i\, F(v) = \mathrm{Im}\, F(v).$$

Together it follows for all $v \in Y$

$$(v, u) = \mathrm{Re}\, (v, u) + i\, \mathrm{Im}\, (v, u) = \mathrm{Re}\, F(v) + i\, \mathrm{Im}\, F(v) = F(v).$$

Assume next that

$$(v, u) = F(v)$$

for all $v \in Y$. We have for all $v \in Y$

$$\begin{aligned}
J_F(u + v) &= \frac{1}{2} (u, u) + \mathrm{Re}\, (v, u) + \frac{1}{2} (v, v) - \mathrm{Re}\, F(u) - \mathrm{Re}\, F(v) \\
&= \frac{1}{2} (u, u) - \mathrm{Re}\, F(u) + \frac{1}{2} (v, v) \geq J_F(u),
\end{aligned}$$

whence

$$J_F(u) = \min_{v \in Y} J_F(u + v) = \min_{w \in Y} J_F(w)\,.$$

■

The linear functional $F : X \to \mathbb{C}$ is called bounded if a constant C exists such that

$$|F(u)| \le C\|u\|$$

holds for all $u \in X$. It is a well known result from functional analysis that the linear functional F is bounded if and only if it is continuous. A proof of this statement is given in Section 7.2.

Theorem 3.6 *Let the the linear functional $F : X \to \mathbb{C}$ be bounded, and let Y be a closed subspace of X. Then the mapping J_F assumes the minimum on Y at a unique $u \in Y$.*

Proof: We use the parallelogram equality

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2,$$

which holds for all $u, v \in X$. Note also that for all $a, b \ge 0$ and all $\varepsilon > 0$

$$0 \le \Big(\sqrt{\varepsilon}\, a - \frac{1}{\sqrt{\varepsilon}}\, b\Big)^2 = \varepsilon a^2 - 2ab + \frac{1}{\varepsilon}\, b^2,$$

whence

$$ab \le \frac{\varepsilon}{2}\, a^2 + \frac{1}{2\varepsilon}\, b^2.$$

Therefore we have with $\varepsilon = \frac{1}{2}$

$$\begin{aligned} J_F(u) &= \frac{1}{2}\|u\|^2 - \operatorname{Re} F(u) \ge \frac{1}{2}\|u\|^2 - |F(u)| \\ &\ge \frac{1}{2}\|u\|^2 - C\|u\| \ge \frac{1}{2}\,\|u\|^2 - \frac{1}{2\varepsilon}C^2 - \frac{\varepsilon}{2}\|u\|^2 \\ &= \frac{1}{4}\,\|u\|^2 - C^2 \ge -C^2\,. \end{aligned}$$

Consequently the infimum of J_F exists on Y and satisfies

$$d = \inf_{v \in Y} J_F(v) \ge -C^2\,.$$

Choose a sequence $\{u_n\}_n \subseteq Y$ such that

$$\lim_{n\to\infty} J_F(u_n) = d\,.$$

The parallelogram equality yields

$$\begin{aligned} \|u_m - u_n\|^2 &= 2\|u_m\|^2 + 2\|u_n\|^2 - 4\|\frac{1}{2}(u_m+u_n)\|^2 \\ &= 4\big(\frac{1}{2}\|u_m\|^2 - \operatorname{Re} F(u_m) + \frac{1}{2}\|u_n\|^2 - \operatorname{Re} F(u_n)\big) \\ &\quad -8\Big(\frac{1}{2}\,\|\frac{1}{2}(u_m+u_n)\|^2 - \operatorname{Re} F\big(\frac{1}{2}(u_m+u_n)\big)\Big) \\ &= 4J_F(u_m) + 4J_F(u_n) - 8J_F\big(\frac{1}{2}(u_n+u_m)\big) \\ &\le 4J_F(u_m) + 4J_F(u_n) - 8d \to 0, \end{aligned}$$

for $m, n \to \infty$. Consequently, $\{u_n\}_n$ is a Cauchy sequence and has a limit u. Since Y is closed, u belongs to Y. From the Cauchy-Schwarz inequality $|(v, w)| \le \|v\| \|w\|$ it follows that the mapping $w \mapsto \|w\|^2 : X \to \mathbb{R}$ is continuous, hence J_F is continuous. We thus obtain

$$\inf_{v \in Y} J_F(v) = \lim_{n\to\infty} J_F(u_n) = J_F(u).$$

Therefore u is the minimum of J_F on Y. To see that the minimum is unique, let u and v be two minima on Y. The calculation above yields

$$\|u - v\|^2 = 4J_F(u) + 4J_F(v) - 8J_F\big(\frac{1}{2}(u + v)\big) \le 4d + 4d - 8d = 0,$$

whence $u = v$. ■

Corollary 3.7 *(i)* **(Riesz representation theorem)** *To every bounded linear mapping $F : X \to \mathbb{C}$ there is a unique $u \in X$ such that*

$$(v, u) = F(v)$$

for all $v \in X$.
(ii) **(Projection theorem)** *Let Y be a closed subspace of X. To every $v \in X$ there is a unique $u \in Y$ such that*

$$\|v - u\| = \min_{w \in Y} \|v - w\|.$$

u is the unique element in Y which satisfies

$$(v - u, w) = 0, \tag{3.1}$$

for all $w \in Y$.

Proof: (i) For the subspace in Theorems 3.5 and 3.6 choose $Y = X$, let u be the minimum of J_F on X, which exists by Theorem 3.6, and apply Theorem 3.5.
(ii) Define the bounded linear functional $F : X \to \mathbb{C}$ by

$$F(w) = (w, v).$$

By Theorem 3.6 the mapping J_F has a unique minimum u on Y. Since

$$\begin{aligned} J_F(w) &= \frac{1}{2}\|w\|^2 - \operatorname{Re} F(w) \\ &= \Big(\frac{1}{2}\|w\|^2 - \operatorname{Re}\,(w, v) + \frac{1}{2}\|v\|^2\Big) - \frac{1}{2}\|v\|^2 = \frac{1}{2}\|v - w\|^2 - \frac{1}{2}\|v\|^2, \end{aligned}$$

u is also the unique minimum of $w \mapsto \|v - w\|$ on Y. By Theorem 3.5, $u \in Y$ is the unique element satisfying $(w, u) = F(w) = (w, v)$ for all $w \in Y$. This implies (3.1). ■

Remark 3.8 The space X' of bounded linear functionals on X is called dual space of the Hilbert space X. The Riesz representation theorem shows that for every $F \in X'$ there is a unique element $TF \in X$ such that $(v, FT) = F(v)$ for all $v \in X$, which means that the mapping $(\cdot, TF)$ is a representation of F. The relation $F \mapsto TF$ defines a mapping $T : X' \to X$. We see immediately that T is linear and injective. T is also surjective: To see this, consider the linear functional $G : X \to \mathbb{C}$ defined by $G(v) = (v, u)$. The Cauchy-Schwarz inequality implies $|G(v)| \leq \|u\| \, \|v\|$. Hence G is a bounded linear functional with $u = TG$. This shows that X' is isomorphic to X.

3.3 Complete orthonormal systems

Definition 3.9 *Let $\{v_m\}_{m=1}^{\infty}$ be a sequence in a Hilbert space X.*
(i) If $(v_m, v_\ell) = 0$ for $m \neq \ell$ and $\|v_m\| = 1$ for all m, then $\{v_m\}_m$ is called a (countable) orthonormal system in X.
(ii) The orthonormal system $\{v_m\}_m$ is called complete if the linear subspace

$$\operatorname{span}\{v_m\}_m = \Big\{ \sum_{m=1}^{k} a_m v_m \;\Big|\; k \in \mathbb{N},\ a_1, \ldots a_k \in \mathbb{C} \Big\}$$

is dense in X.

Theorem 3.10 *Let $\{v_m\}_m$ be an orthonormal system. Equivalent are*

(i) *$\{v_m\}_m$ is complete.*

(ii) *For all $f \in X$ the series $\sum_{m=1}^{\infty} (f, v_m) v_m$ converges to f in X:*

$$f = \sum_{m=1}^{\infty} (f, v_m) v_m \,,$$

i.e.

$$\lim_{k \to \infty} \|f - \sum_{m=1}^{k} (f, v_m) v_m\| = 0.$$

(iii) **(Parseval identity)** *For all $f \in X$ we have*

$$\|f\|^2 = \sum_{m=1}^{\infty} |(f, v_m)|^2.$$

(Marc-Antoine Parseval, 1755 - 1836)

For a **proof** cf. pp. 274, 275 of the book of Alt. $\sum_{m=1}^{\infty} (f, v_m) v_m$ is called Fourier series of f and (f, v_m) is the m^{th} Fourier coefficient.

Theorem 3.11 *An orthonormal system $\{v_m\}_m$ is complete if and only if for all $f \in X$, $f \neq 0$, there is $v_k \in \{v_m\}_m$ such that*

$$(f, v_k) \neq 0.$$

Proof. Let $V = \operatorname{span}\{v_m\}_m$. It is obvious that there is $w \in \overline{V}$ with $(f, w) \neq 0$ if and only if there is $v_k \in \{v_m\}_m$ with $(f, v_k) \neq 0$. Therefore it suffices to show that $\overline{V} = X$ if and only if to all $f \in X$ with $f \neq 0$ there is $w \in \overline{V}$ such that $(f, w) \neq 0$.

Now, if $\overline{V} = X$ then for all $f \in X$, $f \neq 0$, choose $w = f$. This yields $(f, w) = (f, f) > 0$. On the other hand, if $\overline{V} \neq X$ choose $g \in X \setminus \overline{V}$. Since $\overline{V}$ is a closed subspace it follows by Corollary 3.7 (projection theorem) that there is $g_0 \in \overline{V}$ such that $f = g - g_0 \neq 0$ satisfies $(f, w) = 0$ for all $w \in \overline{V}$. Hence, the statement of the theorem follows. ∎

Example 3.12 *For $m \in \mathbb{Z}$ let $v_m : (0, 2\pi) \to \mathbb{C}$ be defined by*

$$v_m(x) = \frac{1}{\sqrt{2\pi}}\, e^{imx}.$$

$\{v_m\}_{m=-\infty}^{\infty}$ is a complete orthonormal system in $L^2(0, 2\pi)$.

Proof. $\{v_m\}_m$ is an orthonormal system, since

$$(v_\ell, v_m) = \int_0^{2\pi} v_\ell(x)\overline{v_m(x)}\, dx = \frac{1}{2\pi} \int_0^{2\pi} e^{i(\ell - m)x}\, dx = \delta_{\ell m},$$

with the Kronecker symbol

$$\delta_{\ell m} = \begin{cases} 1, & \ell = m, \\ 0, & \text{otherwise.} \end{cases}$$

To show that the orthonormal system is complete, we need

Theorem 3.13 (Fejér) *Let $g : \mathbb{R} \to \mathbb{C}$ be continuous and 2π-periodic. For $k, m, n \in \mathbb{Z}$, $m \geq 0$, $n \geq 1$ define*

$$a_k = \frac{1}{2\pi} \int_0^{2\pi} g(x)\, e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}}\, (g, v_k),$$

$$s_m(x) = \sum_{k=-m}^{m} a_k\, e^{ikx},$$

$$\sigma_n(x) = \frac{1}{n} \big(s_0(x) + \ldots + s_{n-1}(x)\big).$$

Then the sequence $\{\sigma_n\}_{n=1}^{\infty}$ converges to g uniformly on $[0, 2\pi]$.
(Leopold Fejér, 1880 – 1959)

With this theorem we can prove that $\{v_m\}_m$ is complete: Let $f \in L^2(0, 2\pi)$ and $\varepsilon > 0$ be given arbitrarily. By a well known result from Lebesgue integration theory, the set of continuous functions on $[0, 2\pi]$ vanishing at $x = 0$ and $x = 2\pi$ is dense in $L^2(0, 2\pi)$. We can therefore choose such a function g with

$$\|f - g\| < \varepsilon.$$

Since g vanishes at the boundary points of the interval $[0, 2\pi]$, it follows that the 2π-periodic extension of g to $\mathbb{R}$ is continuous. By the Theorem of Fejér it thus follows that there is $n \in \mathbb{N}$ with

$$\sup_{0 \le x \le 2\pi} |g(x) - \sigma_n(x)| < \varepsilon.$$

Thus

$$\begin{aligned} \|f - \sigma_n\| &\le \|f - g\| + \|g - \sigma_n\| \\ &\le \varepsilon + \Big(\int_0^{2\pi} |g(x) - \sigma_n(x)|^2 \, dx \Big)^{1/2} \le \varepsilon(1 + \sqrt{2\pi}). \end{aligned}$$

Since σ_n is a linear combination of functions from $\{v_m\}_{m=-\infty}^{\infty}$, we conclude from this estimate that span $\{v_m\}_{m=-\infty}^{\infty}$ is dense in $L^2(0, 2\pi)$. Consequently the orthonormal system is complete. ■

Remark 3.14 From this result and from Theorem 3.10 we see that for every function $f \in L^2(0, 2\pi)$ the Fourier series

$$f(x) = \sum_{m=-\infty}^{\infty} (f, v_m)_{[0,2\pi]} v_m(x) = \sum_{m=-\infty}^{\infty} (f, v_m)_{[0,2\pi]} \frac{1}{\sqrt{2\pi}} e^{imx} = \sum_{m=-\infty}^{\infty} a_m e^{imx},$$

with

$$a_m = \frac{1}{\sqrt{2\pi}} (f, v_m)_{[0,2\pi]} = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(y) \, \overline{v_m(y)} \, dy = \frac{1}{2\pi} \int_0^{2\pi} f(y) e^{-imy} dy.$$

satisfies

$$\lim_{k \to \infty} \Big\| f - \sqrt{2\pi} \sum_{m=-k}^{k} a_m v_m \Big\|_{[0,2\pi]} = 0,$$

where $(g, h)_{[0,2\pi]}$ and $\|g\|_{[0,2\pi]}$ denote the scalar product and norm of $L^2(0, 2\pi)$. Moreover, Parseval's identity implies that

$$\|f\|_{[0,2\pi]}^2 = \sum_{m=-\infty}^{\infty} |(f, v_m)|_{[0,2\pi]}^2 = 2\pi \sum_{m=-\infty}^{\infty} |a_m|^2.$$

The series converges to f with respect to the L^2–norm. If f is continuous on the interval $[0, 2\pi]$, then the convergence is pointwise, but not necessarily uniform. Yet, by the theorem of Fejér, the sequence of mean values of partial sums of the Fourier series converges uniformly.

Remark 3.15 Since e^{imx} is 2π–periodic, the family $\{\frac{1}{\sqrt{2\pi}} e^{imx}\}_{m=-\infty}^{\infty}$ is obviously a complete orthonormal system on every interval $(a, 2\pi + a)$ obtained by translation of the interval $(0, 2\pi)$ by $a \in \mathbb{R}$. This remark holds also for the orthonormal system of the next example, which is often considered on the interval $(-\pi, \pi)$.

Example 3.16 *A complete orthonormal system in $L^2(0, 2\pi)$ of real functions is given by*

$$\left\{ \frac{1}{\sqrt{\pi}} \cos(mx), \ \frac{1}{\sqrt{\pi}} \sin(mx) \mid m = 0, 1, 2, \ldots \right\}.$$

Proof: A well known computation shows that this system is orthonormal. To prove completeness it suffices to remark that for the functions v_m from Example 3.12

$$v_m(x) = \frac{1}{\sqrt{2\pi}} \cos(mx) + i \frac{1}{\sqrt{2\pi}} \sin(mx).$$

Hence, the linear span of this system is equal to the dense subspace $\mathrm{span}\{v_m\}_m$. ■

3.4 Eigenfunctions of the Dirichlet boundary value problem in $\mathbb{R}^1$.

The Helmholtz equation in $\mathbb{R}^1$ is an ordinary differential equation. Therefore the solution of the boundary value problems to the Helmholtz equation is considerably simpler in one space dimension than in higher dimensions. Nevertheless, the solution properties of the one dimensional and higher dimensional problems are similar. Since it is helpful to know these properties when studying higher dimensional problems, we investigate in this section the one-dimensional problem. Thus, let $\Omega = (a, b)$, let $\gamma_a, \gamma_b \in \mathbb{C}$ and $\lambda \in \mathbb{C}$. We search a two times continuously differentiable solution $u : [a, b] \to \mathbb{C}$ of

$$\begin{aligned} &u''(x) + \lambda u(x) = 0, \qquad x \in [a, b], \\ &u(a) = \gamma_a, \quad u(b) = \gamma_b. \end{aligned}$$

For $\lambda = 0$ the general solution of the ordinary differential equation is

$$u(x) = C_1 x + C_2, \quad C_1, C_2 \in \mathbb{C}.$$

The boundary conditions yield the linear system

$$\begin{aligned} C_1 a + C_2 &= \gamma_a, \\ C_1 b + C_2 &= \gamma_b. \end{aligned}$$

It follows that for $\lambda = 0$ the boundary value problem has a unique solution given by

$$u(x) = \frac{\gamma_a - \gamma_b}{a - b} x + \frac{1}{a - b}(a\gamma_b - b\gamma_a).$$

For $\lambda \neq 0$ the general solution of the ordinary differential equation is

$$u(x) = C_1 e^{\sqrt{-\lambda}\,x} + C_2 e^{-\sqrt{-\lambda}\,x}$$

with $C_1, C_2 \in \mathbb{C}$. The boundary conditions imply

$$\begin{aligned} C_1 e^{\sqrt{-\lambda}\,a} + C_2 e^{-\sqrt{-\lambda}\,a} &= \gamma_a \\ C_1 e^{\sqrt{-\lambda}\,b} + C_2 e^{-\sqrt{-\lambda}\,b} &= \gamma_b. \end{aligned}$$

This is a linear system of equations for C_1 and C_2 with the coefficient matrix

$$A = \begin{pmatrix} e^{\sqrt{-\lambda}\,a} & e^{-\sqrt{-\lambda}\,a} \\ e^{\sqrt{-\lambda}\,b} & e^{-\sqrt{-\lambda}\,b} \end{pmatrix}.$$

Therefore the boundary value problem is uniquely solvable for all γ_a, γ_b if and only if $\det A \neq 0$. Now

$$\det A = e^{\sqrt{-\lambda}\,a}\, e^{-\sqrt{-\lambda}\,b} - e^{\sqrt{-\lambda}\,b}\, e^{-\sqrt{-\lambda}\,a} = e^{\sqrt{-\lambda}\,(a-b)}(1 - e^{2\sqrt{-\lambda}\,(b-a)}).$$

Thus, $\det A = 0$ if and only if

$$2\sqrt{-\lambda}\,(b - a) = 2\pi i m, \quad m \in \mathbb{Z},$$

which is equivalent to

$$\lambda = \Big(\frac{\pi m}{b - a}\Big)^2.$$

Together we obtain

Theorem 3.17 *Let the sequence* $\{\lambda_m\}_{m\in\mathbb{N}}$ *be given by*

$$\lambda_m = \Big(\frac{\pi m}{b - a}\Big)^2.$$

(i) *For* $\lambda \in \mathbb{C} \setminus \{\lambda_m \mid m \in \mathbb{N}\}$ *the boundary value problem*

$$u''(x) + \lambda u(x) = 0, \qquad x \in [a, b], \tag{3.2}$$

$$u(a) = \gamma_a, \quad u(b) = \gamma_b, \tag{3.3}$$

is uniquely solvable for all $\gamma_a, \gamma_b \in \mathbb{C}$. *In particular,* $u = 0$ *is the only solution to the homogeneous boundary value problem* ($\gamma_a = \gamma_b = 0$).
(ii) *If there is* $m \in \mathbb{N}$ *such that* $\lambda = \lambda_m$, *then there are pairs* (γ_a, γ_b) *of complex numbers, for which the boundary value problem is not solvable. If a solution exists, then the solution is not unique. In particular, for every* $C \neq 0$ *the function*

$$w(x) = C \sin\big(\sqrt{\lambda_m}\,(x-a)\big) = C \sin\big(\frac{\pi m}{b-a}\,(x-a)\big)$$

is a nonzero solution of the homogeneous boundary value problem.

Proof. it only remains to show that w solves the homogeneous boundary value problem. Yet, obviously we have $w(a) = C\sin(0) = 0$ and $w = C\sin(\pi m) = 0$. ■

Definition 3.18 *The numbers* $\lambda_m = \big(\frac{\pi m}{b-a}\big)^2$, $m \in \mathbb{N}$, *are called eigenvalues of the boundary value problem* (3.2), (3.3). *Every nonvanishing solution of this boundary value problem with* $\lambda = \lambda_m$ *and* $\gamma_a = \gamma_b = 0$ *is called eigenfunction to the eigenvalue* λ_m.

Theorem 3.19 *Let*

$$u_m(x) = \sqrt{\frac{2}{b-a}}\,\sin\Big(\frac{\pi m}{b-a}\,(x-a)\Big). \tag{3.4}$$

$\{u_m\}_{m\in\mathbb{N}}$ *is a complete orthonormal system in* $L^2([a,b])$ *of eigenfunctions to the Dirichlet boundary value problem.*

Proof. By Theorem 3.17 the function u_m is an eigenfunction for the Dirichlet boundary value problem, and a simple computation yields that $\{u_m\}_{m\in\mathbb{N}}$ is orthonormal. To prove completeness, we scale and translate u_m to define the odd function $w_m : [-\pi, \pi] \to \mathbb{C}$ by

$$w_m(x) = \begin{cases} u_m\Big(\dfrac{b-a}{\pi}x + a\Big) = \sqrt{\dfrac{2}{b-a}}\sin(mx), & 0 \le x \le \pi, \\ -w_m(-x) = \sqrt{\dfrac{2}{b-a}}\sin(mx), & -\pi \le x \le 0. \end{cases}$$

By Remark 3.15 and Example 3.16, $\operatorname{span}\{w_m\}_m$ is dense in the space

$$\{f \in L^2(-\pi, \pi) \mid f(x) = -f(-x)\},$$

since for odd functions the Fourier coefficients of the cosine functions vanish. From this we conclude immediately that $\text{span}\{u_m\}_m$ is dense in $L^2([a,b])$. ■

This result suggests to construct solutions of the Dirichlet boundary value problem

$$u''(x) + \lambda u(x) = f(x), \qquad x \in [a,b], \tag{3.5}$$

$$u(a) = u(b) = 0, \tag{3.6}$$

with a given right hand side $f \in L^2([a,b])$ as follows:

Let $\{\lambda_m\}_m$ be the sequence of eigenvalues to the Dirichlet boundary value problem and assume that $\lambda \neq \lambda_m$ for all m. With the complete orthonormal system $\{u_m\}_m$ of eigenfunctions consider the series

$$\sum_{m=1}^{\infty} \frac{1}{\lambda - \lambda_m} (f, u_m) u_m.$$

This series converges in $L^2([a,b])$. To see this, note that

$$\Big\| \sum_{m=k}^{\ell} \frac{1}{\lambda - \lambda_m} (f, u_m) u_m \Big\|^2 = \sum_{m,j=k}^{\ell} \frac{(f, u_m)}{\lambda - \lambda_m} \overline{\frac{(f, u_j)}{\lambda - \lambda_j}} (u_m, u_j) = \sum_{m=k}^{\ell} \Big| \frac{(f, u_m)}{\lambda - \lambda_m} \Big|^2.$$

The Cauchy convergence criterion implies therefore that the series converges, if and only if $\sum_{m=1}^{\infty} |\frac{(f,u_m)}{\lambda - \lambda_m}|^2 < \infty$. From $\lambda_m = \left(\frac{\pi m}{b-a}\right)^2$ and from $\lambda_m \neq \lambda$ for all m we conclude that there is a constant $C > 0$ such that

$$\Big| \frac{1}{\lambda - \lambda_m} \Big| \leq \frac{C}{m^2}$$

for all $m \in \mathbb{N}$. Since the Fourier series $\sum_{m=1}^{\infty} (f, u_m) u_m$ converges to f in $L^2([a,b])$, we thus obtain from this inequality and from Parseval's identity (Theorem 3.10) that

$$\sum_{m=1}^{\infty} \Big| \frac{(f, u_m)}{\lambda - \lambda_m} \Big|^2 \leq \sum_{m=1}^{\infty} \frac{C^2}{m^4} |(f, u_m)|^2 \leq C^2 \sum_{m=1}^{\infty} |(f, u_m)|^2 < \infty.$$

Consequently, the series $\sum_{m=1}^{\infty} \frac{1}{\lambda - \lambda_m} (f, u_m) u_m$ converges. Denote the limit function by u:

$$u = \sum_{m=1}^{\infty} \frac{1}{\lambda - \lambda_m} (f, u_m) u_m.$$

We want to show that u is a solution of the boundary value problem (3.5), (3.6). To this end note that if u is two-times differentiable and if the derivatives can be interchanged with the summation sign it follows that

$$
\begin{aligned}
u'' + \lambda u &= \frac{d^2}{dx^2} \sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda - \lambda_m} u_m + \lambda \sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda - \lambda_m} u_m \\
&= \sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda - \lambda_m} (u_m'' + \lambda u_m) = \sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda - \lambda_m} (\lambda - \lambda_m) u_m = \sum_{m=1}^{\infty} (f, u_m) u_m = f.
\end{aligned}
$$

Therefore (3.5) is fulfilled. Moreover, if in addition the series $\sum_{m=1}^{\infty}(f, u_m)u_m(x)$ converges for all $x \in [a, b]$ to $u(x)$, then

$$
u(a) = \sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda - \lambda_m} u_m(a) = 0, \quad u(b) = \sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda - \lambda_m} u_m(b) = 0,
$$

because of $u_m(a) = u_m(b) = 0$. Consequently the boundary condition (3.6) holds. Thus, under the assumed properties of the series $\sum_{m=1}^{\infty} \frac{(f,u_m)}{\lambda-\lambda_m} u_m$ the limit function u is a solution of the boundary value problem (3.5), (3.6).

However, in general these assumptions are not satisfied for $f \in L^2([a, b])$. Namely, a precise investigation shows that the boundary value problem is solvable in the classical sense only if f satisfies certain regularity properties, for example if f is continuous. Yet, if the boundary value problem has a classical solution, then it coincides with the function u given by the series. From there the idea originates to generalize the notion of a solution of the boundary value problem and to define weak solutions. The weak solution has the property to coincide with the classical solution if it exists. I introduce weak solutions in the following.

3.5 Weak derivatives

First I define weak derivatives. I need the following standard notations:

Definition 3.20 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $m \in \mathbb{N}_0 \cup \{\infty\}$. We set*

(i) $C_m(\Omega) = C_m(\Omega, \mathbb{C}) = \{f : \Omega \to \mathbb{C} \mid D^\alpha f$ *exists and is continuous for all* $\alpha \in \mathbb{N}_0^n$ *such that* $|\alpha| \le m\}$,

(ii) $C_m^*(\Omega) = \{f \in C_m(\Omega) \mid D^\alpha f \in L^2(\Omega)$ *for all* $|\alpha| \le m\}$,

(iii) $C_m(\overline{\Omega}) = \{f \in C_m(\Omega) \mid D^\alpha f$ *can be extended continuously up to the boundary*$\}$,

(iv) For $f \in C_0(\mathbb{R}^n)$ *let* $\operatorname{supp} f = \overline{\{x \in \mathbb{R}^n \mid f(x) \neq 0\}}$ *be the support of* f,

(v) $\overset{\circ}{C}_\infty(\Omega) = \{\varphi \in C_\infty(\mathbb{R}^n) \mid \operatorname{supp} \varphi$ *is a compact subset of* $\Omega\}$.

Of course $C_m(\Omega), C_m^*(\Omega), C_m(\overline{\Omega})$ and $\overset{\circ}{C}_\infty(\Omega)$ are vector spaces. One also uses the notation $C(\Omega) = C_0(\Omega)$.

Theorem 3.21 *The space $\overset{\circ}{C}_\infty(\Omega)$ is a dense subset of $L^2(\Omega)$, i.e. $\overline{\overset{\circ}{C}_\infty(\Omega)} = L^2(\Omega)$.*

A **proof** can be found in the book of H.W. Alt, pp. 74, 75.

Definition 3.22 *Let $v \in L^2(\Omega)$ and $\alpha \in \mathbb{N}_0^n$. If there is a function $w \in L^2(\Omega)$ such that*

$$(-1)^{|\alpha|}(v, D^\alpha \varphi) = (w, \varphi)$$

for all $\varphi \in \overset{\circ}{C}_\infty(\Omega)$, then w is called the α-th weak derivative of v.

Theorem 3.23 *(i) The α-th weak derivative is uniquely determined.*
(ii) For $v \in C_m^*(\Omega)$ and $|\alpha| \le m$ the α-th weak derivative coincides with the classical derivative $D^\alpha v$.

Proof. (i) Let w_1 and w_2 be weak α-th derivatives of $v \in L^2(\Omega)$. Then, for all $\varphi \in \overset{\circ}{C}_\infty(\Omega)$

$$(w_1, \varphi) = (-1)^{|\alpha|}(v, D^\alpha \varphi) = (w_2, \varphi),$$

hence $(w_1 - w_2, \varphi) = 0$. Since $\overline{\overset{\circ}{C}_\infty(\Omega)} = L^2(\Omega)$, there is a sequence $\{\varphi_m\}_m \subseteq \overset{\circ}{C}_\infty(\Omega)$ such that $\lim_{m\to\infty} \|(w_1 - w_2) - \varphi_m\| = 0$. Thus

$$\begin{aligned}(w_1 - w_2, w_1 - w_2) &= \lim_{m\to\infty} [(w_1 - w_2, (w_1 - w_2) - \varphi_m) + (w_1 - w_2, \varphi_m)] \\ &\le \lim_{m\to\infty} \|w_1 - w_2\| \, \|(w_1 - w_2) - \varphi_m\| = 0,\end{aligned}$$

whence $w_1 = w_2$. Here I used Cauchy-Schwarz' inequality. Therefore v has at most one weak derivative.

(ii) To $\varphi \in \overset{\circ}{C}_\infty(\Omega)$ there is a neighborhood of the boundary $\partial\Omega$ where φ vanishes. Thus, for $v \in C_m^*(\Omega)$ it follows by partial integration

$$(-1)^{|\alpha|}(v, D^\alpha \varphi) = (-1)^{|\alpha|} \int_\Omega v(x) D^\alpha \overline{\varphi(x)} \, dx = \int_\Omega D^\alpha v(x) \overline{\varphi(x)} \, dx = (D^\alpha v, \varphi).$$

Consequently, $D^\alpha v \in L^2(\Omega)$ is the weak derivative of v. ■

Because of this theorem one uses the notation $D^\alpha v$ also for weak derivatives of v. Confusion is not possible, since the weak derivative is equal to the classical derivative, if the latter exists.

Examples (a) Let $\Omega = (-1,1)$ and let $v \in L^2\big((-1,1)\big)$ be defined by $v(x) = |x|$. This function has the weak derivative

$$v'(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 \le x < 1. \end{cases}$$

For, if $\varphi \in \overset{\circ}{C}_\infty\big((-1,1)\big)$ then

$$\begin{aligned} -(v,\varphi') &= -\int_{-1}^{1} v(x)\overline{\varphi}'(x)dx = \int_{-1}^{0} x\,\overline{\varphi}'(x)dx - \int_{0}^{1} x\,\overline{\varphi}'(x)dx \\ &= -\int_{-1}^{0} \overline{\varphi}dx + \int_{0}^{1} \overline{\varphi}(x)dx = \int_{-1}^{1} v'(x)\,\overline{\varphi}(x)dx = (v',\varphi). \end{aligned}$$

(b) v does not have a second weak derivative. For, if $v'' \in L^2(\Omega)$ is the second weak derivative then for all $\varphi \in \overset{\circ}{C}_\infty(\Omega)$

$$\begin{aligned} (v'',\varphi) &= (v,\varphi'') = -(v',\varphi') = \int_{-1}^{0} \overline{\varphi}'(x)dx - \int_{0}^{1} \overline{\varphi}'(x)dx \\ &= \overline{\varphi}(0) - \overline{\varphi}(-1) - \overline{\varphi}(1) + \overline{\varphi}(0) = 2\overline{\varphi}(0). \end{aligned}$$

Now choose $\varphi \in \overset{\circ}{C}_\infty\big((-1,1)\big)$ with $\varphi(0) \neq 0$ and define φ_ℓ by $\varphi_\ell(x) = \varphi(\ell x)$, for $\ell \in \mathbb{N}$. Then $\varphi_\ell \in \overset{\circ}{C}_\infty\big((-1,1)\big)$, and by the preceding equation

$$\begin{aligned} 2|\varphi(0)| &= 2|\varphi_\ell(0)| = 2\lim_{\ell\to\infty} |\varphi_\ell(0)| = \lim_{\ell\to\infty} |(v'',\varphi_\ell)| \\ &\le \lim_{\ell\to\infty} \|v''\|\,\|\varphi_\ell\| = \|v''\| \lim_{\ell\to\infty} \Big(\int_{-1}^{1} |\varphi(\ell x)|^2 dx\Big)^{1/2} \\ &= \|v''\| \lim_{\ell\to\infty} \Big(\int_{-\infty}^{\infty} |\varphi(y)|^2 \frac{1}{\ell}\, dy\Big)^{1/2} = 0. \end{aligned}$$

This contradicts $\varphi(0) \neq 0$, hence v cannot have the second weak derivative v''.

3.6 Sobolev spaces

Definition 3.24 *For an open set $\Omega \subseteq \mathbb{R}^n$ and $m \in \mathbb{N}_0$ let*

$$H_m(\Omega) = \{v \in L^2(\Omega) \mid \textit{the weak derivative } D^\alpha v \textit{ exists for all } |\alpha| \le m\}.$$

$H_m(\Omega)$ is called Sobolev space. For $u, v \in H_m(\Omega)$ we define

$$(u,v)_m = (u,v)_{m,\Omega} = \sum_{|\alpha|\le m} (D^\alpha u, D^\alpha v)_\Omega, \quad \|u\|_m = \|u\|_{m,\Omega} = (u,u)_{m,\Omega}^{1/2}.$$

$H_m(\Omega)$ is a vector space. We even have:

Theorem 3.25 *$H_m(\Omega)$ is a Hilbert space with the scalar product $(u,v)_m$ and the norm $\|u\|_m$.*

Proof. It is immediately seen that $(u,v)_m$ has the properties of a scalar product. Therefore it remains to show that $H_m(\Omega)$ is complete. Thus, let $\{u_\ell\}_{\ell=1}^\infty$ be a Cauchy sequence in $H_m(\Omega)$. Since

$$\|u_\ell - u_k\|_m^2 \;=\; (u_\ell - u_k,\, u_\ell - u_k)_m \;=\; \sum_{|\alpha|\le m} \|D^\alpha u_\ell - D^\alpha u_k\|^2,$$

it follows that $\{D^\alpha u_\ell\}_\ell$ is a Cauchy sequence in $L^2(\Omega)$ for $|\alpha| \le m$. Because $L^2(\Omega)$ is complete, $\{D^\alpha u_\ell\}_\ell$ has a limit function $u^{(\alpha)} \in L^2(\Omega)$. I write $u = u^{(0)}$ and show that $u^{(\alpha)} = D^\alpha u$ for all $0 < |\alpha| \le m$. To this end let $\varphi \in \overset{\circ}{C}_\infty(\Omega)$. Then

$$(-1)^{|\alpha|}(u, D^\alpha \varphi) = \lim_{\ell\to\infty} (-1)^{|\alpha|}(u_\ell, D^\alpha\varphi) = \lim_{\ell\to\infty}(D^\alpha u_\ell, \varphi) = (u^{(\alpha)}, \varphi).$$

This implies $u^{(\alpha)} = D^\alpha u$. Consequently, $u \in H_m(\Omega)$ and $\|u - u_\ell\|_m \to 0$ for $\ell \to \infty$, whence $H_m(\Omega)$ is complete. ■

Theorem 3.26 *(i) $C_m^*(\Omega)$ is dense in $H_m(\Omega)$:*

$$H_m(\Omega) = \overline{C_m^*(\Omega)}.$$

(ii) If Ω has Lipschitz boundary, then $C_m(\overline{\Omega})$ is dense in $H_m(\Omega)$:

$$H_m(\Omega) = \overline{C_m(\overline{\Omega})}.$$

A **proof** of this theorem can be found for example in the book of Alt, pp. 108-109, and also in my lecture notes: H.-D. Alber, Variationsrechnung und Sobolevräume, p. 33 and Chapter 3[1].

Definition 3.27 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. The closure of the linear subspace $\overset{\circ}{C}_\infty(\Omega)$ in $H_m(\Omega)$ is denoted by $\overset{\circ}{H}_m(\Omega)$.*

[1] http://www3.mathematik.tu-darmstadt.de/ags/analysis/ag-partielle-differentialgleichungen-und-anwendungen/lehrmaterial.html

$\overset{\circ}{H}_m(\Omega)$ is a closed linear subspace of $H_m(\Omega)$, hence $\overset{\circ}{H}_m(\Omega)$ is complete as a closed subspace of the complete space $H_m(\Omega)$. Therefore $\overset{\circ}{H}_m(\Omega)$ is a Hilbert space with the scalar product $(u,v)_m$ and the norm $\|u\|_m$. In general $\overset{\circ}{H}_m(\Omega)$ is a proper subspace of $H_m(\Omega)$. This subspace consists of all functions of $H_m(\Omega)$, which in a generalized sense vanish on the boundary $\partial\Omega$.

Another important property of Sobolev functions is that if $m > \frac{n}{2}$, then $u \in H_m(\Omega)$ is continuous and all weak derivatives $D^\alpha u$ with $|\alpha| < m - \frac{n}{2}$ are classical, hence $H_m(\Omega) \subseteq C^*_{\lceil m-\frac{n}{2}-1\rceil}(\Omega)$, where $\lceil r\rceil$ denotes the smallest integer greater or equal to r. This property is called Sobolev imbedding theorem.

The investigation of these properties of Sobolev functions is an extended topic. Fortunately, in this introductory course we almost exclusively need those properties of Sobolev functions which immediately follow from the definitions of the Sobolev spaces given above. Yet, to familiarize the reader with Sobolev spaces we prove now two of these properties in the case $\Omega \subseteq \mathbb{R}^1$:

Theorem 3.28 *Let $\Omega = (a,b) \subseteq \mathbb{R}$ be an open interval and $u, v \in H_1\big((a,b)\big)$. Then*

$$|u(y) - u(x)| \leq \|u'\|_{(a,b)}\, |y-x|^{1/2}, \tag{3.7}$$

$$|u(x)| \leq r^{1/2}\, \|u'\|_{(a,b)} + r^{-1/2}\, \|u\|_{(a,b)}\,, \tag{3.8}$$

$$(u', v)_{(a,b)} + (u, v')_{(a,b)} = u(b)v(b) - u(a)v(a), \tag{3.9}$$

for almost all $x, y \in (a,b)$ and for all $0 < r \leq b-a$.

Remark 3.29 This means that there is a set $M \subseteq (a,b)$ with $\operatorname{meas}\big((a,b)\backslash M\big) = 0$, which consequently is dense in $[a,b]$, such that (3.7) and (3.8) hold for all $x, y \in M$. By (3.7), u is Hölder continuous on M with exponent $\frac{1}{2}$. Hence, u is uniformly continuous on M and can be modified on ∂M, such that the modified function $\tilde{u}$ is Hölder continuous on all of $\overline{M} = M \cup \partial M = [a,b]$. There can be no other continuous function in the equivalence class of u. Therefore we can single out this continuous function and identify the equivalence class with $\tilde{u}$. With this identification every $u \in H_1\big((a,b)\big)$ belongs to the space $C_{1/2}([a,b])$ of Hölder continuous functions with exponent $\frac{1}{2}$, and $H_1\big((a,b)\big)$ is embedded in this space. In (3.9) we use this identification, so $u(a), u(b), v(a), v(b)$ are the values of the continuous representatives. (3.9) shows that partial integration is allowed for weak derivatives.

Proof of Theorem 3.28. Choose a sequence $\{u_\ell\}_\ell \subseteq C_1\big([a,b]\big)$ such that $\|u - u_\ell\|_{1,(a,b)} \to 0$ for $\ell \to \infty$. Then $\{u_\ell\}_\ell$ converges in $L^2\big((a,b)\big)$ to u. Thus, by a well known theorem from Lebesgue integration theory we can select a subsequence $\{u_{\ell_k}\}_k$ such that

$$\lim_{k\to\infty} u_{\ell_k}(x) = u(x)$$

for almost all $x \in [a,b]$. Let $x < y$ be two points with this property and let $\varepsilon > 0$. Then there is k_0 such that

$$|u(x) - u_{\ell_k}(x)| < \varepsilon, \quad |u(y) - u_{\ell_k}(y)| < \varepsilon$$

for $k \geq k_0$. The fundamental theorem of calculus yields for $k \geq k_0$

$$\begin{aligned}
|u(y) - u(x)| &\leq |u(y) - u_{\ell_k}(y)| + |u_{\ell_k}(y) - u_{\ell_k}(x)| + |u(x) - u_{\ell_k}(x)| \\
&\leq 2\varepsilon + \Big| \int_x^y u'_{\ell_k}(z)dz \Big| \leq 2\varepsilon + \Big(\int_x^y dz \Big)^{1/2} \Big(\int_x^y |u'_{\ell_k}(z)|^2\, dz \Big)^{1/2} \\
&\leq 2\varepsilon + |y-x|^{1/2}\, \|u'_{\ell_k}\|_{(a,b)} \\
&\leq 2\varepsilon + |y-x|^{1/2} \big(\|u'\|_{(a,b)} + \|u'_{\ell_k} - u'\|_{(a,b)} \big) .
\end{aligned}$$

Because of $\|u'_{\ell_k} - u'\|_{(a,b)} \leq \|u_{\ell_k} - u\|_{1,(a,b)} < \varepsilon$ for $k \geq k_1$ with k_1 sufficiently large, we deduce from this inequality by choosing $k \geq \max(k_0.k_1)$ that

$$|u(y) - u(x)| \leq \varepsilon \big(2 + (b-a)^{1/2} \big) + \|u'\|_{(a,b)}\, |y-x|^{1/2}.$$

Since $\varepsilon > 0$ was arbitrary, (3.7) follows.

To prove (3.8), let (c,d) with $x \in (c,d) \subseteq (a,b)$ be an interval of finite length. We integrate (3.7) with respect to y from c to d and obtain

$$\begin{aligned}
|u(x)|(d-c) &\leq \|u'\|_{(a,b)} \int_c^d |x-y|^{1/2}\, dx + \int_c^d |u(y)| dy \\
&\leq \|u'\|_{(a,b)}\, (d-c)^{3/2} + (d-c)^{1/2} \Big(\int_c^d |u(y)|^2 dy \Big).
\end{aligned}$$

Division by $(d-c)$ yields

$$|u(x)| \leq (d-c)^{1/2}\, \|u'\|_{(a,b)} + (d-c)^{-1/2} \|u\|_{(a,b)} .$$

This implies (3.8) with $r = d - c$.

To prove (3.9) we can assume that u and v are continuous. Choose sequences $\{u_\ell\}_\ell$, $\{v_\ell\}_\ell \subseteq C_1([a,b])$ such that $\|u - u_\ell\|_{1,(a,b)} \to 0$, $\|v - v_\ell\|_{1,(a,b)} \to 0$ for $\ell \to \infty$. From (3.8) we obtain

$$\lim_{\ell\to\infty} |u(x) - u_\ell(x)| \leq \lim_{\ell\to\infty} \big(r^{1/2}\, \|u' - u'_\ell\| + r^{-1/2} \|u - u_\ell\| \big) = 0$$

for almost all $x \in [a,b]$. This relation shows that $\{u_\ell\}_\ell$ and $\{v_\ell\}_\ell$ converge uniformly on M to u and v, respectively, with the set M defined in Remark 3.29. Since u, v, u_ℓ, v_ℓ are continuous, we infer that the function sequences converge pointwise everywhere to u and v. In particular, we have $\lim_{\ell\to\infty} u_\ell(a) = u(a)$, $\lim_{\ell\to\infty} u_\ell(b) = u(b)$, and similarly $\lim_{\ell\to\infty} v_\ell(a) = v(a)$, $\lim_{\ell\to\infty} v_\ell(b) = v_b$. Using the continuity of the scalar product we obtain by partial integration

$$\begin{aligned}(u', v) + (u, v') &= \lim_{\ell\to\infty} \big((u'_\ell, v_\ell) + (u_\ell, v'_\ell)\big) \\ &= \lim_{\ell\to\infty} \big(u_\ell(b)\, v_\ell(b) - u_\ell(a)\, v_\ell(a)\big) = u(b)v(b) - u(a)v(a),\end{aligned}$$

which proves (3.9). ∎

Lemma 3.30 *The orthogonal space*

$$\overset{\circ}{H}{}_1^{\perp}\big((a,b)\big) = \{u \in H_1\big((a,b)\big) \mid (u,v)_1 = 0 \quad \textit{for all } v \in \overset{\circ}{H}_1\big((a,b)\big)\}$$

is given by

$$\overset{\circ}{H}{}_1^{\perp}\big((a,b)\big) = \{C_1 e^x + C_2 e^{-x} \mid C_1, C_2 \in \mathbb{C}\}.$$

Hence, the orthogonal space is of dimension 2.

Proof. u belongs to $\overset{\circ}{H}{}_1^{\perp}\big((a,b)\big)$ if and only if for all $v \in \overset{\circ}{H}_1\big((a,b)\big)$

$$(u', v') = -(u, v).$$

Since $\overset{\circ}{C}_\infty\big((a,b)\big) \subseteq \overset{\circ}{H}_1\big((a,b)\big)$, this equation holds if and only if u has a second weak derivative which satisfies

$$u'' = u.$$

All solutions of this ordinary differential equation are of the form $u(x) = C_1 e^x + C_2 e^{-x}$ with arbitrary constants $C_1, C_2 \in \mathbb{C}$. ∎

Theorem 3.31 $u \in \overset{\circ}{H}_1\big((a,b)\big)$ *if and only if* $u \in H_1\big((a,b)\big)$ *and* $u(a) = u(b) = 0$.

Proof. Let $u \in \overset{\circ}{H}_1\big((a,b)\big)$. By definition of $\overset{\circ}{H}_1\big((a,b)\big)$ there is a sequence $\{u_\ell\}_\ell \subseteq \overset{\circ}{C}_\infty\big((a,b)\big)$ with $\|u - u_\ell\|_1 \to 0$ for $\ell \to \infty$. We apply (3.8) to the difference $u - u_\ell$ and note that $u_\ell(a) = 0$ to obtain

$$|u(a)| \le r^{1/2} \|u' - u'_\ell\|_{(a,b)} + r^{-1/2} \|u - u_\ell\|_{(a,b)} \to 0,$$

for $\ell \to \infty$, whence $u(a) = 0$. In the same way we conclude that $u(b) = 0$. To prove the converse let $u \in H_1\big((a,b)\big)$ satisfy $u(a) = u(b) = 0$. By Lemma 3.30 there is a unique $v \in \overset{\circ}{H}_1\big((a,b)\big)$ and $C_1, C_2 \in \mathbb{C}$ such that

$$u(x) = v(x) + C_1 e^x + C_2 e^{-x}.$$

Since $v(a) = v(b) = 0$, we obtain from this equation by setting $x = a$ and $x = b$ that

$$\begin{aligned} C_1 e^a + C_2 e^{-a} &= 0 \\ C_1 e^b + C_2 e^{-b} &= 0. \end{aligned}$$

This is a system of two linear equations for C_1 and C_2 with determinant of the coefficient matrix $e^a e^{-b} - e^b e^{-a} = e^{a-b}(1 - e^{2(b-a)}) \neq 0$, since $b - a > 0$. Consequently we have $C_1 = C_2 = 0$. Thus, $u = v \in \overset{\circ}{H}_1\big((a,b)\big)$. ■

3.7 Weak solutions of the Dirichlet boundary value problem

We begin with the definition of weak solutions of the Helmholtz equation and of weak solutions to the homogeneous Dirichlet boundary value problem to this equation in n–dimensional space:

Definition 3.32 *(i) Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set, let $\lambda \in \mathbb{C}$ and assume that $f \in L^2(\Omega, \mathbb{C})$.*
(i) A function $u \in H_1(\Omega, \mathbb{C})$ is called weak solution of the partial differential equation

$$\Delta u(x) + \lambda u(x) = f(x) \tag{3.10}$$

in Ω, if for all $\varphi \in \overset{\circ}{C}_\infty(\Omega, \mathbb{C})$ the equation

$$-(\nabla u, \nabla \varphi) + \lambda (u, \varphi) = (f, \varphi) \tag{3.11}$$

holds, where

$$(\nabla u, \nabla \varphi) = \int_\Omega \nabla u(x) \cdot \nabla \overline{\varphi(x)}\, dx = \sum_{i=1}^{3} \int_\Omega \frac{\partial}{\partial x_i} u(x) \frac{\partial}{\partial x_i} \overline{\varphi(x)}\, dx.$$

(ii) A weak solution of the Dirichlet boundary value problem with homogeneous boundary data

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= f(x), \quad x \in \Omega, \\ u\big|_{\partial\Omega} &= 0, \end{aligned}$$

is by definition a weak solution u of the partial differential equation (3.10) *belonging to $\overset{\circ}{H}_1(\Omega, \mathbb{C})$.*

Formally the equation (3.11) is obtained by multiplication of both sides of the equation $\Delta u + \lambda u = f$ by φ, integration and application of the first Green's formula. The advantage is that weak solutions need to have only first derivatives and not second.

Every classical solution of the Helmholtz equation is also a weak solution, but not vice versa. However, if a weak solution belongs to $C_2^*(\Omega)$, then it is also a classical solution.

We return to the case $n = 1$ and consider the Dirichlet boundary value problem (3.5), (3.6) with homogeneous boundary data in the bounded open interval $\Omega = (a, b)$. By Definition 3.32, a function u is a weak solution of this boundary value problem if it belongs to the space $\overset{\circ}{H}_1\big((a,b)\big)$ and satisfies

$$-(u', \varphi') + \lambda(u, \varphi) = (f, \varphi)$$

for all $\varphi \in \overset{\circ}{C}_\infty\big((a,b)\big)$.

Theorem 3.33 *Let $\{\lambda_m\}_{m\in\mathbb{N}}$ be the eigenvalues of the Dirichlet boundary value problem in the bounded interval $(a, b) \subseteq \mathbb{R}$ (Definition 3.18), and let $\{u_m\}_{m\in\mathbb{N}}$ be the sequence of eigenfunctions defined in* (3.4)*. Assume that $\lambda \neq \lambda_m$ for all m. Then the Dirichlet boundary value problem*

$$\begin{aligned} u''(x) + \lambda u(x) &= f(x), \qquad x \in [a, b], \\ u(a) = u(b) &= 0, \end{aligned}$$

has a unique weak solution to every $f \in L^2\big((a,b)\big)$, which is given by

$$u = \sum_{m=1}^{\infty} \frac{1}{\lambda - \lambda_m}\,(f, u_m)u_m\,. \tag{3.12}$$

Proof. Let u be given by (3.12). First it must be shown that u belongs to the space $\overset{\circ}{H}_1\big((a,b)\big)$. Since the eigenfunction u_m satisfies $u_m(a) = u_m(b) = 0$, we infer from Theorem 3.31 that $u_m \in \overset{\circ}{H}_1\big((a,b)\big)$. To prove that $u \in \overset{\circ}{H}_1\big((a,b)\big)$ it therefore suffices to show that $\sum_{m=1}^{\infty} \frac{1}{\lambda-\lambda_m}\,(f, u_m)u_m$ converges in the norm of $H_1\big((a,b)\big)$. To this end it suffices to show that $\sum_{m=1}^{\infty} \frac{(f,u_m)}{\lambda-\lambda_m}\,u_m'$ converges in $L^2\big((a,b)\big)$, because we already proved at the end of Section 3.4 that $\sum_{m=1}^{\infty} \frac{(f,u_m)}{\lambda-\lambda_m}\,u_m$ converges in $L^2\big((a,b)\big)$. Using that the scalar product is a sesquilinear form, we compute

$$\Big\| \sum_{m=\ell}^{k} \frac{(f, u_m)}{\lambda - \lambda_m}\,u_m' \Big\|^2 = \Big(\sum_{m=\ell}^{k} \frac{(f, u_m)}{\lambda - \lambda_m}\,u_m',\ \sum_{j=\ell}^{k} \frac{(f, u_j)}{\lambda - \lambda_j}\,u_j' \Big) = \sum_{j,m=\ell}^{k} \frac{(f, u_m)}{\lambda - \lambda_m}\,\frac{\overline{(f, u_j)}}{\overline{\lambda - \lambda_j}}\,(u_m', u_j')\,. \tag{3.13}$$

By Theorem 3.19 the family $\{u_m\}_{m\in\mathbb{N}}$ is a complete orthonormal system in $L^2\big((a,b)\big)$, whence

$$(u_m', u_j') = -(u_m'', u_j) = (\lambda_m u_m, u_j) = \begin{cases} \lambda_m, & m = j \\ 0, & m \neq j\,, \end{cases}$$

With this relation we conclude from (3.13) that

$$\Big\| \sum_{m=\ell}^{k} \frac{(f,u_m)}{\lambda - \lambda_m}\, u_m' \Big\|^2 = \sum_{m=\ell}^{k} \Big| \frac{(f,u_m)}{\lambda - \lambda_m} \Big|^2 \lambda_m \leq C \sum_{m=\ell}^{k} |(f,u_m)|^2,$$

where the constant $C = \sup_{m\in\mathbb{N}} \frac{\lambda_m}{|\lambda - \lambda_m|^2} < \infty$ is independent of k and ℓ. This inequality and Parseval's identity $\sum_{m=1}^{\infty} |(f,u_m)|^2 = \|f\|^2 < \infty$ together imply that the series $\sum_{m=1}^{\infty} \frac{(f,u_m)}{\lambda-\lambda_m}\, u_m'$ satisfies the Cauchy convergence criterion, hence it converges in the complete space $L^2\big((a,b)\big)$. This proves that $u \in \overset{\circ}{H}_1\big((a,b)\big)$ and that

$$u' = \sum_{m=1}^{\infty} \frac{(f,u_m)}{\lambda - \lambda_m}\, u_m'.$$

In the next step of the proof we use this equation. Namely, for $\varphi \in \overset{\circ}{C}_\infty\big((a,b)\big)$ we have

$$\begin{aligned} -(u',\varphi') &= -\sum_{m=1}^{\infty} \frac{(f,u_m)}{\lambda-\lambda_m}\,(u_m',\varphi') = \sum_{m=1}^{\infty} \frac{(f,u_m)}{\lambda-\lambda_m}\,(u_m'',\varphi) \\ &= -\sum_{m=1}^{\infty} \frac{(f,u_m)}{\lambda-\lambda_m}\,(\lambda_m u_m,\varphi) = \sum_{m=1}^{\infty} \frac{(\lambda-\lambda_m)-\lambda}{\lambda-\lambda_m}\,\big((f,u_m)u_m,\varphi\big) \\ &= -\lambda(u,\varphi) + \Big(\sum_{m=1}^{\infty}(f,u_m)u_m,\varphi\Big) = -\lambda(u,\varphi) + (f,\varphi). \end{aligned}$$

In the last step we used that $f = \sum_{m=1}^{\infty}(f,u_m)u_m$, which follows from the completeness of the system $\{u_m\}_{m\in\mathbb{N}}$. Consequently, u is a weak solution.

It remains to show that u is the only weak solution. Assume that $v \in \overset{\circ}{H}_1\big((a,b)\big)$ is a second weak solution. Then for every $\varphi \in \overset{\circ}{C}_\infty\big((a,b)\big)$

$$-(u'-v',\varphi') + \lambda(u-v,\varphi) = (f-f,\varphi) = 0.$$

Since every eigenfunction u_m belongs to $\overset{\circ}{H}_1\big((a,b)\big)$, we can choose a sequence $\{\varphi_k\}_k \subseteq \overset{\circ}{C}_\infty\big((a,b)\big)$ such that $\|u_m - \varphi_k\|_1 \to 0$ for $k \to \infty$, by definition of $\overset{\circ}{H}_1\big((a,b)\big)$, and obtain from this equation and from the continuity of the scalar product that

$$-(u'-v',u_m') + \lambda(u-v,u_m) = \lim_{k\to\infty}\big[-(u'-v',\varphi_k') + \lambda(u-v,\varphi_k)\big] = 0.$$

Since $u(a) = v(a) = u(b) = v(b) = 0$, we obtain from the partial integration formula (3.9) that

$$\lambda(u - v, u_m) = (u' - v', u_m') = -(u - v, u_m'') = (u - v, \lambda_m u_m) = \lambda_m (u - v, u_m)\,.$$

Since by assumption $\lambda \neq \lambda_m$ it follows from this equation that $(u - v, u_m) = 0$ for all m. Because the orthonormal system $\{u_m\}_m$ is complete, we infer from Theorem 3.11 that $u - v = 0$, whence $u = v$. ∎

For boundary value problems to the Helmholtz equation $\Delta u + \lambda u = f$ in higher dimensions a result holds, which is completely analogous to the result for the boundary value problem to the ordinary differential equation $u'' + \lambda u = f$ discussed here. This will be shown in Sections 8 and 9.

4 Boundary value problems on circular domains

4.1 The Laplace operator in polar coordinates and Fourier expansions

In this section we assume that Ω is a circular ring domain

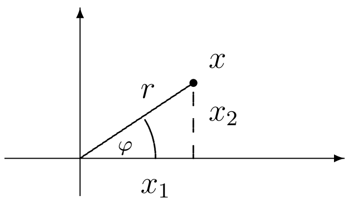
$$\Omega = \{x \in \mathbb{R}^2 \mid R_1 < |x| < R_2\}, \qquad 0 \le R_1 < R_2 \le \infty,$$

or a ball

$$\Omega = \{x \in \mathbb{R}^2 \mid |x| < R\}.$$

To find solutions of $\Delta u(x) + \lambda u(x) = 0$ in such circular domains Ω we use polar coordinates (r, φ) in $\mathbb{R}^2$ and apply separation of variables. To this end we first determine the form of the Laplace operator in polar coordinates. Thus, let $x = (x_1, x_2)$ and

$$r = r(x) = \sqrt{x_1^2 + x_2^2} = |x|,$$
$$\varphi = \varphi(x) = \arctan \frac{x_2}{x_1}.$$



Since for $i = 1, 2$,

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{|x|}, \qquad \frac{\partial \varphi}{\partial x_1} = -\frac{1}{1 + (\frac{x_2}{x_1})^2} \frac{x_2}{x_1^2} = -\frac{x_2}{|x|^2}, \qquad \frac{\partial \varphi}{\partial x_2} = \frac{1}{1 + (\frac{x_2}{x_1})^2} \frac{1}{x_1} = \frac{x_1}{|x|^2},$$

we have

$$\frac{\partial}{\partial x_1} = \frac{\partial r}{\partial x_1}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial x_1}\frac{\partial}{\partial \varphi} = \frac{x_1}{|x|}\frac{\partial}{\partial r} - \frac{x_2}{|x|^2}\frac{\partial}{\partial \varphi}, \qquad \frac{\partial}{\partial x_2} = \frac{x_2}{|x|}\frac{\partial}{\partial r} + \frac{x_1}{|x|^2}\frac{\partial}{\partial \varphi}, \tag{4.1}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial x_1^2} &= \frac{\partial}{\partial x_1}\Big(\frac{x_1}{|x|}\frac{\partial}{\partial r} - \frac{x_2}{|x|^2}\frac{\partial}{\partial \varphi}\Big) \\ &= \Big(\frac{1}{|x|} - \frac{x_1^2}{|x|^3}\Big)\frac{\partial}{\partial r} + \frac{x_1}{|x|}\Big(\frac{x_1}{|x|}\frac{\partial^2}{\partial r^2} - \frac{x_2}{|x|^2}\frac{\partial^2}{\partial r \partial \varphi}\Big) + 2\frac{x_1 x_2}{|x|^4}\frac{\partial}{\partial \varphi} - \frac{x_2}{|x|^2}\Big(\frac{x_1}{|x|}\frac{\partial^2}{\partial \varphi \partial r} - \frac{x_2}{|x|^2}\frac{\partial^2}{\partial \varphi^2}\Big) \\ &= \frac{x_1^2}{|x|^2}\frac{\partial^2}{\partial r^2} + \frac{x_2^2}{|x|^3}\frac{\partial}{\partial r} + 2\frac{x_1 x_2}{|x|^4}\frac{\partial}{\partial \varphi} - 2\frac{x_1 x_2}{|x|^3}\frac{\partial^2}{\partial \varphi \partial r} + \frac{x_2^2}{|x|^4}\frac{\partial^2}{\partial \varphi^2},\end{aligned}$$

as well as

$$\frac{\partial^2}{\partial x_2^2} = \frac{x_2^2}{|x|^2}\frac{\partial^2}{\partial r^2} + \frac{x_1^2}{|x|^3}\frac{\partial}{\partial r} - 2\frac{x_1 x_2}{|x|^4}\frac{\partial}{\partial \varphi} + 2\frac{x_1 x_2}{|x|^3}\frac{\partial^2}{\partial \varphi \partial r} + \frac{x_1^2}{|x|^4}\frac{\partial^2}{\partial \varphi^2}.$$

Thus, if $u(x) = \tilde{u}(r(x), \varphi(x))$ then

$$\begin{aligned}\Delta_x u(x) &= \sum_{i=1}^{2} \frac{\partial^2}{\partial x_i^2}\,\tilde{u}(r(x),\varphi(x)) \\ &= \frac{x_1^2+x_2^2}{|x|^2}\,\frac{\partial^2}{\partial r^2}\,\tilde{u}(r(x),\varphi(x)) + \frac{x_1^2+x_2^2}{|x|^3}\,\frac{\partial}{\partial r}\tilde{u}(r(x),\varphi(x)) + \frac{x_2^2+x_1^2}{|x|^4}\,\frac{\partial^2}{\partial \varphi^2}\tilde{u}(r(x),\varphi(x)) \\ &= \frac{\partial^2}{\partial r^2}\tilde{u}(r(x),\varphi(x)) + \frac{1}{r(x)}\,\frac{\partial}{\partial r}\tilde{u}(r(x),\varphi(x)) + \frac{1}{r(x)^2}\,\frac{\partial^2}{\partial \varphi^2}\,\tilde{u}(r(x),\varphi(x)).\end{aligned}$$

Consequently, the representation of the Laplace operator in polar coordinates is

$$\Delta_{(r,\varphi)} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial \varphi^2}\,.$$

We next expand a two-times continuously differentiable solution u of $\Delta u + \lambda u = 0$ in a Fourier series with respect to φ on every circle $|x| = r$ with $R_1 < r < R_2$. As usual, we drop the tilde and write $u(x) = u(r, \varphi)$. By Remark 3.14 we have

$$u(r,\varphi) = \sum_{m=-\infty}^{\infty} u_m(r) e^{im\varphi}, \tag{4.2}$$

with

$$u_m(r) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\varphi) e^{-im\varphi} d\varphi.$$

If we can interchange partial derivatives up to order 2 with the summation sign, we obtain

$$\begin{aligned}0 = (\Delta+\lambda)u(x) &= \Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\Big) u(r,\varphi) + \lambda u(r,\varphi) \\ &= \sum_{m=-\infty}^{\infty} \Big[\Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r}\Big) u_m(r) + \Big(\lambda - \frac{m^2}{r^2}\Big) u_m(r)\Big] e^{im\varphi}.\end{aligned}$$

Fix r. Sinc the Fourier expansion is unique, the Fourier series vanishes identically for all $0 \le \varphi < 2\pi$ only if all coefficients vanish. It follows that u_m must satisfy the equation

$$\frac{d^2}{dr^2}\,u_m(r) + \frac{1}{r}\,\frac{d}{dr}\,u_m(r) + \Big(\lambda - \frac{m^2}{r^2}\Big) u_m(r) = 0, \tag{4.3}$$

for all $R_1 < r < R_2$ and all $m \in \mathbb{Z}$. This is a linear ordinary differential equation of second order, which we use in the next sections to determine the coefficient functions u_m in the expansion (4.2) of solutions of the potential equation or the Helmholtz equation.

4.2 Solution of the potential equation in circular domains

We first consider the case $\lambda = 0$. In this case the differential equation (4.3) becomes

$$\frac{d^2}{dr^2}\, u_m(r) + \frac{1}{r}\,\frac{d}{dr}\, u_m(r) - \frac{m^2}{r^2} u_m(r) = 0,$$

whose general solution is

$$\begin{aligned} u_0(r) &= C_{01} + C_{02} \ln r \\ u_m(r) &= C_{m1} r^m + C_{m2} r^{-m}, \quad m \neq 0. \end{aligned}$$

The general solution of the potential equation $\Delta u(x) = 0$ in a circular domain $\Omega = \{x \in \mathbb{R}^2 \mid R_1 < x < R_2\}$ is therefore

$$u(x) = u(r,\varphi) = C_{01} + C_{02} \ln r + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} (C_{m1} r^m + C_{m2} r^{-m}) e^{im\varphi},$$

with arbitrary constants $C_{m1}, C_{m2} \in \mathbb{C}$. These coefficients must be determined from boundary conditions and possibly from conditions at infinity (radiation conditions).

Example 4.1 Let $\Omega = \{x \in \mathbb{R}^2 \mid |x| < R\}$ be a ball with center at 0. Consider the Dirichlet boundary value problem

$$\begin{aligned} \Delta u(x) = 0, &\qquad x \in \Omega, \\ u(x) = \mathsf{u}(x), &\qquad x \in \partial\Omega. \end{aligned}$$

We want to find a solution $u \in C^2(\Omega)$. This requires that

$$C_{m1} r^m + C_{m2} r^{-m} = u_m(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r,\varphi) e^{-im\varphi}\, d\varphi$$

and

$$C_{01} + C_{02} \ln r = u_0(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r,\varphi) d\varphi$$

must be bounded at $x = 0$, hence $C_{m2} = 0$ for $m \geq 0$ and $C_{m1} = 0$ for $m < 0$. Thus

$$u(r,\varphi) = C_{01} + \sum_{m=1}^{\infty} r^m (C_{m1} e^{im\varphi} + C_{-m2} e^{-im\varphi}).$$

The Fourier series expansion of the boundary data is

$$\mathsf{u}(\varphi) = \sum_{m=-\infty}^{\infty} a_m e^{im\varphi}.$$

From the boundary condition we obtain

$$u(R,\varphi) = C_{01} + \sum_{m=1}^{\infty} R^m (C_{m1} e^{im\varphi} + C_{-m2} e^{-im\varphi}) = \mathsf{u}(\varphi) = \sum_{m=-\infty}^{\infty} a_m e^{im\varphi}.$$

The uniqueness of the Fourier expansion implies that the coefficients in both series must coincide. We thus infer that

$$a_m = \begin{cases} R^m C_{m1}, & m \geq 0, \\ R^{|m|} C_{m2}, & m < 0. \end{cases}$$

We solve these equations for C_{m1} and C_{m2}. If the series thus constructed converges and can be differentiated twice under the summation sign, then the sum u is a solution of the boundary value problem. This holds in fact true. We formulate the result as a theorem.

Theorem 4.2 *Let* $\Omega = B_R(0)$*, let* $\mathsf{u} \in L^2(\partial\Omega, \mathbb{C})$ *and set*

$$a_m = \frac{1}{2\pi} \int_0^{2\pi} \mathsf{u}(\varphi) e^{-im\varphi} d\varphi.$$

Then

$$u(x) = u(r,\varphi) = \sum_{m=-\infty}^{\infty} a_m \left(\frac{r}{R}\right)^{|m|} e^{im\varphi} \tag{4.4}$$

is the unique solution $u \in C_\infty(\Omega)$ *of*

$$\Delta u(x) = 0, \qquad x \in \Omega,$$
$$L^2 - \lim_{r\to R} u(r,\cdot) = \mathsf{u},$$

where the limit is understood in the L^2*-sense:*

$$\lim_{r\to R} \|u(r,\cdot) - \mathsf{u}(\cdot)\|_{[0,2\pi]} = 0. \tag{4.5}$$

The **proof** of this theorem is postponed to the end of Section 4, where we prove it together with a corresponding result for boundary value problems to the Helmholtz equation with $\lambda \neq 0$, which we derive later.

Example 4.3 Let $\Omega = \{x \in \mathbb{R}^2 \mid |x| > R\}$ be an exterior domain. We want to find a solution of the Dirichlet boundary value problem

$$\Delta u(x) = 0, \qquad x \in \Omega,$$
$$u(x) = \mathsf{u}(x), \qquad x \in \partial\Omega.$$

In this case we cannot conclude that half of the coefficients in the expansion

$$u(x) = C_{01} + C_{02} \ln r + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} (C_{m1} r^m + C_{m2} r^{-m}) e^{im\varphi}$$

must vanish, and the boundary condition is not enough to determine all coefficients uniquely. Therefore the solution of the problem is not unique. To get a unique solution one must pose suitable conditions for the asymptotic behavior of u at infinity. Normally one requires that the solution is bounded:

$$|u(x)| \leq C, \quad x \in \Omega.$$

As above we then obtain $C_{m2} = 0$ for all $m \leq 0$ and $C_{m1} = 0$ for $m > 0$. Thus

$$u(r, \varphi) = C_{01} + \sum_{m=1}^{\infty} r^{-m} (C_{m2} e^{im\varphi} + C_{-m1} e^{-im\varphi}).$$

By the boundary condition we must have

$$u(R, \varphi) = C_{01} + \sum_{m=1}^{\infty} R^{-m} (C_{m2} e^{im\varphi} + C_{-m1} e^{-im\varphi}) = u(\varphi) = \sum_{m=-\infty}^{\infty} a_m e^{im\varphi},$$

for all $\varphi \in [0, 2\pi]$. Because of the uniqueness of the Fourier expansion, both series in this equation must coincide. This yields

$$u(x) = u(r, \varphi) = \sum_{m=-\infty}^{\infty} a_m \left(\frac{r}{R}\right)^{-|m|} e^{im\varphi}.$$

4.3 Solution of the Helmholtz equation in circular domains

We study now the solution of the differential equation (4.3) for $\lambda \neq 0$. By a short computation it follows that the function $u_m(r)$ satisfies the equation (4.3) for $\lambda \neq 0$ if and only if the function $v(r) = u_m(\frac{r}{\sqrt{\lambda}})$ solves the differential equation

$$\frac{d^2}{dr^2} v(r) + \frac{1}{r} \frac{d}{dr} v(r) + \Big(1 - \frac{m^2}{r^2}\Big) v(r) = 0. \tag{4.6}$$

Comparison with equation (A.1) in the appendix shows that this is the Bessel differential equation with integer order $\nu = m$. Since the general solution of (A.1) is given by (A.4), we conclude that for $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ and for $m \in \mathbb{Z}$ the solution u_m of (4.3) can be expressed as the linear combination

$$u_m(r) = C_1 J_m(\sqrt{\lambda} r) + C_2 N_m(\sqrt{\lambda} r),$$

where J_m is the Bessel function and N_m is the Neumann function of integer order. These functions are discussed in Appendix A.

From (4.2) we thus conclude that a solution of the Helmholtz equation $\Delta u + \lambda u = 0$ in circular domains must be of the form

$$u(x) = u(r, \varphi) = \sum_{m=-\infty}^{\infty} \Big(C_{m1} J_m(\sqrt{\lambda} r) + C_{m2} N_m(\sqrt{\lambda} r) \Big) e^{im\varphi}. \tag{4.7}$$

The constants C_{m1}, C_{m2} must be determined from the boundary and radiation conditions.

Example 4.4 Let $\Omega = B_R(0)$ be a ball, let $\lambda \in \mathbb{C}$, $\lambda \neq 0$ and assume that $\mathsf{u} \in L^2(\partial\Omega)$. We want to solve

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, \qquad x \in \Omega \\ u(x) &= \mathsf{u}(x), \qquad x \in \partial\Omega. \end{aligned}$$

Since u must be two times continuously differentiable at $x = 0$, it follows that in the expansion (4.7) of u we must have $C_{m2} = 0$ for all $m \in \mathbb{Z}$, since J_m is regular and N_m is singular at $r = 0$. Thus,

$$u(r, \varphi) = \sum_{m=-\infty}^{\infty} C_{m1} J_m(\sqrt{\lambda} r) e^{im\varphi}.$$

Let

$$\mathsf{u}(\varphi) = \sum_{m=-\infty}^{\infty} a_m e^{im\varphi} \tag{4.8}$$

be the Fourier series of u. The boundary condition requires

$$u(R, \varphi) = \sum_{m=-\infty}^{\infty} C_{m1} J_m(\sqrt{\lambda} R) e^{im\varphi} = \mathsf{u}(\varphi) = \sum_{m=-\infty}^{\infty} a_m e^{im\varphi}.$$

The uniqueness of the Fourier expansion therefore implies that the equation $a_m = C_{m1} J_m(\sqrt{\lambda} R)$ must hold. This equation can only be solved for arbitrary $a_m \in \mathbb{C}$, if $J_m(\sqrt{\lambda} R) \neq 0$. In this case we have

$$C_{m1} = \frac{a_m}{J_m(\sqrt{\lambda} R)}.$$

We thus see that for $\lambda \in \mathbb{C} \setminus \{0\}$

$$\Lambda_\lambda = \{ m \in \mathbb{Z} \mid J_m(\sqrt{\lambda} R) = 0 \} \tag{4.9}$$

is an exceptional set of indices.

Theorem 4.5 *Let* $\Omega = B_R(0) \subseteq \mathbb{R}^2$*, suppose that* $\lambda \in \mathbb{C} \setminus \{0\}$*, and let* $\{a_m\}_{m\in\mathbb{Z}}$ *be the coefficients of the Fourier series* (4.8) *of* $\mathsf{u} \in L^2([0, 2\pi])$*.*

(i) *If* $\Lambda_\lambda = \emptyset$*, then the Dirichlet boundary value problem*

$$\Delta u(x) + \lambda u(x) = 0, \qquad x \in B_R(0) \tag{4.10}$$

$$L^2 - \lim_{r\to R} u(r, \cdot) = \mathsf{u}, \tag{4.11}$$

has a unique solution $u \in C_2(B_R(0))$ *for all* $\mathsf{u} \in L^2(\partial B_R(0))$*. This solution is given by*

$$u(x) = u(r, \varphi) = \sum_{m=-\infty}^{\infty} \frac{a_m}{J_m(\sqrt{\lambda}R)} J_m(\sqrt{\lambda}r)e^{im\varphi}. \tag{4.12}$$

In particular, the only solution to homogeneous boundary data $\mathsf{u} = 0$ *is* $u = 0$*.*

(ii) *If* $\Lambda_\lambda \neq \emptyset$*, then the boundary value problem* (4.10)*,* (4.11) *is only solvable if in the Fourier expansion* (4.8) *of the boundary data we have* $a_m = 0$ *for all* $m \in \Lambda_\lambda$*. On the other hand, if* u *is a solution of the homogeneous boundary value problem* ($\mathsf{u} = 0$)*, then the Fourier expansion is of the form*

$$u(r, \varphi) = \sum_{m\in\Lambda_\lambda} C_m J_m(\sqrt{\lambda}r)e^{im\varphi}. \tag{4.13}$$

Moreover, any nonzero function with such a Fourier expansion, which converges, is a solution of the homogeneous boundary value problem. This holds in particular, if only finitely many C_m *differ from zero. Hence* λ *is an eigenvalue of the boundary value problem* (4.10)*,* (4.11) *if and only if* $\Lambda_\lambda \neq \emptyset$*.*

Statement (ii) in this theorem follows immediately from the construction of the series expansion of the solution given above. The **proof** of statement (i) is postponed to the end of Section 4, where we prove it together with Theorem 4.2. Before we give this proof, we discuss next the set of eigenvalues of the Dirichlet boundary value problem (4.10), (4.11).

If u_1 and u_2 are eigenfunctions to the eigenvalue λ of the Dirichlet problem then also the linear combination $C_1u_1 + C_2u_2$ is an eigenfunction, if this function is not zero. Therefore the set of eigenfunctions together with the zero function forms a vector space V_λ, the eigenspace of λ.

Definition 4.6 *The set* Σ *of eigenvalues is called the spectrum of the boundary value problem* (4.10)*,* (4.11)*.*

For $\lambda \in \Sigma$ *the vector space of eigenfunctions is called the eigenspace to the eigenvalue* λ*. The dimension of the eigenspace* V_λ *is called the geometric multiplicity* d_λ *of* λ*.*

Let $R > 0$ denote the radius of the ball $B_R(0)$, on which the boundary value problem (4.10), (4.11) is defined. For $m \in \mathbb{Z}$ we set

$$N_m(R) = \left\{ \left(\frac{y}{R}\right)^2 \,\middle|\, y \text{ is a non-vanishing zero of } J_m \right\}.$$

We have $N_{-m}(R) = N_m(R)$ for all $m \in \mathbb{Z}$, since by equation (A.3) from the appendix the Bessel function J_{-m} is equal to J_m, hence the sets of zeros coincide.

If M is a set, we denote by $\#M$ the cardinality of M. If M is finite, this is the number of elements of M.

Theorem 4.7 *Let Σ be the spectrum of the boundary value problem* (4.10), (4.11) *on the domain $\Omega = B_R(0)$.*

(i) *The spectrum satisfies*

$$\Sigma = \bigcup_{m \in \mathbb{Z}} N_m(R).$$

(ii) *Σ is contained in the positive real axis and does not have an accumulation point. Hence it is a countable set.*

(iii) *The geometric multiplicity d_λ of every eigenvalue $\lambda \in \Sigma$ is finite and satisfies*

$$2 \le d_\lambda = \#\Lambda_\lambda = \#\{m \in \mathbb{Z} \mid \lambda \in N_m(R)\} < 2R\sqrt{\lambda} + 1. \tag{4.14}$$

Proof: From the definition of $N_m(R)$ it follows that $\lambda \in N_m(R)$, if and only if $\sqrt{\lambda}R$ is a non-vanishing zero of J_m. This holds irrespective of the choice of the branch of the square root $\sqrt{\lambda}$, since by Lemma A.1 with y also $-y$ is a zero of J_m. By definition of the set Λ_λ in (4.9) we thus have

$$\Lambda_\lambda = \{m \in \mathbb{Z} \mid \lambda \in N_m(R)\}. \tag{4.15}$$

Consequently, the relation $\lambda \in \bigcup_{m \in \mathbb{Z}} N_m(R)$ holds, if and only if $\Lambda_\lambda \neq \emptyset$, and by statement (ii) of Theorem 4.5, this last relation holds if and only if $\lambda \in \Sigma \setminus \{0\}$. By Theorem 4.2, the unique solution of the homogeneous Dirichlet problem to the potential equation is $u = 0$, hence $\lambda = 0$ is not an eigenvalue. Therefore λ belongs to the set $\bigcup_{m \in \mathbb{Z}} N_m(R)$, if and only if $\lambda \in \Sigma$. This proves (i).

By Lemma A.1 in the appendix, every zero y of J_m is real and satisfies $y^2 > m^2$, which implies that $N_m(R)$ is contained in the interval $(\frac{m^2}{R^2}, \infty)$. For every positive number b we thus have $\Sigma \cap [0, b] \subseteq \bigcup_{m=-m_b}^{m_b} \big(N_m(R) \cap [0, b]\big)$, where m_b is the largest integer such that $\frac{m_b^2}{R^2} < b$. By Lemma A.1, the set of zeros of J_m does not have an accumulation point, hence $N_m(R)$ does not have an accumulation point. Thus, the set $N_m(R) \cap [0, b]$ is finite and therefore also the finite union $\bigcup_{m=-m_b}^{m_b}\big(N_m(R)\cap[0,b]\big)$. Consequently, $\Sigma\cap[0,b]$ contains only finitely many points for every $b>0$, and therefore Σ does not have accumulation points. This proves (ii).

Since $\lambda\in\mathbb{R}$ does not belong to $N_m(R)$ if $m^2\geq R^2\lambda$, it follows from this inequality and from(4.15) that

$$\#\Lambda_\lambda=\#\{m\in\mathbb{Z}\mid\lambda\in N_m(R)\}<2R\sqrt{\lambda}+1. \tag{4.16}$$

Therefore Λ_λ is a finite set, which by statement (ii) of Theorem 4.5 implies that all eigenfunctions to the eigenvalue λ have the form of (4.13). This means that the eigenspace V_λ is spanned by the functions from the set $\big\{J_m(\sqrt{\lambda}r)e^{im\varphi}\ \big|\ m\in\Lambda_\lambda\big\}$. This set is linearly independent, hence it is a basis of V_λ, whence by (4.16) the geometric multiplicity of λ satisfies $d_\lambda=\#\Lambda_\lambda<2R\sqrt{\lambda}+1$. Furthermore, from $N_m(R)=N_{-m}(R)$, it follows that if Λ_λ is not empty, then it contains at least two elements, hence $d_\lambda\geq 2$. This proves (iii).

■

Remark. It is well known that every Bessel function J_m has infinitely many positive zeros, which by Theorem 4.7 means that the spectrum Σ is a non-empty, countably infinite set. However, we do not study the spectrum here more precisely, because it will be shown later in Sections 8 and 9 for boundary value problems to general linear elliptic differential operators in general bounded domains $\Omega\subseteq\mathbb{R}^n$ that there exist countably infinitely many eigenvalues and that one can choose a complete orthonormal system in $L^2(\Omega,\mathbb{C})$ consisting of eigenfunctions. This general result also applies to boundary value problems for the Helmholtz equation in circular domains. The situation is therefore completely analogous to the situation in one space dimension.

Note however, that in one space dimension all eigenvalues of the Dirichlet problem for the vibrating string, which are the physical eigenfrequencies of the string, are integer multiples of the lowest eigenvalue, whereas the zeros of the Bessel functions, which determine the eigenvalues of the boundary value problems for the circular vibrating membrane, are not ordered in this harmonic way. This has important consequences for musical instruments. Because of these properties a drum can not be used in the same way as a violine.

Moreover, in one space dimension all eigenvalues of the Dirichlet problem are of multiplicity one, whereas in two space dimensions by (4.14) every eigenvalue has at least multiplicity two. This property results from the radial symmetry of the circular disc, which implies that eigenfunctions can be rotated by any angle and still are eigenfunctions. Therefore the shape of the domain reflects itself in the properties of the spectrum. Because of this relation between the multplicity and the symmetry properties of the disk, one can surmise that every eigenvalue has exactly multiplicity two. By (4.14) and by definition of Λ_λ in (4.9) the multiplicity of the eigenvalue λ would be greater than two, if there exist Bessel functions J_n and J_m with $|n| \neq |m|$, for both of which $\sqrt{\lambda}R$ is a zero. It was hypothezised in the 19th-century by the French Mathematician Bourget that if n, m are integers with $|n| \neq |m|$ then the Bessel functions J_n and J_m do not have common zeros other than 0. This hypothesis was proved in 1929 by the German mathematician Carl Ludwig Siegel. This result implies that indeed $d_\lambda = 2$ for every eigenvalue $\lambda \in \Sigma$. The estimate (4.14) is therefore not sharp.

For boundary value problems in general domains one can ask, whether from the knowledge of the spectrum the shape of the domain can be determined. This question is of great interest in mathematics and in applications of physics and engineering. It has be investigated in various situations.

Proof of Theorem 4.2 and Theorem 4.5 (i) We prove these theorems together, since both proofs are based on the same arguments. However, the proof of statement (i) in Theorem 4.5 is only given for real values of λ, since estimates for the Bessel functions are needed, which are derived in the appendix, however only for $\lambda \in \mathbb{R}$. The derivation of these estimates for complex λ involves technicalities, which we avoid.

We write the series representations (4.4) and (4.12) in the unified form

$$u(r, \varphi) = \sum_{m=-\infty}^{\infty} a_m U_m(\lambda, r) e^{im\varphi}, \tag{4.17}$$

where

$$U_m(\lambda, r) = \begin{cases} \left(\frac{r}{R}\right)^{|m|}, & \text{for } \lambda = 0, \\ \frac{J_m(\sqrt{\lambda}r)}{J_m(\sqrt{\lambda}R)}, & \text{for } \lambda \in \mathbb{R} \setminus (\Sigma \cup \{0\}). \end{cases} \tag{4.18}$$

To prove that the series (4.17) defines a solution of the partial differential equation (4.10) for all values $\lambda \in \mathbb{R}$, we need to show that it converges and can be differentiated twice under the summation sign. To verify this, we need a lemma, which we state and prove first.

Lemma 4.8 *Let $R > 0$ and $\lambda \in \mathbb{R} \setminus \Sigma$. Assume that $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$ is a multi-index with $|\alpha| = \alpha_1 + \alpha_2 \leq 2$. Set*

$$\varepsilon_\lambda = \begin{cases} 1, & \text{if } \lambda \leq 0, \\ \frac{1}{2}, & \text{if } \lambda > 0, \end{cases} \qquad \text{and} \qquad m_\lambda = \begin{cases} \max\{\alpha_1, 1\}, & \text{if } \lambda \leq 0, \\ \max\left\{2\alpha_1, \sqrt{\frac{4}{3}\lambda}\, R\right\}, & \text{if } \lambda > 0. \end{cases}$$

Then for all $\rho > 1$ there is a constant $C = C(\alpha, \rho)$ such that for all $m \in \mathbb{Z}$ with $|m| \geq m_\lambda$, all $0 < r \leq R$ and all $0 \leq \varphi < 2\pi$

$$|\partial_r^{\alpha_1} \partial_\varphi^{\alpha_2} U_m(\lambda, r) e^{im\varphi}| \leq K \frac{|m|^{|\alpha|}}{r^{\alpha_1}} \Big(\frac{r}{R}\Big)^{\varepsilon_\lambda |m|} \leq KC \frac{\rho^{|\alpha||m|}}{r^{\alpha_1}} \Big(\frac{r}{R}\Big)^{\varepsilon_\lambda |m|}. \tag{4.19}$$

with $K = \max(K_1, K_2, 2)$, where K_1 and K_2 are the constants given in Lemma A.3.

Proof of Lemma 4.8: Consider first the case $\lambda = 0$. For $|m| \geq m_\lambda$, which implies $|m| - \alpha_1 \geq 0$, we obtain from the definition of U in (4.18) that

$$\begin{aligned} |\partial_r^{\alpha_1} \partial_\varphi^{\alpha_2} U_m(0, r) e^{im\varphi}| &= |m|^{\alpha_2} \frac{|m|!}{(|m| - \alpha_1)!} \frac{1}{R^{\alpha_1}} \Big(\frac{r}{R}\Big)^{|m| - \alpha_1} \\ &\leq \frac{|m|^{\alpha_2} |m|^{\alpha_1}}{R^{\alpha_1}} \Big(\frac{r}{R}\Big)^{|m| - \alpha_1} = \frac{|m|^{|\alpha|}}{r^{\alpha_1}} \Big(\frac{r}{R}\Big)^{|m|}. \end{aligned} \tag{4.20}$$

Now consider the case $\lambda \in \mathbb{R} \setminus \{0\}$. If $|m| \geq m_\lambda$, then we obtain from the definition of m_λ for positive or negative λ that $\varepsilon_\lambda |m| - \alpha_1 \geq 0$, and for $\lambda > 0$ we obtain in addition that $|m| \geq \sqrt{\frac{4}{3}\lambda}\, R$. Therefore the assumptions of Lemma A.3 in the appendix are satisfied if we set $j = \alpha_1$ and choose $|m| \geq m_\lambda$, hence the estimates (A.11) – (A.14) together yield for U defined in (4.18) that

$$|\partial_r^{\alpha_1} \partial_\varphi^{\alpha_2} U_m(\lambda, r) e^{im\varphi}| \leq |m|^{\alpha_2} \Big(\frac{r}{R}\Big)^{\varepsilon_\lambda |m| - \alpha_1} \Big(\frac{|m|}{R}\Big)^{\alpha_1} K = \frac{K}{r^{\alpha_1}} |m|^{|\alpha|} \Big(\frac{r}{R}\Big)^{\varepsilon_\lambda |m|}. \tag{4.21}$$

The estimates (4.20) and (4.21) show that the first inequality sign in (4.19) is valid.

The second inequality sign follows from a standard estimate, which we derive here for completness: Since ln is a concave function, which for $z > 0$ satisfies $\ln' z = \frac{1}{z}$, it follows for all $y > 0$ that $\ln y \leq \frac{1}{z}(y - z) + \ln z$, hence $y \leq e^{\frac{1}{z} y - 1 + \ln z} = \frac{z}{e} e^{\frac{1}{z} y}$. For $\rho > 1$ and $|m| \in \mathbb{N}$ set $\frac{1}{z} = \ln \rho$ and $y = |m|$. This yields $|m| \leq \frac{1}{e \ln \rho} \rho^{|m|}$. We estimate $|m|^{|\alpha|}$ in the middle term of (4.19) with this inequality, from which the second inequality sign in (4.19) results with $C = (e \ln \rho)^{-|\alpha|}$. ■

End of the proof of Theorems 4.2 and 4.5. For the coefficients $\{a_m\}_{m \in \mathbb{Z}}$ of the Fourier series (4.8) of $\mathsf{u} \in L^2([0, 2\pi])$ we have by Remark 3.14 that $2\pi \sum_{m=-\infty}^{\infty} |a_m|^2 = \|\mathsf{u}\|_{[0,2\pi]}^2$, whence $|a_m| \leq \|\mathsf{u}\|_{[0,2\pi]} / \sqrt{2\pi}$, for all $m \in \mathbb{Z}$.

Let r_1 be any number satisfying $0 < r_1 < R$. With ε_λ defined in Lemma 4.8 and with $|\alpha| \leq 2$ choose $\rho > 1$ such that $q = \rho^{|\alpha|} \big(\frac{r_1}{R}\big)^{\varepsilon_\lambda} < 1$. Let $|m| \geq m_\lambda$. By definition of m_λ we have $\varepsilon_\lambda |m| - \alpha_1 \geq 0$, which yields for all $0 \leq r \leq r_1$ that $r^{\varepsilon_\lambda |m| - \alpha_1} \leq r_1^{\varepsilon_\lambda |m| - \alpha_1}$. Therefore

(4.19) yields

$$|a_m\,\partial_r^{\alpha_1}\partial_\varphi^{\alpha_2}\,U_m(\lambda,r)e^{im\varphi}| \le |a_m|KC\frac{\rho^{|\alpha||m|}}{r^{\alpha_1}}\Big(\frac{r}{R}\Big)^{\varepsilon_\lambda|m|}$$
$$\le |a_m|\frac{KC}{r_1^{\alpha_1}}\Big(\rho^{|\alpha|}\Big(\frac{r_1}{R}\Big)^{\varepsilon_\lambda}\Big)^{|m|} \le \frac{\|\mathsf{u}\|_{[0,2\pi]}}{\sqrt{2\pi}}\frac{KC}{r_1^{\alpha_1}}q^{|m|},$$

for all $m \in \mathbb{Z}$ with $|m| \ge m_\lambda$ and for all $0 \le r \le r_1$ and $0 \le \varphi < 2\pi$. This estimate implies that the series of derivatives $\sum_{m=-\infty}^{\infty}\partial_r^{\alpha_1}\partial_\varphi^{\alpha_2}\,a_m U_m(\lambda,r)e^{im\varphi}$ converges uniformly in every closed ball $\overline{B_{r_1}(0)}$ with $r_1 < R$. By a standard result from calculus we thus know that $u : B_R(0) \to \mathbb{C}$ defined in (4.17) is a two times continuously differentiable function, and that the derivatives of this function can be computed by differentiation under the summation sign. Since $U_m(\lambda,r)e^{im\varphi}$ is a solution of the Helmholtz equation, it thus follows that $(\Delta+\lambda)u(r,\varphi) = \sum_{m=-\infty}^{\infty}(\Delta+\lambda)\,a_m U_m(\lambda,r)e^{im\varphi} = 0$. Therefore the partial differential equation (4.10) is satisfied, and it remains to verify the boundary condition (4.11).

To this end let $\varepsilon > 0$ and choose the number $m_0 \in \mathbb{N}$ large enough such that $\sum_{|m|\ge m_0}|a_m|^2 < \varepsilon$, and, if $\lambda > 0$, such that $m_0 \ge \sqrt{\frac{4}{3}\lambda}\,R$. The definition of U_m in (4.18) and the inequalities (A.11) and (A.13) in Lemma A.3 then imply for all $\lambda \in \mathbb{R}\setminus\Sigma$, all $m \in \mathbb{Z}$ with $|m| \ge m_0$, and all $0 \le r \le R$ that $0 \le U_m(\lambda,r) \le \big(\frac{r}{R}\big)^{\frac{1}{2}|m|}$, whence

$$0 \le (1 - U_m(\lambda,r)) \le 1.$$

Since $\lim_{r\to R}U_m(\lambda,r) = 1$, for all $m \in \mathbb{Z}$, it thus follows that

$$\lim_{r\to R}\|u(r,\cdot)-\mathsf{u}\|_{[0,2\pi]}^2 = \lim_{r\to R}\int_0^{2\pi}\Big|\sum_{m=-\infty}^{\infty}a_m(U_m(\lambda,r)-1)e^{im\varphi}\Big|^2\,d\varphi$$
$$= \lim_{r\to R}2\pi\sum_{m=-\infty}^{\infty}|a_m(U_m(\lambda,r)-1)|^2 \le 2\pi\lim_{r\to R}\sum_{|m|<m_0}|a_m|^2(U_m(\lambda,r)-1)^2$$
$$+\,2\pi\lim_{r\to R}\sum_{|m|\ge m_0}|a_m|^2 < 2\pi\varepsilon.$$

This proves (4.11), since $\varepsilon > 0$ was chosen arbitrarily. ■

4.4 The Sobolev space $H_{\frac{1}{2}}$

Introduction to Sections 4.4 – 4.6. The solutions of the Dirichlet problem to the potential and Helmholtz equations, which we constructed in Theorem 4.2 and Theorem 4.5, are infinitely differentiable, so these partial differential equations are satisfied in the classical sense. However, we do not know much about the behavior of the solutions at the boundary $\partial B_R(0)$, since we did not show that the solutions converge pointwise to the boundary data when the boundary is approached from the interior of $B_R(0)$, that is, we did not show that the boundary condition is satisfied in the classical sense; we only showed that the Dirichlet boundary conditions are satisfied in the L^2–sense made precise in (4.5).

Yet, in Theorem 4.2 and Theorem 4.5 we allowed that the boundary data u belong to $L^2(\partial B_R(0))$. This is a big class of functions, and one can surmise that the boundary condition is satisfied in the classical sense when one chooses the boundary data from a smaller class. In fact, in Section 5, where we study boundary value problems in very general domains, we show that the boundary condition is satisfied in the classical sense when the boundary data u are continuous and when additionally the boundary satisfies some conditions.

On the other hand, in Definition 3.32 we defined weak solutions of boundary value problems to the Helmholtz equation, however only to homogeneous boundary data. The idea suggests itself to define weak solutions of boundary value problems to non-zero L^2–boundary data by requiring that the boundary condition is satisfied in the L^2–sense of (4.5). Whereas this is in principle possible, technical difficulties arise with this approach when one studies boundary value problems in general domains Ω, since one has to construct smooth parallel curves to the boundary curve $\partial\Omega$, which converge to the boundary curve, and one must compute the L^2–difference of the values of the solution on these parallel curves and the values of the boundary data on the boundary curve. In the case of the circular domain $B_R(0)$, this difficulty does not exist, since on can simply choose circles with radius smaller than R for the parallel curves.

Another way to formulate a weak boundary condition is opened up by the trace theorem for Sobolev functions. In the case of an open set $\Omega \subseteq \mathbb{R}^2$, this theorem says that if $\omega \subseteq \Omega$ is a smooth curve, than though for general L^2–functions the restriction to the set ω of measure zero is not defined, for functions u in $H_1(\Omega)$ this restriction can be uniquely defined in a natural way. This restriction, which is called the trace of u on ω, belongs to a certain Sobolev space $H_{1/2}(\omega)$. In the case of $B_R(0)$ we take for ω the boundary $\partial B_R(0)$. Since we require in Definition 3.32 that weak solutions of the Helmholtz equation belong to $H_1(B_R)$, such weak solutions have a trace on $\partial B_R(0)$. The weak boundary condition then simply requires that the trace is equal to the given boundary data.

The question arises, whether the solutions constructed in Theorem 4.2 and Theorem 4.5 are weak solutions in this trace sense.

This problem will be studied in the following three sections. In the present section we introduce the Sobolev spaces $H_1(\partial B_R(0))$ and $H_{1/2}(\partial B_R(0))$, in Section 4.5 we prove two trace theorems, and in Section 4.6 we show that the solution constructed in Theorems 4.2 and 4.5 are weak solutions in the trace sense if and only if the boundary data belong to the space $H_{1/2}(\partial B_R(0))$. The proofs are mainly based on Fourier analysis of functions defined in $B_R(0)$ and on the boundary $\partial B_R(0))$.

Sobolev spaces of boundary functions. At the outset we must define the Sobolev space $H_1(\partial B_R(0))$. We cannot immediately carry over the Definition 3.24 of $H_1(\Omega)$, since in that definition Ω is an open set in $\mathbb{R}^n$, whereas $\partial B_R(0)$ is a curve in $\mathbb{R}^2$. Therefore we use the idea suggested by Theorem 3.26 to define $H_1(\partial B_R(0))$ as complete hull of the space of continuously differentiable functions.

To simplify the notation we denote the open ball in $\mathbb{R}^2$ with radius R and center 0 by B_R. If u is a function defined on ∂B_R we denote by $\tilde{u}(\varphi) = u(R\cos\varphi, R\sin\varphi)$ the function transformed to the angle variable φ. The domain of definition of $\tilde{u}$ is $[0, 2\pi)$. Conversely, if $\tilde{u} : [0, 2\pi) \to \mathbb{C}$ is a given function, we denote by u the inversely transformed function defined on ∂B_R.

Since $\varphi \mapsto (R\cos\varphi, R\sin\varphi)$ is a parametrization of the curve ∂B_R, the definition of curve integrals on ∂B_R implies that $u \in L^2(\partial B_R)$ if and only if $\tilde{u} \in L^2([0, 2\pi])$ and

$$\|u\|^2_{\partial B_R} = \int_{\partial B_R} |u(x)|^2 \, d\ell(x) = \int_0^{2\pi} |\tilde{u}(\varphi)|^2 R \, d\varphi = R \, \|\tilde{u}\|^2_{[0,2\pi]}. \tag{4.22}$$

It thus follows that up to the factor $\sqrt{R}$ the mapping $u \mapsto \tilde{u}$ is an isometric isomorphism between the spaces $L^2(\partial B_R)$ and $L^2([0, 2\pi])$. The derivative of functions $u : \partial B_R \mapsto \mathbb{C}$ at $x = (R\cos\varphi, R\sin\varphi)$ is defined by

$$u'(x) = \frac{1}{R} \, \tilde{u}'(\varphi), \tag{4.23}$$

and $C_1(\partial B_R)$ is the space of mappings $u : \partial B_R \to \mathbb{C}$, for which the derivative $u' : \partial B_R \to \mathbb{C}$ is continuous. It follows from this definition that if $u \in C_1(\partial B_R)$, then the transformed function $\tilde{u}$ is periodic with period 2π, hence it belongs to the space

$$C_1^{\text{per}}([0, 2\pi)) = \big\{ \tilde{U}|_{[0,2\pi)} \;\big|\; \tilde{U} \in C_1(\mathbb{R}),\ \tilde{U}(\varphi + 2\pi) = \tilde{U}(\varphi),\ \text{for all } \varphi \in \mathbb{R} \big\}.$$

Conversely, if $\tilde{u} \in C_1^{\text{per}}([0, 2\pi))$, then u belongs to $C_1(\partial B_R)$.

Definition 4.9 *The Sobolev norm* $\| \cdot \|_{H_1(\partial B_R)}$ *on* $C_1(\partial B_R)$ *is defined by*

$$\|u\|_{H_1(\partial B_R)} = \sqrt{R} \, \|\tilde{u}\|_{H_1([0,2\pi])}, \tag{4.24}$$

and the Sobolev space $H_1(\partial B_R)$ *is by definiton the complete hull of the space* $C_1(\partial B_R)$ *with respect to the norm* $\| \cdot \|_{H_1(\partial B_R)}$.

With this norm and up to the factor $\sqrt{R}$, the mapping $u \mapsto \tilde{u}$ is an isometric isomorphism between the normed spaces $\big(C_1(\partial B_R), \|\cdot\|_{H_1(\partial B_R)}\big)$ and $\big(C_1^{\text{per}}([0,2\pi)), \|\cdot\|_{H_1([0,2\pi])}\big)$. This isomorphism can be extended by continuity to an isomorphism between $H_1(\partial B_R)$ and the space obtained by completion of the space $C_1^{\text{per}}([0,2\pi))$ with respect to the norm $\|\cdot\|_{H_1([0,2\pi])}$. Since $C_1^{\text{per}}([0,2\pi))$ is a subspace of the complete space $H_1([0,2\pi])$, this completion is equal to the closure of $C_1^{\text{per}}([0,2\pi))$ in $H_1([0,2\pi])$. For this closed subspace of $H_1([0,2\pi])$ we use the notation $H_1^{\text{per}}([0,2\pi])$, hence

$$H_1^{\text{per}}([0,2\pi]) = \overline{C_1^{\text{per}}([0,2\pi))}.$$

It follows that $u \mapsto \tilde{u} : H_1(\partial B_R) \to H_1^{\text{per}}([0,2\pi])$ is an isomorphism such that (4.24) holds for all $u \in H_1(\partial B_R)$. Since $H_1^{\text{per}}([0,2\pi])$ is a subspace of $L^2([0,2\pi])$, it follows that the inverse image $H_1(\partial B_R)$ of $H_1^{\text{per}}([0,2\pi])$ under the mapping $u \mapsto \tilde{u}$ is a subspace of $L^2(\partial B_R)$.

Every function $\tilde{u} \in H_1^{\text{per}}([0,2\pi])$ belongs to $H_1([0,2\pi])$, hence $\tilde{u}$ has the weak derivative $\tilde{u}' \in L^2([0,2\pi])$ defined in the usual way.

Definition 4.10 *For $u \in H_1(\partial B_R)$ the weak derivative $u' \in L^2(\partial B_R)$ is defined to be $\frac{1}{R}$ times the inverse image of the weak derivative $\tilde{u}' \in L^2([0,2\pi])$ under the mapping $u \mapsto \tilde{u}$. More precisely,*

$$\widetilde{(u')} = \frac{1}{R}\,(\tilde{u})'.$$

This definition extends (4.23) from classical derivatives to weak derivatives. With this definition it follows from (4.22) and (4.24) that

$$\|u\|^2_{H_1(\partial B_R)} = R\,\|\tilde{u}\|^2_{H_1([0,2\pi])} = R\left(\|\tilde{u}\|^2_{[0,2\pi]} + \|\tilde{u}'\|^2_{[0,2\pi]}\right) = \|u\|^2_{\partial B_R} + R^2\,\|u'\|^2_{\partial B_R}. \quad (4.25)$$

We denote the L^2–scalar product on the interval $[0,2\pi]$ by

$$(\tilde{f},\tilde{g})_{[0,2\pi]} = \int_0^{2\pi} \tilde{f}(\varphi)\overline{\tilde{g}(\varphi)}\,d\varphi.$$

As noted earlier, the scalar product $(\tilde{f},\tilde{g}) \mapsto (\tilde{f},\tilde{g})_{[0,2\pi]} : L^2([0,2\pi]) \times L^2([0,2\pi]) \to \mathbb{C}$ is a continuous mapping. This follows from Cauchy-Schwarz' inequality. Because of the periodicity of functions from $H_1^{\text{per}}([0,2\pi])$, we can integrate by parts in the scalar product without boundary terms:

Lemma 4.11 *For $\tilde{f},\tilde{g} \in H_1^{\text{per}}([0,2\pi])$ we have*

$$(\tilde{f}',\tilde{g})_{[0,2\pi]} = -(\tilde{f},\tilde{g}')_{[0,2\pi]}. \quad (4.26)$$

Proof: To $\tilde{f}, \tilde{g} \in H_1^{\text{per}}([0,2\pi])$ choose sequences $\{\tilde{f}_n\}_{n\in\mathbb{N}}, \{\tilde{g}_k\}_{k\in\mathbb{N}} \subseteq C_1^{\text{per}}([0,2\pi))$, which converge to $\tilde{f}$ and $\tilde{g}$, respectively, in the norm $\|\cdot\|_{H_1^{\text{per}}([0,2\pi])}$. This implies that $\tilde{f}_n \to \tilde{f}$, $\tilde{f}'_n \to \tilde{f}'$, $\tilde{g}_k \to \tilde{g}$, $\tilde{g}'_k \to \tilde{g}'$ in $L^2([0,2\pi])$. Using the continuity of the scalar product, we compute

$$(\tilde{f}', \tilde{g})_{[0,2\pi]} = \lim_{n\to\infty} \lim_{k\to\infty} (\tilde{f}'_n, \tilde{g}_k)_{[0,2\pi]} = -\lim_{n\to\infty} \lim_{k\to\infty} (\tilde{f}_n, \tilde{g}'_k)_{[0,2\pi]} = -(\tilde{f}, \tilde{g}')_{[0,2\pi]}.$$

To get the second equality sign we used that the boundary terms cancel when we integrate by parts in $C_1^{\text{per}}([0,2\pi))$. ■

The following theorem gives an interesting characterization of functions in $H_1^{\text{per}}([0,2\pi])$ by Fourier series:

Theorem 4.12 *Let $\tilde{f} \in L^2([0,2\pi])$ have the Fourier series*

$$\tilde{f}(\varphi) = \sum_{m=-\infty}^{\infty} a_m e^{im\varphi}.$$

(i) If $\tilde{f}$ belongs to $H_1^{\text{per}}([0,2\pi])$, then we have

$$\tilde{f}'(\varphi) = \sum_{m=-\infty}^{\infty} ima_m e^{im\varphi} \quad \textit{and} \quad \|\tilde{f}\|^2_{H_1([0,2\pi])} = 2\pi \sum_{m=-\infty}^{\infty} (1+m^2)|a_m|^2. \tag{4.27}$$

(ii) If $\sum_{m=-\infty}^{\infty} m^2|a_m|^2 < \infty$, then $\tilde{f}$ belongs to $H_1^{\text{per}}([0,2\pi])$.

Proof: Let $\tilde{f} \in H_1^{\text{per}}([0,2\pi])$. Then the weak derivative $\tilde{f}' \in L^2([0,2\pi])$ has a Fourier expansion $\tilde{f}' = \sum_{m=-\infty}^{\infty} b_m e^{im\varphi}$. Since the function $\varphi \mapsto e^{im\varphi}$ belongs to $C_1^{\text{per}}([0,2\pi))$, we conclude from (4.26) that

$$2\pi b_m = \int_0^{2\pi} \tilde{f}'(\varphi) e^{-im\varphi} d\varphi = -\int_0^{2\pi} \tilde{f}(\varphi) \frac{d}{d\varphi} e^{-im\varphi} d\varphi = im \int_0^{2\pi} \tilde{f}(\varphi) e^{-im\varphi} d\varphi = 2\pi ima_m\,,$$

whence $b_m = ima_m$, which means that $\tilde{f}' = \sum_{m=-\infty}^{\infty} ima_m e^{im\varphi}$. Parseval's identity (Remark 3.14) thus implies

$$\|\tilde{f}\|^2_{H_1([0,2\pi])} = \|\tilde{f}\|^2_{[0,2\pi]} + \|\tilde{f}'\|^2_{[0,2\pi]} = 2\pi \sum_{m=-\infty}^{\infty} (1+m^2)|a_m|^2.$$

This proves (i). To prove (ii), define for $n \in \mathbb{N}$ the function $\tilde{f}_n \in C_1^{\text{per}}([0,2\pi))$ by

$$\tilde{f}_n(\varphi) = \sum_{m=-n}^{n} a_m e^{im\varphi}.$$

We use the Cauchy convergence criterion to prove that the sequence $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ converges in $H_1^{\mathrm{per}}([0,2\pi])$. To this end note that from the assumption $\sum_{m=-\infty}^{\infty} m^2|a_m|^2 < \infty$ it follows that to every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $n_2 \geq n_1 \geq n_0$ we have by Parseval's identity that

$$\begin{aligned}
\|\tilde{f}_{n_2} - \tilde{f}_{n_1}\|^2_{H_1^{\mathrm{per}}([0,2\pi])} &= \|\tilde{f}_{n_2} - \tilde{f}_{n_1}\|^2_{[0,2\pi]} + \|\tilde{f}'_{n_2} - \tilde{f}'_{n_1}\|^2_{[0,2\pi]} \\
&= \Big\| \sum_{n_1<|m|\leq n_2} a_m e^{im\varphi} \Big\|^2_{[0,2\pi]} + \Big\| \sum_{n_1<|m|\leq n_2} ima_m e^{im\varphi} \Big\|^2_{[0,2\pi]} = \\
&= 2\pi \sum_{n_1<|m|\leq n_2} (1+m^2)|a_m|^2 \leq 2\pi \sum_{n_0\leq|m|} (1+m^2)|a_m|^2 < \varepsilon.
\end{aligned}$$

Therefore $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ converges to a function $\tilde{g} \in H_1^{\mathrm{per}}([0,2\pi])$. Since $\|\tilde{u}\|_{[0,2\pi]} \leq \|\tilde{u}\|_{H_1^{\mathrm{per}}([0,2\pi])}$ holds for all $\tilde{u} \in H_1^{\mathrm{per}}([0,2\pi])$, the sequence $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ converges to $\tilde{g}$ also in the space $L^2([0,2\pi])$. On the other hand, $\{\tilde{f}_n\}_{n\in\mathbb{N}}$ converges to $\tilde{f}$ in $L^2([0,2\pi])$, hence $\tilde{f} = \tilde{g} \in H_1^{\mathrm{per}}([0,2\pi])$. ■

Corollary 4.13 *A function $f \in L^2(\partial B_R)$ belongs to $H_1(\partial B_R)$ if and only if the Fourier coefficients $\{a_m\}_{m\in\mathbb{Z}}$ of the transformed function $\tilde{f}$ satisfy $\sum_{m=-\infty}^{\infty} m^2|a_m|^2 < \infty$, and the norm satisfies*

$$\|f\|^2_{H_1(\partial B_R)} = 2\pi R \sum_{m=-\infty}^{\infty} (1+m^2)|a_m|^2.$$

Proof: Since f belongs to $H_1(\partial B_R)$ if and only if $\tilde{f}$ belongs to $H_1^{\mathrm{per}}([0,2\pi])$, this corollary follows immediately from the definition of the norm $\|\cdot\|_{H_1(\partial B_R)}$ in (4.24) and from Theorem 4.12. ■

Motivated by this result, one defines

Definition 4.14 *(i) Let $0 \leq \alpha \leq 1$. The Sobolev space $H_\alpha^{\mathrm{per}}([0,2\pi])$ consists of all functions $\tilde{f} \in L^2([0,2\pi])$, for which the coefficients in the expansion*

$$\tilde{f}(\varphi) = \sum_{m=-\infty}^{\infty} a_m e^{im\varphi}$$

satisfy $\sum_{m=-\infty}^{\infty} |m|^{2\alpha}|a_m|^2 < \infty$. The norm $\|\cdot\|_{H_\alpha^{\mathrm{per}}([0,2\pi])}$ on this space is defined by

$$\|\tilde{f}\|^2_{H_\alpha^{\mathrm{per}}([0,2\pi])} = 2\pi \sum_{m=-\infty}^{\infty} (1+|m|^{2\alpha})|a_m|^2.$$

(ii) The Sobolev space $H_\alpha(\partial B_R)$ consists of all functions $f \in L^2(\partial B_R)$, for which the transformed function $\tilde{f}$ belongs to $H_\alpha^{\mathrm{per}}([0,2\pi])$. The norm is defined by

$$\|f\|_{H_\alpha(\partial B_R)} = \sqrt{R}\, \|\tilde{f}\|_{H_\alpha^{\mathrm{per}}([0,2\pi])}.$$

Of course, we have $H_0(\partial B_R) = L^2(\partial B_R)$ and $H_0^{\text{per}}([0, 2\pi]) = L^2([0, 2\pi])$. The interest in the spaces H_α with broken index α is motivated by the trace theorem proved in the next section, where the space $H_{1/2}$ appears. These spaces are Hilbert spaces. To give the scalar product on these spaces, let $f, g \in H_\alpha(\partial B_R)$ and let $\tilde{f} = \sum_{m=-\infty}^{\infty} a_m e^{im\varphi}$, $\tilde{g} = \sum_{m=-\infty}^{\infty} b_m e^{im\varphi}$ be the Fourier expansions of the transformed functions. Then the scalar products $(f, g)_{\alpha,\partial B_R}$ and $(\tilde{f}, \tilde{g})_{\alpha,[0,2\pi]}$ are given by

$$(f, g)_{\alpha,\partial B_R} = R\,(\tilde{f}, \tilde{g})_{\alpha,[0,2\pi]} = 2\pi R \sum_{m=-\infty}^{\infty} (1 + |m|^{2\alpha}) a_m \overline{b_m}.$$

The infinite sum on the right hand side exists, since by the Cauchy-Schwarz inequality we have

$$\Big(\sum_{m=-\infty}^{\infty} (1 + |m|^{2\alpha}) |a_m \overline{b_m}| \Big)^2 \leq \sum_{m=-\infty}^{\infty} (1 + |m|^{2\alpha}) |a_m|^2 \sum_{m=-\infty}^{\infty} (1 + |m|^{2\alpha}) |b_m|^2.$$

Obviously we have $(f, f)_{\alpha,\partial B_R} = \|f\|^2_{H_\alpha(\partial B_R)}$ and $(\tilde{f}, \tilde{f})_{\alpha,[0,2\pi]} = \|\tilde{f}\|^2_{H_\alpha^{\text{per}}([0,2\pi])}$.

4.5 The trace theorems

In this section we prove two trace theorems. As preparation we need to study first the Fourier analysis of functions in $H_1(B_R)$ more precisely. We start with some notations.

If $u(x_1, x_2)$ is a function defined on B_R and written in cartesian coordinates, then we denote by $\tilde{u}(r, \varphi) = u(r \cos\varphi, r \sin\varphi)$ the function transformed to polar coordinates and defined on $[0, R) \times [0, 2\pi)$. The function $\tilde{u}$ is constant on the boundary line $\{0\} \times [0, 2\pi)$. Therefore not every function defined on the rectangle $\{[0, R)\} \times [0, 2\pi)$ is obtained by transformation of a function u defined on B_R to polar coordinates.

If u is a function in $C_k(B_R)$, then $\tilde{u}$ is 2π–periodic with respect to the variable φ, hence $\tilde{u}$ belongs to the space

$$C_k^{\text{per}}([0, R) \times [0, 2\pi)) = \{\tilde{U}|_{[0,R)\times[0,2\pi)} \mid \tilde{U} \in C_k([0, R) \times \mathbb{R}),\ \tilde{U}(r, \varphi) = \tilde{U}(r, \varphi + 2\pi)\}.$$

For $u \in C_1(B_R)$ we denote by $\partial_\varphi u$ and $\partial_r u$ the functions obtained by back transformation of $\partial_\varphi \tilde{u}$ and $\partial_r \tilde{u}$ to cartesian coordinates. More precisely, these derivatives are defined by the equations

$$\widetilde{\partial_\varphi u} = \partial_\varphi \tilde{u}, \qquad \widetilde{\partial_r u} = \partial_r \tilde{u}.$$

We define the weighted L^2–space

$$L^2(r; [0, R) \times [0, 2\pi)) = \Big\{ \tilde{f} : [0, R) \times [0, 2\pi) \to \mathbb{C} \;\Big|\; \int_0^{2\pi} \int_0^R |\tilde{f}(r, \varphi)|^2 r\, dr < \infty \Big\}.$$

For $u \in L^2(B_R)$ the transformation theorem implies $\tilde{u} \in L^2(r;[0,R) \times [0,2\pi))$ and

$$\|u\|_{B_R} = \|\tilde{u}\|_{L^2(r;[0,R)\times[0,2\pi))}. \tag{4.28}$$

For $f \in C_1(\overline{B_R})$ and $m \in \mathbb{Z}$ we define the function $f_m \in C_1([0,R])$ by

$$f_m(r) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{f}(r,\varphi)e^{-im\varphi}d\varphi = \frac{1}{\sqrt{2\pi}}\big(\tilde{f}(r,\cdot), v_m\big)_{[0,2\pi]},$$

where we used the notation $v_m(\varphi) = \frac{1}{\sqrt{2\pi}}e^{im\varphi}$. By Remark 3.14 we have for all $r \in [0,R]$ that

$$\lim_{k\to\infty}\Big\|\tilde{f}(r,\cdot) - \sqrt{2\pi}\sum_{m=-k}^{k} f_m(r)v_m\Big\|_{[0,2\pi]} = 0 \quad \text{and} \quad \|\tilde{f}(r,\cdot)\|^2_{[0,2\pi]} = 2\pi\sum_{m=-\infty}^{\infty}|f_m(r)|^2. \tag{4.29}$$

We call $\sqrt{2\pi}\sum_{m=-\infty}^{\infty} f_m(r)v_m(\varphi) = \sum_{m=-\infty}^{\infty} f_m(r)e^{im\varphi}$ the partial Fourier expansion of the function f. Besides (4.29) we moreover have

$$\lim_{k\to\infty}\Big\|\tilde{f} - \sqrt{2\pi}\sum_{m=-k}^{k} f_m(r)v_m\Big\|_{L^2(r;[0,R]\times[0,2\pi])} = 0, \quad \|f\|^2_{B_R} = 2\pi\sum_{m=-\infty}^{\infty}\int_0^R |f_m(r)|^2 r\,dr. \tag{4.30}$$

For the proof consider the following computation, where we use that $\{v_m\}_{m\in\mathbb{Z}}$ is an orthonormal system. We have

$$\begin{aligned}
&\int_0^R\int_0^{2\pi}\Big|\tilde{f}(r,\varphi) - \sqrt{2\pi}\sum_{m=-k}^{k} f_m(r)v_m(\varphi)\Big|^2 r\,d\varphi dr \\
&\quad = \int_0^R\int_0^{2\pi}\Big|\tilde{f}(r,\varphi) - \sum_{m=-k}^{k}\big(\tilde{f}(r,\cdot),v_m\big)_{[0,2\pi]}v_m(\varphi)\Big|^2 r\,d\varphi dr \\
&= \int_0^R\Big(\tilde{f}(r,\cdot) - \sum_{m=-k}^{k}\big(\tilde{f}(r,\cdot),v_m\big)_{[0,2\pi]}v_m\,,\ \tilde{f}(r,\cdot) - \sum_{m=-k}^{k}\big(\tilde{f}(r,\cdot),v_m\big)_{[0,2\pi]}v_m\Big)_{[0,2\pi]} r\,dr \\
&\quad = \int_0^R\Big(\|\tilde{f}(r,\cdot)\|^2_{[0,2\pi]} - \sum_{m=-k}^{k}\big|\big(\tilde{f}(r,\cdot),v_m\big)_{[0,2\pi]}\big|^2\Big) r\,dr \\
&\qquad = \int_0^R\Big(\|\tilde{f}(r,\cdot)\|^2_{[0,2\pi]} - 2\pi\sum_{m=-k}^{k}|f_m(r)|^2\Big) r\,dr.
\end{aligned}$$

From (4.29) we see that for all $r \in [0,R]$ the integrand of the integral on the right hand side of this equation decreases monotonically to zero for $k \to \infty$. The theorem of Beppo Levi thus implies that the integral tends to zero for $k \to \infty$. Since the left hand side of this equation is equal to $\|\tilde{f} - \sqrt{2\pi}\sum_{m=-k}^{k} f_m\, v_m\|^2_{L^2(r;[0,R]\times[0,2\pi])}$ and since by (4.28) the right hand side equals $\|f\|^2_{B_R} - 2\pi \sum_{m=-k}^{k} \int_0^R |f_m(r)|^2 r\,dr$, both equations of (4.30) follow from this relation. ■

Let $\Omega \subseteq \mathbb{R}^n$ be a measurable set and let $u \in H_1(\Omega)$. In the following we write

$$|u|^2_\Omega = \|\nabla_x u\|^2_\Omega = \int_\Omega |\partial_{x_1} u|^2 + |\partial_{x_2} u|^2 dx.$$

$|u|_\Omega$ is a seminorm on $H_1(\Omega)$, but not a norm, since for constant functions u one has $|u|_\Omega = 0$.

Theorem 4.15 *(i) Let $f \in C_1(\overline{B_R})$. Then we have*

$$|f|^2_{B_R} = \int_0^R \int_0^{2\pi} \Big(|\partial_r \tilde{f}|^2 + \frac{1}{r^2}\,|\partial_\varphi \tilde{f}|^2\Big) r\,d\varphi dr = ||\partial_r f||^2_{B_R} + \Big\|\frac{1}{r}\partial_\varphi f\Big\|^2_{B_R}. \tag{4.31}$$

(ii) Let f be a function in $C_1(\overline{B_R})$ with partial Fourier series

$$\tilde{f}(r,\varphi) = \sum_{m=-\infty}^{\infty} f_m(r) e^{im\varphi}.$$

Then the partial Fourier series of the derivatives are given by

$$\partial_r \tilde{f}(r,\varphi) = \sum_{m=-\infty}^{\infty} f'_m(r) e^{im\varphi}, \qquad \partial_\varphi \tilde{f}(r,\varphi) = \sum_{m=-\infty}^{\infty} im f_m(r) e^{im\varphi}. \tag{4.32}$$

The L^2–norms satisfy

$$\|\partial_r f\|^2_{B_R} = 2\pi \sum_{m=-\infty}^{\infty} \int_0^R |f'_m(r)|^2 r\,dr, \qquad \Big\|\frac{1}{r}\,\partial_\varphi f\Big\|^2_{B_R} = 2\pi \sum_{m=-\infty}^{\infty} m^2 \int_0^R |f_m(r)|^2\,\frac{1}{r}\,dr. \tag{4.33}$$

Proof: (i). From $\tilde{f}(r,\varphi) = f(r\cos\varphi, r\sin\varphi)$ we obtain by the chain rule

$$\begin{aligned}\partial_\varphi \tilde{f}(r,\varphi) &= -r\sin\varphi\,\partial_{x_1} f(r\cos\varphi, r\sin\varphi) + r\cos\varphi\,\partial_{x_2} f(r\cos\varphi, r\sin\varphi)\\ &= -r\sin\varphi\,(\widetilde{\partial_{x_1} f})(r,\varphi) + r\cos\varphi\,(\widetilde{\partial_{x_2} f})(r,\varphi),\end{aligned}$$

and similarly

$$\partial_r \tilde{f}(r,\varphi) = \cos\varphi\,(\widetilde{\partial_{x_1} f})(r,\varphi) + \sin\varphi\,(\widetilde{\partial_{x_2} f})(r,\varphi).$$

These equations yield

$$\begin{aligned}|\partial_r \tilde{f}|^2 + \frac{1}{r^2}\,|\partial_\varphi \tilde{f}|^2 &= (\cos\varphi)^2\,|\widetilde{\partial_{x_1} f}|^2 + 2\cos\varphi\sin\varphi\,\mathrm{Re}\,(\widetilde{\partial_{x_1} f}\,\overline{\widetilde{\partial_{x_2} f}}) + (\sin\varphi)^2\,|\widetilde{\partial_{x_2} f}|^2\\ &+ (\sin\varphi)^2\,|\widetilde{\partial_{x_1} f}|^2 - 2\sin\varphi\cos\varphi\,\mathrm{Re}\,(\widetilde{\partial_{x_1} f}\,\overline{\widetilde{\partial_{x_2} f}}) + (\cos\varphi)^2\,|\widetilde{\partial_{x_2} f}|^2 = |\widetilde{\partial_{x_1} f}|^2 + |\widetilde{\partial_{x_2} f}|^2.\end{aligned}$$

Consequently, we have

$$\int_0^R \int_0^{2\pi} \Big(|\partial_r \tilde{f}|^2 + \frac{1}{r^2} |\partial_\varphi \tilde{f}|^2 \Big) r \, d\varphi dr = \int_0^R \int_0^{2\pi} \Big(|\widetilde{\partial_{x_1} f}|^2 + |\widetilde{\partial_{x_2} f}|^2 \Big) r \, d\varphi dr$$

$$= \int_{B_R} |\partial_{x_1} f|^2 + |\partial_{x_2} f|^2 \, dx = |f|^2_{B_R},$$

which is (4.31).

(ii) For $f \in C_1(B_R)$ the function $\tilde{f}$ belongs to $C_1^{\text{per}}([0,R) \times [0,2\pi))$, hence the derivatives $\partial_\varphi \tilde{f}$ and $\partial_r \tilde{f}$ belong to $C_0^{\text{per}}([0,R) \times [0,2\pi))$ and thus have partial Fourier expansions. Let

$$\partial_\varphi \tilde{f}(r,\varphi) = \sum_{m=-\infty}^{\infty} g_m(r) e^{im\varphi} \tag{4.34}$$

be the expansion of the first of these derivatives. The coefficient $g_m(r)$ is given for every $r \in [0,R]$ by the integral stated in Remark 3.14. By integration by parts we obtain from this integral

$$g_m(r) = \frac{1}{2\pi} \int_0^{2\pi} \partial_\varphi \tilde{f}(r,\varphi) e^{-im\varphi} d\varphi = -\frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(r,\varphi) \partial_\varphi e^{-im\varphi} d\varphi$$

$$= im \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(r,\varphi) e^{-im\varphi} d\varphi = im f_m(r), \tag{4.35}$$

where we used that in the integration by parts the boundary terms cancel because of the periodicity of the functions $\varphi \mapsto \tilde{f}(r,\varphi)$ and $\varphi \mapsto e^{-im\varphi}$. The second equation in (4.32) follows from (4.34) and (4.35). To prove the first equation in (4.32) let

$$\partial_r \tilde{f}(r,\varphi) = \sum_{m=-\infty}^{\infty} h_m(r) e^{im\varphi} = \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} \partial_r \tilde{f}(r,\varphi) e^{-im\varphi} d\varphi \, e^{im\varphi} \tag{4.36}$$

be the partial Fourier expansion of $\partial_r \tilde{f}$. Since the function $(r,\varphi) \mapsto \tilde{f}(r,\varphi) e^{-im\varphi}$ is continuously differentiable on the compact set $[0,R] \times [0,2\pi]$ with respect to r, it follows from a standard theorem of Lebesgue integration theory that we can differentiate under the integral sign to obtain

$$f_m'(r) = \partial_r \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(r,\varphi) e^{-im\varphi} d\varphi = \frac{1}{2\pi} \int_0^{2\pi} \partial_r \tilde{f}(r,\varphi) e^{-im\varphi} d\varphi = h_m(r).$$

This equation together with (4.36) yields the first equation in (4.32).

Finally, (4.33) is an immediate consequence of (4.32) and the second equation in (4.30).

■

Theorem 4.16 (Trace theorem I) *There is a uniquely defined linear mapping* $T : H_1(B_R) \to H_{1/2}(\partial B_R)$, *which satisfies* $Tu = u|_{\partial B_R}$ *for all* $u \in C_1(\overline{B_R})$ *and*

$$\|Tu\|_{H_{1/2}(\partial B_R)} \leq C_R \|u\|_{H_1(B_R)}, \qquad \text{for all } u \in H_1(B_R), \tag{4.37}$$

where $C_R = \sqrt{1 + R + 2\max\left(R, \frac{1}{R}\right)}$. *The function* Tu *is called the trace of* u *on the boundary* ∂B_R. *The trace* Tu *is denoted by* $u|_{\partial B_R}$ *for all* $u \in H_1(B_R)$.

Proof: Since by Theorem 3.26 the space $C_1(\overline{B_R})$ is dense in $H_1(B_R)$, it suffices to show that (4.37) holds for all $u \in C_1(\overline{B_R})$, because this inequality implies that the linear mapping $T : C_1(\overline{B_R}) \to C_1(\partial B_R)$ defined by $Tu = u|_{\partial B_R}$ is continuous and can be extended in a unique way to a continuous mapping $T : H_1(B_R) \to H_{1/2}(\partial B_R)$. The extended mapping satisfies (4.37).

Thus, let $u \in C_1(\overline{B_R})$ and let $\tilde{u}(r,\varphi) = \sum_{m=-\infty}^{\infty} u_m(r) e^{im\varphi}$ be the partial Fourier expansion, hence $\widetilde{u|_{\partial B_R}}(\varphi) = \tilde{u}(R,\varphi) = \sum_{m=-\infty}^{\infty} u_m(R) e^{im\varphi}$. With the function

$$k(r) = 2\pi \left(\frac{r}{R}\right)^2, \qquad 0 \leq r \leq R.$$

we thus obtain from Definition 4.14 that

$$\begin{aligned} \|u|_{\partial B_R}\|^2_{H_{1/2}(\partial B_R)} &= 2\pi R \sum_{m=-\infty}^{\infty} (1+|m|)|u_m(R)|^2 = R \sum_{m=-\infty}^{\infty} (1+|m|)\, k(R)\, |u_m(R)|^2 \\ &= R \sum_{m=-\infty}^{\infty} (1+|m|) \int_0^R \partial_r k(r) |u_m(r)|^2 + k(r) \partial_r |u_m(r)|^2 dr. \end{aligned} \tag{4.38}$$

We estimate the right hand side term by term. Note first that

$$\left|\, \partial_r |u_m(r)|^2 \,\right| = \left|\, \partial_r \big(u_m(r) \overline{u_m(r)}\big) \,\right| = \left| 2\mathrm{Re}\, \big(u_m(r) \overline{u_m'(r)}\big) \right| \leq 2 |u_m(r)|\, |u_m'(r)|.$$

From this estimate, from the estimate $k(r) \leq 2\pi$, from Cauchy-Schwarz' inequality, applied first to an integral and then to infinite sums, and from (4.33), (4.31) we obtain that

$$\begin{aligned} \sum_{m=-\infty}^{\infty} \int_0^R |m|\, k(r)\, \partial_r |u_m(r)|^2 dr &\leq 4\pi \sum_{m=-\infty}^{\infty} \int_0^R |m|\, |u_m(r)|\, |u_m'(r)|\, dr \\ &\leq 4\pi \sum_{m=-\infty}^{\infty} \Big(\int_0^R m^2\, |u_m(r)|^2 \frac{1}{r}\, dr \Big)^{\frac{1}{2}} \Big(\int_0^R |u_m'(r)|^2 r\, dr \Big)^{\frac{1}{2}} \\ &\leq 4\pi \Big(\sum_{m=-\infty}^{\infty} \int_0^R m^2\, |u_m(r)|^2 \frac{1}{r}\, dr \Big)^{\frac{1}{2}} \Big(\sum_{m=-\infty}^{\infty} \int_0^R |u_m'(r)|^2 r\, dr \Big)^{\frac{1}{2}} \\ &= 2 \Big\| \frac{1}{r} \partial_\varphi u \Big\|_{B_R} \|\partial_r u\|_{B_R} \leq \Big(\Big\| \frac{1}{r} \partial_\varphi u \Big\|^2_{B_R} + \|\partial_r u\|^2_{B_R} \Big) = |u|^2_{B_R} \leq \|u\|^2_{H_1(B_R)}. \end{aligned} \tag{4.39}$$

In the second last step we used the inequality $2ab \le a^2 + b^2$. Similarly, we obtain from the estimate $k(r) \le 2\pi\left(\frac{r}{R}\right)$, from Cauchy-Schwarz' inequality, and from (4.30), (4.33), (4.31) that

$$\begin{aligned}\sum_{m=-\infty}^{\infty} \int_0^R k(r)\,\partial_r |u_m(r)|^2 dr &\le \frac{4\pi}{R} \sum_{m=-\infty}^{\infty} \int_0^R |u_m(r)|\,|u_m'(r)| r\,dr \\ &\le \frac{4\pi}{R}\Big(\sum_{m=-\infty}^{\infty} \int_0^R |u_m(r)|^2 r\,dr\Big)^{\frac{1}{2}} \Big(\sum_{m=-\infty}^{\infty} \int_0^R |u_m'(r)|^2 r\,dr\Big)^{\frac{1}{2}} = \frac{2}{R}\|u\|_{B_R}\|\partial_r u\|_{B_R} \\ &\le \frac{1}{R}\big(\|u\|_{B_R}^2 + \|\partial_r u\|_{B_R}^2\big) \le \frac{1}{R}\big(\|u\|_{B_R}^2 + |u|_{B_R}^2\big) = \frac{1}{R}\|u\|_{H_1(B_R)}^2. \end{aligned} \tag{4.40}$$

Finally, the estimate $k'(r) = \frac{4\pi r}{R^2} = \frac{4\pi r^2}{R^2}\frac{1}{r} \le 4\pi\frac{1}{r}$ yields

$$\begin{aligned}&\sum_{m=-\infty}^{\infty} (1+|m|) \int_0^R \partial_r k(r)\,|u_m(r)|^2 dr \\ &\qquad \le \sum_{m=-\infty}^{\infty} \Big(\frac{4\pi}{R^2}\int_0^R |u_m(r)|^2 r\,dr + 4\pi\, m^2 \int_0^R |u_m(r)|^2 \frac{1}{r}\,dr\Big) \\ &= \frac{2}{R^2}\|u\|_{B_R}^2 + 2\left\|\frac{1}{r}\partial_\varphi u\right\|_{B_R}^2 \le \frac{2}{R^2}\|u\|_{B_R}^2 + 2\,|u|_{B_R}^2 \le 2\max\big(1, \frac{1}{R^2}\big)\|u\|_{H_1(B_R)}^2. \end{aligned} \tag{4.41}$$

(4.37) is a consequence of (4.38) – (4.41). ■

It is clear that $u \in H_1(B_R)$ has a trace on every circle ∂B_{R_1} with $0 < R_1 \le R$, since the restriction of u to B_{R_1} belongs to $H_1(B_{R_1})$. How much do the traces $u|_{\partial B_{R_1}}$ and $u|_{\partial B_{R_2}}$ of $u \in H_1(B_R)$ differ on nearby surfaces ∂B_{R_1} and ∂B_{R_2}? This difference cannot be estimated immediately, since $u|_{\partial B_{R_1}}$ and $u|_{\partial B_{R_2}}$ have different domains of definition. However, since $\widetilde{u|_{\partial B_{R_1}}}$ and $\widetilde{u|_{\partial B_{R_2}}}$ are both defined on the interval $[0, 2\pi)$, the difference of these functions is defined. Moreover, since by Definition 4.14 we have $H_{1/2}(\partial B_{R_i}) \subseteq L^2(\partial B_{R_i})$ for $i = 1, 2$, it follows that $u|_{\partial B_{R_i}} \in L^2(\partial B_{R_i})$ and $\widetilde{u|_{\partial B_{R_i}}} \in L^2([0, 2\pi])$, so the difference $\widetilde{u|_{\partial B_{R_1}}} - \widetilde{u|_{\partial B_{R_2}}}$ belongs to $L^2([0, 2\pi])$ and can be estimated in the norm of this space. Such an estimate is given in the next theorem.

Theorem 4.17 (Trace theorem II) *Let $0 < R_1 \le R_2 \le R$. Then we have for $u \in H_1(B_R)$ that*

$$\big\|\widetilde{u|_{\partial B_{R_1}}} - \widetilde{u|_{\partial B_{R_2}}}\big\|_{[0,2\pi]} \le \frac{|R_1 - R_2|^{1/2}}{R_1^{1/2}}\|u\|_{H_1(B_R)}. \tag{4.42}$$

Proof: Let $T_{R_i} : H_1(B_R) \to H_{1/2}(\partial B_{R_i})$ be the trace maps and let $\tilde{T}_{R_i} : H_1(B_R) \to H_{1/2}([0, 2\pi])$ be defined by $\tilde{T}_{R_i} u = \widetilde{T_{R_i} u}$. With these mappings (4.42) can be written in the form

$$\|(\tilde{T}_{R_1} - \tilde{T}_{R_2})u\|_{[0,2\pi]} \le \frac{|R_1 - R_2|^{1/2}}{R_1^{1/2}} \|u\|_{H_1(B_R)}. \tag{4.43}$$

It suffices to prove that this estimate holds for $u \in C_1(\overline{B_R})$. To see this, note that from Definition 4.14 and from (4.37) it follows that

$$\|\tilde{T}_{R_i}u\|_{[0,2\pi]} \le \|\tilde{T}_{R_i}u\|_{H^{\text{per}}_{1/2}([0,2\pi])} = \frac{1}{\sqrt{R}} \|T_{R_i}u\|_{H_{1/2}(\partial B_R)} \le \frac{1}{\sqrt{R}} C_R \|u\|_{H_1(B_R)},$$

hence $\tilde{T}_{R_i} : H_1(B_R) \to L^2([0,2\pi])$ is a continuous linear mapping for $i = 1,2$, whence also the difference $\tilde{T}_{R_1} - \tilde{T}_{R_2}$ is continuous. Because of the continuity it follows that if (4.43) holds for all u from the dense subset $C_1(\overline{B_R})$ of $H_1(B_R)$, then it holds also for all $u \in H_1(B_R)$.

Thus, assume that $u \in C_1(\overline{B_R})$ and that $0 < R_1 < R_2$. We employ Cauchy-Schwarz' inequality and (4.28), (4.31) to compute

$$\begin{aligned}
&\|\widetilde{u|_{\partial B_{R_1}}} - \widetilde{u|_{\partial B_{R_2}}}\|^2_{[0,2\pi]} = \int_0^{2\pi} |\tilde{u}(R_1,\varphi) - \tilde{u}(R_2,\varphi)|^2 d\varphi = \int_0^{2\pi} \Big| \int_{R_1}^{R_2} \partial_r \tilde{u}(r,\varphi)\, dr \Big|^2 d\varphi \\
&\le \int_0^{2\pi} (R_2 - R_1) \int_{R_1}^{R_2} |\partial_r \tilde{u}(r,\varphi)|^2 dr d\varphi \le (R_2 - R_1) \int_0^{2\pi} \int_{R_1}^{R_2} |\partial_r \tilde{u}(r,\varphi)|^2 \frac{r}{R_1}\, dr d\varphi \\
&\le \frac{R_2 - R_1}{R_1} \int_0^{2\pi} \int_0^{R} |\partial_r \tilde{u}(r,\varphi)|^2 r\, dr d\varphi = \frac{R_2 - R_1}{R_1} \|\partial_r u\|^2_{B_R} \le \frac{R_2 - R_1}{R_1} \|u\|^2_{H_1(B_R)}.
\end{aligned}$$

The estimate (4.42) results by taking the square root on both sides. ■

4.6 Weak solutions of inhomogeneous boundary value problems

In Definition 3.32 we introduced the notion of weak solutions to homogeneous boundary value problems for the Helmholtz and potential equations. With the trace theorem we can extend this definition to inhomogeneous boundary data. Of course, since we proved the trace theorem only for balls $B_R \subseteq \mathbb{R}^2$, we must restrict ourselves to boundary value problems in balls.

Definition 4.18 *Assume that $\lambda \in \mathbb{C}$, that $\mathsf{u} \in H_{1/2}(\partial B_R)$, and $f \in L^2(B_R)$. A function $u \in H_1(B_R)$ is called weak solution of the Dirichlet boundary value probelm*

$$\Delta u(x) + \lambda u(x) = f(x), \qquad x \in B_R, \tag{4.44}$$

$$u|_{\partial B_R} = \mathsf{u}, \tag{4.45}$$

if u is a weak solution of the partial differential equation (4.44) *and if $u|_{\partial B_R}$ in* (4.45) *is understood to be the trace of u on ∂B_R.*

Since weak solutions u are required to belong to $H_1(B_R)$, it is clear that one is restricted to choose the boundary data u from the space $H_{1/2}(\partial B_R)$, since the trace of u belongs to this space. In Theorem 4.2 and Theorem 4.5 we constructed soutions of the problem (4.44), (4.45) with $f = 0$, but with u from the space $L^2(\partial B_R)$, which is larger than $H_{1/2}(\partial B_R)$. Therefore the solutions u constructed in these theorems will in general not be weak solutions in the sense of Definition 4.18. In particular, if $\mathsf{u} \in L^2(\partial B_R)\backslash H_{1/2}(\partial B_R)$, then u will not belong to $H_1(B_R)$. We thus see that the weak solutions defined above are not the most general solutions of boundary value problems which we can define and prove to exist. However, we have:

Theorem 4.19 *If* $\mathsf{u} \in H_{1/2}(\partial B_R)$, *then the solutions of the boundary value problem* (4.44), (4.45) *with* $f = 0$, *which are constructed in Theorem 4.2 and Theorem 4.5, are weak solutions.*

Proof: We give the proof only for $\lambda \in \mathbb{R}$, since we use the estimates from Lemma 4.8 in the appendix, which we only formulated and proved for real λ. Moreover, we assume for simplicity that λ is not an eigenvalue.

Let

$$\tilde{\mathsf{u}}(\varphi) = \sum_{m=-\infty}^{\infty} a_m e^{im\varphi}$$

be the Fourier series of the boundary data $\mathsf{u} \in H_{1/2}(\partial B_R)$. It must be shown that the solution u of (4.44), (4.45) to these boundary data belongs to $H_1(B_R)$. The solution is given by the series representation (4.17). From this representation and form (4.30), (4.31) and (4.33) we obtain

$$\begin{aligned} \|u\|^2_{H_1(B_R)} &= \|u\|^2_{B_R} + |u|^2_{B_R} \\ &= 2\pi \sum_{m=-\infty}^{\infty} \int_0^R |a_m U_m(\lambda, r)|^2 r + |a_m U'_m(\lambda, r)|^2 r + m^2 |a_m U_m(\lambda, R)|^2 \frac{1}{r}\, dr \end{aligned} \quad (4.46)$$

It must be shown that the infinite sum on the right hand side of this equation is finite. To this end note that the definition of U_m in (4.18) and the asymptotics of the Bessel functions given in (A.8) imply that $U_m(\lambda, r) = O(r^m)$ and $U'_m(\lambda, r) = O(1)$ for $r \to 0$, hence all the integrals in this sum exist. To show that the sum converges it therefore suffices to estimate the terms with $|m| \geq m_\lambda$, where m_λ is the number introduced in

Lemma 4.8. Yet, (4.19) yields

$$2\pi \sum_{|m|\geq m_\lambda} \int_0^R |a_m U_m(\lambda,r)|^2 r + |a_m U_m'(\lambda,r)|^2 r + m^2 |a_m U_m(\lambda,R)|^2 \frac{1}{r}\, dr$$

$$\leq 2\pi K^2 \sum_{|m|\geq m_\lambda} |a_m|^2 \int_0^R \Big(\frac{r}{R}\Big)^{2\varepsilon_\lambda |m|} \Big(r + 2m^2 \frac{1}{r}\Big)\, dr$$

$$= 2\pi K^2 \sum_{|m|\geq m_\lambda} |a_m|^2 \left(\Big(\frac{r^2}{2\varepsilon_\lambda |m| + 2} + \frac{2m^2}{2\varepsilon_\lambda |m|} \Big) \Big(\frac{r}{R}\Big)^{2\varepsilon_\lambda |m|} \right)_{r=0}^{r=R}$$

$$\leq 2\pi K^2 \Big(\frac{R^2}{2\varepsilon_\lambda m_\lambda + 2} + \frac{1}{\varepsilon_\lambda} \Big) \sum_{|m|\geq m_\lambda} (1+|m|)|a_m|^2 \leq K^2 \Big(\frac{R}{2\varepsilon_\lambda m_\lambda + 2} + \frac{1}{\varepsilon_\lambda R} \Big) \|\mathsf{u}\|^2_{H_{1/2}(\partial B_R)}.$$

In the last step we used Definition 4.14. From this estimate and from (4.46) we see that $u \in H_1(B_R)$. It remains to show that the boundary condition (4.45) holds in the sense of traces. For this aim note that by Theorem 4.2 and Theorem 4.5 the solution u belongs to the space $C_1(B_R)$; hence for $r < R$ the trace of u on ∂B_r coincides with the restriction of u to ∂B_r. By these theorems the restriction satisfies $\lim_{r\to R} \|\widetilde{u|_{\partial B_r}} - \tilde{\mathsf{u}}\|_{[0,2\pi]} = 0$. On the other hand, by Theorem 4.17 the traces satisfy $\lim_{r\to R} \|\widetilde{u|_{\partial B_r}} - \widetilde{u|_{\partial B_R}}\|_{[0,2\pi]} = 0$. This implies $\widetilde{u|_{\partial B_R}} = \tilde{\mathsf{u}}$, whence $u|_{\partial B_R} = \mathsf{u}$. ∎

Corollary 4.20 *The trace map $T : H_1(B_R) \to H_{1/2}(\partial B_R)$ is surjective.*

Proof: For every $\mathsf{u} \in H_{1/2}(\partial B_R)$ the solution $u \in H_1(B_R)$ of the Dirichlet boundary value problem (4.44), (4.45) has trace equal to u, hence every function of $H_{1/2}(\partial B_R)$ is the trace of a function in $H_1(B_R)$. ∎

Remark. The space $H_{1/2}(\partial B_R)$ is therefore just the space of traces of functions in $H_1(B_R)$. One could use this as the definition of $H_{1/2}(B_R)$. However, from this abstract definition one does not see what properties the functions in $H_{1/2}(\partial B_R)$ have.

Of course, the trace map is not injective.

5 Maximum principle, subsolutions, Perron's method

In this section we only consider real valued solutions of partial differential equations.

5.1 Maximum principle

Theorem 5.1 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, let $g : \Omega \to \mathbb{R}, f : \Omega \to \mathbb{R}$ and assume that for the function $u \in C(\overline{\Omega}, \mathbb{R})$ the partial derivatives $\frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i^2}$ exist in Ω for $i = 1, \ldots n$ and that u satisfies*

$$\Delta u(x) - g(x)u(x) = f(x), \quad x \in \Omega.$$

(i) *If $g \geq 0$ in Ω, then for all $x \in \Omega$*

$$u(x) \leq \max\big(0, \max_{y \in \partial\Omega} u(y)\big), \qquad \text{if } f \geq 0 \text{ in } \Omega,$$
$$u(x) \geq \min\big(0, \min_{y \in \partial\Omega} u(y)\big), \qquad \text{if } f \leq 0 \text{ in } \Omega.$$

(weak maximum principle)

(ii) *If $g > 0$ in Ω, then for all $x \in \Omega$*

$$u(x) \leq 0 \quad \text{or} \quad u(x) < \max_{y \in \partial\Omega} u(y), \qquad \text{if } f \geq 0 \text{ in } \Omega,$$
$$u(x) \geq 0 \quad \text{or} \quad u(x) > \min_{y \in \partial\Omega} u(y), \qquad \text{if } f \leq 0 \text{ in } \Omega.$$

(strong maximum principle)

Proof: (i) We first consider the case $f \geq 0$. Assume that the statement is false. Then there is $x_0 \in \Omega$ such that $u(x_0) > 0$ and

$$\max_{x \in \overline{\Omega}} u(x) = u(x_0) > \max_{y \in \partial\Omega} u(y).$$

Define $v : \overline{\Omega} \to \mathbb{R}$ by

$$v(x) = u(x) + \varepsilon |x|^2, \quad x \in \overline{\Omega},$$

where $\varepsilon > 0$ is chosen small enough such that

$$v(x_0) = u(x_0) + \varepsilon|x_0|^2 > \max_{y \in \partial\Omega}(u(y) + \varepsilon|y|^2) = \max_{y \in \partial\Omega} v(y),$$
$$\max_{y \in \overline{\Omega}} \varepsilon|y|^2 < u(x_0).$$

v is continuous on the compact set $\overline{\Omega}$ and therefore assumes its maximum in a point $z \in \overline{\Omega}$. By the choice of ε we have $z \notin \partial\Omega$ and

$$u(z) = v(z) - \varepsilon|z|^2 > v(x_0) - u(x_0) = \varepsilon|x_0|^2 \geq 0.$$

Thus, the maximum z belongs to the open set Ω, which implies that

$$\frac{\partial v}{\partial x_i}(z) = 0, \quad \frac{\partial^2 v}{\partial x_i^2}(z) \leq 0, \quad i = 1, \ldots, n,$$

whence

$$\Delta v(z) = \sum_{i=1}^{n} \frac{\partial^2 v}{\partial x_i^2}(z) \leq 0.$$

On the other hand

$$\begin{aligned}\Delta v(z) = \Delta u(z) + \Delta(\varepsilon|x|^2)\big|_{x=z} &= g(z)u(z) + \varepsilon \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} x_i^2 + f(z) \\ &= g(z)u(z) + 2n\varepsilon + f(z) > 0,\end{aligned}$$

because of $g(z) \geq 0, u(z) > 0, f(z) \geq 0$ and $\varepsilon > 0$. This is a contradiction, hence

$$\max_{x \in \overline{\Omega}} u(x) = u(x_0) \leq \max(0, \max_{y \in \partial\Omega} u(y)).$$

If $f \leq 0$ define $w = -u$. Then $\Delta w(x) - g(x)w(x) = -f(x) \geq 0$ for all $x \in \Omega$, hence

$$\begin{aligned}-\min_{x \in \overline{\Omega}} u(x) = \max_{x \in \overline{\Omega}} w(x) &\leq \max\big(0, \max_{y \in \partial\Omega} w(y)\big) \\ &= \max\big(0, -\min_{y \in \partial\Omega} u(y)\big) = -\min\big(0, \min_{y \in \partial\Omega} u(y)\big),\end{aligned}$$

which implies the statement for the minimum.

(ii) Let $g > 0$ and $f \geq 0$, and assume that the statement for the maximum is false. Then there is $x_0 \in \Omega$ such that $u(x_0) > 0$ and

$$u(x_0) = \max_{x \in \overline{\Omega}} u(x) \geq \max_{y \in \partial\Omega} u(y).$$

Consequently, x_0 is a local maximum of u in the open set Ω, hence

$$\frac{\partial u}{\partial x_i}(x_0) = 0, \quad \frac{\partial^2 u}{\partial x_i^2}(x_0) \leq 0, \quad i = 1, \ldots, n,$$

whence

$$0 \geq \Delta u(x_0) = g(x_0)u(x_0) + f(x_0) > 0,$$

which is a contradiction, and the statement for the maximum must be true. The statement for the minimum is proved by considering $-u$. ■

We note some consequences of the maximum principle:

Corollary 5.2 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, let $g : \Omega \to \mathbb{R}_0^+$, $f : \Omega \to \mathbb{R}$ be given. Assume that $u, v \in C(\overline{\Omega}, \mathbb{R})$ and that for $w = u$ and $w = v$ the partial derivatives $\frac{\partial w}{\partial x_i}$, $\frac{\partial^2 w}{\partial x_i^2}$ exist in Ω for $i = 1, \ldots n$ and the equation*

$$\Delta w(x) - g(x)w(x) = f(x), \quad x \in \Omega$$

holds. Then the following statements are true:

(i) *If $u(y) \geq v(y)$ for all $y \in \partial\Omega$ then $u(x) \geq v(x)$ for all $x \in \overline{\Omega}$.*

(ii) *For all $x \in \Omega$,*

$$|u(x) - v(x)| \leq \max_{y \in \partial\Omega} |u(y) - v(y)|.$$

(iii) *Let $g > 0$ and $f = 0$ in Ω. If u assumes the maximum in Ω, then $u \leq 0$. If u assumes the minimum in Ω, then $u \geq 0$.*

Proof: (i) $w = u - v$ satisfies $\Delta w - gw = 0$ in Ω. Hence, by the weak maximum principle

$$0 = \min\big(0, \min_{y \in \partial\Omega} w(y)\big) \leq w(x) = u(x) - v(x), \; x \in \Omega.$$

(ii) Again we apply the weak maximum principle to $w = u - v$ and obtain

$$\begin{aligned}\min_{y \in \partial\Omega}(-|w(y)|) &\leq \min\big(0, \min_{y \in \partial\Omega} w(y)\big) \leq w(x) \\ &\leq \max\big(0, \max_{y \in \partial\Omega} w(y)\big) \leq \max_{y \in \partial\Omega} |w(y)|.\end{aligned}$$

Thus,

$$w(x) \leq \max_{y \in \partial\Omega} |w(y)| \qquad \text{and} \qquad -w(x) \leq -\min_{y \in \partial\Omega}(-|w(y)|) = \max_{y \in \partial\Omega} |w(y)|,$$

whence $|w(x)| \leq \max_{y \in \partial\Omega} |w(y)|$.

(iii) Let $x_0 \in \Omega$ and assume that $u(x_0) = \max_{x \in \overline{\Omega}} u(x)$. By the strong maximum principle this can only be if $u(x_0) \leq 0$. The statement for the minimum is proved in the same way.

■

Corollary 5.3 (Uniqueness) *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, let $g : \Omega \to \mathbb{R}_0^+$, $f : \Omega \to \mathbb{R}$ and let the functions $u, v \in C(\overline{\Omega}, \mathbb{R})$ have the differentiability properties stated in Corollary 5.2. Assume that $u|_{\partial\Omega} = v|_{\partial\Omega}$ and that u and v both satisfy the differential equation*

$$\Delta w(x) - g(x)\, w(x) = f(x), \quad x \in \Omega\,.$$

Then $u = v$.

Proof: The preceding corollary yields

$$|u(x) - v(x)| \le \max_{y \in \partial\Omega} |u(y) - v(y)| = 0, \quad x \in \Omega.$$

■

5.2 Continuity of solutions to the Helmholtz equation up to the boundary

In Section 4 we constructed a solution of the Dirichlet problem

$$\Delta u(x) + \lambda u(x) = 0, \qquad x \in B, \tag{5.1}$$

$$u(x) = \mathsf{g}(x), \qquad x \in \partial B, \tag{5.2}$$

for the Helmholtz equation in a ball $B \subseteq \mathbb{R}^2$, which is infinitely differentiable in the interior of B. For $\mathsf{g} \in L^2(\partial B)$ we proved in Sections 4.2 and 4.3 that u satisfies the boundary condition (5.2) in the L^2–sense, which is defined in (4.5), and for $\mathsf{g} \in H_{1/2}(\partial B)$ we showed in Sections 4.4 – 4.6 that u satisfies the boundary condition in the trace sense. In this section we use the maximum principle to prove that for $\mathsf{g} \in C(\partial B)$ the solution u satisfies the boundary condition in the classical sense, that is we prove that $u \in C(\overline{B})$. This result holds for all values of λ, but since we use the maximum principle for the proof, we must assume that $\lambda \le 0$.

As preparation we show first that the solutions constructed in Section 4 depend continuously on the boundary data in L^2. As we did previously, we denote by B_r the ball in $\mathbb{R}^2$ with center 0 and radius $r > 0$.

Theorem 5.4 *Let $\lambda \le 0$ and assume that $\{\mathsf{w}_m\}_{m\in\mathbb{N}} \subseteq L^2(\partial B_R)$ is a sequence, which converges in $L^2(\partial B_R)$ to 0. Let w_m be the solutions of the boundary value problem* (5.1), (5.2) *to the boundary data $\mathsf{g} = \mathsf{w}_m$. Then for every $0 < r_1 < R$ the sequence $\{w_m\}_{m\in\mathbb{N}}$ converges uniformly to 0 in the ball $\overline{B}_{r_1}$.*

Proof. By (4.17), the solution is given by the series

$$w_m(r,\varphi) = \sum_{n=-\infty}^{\infty} a_n^{(m)} U_n(\lambda, r) e^{in\varphi},$$

where $\{a_n^{(m)}\}_{n\in\mathbb{Z}}$ are the coefficients in the Fourier series of w_m. By Remark 3.14 we have

$$2\pi \sum_{n=-\infty}^{\infty} |a_n^{(m)}|^2 = \|\mathsf{w}_m\|^2_{[0,2\pi]} \to 0, \quad \text{for } m \to \infty. \tag{5.3}$$

Lemma 4.8 yields for $\lambda \leq 0$, $n \neq 0$, $\alpha = 0$ and $0 \leq r \leq R$ that

$$|U_n(\lambda, r)| \leq C_1\big(\frac{r}{R}\big)^{|n|}.$$

We set $C = \max\{C_1, \max_{0\leq r\leq R} |U_0(\lambda, r)|\}$ and conclude from this estimate and from (5.3) by application of Cauchy-Schwarz' inequality that for all $0 \leq r \leq r_1$ and $0 \leq \varphi < 2\pi$

$$\begin{aligned}
|w_m(r,\varphi)| &\leq \sum_{n=-\infty}^{\infty} \big|a_n^{(m)} U_n(\lambda, r)\big| \\
&\leq \Big(\sum_{n=-\infty}^{\infty} |a_n^{(m)}|^2\Big)^{\frac{1}{2}} \Big(\sum_{n=-\infty}^{\infty} |U_n(\lambda, r)|^2\Big)^{\frac{1}{2}} \leq \frac{\|\mathsf{w}_m\|_{[0,2\pi]}}{\sqrt{2\pi}} \Big(\sum_{n=-\infty}^{\infty} C^2 \Big(\frac{r}{R}\Big)^{2|n|}\Big)^{\frac{1}{2}} \\
&\leq \frac{\|\mathsf{w}_m\|_{[0,2\pi]}}{\sqrt{2\pi}} \Big(\frac{2C^2}{1-(\frac{r}{R})^2}\Big)^{\frac{1}{2}} \leq \frac{C}{\sqrt{\pi}}\Big(1-\Big(\frac{r_1}{R}\Big)^2\Big)^{-\frac{1}{2}} \|\mathsf{w}_m\|_{[0,2\pi]}.
\end{aligned}$$

The term on the right hand side of this inequality is independent of r and φ and converges to zero. Therefore the sequence $\{w_m\}_{m\in\mathbb{N}}$ converges to zero, uniformly in the closed ball $\overline{B}_{r_1}$. ∎

Corollary 5.5 *Let $\lambda \leq 0$ and assume that $\{\mathsf{u}_m\}_{m\in\mathbb{N}} \subseteq L^2(\partial B_R)$ is a sequence, which converges in $L^2(\partial B_R)$ to $\mathsf{u} \in L^2(\partial B_R)$. Let u_m and u be the solutions of the boundary value problem* (5.1), (5.2) *to the boundary data $\mathsf{g} = \mathsf{u_m}$ and $\mathsf{g} = \mathsf{u}$, respectively. Then the sequence $\{u_m\}_{m\in\mathbb{N}}$ converges to the solution u, uniformly in every ball $\overline{B}_{r_1}$ with $0 < r_1 < R$.*

Proof. Set $\mathsf{w}_m = \mathsf{u}_m - \mathsf{u}$ and $w_m = u_m - u$. The function w_m solves the boundary value problem (5.1), (5.2) to the boundary data w_m. Since the sequence $\{\mathsf{w}_m\}_{m\in\mathbb{N}}$ converges to zero in $L^2(\partial B_R)$, Theorem 5.4 implies that w_m converges to zero uniformly in $\overline{B}_{r_1}$ for every $0 < r_1 < R$. ∎

Theorem 5.6 *Let $B \subseteq \mathbb{R}^2$ be a bounded open ball and let $\lambda \leq 0$. Then for every $\mathsf{u} \in C(\partial B, \mathbb{R})$ there is a unique solution $u \in C(\overline{B}, \mathbb{R}) \cap C_2(B, \mathbb{R})$ of the Dirichlet problem* (5.1), (5.2) *with boundary data $\mathsf{g} = \mathsf{u}$. The solution coincides with the solution constructed in Section 4 satisfying the boundary condition in the L^2–sense.*

Proof: Without restriction of generality we can assume that $B = B_R(0)$ with $R > 0$. The uniqueness follows from Corollary 5.3. To prove that the solution constructed in Section 4 is continuous up to the boundary, let

$$\mathsf{u}(\varphi) = \sum_{m=-\infty}^{\infty} a_m\, e^{im\varphi}$$

be the Fourier series of the boundary data and define for $m \geq 0$, $n \geq 1$, $\varphi \in [0, 2\pi)$, and $r \geq 0$

$$
\begin{aligned}
s_m(\varphi) &= \sum_{k=-m}^{m} a_k e^{ik\varphi}, \\
\sigma_n(\varphi) &= \frac{1}{n}\big(s_0(\varphi) + \ldots + s_{n-1}(\varphi)\big), \\
v_m(r,\varphi) &= \sum_{k=-m}^{m} a_k U_k(\lambda, r) e^{ik\varphi}, \\
\nu_n(r,\varphi) &= \frac{1}{n}\big(v_0(r,\varphi) + \ldots + v_{n-1}(r,\varphi)\big).
\end{aligned}
$$

where U_n is defined in (4.18). All of these functions are real valued. To see this, note that by Remark 3.14 we have

$$
\begin{aligned}
a_k e^{ik\varphi} + a_{-k} e^{-ik\varphi} &= \frac{1}{2\pi} \int_0^{2\pi} \mathsf{u}(\xi)\Big(e^{ik(\varphi-\xi)} + e^{-ik(\varphi-\xi)}\Big) d\xi \\
&= \frac{1}{\pi} \int_0^{2\pi} \mathsf{u}(\xi) \cos\big(k(\varphi-\xi)\big) d\xi \in \mathbb{R}, \quad \text{for all } k \in \mathbb{Z} \text{ and } \varphi \in [0, 2\pi).
\end{aligned}
$$

From this relation it follows immediately that s_m and σ_n are real valued. To see that v_m is real valued, note that by definition in (4.18) we have $U_{-k} = U_k$ and that by Lemma A.2 in the appendix we have $U_k(\lambda, r) \in \mathbb{R}$ for real λ. Therefore we conclude that

$$
a_k U_k(\lambda, r) e^{ik\varphi} + a_{-k} U_{-k}(\lambda, r) e^{-ik\varphi} = U_k(\lambda, r)\big(a_k e^{ik\varphi} + a_{-k} e^{-ik\varphi}\big) \in \mathbb{R},
$$

which implies that v_m and ν_n are real valued functions.

The function $U_k(\lambda, r) e^{ik\varphi}$ is an infinitely differentiable solution of the Helmholtz equation in all of $\mathbb{R}^2$, hence the finite linear combinations v_m and ν_n are infinitely differentiable solutions of the Helmholtz equation in $\mathbb{R}^2$. Since by (4.18) we have $U_k(\lambda, R) = 1$, we obtain $v_m(R, \varphi) = s_m(\varphi)$, and therefore also $\nu_n(R, \varphi) = \sigma_n(\varphi)$, for all $\varphi \in [0, 2\pi)$.

It follows that ν_n is a solution of the boundary value problem (5.1), (5.2) to the boundary data σ_n. Since ν_n is a finite linear combination of the infinitely differentiable functions $U_k(\lambda, r) e^{ik\varphi}$, which are defined on all of $\mathbb{R}^2$, the function ν_n is infinitely differentiable up to the boundary and therefore, in particular, continuous up to the boundary. ν_n is equal to the solution of this boundary value problem constructed in Section 4, because ν_n and the solutions from Section 4 are represented as a series of functions of the form $b_k U_k(\lambda, r) e^{ik\varphi}$ with uniquely determined coefficients b_k.

Since by assumption $\mathsf{u} \in C(\partial B_R)$, it follows from Theorem 3.13 (Theorem of Fejér), that the sequence $\{\sigma_n\}_{n \in \mathbb{N}}$ converges to the function u uniformly on ∂B_R, whence it also converges to u in $L^2(\partial B_R)$. By Corollary 5.5 this means that the sequence of solutions $\{\nu_n\}_{n\in\mathbb{N}}$ converges to the solution u constructed in Section 4 of the Helmholtz equation to the boundary data u. The convergence is uniform in every ball B_{r_1} with $r_1 < R$, hence $\{\nu_n\}_{n\in\mathbb{N}}$ converges to u pointwise in all of B_R.

On the other hand, since $\lambda \le 0$, we can apply the maximum principle. Statement (ii) of Corollary 5.2 yields

$$\sup_{x\in\overline{B}_R} |\nu_n(x) - \nu_\ell(x)| \le \max_{y\in\partial B_R} |\sigma_n(y) - \sigma_\ell(y)|,$$

which means that the sequence $\{\nu_n\}_{n\in\mathbb{N}}$ converges uniformly on $\overline{B}_R$ to u, hence, as the uniform limit of continuous functions, the solution u belongs to $C(\overline{B}_R, \mathbb{R})$. ■

5.3 Mean value property and improved maximum principle

In this section we derive the mean value property of solutions of the Helmholtz equation and use it to improve the maximum principle for the Helmholtz equation.

Theorem 5.7 *Let $B_R(0) \subseteq \mathbb{R}^2$, let $\lambda \le 0$ and let $u \in C(\overline{B_R(0)}, \mathbb{R}) \cap C_2(B_R(0), \mathbb{R})$ solve*

$$\Delta u(x) + \lambda u(x) = 0, \quad x \in B_R(0).$$

Then

$$u(0) = \frac{1}{2\pi R J_0(\sqrt{\lambda}R)} \int_{|x|=R} u(x)\, ds.$$

Since $J_0(0) = 1$, this formula becomes for $\lambda = 0$

$$u(0) = \frac{1}{2\pi R} \int_{|x|=R} u(x)\, ds.$$

Proof: Let $\lambda < 0$. From the preceding investigations we know that if

$$u(R, \varphi) = \sum_{m=-\infty}^{\infty} a_m e^{im\varphi},$$

then

$$u(r, \varphi) = \sum_{m=-\infty}^{\infty} \frac{a_m}{J_{|m|}(\sqrt{\lambda}R)} J_{|m|}(\sqrt{\lambda}r) e^{im\varphi}, \quad 0 < r \le R.$$

From the power series expansion (A.2) we see that $J_0(0) = 1$ and $J_m(0) = 0$ for $m \ge 1$, hence

$$u(0) = \frac{a_0}{J_0(\sqrt{\lambda}R)} = \frac{1}{2\pi J_0(\sqrt{\lambda}R)} \int_0^{2\pi} u(R, \varphi) d\varphi = \frac{1}{2\pi R J_0(\sqrt{\lambda}R)} \int_{|x|=R} u(x) ds.$$

To prove the statement for $\lambda = 0$ we proceed in the same way, using the Fourier expansion of u. ■

Corollary 5.8 *Let $\Omega \subseteq \mathbb{R}^2$ be a open, connected set, let $\lambda \leq 0$ and let $u \in C_\infty(\Omega, \mathbb{R})$ be a solution of*

$$\Delta u(x) + \lambda u(x) = 0, \quad x \in \Omega.$$

Assume that $x_0 \in \Omega$ exists such that $u(x_0) = 0$. Then, if $u \geq 0$ or $u \leq 0$ in Ω it follows that $u = 0$ in Ω.

Proof: Let

$$M = \{x \in \Omega \mid u(x) = 0\}.$$

By assumption, M is not empty since $x_0 \in \Omega$. We prove that M is closed and open in Ω, which implies $M = \Omega$, since Ω is connected.

Since u is continuous, M is obviously closed. To verify that M is open, let $y \in M$. Since Ω is open there exists a ball $B_R(y)$ with center y contained in Ω. By the mean value property we have for all $0 < r < R$

$$\frac{1}{2\pi r} \int_{|x-y|=r} u(x)\, ds_x = J_0(\sqrt{\lambda} r) u(y) = 0.$$

If $u \geq 0$ or $u \leq 0$ in Ω this can only be if $u(x) = 0$ for all x with $|x - y| = r$. This holds for all $0 < r < R$, hence $u(x) = 0$ for all $x \in B_R(y)$. Thus, $B_R(y) \subseteq M$, hence M is open. The proof is complete. ∎

Corollary 5.9 *Let Ω be a bounded, open, connected set and let $u \in C(\overline{\Omega}, \mathbb{R})$ be a solution of*

$$\Delta u(x) + \lambda u(x) = 0, \;\; x \in \Omega.$$

(i) If $\lambda < 0$, then $u = 0$ or, for all $x \in \Omega$,

$$\min(0, \min_{y \in \partial\Omega} u(y)) < u(x) < \max(0, \max_{y \in \partial\Omega} u(y)).$$

(ii) If $\lambda = 0$, then $u = \text{const}$ or, for all $x \in \Omega$,

$$\min_{y \in \partial\Omega} u(y) < u(x) < \max_{y \in \partial\Omega} u(y).$$

Proof: (i) Combine the strong maximum principle with the preceding result.
(ii) Assume that there is $x_0 \in \Omega$ such that $u(x_0) = \max_{y \in \overline{\Omega}} u(y)$. Then the function $v \in C(\overline{\Omega}) \cap C_\infty(\Omega)$ defined by $v(x) = u(x) - u(x_0)$ satisfies $\Delta v = 0$, $v \leq 0$ and $v(x_0) = 0$. Since x_0 is an interior point of Ω, Corollary 5.8 implies that $v = 0$, hence $u = \text{const} = u(x_0)$. In the same way it follows that if there is $x_1 \in \Omega$ such that $u(x_1) = \min_{y \in \overline{\Omega}} u(y)$, then $u = \text{const} = u(x_1)$. Statement (ii) follows from these results. ∎

5.4 Subsolutions, supersolutions, comparison

Up to now we only know how to solve boundary value problems in circular domains. Our goal is to develop a method to solve boundary value problems in very general domains $\Omega \subseteq \mathbb{R}^2$. To this end we need subsolutions and supersolutions, which we define and discuss in this section.

Thus, let $\Omega \subseteq \mathbb{R}^2$ be a bounded open domain and let $\lambda \leq 0$.

Definition 5.10 *A function $v \in C(\overline{\Omega}, \mathbb{R})$ is called subsolution (supersolution) of the equation $\Delta u + \lambda u = 0$ in $\overline{\Omega}$, if to every open ball B with $\overline{B} \subseteq \Omega$ the uniquely determined solution $u \in C(\overline{B})$ of*

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, \ x \in B \\ u_{|\partial B} &= v_{|\partial B}, \end{aligned}$$

satisfies

$$v(x) \leq u(x), \quad (v(x) \geq u(x)),$$

for all $x \in B$.

Remark 5.11 In the case $\lambda = 0$ a subsolution is also called a subharmonic function, since a solution u of $\Delta u(x) = 0$ is called a harmonic function.

Theorem 5.12 *Let $v_1, \ldots, v_m$ be subsolutions of $\Delta u + \lambda u = 0$. Then*

$$v = \max(v_1, \ldots, v_m)$$

is a subsolution.

Proof: Let $v^{(2)} = \max(v_1, v_2)$, let $\overline{B} \subseteq \Omega$ be a closed ball and let $u_1, u_2, u^{(2)} \in C(\overline{B})$ be solutions of

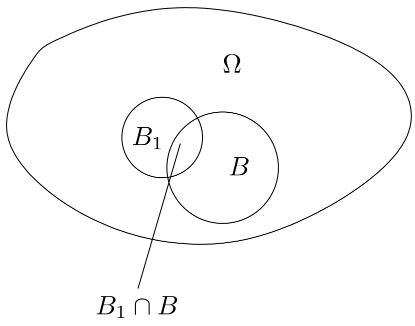
$$\Delta w + \lambda w = 0$$

in B satisfying

$$u_{1|\partial B} = v_{1|\partial B}, \quad u_{2|\partial B} = v_{2|\partial B}, \quad {u^{(2)}}_{|\partial B} = {v^{(2)}}_{|\partial B}.$$

For $y \in \partial B$ we have

$$u^{(2)}(y) = v^{(2)}(y) \geq v_i(y) = u_i(y), \quad i = 1, 2.$$



Ball B_1 in arbitrary position

Using the maximum principle and noting that v_1, v_2 are subsolutions we thus obtain for $x \in \overline{B}$

$$\begin{array}{rcccl} v_1(x) & \leq & u_1(x) & \leq & u^{(2)}(x) \\ v_2(x) & \leq & u_2(x) & \leq & u^{(2)}(x), \end{array}$$

which yields

$$v^{(2)}(x) = \max(v_1(x), v_2(x)) \leq u^{(2)}(x).$$

This shows that $v^{(2)}$ is a subsolution. Since

$$v^{(k)} = \max(v_1, \dots, v_k) = \max(v_k, \max(v_1, \dots, v_{k-1})) = \max(v_k, v^{(k-1)}),$$

the statement follows by induction. ∎

Theorem 5.13 *Let v be a subsolution of $\Delta u + \lambda u = 0$ in Ω, let B be an open ball with $\overline{B} \subseteq \Omega$ and let $u \in C(\overline{B}) \cap C_\infty(B)$ be the solution of*

$$\begin{array}{rcl} \Delta u(x) + \lambda u(x) & = & 0, \quad x \in B \\ u_{|\partial B} & = & v_{|\partial B} \,. \end{array}$$

Then the function $\tilde{v} : \Omega \to \mathbb{R}$,

$$\tilde{v}(x) = \begin{cases} v(x), & x \notin B \\ u(x), & x \in B \end{cases}$$

is a subsolution.

Proof: We have $\tilde{v} \geq v$ since v is a subsolution. Now let B_1 be an open ball with $\overline{B_1} \subseteq \Omega$ and let $\tilde{u} \in C(\overline{B_1})$ be the solution of

$$\begin{array}{rcl} \Delta \tilde{u}(x) + \lambda \tilde{u}(x) & = & 0, \quad x \in B_1 \\ \tilde{u}_{|\partial B_1} & = & \tilde{v}_{|\partial B_1}. \end{array}$$

We are finished if we can show that $\tilde{u} \geq \tilde{v}$ in B_1. By definition of $\tilde{v}$ this holds if

$$\begin{aligned} \tilde{u}(x) &\geq v(x), \quad x \in B_1 \setminus B \qquad (5.4)\\ \tilde{u}(x) &\geq u(x), \quad x \in B_1 \cap B. \qquad (5.5) \end{aligned}$$

To prove these relations, let $\hat{u} \in C(\overline{B_1})$ be the solution of

$$\begin{aligned} \Delta \hat{u}(x) + \lambda \hat{u}(x) &= 0, \quad x \in B_1 \\ \hat{u}_{|\partial B_1} &= v_{|\partial B_1}. \end{aligned}$$

Since v is a subsolution, it follows that $\hat{u} \geq v$ in $\overline{B_1}$. Also, $\tilde{u}$ and $\hat{u}$ solve the Helmholtz equation in B_1 and satisfy

$$\tilde{u}_{|\partial B_1} = \tilde{v}_{|\partial B_1} \geq v_{|\partial B_1} = \hat{u}_{|\partial B_1},$$

hence the maximum principle yields $\tilde{u} \geq \hat{u} \geq v$ in $\overline{B_1}$, which proves (5.4).

To verify (5.5), note that $\partial(B_1 \cap B) = (\overline{B_1} \cap \partial B) \cup (\overline{B} \cap \partial B_1)$ and that

$$\begin{aligned} \tilde{u}_{|\overline{B} \cap \partial B_1} &= \tilde{v}_{|\overline{B} \cap \partial B_1} = u_{|\overline{B} \cap \partial B_1} \\ \tilde{u}_{|\overline{B_1} \cap \partial B} &\geq v_{|\overline{B_1} \cap \partial B} = u_{|\overline{B_1} \cap \partial B}, \end{aligned}$$

where we used (5.4) to get the last relation. Since both $\tilde{u}$ and u satisfy the Helmholtz equation in $B_1 \cap B$, it follows from these relations and from the maximum principle that $\tilde{u} \geq u$ in $B_1 \cap B$. This is (5.5). The proof is complete. ■

Theorem 5.14 (Comparison) *Let $\lambda \leq 0$ and let $v \in C(\overline{\Omega}, \mathbb{R})$ be a subsolution, $w \in C(\overline{\Omega}, \mathbb{R})$ be a supersolution with $v|_{\partial\Omega} \leq w|_{\partial\Omega}$. Then $v \leq w$ in $\overline{\Omega}$.*

Proof: Assume that there is $x_0 \in \Omega$ such that $v(x_0) > w(x_0)$. Then since $h = v - w$ is less or equal to zero on $\partial\Omega$, it follows that h assumes the positive maximum at a point $z \in \Omega$. We can assume that z is a boundary point of the closed set $M = \{x \in \Omega \mid h(x) = \max_{y \in \overline{\Omega}} h(y)\} \subset \Omega$. It follows that every neighborhood of z contain points, where h assumes values smaller than $h(z)$. Therefore we can choose a ball B with center z and with $\overline{B} \subseteq \Omega$ such that ∂B contains such a point. We thus have

$$h(z) > 0, \quad h(z) \geq \max_{x \in \partial B} h(x), \quad h(z) > \min_{x \in \partial B} h(x). \qquad (5.6)$$

Now let $\hat{v}, \hat{w}$ be the solutions of

$$\begin{aligned} \Delta \hat{v}(x) + \lambda \hat{v}(x) &= 0 \text{ in } B, \qquad \hat{v}|_{\partial B} = v|_{\partial B}, \\ \Delta \hat{w}(x) + \lambda \hat{w}(x) &= 0 \text{ in } B, \qquad \hat{w}|_{\partial B} = w|_{\partial B}. \end{aligned}$$

Then $\hat{v} \geq v$ and $w \geq \hat{w}$ in B. Therefore the function

$$u = \hat{v} - \hat{w}$$

satisfies $u = \hat{v} - \hat{w} \geq v - w = h$ in B, hence

$$u(z) \geq h(z) > 0, \tag{5.7}$$

and

$$u|_{\partial B} = \hat{v}|_{\partial B} - \hat{w}|_{\partial B} = v|_{\partial B} - w|_{\partial B} = h|_{\partial B}.$$

This equation and (5.6), (5.7) imply

$$u(z) \geq \max(0, \max_{y \in \partial B} u(y)), \quad u(z) > \min_{y \in \partial B} u(y). \tag{5.8}$$

Since $\Delta u(x) + \lambda u(x) = 0$ in B, it follows in the case $\lambda < 0$ from the first of the inequalities (5.8) and from Corollary 5.9(i) that $u = 0$, which contradicts (5.7). In the case $\lambda = 0$ it follows from the first of the inequalities (5.8) and from Corollary 5.9(ii) that $u = \text{const}$, which contradicts the second of the inequalities (5.8). Consequently, in both cases we must have $v \leq w$ in $\overline{\Omega}$. ■

Theorem 5.15 *Let $\lambda \leq 0$ and let $w \in C(\overline{\Omega}) \cap C_\infty(\Omega)$ be a solution of the potential equation $\Delta w = 0$. If w is non-negative, then w is a supersolution, if w is non-positive, then w is a subsolution of the equation $\Delta u + \lambda u = 0$.*

Proof: For a ball B with $\overline{B} \subseteq \Omega$ let u be the solution of

$$\Delta u(x) + \lambda u(x) = 0, \qquad x \in B$$

$$u|_{\partial B} = w|_{\partial B}.$$

Since $\Delta w = 0$, the function $h = w - u$ satisfies

$$\Delta h(x) + \lambda h(x) = \lambda w(x) \begin{cases} \leq 0, & \text{if } w \geq 0, \\ \geq 0, & \text{if } w \leq 0. \end{cases}$$

Since also $h|_{\partial B} = 0$, we conclude from the maximum principle in the first case that $h \geq 0$, hence $u \leq w$, which shows that w is a supersolution. In the second case the maximum principle yields $h \leq 0$, hence $u \geq w$, which implies that w is a subsolution. ■

Corollary 5.16 (Maximum principle for sub- and supersolutions) *Let* $\lambda \le 0$.
(i) Any non-negative constant function is a supersolution and any non-positive constant function is a subsolution of $\Delta u + \lambda u = 0$.
(ii) If v *is a subsolution and* w *is a supersolution, then*

$$\begin{aligned} v(x) &\le \max\big(0, \max_{y\in\partial\Omega} v(y)\big), \\ w(x) &\ge \min\big(0, \min_{y\in\partial\Omega} w(y)\big). \end{aligned}$$

Proof: (i) A non-negative constant function is a supersolution and a non-positive constant function is a subsolution, since they satisfy the potential equation.
(ii) For a subsolution v define $\hat{v} : \Omega \to \mathbb{R}$ by

$$\hat{v}(x) = \text{const} = \max\big(0, \max_{y\in\partial\Omega} v(y)\big), \quad x \in \Omega.$$

Then $\hat{v} \ge 0$ is a supersolution satisfying $v|_{\partial\Omega} \le \hat{v}|_{\partial\Omega}$, whence $v \le \hat{v}$, by Theorem 5.14.
Similarly, for a supersolution w define $\hat{w} : \Omega \to \mathbb{R}$ by

$$\hat{w}(x) = \text{const} = \min\big(0, \min_{y\in\partial\Omega} w(y)\big), \quad x \in \Omega.$$

Then $\hat{w} \le 0$ is a subsolution satisfying $\hat{w}|_{\partial\Omega} \le w|_{\partial\Omega}$, whence $\hat{w} \le w$. ∎

5.5 Perron's method

For a bounded open set $\Omega \subseteq \mathbb{R}^2$, for $\lambda \le 0$ and for a function $f \in C(\partial\Omega, \mathbb{R})$ define

$$\mathcal{S}_f = \{v \in C(\overline{\Omega}) \mid v \text{ is a subsolution of } \Delta u + \lambda u = 0 \text{ with } v|_{\partial\Omega} \le f\}.$$

Note that by the preceding corollary every $v \in \mathcal{S}_f$ satisfies

$$v(x) \le \max\big(0, \max_{y\in\partial\Omega} f(y)\big).$$

Theorem 5.17 (Perron) *If* $\mathcal{S}_f \ne \emptyset$ *then*

$$u_f(x) = \sup_{v\in\mathcal{S}_f} v(x), \quad x \in \Omega,$$

satisfies $u_f \in C_\infty(\Omega)$ *and*

$$\Delta u_f(x) + \lambda u_f(x) = 0, \quad x \in \Omega.$$

(Oskar Perron, 1880 – 1975)

Proof: The proof is in two steps. In the first step we construct in a neighborhood of an arbitrary $y \in \Omega$ a solution u of the Helmholtz equation with $u(y) = u_f(y)$. In the second step we show that $u = u_f$ in this neighborhood, hence u_f is a solution of the Helmholtz equation in this neighborhood. This proves the theorem, since y was arbitrary.

I.) Let $y \in \Omega$ and choose a sequence $\{v_m\}_{m=1}^\infty \subseteq \mathcal{S}_f$ with $\lim_{m\to\infty} v_m(y) = u_f(y)$. We can assume that

$$v_1 \leq v_2 \leq v_3 \leq \ldots \,. \tag{5.9}$$

Otherwise we consider the sequence $\{\overline{v}_m\}_{m=1}^\infty$ defined by

$$\overline{v}_m = \max(v_1, \ldots, v_m).$$

This is a monotonically increasing sequence of subsolutions with $\overline{v}_m \in \mathcal{S}_f$ and $\overline{v}_m \geq v_m$, hence $\lim_{m\to\infty} \overline{v}_m(y) = u_f(y)$. Thus, let (5.9) be satisfied.

We choose an open bounded ball B with $y \in B$ and $\overline{B} \subseteq \Omega$. Consider the subsolution

$$w_m(x) = \begin{cases} v_m(x), & x \in \Omega\backslash B, \\ u_m(x), & x \in B, \end{cases}$$

where $u_m \in C(\overline{B}) \cap C_\infty(B)$ is the solution of

$$\Delta u_m(x) + \lambda u_m(x) = 0, \qquad x \in B$$
$$u_m\big|_{\partial B} = v_m\big|_{\partial B}.$$

Then $w_m \in \mathcal{S}_f$ satisfies $v_m \leq w_m$. Moreover, since $u_m\big|_{\partial B} = v_m\big|_{\partial B}$ it follows that $\{u_m\big|_{\partial B}\}_m$ is monotonically increasing, hence the maximum principle implies that $\{u_m\}_m$ and therefore also $\{w_m\}_m$ is monotonically increasing. Because the last sequence is bounded above by the function u_f, it follows that $\{w_m\}_m$ converges pointwise everywhere on $\overline{\Omega}$ to a limit function. Let u be the restriction of this limit function to $\overline{B}$. The function u is the pointwise limit of $\{u_m\}_m$ and satisfies

$$u \leq u_f, \quad u(y) = u_f(y) \tag{5.10}$$
$$\Delta u(x) + \lambda u(x) = 0, \quad x \in B. \tag{5.11}$$

(5.10) follows from

$$u_f(y) = \lim_{m\to\infty} v_m(y) \leq \lim_{m\to\infty} w_m(y) = \lim_{m\to\infty} u_m(y) \leq u_f(y).$$

To see (5.11) note that $\big(u(x) - u_m(x)\big)^2$ decreases pointwise monotonically to zero. The monotone convergence theorem of Beppo Levi therefore yields

$$\lim_{m\to\infty} \|u - u_m\|^2_{\partial B} = \lim_{m\to\infty} \int_{\partial B} \big(u(x) - u_m(x)\big)^2 dx = 0.$$

Since u_m satisfies the Helmholtz equation in B, it thus follows from Theorem 5.4 that $\{u_m\}_m$ converges uniformly in every compact subset of B to the solution of the Helmholtz equation with Dirichlet boundary data given by $u|_{\partial B}$. This solution equals u, since the pointwise and uniform limit functions coincide, whence (5.11) holds.

II.) To verify that $u|_B = u_f|_B$ let $y_1 \in B$ and choose as above a monotonically increasing sequence $\{v'_m\}_m \subseteq \mathcal{S}_f$ satisfying $\lim_{m\to\infty} v'_m(y_1) = u_f(y_1)$. We can assume that

$$v'_m \geq w_m\,, \tag{5.12}$$

since we otherwise replace v'_m by $\max(w_m, v'_m)$. With the sequence $\{v'_m\}_m$ we construct in the same way as above a solution $u' \in C(\overline{B})$ of the Helmholtz equation in the Ball B satisfying $u'(y_1) = u_f(y_1)$. Because u is the pointwise limit of $\{w_m\}$ and because $u' \geq v'_m$ for every m, it follows from (5.12) that

$$u' \geq u,$$

which implies that $u_f(y) = u(y) \leq u'(y) \leq u_f(y)$, whence $u'(y) = u(y)$. Therefore $u' - u$ is a solution of the Helmholtz equation in B satisfying $(u' - u) \geq 0$ and $(u' - u)(y) = 0$. Since y is an interior point of B, we conclude from Corollary 5.8 that $u' - u = 0$ in B, thence

$$u_f(y_1) = u'(y_1) = u(y_1).$$

Since $y_1 \in B$ was arbitrary, we obtain $u_f|_B = u|_B$, hence u_f is a solution of the Helmholtz equation in the neighborhood B of y. Since $y \in \Omega$ was arbitrary, we conclude that u_f is a solution of the Helmholtz equation in Ω, as asserted by the theorem. ∎

5.6 Boundary value problems, regular points

Corollary 5.18 *Let the assumptions of Theorem 5.17 be satisfied. If $w \in C(\overline{\Omega}, \mathbb{R})$ is a supersolution with $f \leq w|_{\partial\Omega}$, then the solution u constructed in this theorem satisfies*

$$u(x) \leq w(x), \quad x \in \Omega.$$

Proof: Any subsolution $v \in \mathcal{S}_f$ satisfies $v|_{\partial\Omega} = f \leq w|_{\partial\Omega}$, hence $v \leq w$ on $\overline{\Omega}$, by comparison. Consequently

$$u(x) = \sup_{v \in \mathcal{S}_f} v(x) \leq w(x), \quad x \in \Omega.$$

■

Corollary 5.19 *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set, let $\lambda \leq 0$ and let $f \in C(\partial\Omega, \mathbb{R})$. If there is a subsolution $v \in C(\overline{\Omega})$ and a supersolution $w \in C(\overline{\Omega})$ satisfying*

$$v|_{\partial\Omega} = f = w|_{\partial\Omega},$$

then there is a unique solution $u \in C(\overline{\Omega}) \cap C_\infty(\Omega)$ of

$$\begin{aligned} \Delta u(x) + \lambda u(x) = 0, &\qquad x \in \Omega \\ u|_{\partial\Omega} = f. & \end{aligned}$$

The solution satisfies $v \leq u \leq w$ on $\overline{\Omega}$.

Proof: By assumption the set $\mathcal{S}_f$ of all subsolutions $\hat{v}$ satisfying $\hat{v}|_{\partial\Omega} \leq f$ contains the function v, hence is nonempty. Consequently, by Theorem 5.17 there is a solution $u \in C_\infty(\Omega)$ of

$$\Delta u(x) + \lambda u(x) = 0, \quad x \in \Omega,$$

which by Corollary 5.18 satisfies $v \leq u \leq w$ on Ω. Extend u from Ω to a function on $\overline{\Omega}$ by defining

$$u(x) = f(x), \quad x \in \partial\Omega.$$

To see that the extended function satisfies $u \in C(\overline{\Omega})$, let $x \in \partial\Omega$. Since $v, w \in C(\overline{\Omega})$ satisfy $v|_{\partial\Omega} = w|_{\partial\Omega} = f$, we obtain

$$f(x) = \lim_{\substack{y \to x \\ y \in \overline{\Omega}}} v(y) \leq \lim_{\substack{y \to x \\ y \in \overline{\Omega}}} u(y) \leq \lim_{\substack{y \to x \\ y \in \overline{\Omega}}} w(y) = f(x),$$

whence

$$\lim_{\substack{y \to x \\ y \in \overline{\Omega}}} u(y) = f(x).$$

Consequently, $u \in C(\overline{\Omega})$. Uniqueness of the solution follows from Corollary 5.3. ■

Example 5.20 Let $\lambda < 0$, let $a, b > 0$, let

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid |x_1| < a,\ |x_2| < b\}$$

and let $f \in C(\partial\Omega)$ be defined by

$$f(x) = c$$

with a constant $c > 0$. Then there is a unique solution $u \in C(\overline{\Omega}) \cap C_\infty(\Omega)$ of

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, \quad x \in \Omega, \\ u\big|_{\partial\Omega} &= f. \end{aligned}$$

For, $w(x) = c$ is a supersolution with $w\big|_{\partial\Omega} = f$, cf. Corollary 5.16. To construct a subsolution v with $v\big|_{\partial\Omega} = f$, note that

$$v_1(x_1, x_2) = \frac{c}{e^{\sqrt{-\lambda}a} + e^{-\sqrt{-\lambda}a}} \left(e^{\sqrt{-\lambda}x_1} + e^{-\sqrt{-\lambda}x_1}\right)$$

is a solution, hence a subsolution with

$$v_1(x_1, x_2) \begin{cases} = c, & |x_1| = a, \quad |x_2| \le b \\ \le c, & |x_1| \le a_1, \quad |x_2| = b. \end{cases}$$

Also

$$v_2(x_1, x_2) = \frac{c}{e^{\sqrt{-\lambda}b} + e^{-\sqrt{-\lambda}b}} \left(e^{\sqrt{-\lambda}x_2} + e^{-\sqrt{-\lambda}x_2}\right)$$

is a solution, hence a subsolution with

$$v_2(x_1, x_2) \begin{cases} \le c, & |x_1| = a, \quad |x_2| \le b \\ = c, & |x_1| \le a_1, \quad |x_2| = b. \end{cases}$$

Consequently

$$v(x) = \max(v_1(x), v_2(x))$$

is a subsolution with $v\big|_{\partial\Omega} = f$. Corollary 5.19 thus implies that the boundary value problem has a unique solution.

For an arbitrary domain and for arbitrarily given boundary data it is difficult to find sub- and supersolutions with boundary values equal to the given data. In the following we show that it suffices to find sub- and supersolutions, which satisfy the boundary condition locally.

Definition 5.21 *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set. We call $x_0 \in \partial\Omega$ a regular boundary point, if to every $\xi \in \mathbb{R}$, $\delta > 0$ and $r > |\xi|$ there is a supersolution $w \in C(\overline{\Omega})$ and a* *subsolution* $v \in C(\overline{\Omega})$ *satisfying* $w(x_0) = v(x_0) = \xi$ *and*

$$w(x) \geq \begin{cases} \xi, & x \in \overline{\Omega} \cap B_\delta(x_0), \\ r, & x \in \overline{\Omega} \setminus B_\delta(x_0), \end{cases}$$

$$v(x) \leq \begin{cases} \xi, & x \in \overline{\Omega} \cap B_\delta(x_0), \\ -r, & x \in \overline{\Omega} \setminus B_\delta(x_0). \end{cases}$$

Theorem 5.22 *Let* $\Omega \subseteq \mathbb{R}^2$ *be a bounded open domain, let* $\lambda \leq 0$ *and let* $f \in C(\partial\Omega, \mathbb{R})$. *If* $x_0 \in \partial\Omega$ *is a regular point, then the solution* u *of*

$$\Delta u(x) + \lambda u(x) = 0, \quad x \in \Omega$$

constructed by Perron's method is continuous at x_0 *and satisfies*

$$u(x_0) = f(x_0).$$

Proof: Let $\varepsilon > 0$. Then there is $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon$$

for all $x \in \partial\Omega$ with $|x - x_0| < \delta$. By assumption there is a supersolution w and a subsolution v satisfying $w(x_0) = f(x_0) + \varepsilon, v(x_0) = f(x_0) - \varepsilon$ and

$$\begin{aligned} w(x) &\geq w(x_0) = f(x_0) + \varepsilon, && \text{for } |x - x_0| < \delta, \\ w(x) &\geq \sup_{y \in \partial\Omega} |f(y)|, && \text{for } |x - x_0| \geq \delta, \\ v(x) &\leq v(x_0) = f(x_0) - \varepsilon, && \text{for } |x - x_0| < \delta, \\ v(x) &\leq -\sup_{y \in \partial\Omega} |f(y)|, && \text{for } |x - x_0| \geq \delta. \end{aligned}$$

These conditions imply

$$\begin{aligned} w(x) &\geq f(x_0) + \varepsilon &&\geq f(x), \quad x \in \partial\Omega \cap B_\delta(x_0), \\ w(x) &\geq \sup_{y \in \partial\Omega} |f(y)| &&\geq f(x), \quad x \in \partial\Omega \setminus B_\delta(x_0), \\ v(x) &\leq f(x_0) - \varepsilon &&\leq f(x), \quad x \in \partial\Omega \cap B_\delta(x_0), \\ v(x) &\leq -\sup_{y \in \partial\Omega} |f(y)| &&\leq f(x), \quad x \in \partial\Omega \setminus B_\delta(x_0), \end{aligned}$$

hence $v|_{\partial\Omega} \leq f \leq w|_{\partial\Omega}$. Therefore we have $v \in \mathcal{S}_f$, hence $\mathcal{S}_f \neq \emptyset$. This implies that the solution u exists and satisfies

$$v(x) \leq u(x) \leq w(x),$$

for all $x \in \overline{\Omega}$. Thus,

$$\begin{aligned}
\limsup_{\substack{x\to x_0\\ x\in\overline{\Omega}}} u(x) &\leq \limsup_{\substack{x\to x_0\\ x\in\overline{\Omega}}} w(x) = \lim_{\substack{x\to x_0\\ x\in\overline{\Omega}}} w(x) = f(x_0) + \varepsilon \\
\liminf_{\substack{x\to x_0\\ x\in\overline{\Omega}}} u(x) &\geq \liminf_{\substack{x\to x_0\\ x\in\overline{\Omega}}} v(x) = \lim_{\substack{x\to x_0\\ x\in\overline{\Omega}}} v(x) = f(x_0) - \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we infer that

$$\limsup_{\substack{x\to x_0\\ x\in\overline{\Omega}}} u(x) = \liminf_{\substack{x\to x_0\\ x\in\overline{\Omega}}} u(x) = f(x_0),$$

which implies

$$\lim_{\substack{x\to x_0\\ x\in\overline{\Omega}}} u(x) = f(x_0).$$

Therefore u is continuous at x_0. ■

Corollary 5.23 *Let $\lambda \leq 0$, let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set and let $f \in C(\partial\Omega, \mathbb{R})$. If every point of $\partial\Omega$ is regular, then there is a unique solution $u \in C(\overline{\Omega}, \mathbb{R})$ of*

$$\begin{aligned}
\Delta u(x) + \lambda u(x) &= 0, \quad x \in \Omega \\
u|_{\partial\Omega} &= f.
\end{aligned}$$

This result follows immediately from the preceding theorem. It remains to find a criterion for regular boundary points.

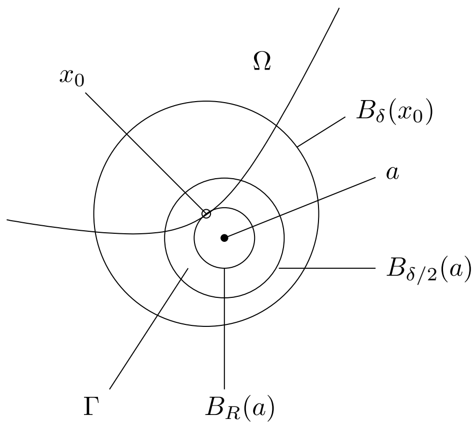
Theorem 5.24 *Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set and let $x_0 \in \partial\Omega$. Assume that $\lambda \leq 0$. If there is an open ball $B \subseteq \mathbb{R}^2 \backslash \Omega$ such that*

$$\overline{B} \cap \overline{\Omega} = \{x_0\},$$

then x_0 is a regular boundary point.

Proof: Let $\xi \in \mathbb{R}$, $r > |\xi|$ and $\delta > 0$ be given. We must construct suitable super- and subsolutions. We first assume that $\xi \leq 0$. Let $B = B_R(a)$ be the ball with

$$\overline{B_R(a)} \cap \overline{\Omega} = \{x_0\}.$$



Ball with center a intersecting $\partial\Omega$ at x_0

Without restriction of generality we can assume that $R < \frac{\delta}{2}$. Otherwise we shrink B until this estimate holds. This estimate for R and the equation $\overline{B_R(a)} \cap \overline{\Omega} = \{x_0\}$ imply

$$B_R(a) \subseteq B_{\delta/2}(a) \subseteq B_\delta(x_0).$$

For simplicity we write

$$\Gamma = B_{\delta/2}(a) \backslash \overline{B_R(a)}.$$

I.) To construct a supersolution we let $u \in C(\overline{\Gamma}) \cap C_\infty(\Gamma)$ be the solution of

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, & x &\in \Gamma, \\ u(x) &= r, & x &\in \partial B_{\delta/2}(a), \\ u(x) &= \xi, & x &\in \partial B_R(a). \end{aligned}$$

This solution is given by

$$u(x) = \begin{cases} C_1 J_0(\sqrt{\lambda}|x-a|) + C_2 N_0(\sqrt{\lambda}|x-a|), & \text{for } \lambda < 0, \\ C_1 + C_2 \ln|x-a|, & \text{for } \lambda = 0, \end{cases}$$

with suitable constants C_1 and C_2, which can be determined from the boundary conditions. We can use this formula to extend u to the region $\mathbb{R}^2 \backslash B_R(a)$, which contains Ω as a subset. The extended function satisfies the Helmholtz equation in the whole domain of definition. Therefore, for $y \in \mathbb{R}^2$ with $|y-a| > \delta/2$ the radial symmetry of u and the maximum principle applied to the region $\Gamma_y = B_{|y-a|}(a) \backslash \overline{B_R(a)}$ yield for all $x \in \Gamma_y$ that

$$\min\big(0, \xi, u(y)\big) = \min(0, \min_{\partial\Gamma_y} u) \leq u(x) \leq \max(0, \max_{\partial\Gamma_y} u) = \max\big(0, \xi, u(y)\big). \qquad (5.13)$$

We can insert $x \in \partial B_{\delta/2}(a) \subseteq \Gamma_y$ into this inequality. Since u has the value r on $\partial B_{\delta/2}(a)$, the second inequality in (5.13) can only hold if

$$u(y) \geq r, \quad \text{for all } y \in \mathbb{R}^2 \backslash \partial B_{\delta/2}(a). \tag{5.14}$$

This implies $\min\big(0, \xi, u(y)\big) = \xi \leq 0$, whence, the first inequality in (5.13) yields

$$u(x) \geq \xi, \quad \text{for all } x \in \mathbb{R}^2 \backslash B_R(a). \tag{5.15}$$

Now define $w = u|_{\overline{\Omega}}$. Then w is a solution of the Helmholtz equation, hence it is a supersolution. Moreover, $x_0 \in \partial B_R(a)$ implies

$$w(x_0) = u(x_0) = \xi.$$

If we note that $\overline{\Omega} \subseteq \mathbb{R}^2 \backslash B_R(a)$ and $\overline{\Omega} \backslash B_\delta(x_0) \subseteq \mathbb{R}^2 \backslash B_{\delta/2}(a)$, we obtain from (5.14) and (5.15) that

$$w \geq \xi \text{ on } \overline{\Omega}, \quad \text{and} \quad w \geq r \text{ on } \overline{\Omega} \backslash B_\delta(x_0).$$

Therefore w satisfies all conditions required from the supersolution in Definition 5.21.

II.) To construct a subsolution v let $u \in C(\overline{\Gamma}) \cap C_\infty(\Gamma)$ be the solution of

$$\begin{aligned} \Delta u(x) = 0, &\qquad x \in \Gamma, \\ u(x) = -r, &\qquad x \in \partial B_{\delta/2}(a), \\ u(x) = \xi, &\qquad x \in \partial B_R(a). \end{aligned}$$

This solution has the form

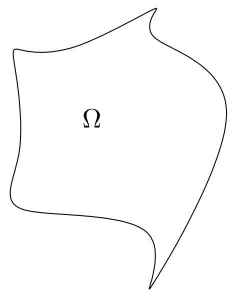
$$u(x) = C_1 + C_2 \ln |x - a|$$

with constants C_1, C_2 uniquely determined by the boundary conditions. We use this formula to extend u to the region $\mathbb{R}^2 \backslash B_R(a)$. The extended function satisfies the potential equation in the whole domain of definition. Choose $y \in \mathbb{R}^2$ with $|y - a| > \delta/2$. Since u is radially symmetric with respect to a, it has the constant value $u(y)$ on the circle $\partial B_{|y-a|}(a)$, whence Corollary 5.9(ii) implies for all $x \in \Gamma_y$ that

$$\min(\xi, u(y)) = \min_{\partial \Gamma_y} u < u(x) < \max_{\partial \Gamma_y} u = \max(\xi, u(y)). \tag{5.16}$$

This inequality must hold for $x \in \partial B_{\delta/2}(a) \subseteq \Gamma_y$. For such x we have $u(x) = -r$. The first inequality in (5.16) can therefore only hold if $u(y) < -r$. This implies $\max(\xi, u(y)) = \xi$, so the second inequality in (5.16) yields that $u(x) < \xi$ for all $x \in \Gamma_y$. Since y was an arbitrary point outside of the ball $\overline{B_{\delta/2}(a)}$, we obtain

$$u \leq \xi \text{ on } \mathbb{R}^2 \backslash B_R(a), \qquad u \leq -r, \text{ on } \mathbb{R}^2 \backslash B_{\delta/2}(a).$$



A domain satisfying the conditions of Example 5.26

Since $\xi \le 0$ by assumption, it follows that $u \le 0$. Consequently $v = u|_{\overline{\Omega}}$ is a non-positive solution of the potential equation, hence v is a subsolution, by Theorem 5.15. Since $x_0 \in \partial B_R(a)$, we have $v(x_0) = \xi$. From the inequalities above it is immediately seen that v satisfies all conditions required in Definition 5.21 from the subsolution.

III.) It remains to construct a supersolution and a subsolution in the case $\xi > 0$. To this end let $\hat{w}$ and $\hat{v}$ be the super- and subsolution to the value $-\xi$ constructed in the preceding part of the proof. Since the negative of a supersolution is a subsolution and the negative of a subsolution is a supersolution, it follows that

$$w = -\hat{v}, \quad v = -\hat{w}$$

are a supersolution and a subsolution, respectively, satisfying the estimates required for a regular point. Consequently x_0 is a regular point. ∎

Example 5.25 Let $\Omega \subseteq \mathbb{R}^2$ be a bounded, open and convex set. Then to every point $x \in \partial\Omega$ there is a ball B such that $\overline{\Omega} \cap \overline{B} = \{x\}$, hence every boundary point is regular. Therefore the Dirichlet boundary value problem

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, \quad x \in \Omega \\ u|_{\partial\Omega} &= f, \end{aligned}$$

with $\lambda \le 0$ has a unique solution $u \in C(\overline{\Omega}) \cap C_\infty(\Omega)$ to every $f \in C(\partial\Omega)$.

Example 5.26 Let $\Gamma \subseteq \partial\Omega$ be a finite subset such that $\partial\Omega$ is two times continuously differentiable at every point of $\partial\Omega\backslash\Gamma$. Assume that through every point y of Γ a straight line ℓ is passing such that Ω is locally on one side of ℓ at y. Then to every point $x \in \partial\Omega$ there is a ball B with $\overline{B} \cap \overline{\Omega} = \{x\}$, hence every point of $\partial\Omega$ is regular, and the Dirichlet problem can be uniquely solved. In particular, the Dirichlet problem can be uniquely solved if $\partial\Omega \in C_2$.

6 Fundamental solution and Green's function

6.1 Convolution integrals

Theorem 6.1 *Let $1 \le p < \infty$, $\varphi \in L^1(\mathbb{R}^n, \mathbb{C})$ and $f \in L^p(\mathbb{R}^n, \mathbb{C})$. Then for almost all $x \in \mathbb{R}^n$ the integral*

$$F(x) = \int_{\mathbb{R}^n} \varphi(x-y) f(y) dy = \int_{\mathbb{R}^n} \varphi(y) f(x-y) dy$$

exists. The function F defined by this integral belongs to $L^p(\mathbb{R}^n, \mathbb{C})$ and satisfies

$$\|F\|_{L^p(\mathbb{R}^n)} \le \|\varphi\|_{L^1(\mathbb{R}^n)} \ \|f\|_{L^p(\mathbb{R}^n)}.$$

Here

$$\|u\|_{L^p(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} |u(x)|^p dx \Big)^{1/p}$$

denotes the norm in the Banach space $L^p(\mathbb{R}^n, \mathbb{C})$.

Proof: We have

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\varphi(y)| \, |f(x-y|^p dx \, dy &= \int_{\mathbb{R}^n} |\varphi(y)| \int_{\mathbb{R}^n} |f(x-y)|^p dx \, dy \\ &= \int_{\mathbb{R}^n} |\varphi(y)| dy \int_{\mathbb{R}^n} |f(x)|^p dx = \|\varphi\|_{L^1(\mathbb{R}^n)} \, \|f\|^p_{L^p(\mathbb{R}^n)}. \quad (6.1) \end{aligned}$$

Thus, Tonelli's theorem yields $\big((x,y) \mapsto |\varphi(y)| |f(x-y)|^p\big) \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$, whence Fubini's theorem implies that $\big(y \mapsto |\varphi(y)| \, |f(x-y)|^p\big) \in L^1(\mathbb{R}^n)$ for almost all $x \in \mathbb{R}^n$, which means that

$$\big(y \mapsto |\varphi(y)|^{1/p} |f(x-y)|\big) \in L^p(\mathbb{R}^n), \quad \text{for almost all } x. \quad (6.2)$$

For $p = 1$ we therefore get from (6.1)

$$\int_{\mathbb{R}^n} |F(x)| dx \le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\varphi(y)| \, |f(x-y)| dy dx \le \|\varphi\|_{L^1(\mathbb{R}^n)} \|f\|_{L^1(\mathbb{R}^n)}.$$

This completes the proof for $p = 1$. For $1 < p < \infty$ let $1 < q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Since $\varphi \in L^1(\mathbb{R})$ implies $|\varphi|^{1/q} \in L^q(\mathbb{R}^n)$, we conclude from (6.2) and from Hölder's inequality that

$$\big(y \mapsto |\varphi(y) f(x-y)| = |\varphi(y)|^{1/q} |\varphi(y)|^{1/p} |f(x-y)|\big) \in L^1(\mathbb{R}^n),$$

for almost all $x \in \mathbb{R}^n$, and

$$\begin{aligned} |F(x)|^p &\le \Big(\int_{\mathbb{R}^n} |\varphi(x-y) f(y)| \, dy \Big)^p = \Big(\int_{\mathbb{R}^n} |\varphi(y) f(x-y)| \, dy \Big)^p \\ &= \Big(\int_{\mathbb{R}^n} |\varphi(y)|^{1/q} |\varphi(y)|^{1/p} |f(x-y)| \, dy \Big)^p \le \|\varphi\|_1^{p/q} \int_{\mathbb{R}^n} |\varphi(y)| \, |f(x-y)|^p dy. \end{aligned}$$

This inequality and (6.1) together imply

$$\int_{\mathbb{R}^n} \Big|F(x)|^p dx \le \|\varphi\|_{L^1(\mathbb{R}^n)}^{p/q} \|\varphi\|_{L^1(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}^p = \big(\|\varphi\|_{L^1(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}\big)^p.$$

■

Remark: One uses the notation

$$F(x) = \int_{\mathbb{R}^n} \varphi(x-y)f(y)dy = (\varphi * f)(x).$$

The operator $*$ is called convolution. With this notation the inequality just proved is

$$\|\varphi * f\|_{L^p(\mathbb{R}^n)} \le \|\varphi\|_{L^1(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.$$

Consequently $f \mapsto \varphi * f : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a linear and continuous mapping with norm not greater than $\|\varphi\|_{L^1(\mathbb{R}^n)}$.

6.2 Fundamental solutions

Fundamental solutions can be used to construct solutions of linear partial differential equations in any space dimension. Here we only discuss the fundamental solutions for the Helmholtz equation in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition 6.2 *Let $\lambda \in \mathbb{C}$ and $n = 2, 3$. The fundamental solution $F : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ of the Helmholtz equation $\Delta u + \lambda u = 0$ is defined by $F(x) = \tilde{F}(|x|)$, where the function $\tilde{F} : (0, \infty) \to \mathbb{R}$ is given by*

$$\tilde{F}(r) = \begin{cases} -\dfrac{1}{2\pi} \ln r, & \text{if } n = 2,\ \lambda = 0, \\[2mm] \dfrac{i}{4} H_0^{(1)}(\sqrt{\lambda} r), & \text{if } n = 2,\ \lambda \neq 0, \\[2mm] \dfrac{e^{i\sqrt{\lambda} r}}{4\pi r}, & \text{if } n = 3,\ \lambda \in \mathbb{C}. \end{cases}$$

Here $H_0^{(1)}$ is the Hankel function of first kind and order 0, which is defined in (A.9) in the appendix. For the square root we take the branch satisfying $0 \le \arg \sqrt{\lambda} < \pi$.

As the name says, the fundamental solutions are solutions of the Helmholtz equation, which is shown by the next lemma.

Lemma 6.3 *The fundamental solution is infinitely differentiable on the set $\mathbb{R}^n \setminus \{0\}$ and satisfies $\Delta F(x) + \lambda F(x) = 0$ for $x \in \mathbb{R}^n \setminus \{0\}$.*

Proof: From the definition it is obvious that F is infinitely differentiable on $\mathbb{R}^n \setminus \{0\}$. To show that the Helmholtz equation is satisfied note that the chain rule yields for $r = r(x) = |x| > 0$ that

$$(\Delta + \lambda)\, F(x) = (\Delta + \lambda)\, \tilde{F}\big(r(x)\big) = \left[\sum_{i=1}^{n}\Big(\Big(\frac{\partial r(x)}{\partial x_i}\Big)^2 \frac{\partial^2}{\partial r^2} + \frac{\partial^2 r(x)}{\partial x_i^2}\frac{\partial}{\partial r}\Big) + \lambda\right]\tilde{F}(r)$$

$$= \left[\sum_{i=1}^{n}\Big(\Big(\frac{x_i}{|x|}\Big)^2 \frac{\partial^2}{\partial r^2} + \Big(\frac{1}{|x|} - \frac{x_i^2}{|x|^3}\Big)\frac{\partial}{\partial r}\Big) + \lambda\right]\tilde{F}(r) = \Big(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \lambda\Big)\tilde{F}(r) = 0,$$

where the last equality sign follows from the definition of $\tilde{F}$. ■

Fundamental solutions are singular at $x = 0$. Yet, the singularities of the fundamental solution and of the first derivative of the fundamental solution are integrable. We next study these integrability properties in the case $n = 3$, but analogous considerations are valid for $n = 2$.

To this end note that from the definition of the fundamental solution in $\mathbb{R}^3$ we obtain the estimates

$$|F(x)| = \Big|\frac{e^{i\sqrt{\lambda}|x|}}{4\pi|x|}\Big| \leq \frac{1}{4\pi|x|}e^{-\mathrm{Im}\,\sqrt{\lambda}|x|}, \tag{6.3}$$

$$\Big|\frac{\partial}{\partial x_i}F(x)\Big| = \Big|\Big(i\sqrt{\lambda} - \frac{1}{|x|}\Big)\frac{x_i}{|x|}\frac{e^{i\sqrt{\lambda}|x|}}{4\pi|x|}\Big| \leq \frac{1}{4\pi|x|}\Big(\sqrt{|\lambda|} + \frac{1}{|x|}\Big)e^{-\mathrm{Im}\,\sqrt{\lambda}|x|}. \tag{6.4}$$

Since by our choice of the square root we have $\mathrm{Im}\,\sqrt{\lambda} \geq 0$ for all $\lambda \in \mathbb{C}$, hence $e^{-\mathrm{Im}\,\sqrt{\lambda}|x|} \leq 1$, it follows from these estimates that

$$|F(x)| \leq \frac{1}{4\pi|x|}, \qquad \Big|\frac{\partial}{\partial x_i}F(x)\Big| \leq \frac{1 + R\sqrt{|\lambda|}}{4\pi|x|^2}, \qquad \text{for all } R > 0 \text{ and } x \in B_R(0), \tag{6.5}$$

where $B_R(0)$ denotes the ball in $\mathbb{R}^3$ with center 0 and radius R. These estimates imply that F and $\frac{\partial}{\partial x_i}F$ are integrable over any ball $B_R(0)$, but the estimates do not imply that these functions are integrable over all of $\mathbb{R}^3$. Yet, for $\lambda \in \mathbb{C} \setminus [0, \infty)$ the choice of the square root implies that $\mathrm{Im}\,\sqrt{\lambda} > 0$, hence for these values of λ the term $e^{-\mathrm{Im}\,\sqrt{\lambda}|x|}$ decays exponentially for $|x| \to \infty$, whence by (6.3) and (6.4) also the functions F and $\frac{\partial}{\partial x_i}F$ decay exponentially at infinity, which implies that both functions are integrable over $\mathbb{R}^3$. We thus obtain

Corollary 6.4 *For every $\lambda \in \mathbb{C}$ the fundamental solution F of the Helmholtz equation in $\mathbb{R}^3$ and the derivatives $\frac{\partial}{\partial x_i}F$, $i = 1, \ldots, 3$, belong to the space $L^1(B_R(0), \mathbb{R})$ for all $R > 0$. For $\lambda \in \mathbb{C} \setminus [0, \infty)$ the functions F and $\frac{\partial}{\partial x_i}F$ decay exponentially at infinity and belong to $L^1(\mathbb{R}^3, \mathbb{R})$.*

Theorem 6.5 *Assume that $\lambda \in \mathbb{C}\backslash[0,\infty)$. For $f \in L^2(\mathbb{R}^3,\mathbb{C})$ set*

$$u(x) = -\int_{\mathbb{R}^3} F(x-y)f(y)dy = -(F * f)(x), \quad x \in \mathbb{R}^3.$$

Then $u \in H_1(\mathbb{R}^3,\mathbb{C})$ is a weak solution of the equation

$$\Delta u + \lambda u = f. \tag{6.6}$$

Proof: In the first step of the proof we show that $u = -F * f$ belongs to $H_1(\mathbb{R}^3)$. To this end we invoke Theorem 6.1, which together with $F \in L^1(\mathbb{R}^3)$, $\frac{\partial}{\partial x_i}F \in L^1(\mathbb{R}^3)$, and $f \in L^2(\mathbb{R}^3)$ implies

$$u = -F * f \in L^2(\mathbb{R}^3), \quad v_i = -\big(\frac{\partial}{\partial x_i}F\big) * f \in L^2(\mathbb{R}^3), \quad \text{for } i = 1,2,3.$$

v_i is the weak derivative $\frac{\partial}{\partial x_i}u$ of u. To verify this let $\varphi \in \overset{\circ}{C}_\infty(\mathbb{R}^3)$. For $r > 0$ and $x \in \partial B_r(0)$ we denote by $n(x) = (n_1(x), n_2(x), n_3(x))$ the interior unit normal vector to $\partial B_r(y)$. With these notations and with Gauß' theorem we compute

$$\begin{aligned}
(u,\varphi_{x_i}) &= -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} F(x-y)f(y)\,dy\,\frac{\partial}{\partial x_i}\overline{\varphi(x)}\,dx \\
&= -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} F(x-y)\frac{\partial}{\partial x_i}\overline{\varphi(x)}\,dx\,f(y)\,dy \\
&= -\int_{\mathbb{R}^3}\lim_{r\to 0}\int_{\mathbb{R}^3\backslash B_r(y)} F(x-y)\frac{\partial}{\partial x_i}\overline{\varphi(x)}\,dx\,f(y)\,dy \\
&= \int_{\mathbb{R}^3}\lim_{r\to 0}\Big(\int_{\mathbb{R}^3\backslash B_r(y)} \frac{\partial}{\partial x_i}F(x-y)\overline{\varphi(x)}\,dx - \int_{\partial B_r(y)} n_i(x)F(x-y)\overline{\varphi(x)}\,dS_x\Big)f(y)\,dy \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\partial}{\partial x_i}F(x-y)\overline{\varphi(x)}\,dx\,f(y)\,dy \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\partial}{\partial x_i}F(x-y)f(y)\,dy\,\overline{\varphi(x)}\,dx = -(v_i,\varphi),
\end{aligned} \tag{6.7}$$

where we used that

$$\begin{aligned}
\lim_{r\to 0}\Big|\int_{\partial B_r(y)} n_i(x)F(x-y)\,\overline{\varphi(x)}\,dS_x\Big| &= \lim_{r\to 0}\Big|\int_{\partial B_r(y)} n_i(x)\frac{e^{i\sqrt{\lambda}r}}{4\pi r}\overline{\varphi(x)}\,dS_x\Big| \\
&\le \lim_{r\to 0}\big(\sup_{x\in\mathbb{R}^3}|\varphi(x)|\big)\frac{1}{4\pi}\int_{\partial B_r(y)}\frac{1}{r}\,dS_x = \sup_{\mathbb{R}^3}|\varphi(x)|\lim_{r\to 0} r = 0.
\end{aligned}$$

(6.7) implies that $v_i = \frac{\partial}{\partial x_i}u$ for $i = 1,2,3$, hence $u \in H_1(\mathbb{R}^3)$.

In the second step of the proof we must show that u solves the Helmholtz equation (6.6) in all of $\mathbb{R}^3$ weakly. By Definition 3.32, the function u is a weak solution of (6.6) if

$$-(\nabla u, \nabla\varphi) + \lambda(u,\varphi) = (f,\varphi) \tag{6.8}$$

holds for all $\varphi \in \overset{\circ}{C}_\infty(\mathbb{R}^3, \mathbb{C})$. To verify (6.8), we compute with the first Green's formula

$$
\begin{aligned}
&-(\nabla u, \nabla \varphi) + \lambda(u, \varphi) \\
&\qquad = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \big(\nabla_x F(x-y)\cdot \nabla_x \overline{\varphi(x)} - \lambda F(x-y)\overline{\varphi(x)}\big) dx f(y)\, dy \\
&\qquad = \int_{\mathbb{R}^3} \lim_{r\to 0} \int_{\mathbb{R}^3\setminus B_r(y)} \Big(\nabla_x F(x-y)\cdot \nabla_x \overline{\varphi(x)} - \lambda F(x-y)\overline{\varphi(x)}\Big)\, dx f(y)\, dy \\
&= \int_{\mathbb{R}^3} \lim_{r\to 0} \Big(\int_{\mathbb{R}^3\setminus B_r(y)} (-\Delta_x - \lambda) F(x-y)\overline{\varphi(x)}\, dx + \int_{\partial B_r(y)} \frac{\partial}{\partial n_x} F(x-y) \overline{\varphi(x)}\, dS_x \Big) f(y)\, dy \\
&\qquad = -\int_{\mathbb{R}^3} \lim_{r\to 0} \int_{\partial B_r(y)} \frac{\partial}{\partial r} \frac{e^{i\sqrt{\lambda} r}}{4\pi r} \overline{\varphi(x)}\, dS_x f(y)\, dy. \qquad (6.9)
\end{aligned}
$$

Now

$$
\begin{aligned}
&-\lim_{r\to 0} \int_{|x-y|=r} \frac{\partial}{\partial r}\frac{e^{i\sqrt{\lambda} r}}{4\pi r} \overline{\varphi(x)}\, dS_x \\
&\qquad = -\lim_{r\to 0} \frac{e^{i\sqrt{\lambda} r}}{4\pi r}\Big(i\sqrt{\lambda} - \frac{1}{r}\Big) \int_{|x-y|=r} \overline{\varphi(y)} + \big(\overline{\varphi(x) - \varphi(y)}\big)\, dS_x \\
&= -\lim_{r\to 0} \frac{e^{i\sqrt{\lambda} r}}{4\pi r}\Big(i\sqrt{\lambda} - \frac{1}{r}\Big)\Big(4\pi r^2 \overline{\varphi(y)} + \int_{|x-y|=r} \overline{\varphi(x) - \varphi(y)}\, dS_x\Big) = \overline{\varphi(y)}, \qquad (6.10)
\end{aligned}
$$

since the continuity of φ implies

$$
\begin{aligned}
&\lim_{r\to 0} \left| \frac{e^{i\sqrt{\lambda} r}}{4\pi r}\big(i\sqrt{\lambda} - \frac{1}{r}\big) \int_{|x-y|=r} \overline{\varphi(x) - \varphi(y)}\, dS_x \right| \\
&\qquad \le \lim_{r\to 0} \sup_{x\in \partial B_r(y)} |\varphi(x) - \varphi(y)| \frac{1}{4\pi r}(|\sqrt{\lambda}| + \frac{1}{r}) 4\pi r^2 = 0. \qquad (6.11)
\end{aligned}
$$

We combine (6.10) with (6.9) and obtain (6.8). Consequently, u is a weak solution. ■

Remark 6.6 If f is more regular then

$$
u(x) = -\int_{\mathbb{R}^3} F(x-y)\, f(y)\, dy
$$

is not only a weak solution, but also a classical solution of $\Delta u + \lambda u = 0$. For, a weak solution u is a classical solution if $u \in C_2(\mathbb{R}^3)$. This regularity of u is obtained for example if $f \in C_1(\mathbb{R}^3)$ and

$$
|f(x)|,\ |\frac{\partial}{\partial x_1} f(x)|, \ldots, |\frac{\partial}{\partial x_3} f(x)| \le C
$$

for all $x \in \mathbb{R}^3$, with a suitable constant C. To see this, suppose that f satisfies these conditions and that $\mathrm{Re}\, i\sqrt{\lambda} < 0$. Then

$$\begin{aligned} \frac{\partial}{\partial x_i} u(x) &= -\int_{\mathbb{R}^3} \frac{\partial}{\partial x_i} F(x-y) f(y)\, dy \\ &= \int_{\mathbb{R}^3} \frac{\partial}{\partial y_i} F(x-y) f(y)\, dy = -\int_{\mathbb{R}^3} F(x-y) \frac{\partial}{\partial y_i} f(y)\, dy \end{aligned}$$

and

$$\frac{\partial^2}{\partial x_j \partial x_i} u(x) = -\int_{\mathbb{R}^3} \frac{\partial}{\partial x_j} F(x-y) \frac{\partial}{\partial y_i} f(y)\, dy,$$

and some technical considerations show that these derivatives exist in the classical sense.

6.3 Green's function

Convolution of the right hand side of the Helmholtz equation with the fundamental solution yields a solution in the whole space $\mathbb{R}^3$, but solutions of boundary value problems cannot be obtained in this way, since there is no possibility to account for the boundary condition. To construct solutions of boundary value problems the convolution integral must be replaced by an integral representation of the solution, which does not only involve the right hand side of the Helmholtz equation, but also the boundary data. To find such an integral representation suppose that $\lambda \in \mathbb{C}$ and that the boundary value problem

$$\begin{aligned} \Delta u(x) + \lambda u(x) = f(x), \quad x \in \Omega, \\ u(x) = g(x), \quad x \in \partial\Omega, \end{aligned}$$

has a solution u in the domain $\Omega \subseteq \mathbb{R}^3$. Assuming that the solution is regular enough such that Green's formula can be applied we obtain with the fundamental solution F

$$\begin{aligned} \int_\Omega F(x-y) f(y)\, dy &= \int_\Omega F(x-y)(\Delta_y + \lambda) u(y)\, dy \\ &= \lim_{r\to 0} \int_{\Omega\setminus B_r(x)} F(x-y)(\Delta_y + \lambda) u(y)\, dy \\ &= \lim_{r\to 0} \Bigg(\int_{\Omega\setminus B_r(x)} (\Delta_y + \lambda) F(x-y) u(y)\, dy \\ &\qquad + \int_{\partial\Omega} F(x-y) \frac{\partial}{\partial n_y} u(y) - \frac{\partial}{\partial n_y} F(x-y) u(y)\, dS_y \\ &\qquad\quad + \int_{\partial B_r(x)} F(x-y) \frac{\partial}{\partial n_y} u(y) - \frac{\partial}{\partial n_y} F(x-y) u(y)\, dS_y \Bigg) \\ &= \int_{\partial\Omega} F(x-y) \frac{\partial}{\partial n_y} u(y) - \frac{\partial}{\partial n_y} F(x-y) u(y)\, dS_y - u(x), \end{aligned} \tag{6.12}$$

since $(\Delta_y + \lambda)F(x-y) = 0$ in the domain $\Omega \setminus B_r(x)$, and since

$$\lim_{r\to 0} \int_{\partial B_r(x)} F(x-y)\frac{\partial}{\partial n_y} u(y)\, dS_y = 0, \tag{6.13}$$

$$\lim_{r\to 0} \int_{\partial B_r(x)} \frac{\partial}{\partial n_y} F(x-y) u(y)\, dS_y = u(x), \tag{6.14}$$

which limit relations are proved as in (6.10), (6.11). From (6.12) we thus obtain

$$u(x) = -\int_\Omega F(x-y)f(y)\, dy + \int_{\partial\Omega} F(x-y)\frac{\partial}{\partial n_y} u(y) - \frac{\partial}{\partial n_y} F(x-y)u(y)\, dS_y.$$

This is a representation formula for the solution in terms of the right hand side f, the boundary values $u\big|_{\partial\Omega}$, and the derivative $\frac{\partial u}{\partial n_y}$ of the solution in direction of the exterior unit normal vector n_y to $\partial\Omega$.

Since $u\big|_{\partial\Omega}$ is equal to the given data g, we can insert $g(y)$ for $u(y)$ in the boundary integral on the right hand side of the representation formula. However, the normal derivative $\frac{\partial u}{\partial n_y}$ appearing also in the boundary integral is unknown and can only be determined by solving the boundary value problem. To avoid this problem and to obtain a representation formula which only involves given data, one must replace the fundamental solution F by the *Green's function for the Dirichlet boundary value problem*

$$G(x, y) = F(x-y) + w(x, y),$$

where $w : \Omega \times \overline{\Omega} \to \mathbb{C}$ is defined as follows: For $x \in \Omega$ let $v_x : \overline{\Omega} \to \mathbb{C}$ be the solution of

$$\Delta v_x(y) + \lambda v_x(y) = 0, \qquad y \in \Omega,$$
$$v_x(y) = -F(x-y), \qquad y \in \partial\Omega,$$

and set

$$w(x, y) = v_x(y).$$

The function G satisfies for every $\varphi \in C(\Omega)$ and $x \in \Omega$

(i) $(\Delta_y + \lambda)G(x, y) = 0, \quad y \in \Omega, \quad x \neq y,$

(ii) $G(x, y) = 0, \quad y \in \partial\Omega,$

(iii) $\displaystyle\lim_{r\to 0} \int_{\partial B_r(x)} \varphi(y)G(x, y) dS_y$
$\displaystyle= \lim_{r\to 0} \int_{\partial B_r(x)} \varphi(y)F(x-y) dS_y + \lim_{r\to 0} \int_{\partial B_r(x)} \varphi(y)w(x, y) dS_y = 0,$

(iv) $\lim_{r\to 0}\int_{\partial B_r(x)} \varphi(y)\frac{\partial}{\partial n_y}G(x,y)dS_y$
$$= \lim_{r\to 0}\int_{\partial B_r(x)} \varphi(y)\frac{\partial}{\partial n_y}F(x-y)dS_y + \lim_{r\to 0}\int_{\partial B_r(x)} \varphi(y)\frac{\partial}{\partial n_y}w(x,y)dS_y = \varphi(x),$$

where $\frac{\partial}{\partial n_y}$ is the derivative in direction of the interior unit normal vector n_y to $\partial B_r(x)$ at $y \in \partial B_r(x)$. The limit relations in (iii) and (iv) follow from (6.13), (6.14) with u replaced by φ.

If we repeat the computation (6.12) with $F(x-y)$ replaced by $G(x,y)$ and use the Dirichlet boundary condition $u|_{\partial\Omega} = g$, then we obtain from the properties (i) – (iv) for the solution u of the Dirichlet boundary value problem for the Helmholtz equation that

$$u(x) = -\int_\Omega G(x,y)f(y)dy - \int_{\partial\Omega} \frac{\partial}{\partial n_y}G(x,y)g(y)dS_y, \qquad (6.15)$$

in which formula the boundary integral term containing $\frac{\partial}{\partial n_y}u$ does not appear because of $G(x,y) = 0$ for $y \in \partial\Omega$.

(6.15) is a representation formula for the solution of the Dirichlet boundary value problem in terms of the given data. Note however, that the determination of G requires to solve the Dirichlet boundary value problem in Ω. Therefore G can only be constructed if it is known in advance that the Dirichlet boundary value problem has a solution. The Green's function cannot be used to answer the questions of existence and uniqueness of boundary value problems. But if G can be determined explicitly it offers a means to represent, to compute and to study properties of the solution.

The *Green's function for the Neumann boundary value problem*

$$\Delta u(x) + \lambda u(x) = f(x), \quad x \in \Omega,$$
$$\frac{\partial}{\partial n}u(x) = g(x), \quad x \in \partial\Omega,$$

where $\lambda \in \mathbb{C}$ is a given constant, is defined analogously. Set

$$G(x,y) = F(x-y) + w(x,y),$$

where $w : \Omega \times \overline{\Omega} \to \mathbb{C}$ is defined as follows: For $x \in \Omega$ let $v_x : \overline{\Omega} \to \mathbb{C}$ be the solution of

$$\Delta v_x(y) + \lambda v_x(y) = 0, \quad \text{for } y \in \Omega,$$
$$\frac{\partial}{\partial n_y}v_x(y) = -\frac{\partial}{\partial n_y}F(x-y), \quad \text{for } y \in \partial\Omega,$$

and define

$$w(x,y) = v_x(y).$$

The function G satisfies

$$\frac{\partial}{\partial n_y} G(x,y) = 0, \quad y \in \partial\Omega,$$

from which we obtain the representation formula

$$u(x) = -\int_\Omega G(x,y)f(y)dy + \int_{\partial\Omega} G(x,y)g(y)dS_y$$

for the solution of the Neumann boundary value problem.

6.4 The Green's function for the potential equation in a ball. Poisson's representation formula.

It is possible to determine the Green's function explicitly in some cases. Here we derive the Green's function for a ball $B_R(0)$ in $\mathbb{R}^n$ with $n = 2$ and $n = 3$. To this end let $R > 0$ and consider the Kelvin transformation $K : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}$ defined by

$$K(x) = \Big(\frac{R}{|x|}\Big)^2 x.$$

(Reflection at the sphere with radius R.)

Lemma 6.7 *For $x, y \in \mathbb{R}^n$ with $0 < |x| < R$, $|y| = R$ we have*

$$|y - K(x)| = \frac{R}{|x|}\,|y - x|.$$

Proof: We have

$$\begin{aligned} |y - K(x)|^2 &= |K(x)|^2 + |y|^2 - 2y \cdot K(x) \\ &= \Big(\frac{R}{|x|}\Big)^4 |x|^2 + R^2 - 2\Big(\frac{R}{|x|}\Big)^2 x \cdot y = \Big(\frac{R}{|x|}\Big)^2 R^2 + \Big(\frac{R}{|x|}\Big)^2 |x|^2 - 2\Big(\frac{R}{|x|}\Big)^2 x \cdot y \\ &= \Big(\frac{R}{|x|}\Big)^2 (|y|^2 + |x|^2 - 2x \cdot y) = \Big(\frac{R}{|x|}\Big)^2 |y - x|^2. \end{aligned}$$

■

Theorem 6.8 *The Green's function to the Dirichlet problem for the potential equation in the ball $B_R(0) \subseteq \mathbb{R}^3$ is*

$$G(x,y) = \frac{1}{4\pi|x-y|} + w(x,y),$$

with

$$w(x,y) = \begin{cases} -\dfrac{R}{4\pi|x|}\,\dfrac{1}{\big|y - \big(\frac{R}{|x|}\big)^2 x\big|}, & 0 < |x| < R, \\ -\dfrac{1}{4\pi R}, & x = 0. \end{cases}$$

Proof: For $0 < |x| < R$ we have $\left|\left(\frac{R}{|x|}\right)^2 x\right| = R\,\frac{R}{|x|} > R$. Thus, for all $x \in B_R(0)$

$$y \mapsto w(x,y) \in C_\infty\big(\overline{B_R(0)}\big),$$

and

$$\Delta_y w(x,y) = 0\,.$$

Also, for $0 < |x| < R$ and $y \in \partial B_R(0)$ we have

$$w(x,y) = -\frac{1}{4\pi\,\frac{|x|}{R}\,|y-K(x)|} = -\frac{1}{4\pi|x-y|} = -F(x-y)\,.$$

Clearly, for $x = 0$ and $y \in \partial B_R(0)$

$$w(x,y) = -\frac{1}{4\pi R} = -\frac{1}{4\pi|y|} = -F(x-y)\,.$$

Consequently, w satisfies

$$\begin{aligned} \Delta_y w(x,y) &= 0, && (x,y) \in B_R(0) \times B_R(0) \\ w(x,y) &= -F(x-y), && (x,y) \in B_R(0) \times \partial B_R(0), \end{aligned}$$

hence

$$G(x,y) = F(x-y) + w(x,y)$$

is the Green's function. ■

Corollary 6.9 *Let $B_R(0) \subseteq \mathbb{R}^3$ and let $u \in C_1\big(\overline{B_R(0)}\big) \cap C_2\big(B_R(0)\big)$ be a solution of the Dirichlet problem*

$$\begin{aligned} \Delta u(x) = 0, &\quad x \in B_R(0), \\ u(x) = g(x), &\quad x \in \partial B_R(0). \end{aligned}$$

Then this solution is given by the Poisson representation formula

$$u(x) = \frac{1}{4\pi R}\int_{|y|=R} \frac{R^2 - |x|^2}{|x-y|^3}\, g(y)\, dS_y\,.$$

Proof: If $u \in C_1\big(\overline{B_R(0)}\big) \cap C_2\big(B_R(0)\big)$, then the derivation of the representation formula (6.15) is valid, hence

$$u(x) = -\int_{\partial B_R(0)} \frac{\partial}{\partial n_y} G(x,y)\, g(y) dS_y. \tag{6.16}$$

Now for $0 < |x| < R$ and $|y| = R$

$$\begin{aligned}
\frac{\partial}{\partial n_y} G(x,y) &= \frac{y}{|y|} \cdot \nabla_y \Big(\frac{1}{4\pi|x-y|} - \frac{R}{4\pi|x|} \frac{1}{|y - \big(\frac{R}{|x|}\big)^2 x|} \Big) \\
&= -\frac{1}{4\pi|x-y|^2} \frac{(y-x)}{|x-y|} \cdot \frac{y}{|y|} + \frac{R}{4\pi|x|} \frac{1}{|y - \big(\frac{R}{|x|}\big)^2 x|^2} \frac{y - \big(\frac{R}{|x|}\big)^2 x}{|y - \big(\frac{R}{|x|}\big)^2 x|} \cdot \frac{y}{|y|} \\
&= \frac{-1}{4\pi|x-y|^3} (y-x) \cdot \frac{y}{|y|} + \frac{|x|^2}{4\pi R^2} \frac{1}{|x-y|^3} \Big(y - \big(\frac{R}{|x|}\big)^2 x \Big) \cdot \frac{y}{|y|} \\
&= \frac{1}{4\pi|x-y|^3} \Big(-(y-x) \cdot \frac{y}{|y|} + \big(\frac{|x|}{R}\big)^2 \big(y - \big(\frac{R}{|x|}\big)^2 x \big) \cdot \frac{y}{|y|} \Big) \\
&= \frac{1}{4\pi|x-y|^3} \big(-|y| + \frac{|x|^2}{R^2} |y| \big) = -\frac{1}{4\pi R} \frac{R^2 - |x|^2}{|x-y|^3}.
\end{aligned}$$

Insertion into (6.16) yields the formula claimed in the lemma. ∎

For $\Omega \subseteq \mathbb{R}^2$ the Green's function to the Dirichlet problem for the potential equation is defined by

$$G(x,y) = -\frac{1}{2\pi} \ln|x-y| + w(x,y),$$

where for every $x \in \Omega$

$$\begin{aligned}
\Delta_y w(x,y) &= 0, \qquad y \in \Omega, \\
w(x,y) &= \frac{1}{2\pi} \ln|x-y|, \qquad y \in \partial\Omega.
\end{aligned}$$

By the same method as in the three-dimensional case one obtains:

Theorem 6.10 (i) *The Green's function for the Dirichlet problem to the potential equation in the circle $B_R(0) \subseteq \mathbb{R}^2$ is*

$$G(x,y) = -\frac{1}{2\pi} \ln \frac{R|x-y|}{|x|\,|y - \big(\frac{R}{|x|}\big)^2 x|}.$$

(ii) *Let $u \in C_1(\overline{B_R(0)}) \cap C_2(B_R(0))$ be a solution of*

$$\Delta u(x) = 0, \quad x \in B_R(0), \tag{6.17}$$

$$u(x) = g(x), \quad x \in \partial B_R(0). \tag{6.18}$$

Then u is given by the Poisson representation formula

$$u(x) = \frac{1}{2\pi R} \int_{|y|=R} \frac{R^2 - |x|^2}{|x-y|^2} g(y)\, ds_y = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - \rho^2)}{R^2 - 2R\rho\cos(\varphi - \vartheta) + \rho^2} g(\vartheta)\, d\vartheta\,,$$

where $x = (\rho, \varphi)$, in polar coordinates.

Remark 6.11 Under the assumptions of Theorem 6.10 the boundary data g are continuously differentiable. However, by Theorem 5.6 we know that the Dirichlet problem (6.17), (6.18) in $\mathbb{R}^2$ has a unique solution for every continuous function g. In fact, using Theorem 5.4 it is not difficult to prove by approximation of a given continuous function g by continuously differentiable functions that the Poisson representation formula also holds if g is only continuous.

Up to now we have not shown that the Dirichlet problem for the potential equation in a ball $B_R(0) \subseteq \mathbb{R}^3$ has a solution. Since the integral

$$u(x) = \frac{1}{4\pi R} \int_{|y|=R} \frac{R^2 - |x|^2}{|x-y|^3}\, g(y) dS_y$$

exists for $x \in B_R(0)$ and for every function $g \in C\big(\partial B_R(0)\big)$, one surmises that the solution of the Dirichlet problem in a three-dimensional ball to given continuous boundary data g exists and is given by this integral formula. This is true and can be proved directly by showing that the function u given by the formula is twice continuously differentiable in $B_R(0)$ and satisfies

$$\Delta u(x) = 0, \qquad x \in B_R(0),$$

$$\lim_{\substack{x \to x_0 \\ x \in B_R(0)}} u(x) = g(x_0), \quad \text{for all } x_0 \in \partial B_R(0).$$

The proof of the first assertion is obvious, but the second assertion is difficult to verify, since the denominator of the integrand in the Poisson representation formula tends to zero if x converges to a boundary point. We do not analyse this boundary behavior here, but investigate a similar integral in Section 7, where we prove a general existence result for the Dirichlet problem in bounded domains in $\mathbb{R}^3$.

6.5 Green's function for the Helmholtz equation in the half space

As last example we determine the Green's function for the Dirichlet problem to the Helmholtz equation in the upper half space $H = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$. We have

$$\partial H = \{(x_1, x_2, 0) \mid (x_1, x_2) \in \mathbb{R}^2\} \cong \mathbb{R}^2.$$

To $g \in \overset{\circ}{C}(\partial H)$ and $\lambda \in \mathbb{C}$ we want to find a solution $u \in C_2(H) \cap C(\overline{H})$ of the Dirichlet problem

$$\Delta u(x) + \lambda u(x) = 0, \qquad x \in H, \tag{6.19}$$

$$u(x) = g(x), \qquad x \in \partial H. \tag{6.20}$$

Lemma 6.12 *The Green's function for the Dirichlet problem* (6.19), (6.20) *in the upper half space H is given by*

$$G(x,y) = \frac{1}{4\pi}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} - \frac{1}{4\pi}\frac{e^{i\sqrt{\lambda}|\hat{x}-y|}}{|\hat{x}-y|}, \tag{6.21}$$

where $x = (x_1, x_2, x_3) \in H$, $\hat{x} = (x_1, x_2, -x_3) \in -H$, and $y \in \overline{H}$ with $x \neq y$.

Proof: For $x \in H$ the function $y \mapsto w(x,y) = -\frac{1}{4\pi}\frac{e^{i\sqrt{\lambda}|\hat{x}-y|}}{|\hat{x}-y|}$ belongs to the space $C(\overline{H}) \cap C_\infty(H)$ and obviously satisfies

$$(\Delta_y + \lambda)w(x,y) = -(\Delta_y + \lambda)\frac{1}{4\pi}\frac{e^{i\sqrt{\lambda}|\hat{x}-y|}}{|\hat{x}-y|} = 0, \qquad y \in H.$$

Moreover, for $x \in H$ and $y = (y_1, y_2, 0) \in \partial H$ we obtain

$$|\hat{x} - y| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (-x_3)^2} = |x-y|\,,$$

hence

$$G(x,y) = \frac{1}{4\pi}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} + w(x,y) = \frac{1}{4\pi}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} - \frac{1}{4\pi}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} = 0.$$

Therefore G has the properties required from the Green's function. ■

Corollary 6.13 *Let $\lambda \in \mathbb{C} \setminus [0,\infty)$ and let $u \in C_1(\overline{H}) \cap C_2(H) \cap L^\infty(H)$ with $\nabla_x u \in L^\infty(H)$ be a solution of the Dirichlet problem* (6.19), (6.20). *Then we have for $x \in H$ that*

$$u(x) = -\frac{1}{2\pi}\int_{\partial H} \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}\, g(y)dy = \frac{1}{2\pi}\int_{\partial H}\frac{\partial}{\partial y_3}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}\, g(y)dy. \tag{6.22}$$

Remark 6.14 The assumptions $u \in L^\infty(H)$ and $\nabla_x u \in L^\infty(H)$ mean that there exists a constant C such that

$$|u(x)|,\ |\nabla u(x)| \leq C, \qquad \text{for all } x \in \overline{H}. \tag{6.23}$$

This shows that the assumptions impose conditions for the behavior of $u(x)$ and $\nabla_x u(x)$ if $|x| \to \infty$.

Proof: In the following computation we need estimates for the absolute values $|G(x,y)|$ and $|\nabla_y G(x,y)|$ if the pair $(x,y) \in H \times \overline{H}$ satisfies $|y| \geq \max(1, 2|x|)$. To derive these estimates, we infer from the inverse triangle inequality that

$$|\hat{x} - y| \geq |x-y| \geq \big||y| - |x|\big| \geq \frac{1}{2}|y| \geq \frac{1}{2}.$$

Noting that the assumption $\lambda \in \mathbb{C} \setminus [0, \infty)$ and our choice of the branch of the square root together imply $\operatorname{Re} i\sqrt{\lambda} = -\operatorname{Im} \sqrt{\lambda} > 0$, we obtain from (6.21) and from the above estimate by a short computation that

$$|G(x, y)| \le \frac{e^{-\operatorname{Im}\sqrt{\lambda}|x-y|}}{4\pi|x-y|} + \frac{e^{-\operatorname{Im}\sqrt{\lambda}|\hat{x}-y|}}{4\pi|\hat{x}-y|} \le \frac{1}{\pi} e^{-\operatorname{Im}\sqrt{\lambda}/2\,|y|}, \tag{6.24}$$

and that with $C_1 = \frac{\sqrt{|\lambda|}+2}{\pi}$

$$\begin{aligned} \big|\nabla_y G(x, y)\big| \le \big|i\sqrt{\lambda} - \frac{1}{|x-y|}\big| \frac{e^{-\operatorname{Im}\sqrt{\lambda}|x-y|}}{4\pi|x-y|} + \big|i\sqrt{\lambda} - \frac{1}{|\hat{x}-y|}\big| \frac{e^{-\operatorname{Im}\sqrt{\lambda}|\hat{x}-y|}}{4\pi|\hat{x}-y|} \\ \le C_1 e^{-\operatorname{Im}\sqrt{\lambda}/2\,|y|}. \end{aligned} \tag{6.25}$$

Now let $x \in H$. We want to determine a representation formula for $u(x)$. To this end choose $R > |x|$, define $H_R = \{z \in H \mid |z| < R\}$, and observe that the equation (6.12) remains valid if we insert the Green's function G in place of the fundamental solution F and if we integrate over H_R instead of Ω. Noting that in the present case the right hand side f of the Helmholtz equation vanishes identically, we obtain the representation

$$\begin{aligned} u(x) &= -\int_{\partial H_R} \frac{\partial}{\partial n_y} G(x, y)u(y) - G(x, y) \frac{\partial}{\partial n_y} u(y) dS_y \\ &= -\int_{\substack{y \in \partial H \\ |y|<R}} \frac{\partial}{\partial n_y} G(x, y)u(y)dS_y - \int_{\substack{|y|=R \\ y_3>0}} \frac{\partial}{\partial n_y} G(x, y)u(y) - G(x, y) \frac{\partial}{\partial n_y} u(y)dS_y. \end{aligned} \tag{6.26}$$

This formula holds for all $R > |x|$. Therefore we can let R tend to infinity. To compute the limit of the right hand side of (6.26), note that if $R > \max(1, 2|x|)$ then for the pair (x, y) with $|y| = R$ and $y_3 > 0$ the estimates (6.24) and (6.25) hold. Together with (6.23) we thus conclude

$$\begin{aligned} \lim_{R\to\infty} \Big| \int_{\substack{|y|=R \\ y_3>0}} \frac{\partial}{\partial n_y} G(x, y)u(y) - G(x, y) \frac{\partial}{\partial n_y} u(y)\, dS_y \Big| \\ \le \lim_{R\to\infty} \int_{\substack{|y|=R \\ y_3>0}} (C_1 e^{-\operatorname{Im}\sqrt{\lambda}/2\,R}\, C + \frac{1}{\pi} e^{-\operatorname{Im}\sqrt{\lambda}/2\,R}\, C) dS_y = 0. \end{aligned}$$

This relation and (6.26) yield

$$u(x) = \lim_{R\to\infty} u(x) = -\lim_{R\to\infty} \int_{\substack{y \in \partial H \\ |y|<R}} \frac{\partial}{\partial n_y} G(x, y)u(y)\, dS_y = -\int_{y \in \partial H} \frac{\partial}{\partial n_y} G(x, y)g(y)\, dS_y. \tag{6.27}$$

It remains to compute the term $\frac{\partial}{\partial n_y}G(x,y)$ for $y=(y_1,y_2,0)\in\partial H$. To this end note that

$$\frac{\partial}{\partial n_y}|\hat{x}-y| = -\frac{\partial}{\partial y_3}|\hat{x}-y| = -\left(\frac{y_3+x_3}{|\hat{x}-y|}\right)\Big|_{y_3=0}$$
$$= -\frac{x_3}{|x-y|} = \frac{\partial}{\partial y_3}|x-y| = -\frac{\partial}{\partial n_y}|x-y|\,.$$

From this equation and from (6.21) we obtain for $y\in\partial H$ that

$$\frac{\partial}{\partial n_y}G(x,y) = \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|} - \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|\hat{x}-y|}}{4\pi|\hat{x}-y|}$$
$$= \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|} + \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|} = \frac{1}{2\pi}\frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}\,.$$

Insertion of this equation into the representation formula (6.27) yields the statement of the corollary. ■

Remark 6.15 We did not show that a solution to (6.19), (6.20) satisfying the assumptions of Corollary 6.13 exists. Yet, the integral in the representation formula (6.22) is well defined for all $g\in\overset{\circ}{C}(\partial H)$, and for $\lambda\in\mathbb{C}\setminus[0,\infty)$ the term $\frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}$ decays exponentially for $y\in\operatorname{supp}(g)$ and $|x|\to\infty$. This suggests the conjecture that the formula (6.22) yields a solution u of (6.19), (6.20) for every $\lambda\in\mathbb{C}\setminus[0,\infty)$ and for all $g\in\overset{\circ}{C}(\partial H)$, which decays exponentially for $|x|\to\infty$, and therefore satisfies the boundedness condition (6.23). The conjecture is in fact true. For the proof one must investigate the limit behavior of the integral $\frac{1}{2\pi}\int_{\partial H}\frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}g(y)dy$ when $x\in H$ tends to the boundary ∂H. For this investigation we refer to the next section.

7 Integral equation method

7.1 Introduction

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded open set. We want to prove that the Dirichlet and Neumann boundary value problems for the Helmholtz equation in Ω and in the complement $\mathbb{R}^3 \setminus \overline{\Omega}$ can be solved uniquely, if the boundary $\partial\Omega$ is sufficiently smooth. The Green's function method cannot be used for this, since for such general domains the Green's functions cannot be determined explicitly. Instead, one uses the method of boundary integral equations. In this section we explain the idea of this method.

Consider the Dirichlet problem

$$\Delta u(x) + \lambda u(x) = 0, \quad x \in \Omega \tag{7.1}$$

$$u(x) = g(x), \quad x \in \partial\Omega. \tag{7.2}$$

If the boundary $\partial\Omega$ is smooth, then in sufficiently small neighborhoods of every point $x_0 \in \partial\Omega$ the boundary will be almost planar. This suggets to represent the solution of (7.1), (7.2) by using the representation formula (6.22) for the Dirichlet problem in the half space, for which the boundary is a plane. We thus try to find the solution u in the form

$$u(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y)\, dS_y. \tag{7.3}$$

with a suitable function $v \in C(\partial\Omega, \mathbb{C})$. In this equation u is called *double layer potential*, v is called the *boundary layer*. As will be shown, for every boundary layer v the double layer potential u satisfies the Helmholtz equation in $\mathbb{R}^3 \setminus \partial\Omega$. Therefore u is a solution of the Dirichlet boundary value problem, if v can be chosen such that the boundary condition (7.2) is satisfied. To see how such a function v can be determined, note that if Ω is equal to the upper half space H, then by Corollary 6.13 the double layer potential u with the boundary layer $v = -g$ satisfies the boundary condition $u|_{\partial H} = g$. This means that in the case of the half space problem the double layer potential satisfies the limit relation

$$\lim_{\substack{x \to x_0 \in \partial H \\ x \in H}} u(x) = -v(x_0) \tag{7.4}$$

for all $x_0 \in \partial\Omega$. One cannot expect that u given by (7.3) satisfies this simple limit relation also for curved boundaries $\partial\Omega$. Instead, we shall show that for general boundaries $\partial\Omega$ a correction term appears on the right hand side of (7.4). This correction term is given by the *jump relation*

$$\lim_{\substack{x\to x_0\\ x\in\Omega}} u(x) = -v(x_0) + \frac{1}{2\pi}\int_{\partial\Omega} \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x_0-y|}}{|x_0-y|}v(y)\,dS_y, \quad x_0\in\partial\Omega. \tag{7.5}$$

From this jump relation we see that u is a solution of the Dirichlet boundary value problem (7.1), (7.2), if the boundary layer v satisfies

$$-v(x) + \frac{1}{2\pi}\int_{\partial\Omega} \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}v(y)\,dS_y = g(x), \tag{7.6}$$

for all $x \in \partial\Omega$. This is an integral equation for the unknown function $v \in C(\partial\Omega,\mathbb{C})$ with the given right hand side $g \in C(\partial\Omega,\mathbb{C})$. If a solution v of this integral equation can be determined to a given function g, then the double layer potential u defined in (7.3) with v as the boundary layer is a solution of the Dirichlet boundary value problem to the boundary data g. Thus, if this integral equation is solvable for every $g \in C(\partial\Omega,\mathbb{C})$, the Dirichlet boundary value problem is solvable for all continuous boundary data. Therefore we must study under what conditions the boundary integral equation has a solution.

For $x \in \partial\Omega$ we write

$$(Kv)(x) = \frac{1}{2\pi}\int_{\partial\Omega} \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}v(y)\,dS_y. \tag{7.7}$$

With this notation the integral equation (7.6) can be written in the short form

$$(-I+K)\,v = g, \tag{7.8}$$

where I denotes the identity operator. The solution of the Dirichlet boundary value problem for the Helmholtz equation is thus reduced to the determination of the inverse of the linear operator $-I+K$.

To solve the Neumann boundary value problem

$$\Delta u(x) + \lambda u(x) = 0, \quad x\in\Omega, \tag{7.9}$$

$$\frac{\partial}{\partial n}u(x) = g(x), \quad x\in\partial\Omega, \tag{7.10}$$

we represent the solution by a *single layer potential*

$$u(x) = \frac{1}{2\pi}\int_{\partial\Omega} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}v(y)\,dS_y, \quad x\in\Omega. \tag{7.11}$$

The single layer potential satisfies the Helmholtz equation in $\mathbb{R}^3\setminus\partial\Omega$ and is continuous across the boundary $\partial\Omega$, the normal derivative satisfies a jump relation. To state this relation, we define for $x\in\partial\Omega$

$$\frac{\partial}{\partial n}u(x+) = \lim_{\substack{s\to 0\\ s>0}}\frac{\partial}{\partial s}u(x+sn_x), \quad \frac{\partial}{\partial n}u(x-) = \lim_{\substack{s\to 0\\ s<0}}\frac{\partial}{\partial s}u(x+sn_x). \tag{7.12}$$

The jump relation is

$$\frac{\partial}{\partial n}u(x\pm) = \mp v(x) + \frac{1}{2\pi}\int_{\partial\Omega}\frac{\partial}{\partial n_x}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}v(y)\,dS_y, \quad x \in \partial\Omega, \tag{7.13}$$

where $\frac{\partial}{\partial n_x}$ denotes the derivative with respect to the variable x in the direction of the exterior normal vector n_x at $x \in \partial\Omega$. Note that $\frac{\partial}{\partial n}u(x-)$ is the limit of the normal derivative from the interior of Ω. Therefore the Neumann boundary condition (7.10) is satisfied if the boundary layer v satisfies the integral equation

$$v + K'v = g,$$

with the operator K' defined by

$$(K'v)(x) = \frac{1}{2\pi}\int_{\partial\Omega}\frac{\partial}{\partial n_x}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}v(y)\,dS_y. \tag{7.14}$$

Thus, the Neumann problem is solvable for all continuous boundary data if the operator $(I + K') : C(\partial\Omega, \mathbb{C}) \to C(\partial\Omega, \mathbb{C})$ is invertible.

To prove that that the Dirichlet and Neumann boundary value problems are solvable it therefore suffices to show that the inverses $(-I + K)^{-1}$ and $(I + K')^{-1}$ exist. To prove that these inverses exist, we must study the linear operators K and K'. Though these operators are defined in (7.7) and (7.14) by complicated looking integral expressions, it turns out that these operators have good properties and define compact operators

$$K : X \to X, \qquad K' : X \to X,$$

where $X = C(\partial\Omega, \mathbb{C})$ denotes the Banach space of continuous functions on $\partial\Omega$ equipped with the supremum norm. An important result from functional analysis gives a criterion for the invertibility of operators of the form $\pm I + \mathcal{K}$ when $\mathcal{K}$ is a compact operator on a Banach space. In Section 7.2 we give the definition of compact operators and we sketch the derivation of the criterion. The integral equation method now proceeds to show that this criterion is satisfied, which finishes the existence proofs for the Dirichlet and Neumann problems.

The boundary integral equation method is clear-cut, but it is connected with difficulties stemming from the necessary investigation of the integral expressions in the jump relations and in the equations defining the operators K and K'. For neither of these integrals it is obvious that they exist, since the denominators of the integrands in these integrals vanishes at $y = x$; the integrands might thus have non-integrable singularities. The proof, that this is not the case and that the integrands are in fact quite well behaved is technical.

Before we enter these technical investigations, we first prove in Section 7.2 the spectral theorem for compact operators on a Banach space, from which the invertibility criterion mentioned above derives. In Section 7.3 we derive basic estimates needed in Section 7.4, where we study the double layer potential. The investigation of the single layer potential in Section 7.5 is less technical than the study of the double layer potential, since we can use results derived for the double layer potential. In Sections 7.6 and 7.7 we prove that the Neumann and Dirichlet boundary value problems are uniquely solvable by verifiying that the operators K' and K are compact and that the invertibility criterion from Section 7.2 is satisfied.

7.2 The spectral theorem of compact operators on a Banach space

In this section X denotes an abstract Banach space with norm $\|\cdot\|$.

Definition 7.1 *A linear operator $T : X \to X$ is called bounded if there is a constant C such that*

$$\|Tx\| \leq C\|x\|$$

for all $x \in X$.

Theorem 7.2 *A linear operator $T : X \to X$ is bounded if and only if it is continuous.*

Proof: If T is continuous at 0 it follows that there is $\delta > 0$ such that

$$\|Tx\| \leq 1$$

for all $x \in X$ with $\|x\| \leq \delta$. Since for every $y \in X$, $y \neq 0$ we have

$$\|\delta \frac{y}{\|y\|}\| = \delta \frac{\|y\|}{\|y\|} = \delta,$$

it follows that

$$\|Ty\| = \|T\big(\frac{\|y\|}{\delta}\,\delta\,\frac{y}{\|y\|}\big)\| = \frac{\|y\|}{\delta}\,\|T\big(\delta\,\frac{y}{\|y\|}\big)\| \leq \frac{1}{\delta}\,\|y\|\,.$$

This proves that T is bounded.

On the other hand, assume that T satisfies

$$\|Tx\| \leq C\|x\|$$

for all $x \in X$. Let $y \in X$, $\varepsilon > 0$, and set $\delta = \frac{\varepsilon}{C}$. Then for all $z \in X$ with $\|z - y\| \leq \delta$ it follows

$$\|T(z) - T(y)\| = \|T(z - y)\| \leq C\|z - y\| \leq \varepsilon,$$

hence T is continuous at y. Since y was arbitrary, T is continuous on X. ∎

Definition 7.3 *A linear operator $T : X \to X$ is called compact if to every bounded sequence $\{x_n\}_n \subseteq X$ the sequence of images $\{Tx_n\}_n$ has a subsequence, which converges in X.*

Lemma 7.4 *A compact operator is bounded.*

Proof: If the compact operator T would not be bounded then there would exist a sequence $\{x_n\}_n \subseteq X$ with $\|x_n\| = 1$ and $\|Tx_n\| \geq n$, for all $n \in \mathbb{N}$. The sequence $\{Tx_n\}_n$ would not have a convergent subsequence, hence T is not compact. ∎

Remember that for a linear operator $T : X \to X$ a number $\lambda \in \mathbb{C}$ with the property that there is $x \in X$, $x \neq 0$ satisfying $Tx - \lambda x = 0$ is called eigenvalue of T. The element x is called eigenvector. The set

$$E = \{x \in X \mid Tx - \lambda x = 0\}$$

is a linear subspace of X called eigenspace of the eigenvalue λ. The dimension of E is called the multiplicity of λ.

Definition 7.5 *Let $T : X \to X$ be a bounded operator. The resolvent set $\rho(T)$ of T consists of all points $\lambda \in \mathbb{C}$, which are not eigenvalues and for which the operator $(T - \lambda I) : X \to X$ is surjective. Here I is the identity. The complement $\Sigma(T) = \mathbb{C} \backslash \rho(T)$ is called spectrum of T.*

Clearly, $\lambda \in \rho(T)$ if and only if $T - \lambda I$ is injective and surjective. Hence λ belongs to the resolvent set if and only if $(T - \lambda I)^{-1}$ exists.

Theorem 7.6 *Let $T : X \to X$ be compact. $\Sigma(T)$ is a countable set with no accumulation point different from zero. Each nonzero $\lambda \in \Sigma(T)$ is an eigenvalue of T with finite multiplicity. If X has infinite dimension, then 0 belongs to $\Sigma(T)$.*

I only give part of the proof. The complete proof can be found for example in the book of Alt, pp. 363.

Proof: I.) First I show that the eigenvalues of T do not accumulate at a point $\lambda \neq 0$. Otherwise there would exist a sequence $\{\lambda_n\}_n$ of distinct eigenvalues of T with eigenvectors x_n such that $0 \neq \lambda_n \to \lambda \neq 0$. Let M_n be the subspace spanned by the n vectors $x_1, \ldots, x_n$. The space M_n is invariant under T; for if $x \in M_n$ then $x = c_1 x_1 + \ldots + c_n x_n$, hence

$$Tx = T(c_1 x_1 + \ldots + c_n x_n) = \lambda_1 c_1 x_1 + \ldots + \lambda_n c_n x_n \in M_n ,$$

thus $T(M_n) \subseteq M_n$.

Since eigenvectors to distinct eigenvalues are linearly independent, the vectors $x_1, x_2, \ldots$ are linearly independent. Therefore M_{n-1} is a proper subspace of M_n and there is $y_n \in M_n$ such that $\|y_n\| = 1$ and $\operatorname{dist}(y_n, M_{n-1}) = 1$. This holds since M_n is isomorphic to $\mathbb{R}^n$. With the sequence $\{y_n\}_n$ thus defined I show that $\{\lambda_n^{-1} T y_n\}_n$ contains no Cauchy sequence, contradicting the assumption that T is compact. (Note that $\{\lambda_n^{-1} y_n\}_n$ is a bounded sequence.) We have for $m < n$

$$\lambda_n^{-1} T y_n - \lambda_m^{-1} T y_m = y_n - \big(\lambda_m^{-1} T y_m - \lambda_n^{-1}(T - \lambda_n) y_n\big)$$

where the second term on the right belongs to M_{n-1} because $y_m \in M_{n-1}$, M_{n-1} is invariant under T and $(T - \lambda_n) y_n \in M_{n-1}$. Since $\operatorname{dist}(y_n, M_{n-1}) = 1$, it follows that each element of the sequence $\{\lambda_n^{-1} T y_n\}_n$ has distance ≥ 1 from any other one, showing that no subsequence of this sequence can be convergent.

II.) If there would be an eigenvalue $\lambda \neq 0$ of infinite multiplicity we could derive a contradiction by exactly the same arguments, defining λ_n by $\lambda_n = \lambda$ for all n and choosing for $\{x_n\}_n$ a sequence of linearly independent eigenvectors to λ.

It remains to show that if $\lambda \neq 0$ is not an eigenvalue it belongs to $\rho(T)$, hence the range $R(T - \lambda I)$ is equal to X. To this end it is shown that $R(T - \lambda I)$ is closed and that there is no nontrivial complementary space. For the details I refer to the book of Alt. ■

Corollary 7.7 *If $T : X \to X$ is a compact operator and if $-\lambda \neq 0$ is not an eigenvalue of T, then the inverse $(\lambda I + T)^{-1}$ exists.*

Proof: If $-\lambda \in \mathbb{C}$ with $\lambda \neq 0$ is not an eigenvalue of T, then λ is not an eigenvalue of $-T$, hence by Theorem 7.6 we have $\lambda \in \rho(-T)$. Thus, $(\lambda + T)^{-1} = (\lambda - (-T))^{-1}$ exists. ■

7.3 Basic differential geometry and estimates for the double layer kernel

Because the double layer potential (7.3) is an integral over the surface $\partial\Omega$ and because the normal vector to this surface appears in the integrand, the properties of the double layer potential strongly depend on the differential geometric properties of $\partial\Omega$. Therefore we introduce in this section some basic differential geometric notions and results and use these tools to prove estimates, which follow from the geometry of $\partial\Omega$. Using the geometric estimates, we proceed to prove kernel and integral estimates, which are applied in the following section to study the operator K and to prove the jump relations.

The proofs of the integral estimates are technical. Therefore we advice the reader after having read the definitions of the standard coordinate neighborhood and the theorems on the tubular neighborhood to jump to Section 7.4 and to return to the present section when the integral estimates are needed. Though the proofs of the integral estimates are technical, one must know them to understand why the operator K is well defined and why the jump relations hold.

Let $x_0 \in \partial\Omega$ and let $B_R(x_0)$ be the open ball with radius R and center x_0. The set $V_R(x_0) = B_R(x_0) \cap \partial\Omega$ is called a coordinate neighborhood of x_0 in $\partial\Omega$, if there is an open set $U_R \subseteq \mathbb{R}^2$ and a continuously differentiable function $\phi : U_R \to V_R(x_0) \subseteq \mathbb{R}^3$ with continuous inverse $\phi^{-1} : V_R(x_0) \to U_R$, such that for every $\xi = (\xi_1, \xi_2) \in U_R$ the rank of the Jacobi matrix $\phi'(\xi) = \nabla_\xi \phi(\xi) \in \mathbb{R}^{3\times 2}$ is two. The mapping ϕ is called parametrization of $V_R(x_0)$, and ϕ^{-1} is called coordinate map on $V_R(x_0)$. The boundary $\partial\Omega$ is of class C_2, if every $x_0 \in \partial\Omega$ has a coordinate neighborhood with two times continuously differentiable parametrization.

Definition 7.8 (Standard coordinate neighborhood) *We say that ϕ is a standard parametrization of $V_R(x_0)$ with bound M_{x_0}, if $0 \in U_R$ and if there are orthogonal unit tangent vectors $\tau_1, \tau_2 \in \mathbb{R}^3$ to $\partial\Omega$ at x_0, such that for all $\xi \in U_R$*

$$\phi(\xi) = \xi_1\tau_1 + \xi_2\tau_2 + \varphi(\xi)\nu + x_0\,, \tag{7.15}$$

where $\nu = -n_{x_0}$ is the interior unit normal vector to $\partial\Omega$ at x_0, and where $\varphi \in C_2(U_R, \mathbb{R}^3)$ satisfies

$$\varphi(0) = 0, \qquad \nabla_\xi\varphi(0) = 0, \tag{7.16}$$

and

$$\sup_{\xi\in U_R} \|\nabla^2\varphi(\xi)\| \le M_{x_0}. \tag{7.17}$$

$V_R(x_0)$ is called standard coordinate neighborhood of x_0, if $V_R(x_0)$ has a standard parametrization. We call R the size of $V_R(x_0)$.

We say that the standard coordinate neighborhood $V_R(x_0)$ is star shaped, if U_R is star shaped with respect to 0, and if for all unit vectors $e \in \mathbb{R}^2$

$$\sup_{0<t<t_e} |\phi(te) - x_0| = R \tag{7.18}$$

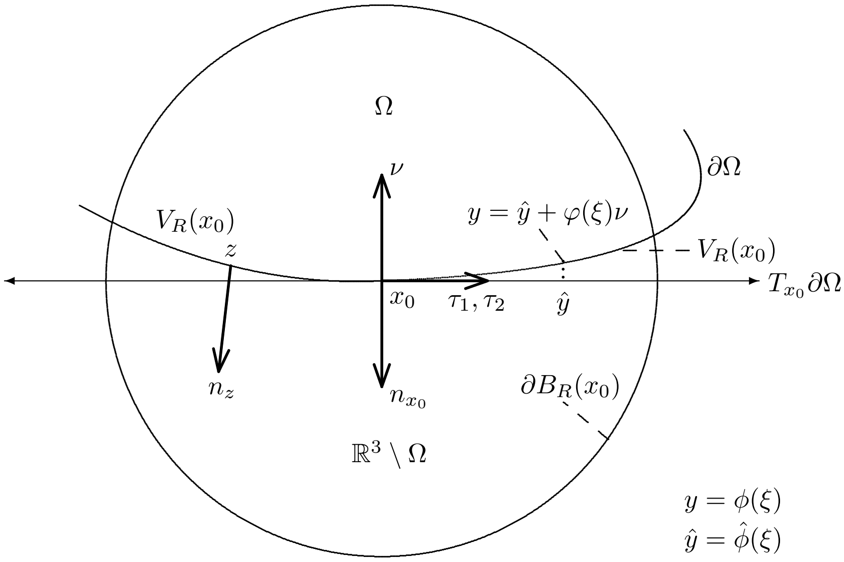


Figure 1: The standard coordinate neighborhood $V_R(x_0) = \partial\Omega \cap B_R(x_0)$ of x_0

holds, where $t_e = \sup\{t \geq 0 \mid te \in U_R\}$.

$\nabla^2\varphi(\xi)$ in (7.17) denotes the Hesse matrix of the second partial derivatives of φ at ξ, and the norm is the operator norm, which for a 2×2–matrix A is defined by $\|A\| = \sup_{\eta\in\mathbb{R}^2, \|\eta\|=1} \|A\eta\|$. Condition (7.18) means that for any ray in $\mathbb{R}^2$ starting at 0 the values of ϕ approach the boundary $\partial B_R(x_0)$ along this ray.

The standard parametrizatiopn ϕ of a standard coordinate neighborhood is not unique, but depends on the pair (τ_1, τ_2) of orthogonal unit tangent vectors chosen. However, two different parametrizations obtained by different choices of the pairs of unit tangent vectors are related by an orthogonal coordinate transformation, from which it follows that the corresponding Hesse matrices are related by an orthogonal tansformation. Since the operator norm is invariant under an orthogonal matrix transformation, every upper bound for the norm of the Hesse matrix of the first parametrization is also an upper bound for the Hesse matrix of the second transformation. Therefore the bound does not depend on the choice of the standard parametrization.

If $V_R(x_0) = B_R(x_0) \cap \Omega$ is a standard coordinate neighborhood with bound M_{x_0} and if $0 < \rho \leq R$, then $V_\rho(x_0) = B_\rho(x_0) \cap \Omega$ is also a standard coordinate neighborhood with bound M_{x_0}. However, if $V_R(x_0)$ is star shaped, then $V_\rho(x_0)$ is not necessarily star shaped.

If the point x_0 is understood, we sometimes write in the following V_R instead of $V_R(x_0)$.

We assume in the following that there are $0 < \hat{R} \leq 1$ *and* $\hat{M} > 0$ *such that every point* $x_0 \in \partial\Omega$ *has a star shaped standard coordinate neighborhood of size* $\hat{R}$ *with bound* $\hat{M}$. It can be shown that if Ω is bounded with boundary $\partial\Omega$ of class C_2, then this assumption is satisfied automatically, but we omit the technical proof.

Definition 7.9 (Projection to the tangent plane) *Let $x_0 \in \partial\Omega$ with standard coordinate neighborhood $V_{\hat{R}}$ of size $\hat{R}$, and let*

$$\xi \mapsto \phi(\xi) = \xi_1\tau_1 + \xi_2\tau_2 + \varphi(\xi)\nu + x_0 : U_{\hat{R}} \to V_{\hat{R}}$$

be a standard parametrization. The tangent plane to $\partial\Omega$ at x_0 is denoted by $T_{x_0}\partial\Omega$. We define a parametrization $\hat{\phi} : \mathbb{R}^2 \to T_{x_0}\partial\Omega$ of the tangent plane by

$$\hat{\phi}(\xi) = \xi_1\tau_1 + \xi_2\tau_2 + x_0.$$

We call $\hat{\phi}$ the parametrization associated to ϕ. For $y = \phi(\xi) \in V_{\hat{R}}$ we define the point $\hat{y} \in T_{x_0}\partial\Omega$ by

$$\hat{y} = \hat{\phi}(\xi),$$

and call $\hat{y}$ the projection of y to the tangent plane.

The mapping $y \mapsto \hat{y}$ has a coordinate independent representation. To see this, remember that $\nu = -n_{x_0}$, hence

$$-(y - x_0) \cdot n_{x_0} = (\xi_1\tau_1 + \xi_2\tau_2 + \varphi(\xi)\nu) \cdot \nu = \varphi(\xi).$$

This yields

$$\hat{y} = \hat{\phi}(\xi) + \varphi(\xi)\nu - \big((y - x_0) \cdot n_{x_0}\big)n_{x_0} = y - \big((y - x_0) \cdot n_{x_0}\big)n_{x_0}, \tag{7.19}$$

which shows that $\hat{y}$ is indeed obtained by orthogonal projection of y to the tangent plane $T_{x_0}\partial\Omega$.

A standard neighborhood and the projection to the tangent plane is depicted in Figure 1.

Theorem 7.10 (Tubular neighborhood) *Let $\Omega \subseteq \mathbb{R}^3$ be bounded with boundary $\partial\Omega$ of class C_2. Then there is a number $s_0 > 0$ such that the mapping $\psi_{\text{tub}} : (-s_0, s_0) \times \partial\Omega \to \mathbb{R}^3$ defined by*

$$\psi_{\text{tub}}(s, x) = x - sn_{x_0}$$

is continuously differentiable and invertible with continuously differentiable inverse $\psi_{\text{tub}}^{-1} : \partial_{s_0}\Omega \to (-s_0, s_0) \times \partial\Omega$, where

$$\partial_{s_0}\Omega = \psi\big((-s_0, s_0) \times \partial\Omega\big).$$

The set $\partial_{s_0}\Omega$ is called tubular neighborhood of $\partial\Omega$ of width s_0.

A **proof** of this theorem can be found in *Manfredo P. do Carmo, Differentiable geometry of curves and surfaces, Prentice Hall 1976, p. 110-111.*

Geometrically, Theorem 7.10 means that for all $x, y \in \partial\Omega$ with $x \neq y$ the line through x in direction of the normal vector n_x and the line through y in the direction of the normal vector n_y do not intersect within $\partial_{s_0}\Omega$. Or, viewed differently, this theorem means that to every $z \in \partial_{s_0}\Omega$ the image $(s, z_0) = \psi_{\text{tub}}^{-1}(z)$ is the unique pair with $z_0 \in \partial\Omega$ and $s \in (-s_0, s_0)$ such that z can be represented in the form

$$z = z_0 - s n_{x_0}.$$

The next theorem contains more detailed information on tubular neighborhoods.

Theorem 7.11 *Let $\partial_{s_0}\Omega$ be a tubular neighborhood of $\partial\Omega$. Then we have*

$$\partial_{s_0}\Omega = \{z \in \mathbb{R}^3 \mid \text{dist}(z, \partial\Omega) < s_0\}.$$

For $z \in \partial_{s_0}\Omega$ let $(s, z_0) = \psi_{\text{tub}}^{-1}(z)$. Then z_0 is the unique point from $\partial\Omega$, which satisfies

$$|z - z_0| = \min_{x \in \partial\Omega} |z - x|.$$

Moreover, we have $|s| = |z - z_0|$.

The **proof** of this theorem will be included in a later version of these lecture notes.

In the following we assume that there is a tubular neighborhood $\partial_{\hat{R}}\Omega$, whose width is equal to the size $\hat{R}$ of the standard star shaped coordinate neighborhoods. For bounded Ω with boundary of class C_2 this assumption is always satisfied, but again we omit the technical proof.

To simplify the notation, we often denote for $x \in \partial\Omega$ and $|s| < \hat{R}$ the image $\psi_{\text{tub}}(s, x)$ by x_s, hence

$$x_s = x - s n_x\,.$$

This implies $x_0 = x$. If $x_0 \in \partial\Omega$ is given, we therefore also write $x_s = \psi_{\text{tub}}(s, x_0)$ instead of $x_{0,s}$. No confusion will arise.

We are now in a position to state and prove some geometric estimates, which are employed in the investigation of the integral kernels of the double layer potential. In the proof we need that the exterior unit normal vector at $y = \phi(\xi) \in V_{\hat{R}}(x_0)$ is given in a standard parametrization by

$$n_y = \frac{\partial_{\xi_1}\varphi(\xi)\tau_1 + \partial_{\xi_2}\varphi(\xi)\tau_2 - \nu}{\sigma(\xi)}, \tag{7.20}$$

with

$$\sigma(\xi) = \sigma(\phi^{-1}(y)) = \sqrt{1 + |\nabla_\xi \varphi(\xi)|^2}. \tag{7.21}$$

The fraction $\frac{1}{\sigma(\xi)}$ can be expressed independently of the parametrization. Namely, (7.20) yields

$$n_{x_0} \cdot n_y = -\nu \cdot n_y = \frac{1}{\sigma(\xi)} = \frac{1}{\sigma(\phi^{-1}(y))}, \qquad y \in V_{\hat{R}}(x_0). \tag{7.22}$$

Lemma 7.12 (Geometric estimates) *For $x_0 \in \partial\Omega$ let $V_{\hat{R}}(x_0)$ be the star shaped standard coordinate neighborhood of x_0 with a standard parametrization $\phi : U_{\hat{R}} \to V_{\hat{R}}(x_0)$ bounded by $\hat{M}$. For $0 < \rho \leq \hat{R} \leq 1$ let $U_\rho = \phi^{-1}(V_\rho(x_0))$.*

(i) For $x_s = x_0 - sn_{x_0} \in \partial_{\hat{R}}\Omega$ with $(s, x_0) = \psi_{\text{tub}}^{-1}(x_s)$, for $y \in \partial\Omega \setminus V_{\hat{R}}(x_0)$ and for the projection $\hat{y}$ of y to the tangent plane $T_{x_0}\partial\Omega$ we have

$$|\hat{y} - x_0| \leq |\hat{y} - x_s|, \quad |\hat{y} - x_0| \leq |y - x_0|, \quad |\hat{y} - x_0| \leq |y - x_s|, \tag{7.23}$$

$$|\hat{y} - x_0| = |\xi|. \tag{7.24}$$

(ii) The parametrizaton satisfies for all $\xi \in U_{\hat{R}}$

$$|\varphi(\xi)| \leq \frac{1}{2}\hat{M}|\xi|^2 \leq \frac{1}{2}\hat{M}, \qquad |\nabla\varphi(\xi)| \leq \hat{M}|\xi| \leq \hat{M}. \tag{7.25}$$

(iii) For $r > 0$ let $\hat{U}_r$ be the open ball in $\mathbb{R}^2$ with center 0 and radius r. For $0 < \rho \leq \hat{R}$ set $\rho' = \frac{\rho}{\sqrt{1+\frac{1}{4}(\hat{M}\rho)^2}}$. Then we have

$$\hat{U}_{\rho'} \subseteq U_\rho \subseteq \hat{U}_\rho. \tag{7.26}$$

(iv) $x_s = x_0 - sn_{x_0} \in \partial_{\rho/2}\Omega$ and $y \in \partial\Omega \setminus V_\rho(x_0)$ satisfy

$$|y - x_s| \geq \rho/2. \tag{7.27}$$

$$\left| \frac{y - x_0}{|y - x_s|} \cdot n_y \right| \leq 2. \tag{7.28}$$

(v) For $x_s = x_0 - sn_{x_0} \in \partial_{\hat{R}}\Omega$, for $y \in V_{\hat{R}}(x_0)$ with $y \neq x_0$, and for the projection $\hat{y}$ of y to the tangent plane $T_{x_0}\partial\Omega$ we have

$$\left| \frac{1}{|y - x_s|^3} - \frac{1}{|\hat{y} - x_s|^3} \right| \leq \frac{\hat{M}(3 + \hat{M})}{2|\hat{y} - x_s|^2}, \tag{7.29}$$

$$\left| \frac{y - x_0}{|y - x_s|} \cdot n_y \right| \leq \frac{3}{2}\hat{M}|y - x_s|. \tag{7.30}$$

The geometric relations between the distances $|y - x_s|$, $|\hat{y} - x_s|$, $|\hat{y} - x_0|$ is depicted in Figure 2.

Proof: (i) The line segment connecting x_0 to $\hat{y}$ belongs to the tangent plane $T_{x_0}\partial\Omega$. By definition of $\hat{y}$ and x_0, this line segment is equal to the orthogonal projections of the line segments connecting x_s to $\hat{y}$, x_0 to y, and x_s to y. The estimates (7.23) follow immediately from these geometric relations. (7.24) is an immediate consequence of the definition of $\hat{y}$ in Definition 7.9, which yields $|\hat{y} - x_0| = |\xi_1\tau_1 + \xi_2\tau_2| = |\xi|$.

(ii) Since $V_\rho(x_0) \subseteq B_\rho(x_0)$, it follows from (7.24) and from the second estimate of (7.23) for all $\xi \in U_\rho$ that

$$|\xi| = |\hat{y} - x_0| \le |y - x_0| < \rho. \tag{7.31}$$

By assumption, $U_{\hat{R}}$ is star shaped with respect to 0. Taylor's formula and the barrier theorem *(see Corollary 4.15 in: H.-D. Alber, Lecture Notes Analysis II)* therefore yield together with (7.16) and with the constant M_{x_0} in (7.17) replaced by $\hat{M}$, that for all $\xi \in U_{\hat{R}}$

$$|\varphi(\xi)| = \frac{1}{2}\left|\xi \cdot \nabla^2\varphi(\xi^{(1)})\xi\right| \le \frac{1}{2}\|\nabla^2\varphi(\xi)\||\xi|^2 \le \frac{1}{2}\hat{M}|\xi|^2 < \frac{1}{2}\hat{M},$$

$$|\nabla\varphi(\xi)| \le \left\|\nabla\big(\nabla\varphi(\xi^{(2)})\big)\right\| |\xi| = \left\|\nabla^2\varphi(\xi^{(2)})\right\| |\xi| \le \hat{M}|\xi| < \hat{M}.$$

The last inequality in both estimates is obtained from (7.31) by setting $\rho = \hat{R}$ and noting that by assumption $\hat{R} \le 1$. This proves (7.25).

(iii) The inequality (7.31) immediately yields $U_\rho \subseteq \hat{U}_\rho$. To verify (7.26) it thus remains to prove the relation $\hat{U}_{\rho'} \subseteq U_\rho$. To this end assume that $\xi \in \hat{U}_{\rho'}$. From the first relation in (7.25) it follows for all $\alpha \in [0, 1]$ that

$$\begin{aligned}|\phi(\alpha\xi) - x_0|^2 &= |\alpha\xi_1\tau_1 + \alpha\xi_2\tau_2 + \varphi(\alpha\xi)\nu|^2 = |\alpha\xi|^2 + \varphi(\alpha\xi)^2\\ &< (\alpha\rho')^2 + \Big(\frac{1}{2}\hat{M}(\alpha\rho')^2\Big)^2 \le (\alpha\rho')^2\Big(1 + \frac{1}{4}(\hat{M}\rho)^2\Big)\\ &= \alpha^2\frac{\rho^2}{1 + \frac{1}{4}(\hat{M}\rho)^2}\Big(1 + \frac{1}{4}(\hat{M}\rho)^2\Big) = (\alpha\rho)^2 \le \hat{R}^2.\end{aligned} \tag{7.32}$$

Since the coordinate neighborhood $V_{\hat{R}}(x_0)$ is star shaped, condition (7.18) holds with R replaced by $\hat{R}$ and with $e = \frac{\xi}{|\xi|}$. It thus follows from this condition and from (7.32) that we can find $t > 0$ with $t\frac{\xi}{|\xi|} \in U_{\hat{R}}$ such that

$$|\phi(\alpha\xi) - x_0| < \left|\phi\Big(t\frac{\xi}{|\xi|}\Big) - x_0\right| < \hat{R}.$$

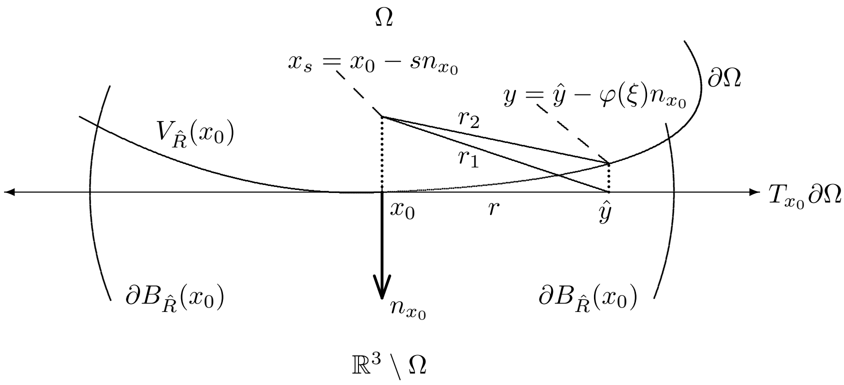


Figure 2: The distances r, r_1, and r_2.

This inequlity holds for all $0 \le \alpha \le 1$, which can only be true if $\frac{t}{|\xi|} > 1$. Therefore ξ belongs to the line segment in $\mathbb{R}^2$ connecting 0 with $t\frac{\xi}{|\xi|}$. Since by assumption $U_{\hat{R}}$ is star shaped with respect to zero, this line segment belongs to $U_{\hat{R}}$, which implies $\xi \in U_{\hat{R}}$, and this in turn implies $\phi(\xi) \in V_{\hat{R}}(x_0) \subseteq \partial\Omega$. Invoking (7.32) with $\alpha = 1$, we therefore conclude that

$$\phi(\xi) \in \partial\Omega \cap B_\rho(x_0) = V_\rho(x_0),$$

which means that $\xi \in U_\rho$. Since ξ was an arbitrary point in $\hat{U}_\rho$, we obtain $\hat{U}_\rho \subseteq U_\rho$. This conludes the proof of (7.26).

(iv) From $V_\rho = B_\rho(x_0) \cap \partial\Omega$ we obtain

$$\partial\Omega \setminus V_\rho = \partial\Omega \setminus (B_\rho(x_0) \cap \partial\Omega) = \partial\Omega \setminus B_\rho(x_0) \subseteq \mathbb{R}^3 \setminus B_\rho(x_0),$$

Therefore $y \in \partial\Omega \setminus V_\rho$ belongs to the complement of the ball $B_\rho(x_0)$. Moreover, if $x_s = x_0 - sn_{x_0}$ belongs to $\partial_{\rho/2}\Omega$, then we have $|s| < \rho/2$, hence x_s belongs to the ball $B_{\rho/2}(x_0)$. Consequently, we have $|y - x_s| \ge \rho/2$. This proves (7.27).

For $x_s \in \partial_{\rho/2}\Omega$ we have $|s| < \rho/2$. The triangle inequality together with (7.27) thus yields

$$|y - x_0| = |y - x_s + sn_{x_0}| \le |y - x_s| + |sn_{x_0}| = |y - x_s| + |s| \le 2|y - x_s|.$$

(7.28) is an immediate consequence of this inequality.

(v) To verify (7.29) we set

$$r = |\hat{y} - x_0|, \quad r_1 = |\hat{y} - x_s|, \quad r_2 = |y - x_s|,$$

to simplify the notation. By successive application of the inverse triangle inequality, the

Definition 7.9 of $\hat{y}$, the first estimate of (7.25), and equation (7.24) we obtain

$$\begin{aligned}|r_2 - r_1| &= \big||y - x_s| - |\hat{y} - x_s|\big| \le |(y - x_s) - (\hat{y} - x_s)| = |y - \hat{y}| \\ &= |\varphi(\xi)\nu| = |\varphi(\xi)| \le \frac{1}{2}\hat{M}|\xi|^2 = \frac{1}{2}\hat{M}|\hat{y} - x_0|^2 = \frac{1}{2}\hat{M}r^2. \quad (7.33)\end{aligned}$$

By (7.23), we have $r \le r_1$ and $r \le r_2$. Using these two estimates and (7.33), we compute

$$\begin{aligned}\frac{|r_1 - r_2|}{r_2^2} &\le \frac{\frac{1}{2}\hat{M}r^2}{r_2^2} \le \frac{1}{2}\hat{M}, \\ \frac{r_1}{r_2} \le \frac{r_2 + |r_1 - r_2|}{r_2} &\le 1 + \frac{\frac{1}{2}\hat{M}r^2}{r} = 1 + \frac{1}{2}\hat{M}r, \\ \frac{r_2}{r_1} \le \frac{r_1 + |r_2 - r_1|}{r_1} &\le 1 + \frac{\frac{1}{2}\hat{M}r^2}{r} = 1 + \frac{1}{2}\hat{M}r.\end{aligned}$$

Thus,

$$\begin{aligned}\Big|\frac{1}{r_2^3} - \frac{1}{r_1^3}\Big| = \Big|\frac{(r_1 - r_2)(r_1^2 + r_1 r_2 + r_2^2)}{r_1^3 r_2^3}\Big| &= \frac{|r_1 - r_2|}{r_2^2}\Big(\frac{r_1}{r_2} + 1 + \frac{r_2}{r_1}\Big)\frac{1}{r_1^2} \\ &\le \frac{1}{2}\hat{M}(3 + \hat{M}r)\frac{1}{r_1^2} < \frac{\hat{M}(3 + \hat{M})}{2r_1^2},\end{aligned}$$

where in the last step we used the second inequality of (7.23), which implies $r = |\hat{y} - x_0| \le |y - x_0| < \hat{R} \le 1$. This proves (7.29).

To verify (7.30) let $y = \xi_1\tau_1 + \xi_2\tau_2 + \varphi(\xi)\nu + x_0 \in V_{\hat{R}}(x_0)$. With the parametric representation (7.20) of n_y and with the estimates (7.25), the equation (7.24), and the last estimate in (7.23), we compute

$$\begin{aligned}|(y - x_0) \cdot n_y| &= \big|\big(\xi_1\tau_1 + \xi_2\tau_2 + \varphi(\xi)\nu\big) \cdot \big(\partial_{\xi_1}\varphi(\xi)\tau_1 + \partial_{\xi_2}\varphi(\xi)\tau_2 - \nu\big)\big|\sigma(\xi)^{-1} \\ &= |\nabla_\xi\varphi(\xi) \cdot \xi - \varphi(\xi)|\sigma(\xi)^{-1} \le \hat{M}|\xi|^2 + \frac{1}{2}\hat{M}|\xi|^2 = \frac{3}{2}\hat{M}|\hat{y} - x_0|^2 \le \frac{3}{2}\hat{M}|y - x_s|^2.\end{aligned}$$

We also noted that (7.21) implies $\sigma(\xi)^{-1} \le 1$. The estimate (7.30) is a consequence of this inequality. ■

Next we study the integral kernel of the double layer potential (7.3).

Lemma 7.13 (Kernel estimates) *Let* $\lambda \in \mathbb{C}$ *and* $0 < \rho \le \hat{R} \le 1$. *For* $t \ge 0$ *define*

$$\begin{aligned}h(t) &= e^{i\sqrt{\lambda}t}(1 - i\sqrt{\lambda}t), \\ h_\infty = \max_{0 \le t \le \operatorname{diam}(\partial_{\hat{R}}\Omega)} |h(t)|, &\qquad \hat{h}_\infty = \max_{0 \le t \le \operatorname{diam}(\partial_{\hat{R}}\Omega)} |h'(t)t - 3h(t)|.\end{aligned}$$

(i) For $x_s = x_0 - sn_{x_0} \in \partial_{\hat{R}}\Omega$ and $y \in \partial\Omega$ with $y \neq x_s$ we have

$$\frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|} = -h(|x_s-y|)\frac{s\,(n_{x_0}\cdot n_y)}{|x_s-y|^3} + g(x_s,y), \tag{7.34}$$

where $g : \partial_{\hat{R}}\Omega \times \partial\Omega \to \mathbb{R}$ is defined by

$$g(x_s,y) = \begin{cases} \dfrac{h(|x_s-y|)}{|x_s-y|^2}\dfrac{x_0-y}{|x_s-y|}\cdot n_y, & \text{for } x_s \neq y, \\ 0, & \text{for } x_s = y, \end{cases} \tag{7.35}$$

and satisfies

$$|g(x_s,y)| \leq \begin{cases} \frac{3\hat{M}h_\infty}{2|x_0-\hat{y}|}, & x_s \in \partial_{\hat{R}}\Omega, \quad y \in V_{\hat{R}}(x_0), \quad x_s \neq y, \\ \frac{8h_\infty}{\rho^2}, & x_s \in \partial_{\rho/2}\Omega, \quad y \in \partial\Omega \setminus V_\rho(x_0). \end{cases} \tag{7.36}$$

(ii) For $z \in \partial_{\hat{R}}\Omega$ and $y \in \partial\Omega$ with $z \neq y$ we have

$$\Big|\nabla_z \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|z-y|}}{|z-y|}\Big| \leq \frac{\hat{h}_\infty + h_\infty}{|z-y|^3}. \tag{7.37}$$

Proof: Equation (7.34) with g defined in (7.35) follows by a direct computation. The estimate (7.36) for the case $x_s \in \partial_{\hat{R}}\Omega$, $y \in V_{\hat{R}}(x_0)$ is an immediate consequence of (7.30) and of the last inequality in (7.23). The estimate (7.36) for the case $x_s \in \partial_{\rho/2}\Omega$, $y \in \partial\Omega \setminus V_\rho(x_0)$ follows from (7.27) and (7.28). Finally, a direct computation yields with the abbreviation $w = z - y$ that

$$\nabla_z \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|z-y|}}{|z-y|} = \frac{h'(|w|)|w| - 3h(|w|)}{|w|^3}\frac{w\cdot n_y}{|w|}\frac{w}{|w|} + \frac{h(|w|)}{|w|^3}n_y,$$

from which (7.37) results by taking absolute values. ■

Since $h(0) = 1$, equation (7.34) and the estimate (7.36) show that $h(|x_s-y|)\frac{s\,(n_{x_0}\cdot n_y)}{|x_s-y|^3}$ is the most singular term in the integrand of the double layer potential. The validity of the jump relations for the double layer potential follows from the presence of this term. This is seen form the estimate (7.43) in the next lemma, which shows that the integral of this term jumps at 0 when s varies. The investigation of the double layer potential in the next section rests mainly on (7.43) and on the other estimates stated now.

Lemma 7.14 (Integral estimates) *Let $0 < \rho \leq \hat{R} \leq 1$ and set*

$$C_1 = \sqrt{1 + \frac{1}{4}\hat{M}^2}, \quad C_2 = \frac{\hat{M}(3+\hat{M})}{2}, \quad C_3 = \frac{\hat{M}^2 C_1^3}{8}, \quad C_4 = \hat{M}\sqrt{1+\hat{M}^2}. \tag{7.38}$$

(i) Let $z = z_0 - s_z n_{z_0} \in \partial_{\rho/3}\Omega$ and $w = w_0 - s_w n_{w_0} \in \partial_{\rho/3}\Omega$. Suppose that $|z - w| < \rho/3$. Then we have with $|\partial\Omega| = \int_{\partial\Omega} dy$

$$\int\limits_{\partial\Omega\setminus(V_\rho(z_0)\cap V_\rho(w_0))} \left| \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|z-y|}}{|z-y|} - \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|w-y|}}{|w-y|} \right| dS_y \leq 27\,|\partial\Omega|\, \frac{\hat{h}_\infty + h_\infty}{\rho^3}\, |z-w|. \quad (7.39)$$

(ii) For $x_s = x_0 - s n_{x_0} \in \partial_{\hat{R}}\Omega$ we have

$$\int_{V_\rho(x_0)} |g(x_s, y)|\, dS_y \leq 3\pi\, C_4 h_\infty\, \rho \leq 3\pi\, C_4 h_\infty. \quad (7.40)$$

(iii) For $x_s = x_0 - s n_{x_0} \in \partial_{\hat{R}}\Omega$ set

$$I(x_s, \rho) = \frac{1}{2\pi} \int_{V_\rho(x_0)} \frac{s\, n_{x_0} \cdot n_y}{|x_s - y|^3}\, dS_y\,. \quad (7.41)$$

Then we have for $\mathrm{sgn}(s) = \frac{s}{|s|}$ *and $s \neq 0$ that*

$$|I(x_s, \rho)| \leq 1 + 2C_2 + C_3, \quad (7.42)$$

$$\big|I(x_s, \rho) - \mathrm{sgn}(s)\big| \leq C_1 \frac{|s|}{\rho} + C_2(|s| + \rho) + C_3 |s| \rho. \quad (7.43)$$

Proof: (i) We apply the mean value theorem and the estimate (7.37) to obtain with a suitable point x^* on the line segment connecting z and w that

$$\left| \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|z-y|}}{|z-y|} - \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|w-y|}}{|w-y|} \right| = \left| \Big(\nabla_x \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} \Big)\Big|_{x=x*} \cdot (z-w) \right|$$
$$\leq \frac{\hat{h}_\infty + h_\infty}{|x^* - y|^3}\, |z-w|. \quad (7.44)$$

In this computation it is justified to apply (7.37), since x^* belongs to the set $\partial_{\hat{R}}\Omega$. To see this, note that since $|z - w| \leq \rho/3$ and since x^* lies on the line segment connecting z and w, we have $|z - x^*|, |w - x^*| < \rho/3$. Because z belongs to $\partial_{\rho/3}\Omega$, we conclude from the first of these estimates that indeed $x^* \in \partial_{2\rho/3}\Omega \subseteq \partial_{\hat{R}}\Omega$. Moreover, we conclude from these estimates that

$$|z_0 - x^*| \leq |z_0 - z| + |z - x^*| < 2\rho/3, \quad |w_0 - x^*| \leq |w_0 - w| + |w - x^*| < 2\rho/3,$$

which means that $B_{\rho/3}(x^*) \subseteq B_\rho(z_0) \cap B_\rho(w_0)$. From this relation and from the equation

$$\partial\Omega \setminus \big(V_\rho(z_0) \cap V_\rho(w_0)\big) = \partial\Omega \setminus \Big(\big(\partial\Omega \cap B_\rho(z_0)\big) \cap \big(\partial\Omega \cap B_\rho(w_0)\big) \Big) = \partial\Omega \setminus \big(B_\rho(z_0) \cap B_\rho(w_0)\big).$$

we infer that $y \in \partial\Omega \setminus \big(V_\rho(z_0) \cap V_\rho(w_0)\big)$ satisfies $|y - x^*| \geq \rho/3$. We estimate the right hand side of (7.44) by this inequality. Integration of the resulting estimate yields (7.39).

(ii) We estimate g by (7.36) and obtain with the parametrization $\phi : U_\rho \to V_\rho$, with the mapping $\hat{\phi}$ introduced in Definition 7.9, and with σ given in (7.21) by definition of the surface integral

$$\int_{V_\rho} |g(x_s, y)| dS_y \leq \int_{V_\rho} \frac{3\hat{M}h_\infty}{2|x_0 - \hat{y}|} dS_y = \int_{U_\rho} \frac{3\hat{M}h_\infty}{2|x_0 - \hat{\phi}(\xi)|} \sigma(\xi)\, d\xi$$

$$\leq \sqrt{1 + \hat{M}^2} \int_{U_\rho} \frac{3\hat{M}h_\infty}{2|\xi|}\, d\xi, \quad (7.45)$$

where in the last step we used (7.25) to estimate σ. The relation (7.26) yields

$$\int_{U_\rho} \frac{1}{|\xi|}\, d\xi \leq \int_{\hat{U}_\rho} \frac{1}{|\xi|}\, d\xi = 2\pi \int_0^\rho \frac{1}{r} r\, dr = 2\pi\rho.$$

We estimate the right hand side of (7.45) with this inequality to obtain (7.40).

(iii) Note that by (7.26) the ball $\hat{U}_{\rho'} \subseteq \mathbb{R}^2$ with radius $\rho' = \frac{\rho}{\sqrt{1+\frac{1}{4}(\hat{M}\rho)^2}}$ is a subset of U_ρ. Since by (7.22) we have $n_{x_0} \cdot n_y = \frac{1}{\sigma(\xi)}$, the definition of surface integrals therefore yields

$$I(x_s, \rho) = \frac{1}{2\pi} \int_{V_\rho} \frac{s\,(n_{x_0} \cdot n_y)}{|x_s - y|^3} dS_y = \frac{1}{2\pi} \int_{U_\rho} \frac{s}{|x_s - \phi(\xi)|^3}\, d\xi$$

$$= \frac{1}{2\pi} \int_{\hat{U}_{\rho'}} \frac{s}{|x_s - \hat{\phi}(\xi)|^3}\, d\xi + \mathcal{R}_1(x_s, \rho) + \mathcal{R}_2(x_s, \rho), \quad (7.46)$$

where

$$\mathcal{R}_1(x_s, \rho) = \frac{s}{2\pi} \int_{\hat{U}_{\rho'}} \Big(\frac{1}{|x_s - \phi(\xi)|^3} - \frac{1}{|x_s - \hat{\phi}(\xi)|^3} \Big)\, d\xi, \quad (7.47)$$

$$\mathcal{R}_2(x_s, \rho) = \frac{1}{2\pi} \int_{U_\rho \setminus \hat{U}_{\rho'}} \frac{s}{|x_s - \phi(\xi)|^3}\, d\xi. \quad (7.48)$$

The first term on the right hand side of (7.46) can be computed explicitly: For $\xi \in U_{\hat{R}}$ we have by Definition 7.9

$$|x_s - \hat{\phi}(\xi)| = |(x_0 + s\nu) - (\xi_1\tau_1 + \xi_2\tau_2 + x_0)| = \sqrt{|\xi|^2 + s^2}, \quad (7.49)$$

whence

$$\frac{1}{2\pi} \int_{\hat{U}_{\rho'}} \frac{s}{|x_s - \hat{\phi}(\xi)|^3}\, d\xi = \frac{1}{2\pi} \int_{\hat{U}_{\rho'}} \frac{s}{\sqrt{|\xi|^2 + s^2}^3}\, d\xi = \int_0^{\rho'} \frac{s}{\sqrt{r^2 + s^2}^3}\, r\, dr$$

$$= -\frac{s}{\sqrt{r^2 + s^2}}\Big|_{r=0}^{r=\rho'} = \operatorname{sgn}(s)\Big(1 - \frac{|s|}{\sqrt{\rho'^2 + s^2}}\Big),$$

from which we conclude that

$$\left| \frac{1}{2\pi} \int_{\hat{U}_{\rho'}} \frac{s}{|x_s - \hat{\phi}(\xi)|^3}\, d\xi \right| \leq 1, \tag{7.50}$$

$$\left| \frac{1}{2\pi} \int_{\hat{U}_{\rho'}} \frac{s}{|x_s - \hat{\phi}(\xi)|^3}\, d\xi - \operatorname{sgn}(s) \right| \leq \frac{|s|}{\sqrt{\rho'^2 + s^2}} \leq \frac{|s|}{\rho'} \leq \frac{|s|}{\rho} \sqrt{1 + \frac{1}{4}\hat{M}^2}, \tag{7.51}$$

where the last inequality sign in (7.51) follows from $\frac{1}{\rho'} \leq \frac{\sqrt{1+\frac{1}{4}(\hat{M}\hat{R})^2}}{\rho} \leq \frac{\sqrt{1+\frac{1}{4}\hat{M}^2}}{\rho}$. To estimate $\mathcal{R}_1$ note that for $\xi \in \hat{U}_{\rho'} \subseteq U_\rho \subseteq U_{\hat{R}}$ the equation (7.49) yields $|x_s - \hat{\phi}(\xi)| \leq |s| + \rho$, whence $1 < \frac{|s|+\rho}{|x_s - \hat{\phi}(\xi)|}$. Now use (7.29) and this inequality to estimate the integrand in (7.47). The resulting integral is estimated by (7.50). We obtain

$$|\mathcal{R}_1(x_s, \rho)| \leq \frac{|s|}{2\pi} \int_{\hat{U}_{\rho'}} \frac{\hat{M}(3+\hat{M})(|s|+\rho)}{2|x_s - \hat{\phi}(\xi)|^3}\, d\xi \leq \frac{\hat{M}(3+\hat{M})}{2}(|s|+\rho). \tag{7.52}$$

To estimate $\mathcal{R}_2$ note that (7.26) yields for the measure $|U_\rho \setminus \hat{U}_{\rho'}|$ that

$$|U_\rho \setminus \hat{U}_{\rho'}| \leq |\hat{U}_\rho \setminus \hat{U}_{\rho'}| = \pi(\rho^2 - \rho'^2) = \pi\big(\rho^2 - \frac{\rho^2}{1 + \frac{1}{4}(\hat{M}\rho)^2}\big) = \frac{\frac{\pi}{4}\hat{M}^2\rho^4}{1 + \frac{1}{4}(\hat{M}\rho)^2} \leq \frac{\pi}{4}\hat{M}^2\rho^4.$$

Since by (7.23) and (7.24) for $\xi \in U_\rho \setminus \hat{U}_{\rho'}$ the estimate

$$|x_s - \phi(\xi)| = |x_s - y| \geq |x_0 - \hat{y}| = |\xi| \geq \rho' \geq \frac{\rho}{\sqrt{1 + \frac{1}{4}(\hat{M}\hat{R})^2}} \geq \frac{\rho}{\sqrt{1 + \frac{1}{4}\hat{M}^2}}$$

holds, we conclude from (7.48) and from the last two inequalities that

$$|\mathcal{R}_2(x_s, \rho)| \leq \frac{1}{2\pi} \frac{\pi}{4} \hat{M}^2 \rho^4 \frac{\big(1 + \frac{1}{4}\hat{M}^2\big)^{3/2}}{\rho^3} |s| = \frac{\hat{M}^2\big(1 + \frac{1}{4}\hat{M}^2\big)^{3/2}}{8} |s|\rho. \tag{7.53}$$

To prove the estimate (7.43) note that by (7.46), the term $|I(x_s, \rho)| - \operatorname{sgn}(s)|$ can be estimated by the sum of the terms on the left hand sides of the inequalities (7.51), (7.52), and (7.53), whence it can be estimated by the sum of the terms on the right hand sides of these inequalities. To obtain the inequality (7.42), we estimate the terms on the right hand side of (7.46) by (7.50), (7.52), and (7.53) and note that $|s|, \rho < \hat{R} \leq 1$. ■

7.4 The double layer potential

We use now the estimates from the preceding section to study the double layer potential. Here and in the remainder of Section 7 the Banach space $C(\partial\Omega, \mathbb{C})$ equipped with the norm

$$\|v\|_\infty = \sup_{x \in \partial\Omega} |v(x)|.$$

is denoted by X.

We start by studying the operator K. Observe first that the integral in the definition (7.7) of K exists. This is seen from the equation (7.34), since the left hand side of this equation is equal to the integral kernel in (7.7) if we set $x_s = x_0 \in \partial\Omega$, hence $s = 0$. Yet, for $s = 0$ the right hand side of (7.34) is equal to $g(x_0, y)$, and by (7.36) the function $y \mapsto g(x_0, y)$ is integrable over $\partial\Omega$. Thus, for every $v \in X$ the function $x \mapsto (Kv)(x) : \partial\Omega \to \mathbb{C}$ is well defined and belongs to the vector space $F(\partial\Omega, \mathbb{C})$ of functions on $\partial\Omega$ with values in $\mathbb{C}$. Obviously, (Kv) depends linearly on v, hence $K : X \to F(\partial\Omega, \mathbb{C})$ is a linear operator.

Theorem 7.15 (Boundedness of K and Hölder continuity of Kv)
(i) K defined in (7.7) is a bounded linear operator on the Banach space X.
(ii) Let h_∞ and $\hat{h}_\infty$ be the constants defined in Lemma 7.13 and let $C_4 = \hat{M}\sqrt{1+\hat{M}^2}$. For $v \in X$ and $z, w \in \partial\Omega$ the estimate

$$\big|(Kv)(z) - (Kv)(w)\big| \le C_K |z-w|^{\frac{1}{4}} \, \|v\|_\infty \,, \tag{7.54}$$

holds, where

$$C_K = \max\left(\frac{27\,|\partial\Omega|\,(\hat{h}_\infty + h_\infty)}{2\pi} + 3C_4 h_\infty \,,\; \Big(\frac{3}{2}C_4 + \frac{4|\partial\Omega|}{\pi\hat{R}^2}\Big)\frac{2h_\infty}{\min(\hat{R}, 3^{-\frac{1}{3}}} \right).$$

Remark. Since the constant C_K does not depend on $x \in \partial\Omega$, the inequality (7.54) means that the function $(Kv) : \partial\Omega \to \mathbb{C}$ is uniformly Hölder continuous with exponent $1/4$.

Proof: For $v \in C(\partial\Omega, \mathbb{C})$ and $x \in \partial\Omega$ we conclude from (7.36) and (7.40) that

$$\begin{aligned} |(Kv)(x)| \le \frac{\|v\|_\infty}{2\pi} \left(\int_{V_{\hat{R}}(x_0)} |g(x,y)|\, dS_y + \int_{\partial\Omega \setminus V_{\hat{R}}(x_0)} |g(x,y)|\, dS_y \right) \\ \le \Big(\frac{3}{2} C_4\, h_\infty + \frac{4h_\infty|\partial\Omega|}{\pi\hat{R}^2} \Big) \|v\|_\infty = \hat{C}_K \|v\|_\infty \,, \end{aligned} \tag{7.55}$$

where $\hat{C}_K = \frac{3}{2} C_4\, h_\infty + \frac{4h_\infty|\partial\Omega|}{\pi\hat{R}^2}$. This inequality implies that $K : X \to B(\partial\Omega, \mathbb{C})$ is a bounded linear operator, where $B(\partial\Omega, \mathbb{C})$ denotes the Banach space of complex valued functions on $\partial\Omega$, equipped with the supremum norm. Statement (i) follows from this result if we prove the estimate (7.54), since (7.54) implies $Kv \in C(\partial\Omega, \mathbb{C})$.

To prove (7.54), let $z, w \in \partial\Omega$ satisfy $|z - w| < \min(\hat{R}^4, 3^{-\frac{4}{3}})$ and define $\rho = |z-w|^{\frac{1}{4}}$. Then we have

$$|z-w| = |z-w|^{\frac{3}{4}}\, \rho < \big(3^{-\frac{4}{3}}\big)^{\frac{3}{4}} \rho = \rho/3, \qquad \rho = |z-w|^{\frac{1}{4}} < \big(\hat{R}^4\big)^{\frac{1}{4}} = \hat{R}.$$

Because of these relations, in the following computation the assumptions of the estimates (7.39) and (7.40) are satisfied, and we obtain for $v \in X$ with $(\partial\Omega)_1 = \partial\Omega \setminus (V_\rho(z) \cap V_\rho(w))$ and $(\partial\Omega)_2 = V_\rho(z) \cap V_\rho(w)$ that

$$|(Kv)(z) - (Kv)(w)| \le \frac{1}{2\pi}\|v\|_\infty \int_{\partial\Omega} \left| \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|z-y|}}{|z-y|} - \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|w-y|}}{|w-y|} \right| dS_y$$

$$\le \frac{\|v\|_\infty}{2\pi} \left(\int_{(\partial\Omega)_1} \left| \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|z-y|}}{|z-y|} - \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|w-y|}}{|w-y|} \right| dS_y + \int_{(\partial\Omega)_2} |g(z,y)| + |g(w,y)|\, dS_y \right)$$

$$\le \frac{\|v\|_\infty}{2\pi} \left(27\,|\partial\Omega|\, \frac{\hat{h}_\infty + h_\infty}{\rho^3}\, |z-w| + 6\pi\, C_4\, h_\infty\, \rho \right) \le C_K\, \|v\|_\infty\, |z-w|^{\frac{1}{4}}. \quad (7.56)$$

For $|z-w| \ge \min(\hat{R}^4, 3^{-\frac{4}{3}})$ we obtain from (7.55)

$$|(Kv)(z) - (Kv)(w)| \le 2\|Kv\|_\infty \Big(\frac{|z-w|}{\min(\hat{R}^4, 3^{-\frac{4}{3}})} \Big)^{\frac{1}{4}} \le \frac{2\hat{C}_K}{\min(\hat{R}, 3^{-\frac{1}{3}})} \|v\|_\infty\, |z-w|^{\frac{1}{4}}.$$

This inequality and (7.56) together yield (7.54). ∎

Theorem 7.16 (Jump relations) *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with $\partial\Omega \in C_2$, and let $n_y \in \mathbb{R}^3$ be the exterior unit normal vector to $\partial\Omega$ at $y \in \partial\Omega$. Assume that $\lambda \in \mathbb{C}$ and $v \in C(\partial\Omega, \mathbb{C})$. For $x \in \mathbb{R}^3 \backslash \partial\Omega$ set*

$$w(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}\, v(y) dS_y. \quad (7.57)$$

(i) Then w belongs to $\in C_\infty(\mathbb{R}^3 \setminus \partial\Omega)$ and satisfies

$$\Delta w(x) + \lambda w(x) = 0, \quad x \in \mathbb{R}^3 \backslash \partial\Omega.$$

(ii) For $x_0 \in \partial\Omega$ we have the jump relations

$$\lim_{\substack{x \to x_0 \\ x \in \Omega}} w(x) = -v(x_0) + (Kv)(x_0), \quad (7.58)$$

$$\lim_{\substack{x \to x_0 \\ x \in \mathbb{R}^3 \backslash \overline{\Omega}}} w(x) = v(x_0) + (Kv)(x_0). \quad (7.59)$$

Proof: (i) Since for all $\alpha \in \mathbb{N}_0^3$

$$\Big((x,y) \mapsto \frac{\partial^{|\alpha|}}{\partial x^\alpha} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} \Big) \in C\big((\mathbb{R}^3 \backslash \partial\Omega) \times \partial\Omega, \mathbb{C} \big),$$

it follows as usual that $w \in C_\infty(\mathbb{R}^3 \backslash \partial\Omega, \mathbb{C})$ with

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} w(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{\partial^{|\alpha|}}{\partial x^\alpha} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}\, v(y) dS_y$$

for $x \in \mathbb{R}^3 \backslash \partial\Omega$. In particular, this implies

$$(\Delta + \lambda)w(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} (\Delta_x + \lambda) \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y) dS_y = 0,$$

since $\frac{1}{4\pi} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}$ is the fundamental solution of the Helmholtz equation. This proves (i).

(ii) We first show that the limit relations

$$\lim_{s \searrow 0} w(x_0 \mp s n_{x_0}) = \mp v(x_0) + (Kv)(x_0) \tag{7.60}$$

hold, which means that the limit relations (7.58) and (7.59) are valid if x tends to x_0 along the directions of the normal n_{x_0}.

In the following considerations we assume that $0 < \rho \leq \hat{R}$. To prepare the proof of (7.60), observe that (7.34) yields for $x_s = x_0 - s n_{x_0} \in \partial_{\hat{R}}\Omega$ with $s \neq 0$ and for $y \in V_\rho(x_0)$

$$\begin{aligned} &\frac{1}{2\pi}\Big(\frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|} - \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x_0-y|}}{|x_0-y|} \Big) v(y) \\ &\qquad = -\frac{1}{2\pi} \frac{s\, n_{x_0} \cdot n_y}{|x_s-y|^3} \big(v(x_0) + H(x_s,y) \big) + \frac{1}{2\pi} \big(g(x_s,y) - g(x_0,y) \big) v(y), \end{aligned} \tag{7.61}$$

where we use the abbreviation

$$H(x_s, y) = h(|x_s - y|)v(y) - v(x_0).$$

With the constant $h_0 = 2\sqrt{|\lambda|} e^{\sqrt{|\lambda|} \operatorname{diam}(\partial_{\hat{R}}\Omega)}$ and with the constant h_∞ defined in Lemma 7.13 the function H satisfies

$$\sup_{y \in V_\rho(x_0)} |H(x_s, y)| \leq h_0 \|v\|_\infty (|s| + \rho) + h_\infty \sup_{y \in V_\rho(x_0)} |v(y) - v(x_0)|. \tag{7.62}$$

To see this, note that for $0 \leq t \leq \operatorname{diam}(\partial_{\hat{R}}\Omega)$

$$|h(t) - 1| = \big| e^{i\sqrt{\lambda}t}(1 - i\sqrt{\lambda}t) - 1 \big| \leq |e^{i\sqrt{\lambda}t} - 1| + |\sqrt{\lambda}|t \leq \sqrt{|\lambda|} t e^{\sqrt{|\lambda|}t} + \sqrt{|\lambda|} t \leq h_0 t,$$

whence, with $t = |x_s - y|$,

$$|H(x_s, y)| \leq |h(t) - 1|\, |v(x_0)| + |h(t)|\, |v(y) - v(x_0)| \leq h_0 t \|v\|_\infty + h_\infty |v(y) - v(x_0)|.$$

(7.62) follows from this inequality if we note that $t = |x_s - y| \leq |x_s - x_0| + |x_0 - y| \leq |s| + \rho$.

We proceed now to prove the relation (7.60). This relation is a consequence of a chain of an equality and two inequalities. We state this chain first and explain afterwards how to obtain it:

$$
\begin{aligned}
&\big| -\operatorname{sgn}(s)v(x_0) + (Kv)(x_0) - w(x_s)\big| \\
&= \left| -\operatorname{sgn}(s)v(x_0) - \left(\int_{V_\rho(x_0)} + \int_{\partial\Omega\setminus V_\rho(x_0)}\right) \frac{1}{2\pi}\Big(\frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|} - \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x_0-y|}}{|x_0-y|}\Big) v(y)\,dS_y \right| \\
&\qquad \le |v(x_0)|\,|I(x_s,\rho) - \operatorname{sgn}(s)| + |I(x_s,\rho)| \sup_{y\in V_\rho(x_0)} |H(x_s,y)| \\
&\qquad\qquad + |J(x_s,\rho)| + 27\,|\partial\Omega| \frac{\hat{h}_\infty + h_\infty}{2\pi\rho^3}\,|s|\,\|v\|_\infty \\
&\le |v(x_0)|\Big(C_1\frac{|s|}{\rho} + C_2(|s|+\rho) + C_3|s|\rho\Big) + 3C_4 h_\infty\|v\|_\infty\rho + 27\,|\partial\Omega|(\hat{h}_\infty + h_\infty)\|v\|_\infty\frac{|s|}{2\pi\rho^3} \\
&\qquad + (1+2C_2+C_3)\Big(h_0\|v\|_\infty(|s|+\rho) + h_\infty \sup_{y\in V_\rho(x_0)}|v(y)-v(x_0)|\Big). \qquad (7.63)
\end{aligned}
$$

The equality sign follows from the definitions of w in (7.57) and v in (7.7). The first inequality sign is obtained by inserting (7.61) for the integrand of the integral over the set $V_\rho(x_0)$, and by estimating the integral over the set $\partial\Omega\setminus V_\rho(x_0)$ by (7.39). In (7.39) we insert x_s for z and x_0 for w. Since $|x_s - x_0| = |s|$, the assumptions for (7.39) are satisfied, if we choose $|s| < \rho/3$. The function $I(x_s,\rho)$ is defined in (7.41). We use the notation $J(x_s,\rho) = \frac{1}{2\pi}\int_{V_\rho(x_0)}(g(x_s,y) - g(x_0,y))v(y)\,dS_y$. Finally, the second inequality sign is a direct consequence of the estimates (7.42), (7.43), (7.40), and (7.62).

Let $\varepsilon > 0$ be given. We choose the size ρ of the standard coordinate neighborhood $V_\rho(x_0)$ as a function of $|s|$. Namely, we set $\rho = |s|^{1/4}$. This is possible if $|s| < \min(3^{-4/3}, \hat{R}^4)$, since this implies $|s| < |s|^{1/4}/3 = \rho/3$ and $\rho < \hat{R}$, hence (7.63) is valid. With this choice of ρ all the terms on the right hand side of the inequality (7.63) tend to zero for $s \to 0$ with the power $|s|^{1/4}$ or with higher powers of $|s|$, with the exception of the very last term on the right hand side. This last term depends on the properties of v. To deal with this term we observe that since v is continuous and $\partial\Omega$ is compact, the function v is uniformly continuous. Therefore we can choose $\delta_1 > 0$ sufficiently small such that for all $x_0 \in \partial\Omega$ and all $|s| < \delta_1$ the estimate

$$
(1+2C_2+C_3)\,h_\infty \sup_{y\in V_{|s|^{1/4}}(x_0)} |v(y)-v(x_0)| < \frac{1}{2}\varepsilon,
$$

holds. Now choose $\delta_2 > 0$ such that for all $|s| < \delta_2$ the sum of the other terms on the right hand side of (7.63) is less than $\frac{1}{2}\varepsilon$, and set $\delta = \min(\delta_1, \delta_2, 3^{-4/3}, \hat{R}^4)$. It follows that for all $s \in \mathbb{R}$ with $|s| < \delta$ the inequality (7.63) is valid and the left hand side is less than ε. Since $\varepsilon > 0$ was chosen arbitrarily, this means that the limit relations (7.60) hold.

It remains to show that the limit relations (7.58) and (7.59) are valid not only if x tends to x_0 along the normal direction, but in an arbitrary way. To verify this, note first that the limits (7.60) are uniform with respect to $x_0 \in \partial\Omega$, because the choice of δ is independent of x_0. For, the choice of δ_1 only depends on C_2, C_3 and h_∞, and the choice of δ_2 depends on $C_1, \ldots, C_4$, h_∞, $\hat{h}_\infty$, and h_0. The definitions of h_∞, $\hat{h}_\infty$, $C_1, \ldots, C_4$ in Lemma 7.13 and Lemma 7.14, and the definition of h_0 given above show that these constants only depend on λ and $\hat{M}$, hence all the constants are independent of x_0, so δ is chosen independently of x_0.

After this preparation, we proceed to prove (7.58). To this end fix $x_0 \in \partial\Omega$ and let $\varepsilon > 0$ be given. Since v is continuous and since by Theorem 7.15 the mapping Kv is Hölder continuous, it follows that the function $(-v + Kv) : \partial\Omega \to \mathbb{C}$ is continuous. Therefore there is a neighborhood $\hat{V} \subseteq \partial\Omega$ of x_0, which is not necessarily a standard coordinate neighborhood, such that

$$|(-v + Kv)(y) - (-v + Kv)(x_0)| < \frac{\varepsilon}{2} \qquad \text{for all } y \in \hat{V}. \tag{7.64}$$

Let $\psi_{\text{tub}}^{-1} : \partial_{\hat{R}}\Omega \to (-\hat{R}, \hat{R}) \times \partial\Omega$ be the inverse mapping to the tubular neighborhood mapping introduced in Theorem 7.10. For $z \in \partial_{\hat{R}}\Omega$ we write $(s_z, z_0) = \psi_{\text{tub}}^{-1}(z)$. With this notation we define the mapping $\hat{\psi} : \partial_{\hat{R}}\Omega \to \partial\Omega$ by $\hat{\psi}(z) = z_0$. Since $z = z_0 - s_z n_{z_0}$ and since the limits (7.60) are uniform, we can choose $\delta > 0$ with $\delta \leq \hat{R}$ such that

$$\big|w(z) - (-v + Kv)(z_0)\big| < \frac{\varepsilon}{2}, \qquad \text{for all } z \in \{\partial_{\hat{R}}\Omega \mid |z - z_0| < \delta\} \cap \Omega = \partial_\delta\Omega \cap \Omega. \tag{7.65}$$

Since by Theorem 7.10 the mapping ψ_{tub}^{-1} is continuous, also the mapping $\hat{\psi}$ is continuous and satisfies $\hat{\psi}(x_0) = x_0$. Thus, there is a neighborhood $W \subseteq \partial_\delta\Omega$ of x_0 such that $\hat{\psi}(W) = \hat{V}$. By Theorem 7.11 we have $\partial_\delta\Omega = \{z \in \mathbb{R}^3 \mid \text{dist}(z, \partial\Omega) < \delta\}$, which means that W is a neighborhood of x_0 in $\mathbb{R}^3$. For z from the one-sided neighborhood $W \cap \Omega$ of x_0 in Ω we have $z_0 = \hat{\psi}(z) \in \hat{V}$, whence (7.64) and (7.65) yield

$$\big|w(z) - (-v + Kv)(x_0)\big| \leq \big|w(z) - (-v + Kv)(z_0)\big| + |(-v + Kv)(z_0) - (-v + Kv)(x_0)| < \varepsilon.$$

Since $\varepsilon > 0$ was chosen arbitrarily, this proves (7.58). The relation (7.59) is obtained in the same way if we replace $-v + Kv$ by $v + Kv$ and consider the one sided neighborhood $W \cap (\mathbb{R}^3 \setminus \overline{\Omega})$ instead of $W \cap \Omega$. ■

7.5 The single layer potential

We next study the single layer potential (7.11). Just as in the previous section it can be shown that to $v \in C(\partial\Omega, \mathbb{C})$ there are constants C, c' such that for all $x, y \in \partial\Omega$ the estimate

$$\Big|\frac{\partial}{\partial n_x}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}\Big| \le \frac{Ce^{c'|x-y|}}{|x-y|}$$

holds. It follows that for every $v \in C(\partial\Omega,\mathbb{C})$ and all $x \in \partial\Omega$ the integral in (7.14) exists. (7.14) thus defines a function $(K'v) : \partial\Omega \to \mathbb{C}$.

Theorem 7.17 (Hölder continuity and boundedness of K')
(i) There is a constant $C_{K'}$ such that for all $v \in C(\partial\Omega,\mathbb{C})$ and $x^{(1)}, x^{(2)} \in \partial\Omega$

$$\big|(K'v)(x^{(1)}) - (K'v)(x^{(2)})\big| \le C_{K'}\big|x^{(1)} - x^{(2)}\big|^{\frac{1}{4}}\,\|v\|_\infty.$$

(ii) The linear operator $K' : X \to X$ is bounded.

The **proof** is similar to the proof of Theorem 7.15. Therefore we omit it.

Theorem 7.18 (Jump relations) *Assume that $v \in C(\partial\Omega,\mathbb{C})$. For $x \in \mathbb{R}^3 \setminus \partial\Omega$ set*

$$\omega(x) = \frac{1}{2\pi}\int_{\partial\Omega}\frac{e^{i\sqrt{\lambda}\,|x-y|}}{|x-y|}\,v(y)\,dS_y.$$

Then
(i) *ω belongs to $C(\mathbb{R}^3,\mathbb{C}) \cap C_\infty(\mathbb{R}^3 \setminus \partial\Omega,\mathbb{C})$ and satisfies*

$$(\Delta + \lambda)\omega(x) = 0, \quad x \in \mathbb{R}^3 \setminus \partial\Omega.$$

(ii) *At $x \in \partial\Omega$ the one sided derivatives $\frac{\partial\omega}{\partial n}(x\pm)$ defined in (7.12) exist and satisfy*

$$\frac{\partial\omega}{\partial n}(x\pm) = \mp v(x) + \frac{1}{2\pi}\int_{\partial\Omega}\frac{\partial}{\partial n_x}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}\,v(y)dS_y\,.$$

To prove this theorem we need a lemma.

Lemma 7.19 *For $x \in \partial\Omega$ and $s \in \mathbb{R}$ let $x_s = x + sn_x$. Then*

$$\lim_{s\to 0}\int_{\partial\Omega}\Big(\frac{\partial}{\partial s}\frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|} + \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|}\Big)v(y)dS_y$$
$$= \int_{\partial\Omega}\Big(\frac{\partial}{\partial n_x}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} + \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}\Big)v(y)dS_y. \qquad (7.66)$$

Proof: Since

$$\lim_{s\to 0}\frac{\partial}{\partial s}\frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|} = \frac{\partial}{\partial n_x}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|},$$

for all $y \in \partial\Omega \setminus \{x\}$, it suffices to show that on the left hand side of (7.66) the limit can be interchanged with the integral. To verify this we constuct a majorant for the integrand on the left hand side of (7.66), which is independent of s. By the Lebesgue convergence theorem, the existence of such a majorant implies that the limit can be interchanged with the integral.

Note first that to every r_0 there is a constant c_1 such that $|\frac{\partial}{\partial r}\frac{e^{i\sqrt{\lambda}r}}{r}| \leq \frac{c_1}{r^2}$ for all $0 < r \leq r_0$. Together with this estimate it thus follows for all $s \in \mathbb{R}$ and all $y \in \partial\Omega$ with $y \neq x$ that

$$\Big|\frac{\partial}{\partial s}\frac{e^{i\sqrt{\lambda}|x+sn_x-y|}}{|x+sn_x-y|} + \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|}\Big|$$

$$= \Big|\Big[\frac{\partial}{\partial r}\frac{e^{i\sqrt{\lambda}r}}{r}\Big]_{r=|x_s-y|}\frac{(x_s-y)\cdot(n_x-n_y)}{|x_s-y|}\Big| \leq \frac{c_1}{|x_s-y|^2}|n_x-n_y|. \tag{7.67}$$

Since $\partial\Omega$ is bounded and of class C_2, there is a constant c_2 such that for all $y \in \partial\Omega$

$$|n_x - n_y| \leq c_2|x-y|. \tag{7.68}$$

We choose $s_0 > 0$ small enough such that the line segment $\{x + sn_x \mid |s| \leq s_0\}$ interescts $\partial\Omega$ only in the point x. It then follows by standard considerations that there is a constant $c_3 > 0$ such that for all $|s| \leq s_0$ we have

$$|x-y| \leq c_3|x_s - y|.$$

From this estimate and from (7.67), (7.68) we conclude that

$$\Big|\frac{\partial}{\partial s}\frac{e^{i\sqrt{\lambda}|x+sn_x-y|}}{|x+sn_x-y|} + \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|}\Big| \leq \frac{c_1c_2|x-y|}{c_3^{-1}|x-y|^2} = \frac{C}{|x-y|},$$

with the constant $C = c_1c_2c_3$ independent of s. The function $\frac{C}{|x-y|}$ is integrable over the two-dimensional manifold $\partial\Omega$, hence it is a majorant for the integrand on the left hand side of (7.66). ∎

Proof of Theorem 7.18: The proof of (i) is standard and we omit it. In the proof of (ii) we restrict ourselves to the verification of the formula for $\frac{\partial\omega}{\partial n}(x-)$. The other formula is proved in the same way.

From Lemma 7.19 and from the jump relation (7.58) we conclude that

$$\begin{aligned}
\frac{\partial \omega}{\partial n}(x-) &= \lim_{\substack{s\to 0\\ s<0}} \frac{1}{2\pi}\int_{\partial\Omega} \frac{\partial}{\partial s}\frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|} v(y)dS_y \\
&= \lim_{\substack{s\to 0\\ s<0}} \frac{1}{2\pi}\int_{\partial\Omega} \Big(\frac{\partial}{\partial s}\frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|} + \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|}\Big) v(y)dS_y \\
&\qquad - \lim_{\substack{s\to 0\\ s<0}} \frac{1}{2\pi}\int_{\partial\Omega} \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x_s-y|}}{|x_s-y|} v(y)dS_y \\
&= \frac{1}{2\pi}\int_{\partial\Omega} \Big(\frac{\partial}{\partial n_x}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} + \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}\Big) v(y)dS_y \\
&\qquad + v(x) - \frac{1}{2\pi}\int_{\partial\Omega} \frac{\partial}{\partial n_y}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y)dS_y \\
&= v(x) + \frac{1}{2\pi}\int_{\partial\Omega} \frac{\partial}{\partial n_x}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(y)dS_y.
\end{aligned}$$

■

7.6 Solution of the Neumann problem

Here we show that the integral equations $(I + K')v = f$ and $(-I + K')v = f$ can be solved, which implies that the interior and exterior Neumann boundary value problems have solutions. The proof of the invertibility is based on Corollary 7.7. To this end I show first that K' is a compact operator on the Banach space X.

Definition 7.20 *Let $\Gamma \subseteq \mathbb{R}^n$. A sequence $\{v_m\}_{m\in\mathbb{N}}$ of functions $v_m : \Gamma \to \mathbb{C}$ is called uniformly equicontinuous, if to every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$|v_m(x) - v_m(y)| < \varepsilon$$

for all $m \in \mathbb{N}$ and all $x, y \in \Gamma$ with $|x - y| < \delta$.

Theorem 7.21 (Arzela-Ascoli) *Let $\{v_m\}_{m\in\mathbb{N}}$ be a bounded, uniformly equicontinuous sequence of functions on a compact set $\Gamma \subseteq \mathbb{R}^n$. Then there is a uniformly convergent subsequence $\{v_{m_\ell}\}_{\ell\in\mathbb{N}}$.*
(Cesare Arzelà (1847–1912), Giulio Ascoli (1843–1896))

Corollary 7.22 *The operator $K' : X \to X$ is compact.*

Proof: Let $\{v_m\}_{m\in\mathbb{N}}$ be a bounded sequence in X. By Theorem 7.17(ii) the operator $K' : X \to X$ is bounded, which implies that also the sequence $\{K'v_m\}_{m\in\mathbb{N}}$ is bounded in X. Let $\varepsilon > 0$, set $C = \sup_{m\in\mathbb{N}} \|v_m\|_\infty + 1$ and let $C_{K'} > 0$ be the constant from Theorem 7.17(i). This theorem implies for all $x^{(1)}, x^{(2)} \in \partial\Omega$ with

$$|x^{(1)} - x^{(2)}| < \delta = \Big(\frac{\varepsilon}{CC_{K'}}\Big)^4$$

and for all $m \in \mathbb{N}$ that

$$|(K'v_m)(x^{(1)}) - (K'v_m)(x^{(2)})| \le C_{K'}|x^{(1)} - x^{(2)}|^{1/4}\, \|v_m\|_\infty \le C_{K'}C\Big(\frac{\varepsilon}{CC_{K'}}\Big) = \varepsilon\,.$$

Thus, $\{K'v_m\}_{m\in\mathbb{N}}$ is a bounded, uniformly equicontinuous sequence of functions on the compact set $\partial\Omega$. From Theorem 7.21 we thus conclude that this sequenc has a subsequence converging with respect to the norm $\|\cdot\|_\infty$ of $C(\partial\Omega, \mathbb{C})$, which means that the subsequence converges in X. Hence, by Definition 7.3, the operator K' is compact. ∎

Lemma 7.23 *Let* $\lambda \in \mathbb{C} \setminus [0, \infty)$*. Then* 1 *and* -1 *are not eigenvalues of* K'*.*

Proof: We prove first that -1 is not an eigenvalue. To this end it suffices to show if $v \in C(\partial\Omega, \mathbb{C})$ satisfies

$$(I + K')v = 0, \tag{7.69}$$

then $v = 0$, since this implies that the kernel of $K' - (-1)I$ is equal to $\{0\}$. To verify that v vanishes, we define for $x \in \mathbb{R}^3$ the single layer potential

$$u(x) = \frac{1}{2\pi}\int_{\partial\Omega} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}\, v(y) dS_y\,. \tag{7.70}$$

Equation (7.69) and the jump relations from Theorem 7.18(ii) imply that u solves the boundary value problem

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, \quad x \in \Omega, \\ \frac{\partial}{\partial n} u(x) &= 0, \quad x \in \partial\Omega. \end{aligned}$$

The first Green's formula yields

$$\begin{aligned} 0 &= \int_{\partial\Omega} \frac{\partial}{\partial n} u(x)\, \overline{u(x)}\, dS = \int_\Omega \Delta u(x)\, \overline{u(x)} + \nabla u(x) \cdot \overline{\nabla u(x)}\, dS \\ &= \int_\Omega (-\lambda |u(x)|^2 + |\nabla u(x)|^2) dx \\ &= -i\,\mathrm{Im}\,\lambda \int_\Omega |u(x)|^2 dx + \int_\Omega |\nabla u(x)|^2 - \mathrm{Re}\,\lambda |u(x)|^2 dx\,. \end{aligned}$$

Since $\operatorname{Im}\lambda \neq 0$ or $\operatorname{Re}\lambda < 0$ it follows from this equation that $u \equiv 0$ in Ω. Since the single layer potential u is continuous on $\mathbb{R}^3$ it thus follows that u is also a solution of the boundary value problem

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, \quad x \in \mathbb{R}^3 \backslash \overline{\Omega} \\ u(x) &= 0, \quad x \in \partial(\mathbb{R}^3 \backslash \overline{\Omega}). \end{aligned}$$

To apply the Green's formula in $\mathbb{R}^3 \setminus \overline{\Omega}$ note that for $\lambda \in \mathbb{C}\backslash[0,\infty)$ we have by our choice of the square root that $\operatorname{Re} i\sqrt{\lambda} < 0$, whence

$$|u(x)| \;\leq\; \frac{1}{2\pi}\int_{\partial\Omega} \frac{e^{\operatorname{Re} i\sqrt{\lambda}\,|x-y|}}{|x-y|}\,|v(y)|dS_y \leq \frac{e^{\operatorname{Re} i\sqrt{\lambda}\,\operatorname{dist}(x,\partial\Omega)}}{\operatorname{dist}(x,\partial\Omega)}\,\frac{1}{2\pi}\int_{\partial\Omega} |v(y)|dS_y\,.$$

Therefore $|u(x)|$ decreases exponentially for $|x| \to \infty$. Because

$$\nabla u(x) = \frac{1}{2\pi}\int_{\partial\Omega} \nabla_x \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}\, v(y)dS_y\,,$$

it follows in the same way that also $|\nabla u(x)|$ decreases exponentially for $|x| \to \infty$. Therefore the first Green's formula yields

$$\begin{aligned} &\int_{\mathbb{R}^3\backslash\overline{\Omega}} (-\lambda|u(x)|^2 + |\nabla u(x)|^2)dx \\ &= \lim_{R\to\infty} \int_{\substack{\mathbb{R}^3\backslash\overline{\Omega} \\ |x|<R}} \big(-\lambda|u(x)|^2 + |\nabla u(x)|^2\big)dx \\ &= \lim_{R\to\infty} \Big(\int_{\partial\Omega} \frac{\partial}{\partial n}\, u(x)\,\overline{u(x)}dS + \int_{|x|=R} \frac{\partial}{\partial n} u(x)\,\overline{u(x)}\,dS \Big) = 0\,. \end{aligned}$$

As above it follows from this equation that $u = 0$ in $\mathbb{R}^3\backslash\overline{\Omega}$, whence $u \equiv 0$ in $\mathbb{R}^3$. Again using the jump relations from Theorem 7.18, we now conclude for all $x \in \partial\Omega$ that

$$\begin{aligned} 0 &= \frac{\partial u}{\partial n_x}(x+) - \frac{\partial u}{\partial n_x}(x-) \\ &= -v(x) + \frac{1}{2\pi}\int_{\partial\Omega} \frac{\partial}{\partial n_x}\,\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}\, v(y)dS_y \\ &\quad -v(x) - \frac{1}{2\pi}\int_{\partial\Omega} \frac{\partial}{\partial n_x}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}\, v(y)dS_y = -2v(x)\,, \end{aligned}$$

hence $v = 0$. Therefore -1 is not an eigenvalue.

To prove that 1 is not an eigenvalue we assume that $v \in C(\partial\Omega, \mathbb{C})$ satisfies

$$(-I + K')v = 0.$$

We insert v into (7.70). The jump relations for single layer potentials then imply that u solves the boundary value problem

$$\begin{aligned}\Delta u(x)+\lambda u(x) &= 0, \quad x\in\mathbb{R}^3\backslash\overline{\Omega},\\ \frac{\partial}{\partial n}u(x) &= 0, \quad x\in\partial(\mathbb{R}^3\backslash\overline{\Omega}).\end{aligned}$$

Proceeding as above we conclude from this that $u=0$ in $\mathbb{R}^3\backslash\overline{\Omega}$ and in Ω, from which we infer by the jump relations that $v=0$. Therefore 1 is not an eigenvalue. ∎

Corollary 7.24 *Let $\Omega\subseteq\mathbb{R}^3$ be a bounded open set with $\partial\Omega\in C_2$. Suppose that $\lambda\in\mathbb{C}\backslash[0,\infty)$. Then the interior Neumann boundary value problem*

$$\begin{aligned}\Delta u(x)+\lambda u(x) &= 0, \quad x\in\Omega,\\ \frac{\partial}{\partial n}u(x-) &= f(x), \quad x\in\partial\Omega\end{aligned}$$

and the exterior Neumann boundary value problem

$$\begin{aligned}\Delta u(x)+\lambda u(x) &= 0, \quad x\in\mathbb{R}^3\backslash\overline{\Omega},\\ \frac{\partial}{\partial n}u(x+) &= f(x), \quad x\in\partial(\mathbb{R}^3\backslash\overline{\Omega})\\ |u(x)|,|\nabla u(x)| &= O(e^{\mathrm{Re}\, i\sqrt{\lambda}|x|}), \quad |x|\to\infty\end{aligned}\tag{7.71}$$

have unique solutions for all $f\in C(\partial\Omega,\mathbb{C})=C(\partial(\mathbb{R}^3\backslash\overline{\Omega}),\mathbb{C})$. The solutions are given by the single layer potentials

$$u(x)=\frac{1}{2\pi}\int_{\partial\Omega}\frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}v(y)\,dS_y, \quad x\in\Omega,\tag{7.72}$$

where v satisfies the integral equation $(I+K')v=f$ for the interior problem and $(-I+K')v=f$ for the exterior problem.

Proof: Since by Corollary 7.22 the operator K' is compact and since by Lemma 7.23 the number -1 is not an eigenvalue of this operator, it follows from Corollary 7.7 that the mapping $I+K'=-(-1)+K':X\to X$ is invertible. Consequently, the boundary integral equation $(I+K')v=f$ has a unique solution $v\in C(\partial\Omega,\mathbb{C})$. With this v as boundary layer the single layer potential u from (7.72) is a solution of the interior Neumann boundary value problem. To prove that the solution is unique let $\hat{u}$ be another solution of the same problem. Then $w=u-\hat{u}$ satisfies

$$\begin{aligned}\Delta w(x)+\lambda w(x) &= 0, \quad x\in\Omega\\ \frac{\partial}{\partial n}w(x) &= 0, \quad x\in\partial\Omega,\end{aligned}$$

hence the first Green's formula yields

$$\begin{aligned} 0 &= \int_{\partial\Omega} \frac{\partial}{\partial n} w(x)\overline{w}(x)\, dS = \int_\Omega \Delta w(x)\overline{w(x)} + |\nabla w(x)|^2\, dx \\ &= \int_\Omega \big(-\lambda |w(x)|^2 + |\nabla w(x)|^2 \big)\, dx \\ &= -i \operatorname{Im}\lambda \int_\Omega |w(x)|^2\, dx + \int_\Omega |\nabla w(x)|^2 - \operatorname{Re}\lambda |w(x)|^2\, dx. \end{aligned}$$

This implies $w = 0$, hence $u = \hat{u}$. Therefore the solution is unique.

A solution u of the exterior Neumann boundary value problem is obtained if we insert the unique solution v of the boundary integral equation $(-I + K')v = f$ into (7.72). As in the proof of Lemma 7.23 we see that u defined in this way satisfies the radiation condition (7.71). To prove uniqueness of the solution suppose that $\hat{u}$ is a second solution. We apply the first Green's formula to $w = u - \hat{u}$ in the exterior domain $\mathbb{R}^3 \backslash \overline{\Omega}$ as in the proof of Lemma 7.23, noting that $w(x)$ decreases exponentially for $|x| \to \infty$, because both u and $\hat{u}$ satisfy the radiation condition. As above, we conclude that $w = 0$. ■

7.7 Solution of the Dirichlet problem

To solve the interior and exterior Dirichlet problems we must show that the boundary integral equations $(-I + K)v = f$ and $(I + K)v = f$ are solvable.

Theorem 7.25 *The operator $K : X \to X$ is compact.*

This theorem is proved in the same way as Corollary 7.22 using Theorem 7.15 instead of Theorem 7.17.

Lemma 7.26 *Let $\lambda \in \mathbb{C} \setminus [0, \infty)$. Then 1 and -1 are not eigenvalues of K.*

Proof: For $u, v \in C(\partial\Omega, \mathbb{C})$ we write

$$\langle u, v \rangle_{\partial\Omega} = \int_{\partial\Omega} u(x)v(x) dS_x.$$

By interchanging the order of integration we obtain

$$\begin{aligned} \langle K'u, v \rangle_{\partial\Omega} = \int_{\partial\Omega} \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_x} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} u(y) dS_y\, v(x) dS_x \\ = \int_{\partial\Omega} u(y)\, \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_x} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|} v(x) dS_x\, dS_y = \langle u, Kv \rangle_{\partial\Omega}\,. \quad (7.73) \end{aligned}$$

Now let $\mu = 1$ or $\mu = -1$ and assume that $v \in C(\partial\Omega, \mathbb{C})$ satisfies

$$(-\mu I + K)v = 0.$$

From (7.73) we conclude for all $u \in C(\partial\Omega, \mathbb{C})$ that

$$\langle(-\mu I + K')u, v\rangle_{\partial\Omega} = \langle u, (-\mu I + K)v\rangle_{\partial\Omega} = 0.$$

Since by Lemma 7.23 neither 1 nor -1 is an eigenvalue of the compact operator K', it follows from Corollary 7.7 that the mapping $(-\mu I + K') : C(\partial\Omega, \mathbb{C}) \to C(\partial\Omega, \mathbb{C})$ is surjective. Therefore there is $u \in C(\partial\Omega, \mathbb{C})$ such that $(-\mu I + K')u = \overline{v}$, where $\overline{v} \in C(\partial\Omega, \mathbb{C})$ denotes the complex conjugate function of v. Thus,

$$\int_{\partial\Omega} |v(x)|^2 dS_x = \langle \overline{v}, v\rangle_{\partial\Omega} = \langle(-\mu I + K')u, v\rangle_{\partial\Omega} = 0.$$

Consequently v must be equal to zero and cannot be an eigenfunction. This implies that μ is not an eigenvalue of K. ∎

Corollary 7.27 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded open set with $\partial\Omega \in C_2$. Suppose that $\lambda \in \mathbb{C}\backslash[0, \infty)$. Then the interior Dirichlet boundary value problem*

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, \quad x \in \Omega \\ u(x) &= f(x), \quad x \in \partial\Omega, \end{aligned}$$

and the exterior Dirichlet boundary value problem

$$\begin{aligned} \Delta u(x) + \lambda u(x) &= 0, \quad x \in \mathbb{R}^3\backslash\Omega \\ u(x) &= f(x), \quad x \in \partial(\mathbb{R}^3\backslash\overline{\Omega}), \\ |u(x)|, |\nabla u(x)| &= O(e^{\mathrm{Re}\, i\sqrt{\lambda}|x|}), \quad |x| \to \infty \end{aligned}$$

have unique solutions for all $f \in C(\partial\Omega, \mathbb{C})$. The solutions are given by the double layer potentials

$$u(x) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{e^{i\sqrt{\lambda}|x-y|}}{|x-y|}\, v(y)\, dS_y,$$

where v satisfies the integral equation $(-I + K)v = f$ for the interior problem and $(I + K)v = f$ for the exterior problem.

This corollary is proved as the corresponding result for the Neumann problem.

8 Hilbert space methods

8.1 Elliptic differential operators, weak solutions

Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let

$$Lu(x) = \sum_{\substack{|\alpha|\leq 1\\ |\beta|\leq 1}} D^\alpha\big(a_{\alpha\beta}(x)\, D^\beta u(x)\big), \quad x \in \Omega$$

be a linear differential operator of second order with coefficient functions

$$\alpha_{\alpha\beta} : \Omega \to \mathbb{C}, \quad \alpha, \beta \in \mathbb{N}_0^n,\ |\alpha|, |\beta| \leq 1.$$

The sum

$$\sum_{|\alpha+\beta|=2} a_{\alpha\beta}(x) D^{\alpha+\beta} u(x)$$

is called principle part of this operator.

Definition 8.1 (i) The operator L is called elliptic if for all $\xi \in \mathbb{R}^n$, $\xi \neq 0$ and all $x \in \Omega$

$$\sum_{|\alpha+\beta|=2} a_{\alpha\beta}(x)\xi^{\alpha+\beta} \neq 0.$$

(ii) L is called strongly elliptic if to every $x \in \Omega$ there is $\eta > 0$ such that for all $\xi \in \mathbb{R}^n$

$$\mathrm{Re}\Big(\sum_{|\alpha+\beta|=2} a_{\alpha\beta}(x)\xi^{\alpha+\beta}\Big) \geq \eta|\xi|^2.$$

η is the ellipticity constant.
(iii) L is uniformly strongly elliptic if L is strongly elliptic with an ellipticity constant which can be chosen independent of $x \in \Omega$.

Example: Choose

$$a_{\alpha\beta}(x) = \begin{cases} 1, & \text{if } \alpha = \beta,\ |\alpha| = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\sum_{\substack{|\alpha|\leq 1\\ |\beta|\leq 1}} D^\alpha\big(a_{\alpha\beta}(x) D^\beta u(x)\big) = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} u(x) = \Delta u(x).$$

For this operator we have

$$\sum_{|\alpha|=|\beta|=1} a_{\alpha\beta}(x)\xi^{\alpha+\beta} = \sum_{i=1}^n \xi_i^2 = |\xi|^2,$$

consequently Δ is uniformly strongly elliptic with ellipticity constant $\eta = 1$.

In the following I assume that $a_{\alpha\beta} : \Omega \to \mathbb{R}$ is measurable and bounded for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ with $|\alpha|, |\beta| \leq 1$. The operator

$$Lu = \sum_{\substack{|\alpha|\leq 1\\|\beta|\leq 1}} D^\alpha (a_{\alpha\beta} D^\beta u)$$

is in divergence form. For such operators the Definition 3.32 of weak solutions for the Helmholtz equation can be generalized immediately:

Definition 8.2 Let $f \in L^2(\Omega, \mathbb{C})$ and $\lambda \in \mathbb{C}$.
(i) The function $u \in H_1(\Omega, \mathbb{C})$ is called weak solution of the partial differential equation

$$\sum_{\substack{|\alpha|\leq 1\\|\beta|\leq 1}} D^\alpha \big(a_{\alpha\beta}(x) D^\beta u(x)\big) + \lambda u(x) = f(x)$$

in Ω, if for all $\varphi \in \overset{\circ}{C}_\infty(\Omega, \mathbb{C})$ the equation

$$\sum_{\substack{|\alpha|\leq 1\\|\beta|\leq 1}} (-1)^{|\alpha|} \int_\Omega a_{\alpha\beta}(x) D^\beta u(x)\, D^\alpha\, \overline{\varphi(x)}\, dx + \lambda \int_\Omega u(x) \overline{\varphi(x)}\, dx = \int_\Omega f(x)\, \overline{\varphi(x)} dx$$

holds.

(ii) Let $g \in H_1(\Omega, \mathbb{C})$. Then $u \in H_1(\Omega, \mathbb{C})$ is called weak solution of the Dirichlet boundary value problem

$$\sum_{\substack{|\alpha|\leq 1\\|\beta|\leq 1}} D^\alpha \big(a_{\alpha\beta}(x)\, D^\beta u(x)\big) + \lambda u(x) = f(x), \quad x \in \Omega$$

$$u\big|_{\partial\Omega} = g\big|_{\partial\Omega},$$

if u is a weak solution of the partial differential equation and if $u - g \in \overset{\circ}{H}_1(\Omega, \mathbb{C})$.

In the following I write for $u, v \in H_1(\Omega, \mathbb{C})$

$$B(u, v) = \int_\Omega \sum_{\substack{|\alpha|\leq 1\\|\beta|\leq 1}} (-1)^{|\alpha|} a_{\alpha\beta}(x) D^\beta u(x)\, D^\alpha \overline{v(x)}\, dx. \tag{8.1}$$

With this definition it follows that $u \in H_1(\Omega, \mathbb{C})$ is a weak solution of

$$\sum_{\substack{|\alpha|\leq 1\\|\beta|\leq 1}} D^\alpha (a_{\alpha\beta} D^\beta u) + \lambda u = f$$

if and only if

$$B(u,\varphi) + \lambda(u,\varphi)_\Omega = (f,\varphi)_\Omega \tag{8.2}$$

for all $\varphi \in \overset{\circ}{C}_\infty(\Omega)$.

For the Laplace operator $L = \Delta$ we have

$$B(u,v) = -(\nabla u, \nabla v)_\Omega.$$

Insertion of this expression into (8.2) shows that for the Helmholtz equation Definition 3.32 of weak solutions coincides with Definition 8.2.

8.2 Coercivity of sesquilinear forms to elliptic operators

Definition 8.3 Let X be a vector space over $\mathbb{C}$ with norm $\|u\|$, and let $(u,v) \mapsto [u,v] : X \times X \to \mathbb{C}$ be a mapping. This mapping is called

1. a sesquilinear form, if

$$[\lambda u + \mu v, w] = \lambda[u,w] + \mu[v,w], \qquad [u, \lambda v + \mu w] = \overline{\lambda}[u,v] + \overline{\mu}[u,w],$$

2. symmetric, if $[u,v] = \overline{[v,u]}$,

3. bounded, if $|[u,v]| \leq K\|u\|\,\|v\|$,

4. strictly coercive, if there is $c > 0$ such that $[u,u] \geq c\|u\|^2$ for all $u \in X$.

The simplest example of a symmetric, bounded, strictly coercive sesquilinear form is the scalar product (u,v) on a Hilbert space. The mapping $(u,v) \mapsto B(u,v) : H_1(\Omega,\mathbb{C}) \times H_1(\Omega,\mathbb{C}) \to \mathbb{C}$ defined in (8.1) is linear in the first argument and antilinear in the second argument, hence B and of course also $-B$ are sesquilinear forms. In this section we study the coercivity of $-B$, which is a slightly weaker property than strict coercivity. In the formulation of the respective result we write for $u \in H_1(\Omega,\mathbb{C})$

$$|u|_{1,\Omega} = (\nabla u, \nabla u)_\Omega^{1/2}.$$

With this notation one has

$$\|u\|_{1,\Omega}^2 = \|u\|_\Omega^2 + |u|_{1,\Omega}^2\,.$$

Theorem 8.4 *Let $a_{\alpha\beta} : \Omega \to \mathbb{C}$ be bounded measurable with*

$$\begin{aligned} a_{\alpha\beta}(x) &= (-1)^{|\alpha+\beta|}\,\overline{a_{\beta\alpha}(x)}\,, && \text{if } |\alpha| + |\beta| \leq 1, \\ a_{\alpha\beta}(x) &= a_{\beta\alpha}(x) \in \mathbb{R}\,, && \text{if } |\alpha| = |\beta| = 1, \end{aligned}$$

and assume that

$$Lu = \sum_{|\alpha|,|\beta|\le 1} D^\alpha(a_{\alpha\beta}D^\beta u)$$

is uniformly strongly elliptic with ellipticity constant $\eta > 0$*. Then* $\hat{B}(u,v) = -B(u,v)$ *is a symmetric and bounded sesquilinear form on* $H_1(\Omega,\mathbb{C})$*, which satisfies*

$$\hat{B}(u,u) \ge c_1|u|_{1,\Omega}^2 - c_2\|u\|_\Omega^2, \quad \textit{for all } u \in H_1(\Omega,\mathbb{C}), \tag{8.3}$$

where

$$c_1 = \frac{\eta}{2}, \quad c_2 = \frac{K^2}{2\eta} + K, \quad K = \sum_{|\alpha+\beta|\le 1} \|a_{\alpha\beta}\|_\infty \,.$$

Definition 8.5 A sesquilinear form $\hat{B}$ satisfying (8.3) with suitable constants $c_1 > 0$ and $c_2 \ge 0$ is called coercive on $H_1(\Omega,\mathbb{C})$.

Remark 8.6 Since $a_{\alpha\beta} \in \mathbb{R}$ for $|\alpha| = |\beta| = 1$ the condition of strong ellipticity is

$$\sum_{|\alpha|=|\beta|=1} a_{\alpha\beta}\xi^{\alpha+\beta} \ge \eta|\xi|^2 \,.$$

Proof of the theorem: For $u, v \in H_1(\Omega,\mathbb{C})$ we have

$$\begin{aligned} B(u,v) &= \sum_{|\alpha|,|\beta|\le 1} (-1)^{|\alpha|}(a_{\alpha\beta}D^\beta u, D^\alpha v) \\ &= \sum_{|\alpha|,|\beta|\le 1} (-1)^{|\alpha|+|\alpha|+|\beta|}(\,\overline{a_{\beta\alpha}}\, D^\beta u, D^\alpha v) \\ &= \sum_{|\alpha|,|\beta|\le 1} (-1)^{|\beta|}(D^\beta u, a_{\beta\alpha}D^\alpha v) = \overline{B(v,u)}\,. \end{aligned}$$

Thus, B is symmetric. Also we have

$$|B(u,v)| \le \sum_{|\alpha|,|\beta|\le 1} \|a_{\alpha\beta}\|_\infty \,|(D^\beta u, D^\alpha v)| \le C\|u\|_{1,\Omega}\,\|v\|_{1,\Omega}\,.$$

Thus B is bounded.

To see that $-B$ is coercive define

$$B'(u,v) = -\sum_{|\alpha|=|\beta|=1} (a_{\alpha\beta}D^\beta u, D^\alpha v)\,.$$

The above calculation shows that B' is symmetric. Since $a_{\alpha\beta}(x) \in \mathbb{R}$ for $|\alpha| = |\beta| = 1$ we thus obtain for real valued functions u, v that

$$B'(u,v) = \overline{B'(v,u)} = B'(v,u).$$

Thus, if $u \in H_1(\Omega, \mathbb{C})$ and $u_1 = \operatorname{Re} u$, $u_2 = \operatorname{Im} u$, it follows

$$\begin{aligned}
-B'(u,u) &= -B'(u_1 + iu_2, u_1 + iu_2) = -B'(u_1,u_1) - B'(u_2,u_2) \\
&\quad -iB'(u_2,u_1) + iB'(u_1,u_2) = -B'(u_1,u_1) - B'(u_2,u_2) \\
&= \int_\Omega \sum_{\substack{|\alpha|=1\\|\beta|=1}} a_{\alpha\beta}(x)\big(D^\beta u_1(x) D^\alpha u_1(x) + D^\beta u_2(x) D^\alpha u_2(x)\big) dx \\
&= \int_\Omega \Big(\sum_{\substack{|\alpha|=1\\|\beta|=1}} a_{\alpha\beta}(x)\big(\nabla u_1(x)\big)^{\alpha+\beta} + \sum_{\substack{|\alpha|=1\\|\beta|=1}} a_{\alpha\beta}(x)\big(\nabla u_2(x)\big)^{\alpha+\beta}\Big) dx \\
&\geq \int_\Omega \eta|\nabla u_1(x)|^2 + \eta|\nabla u_2(x)|^2 dx = \eta|u|^2_{1,\Omega}\,.
\end{aligned}$$

This together with $ab \leq \frac{\varepsilon}{2}\,a^2 + \frac{1}{2\varepsilon}\,b^2$ yields

$$\begin{aligned}
-B(u,u) &= -B'(u,u) - \sum_{|\alpha+\beta|=1} (-1)^{|\alpha|}(a_{\alpha\beta}D^\beta\, u, D^\alpha\, u) - (a_{00}u,u) \\
&\geq \eta|u|^2_{1,\Omega} - \sum_{|\alpha+\beta|=1} \|a_{\alpha\beta}\|_\infty |u|_{1,\Omega}\|u\|_{0,\Omega} - \|a_{00}\|_\infty \|u\|^2_{0,\Omega} \\
&\geq \eta|u|^2_{1,\Omega} - \frac{\varepsilon}{2}|u|^2_{1,\Omega} - \frac{1}{2\varepsilon}K^2\|u\|^2_{0,\Omega} - K\|u\|^2_{0,\Omega}.
\end{aligned}$$

Choosing $\varepsilon = \eta$ shows that $\hat{B} = -B$ is coercive with the constant c_2 given in the theorem. The proof is complete.

8.3 Existence of weak solutions to elliptic equations

The coercivity of the sesquilinear form $-B$ allows to prove that boundary value problems to elliptic operators have weak solutions. To show this we reformulate Definition 8.2 of weak solutions slightly.

Assume that the operator $L = \sum_{\substack{|\alpha|\leq 1\\|\beta|\leq 1}} D^\alpha(a_{\alpha\beta}D^\beta)$ satisfies the assumption of Theorem 8.4, let $\lambda \in \mathbb{R}$ and let $f \in L^2(\Omega, \mathbb{C})$. By Definition 8.2 b.) the function $u \in \overset{\circ}{H}_1(\Omega)$ is a weak solution of the homogeneous Dirichlet boundary value problem

$$Lu + \lambda\, u = f \quad \text{in } \Omega, \tag{8.4}$$

$$u\big|_{\partial\Omega} = 0, \tag{8.5}$$

if for all $v \in \overset{\circ}{C}_\infty(\Omega, \mathbb{C})$

$$B(u,v) + \lambda(u,v)_\Omega = (f,v)_\Omega \tag{8.6}$$

holds. The sesquilinear form B is bounded on $\overset{\circ}{H}_1(\Omega, \mathbb{C})$, hence it is continuous in both arguments. Therefore, since $\overset{\circ}{C}_\infty(\Omega, \mathbb{C})$ is dense in $\overset{\circ}{H}_1(\Omega, \mathbb{C})$, equation (8.6) holds for all $v \in \overset{\circ}{C}_\infty(\Omega, \mathbb{C})$ if and only if it holds for all $v \in \overset{\circ}{H}_1(\Omega, \mathbb{C})$. Using that λ is real, we conclude that $u \in \overset{\circ}{H}_1(\Omega, \mathbb{C})$ is a weak solution of the homogeneous Dirichlet boundary value problem (8.4), (8.5) if and only if

$$B(v, u) + \lambda(v, u)_\Omega = (v, f)_\Omega \tag{8.7}$$

holds for all $v \in \overset{\circ}{H}_1(\Omega, \mathbb{C})$. We have thus reduced the problem of the existence of weak solutions to an abstract problem for symmetric sesquilinear forms B on the Hilbert space $\overset{\circ}{H}_1(\Omega)$. Accordingly, the existence proof is based on the coercivity of B and on the following easy result:

Lemma 8.7 *Let $[u, v]$ be a symmetric, bounded, strictly coercive sequilinear form on a Banach space X over $\mathbb{C}$. Then $[u, v]$ is a scalar product on X. The associated norm $|\!|\!| u |\!|\!| = [u, u]^{1/2}$ is equivalent to the norm $\|u\|$. The space X is complete with respect to the norm $|\!|\!| u |\!|\!|$, whence X is a Hilbert space with the scalar product $[u, v]$.*

Proof: Obviously every symmetric, strictly coercive sesquilinear form is a scaler product. From the boundedness and the strict coercivity we obtain

$$c\|u\|^2 \leq [u, u] = |\!|\!| u |\!|\!|^2 \leq K\|u\|^2, \tag{8.8}$$

which means that $\|\cdot\|$ and $|\!|\!| \cdot |\!|\!|$ are equivalent norms. If $\{u_n\}_{n=1}^\infty$ is a Cauchy sequence with respect to the norm $|\!|\!|\cdot|\!|\!|$, then (8.8) implies that $\{u_n\}_{n=1}^\infty$ is also a Cauchy sequence with respect to the norm $\|\cdot\|$. Since X is complete with respect to this norm, there is a limit element $u \in X$ of this Cauchy sequence. From (8.8) we obtain

$$\lim_{n\to\infty} |\!|\!| u - u_n |\!|\!| \leq \lim_{n\to\infty} K^{\frac{1}{2}} \|u - u_n\| = 0,$$

hence u is also the limit of $\{u_n\}_{n=1}^\infty$ with respect to the norm $|\!|\!|\cdot|\!|\!|$. Therefore X is complete with respect to this norm.

Corollary 8.8 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, let*

$$L = \sum_{|\alpha|,|\beta|\leq 1} D^\alpha(a_{\alpha\beta}D^\beta)$$

satisfy the assumptions of Theorem 8.4, and let $\lambda < -c_2$ with

$$c_2 = \frac{K^2}{2\eta} + K, \quad K = \sum_{|\alpha+\beta|\leq 1} \|a_{\alpha\beta}\|_\infty,$$

where $\eta > 0$ is the ellipticity constant of L. Then the homogeneous Dirichlet boundary value problem (8.4), (8.5) *has a unique weak solution $u \in \overset{\circ}{H}_1(\Omega, \mathbb{C})$ for all $f \in L^2(\Omega, \mathbb{C})$. This solution satisfies*

$$\|u\|_{1,\Omega} \le \max\big(\frac{2}{\eta}, \frac{1}{-\lambda - c_2}\big)\|f\|_\Omega.$$

Proof: Define the sesquilinear form $[u, v]$ on $\overset{\circ}{H}_1(\Omega, \mathbb{C}) \times \overset{\circ}{H}_1(\Omega, \mathbb{C})$ by

$$[u, v] = -B(u, v) - \lambda(u, v)_\Omega. \tag{8.9}$$

Theorem 8.4 implies that this sesquilinear form is symmetric, bounded and satisfies for $u \in \overset{\circ}{H}_1(\Omega, \mathbb{C})$

$$[u, u] = -B(u, u) - \lambda(u, u)_\Omega \ge \frac{\eta}{2}\,|u|^2_{1,\Omega} - (c_2 + \lambda)\|u\|^2_\Omega \ge c\,\|u\|^2_{1,\Omega}, \tag{8.10}$$

with $c = \min(\frac{\eta}{2}, -\lambda - c_2) > 0$. Thus, $[u, v]$ is strictly coercive. Consequently, by Lemma 8.7 this sesquilinear form is a scalar product on $\overset{\circ}{H}_1(\Omega, \mathbb{C})$ with norm

$$|\!|\!| u |\!|\!|^2 = [u, u] = -B(u, u) - \lambda(u, u)_\Omega\,.$$

Moreover, the linear form $h : \overset{\circ}{H}_1(\Omega) \to \mathbb{C}$ defined by

$$h(v) = -(v, f)_\Omega$$

is bounded because (8.10) yields

$$|h(v)| \le \|f\|_\Omega\,\|v\|_\Omega \le \|f\|_\Omega\,\|v\|_{1,\Omega} \le \|f\|_\Omega\,c^{-\frac{1}{2}}\,|\!|\!| v |\!|\!|.$$

The Riesz representation theorem (Corollary 3.7) thus implies that there is a unique function $u \in \overset{\circ}{H}_1(\Omega, \mathbb{C})$ satisfying

$$[v, u] = h(v)$$

for all $v \in \overset{\circ}{H}_1(\Omega, \mathbb{C})$. By definition of $[u, v]$ this equation is equivalent to (8.7). Consequently, u is the unique weak solution of the boundary value problem. This solution satisfies

$$c\|u\|^2_{1,\Omega} \le [u, u] = h(u) \le \|f\|_\Omega\,\|u\|_{1,\Omega},$$

hence $\|u\|_{1,\Omega} \le \frac{1}{c}\|f\|_\Omega$. This proves the corollary.

Let $f \in L^2(\Omega)$ and $g \in H_1(\Omega)$. By definition, $u \in H_1(\Omega)$ is a weak solution of the inhomogeneous Dirichlet boundary value problem

$$Lu + \lambda u = f, \tag{8.11}$$

$$u\big|_{\partial\Omega} = g\big|_{\partial\Omega}, \tag{8.12}$$

if $w = u - g \in \overset{\circ}{H}_1(\Omega)$ and

$$B(v, u) + \lambda(v, u)_\Omega = (v, f)_\Omega$$

for all $v \in \overset{\circ}{H}_1(\Omega)$. This implies that w satisfies

$$B(v, w) + \lambda(v, w)_\Omega = (v, f - \lambda g)_\Omega - B(v, g)$$

for all $v \in \overset{\circ}{H}_1(\Omega)$. On the other hand, if $w \in \overset{\circ}{H}_1(\Omega)$ satisfies this equation for all $v \in \overset{\circ}{H}_1(\Omega)$, then $u = w + g$ is a weak solution of the problem (8.11), (8.12).

Corollary 8.9 *Let the assumptions of Corollary 8.8 be satisfied. Then for all $\lambda < -c_2$, all $f \in L^2(\Omega)$ and $g \in H_1(\Omega)$ there is a unique weak solution $u \in H_1(\Omega)$ of the inhomogeneous Dirichlet boundary value problem* (8.11), (8.12).

Proof: Let the linear form $h : \overset{\circ}{H}_1(\Omega) \to \mathbb{C}$ be defined by

$$h(v) = (v, \lambda g - f) + B(v, g).$$

The function $u = g + w$ is a weak solution of the Dirichlet boundary value problem if and only if $w \in \overset{\circ}{H}_1(\Omega)$ satisfies

$$[v, w] = h(v)$$

for all $v \in \overset{\circ}{H}(\Omega)$, where $[v, w]$ is the sesquilinear form defined in (8.9). From the boundedness of B we have

$$\begin{aligned} |h(v)| &\leq \|\lambda g - f\|_\Omega \, \|v\|_\Omega + K\|g\|_{1,\Omega} \, \|v\|_{1,\Omega} \\ &\leq \big(\|\lambda g - f\|_\Omega + K\|g\|_{1,\Omega}\big) \, \|v\|_{1,\Omega} \leq \big(\|\lambda g - f\|_\Omega + K\|g\|_{1,\Omega}\big) \, c^{-\frac{1}{2}} \, |\!|\!| u |\!|\!|, \end{aligned}$$

where in the last step we used (8.10). Therefore h is a bounded linear form on the Hilbert space $\overset{\circ}{H}_1(\Omega)$ equipped with the scalar product $[u, v]$. Consequently, by Corollary 3.7 applied to this Hilbert space there is a unique solution $w \in \overset{\circ}{H}_1(\Omega)$.

Example 8.10 The operator $L = \Delta$ does not have lower order terms, hence $c_2 = 0$. Therefore there is a unique weak solution of

$$\begin{aligned} \Delta u + \lambda u &= f, \\ u\big|_{\partial\Omega} &= g\big|_{\partial\Omega} \end{aligned}$$

for all $\lambda < 0$, $f \in L^2(\Omega)$, $g \in H_1(\Omega)$.

9 Eigenvalue problems, spectral theory

9.1 The Friedrichs' extension of the operator L

Let $L = \sum_{|\alpha|,|\beta|\leq 1} D^\alpha a_{\alpha\beta} D^\beta$ with $a_{\alpha\beta} : \Omega \to \mathbb{C}$ bounded measurable, and let $f \in L^2(\Omega)$. By definition $u \in H_1(\Omega, \mathbb{C})$ is a weak solution of $Lu = f$ if and only if

$$B(u, \varphi) = (f, \varphi)_\Omega$$

for all $\varphi \in \overset{\circ}{C}_\infty(\Omega)$. In this point of view $L = \sum D^\alpha a_{\alpha\beta} D^\beta$ is merely a symbolic expression. Yet, we can attach a precise meaning to L and define it as an operator on the Hilbert space $L^2(\Omega, \mathbb{C})$ as follows: The domain of definition $D(L)$ of L is given by

$$D(L) = \Big\{ u \in H_1(\Omega) \;\Big|\; \underset{f \in L^2(\Omega)}{\exists} \; \underset{\varphi \in \overset{\circ}{C}_\infty(\Omega)}{\forall} : B(u, \varphi) = (f, \varphi)_\Omega \Big\} .$$

It is immediately seen that $D(L)$ is a linear subspace of $H_1(\Omega)$. Note that if $u \in D(L)$ then the function $f \in L^2(\Omega)$ satisfying $B(u, \varphi) = (f, \varphi)_\Omega$ for all $\varphi \in \overset{\circ}{C}_\infty(\Omega)$, which exists by definition, is unique. For, if $g \in L^2(\Omega)$ is a second such function then

$$(f, \varphi)_\Omega = B(u, \varphi) = (g, \varphi)_\Omega ,$$

whence $(f - g, \varphi)_\Omega = 0$ for all $\varphi \in \overset{\circ}{C}_\infty(\Omega)$. Since $\overset{\circ}{C}_\infty(\Omega)$ is dense in $L^2(\Omega)$, this equation implies $f = g$. Therefore for $u \in D(L)$ we can define

$$Lu := f .$$

This defines a linear operator $L : D(L) \subseteq L^2(\Omega) \to L^2(\Omega)$. For this operator the equation $Lu = f$ holds if and only if u is a weak solution of this equation in the above sense.

We obtain an operator L_D "adapted" to the homogeneous Dirichlet problem if we restrict this operator to the set $\overset{\circ}{H}_1(\Omega) \cap D(L)$:

$$L_D = L\big|_{\overset{\circ}{H}_1(\Omega) \cap D(L)} .$$

This operator has the following property: For $f \in L^2(\Omega)$ the equation

$$L_D u = f$$

holds if and only if u is a solution of

$$\begin{aligned} \sum_{\substack{|\alpha|\leq 1 \\ |\beta|\leq 1}} D^\alpha (a_{\alpha\beta} D^\beta u) &= f, \\ u\big|_{\partial\Omega} &= 0 \end{aligned}$$

in the weak sense.

If in particular $a_{\alpha\beta} \in C_1(\Omega)$ for all $|\alpha|, |\beta| \leq 1$, then for $u \in \overset{\circ}{C}_\infty(\Omega)$ the expression $\sum_{|\alpha|,|\beta|\leq 1} D^\alpha(a_{\alpha\beta}D^\beta u)$ can be computed in the classical sense. By partial integration it thus follows for $u, \varphi \in \overset{\circ}{C}_\infty(\Omega)$ that

$$B(u,\varphi) = \Big(\sum_{|\alpha|,|\beta|\leq 1} D^\alpha(a_{\alpha\beta}D^\beta u), \varphi \Big)_\Omega .$$

Since $f = \Sigma D^\alpha(a_{\alpha\beta}D^\beta u) \in L^2(\Omega)$ and since $\overset{\circ}{C}_\infty(\Omega) \subseteq \overset{\circ}{H}_1(\Omega) \subseteq H_1(\Omega)$, it follows by definition of L and L_D that

$$Lu = L_D u = \sum_{|\alpha|,|\beta|\leq 1} D^\alpha(a_{\alpha\beta}D^\beta)u$$

for $u \in \overset{\circ}{C}_\infty(\Omega)$. Consequently, on $\overset{\circ}{C}_\infty(\Omega)$ the operators L and L_D coincide with the classical differential operator $\sum_{|\alpha|,|\beta|\leq 1} D^\alpha(a_{\alpha\beta}D^\beta)$, both are extensions of this operator. L_D is called the Friedrichs' extension of $\sum_{|\alpha|,|\beta|\leq 1} D^\alpha(a_{\alpha\beta}D^\beta)$ on $\overset{\circ}{C}_\infty(\Omega)$.

Corollary 8.8 implies that for $\lambda < -c_2$ and $f \in L^2(\Omega)$ there is a unique solution u of

$$-L_D u - \lambda u = f ,$$

and this solution satisfies

$$\|u\|_\Omega \leq \|u\|_{1,\Omega} \leq \max\Big(\frac{2}{\eta}, \frac{1}{-\lambda - c_2}\Big)\|f\|_\Omega .$$

This means that the inverse operator

$$(-L_D - \lambda)^{-1} : L^2(\Omega) \to D(L) \subseteq L^2(\Omega)$$

exists, and that this operator satisfies for all $f \in L^2(\Omega)$

$$\|u\|_\Omega = \|(-L_D - \lambda)^{-1} f\|_\Omega \leq \max\Big(\frac{2}{\eta}, \frac{1}{-\lambda - c_2}\Big)\|f\|_\Omega .$$

Therefore this operator is bounded. Consequently

$$(-\infty, -c_2) \subseteq \rho(-L_D) .$$

Thus, the spectrum $\Sigma(-L_D)$ belongs to the complement of $(-\infty, -c_2)$ in $\mathbb{C}$. In the following the spectrum will be determined precisely.

9.2 Existence of eigenvalues in bounded domains

The results of this section are based on the following fundamental result:

Theorem 9.1 (Rellich selection theorem.) *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open subset. Every bounded sequence in $\overset{\circ}{H}_1(\Omega)$ has a subsequence, which converges in the norm of $L^2(\Omega)$.*

We omit the proof. It can be found for example in the books of Alt and Leis.

In this section we always assume that $a_{\alpha\beta} : \Omega \to \mathbb{C}$ are bounded measurable with

$$\begin{aligned} a_{\alpha\beta}(x) &= (-1)^{|\alpha+\beta|}\,\overline{a_{\beta\alpha}(x)}, && \text{if } |\alpha| + |\beta| \le 1, \\ a_{\alpha\beta}(x) &= a_{\beta\alpha} \in \mathbb{R}, && \text{if } |\alpha| = |\beta| = 1, \end{aligned}$$

and that

$$Lu(x) = \sum_{|\alpha|,|\beta|\le 1} D^\alpha\big(a_{\alpha\beta}(x) D^\beta u(x)\big),$$

is a uniformly elliptic operator with ellipticity constant η.

Let $\lambda \in \mathbb{C}$ be an eigenvalue and $u \in D(L_D)$ be an eigenfunction of $-L_D$, hence

$$(-L_D - \lambda)u = 0\,.$$

By definition this holds if and only if $u \in \overset{\circ}{H}_1(\Omega)$ satisfies

$$B(u, \varphi) + \lambda(u, \varphi)_\Omega = 0$$

for all $\varphi \in \overset{\circ}{C}_\infty(\Omega)$. This is equivalent to

$$B(u, v) + \lambda(u, v)_\Omega = 0$$

for all $v \in \overset{\circ}{H}_1(\Omega)$.

Lemma 9.2 *Every eigenvalue of $-L_D$ is real. Eigenfunctions u_1 and u_2 to distinct eigenvalues are orthogonal:*

$$(u_1, u_2)_\Omega = B(u_1, u_2) = 0.$$

Proof: Let $u \in \overset{\circ}{H}_1(\Omega)$ be an eigenfunction to the eigenvalue λ. Then

$$B(u, u) + \lambda(u, u)_\Omega = 0.$$

The symmetry of B implies $B(u,u) \in \mathbb{R}$, hence

$$\lambda = -\frac{B(u,u)}{\|u\|_\Omega^2} \in \mathbb{R}.$$

If u_1 and u_2 are eigenfunctions to the distinct eigenvalues λ_1, λ_2, then

$$\lambda_1 (u_1, u_2)_\Omega = -B(u_1, u_2) = -\overline{B(u_2, u_1)} = \lambda_2 \overline{(u_2, u_1)_\Omega} = \lambda_2 (u_1, u_2)_\Omega,$$

hence $(\lambda_1 - \lambda_2)(u_1, u_2)_\Omega = 0$. Since $\lambda_1 - \lambda_2 \neq 0$, this yields $(u_1, u_2)_\Omega = 0$, whence

$$B(u_1, u_2) = -\lambda_1 (u_1, u_2)_\Omega = 0.$$

The proof is complete.

Let M be a finite dimensional linear subspace of $\overset{\circ}{H}_1(\Omega)$ spanned by eigenfunctions of $-L_D$. We denote by $M^\perp$ the linear space of all functions in $\overset{\circ}{H}_1(\Omega)$, which are orthogonal to M with respect to the scalar product (u,v). Since $|(u,v)| \leq \|u\|_\Omega \|v\|_\Omega \leq \|u\|_1 \|v\|_1$, the scalar product (u,v) is continuous with respect to the norm $\|u\|_1$, whence $M^\perp$ is closed. We allow $M = \emptyset$, in which case $M^\perp = \overset{\circ}{H}_1(\Omega)$.

Theorem 9.3 *If $u \in M^\perp$ with $\|u\|_\Omega = 1$ exists such that*

$$-B(u,u) = \min_{\substack{v \in M^\perp \\ \|v\|_\Omega = 1}} (-B(v,v)),$$

then $\lambda = -B(u,u)$ is an eigenvalue of $-L_D$ and u is an eigenfunction to this eigenvalue.

Proof: Note that $\|\frac{v}{\|v\|_\Omega}\|_\Omega = 1$ for all $v \neq 0$, hence

$$\min_{\substack{v \in M^\perp \\ \|v\|_\Omega = 1}} (-B(v,v)) = \min_{v \in M^\perp} \Big(-B\big(\frac{v}{\|v\|_\Omega}, \frac{v}{\|v\|_\Omega}\big)\Big) = \min_{v \in M^\perp} \big(-\frac{B(v,v)}{\|v\|_\Omega^2}\big).$$

It follows that $-\frac{B(u,u)}{\|u\|_\Omega^2} = -B(u,u) \leq -\frac{B(v,v)}{\|v\|_\Omega^2}$ for all v, hence $\lambda \mapsto -\frac{B(u+\lambda v, u+\lambda v)}{\|u+\lambda v\|_\Omega^2} : \mathbb{R} \to \mathbb{R}$ has the minimum at $\lambda = 0$ for all $v \in M^\perp$. Thus

$$\begin{aligned} 0 &= \frac{d}{d\lambda} \frac{B(u+\lambda v, u+\lambda v)}{\|u+\lambda v\|_\Omega^2}\Big|_{\lambda=0} \\ &= \frac{d}{d\lambda} \frac{B(u,u) + \lambda 2 \operatorname{Re} B(u,v) + \lambda^2 B(v,v)}{\|u\|_\Omega^2 + \lambda 2 \operatorname{Re}(u,v) + \lambda^2 \|v\|_\Omega^2}\Big|_{\lambda=0} \\ &= \frac{2 \operatorname{Re} B(u,v) \|u\|_\Omega^2 - B(u,u) 2 \operatorname{Re}(u,v)}{\|u\|_\Omega^2} = 2 \operatorname{Re}\big(B(u,v) + \lambda (u,v)\big). \end{aligned}$$

Since $M^\perp$ is a linear space it follows that $iv \in M^\perp$ if $v \in M^\perp$. Thus

$$\begin{aligned}\operatorname{Im}\big(B(u,v)+\lambda(u,v)\big) &= \operatorname{Re}\big(-i\,B(u,v)-i\,\lambda(u,v)\big)\\ &= \operatorname{Re}\big(B(u,iv)+\lambda(u,iv)\big)=0,\end{aligned}$$

hence

$$B(u,v)+\lambda(u,v)=0 \tag{9.1}$$

for all $v \in M^\perp$. Let w be one of the finitely many eigenfunctions which span M, and let μ be the eigenvalue to w. Then

$$B(w,v)+\mu(w,v)_\Omega = 0$$

for all $v \in \overset{\circ}{H}_1(\Omega)$. Since $u \in M^\perp$ we have $(w,u)_\Omega = 0$, thus $B(w,u)=0$, whence $B(u,v)=(u,v)_\Omega = 0$ for all $v \in M$. Together with (9.1) this implies $B(u,v)+\lambda(u,v)=0$ for all $v \in \overset{\circ}{H}_1(\Omega)$, whence λ is an eigenvalue and u is an eigenfunction.

Theorem 9.4 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. The function $-B(v,v)$ assumes a minimum on the set $\{v \in M^\perp \mid \|v\|_\Omega = 1\}$, which is not smaller than $-c_2$, where $c_2 \geq 0$ is the constant from the coercivity estimate proved in Theorem 8.4. (The minimum is not unique.)*

Proof: It has been shown in Theorem 8.4 that $-B$ is coercive. This implies for all $u \in H_1(\Omega)$ with $\|u\|_\Omega = 1$ that

$$-B(u,u) \geq \frac{\eta}{2}\,|u|_{1,\Omega}^2 - c_2\|u\|_\Omega^2 \geq -c_2\,,$$

consequently the infimum

$$\lambda = \inf_{\substack{v\in M^\perp\\ \|v\|_\Omega=1}} \big(-B(v,v)\big) \geq -c_2$$

exists, and we can select a sequence $\{u_k\}_k \subseteq \{v \in M^\perp \mid \|v\|_\Omega = 1\}$ satisfying

$$\lim_{k\to\infty} -B(u_k,u_k) = \lambda.$$

The coercivity implies

$$\frac{\eta}{2}\,|u_k|_{1,\Omega}^2 \leq -B(u_k,u_k) + c_2 \to \lambda + c_2\,,$$

which yields

$$\|u_k\|_1 = \big(\|u_k\|_\Omega^2 + |u_k|_{1,\Omega}^2\big)^{1/2} \leq c$$

with a suitable constant C. Hence $\{u_k\}_k$ is bounded in $\overset{\circ}{H}_1(\Omega)$. In general, the sequence $\{u_k\}_k$ does not converge. However, we can select a convergent subsequence: Let $|\!|\!| u |\!|\!|^2 = -B(u,u)$. Since $B(u,v)$ is a sesquilinear form, the parallelogram equality holds:

$$|\!|\!| u+v |\!|\!|^2 + |\!|\!| u-v |\!|\!|^2 = 2|\!|\!| u |\!|\!|^2 + 2|\!|\!| v |\!|\!|^2.$$

Thus

$$\begin{aligned} |\!|\!| u_\ell - u_k |\!|\!|^2 &= 2|\!|\!| u_\ell |\!|\!|^2 + 2|\!|\!| u_k |\!|\!|^2 - |\!|\!| u_\ell + u_k |\!|\!|^2 \\ &= |\!|\!| u_\ell |\!|\!|^2 + 2|\!|\!| u_k |\!|\!|^2 - \|u_\ell + u_k\|_\Omega^2 \, |\!|\!| \frac{u_\ell + u_k}{\|u_\ell + u_k\|_\Omega} |\!|\!|^2 \\ &\le 2|\!|\!| u_\ell |\!|\!|^2 + 2|\!|\!| u_k |\!|\!|^2 - \lambda^2 \|u_\ell + u_k\|_\Omega^2 \,. \end{aligned} \tag{9.2}$$

Here we used that $\frac{u_\ell+u_k}{\|u_\ell+u_k\|_\Omega} \in \{v \in M^\perp \mid \|v\|_\Omega = 1\}$, whence $|\!|\!| \frac{u_\ell+u_k}{\|u_\ell+u_k\|_\Omega} |\!|\!| \ge \lambda$.

Since $\{u_k\}_k$ is bounded in $\overset{\circ}{H}_1(\Omega)$ and since Ω is bounded, there is a subsequence $\{u_{k_s}\}_s$ converging in $L^2(\Omega)$, by the Rellich selection theorem. Let $u \in L^2(\Omega)$ be the limit function. Denoting the subsequence by $\{u'_k\}_k$, for simplicity, we obtain from the continuity of the norm that $\|u\|_\Omega = 1$ and $\|u'_k + u'_\ell\|_\Omega \to \|2u\|_\Omega = 2$, for $k, \ell \to \infty$. The inequality (9.2) together with the coercivity of B thus yields for $k, \ell \to \infty$ that

$$\frac{\eta}{2}\, |u'_\ell - u'_k|_{1,\Omega}^2 \le |\!|\!| u'_\ell - u'_k |\!|\!|^2 + c_2 \|u'_\ell - u'_k\|_\Omega^2 \;\to\; 2\lambda^2 + 2\lambda^2 - 4\lambda^2 = 0\,.$$

Consequently $\{u'_k\}_k$ converges in $\overset{\circ}{H}_1(\Omega)$ with limit function u, since the limits in $L^2(\Omega)$ and $\overset{\circ}{H}_1(\Omega)$ coincide. From the continuity of B on $\overset{\circ}{H}_1(\Omega) \times \overset{\circ}{H}_1(\Omega)$ we thus conclude

$$-B(u,u) = \lim_{k\to\infty} -B(u'_k, u'_k) = \lambda = \inf_{\substack{v \in M^\perp \\ \|v\|_\Omega = 1}} -B(v,v)\,.$$

From the closedness of $M^\perp$ we conclude that $u \in \{v \in M^\perp \mid \|v\|_\Omega = 1\}$, hence u is a minimum of $-B(v,v)$ on this set.

Corollary 9.5 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and let $M \subseteq \overset{\circ}{H}_1(\Omega)$ be a finite dimensional linear space spanned by eigenfunctions of $-L_D$, or let $M = \emptyset$. Then there is an eigenvalue λ of $-L_D$ and an eigenfunction $u \in M^\perp$ to λ, which satisfy $\|u\|_\Omega = 1$ and*

$$\lambda = -B(u,u) = \min_{\substack{v \in M^\perp \\ \|v\|_\Omega = 1}} -B(v,v) \ge -c_2.$$

Proof: Combination of the preceding two theorems.

9.3 Spectral theorem and resolvent set

Also in this section we assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded open set and that

$$Lu(x) = \sum_{|\alpha|,|\beta|\le 1} D^\alpha\big(a_{\alpha\beta}(x)\, D^\beta u(x)\big)$$

is a uniformly elliptic operator with bounded, measurable coefficient functions $a_{\alpha\beta} : \Omega \to \mathbb{C}$ satisfying

$$\begin{aligned} a_{\alpha\beta}(x) &= (-1)^{|\alpha+\beta|}\,\overline{a_{\beta\alpha}(x)}, \qquad && |\alpha+\beta| \le 1 \\ a_{\alpha\beta}(x) &= a_{\beta\alpha}(x) \in \mathbb{R}, && |\alpha| = |\beta| = 1\,. \end{aligned}$$

Theorem 9.6 (Spectral theorem for $-L_D$) *There is a countably infinite sequence $\{\lambda_m\}_m \subseteq \mathbb{R}$ of eigenvalues of $-L_D$ satisfying*

$$-c_2 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_m \le \ldots \to \infty, \quad m \to \infty\,,$$

where the eigenvalues are repeated according to multiplicity. Moreover, there is a sequence $\{u_m\}_m \subseteq \overset{\circ}{H}_1(\Omega)$ of corresponding eigenfunctions, which form a complete orthonormal system in $L^2(\Omega)$.

Proof: We construct the sequences $\{\lambda_m\}_m$ and $\{u_m\}_m$ by induction: If $\lambda_1, \ldots, \lambda_m$ and $u_1, \ldots, u_m$ are already constructed, let M_m be the space spanned by $u_1, \ldots, u_m$. Define λ_{m+1} and $u_{m+1} \in M_m^\perp$ to be the eigenvalue and eigenfunction of $-L_D$ satisfying $\|u_{m+1}\|_\Omega = 1$ and

$$\lambda_{m+1} = -B(u_{m+1}, u_{m+1}) = \min_{\substack{v \in M_m^\perp \\ \|v\|_\Omega = 1}} -B(v,v)\,,$$

which exist according to Corollary 9.5. The corollary also yields $\lambda_m \ge -c_2$. Since $M_{m+1}^\perp \subseteq M_m^\perp$ it follows that

$$\lambda_{m+1} = \min_{\substack{v \in M_m^\perp \\ \|v\|_\Omega = 1}} -B(v,v) \le \min_{\substack{v \in M_{m+1}^\perp \\ \|v\|_\Omega = 1}} -B(v,v) = \lambda_{m+2}.$$

Moreover, $\lambda_m \to \infty$ for $m \to \infty$. Otherwise there would exist $C > 0$ with $\lambda_m \le C$ for all m. The coercivity of B yields

$$\begin{aligned} \frac{\eta}{2}\,|u_m|_{1,\Omega}^2 &\le -B(u_m,u_m) + c_2\|u_m\|_\Omega^2 = \lambda_m(u_m,u_m)_\Omega + c_2 \\ &= \lambda_m + c_2 \le C + c_2, \end{aligned}$$

whence $\{u_m\}_m$ is bounded in $\overset{\circ}{H}_1(\Omega)$. By the Rellich selection theorem we could select a subsequence converging in $L^2(\Omega)$. However, such a subsequence does not exist, since $(u_\ell, u_m) = 0$ implies

$$\|u_\ell - u_m\|_\Omega^2 = \|u_\ell\|_\Omega^2 + \|u_m\|_\Omega^2 = 2\,,$$

whenever $\ell \neq m$. Therefore $\lambda_m \to \infty$.

By construction, $\{u_m\}_m$ is an orthonormal system in $L^2(\Omega)$. If it is not complete there is $f \in L^2(\Omega)$ different from zero such that

$$(u_m, f)_\Omega = 0$$

for all m.

In Theorem 8.4 we proved that the sesquilinear from $-B(u,v)+(c_2+1)(u,v)$ is strictly coercive, which implies that there is $w \in \overset{\circ}{H}_1(\Omega)$, $w \neq 0$, such that

$$-B(v,w) + (c_2+1)(v,w) = (v,f)$$

for all $v \in \overset{\circ}{H}_1(\Omega)$. For the eigenfunctions u_m we thus obtain

$$(\lambda_m + c_2 + 1)(u_m, w) = -B(u_m, w) + (c_2+1)(u_m, w) = (u_m, f) = 0.$$

Because $\lambda_m + c_2 + 1 \geq 1$, it follows that $(u_m, w) = 0$ for all m, thus $w \in M_k^\perp$ for all k. Setting $w' = \frac{w}{\|w\|_\Omega}$, we obtain for all k

$$\lambda_{k+1} = \min_{\substack{v \in M_k^\perp \\ \|v\|_\Omega = 1}} -B(v,v) \leq -B(w', w')\,.$$

This contradicts $\lambda_k \to \infty$ for $k \to \infty$. Consequently the orthonormal system $\{u_m\}_m$ is complete in $L^2(\Omega)$.

Corollary 9.7 *Let $\{\lambda_m\}_m$ be the eigenvalues constructed in the preceding theorem and let $\{u_m\}_m$ be the complete orthonormal system of eigenfunctions.*

(i) *$u \in L^2(\Omega)$ belongs to $\overset{\circ}{H}_1(\Omega)$ if and only if $\sum_{m=1}^\infty \lambda_m |(u,u_m)|^2 < \infty$. If u belongs to $\overset{\circ}{H}_1(\Omega)$, it holds that*

$$\lim_{\ell \to \infty} \|u - \sum_{m=1}^{\ell} (u, u_m) u_m\|_{1,\Omega} = 0. \tag{9.3}$$

(ii) *For $u \in \overset{\circ}{H}_1(\Omega)$ we have*

$$-B(u,u) = \sum_{m=1}^\infty \lambda_m |(u,u_m)|^2. \tag{9.4}$$

(iii) *$u \in L^2(\Omega)$ belongs to $D(L_D)$ if and only if $\sum_{m=1}^{\infty} \lambda_m^2 \, |(u,u_m)|^2 < \infty$. If u belongs to $D(L_D)$, it holds that*

$$-L_D u = \sum_{m=1}^{\infty} \lambda_m (u,u_m) u_m. \tag{9.5}$$

Proof: (i) Let $u \in L^2(\Omega)$ be given. For $k \leq \ell$ define

$$u_{k\ell} = \sum_{m=k}^{\ell} (u,u_m) u_m.$$

The function $u_{k\ell}$ belongs to $\overset{\circ}{H}_1(\Omega)$, and since the orthonormal system is complete in $L^2(\Omega)$, it follows that the sequence $\{u_{1\ell}\}_{\ell\in\mathbb{N}}$ converges to u in the norm of $L^2(\Omega)$.

We first show that if $\sum_{m=1}^{\infty} \lambda_m |(u,u_m)|^2 < \infty$ holds, then u belongs to $\overset{\circ}{H}_1(\Omega)$ and $\{u_{1\ell}\}_{\ell\in\mathbb{N}}$ converges to u in the norm of $\overset{\circ}{H}_1(\Omega)$. To this end let c_2 be the constant from (8.3) and choose $\lambda > c_2$. Then (8.3) implies for $u \in \overset{\circ}{H}_1(\Omega)$ that

$$-B(u,u) + \lambda(u,u) \geq c_1 |u|_{1,\Omega}^2 + (\lambda - c_2)\|u\|^2 \geq c\,\|u\|_{1,\Omega}^2$$

with $c = \min(c_1, \lambda - c_2) > 0$, whence

$$\begin{aligned} c\,\|u_{k,\ell}\|_{1,\Omega}^2 &\leq -B(u_{k\ell},u_{k\ell}) + \lambda(u_{k\ell},u_{k\ell}) \\ &= \sum_{m,s=k}^{\ell} (u,u_m)\,\overline{(u,u_s)}\,\big(-B(u_m,u_s) + \lambda(u_m,u_s)\big) \\ &= \sum_{m,s=k}^{\ell} (u,u_m)\,\overline{(u,u_s)}\,(\lambda_m + \lambda)\,(u_m,u_s) = \sum_{m=k}^{\ell} (\lambda_m + \lambda)\,|(u,u_m)|^2. \end{aligned}$$

This inequality implies that if $\sum_{m=1}^{\infty} \lambda_m |(u,u_m)|^2 < \infty$, then $\{u_{1\ell}\}_{\ell\in\mathbb{N}}$ is a Cauchy sequence in the norm of $\overset{\circ}{H}_1(\Omega)$, hence converges in $\overset{\circ}{H}_1(\Omega)$. Since the limit in $\overset{\circ}{H}_1(\Omega)$ coincides with the limit u in $L^2(\Omega)$, we obtain $u \in \overset{\circ}{H}_1(\Omega)$ and

$$\lim_{\ell\to\infty} \|u - u_{1\ell}\|_{1,\Omega} = 0,$$

which is (9.3).

Assume on the other hand that $u \in \overset{\circ}{H}_1(\Omega)$. To show that $\sum_{m=1}^{\infty} \lambda_m |(u,u_m)|^2 < \infty$, we compute similarly as above

$$\begin{aligned} 0 &\leq -B(u - u_{1\ell}, u - u_{1\ell}) + \lambda(u - u_{1\ell}, u - u_{1\ell}) \\ &= -B(u,u) + \lambda(u,u) - \sum_{m=1}^{k} (\lambda_m + \lambda)\,|(u,u_m)|^2, \end{aligned}$$

hence

$$\sum_{m=1}^{\infty} \lambda_m |(u, u_m)|^2 \leq -B(u,u) + \lambda\Big((u,u) - \sum_{m=1}^{\infty} |(u,u_m)|^2\Big) = -B(u,u).$$

Since $\lambda_m \leq \lambda_{m+1} \leq \ldots \to \infty$, this proves that the series $\sum_{m=1}^{\infty} \lambda_m |(u,u_m)|^2$ converges.

(ii) Since the bilinear form $B : \overset{\circ}{H}_1(\Omega) \times \overset{\circ}{H}_1(\Omega) \to \mathbb{C}$ is continuous, we compute with (9.3) that

$$\begin{aligned} - B(u,u) = -B\Big(\sum_{m=1}^{\infty}(u,u_m)u_m, \sum_{m=1}^{\infty}(u,u_m)u_m\Big) \\ = -\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}(u,u_m)\,\overline{(u,u_s)}\,B(u_m,u_s) \\ = \sum_{m,s=1}^{\infty} (u,u_m)\,\overline{(u,u_s)}\,\lambda_m(u_m,u_s) = \sum_{m=1}^{\infty}\lambda_m|(u,u_m)|^2. \end{aligned}$$

This proves (9.4).

(iii) By definition, $u \in D(L_D)$ if and only if $u \in \overset{\circ}{H}_1(\Omega)$ and there is $f \in L^2(\Omega)$ such that

$$-B(v,u) = (v,f), \tag{9.6}$$

for all $v \in \overset{\circ}{H}_1(\Omega)$. If (9.6) holds, we have $-L_D u = f$. For $v \in \overset{\circ}{H}_1(\Omega)$ it follows from statement (i) that $\sum_{m=1}^{\infty}(v,u_m)u_m$ converges to v in the norm of $\overset{\circ}{H}_1(\Omega)$. Since $B : \overset{\circ}{H}_1(\Omega) \times \overset{\circ}{H}_1(\Omega) \to \mathbb{C}$ is continuous, we obtain

$$-B(v,u) = -\sum_{m=1}^{\infty}(v,u_m)\,B(u_m,u) = \sum_{m=1}^{\infty}(v,u_m)\,\lambda_m(u_m,u).$$

Since the orthonormal system is complete in $L^2(\Omega)$, we moreover have

$$(v,f) = \sum_{m=1}^{\infty}(v,u_m)\,(u_m,f).$$

Thus (9.6) holds if and only if

$$\sum_{m=1}^{\infty}\big(\lambda_m(u_m,u) - (u_m,f)\big)(v,u_m) = 0$$

for all $v \in \overset{\circ}{H}_1(\Omega)$. Setting $v = u_k$ shows that (9.6) holds if and only if

$$\lambda_k(u_k,u) = (u_k,f), \quad \text{for all } k \in \mathbb{N}. \tag{9.7}$$

Thus, $u \in D(L_D)$ if and only if $u \in \overset{\circ}{H}_1(\Omega)$ and there is $f \in L^2(\Omega)$ satisfying (9.7).

If $u \in D(L_D)$ we conclude from (9.7) that

$$\sum_{m=1}^{\infty} \lambda_m^2 \, |(u, u_m)|^2 = \sum_{m=1}^{\infty} |(f, u_m)|^2 < \infty$$

and

$$-L_D u = f = \sum_{m=1}^{\infty} (f, u_m) u_m = \sum_{m=1}^{\infty} \lambda_m (u, u_m) u_m.$$

This proves (9.5). On the other hand, if $\sum_{m=1}^{\infty} \lambda_m^2 \, |(u, u_m)|^2 < \infty$ it follows that also $\sum_{m=1}^{\infty} \lambda_m \, |(u, u_m)|^2 < \infty$. By statement (i) this means that $u \in \overset{\circ}{H}_1(\Omega)$. Define a function $f \in L^2(\Omega)$ by $f = \sum_{m=1}^{\infty} \lambda_m (u, u_m) u_m$. Since this function satisfies (9.7), we infer that $u \in D(L_D)$. ∎

Corollary 9.8 *To every* $\lambda \in \mathbb{C} \backslash \{\lambda_m\}_m$ *and every* $f \in L^2(\Omega, \mathbb{C})$ *there is a unique solution* u *of*

$$-L_D \, u - \lambda \, u = f$$

given by

$$u = \sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda_m - \lambda} \, u_m.$$

Consequently $\rho(-L_D) = \mathbb{C} \backslash \{\lambda_m\}_m$, $\Sigma(-L_D) = \{\lambda_m\}_m$.

Remark 9.9 This result means, of course, that the Dirichlet problem

$$\sum_{|\alpha|, |\beta| \le 1} D^\alpha (a_{\alpha\beta} D^\beta \, u) + \lambda \, u = f$$

$$u\big|_{\partial\Omega} = 0$$

has a unique weak solution for all $\lambda \neq \lambda_m$ and all $f \in L^2(\Omega)$.

Proof: From $u = \sum_{m=1}^{\infty} \frac{(f, u_m)}{\lambda_m - \lambda} \, u_m$ it follows

$$(u, u_m) = \frac{(f, u_m)}{\lambda_m - \lambda},$$

hence

$$\sum_{m=1}^{\infty} |\lambda_m|^2 \, |(u, u_m)|^2 = \sum_{m=1}^{\infty} \left| \frac{\lambda_m}{\lambda_m - \lambda} \right|^2 |(f, u_m)|^2 \le C \sum_{m=1}^{\infty} |(f, u_m)|^2 < \infty.$$

Corollary 9.7 thus shows that $u \in D(L_D)$ and

$$-L_D\, u - \lambda\, u = \sum_{m=1}^{\infty} (\lambda_m - \lambda)(u, u_m) u_m = \sum_{m=1}^{\infty} (f, u_m) u_m = f.$$

The solution is unique since $-L_D - \lambda$ is injective. For,

$$0 = -L_D\, v - \lambda\, v = \sum_{m=1}^{\infty} (\lambda_m - \lambda)(v, u_m) u_m$$

yields together with $\lambda_m - \lambda \neq 0$ that $(v, u_m) = 0$ for all m, hence $v = 0$.

10 Linear hyperbolic equations of second order

10.1 Hyperbolic differential operators

The wave equation $\frac{\partial^2}{\partial t^2} u(x,t) = c\Delta_x u(x,t)$ is a hyperbolic equation. We now show that the spectral theorem from Section 9 can be used to prove existence of solutions for the wave equation and other hyperbolic equations.

Let $Lu(x) = \sum_{\substack{|\alpha|\le 1\\|\beta|\le 1}} D^\alpha\big(a_{\alpha\beta}(x)D^\beta u(x)\big)$ be a linear differential operator of second order. In the remainder I always assume that the coefficients of the principal part

$$L'u(x) = \sum_{\substack{|\alpha|=1\\|\beta|=1}} a_{\alpha\beta}(x)D^{\alpha+\beta}u(x)$$

are real valued functions:

$$a_{\alpha\beta}(x) \in \mathbb{R}, \quad \text{for } |\alpha| = |\beta| = 1\,.$$

One uses the set of zeros of the "principal symbol"

$$p(x,\xi) = \sum_{|\alpha|=|\beta|=1} a_{\alpha\beta}(x)\xi^{\alpha+\beta}, \quad \xi \in \mathbb{R}^n,$$

of the differential operator L to classify the operator. An operator, whose set of zeros only consists of $0 \in \mathbb{R}^n$ is elliptic.

A subset M of $\mathbb{R}^n$ is called conic with vertex at 0 if $\xi \in M$ implies $\mu\xi \in M$ for all $\mu \ge 0$. Since $p(x,\xi)$ is homogeneous of order 2 with respect to ξ, it follows that if ξ is a zero of $p(x,\xi)$, then $\mu\xi$ is a zero for all $\mu \in \mathbb{R}$, hence the set of zeros of $p(x,\xi)$ is a conic subset of $\mathbb{R}^n$ symmetric with respect to the vertex 0.

The operator L is called hyperbolic if the set of zeros of the principal symbol p is a double cone. This is made precise in the following

Definition 10.1 The operator L is hyperbolic at $x \in \mathbb{R}^n$, if there is a vector $\theta \neq 0$ such that every line in $\mathbb{R}^n$ parallel to θ, not passing through the origin, intersects the set $\{\xi \mid p(x,\xi) = 0\}$ in precisely two distinct points.

Example 10.2 Let $\sum_{|\alpha|,|\beta|\le 1} D_x^\alpha\big(a_{\alpha\beta}(x)D_x^\beta\big)$, $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$, be an elliptic operator satisfying

$$\sum_{|\alpha|=|\beta|=1} a_{\alpha\beta}(x)\xi^{\alpha+\beta} > 0, \quad \xi \in \mathbb{R}^n,\, \xi \neq 0.$$

Then

$$Lu(x,t) = \frac{\partial^2}{\partial t^2} u(x,t) - \sum_{\substack{|\alpha|\le 1\\ |\beta|\le 1}} D_x^\alpha\big(a_{\alpha\beta}(x) D_x^\beta u(x,t)\big)$$

is a hyperbolic operator. To see this note that with $x, \xi \in \mathbb{R}^n$, $t, \zeta \in \mathbb{R}$ the principal symbol is

$$p(x,t,\xi,\zeta) = \zeta^2 - \sum_{\substack{|\alpha|=1\\ |\beta|=1}} a_{\alpha\beta}(x)\xi^{\alpha+\beta}.$$

Set $\theta = (0,\dots,0,1) \in \mathbb{R}^{n+1}$. Every line in $\mathbb{R}^{n+1}$ parallel to θ and not passing through the origin is of the form $\zeta \mapsto (\xi,\zeta)$ with $\xi \in \mathbb{R}^n$, $\xi \neq 0$. For such ξ the equation

$$p(x,t,\xi,\zeta) = \zeta^2 - \sum_{|\alpha|=|\beta|=1} a_{\alpha\beta}(x)\xi^{\alpha+\beta} = 0$$

has the two distinct solutions $\zeta = \pm\sqrt{\sum_{|\alpha|=|\beta|=1} a_{\alpha\beta}(x)\xi^{\alpha+\beta}}$. In particular, the operator

$$\partial_t^2 - c\Delta_x = \frac{\partial^2}{\partial t^2} - c\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$$

is hyperbolic for every constant $c > 0$. Therefore the wave equation

$$\frac{\partial^2}{\partial t^2} u(x,t) = c\Delta_x u(x,t)$$

is a hyperbolic equation. ($\partial_t^2 - c\Delta_x$ is sometimes called d'Alembert operator, after Jean-Baptiste le Rond d'Alembert, 1717 – 1783.)

10.2 Energy estimate for the wave equation, uniqueness of solutions

Let $\Omega \subseteq \mathbb{R}^n$ be an open set, let $f : \Omega \times [0,\infty) \to \mathbb{C}$ be a bounded continuous function.

Theorem 10.3 *Let $u : C_2(\Omega \times [0,\infty), \mathbb{C}) \cap C(\overline{\Omega} \times [0,\infty), \mathbb{C})$ be a solution of the initial-boundary value problem*

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} u(x,t) &= c\Delta_x u(x,t) + f(x,t), && (x,t) \in \Omega \times (0,\infty)\\
u(x,t) &= 0, && (x,t) \in \partial\Omega \times [0,\infty)\\
u(x,0) &= u^{(0)}(x),\ u_t(x,0) = u^{(1)}(x), && x \in \overline{\Omega}.
\end{aligned}$$

with a constant $c > 0$. Then u satisfies the energy estimate

$$E(u,t)^{1/2} \le E(u,0)^{1/2} + \int_0^t \|f(t)\|_\Omega \, dt,$$

where the energy $E(u,t)$ is defined by

$$E(u,t) = \int_\Omega \frac{1}{2}|u_t(x,t)|^2 + \frac{c}{2}\,|\nabla_x u(x,t)|^2 dx.$$

Proof: Since u is two times continuously differentiable we have

$$\begin{aligned}
\frac{d}{dt}\,E(u,t) &= \frac{d}{dt}\int_\Omega \frac{1}{2}|u_t(x,t)|^2 + \frac{c}{2}\,|\nabla_x u(x,t)|^2 dx \\
&= \int_\Omega \mathrm{Re}\,\Big(u_{tt}(x,t)\,\overline{u_t(x,t)} + c\nabla_x u(x,t)\cdot\nabla_x\,\overline{u_t(x,t)}\Big)dx \\
&= \int_\Omega \mathrm{Re}\,\Big(\big(u_{tt}(x,t) - c\Delta_x u(x,t)\big)\,\overline{u_t(x,t)}\Big)dx \\
&= \mathrm{Re}\int_\Omega f(x,t)\,\overline{u_t(x,t)}\,dx \le \|f(t)\|_\Omega\,\|u_t(t)\|_\Omega \le 2\|f(t)\|_\Omega E(u,t)^{1/2}.
\end{aligned}$$

Now

$$\frac{d}{dt}E(u,t) = \frac{d}{dt}\big(E(u,t)^{1/2}\big)^2 = 2E(u,t)^{1/2}\,\frac{d}{dt}\,E(u,t)^{1/2}.$$

Combination of these relations yields

$$\frac{d}{dt}\,E(u,t)^{1/2} \le \|f(t)\|_\Omega\,.$$

Integration yields the stated estimate.

Corollary 10.4 *The initial-boundary value problem*

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}u(x,t) &= c\Delta_x u(x,t) + f(x,t) \\
u(x,t) &= 0, \quad (x,t)\in\partial\Omega\times[0,\infty) \\
u(x,0) &= u^{(0)}(x),\ u_t(x,0) = u^{(1)}(x), \quad x\in\overline{\Omega}
\end{aligned}$$

has at most one solution $u \in C_2(\Omega\times[0,\infty),\mathbb{C}) \cap C(\overline{\Omega}\times[0,\infty),\mathbb{C})$.

Proof: Let u and v be two solutions. Then the difference $w = u - v$ satisfies

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}w(x,t) &= c\Delta_x w(x,t) \\
w(x,t) &= 0, \qquad (x,t)\in\partial\Omega\times[0,\infty) \\
w(x,0) &= w_t(x,0) = 0, \qquad x\in\overline{\Omega}.
\end{aligned}$$

Form the energy estimate it thus follows

$$\int_\Omega \frac{1}{2}|w_t(x,t)|^2 + \frac{c}{2}|\nabla_x w(x,t)|^2 dx \le E(w,0) = 0.$$

Consequently $w_t(x,t) = 0$ for all $(x,t)\in\Omega\times[0,\infty)$ and $w(x,0) = 0$ for all $x\in\Omega$, whence

$$w(x,t) = \int_0^t w_t(x,\tau)d\tau = 0,$$

for all $(x,t)\in\overline{\Omega}\times[0,\infty)$. Thus, $u = v$.

10.3 Existence of weak solutions of initial-boundary value problems to hyperbolic equations

In the following I consider hyperbolic differential operators of the form $\frac{\partial^2}{\partial t^2} - L$, where

$$L = \sum_{\substack{|\alpha|\leq 1\\ |\beta|\leq 1}} D_x^\alpha \big(a_{\alpha\beta}(x) D_x^\beta \big)$$

is a uniformly strongly elliptic differential operator with bounded measurable coefficient functions $a_{\alpha\beta} : \Omega \to \mathbb{C}$ satisfying

$$\begin{aligned} a_{\alpha\beta}(x) &= (-1)^{|\alpha+\beta|}\overline{a_{\beta\alpha}(x)}, \qquad |\alpha+\beta| \leq 1 \\ a_{\alpha\beta}(x) &= a_{\beta\alpha}(x) \in \mathbb{R}, \qquad |\alpha| = |\beta| = 1. \end{aligned}$$

$\Omega \subseteq \mathbb{R}^n$ is a bounded open set, for $T > 0$

$$Z_T = \Omega \times (0, T).$$

denotes a cylindric subset of $\mathbb{R}^{n+1}$, and for $u : Z_T \to \mathbb{C}$ and $0 < t < T$ the function $u(t) : \Omega \to \mathbb{C}$ is defined by

$$\big(u(t)\big)(x) = u(x, t).$$

The goal of this section is to show that the initial-boundary value problem

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u(x,t) &= Lu(x,t) + f(x,t), && (x,t) \in Z_T \\ u(x,t) &= 0, && (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) &= u^{(0)}(x),\ u_t(x,0) = u^{(1)}(x), && x \in \Omega, \end{aligned}$$

has a weak solution. In order to give the definition of weak solutions inhomogeneous Sobolev spaces must be introduced:

Definition 10.5 For $T > 0$, $m \in \mathbb{N}$ let

$$H_m^{(t)}(Z_T, \mathbb{C}) = \{u \in L^2(Z_T, \mathbb{C}) \mid \frac{\partial^k}{\partial t^k} u \in L^2(Z_T, \mathbb{C}), k \leq m\}.$$

$H_m^{(t)}(Z_T)$ is a Hilbert space with the scalar product

$$(u, v)_m^{(t)} = \sum_{k=0}^{m} \Big(\frac{\partial^k}{\partial t^k} u,\ \frac{\partial^k}{\partial t^k} v \Big)_{Z_T}$$

and the norm $\|u\|_m^{(t)} = \big((u,u)_m^{(t)}\big)^{1/2}$.

Theorem 10.6 (Sobolev embedding theorem.) *Let* $T > 0$ *and* $0 \leq \tau \leq T$. *Then there is a unique continuous linear mapping* $B_\tau : H_1^{(t)}(Z_T, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$ *satisfying*

$$(B_\tau u)(x) = u(x, \tau),$$

for all $u \in C\big(\Omega \times [0,T]\big) \cap H_1^{(t)}(Z_T, \mathbb{C})$.

A proof can be found in the book of Alt, p.249.

Definition 10.7 The function $B_\tau u$ is called the trace of the mapping $u \in H_1^{(t)}(Z_T)$ on $\Omega \times \{\tau\}$ and is denoted by $u|_{\Omega \times \{\tau\}}$.

As in Section 9 let $L_D : D(L_D) \subseteq L^2(\Omega) \to L^2(\Omega)$ denote the Friedrichs' extension of the operator $\sum_{|\alpha|,|\beta| \leq 1} D^\alpha (a_{\alpha\beta} D^\beta)$ in Ω.

Definition 10.8 Let $T > 0$, $f \in L^2(Z_T, \mathbb{C})$, $u^{(0)} \in D(L_D)$, $u^{(1)} \in \overset{\circ}{H}_1(\Omega, \mathbb{C})$. A function $u : Z_T \to \mathbb{C}$ is a weak solution of the Dirichlet initial-boundary value problem

$$\begin{aligned} \partial_t^2 u(x,t) &= Lu(x,t) + f(x,t), && (x,t) \in Z_T, \\ u(x,t) &= 0, && (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) &= u^{(0)}(x),\ u_t(x,0) = u^{(1)}(x), && x \in \Omega. \end{aligned} \tag{10.1}$$

if

1. $u \in H_2^{(t)}(Z_T)$,
2. $u(t) \in D(L_D)$, for almost all $t \in (0,T)$,
3. $\partial_t^2 u(t) = L_D u(t) + f(t)$, for almost all $t \in (0,T)$,
4. $u|_{\Omega \times \{0\}} = u^{(0)}$, $u_t|_{\Omega \times \{0\}} = u^{(1)}$.

Theorem 10.9 *Let* $\Omega \subseteq \mathbb{R}^n$ *be an open bounded set. To every* $f \in H_1(Z_T, \mathbb{C})$, $u^{(0)} \in D(L_D)$ *and* $u^{(1)} \in \overset{\circ}{H}_1(\Omega, \mathbb{C})$ *there is a weak solution of the Dirichlet initial-boundary value problem* (10.1). *This solution is given by*

$$u(x,t) = \sum_{m=1}^{\infty} \alpha_m(t) u_m(x),$$

where $\{u_m\}_{m=1}^{\infty}$ *is the complete orthonormal system of eigenfunctions of the operator* $-L_D$*, and where* $\alpha_m : [0,\infty) \to \mathbb{C}$ *is the solution of the initial value problem*

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\alpha_m(t) + \lambda_m \alpha_m(t) &= \big(f(t), u_m\big)_\Omega \\ \alpha_m(0) &= \big(u^{(0)}, u_m\big)_\Omega \\ \frac{\partial}{\partial t}\alpha_m(0) &= \big(u^{(1)}, u_m\big)_\Omega. \end{aligned}$$

Here λ_m *is the eigenvalue to* u_m*.*

Clearly, this implies for $\lambda_m > 0$

$$\begin{aligned} \alpha_m(t) &= \cos(\sqrt{\lambda_m}t)(u^{(0)}, u_m)_\Omega + \frac{1}{\sqrt{\lambda_m}}\sin(\sqrt{\lambda_m}t)(u^{(1)}, u_m)_\Omega \\ &\quad + \int_0^t \frac{1}{\sqrt{\lambda_m}} \sin\big(\sqrt{\lambda_m}(t-\tau)\big)\big(f(\tau), u_m\big)_\Omega d\tau. \end{aligned}$$

For $\lambda_m = 0$ we obtain

$$\alpha_m(t) = (u^{(0)}, u_m)_\Omega + (u^{(1)}, u_m)_\Omega + \int_0^t (t-\tau)\big(f(\tau), u_m\big)_\Omega d\tau,$$

and for $\lambda_m < 0$

$$\begin{aligned} \alpha_m(t) &= \cosh(\sqrt{-\lambda_m}t)(u^{(0)}, u_m)_\Omega + \frac{1}{\sqrt{-\lambda_m}}\sinh(\sqrt{-\lambda_m}t)(u^{(1)}, u_m)_\Omega \\ &\quad + \int_0^t \frac{1}{\sqrt{-\lambda_m}} \sinh\big(\sqrt{-\lambda_m}(t-\tau)\big)\big(f(\tau), u_m\big)_\Omega d\tau\,. \end{aligned}$$

Since $-c_2 \leq \lambda_1 \leq \ldots \leq \lambda_m \leq \ldots \to \infty$ for $m \to \infty$ there are only finitely many eigenvalues $\lambda_m \leq 0$.

Proof: From the explicit expression for α_m given above we obtain for $s = 0, 1, 2$

$$\begin{aligned} |\partial_t^s \alpha_m(t)| &\leq C(t)\Big(\sqrt{1+|\lambda_m|}\Big)^s \Big(|(u^{(0)}, u_m)_\Omega| \\ &\quad + \frac{1}{\sqrt{1+|\lambda_m|}}|(u^{(1)}, u_m)_\Omega| + \frac{1}{\sqrt{1+|\lambda_m|}}\int_0^t |\big(f(\tau), u_m\big)_\Omega| d\tau\Big) \\ &\quad + \delta_{s2}\,|\big(f(t), u_m\big)_\Omega|\,, \end{aligned} \tag{10.2}$$

where

$$\delta_{s2} = \begin{cases} 0, & s \neq 2 \\ 1, & s = 2, \end{cases}$$

and

$$C(t)=\begin{cases}\hat{C}e^{\sqrt{-\lambda_1}t}, & \text{if } \lambda_1<0\\ \hat{C}(1+t), & \text{if } \lambda_1=0\\ \hat{C}, & \text{if } \lambda_1>0,\end{cases}$$

with a suitable constant $\hat{C}$. Using the Cauchy-Schwarz inequality, which yields

$$|a+b+c+d|^2\le 4(a^2+b^2+c^2+d^2)$$

and

$$\Big(\int_0^t |(f(\tau),u_m)|\,d\tau\Big)^2\le t\int_0^t |(f(\tau),u_m)|^2d\tau,$$

we obtain from (10.2)

$$\begin{aligned}|\partial_t^s\alpha_m(t)|^2 \;\le\;& 4C(t)^2(1+|\lambda_m|)^s\Big(|(u^{(0)},u_m)_\Omega|^2\\ &+\frac{1}{1+|\lambda_m|}\,|(u^{(1)},u_m)_\Omega|^2+\frac{t}{1+|\lambda_m|}\int_0^t |\big(f(\tau),u_m\big)|^2d\tau\Big)\\ &+4\delta_{s2}\,|\big(f(t),u_m\big)_\Omega|^2. \end{aligned}\qquad (10.3)$$

The assumptions $u^{(0)}\in D(L_D)$ and $u^{(1)}\in \overset{\circ}{H}_1(\Omega)$ imply

$$\begin{aligned}\sum_{m=1}^\infty \big(1+|\lambda_m|^2\big)\,|(u^{(0)},u_m)_\Omega|^2 &< \infty,\\ \sum_{m=1}^\infty \big(1+|\lambda_m|\big)\,|(u^{(1)},u_m)_\Omega|^2 &< \infty,\end{aligned}$$

cf. Corollary 9.7. Moreover, from $f\in H_1(Z_T)$ it follows by the theorem of Fubini that $f(t)\in H_1(\Omega)$ for almost all $t>0$. Thus, Corollary 9.7 (a) yields

$$\begin{aligned}&\sum_{m=1}^\infty(1+|\lambda_m|)\int_0^t |\big(f(\tau),u_m\big)_\Omega|^2d\tau\\ &\quad\le\int_0^t K_1\Big(B\big(f(\tau),f(\tau)\big)+\|f(\tau)\|_\Omega^2\Big)d\tau\\ &\quad\le K_2\int_0^t \|f(\tau)\|_{1,\Omega}^2d\tau\le K\|f\|_{1,Z_t}^2<\infty,\end{aligned}$$

with suitable constants $K_1,K_2>0$. From these estimates and from (10.3) we obtain

$$\sum_{m=1}^\infty |\lambda_m|^2\,|\alpha_m(t)|^2 \;\le\; C_1C(t)^2(1+t), \qquad (10.4)$$

$$\begin{aligned}\sum_{m=1}^\infty |\partial_t^s\alpha_m(t)|^2 &\le C_1C(t)^2(1+t)+4\delta_{s2}\sum_{m=1}^\infty |\big(f(t),u_m\big)_\Omega|^2\\ &= C_1C(t)^2(1+t)+4\delta_{s2}\|f(t)\|_\Omega^2\,.\end{aligned}$$

Thus, for $s = 0, 1, 2$

$$\begin{aligned}\int_0^T \sum_{m=1}^{\infty} |\partial_t^s \alpha_m(t)|^2 dt &\leq \int_0^T C_1 C(t)^2 (1+t) + 4\delta_{s2} \|f(t)\|_\Omega^2 dt \\ &\leq C_1 C(T)^2 (1+T) T + 4\delta_{s2} \|f\|_{Z_T}^2 ,\end{aligned} \tag{10.5}$$

where we used that $C(t)$ is an increasing function. Since $u_\ell = \sum_{m=1}^{\ell} \alpha_m u_m$ satisfies for $k \leq \ell$

$$\begin{aligned}\|\partial_t^s (u_\ell - u_k)\|_{Z_T}^2 &= (\partial_t^s (u_\ell - u_k), \partial_t^s (u_\ell - u_k))_{Z_T} = \sum_{m,j=k}^{\ell} (\partial_t^s \alpha_m u_m, \partial_t^s \alpha_j u_j)_{Z_T} \\ &= \sum_{m,j=k}^{\ell} \int_0^T \partial_t^s \alpha_m \partial_t^s \overline{\alpha_j} \, dt \, (u_m, u_j)_\Omega = \sum_{m=k}^{\ell} \int_0^T |\partial_t^s \alpha_m(t)|^2 dt ,\end{aligned}$$

we obtain from (10.5) that $\{\partial_t^s u_\ell\}_\ell$ is a Cauchy sequence in $L^2(Z_T)$ with

$$\lim_{\ell \to \infty} \partial_\ell^s u_\ell = \sum_{m=1}^{\infty} \partial_t^s \alpha_m u_m ,$$

for $s = 0, 1, 2$. This means that $\{u_\ell\}_\ell$ converges in the space $H_2^{(t)}(Z_T, \mathbb{C})$ and that the limit function $u \in H_2^{(t)}(Z_T, \mathbb{C})$ satisfies

$$\partial_t^s u = \sum_{m=1}^{\infty} \partial_t^s \alpha_m u_m . \tag{10.6}$$

Also, since $u(t) = \sum_{m=1}^{\infty} \alpha_m(t) u_m$ implies $\big(u(t), u_m\big)_\Omega = \alpha_m(t)$, we infer from (10.4) that

$$\sum_{m=1}^{\infty} |\lambda_m|^2 |\big(u(t), u_m\big)_\Omega|^2 < \infty ,$$

whence $u(t) \in D(L_D)$ for all $t \geq 0$ and

$$-L_D u(t) = \sum_{m=1}^{\infty} \lambda_m \big(u(t), u_m\big)_\Omega u_m , \tag{10.7}$$

by Corollary 9.7. Summing up, we conclude from (10.6) and (10.7) that

$$\begin{aligned}\partial_t^2 u(t) &= \partial_t^2 \sum_{m=1}^{\infty} \alpha_m(t) u_m = \sum_{m=1}^{\infty} \partial_t^2 \alpha_m(t) u_m \\ &= \sum_{m=1}^{\infty} \Big(-\lambda_m \alpha_m(t) + \big(f(t), u_m\big)_\Omega \Big) u_m \\ &= -\sum_{m=1}^{\infty} \lambda_m \big(u(t), u_m\big)_\Omega u_m + \sum_{m=1}^{\infty} \big(f(t), u_m\big)_\Omega u_m \\ &= L_D u(t) + f(t) .\end{aligned}$$

Thus, the first three conditions of Definition 10.8 are satisfied. To verify the last condition note that $\alpha_m(0) = \big(u^{(0)}, u_m\big)_\Omega$ and $\frac{\partial}{\partial t}\,\alpha_m(0) = (u^{(1)}, u_m)_\Omega$ yield

$$
\begin{aligned}
u\big|_{\Omega\times\{0\}} &= \sum_{m=1}^{\infty} \alpha_m(0) u_m = \sum_{m=1}^{\infty} (u^{(0)}, u_m)_\Omega \, u_m = u^{(0)}, \\
\partial_t u\big|_{\Omega\times\{0\}} &= \sum_{m=1}^{\infty} \partial_t \alpha(0) u_m = \sum_{m=1}^{\infty} (u^{(1)}, u_m)_\Omega \, u_m = u^{(1)}\,.
\end{aligned}
$$

This completes the proof.

A Appendix: Bessel, Neumann and Hankel functions

Bessel and Neumann functions are by definition solutions of Bessel's differential equation

$$\frac{d^2}{dx^2}v(x) + \frac{1}{x}\frac{d}{dx}v(x) + (1 - \frac{\nu^2}{x^2})v(x) = 0. \tag{A.1}$$

Here $\nu \in \mathbb{C}$ is a constant and $x \in \mathbb{C}$. This equation cannot be solved by elementary functions. Instead, Bessel- and Neumann functions belong to a class of functions called special functions of mathematical physics. The Bessel function or cylinder function of order $\nu \in \mathbb{C}$, $\nu \neq -1, -2, -3, \dots$, is

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k},$$

where Γ is the Gamma function. The power series converges for all $x \in \mathbb{C}$. If ν is not a nonnegative integer, then the term $\left(\frac{x}{2}\right)^\nu$ and therefore also the function J_ν are only defined on the set $\mathbb{C} \setminus (-\infty, 0]$. To be precise, $\left(\frac{x}{2}\right)^\nu$ and J_ν are defined on a Riemannian manifold. If ν is equal to a nonnegative integer m, then the formula for the Bessel function becomes

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(m+k)!} \left(\frac{x}{2}\right)^{2k+m}, \tag{A.2}$$

where we used the equation

$$\Gamma(\ell+1) = \ell!\,,$$

which holds for integers $\ell \geq 0$. The properties of Bessel function of negative integer order can be read off from the equation

$$J_{-m}(x) = (-1)^m J_m(x), \tag{A.3}$$

which holds for all $m \in \mathbb{Z}$. We see that for every $m \in \mathbb{Z}$ the Bessel function J_m is represented by a power series converging on all of $\mathbb{C}$. Hence, J_m is an entire function.

Since Bessel's equation is a linear differential equation of second order there must exist other solutions of Bessel's equation which are linearly independent of J_ν. In fact, if ν is not an integer, then one sees immediately that also $J_{-\nu}$ is a solution of Bessel's equation, which is linearly independent of J_ν. Hence, also the Neumann function

$$N_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

is a solution of Bessel's equation linearly independent of J_ν. If $\nu = m$ is an integer this formula cannot be used to define N_m, since the denominator vanishes. Instead, in this case the Neumann function is

$$N_m(x) = \lim_{\nu \to m} N_\nu(x).$$

The general solution of Bessel's differential equation is thus given by

$$v(x) = C_1 J_\nu(x) + C_2 N_\nu(x), \tag{A.4}$$

with arbitrary constants $C_1, C_2 \in \mathbb{C}$.

If m is a nonnegative integer, then a series representation for the Neumann function of order m is

$$\begin{aligned} N_m(x) &= -\frac{1}{\pi} \sum_{k=0}^{m-1} \frac{(m-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-m} \\ &\quad + \frac{2}{\pi} \ln\left(\frac{x}{2}\right) J_m(x) \\ &\quad - \frac{1}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{\psi(k+1) + \psi(m+k+1)}{k!(m+k)!} \left(\frac{x}{2}\right)^{2k+m}, \end{aligned} \tag{A.5}$$

where J_m is the Bessel function and where the ψ–function is defined by

$$\begin{aligned} \psi(1) &= -\gamma, \\ \psi(m) &= -\gamma + \sum_{k=1}^{m-1} k^{-1}, \quad m \geq 2. \end{aligned}$$

Here

$$\gamma = \lim_{m \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{m} - \ln(m)\right) = 0.5772156649\ldots$$

denotes the Euler constant. In particular, for $m = 0$ we obtain

$$\begin{aligned} N_0(x) &= \frac{2}{\pi} \left(\ln\left(\frac{x}{2}\right) + \gamma\right) J_0(x) \\ &\quad + \frac{2}{\pi} \left(\frac{1}{(1!)^2} \left(\frac{x}{2}\right)^2 - \frac{1 + \frac{1}{2}}{(2!)^2} \left(\frac{x}{2}\right)^4 + \frac{1 + \frac{1}{2} + \frac{1}{3}}{(3!)^2} \left(\frac{x}{2}\right)^6 - \ldots \right). \end{aligned} \tag{A.6}$$

The properties of the Neumann functions of negative integer order follow from the properties of the Neumann functions of positive integer order using the equation

$$N_{-m}(x) = (-1)^m N_m(x), \tag{A.7}$$

which holds for all $m \in \mathbb{Z}$.

The Bessel functions $J_m(x)$ of integer order m are analytic at $x = 0$, hence they are infinitely differentiable at this point. The asymptotic behavior of these functions at 0 can be read off from the series representation (A.2) and from (A.3). One obtains

$$\begin{aligned} J_0(0) = 1, \quad J_0'(x) = O(x), \qquad & x \to 0, \\ J_m(x) = O(x^{|m|}), \quad J_m'(x) = O(x^{|m|-1}), \qquad & |m| \geq 1,\ x \to 0, \end{aligned} \tag{A.8}$$

The Neumann functions $N_m(x)$ of integer order have singularities at $x = 0$. The asymptotic behavior of these functions at $x = 0$ can be read off from (A.5), (A.6), and (A.7). If we note (A.8), we obtain for $x \to 0$ that

$$\begin{aligned} N_0(x) &= \frac{2}{\pi} \ln(x) + O(1), \\ N_{\pm 1}(x) &= \mp \frac{2}{\pi x} + O(x \ln(x)), \\ N_m(x) &= \begin{cases} -\frac{(m-1)!}{\pi} \left(\frac{2}{x}\right)^m + O(x^{-m+2}), & m \geq 2, \\ (-1)^{m+1} \frac{(m-1)!}{\pi} \left(\frac{2}{x}\right)^m + O(x^{-m+2}), & m \leq -2. \end{cases} \end{aligned}$$

Besides the functions J_ν and N_ν often the linear combinations

$$\begin{aligned} H_\nu^{(1)}(x) &= J_\nu + iN_\nu(x), \\ H_\nu^{(2)}(x) &= J_\nu - iN_\nu(x), \end{aligned} \tag{A.9}$$

are used. These functions are called Hankel functions of first and second kind.

In the following three lemmas we derive properties of Bessel functions and some estimates for these functions, which are used in Sections 4 and 5.

Lemma A.1 *Let $m \in \mathbb{Z}$.*
(i) *The zeros of J_m do not have an accumulation point in $\mathbb{C}$. Hence, the set of zeros is countable. If $y \in \mathbb{C}$ is a zero of J_m, then also $-y$ is a zero.*
(ii) *Assume that $y \in \mathbb{C} \setminus \{0\}$ is a zero of J_m. Then y is real and satisfies*

$$y^2 > m^2.$$

Proof: (i) For $m \in \mathbb{Z}$ the Bessel function J_m is entire. Therefore, if the zeros would accumulate in $\mathbb{C}$ we would have $J_m \equiv 0$. Consequently the zeros do not have an accumulation point. By (A.2) and (A.3) we have $J_m(-x) = (-1)^m J_m(x)$, which shows that with y also $-y$ is a zero.

(ii) Assume that $y \in \mathbb{C} \setminus \{0\}$ is a zero of J_m. Set

$$u(r) = J_m(yr).$$

This function satisfies $u(1) = 0$, and by a short computation we obtain from the Bessel differential equation (A.1) that

$$u''(r) + \frac{1}{r}u'(r) + \Big(y^2 - \frac{m^2}{r^2}\Big)u(r) = 0.$$

Multiply this equation by r and observe that $ru''(r) + u'(r) = (ru(r)')'$ to obtain

$$\big(ru'(r)\big)' + \Big(y^2 - \frac{m^2}{r^2}\Big)ru(r) = 0.$$

We multiply this equation by $\overline{u(r)}$ and integrate over the real interval $[0, 1]$. Thus,

$$\int_0^1 \big(ru'(r)\big)'\,\overline{u(r)} + \Big(y^2 - \frac{m^2}{r^2}\Big)r|u(r)|^2 dr = 0.$$

Partial integration yields

$$-\int_0^1 r\,u'(r)\big(\overline{u(r)}\big)' dr + \int_0^1 \Big(y^2 - \frac{m^2}{r^2}\Big)r\,|u(r)|^2 dr = \big(r\,u'(r)\,\overline{u(r)}\big)\Big|_{r=0}^{r=1} = 0.$$

Since $u'(r)\big(\overline{u(r)}\big)' = u'(r)\overline{u'(r)} = |u'(r)|^2$, it follows

$$\int_0^1 \left(-|u'(r)|^2 + \Big(y^2 - \frac{m^2}{r^2}\Big)\,|u(r)|^2\right) r\,dr = 0. \tag{A.10}$$

Since the imaginary part of this integral is

$$(\operatorname{Im} y^2)\int_0^1 |u(r)|^2 r\,dr = 0,$$

and since $\int_0^1 |u(r)|^2 r\,dr > 0$, it follows that Im $y^2 = 0$, hence $y^2 \in \mathbb{R}$. Moreover, we must have $y^2 > m^2$, because otherwise it holds

$$\int_0^1 \left(-|u'(r)|^2 + \Big(y^2 - \frac{m^2}{r^2}\Big)|u_m(r)|^2\right) r\,dr$$
$$\leq \int_0^1 \left(-|u'(r)|^2 + \Big(m^2 - \frac{m^2}{r^2}\Big)\,|u(r)|^2\right) r\,dr < 0,$$

which contradicts (A.10). The proof is complete. ■

Lemma A.2 *Let $m \in \mathbb{Z}$, $\lambda \neq 0$ and $R > 0$. If λ is real and if $\sqrt{\lambda}R$ is not a zero of J_m, then we have*

$$\frac{J_m(\sqrt{\lambda}r)}{J_m(\sqrt{\lambda}R)} \in \mathbb{R}, \qquad \textit{for all } r \geq 0.$$

Proof: By (A.3) we have $\frac{J_{-m}(\sqrt{\lambda}r)}{J_{-m}(\sqrt{\lambda}R)} = \frac{J_m(\sqrt{\lambda}r)}{J_m(\sqrt{\lambda}R)}$. We can therefore assume that $m \in \mathbb{N}_0$. Define the analytic function $w : \mathbb{C} \to \mathbb{C}$ by

$$w(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{k!(m+k)!} \left(\frac{x}{2}\right)^{2k+m}, \quad x \in \mathbb{C}.$$

Since the coefficients of this power series are real, this function satisfies $w(r) \in \mathbb{R}$ for all $r \geq 0$, and by (A.2) we have $J_m(\sqrt{\lambda}x) = \sqrt{\lambda}^m w(x)$ for all $x \in \mathbb{C}$. This yields $\frac{J_m(\sqrt{\lambda}r)}{J_m(\sqrt{\lambda}R)} = \frac{w(r)}{w(R)} \in \mathbb{R}$, for all $r \geq 0$. ■

Lemma A.3 *Let $m \in \mathbb{Z} \setminus \{0\}$, let $R > 0$, and $\lambda \in \mathbb{R}$.*
(i) If $\lambda < 0$, then we have for $0 \leq r \leq R$ that

$$0 < \frac{J_m(\sqrt{\lambda}r)}{J_m(\sqrt{\lambda}R)} \leq \left(\frac{r}{R}\right)^{|m|}, \tag{A.11}$$

and for $0 < r \leq R$ and $j = 1, 2$ that

$$\left| \frac{\frac{d^j}{dr^j} J_m(\sqrt{\lambda}r)}{J_m(\sqrt{\lambda}R)} \right| \leq \left(\frac{r}{R}\right)^{|m|-j} \left(\frac{|m|}{R}\right)^j K_j, \tag{A.12}$$

where $K_1 = \sqrt{1 + 2|\lambda|R^2}$ and $K_2 = K_1 + 1 + |\lambda|R^2$.
(ii) If $\lambda > 0$, then we have for $0 \leq r \leq R$ and for $m \in \mathbb{Z}$ with $|m| > \sqrt{\frac{4}{3}\lambda}\, R$ that

$$0 < \frac{J_m(\sqrt{\lambda}r)}{J_m(\sqrt{\lambda}R)} \leq \left(\frac{r}{R}\right)^{\frac{1}{2}|m|}, \tag{A.13}$$

and for $0 < r \leq R$, $j = 1, 2$ and $m \in \mathbb{Z}$ with $|m| > \sqrt{\frac{4}{3}\lambda}\, R$ that

$$\left| \frac{\frac{d^j}{dr^j} J_m(\sqrt{\lambda}r)}{J_m(\sqrt{\lambda}R)} \right| \leq \left(\frac{r}{R}\right)^{\frac{1}{2}|m|-j} \left(\frac{|m|}{R}\right)^j \hat{K}_j, \tag{A.14}$$

where $\hat{K}_1 = 1$ and $\hat{K}_2 = 2$.

Proof: From (A.3) we see that all the statements of the lemma are valid for negative m if they are valid for positive m. We therefore assume in the following that $m \in \mathbb{N}$. Set

$$u(r) = \frac{J_m(\sqrt{\lambda}r)}{J_m(\sqrt{\lambda}R)}. \tag{A.15}$$

By Lemma A.2 this function satisfies $u(r) \in \mathbb{R}$ for all $r \geq 0$. Moreover, we have $u(R) = 1$ and since (A.2) shows that $J_m(0) = 0$, we also have $u(0) = 0$. Using that J_m is a solution of (A.1) with $\nu = m$, a short computation yields that

$$u''(r) + \frac{1}{r}u'(r) + (\lambda - \frac{m^2}{r^2})u(r) = 0. \tag{A.16}$$

We multiply this equation by r. Since $ru''(r) + u'(r) = (ru'(r))'$, this yields

$$(ru')' + (\lambda - \frac{m^2}{r^2})ru = 0,$$

where we dropped the argument of u to simplify the notation. After multiplication of this equation with ru', the resulting equation can be written as

$$\Big(\frac{1}{2}(ru')^2\Big)' + (\lambda - \frac{m^2}{r^2})\Big(\frac{1}{2}(ru)^2\Big)' = (\lambda - \frac{m^2}{r^2})ru^2,$$

hence

$$\frac{d}{dr}\Big(\frac{1}{2}(ru')^2 + (\lambda - \frac{m^2}{r^2})\frac{1}{2}(ru)^2\Big) = \frac{m^2}{r^3}(ru)^2 + (\lambda - \frac{m^2}{r^2})ru^2 = \lambda ru^2.$$

Since $u(0) = 0$ and $(ru'(r))|_{r=0} = 0$, integration of this equation over the interval $[0, r]$ yields

$$\big(ru'(r)\big)^2 = (\frac{m^2}{r^2} - \lambda)\big(ru(r)\big)^2 + 2\lambda \int_0^r su(s)^2\, ds. \tag{A.17}$$

We consider now the cases $\lambda < 0$ and $\lambda > 0$ separately.

(i) Assume first that $\lambda < 0$. In this case the function $r \mapsto u(r)^2$ is monotonically increasing. To see this, suppose that this is not true. Since $u(0) = 0$, this implies that there is $a > 0$ such that u^2 assumes the maximum on the interval $[0, a]$ at a point $0 < r_0 < a$, whence $u^2(r_0) > 0$ and $0 = \big(u^2(r_0)\big)' = 2u(r_0)u'(r_0)$, thence $u'(r_0) = 0$. With

$$2\lambda \int_0^{r_0} su(s)^2\, ds \ge 2\lambda u(r_0)^2 \int_0^{r_0} s\, ds = \lambda\big(r_0 u(r_0)\big)^2 \tag{A.18}$$

we thus infer from (A.17) that

$$0 = (\frac{m^2}{r_0^2} - \lambda)\big(r_0 u(r_0)\big)^2 + 2\lambda \int_0^{r_0} su(s)^2\, ds \ge \frac{m^2}{r_0^2}\big(r_0 u(r_0)\big)^2 > 0.$$

Therefore $r \mapsto u(r)^2$ is increasing. As in (A.18), we thus obtain for all $r > 0$ that $2\lambda \int_0^r su(s)^2\, ds \ge \lambda\big(ru(r)\big)^2$, hence (A.17) implies

$$\big(ru'(r)\big)^2 = (\frac{m^2}{r^2} - \lambda)\big(ru(r)\big)^2 + 2\lambda \int_0^r su(s)^2\, ds \ge \frac{m^2}{r^2}\big(ru(r)\big)^2,$$

whence

$$\Big|\frac{u'(r)}{u(r)}\Big| \ge \frac{m}{r}.$$

Since $u(0) = 0$ and $u(R) = 1$ we see that $u'(r)$ and $u(r)$ must be positive, hence we can drop the absolute value signs on the left hand side of this inequality, which implies that

$$\big(\ln u(r)\big)' \ge \frac{m}{r}.$$

We integrate this inequality over the interval $[r, R]$ and note that $u(R) = 1$ to obtain

$$\ln \frac{1}{u(r)} = -\ln u(r) \geq m(\ln R - \ln r) = \ln \Big(\frac{R}{r}\Big)^m,$$

hence,

$$0 < u(r) \leq \Big(\frac{r}{R}\Big)^m, \tag{A.19}$$

which together with the definition of u in (A.15) proves (A.11).

To prove (A.12) for $j = 1$, devide (A.17) by r^2 and use inequality (A.19) to estimate $u(r)$ and $u(s)$ in the resulting equation. This yields the estimate

$$\begin{aligned} u'(r)^2 &\leq \big(\frac{m^2}{r^2} - \lambda\big)\Big(\frac{r}{R}\Big)^{2m} + \frac{2|\lambda|}{r^2}\int_0^r s\Big(\frac{s}{R}\Big)^{2m} ds = \Big(\frac{m^2}{r^2} - \lambda - \frac{2\lambda}{2m+2}\Big)\Big(\frac{r}{R}\Big)^{2m} \\ &= \Big(\frac{m^2}{R^2} - \lambda\frac{(2m+4)r^2}{(2m+2)R^2}\Big)\Big(\frac{r}{R}\Big)^{2m-2} \leq \Big(\frac{m^2}{R^2} - \lambda\frac{(m+2)}{(m+1)}\Big)\Big(\frac{r}{R}\Big)^{2m-2} \\ &\leq \frac{m^2}{R^2}\Big(1 + 2|\lambda|R^2\Big)\big(\frac{r}{R}\Big)^{2m-2}, \end{aligned} \tag{A.20}$$

which together with (A.15) proves (A.12) for $j = 1$. To prove (A.12) for $j = 2$, use (A.19) and (A.20) to estimate the terms $u(r)$ and $u'(r)$ in (A.16). This yields

$$\begin{aligned} |u''(r)| &\leq \frac{1}{r}\frac{m}{R}\sqrt{1 + 2|\lambda|R^2}\Big(\frac{r}{R}\Big)^{m-1} + \Big(\frac{m^2}{r^2} - \lambda\Big)\Big(\frac{r}{R}\Big)^m \\ &= \Big(\frac{m}{R^2}\sqrt{1 + 2|\lambda|R^2} + \frac{m^2}{R^2} - \lambda\frac{r^2}{R^2}\Big)\Big(\frac{r}{R}\Big)^{m-2} = \frac{m^2}{R^2}\Big(\frac{\sqrt{1 + 2|\lambda|R^2}}{m} + 1 + \frac{|\lambda|r^2}{m^2}\Big)\Big(\frac{r}{R}\Big)^{m-2} \\ &\leq \frac{m^2}{R^2}\big(\sqrt{1 + 2|\lambda|R^2} + 1 + |\lambda|R^2\big)\Big(\frac{r}{R}\Big)^{m-2}. \end{aligned}$$

This estimate together with (A.15) proves (A.12) for $j = 2$.

(ii) Now assume that $\lambda > 0$ and $0 < \sqrt{\lambda}R < m$. Then (A.17) yields

$$\big(ru'(r)\big)^2 - \big(\frac{m^2}{r^2} - \lambda\big)\big(ru(r)\big)^2 = 2\lambda\int_0^r su(s)^2\, ds \geq 0. \tag{A.21}$$

In Lemma A.1 we showed that J_m does not have a zero on the interval $(0, m)$. By definition of u in (A.15), this means that u is different from zero on the intervall $(0, \lambda^{-1/2}m)$. Consequently, (A.21) implies for $0 < r \leq R$ that

$$\Big|\frac{u'(r)}{u(r)}\Big| \geq \sqrt{\frac{m^2}{r^2} - \lambda}.$$

Since the right hand side of this inequality is positive for r from the interval $(0, \lambda^{-1/2}m)$, the derivative $u'(r)$ does not vanish on this interval. Since $u(0) = 0$ and $u(R) = 1$, this derivative must be positive, hence $u(r)$ must be positive on this interval. Therefore we can drop the absolute value signs on the left hand side of the last inequality, which yields that

$$\big(\ln u(r)\big)' \geq \sqrt{\frac{m^2}{r^2} - \lambda}.$$

We integrate this inequality over the interval $[r, R]$ and use $u(R) = 1$ to obtain

$$-\ln u(r) \geq \int_r^R \sqrt{\frac{m^2}{s^2} - \lambda}\, ds \geq \int_r^R \frac{m}{s}\sqrt{1 - \lambda\frac{R^2}{m^2}} ds = m\sqrt{1 - \lambda\frac{R^2}{m^2}} \ln\frac{R}{r},$$

hence

$$0 < u(r) \leq \Big(\frac{r}{R}\Big)^{m\sqrt{1-\lambda\left(\frac{R}{m}\right)^2}} \leq \Big(\frac{r}{R}\Big)^{\frac{1}{2}m}. \tag{A.22}$$

The last inequality sign holds for $m \geq \sqrt{\frac{4}{3}\lambda}\, R$, since in this case we have $\sqrt{1 - \lambda\left(\frac{R}{m}\right)^2} \geq \frac{1}{2}$. The inequality (A.22) together with (A.15) yields (A.13).

To prove (A.14) for $j = 1$, devide (A.17) by r^2 and use inequality (A.22) to estimate $u(r)$ and $u(s)$ in the resulting equation. We obtain

$$\begin{aligned} u'(r)^2 \leq \big(\frac{m^2}{r^2} - \lambda\big)\Big(\frac{r}{R}\Big)^m + \frac{2\lambda}{r^2}\int_0^r s\Big(\frac{s}{R}\Big)^m ds = \Big(\frac{m^2}{r^2} - \lambda + \frac{2\lambda}{m+2}\Big)\Big(\frac{r}{R}\Big)^m \\ \leq \frac{m^2}{r^2}\Big(\frac{r}{R}\Big)^m = \frac{m^2}{R^2}\Big(\frac{r}{R}\Big)^{m-2}. \end{aligned} \tag{A.23}$$

This inequality and (A.15) together yield (A.14) for $j = 1$. To prove (A.14) for $j = 2$, we use (A.22) and (A.23) to estimate the terms $u(r)$ and $u'(r)$ in (A.16) and obtain

$$|u''(r)| \leq \frac{1}{r}\frac{m}{R}\Big(\frac{r}{R}\Big)^{\frac{1}{2}m-1} + \big(\frac{m^2}{r^2} - \lambda\big)\Big(\frac{r}{R}\Big)^{\frac{1}{2}m} \leq \frac{m + m^2}{R^2}\Big(\frac{r}{R}\Big)^{\frac{1}{2}m-2} \leq 2\frac{m^2}{R^2}\Big(\frac{r}{R}\Big)^{\frac{1}{2}m-2}.$$

(A.14) with $j = 2$ follows from this estimate and from (A.15). ∎