Small deviations for stable processes via compactness properties of the parameter set

Frank Aurzada*

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Abstract

We establish a general lower bound for the small deviations of α-stable processes in terms of the metric entropy behaviour w.r.t. the Dudley metric. This generalises work by Talagrand (1993) for Gaussian processes and yields new bounds for stable self-similar processes.

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*Address: Technische Universität Berlin, Institut für Mathematik, Sekr. MA 7-5, Str. des 17. Juni 136, 10623 Berlin, Germany. Email: `aurzada@math.tu-berlin.de`

1 Introduction

While large deviation problems are well-studied for many classes of processes, the investigation of small deviation problems still lacks *general* tools to tackle whole classes of processes. Only for Gaussian processes several general tools are known. In this note, we generalise one of those tools to the symmetric α-stable case.

In the investigation of Gaussian and stable processes, the metric entropy approach has turned out far-reaching, cf. Samorodnitsky and Taqqu (1994). We generalise a result of Talagrand (1993) for Gaussian processes in this direction. The result and its proof can be found in a convenient formulation in Ledoux (1996), p. 256–258.

Note that there are some results concerning the relation between the entropy of the operator related to a Gaussian or stable process and the process' small deviation probabilities, cf. Kuelbs and Li (1993), Li and Linde (1999) for the Gaussian, and Li and Linde (2004), Aurzada (2006) for the stable case. However, here we relate the entropy of the parameter set w.r.t. the Dudley metric to the small deviation probabilities.

Lifshits and Simon (2005) undertook a detailed investigation of the small deviation problem for a very important example of stable processes under various kinds of norms. However, results that apply to a whole class of processes almost do not exist except for those of Samorodnitsky (1998), where self-similar processes are treated. We extend the results for these examples.

In Section 2, we give the main result and its proof. Some applications are listed in Section 3. In particular, we apply the results to the class of stable self-similar processes with stationary increments, we investigate the question for stable processes with integral representation, and we treat the case of weighted sums of independent stable random variables.

2 Result

Let $X = (X_t)_{t \in T}$ be a real-valued symmetric α-stable process with parameter set $T \neq \emptyset$; i.e. assume that $(X_{t_1}, \ldots, X_{t_n})$ is a symmetric α-stable vector, for all choices $t_1, \ldots, t_n \in T$, cf. Samorodnitsky and Taqqu (1994). Since we investigate the supremum of the process, we have to assume that it is measurable; i.e. we assume that there exists a separable version.

Let us define the Dudley metric related to X by letting $d_X(t, s)$ denote the scale parameter of the random variable $X_t - X_s$. We recall from Property 1.2.17 in Samorodnitsky and Taqqu (1994) that $d_X(t,s)$ equals $(\mathbb{E}|X_t - X_s|^r)^{1/r}$ up to a constant depending on $r < \alpha$ and α.

We assume that (T, d_X) is a precompact quasi-metric space. Then the covering numbers are defined in the usual way:

$$N(T, d_X, \varepsilon) := \min\{n \in \mathbb{N} \mid \exists t_1, \ldots, t_n \in T \ \forall t \in T \ \exists i \ : \ d_X(t, t_i) \leq \varepsilon\}.$$

Furthermore, we use $f(\varepsilon) \preceq g(\varepsilon)$ if $\limsup_{\varepsilon \to 0} f(\varepsilon)/g(\varepsilon) < \infty$; and $f(\varepsilon) \approx g(\varepsilon)$ if $f(\varepsilon) \preceq g(\varepsilon)$ and $g(\varepsilon) \preceq f(\varepsilon)$ hold.

With the above notation, we can now state our main theorem.

Theorem 1. *Let*

$$N(T, d_X, \varepsilon) \leq \psi(\varepsilon), \qquad \text{for all } 0 < \varepsilon < \varepsilon_0, \tag{1}$$

where $\psi : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ and there are constants c_1, c_2 with $1 < c_1 \leq c_2 < 2^\alpha$ (for $\alpha = 2$, only $1 < c_1 \leq c_2 < \infty$ is required) such that

$$c_1 \psi(\varepsilon) \leq \psi(\varepsilon/2) \leq c_2 \psi(\varepsilon), \qquad \text{for all } 0 < \varepsilon < \varepsilon_1.$$

Then

$$-\log \mathbb{P}\left(\sup_{t,s \in T} |X_t - X_s| \leq \varepsilon\right) \preceq \psi(\varepsilon).$$

Remark: For $1 \leq \alpha < 2$, a particular consequence of the assumptions is that the process is almost surely bounded, by a version of Dudley's Theorem for stable processes (cf. Samorodnitsky and Taqqu (1994), Theorem 12.2.1). For $0 < \alpha < 1$, our Theorem 1 provides a new Dudley-type theorem, since it implies boundedness, by the Zero-one law in Corollary 9.5.5 in Samorodnitsky and Taqqu (1994).

Proof of Theorem 1: The case $\alpha = 2$ was treated by Ledoux (1996); we thus concentrate on the proof for $\alpha < 2$.

Let $h := \log c_2 / \log 2$. Then, by assumption, $0 < h < \alpha$ and thus

$$c_1 \psi(\varepsilon) \leq \psi(\varepsilon/2) \leq 2^h \psi(\varepsilon). \tag{2}$$

Since (T, d_X) is assumed to be precompact, $D(T) := \sup_{t,s} d_X(t,s) < \infty$. Let n_0 be the largest number in $\mathbb{Z}$ such that $2^{-n_0} \geq D(T)$.

We proceed as in Ledoux (1996), p. 257, where the Gaussian case is treated, with some modifications inspired by Lifshits and Simon (2005). Analogously to the former, we let $N(n) := N(T, d_X, 2^{-n})$ and $T_n \subseteq T$ be a subset of cardinality $N(n)$ with $d_X(T, T_n) \le 2^{-n}$. Furthermore, let $s_{n-1}(t)$ denote a point in T_{n-1} with $d_X(t, s_{n-1}(t)) \le 2^{-n+1}$ and define $\mathcal{Y}_n := \{X_t - X_{s_{n-1}(t)}, t \in T_n\}$. Then the scale parameter of any of the stable random variables $Y_n \in \mathcal{Y}_n$ is less than 2^{-n+1}. Furthermore, the cardinality of $\mathcal{Y}_n$ is at most $N(n)$. By separability, for every t there are appropriate $Y_n \in \mathcal{Y}_n$ such that

$$X_t = X_{t_0} + \sum_{n > n_0} Y_n. \tag{3}$$

Since $h < \alpha$, we can find a δ with $0 < \delta < 1 - h/\alpha$. Fix $q > n_0$ and let

$$b_n := 2^{-n+1+(n-q)-\delta|n-q|}.$$

Then it is elementary to see that $\sum_{n>n_0} b_n \le c_\delta 2^{-1}\, 2^{-q}$, for some constant c_δ only depending on δ. Therefore, using (3),

$$\begin{aligned}
\mathbb{P}\left(\sup_{t,s\in T} |X_t - X_s| \le c_\delta 2^{-q}\right) &\ge \mathbb{P}\left(\forall n > n_0 \, \forall Y_n \in \mathcal{Y}_n, |Y_n| \le b_n\right) \\
&\ge \prod_{n>n_0} \mathbb{P}\left(|g| \le b_n 2^{n-1}\right)^{N(n)},
\end{aligned}$$

where g is a standard symmetric α-stable random variable and we used Šidák's Inequality (cf. Lemma 2.1 in Samorodnitsky (1998)).

Taking logarithms and using (1), we obtain

$$\begin{aligned}
&\log \mathbb{P}\left(\sup_{t,s\in T} |X_t - X_s| \le c_\delta 2^{-q}\right) \\
&\ge \sum_{n=n_0+1}^{q-1} \psi(2^{-n}) \log \mathbb{P}\left(|g| \le b_n 2^{n-1}\right) + \sum_{n\ge q} \psi(2^{-n}) \log \mathbb{P}\left(|g| \le b_n 2^{n-1}\right).
\end{aligned} \tag{4}$$

Let us estimate the first sum using (2) and the behaviour of the distribution of g at the origin:

$$\begin{aligned}
\sum_{n_0+1}^{q-1} \psi(2^{-n}) \log \mathbb{P}\left(|g| \le b_n 2^{n-1}\right) &\ge \sum_{n=1}^{q-1} c_1^{n-q} \psi(2^{-q}) \log \mathbb{P}\left(|g| \le 2^{(1+\delta)(n-q)}\right) \\
&\ge \psi(2^{-q}) \sum_{n=1}^{q-1} c_1^{n-q} \log\left(c 2^{(1+\delta)(n-q)}\right).
\end{aligned}$$

Since $c_1 > 1$, the latter term can be estimated by $-C_1\psi(2^{-q})$, where $C_1 > 0$ is some constant depending on δ, c, and c_1 only. Let us come to the second term in (4) now. Here we use (2) and the tail behaviour of g:

$$\sum_{n\geq q}\psi(2^{-n})\log\mathbb{P}\left(|g|\leq b_n 2^{n-1}\right)=\sum_{n\geq q}\psi(2^{-n})\log\left(1-\mathbb{P}\left(|g|>2^{(1-\delta)(n-q)}\right)\right)$$
$$\geq -C\sum_{n\geq q}2^{h(n-q)}\psi(2^{-q})\mathbb{P}\left(|g|>2^{(1-\delta)(n-q)}\right)$$
$$\geq -C'\psi(2^{-q})\sum_{n\geq q}2^{h(n-q)}2^{-\alpha(1-\delta)(n-q)}.$$

Since $h-\alpha(1-\delta)<0$, the latter term can be estimated by $-C_2\psi(2^{-q})$, where $C_2 > 0$ is some constant depending on δ, α, and h only.

Thus we have seen that

$$\log\mathbb{P}\left(\sup_{t,s\in T}|X_t-X_s|\leq c_\delta 2^{-q}\right)\geq -C\psi(2^{-q}),$$

for all $q > n_0$ and some constant $C = C(\alpha, c, c_1, h) > 0$. Using the standard argument from Ledoux (1996), p. 259, this implies the assertion. ■

The easiest case is the one where ψ behaves as a polynomial. In this case, we immediately obtain the following corollary.

Theorem 2. *Let* $N(T, d_X, \varepsilon) \leq M\varepsilon^{-h}$, *for all* $0 < \varepsilon < \varepsilon_0$, *where* $M > 0$ *and* $0 < h < \alpha$. *Then* $-\log\mathbb{P}\left(\sup_{t,s\in T}|X_t - X_s| \leq \varepsilon\right) \preceq \varepsilon^{-h}$.

Note that the assumption on h in the above corollary cannot be relaxed, since there are unbounded stable processes with $N(T, d_X, \varepsilon) \approx \varepsilon^{-\alpha}$, cf. Samorodnitsky and Taqqu (1994).

However, one can ask for a relaxation of the conditions on ψ in Theorem 1: on the one hand, the case of slowly varying ψ (i.e. $c_1 = 1$) is an interesting open question even in the Gaussian case. On the other hand, it is not clear what happens if the process is bounded but only $c_2 = 2^\alpha$ is possible. These situations will be investigated in a future work by Aurzada and Lifshits (2007).

3 Applications

3.1 Self-similar processes

We can now easily obtain the following general result for processes with parameter set $T = [a, b]$.

Corollary 1. *Let $d_X(t, s) \le c|t - s|^H$ with $H > 1/\alpha$ and some $c > 0$. Then*

$$- \log \mathbb{P} \left(\sup_{t,s \in T} |X_t - X_s| \le \varepsilon \right) \preceq \varepsilon^{-1/H}. \tag{5}$$

Similarly, one can argue with other parameter sets that already possess a metric d satisfying $d_X \le d$.

Of great importance are symmetric stable H-self-similar processes with stationary increments (H-sssi processes), cf. Chapter 7 in Samorodnitsky and Taqqu (1994). Samorodnitsky (1998) carried out a study of their small deviation probabilities. We improve the lower bound by removing the unnecessary logarithmic term, as conjectured by Samorodnitsky.

Corollary 2. *If X is an H-sssi process with $H > 1/\alpha$ then (5) holds.*

Proof: By Corollary 7.3.4 in Samorodnitsky and Taqqu (1994), $d_X(t, s) = c|t - s|^H$. ∎

This bound cannot be improved in general, as shown by Samorodnitsky (1998), which means that the general bound in Theorem 1 cannot be improved either.

One of the most important examples of stable H-sssi processes is the so-called linear fractional α-stable motion (LFSM), which is one possible stable generalisation of the fractional Brownian motion. By the LFSM we mean the process $L_{\alpha,H}(a, b; t)$ from Definition 7.4.1 in Samorodnitsky and Taqqu (1994).

A detailed study of the small deviation problem of $L_{\alpha,H}(1, 0; t)$ can be found in Lifshits and Simon (2005). There, upper and lower bounds were obtained under various kinds of norms; in particular, under the supremum norm for $H > 1/\alpha$. Also of particular interest is the so-called well-balanced LFSM $L_{\alpha,H}\,(1, 1; t)$.

The above result on H-sssi processes yields a lower bound for the small deviations of $L_{\alpha,H}(a, b; t)$ under the supremum norm for all a, b and $H > 1/\alpha$.

The respective upper bound can be established with help of Theorem 4.5 in Li and Linde (2004). This clarifies the small deviation rate for all LFSMs.

3.2 Hölder continuous kernels

Let us consider the case of symmetric α-stable processes given by an integral representation:

$$X_t = \int_S K(t,x)\, dM(x), \qquad t \in T,$$

where M is a symmetric α-stable random measure with finite control measure m. In this case, the distance d_X can be calculated by

$$d_X(t,s) = \left(\int_S |K(t,x) - K(s,x)|^\alpha \, dm(x) \right)^{1/\alpha}.$$

Given the kernel K, it is usually easy to calculate d_X using the above formula. One example is as follows.

Corollary 3. *Let the kernel K be H-Hölder with $1/\alpha < H \leq 1$, i.e. $|K(t,x) - K(s,x)| \leq C|t-s|^H$, for all $t, s \in T$, $x \in S$. Then (5) holds.*

3.3 Sequences of independent random variables

Let $\theta_1, \theta_2, \ldots$ be i.i.d. standard symmetric α-stable random variables. We consider a function $S : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ that is regularly varying at infinity and strictly decreasing with $S(x) \to 0$ for $x \to \infty$. Then we study the small deviation problem for the stochastic process $(X_n)_{n\in\mathbb{N}} := (S(n)\theta_n)_{n\in\mathbb{N}}$.

It is easy to calculate that $d_X(n,m) = (S(n)^\alpha + S(m)^\alpha)^{1/\alpha}$, for $n \neq m$, and 0, for $n = m$. Using this, one obtains that $N(\mathbb{N}, d_X, \varepsilon) \leq S^{-1}(\varepsilon/c)$, for some constant $c > 0$. If S is regularly varying at infinity with exponent less than $-1/\alpha$, $\psi(\varepsilon) := S^{-1}(\varepsilon/c)$ satisfies the conditions of Theorem 1. We thus obtain that

$$-\log \mathbb{P}\left(\sup_{n\in\mathbb{N}} |S(n)\theta_n| \leq \varepsilon \right) \preceq S^{-1}(\varepsilon).$$

The respective upper bound was obtained in Theorem 3.1 in Aurzada (2007) in a more general context w.r.t. the law of the θ_n, but under the additional condition that S is differentiable (which is not a restriction for the problem we are considering) and that S' is ultimately increasing.

Corollary 4. *Let $\theta_1, \theta_2, \ldots$ be i.i.d. standard symmetric α-stable random variables and let S be a strictly decreasing function such that S' exists and is ultimately increasing. Assume furthermore that S is regularly varying at infinity with exponent less than $-1/\alpha$. Then*

$$-\log \mathbb{P}\left(\sup_{n\in\mathbb{N}} |S(n)\theta_n| \leq \varepsilon\right) \approx S^{-1}(\varepsilon).$$

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