UNIFIED GREEDY APPROXIMABILITY BEYOND SUBMODULAR MAXIMIZATION*

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Abstract. We consider classes of objective functions of cardinality-constrained maximization problems for which the greedy algorithm guarantees a constant approximation. We propose the new class of γ-α-augmentable functions and prove that it encompasses several important subclasses, such as functions of bounded submodularity ratio, α-augmentable functions, and weighted rank functions of an independence system of bounded rank quotient – as well as additional objective functions for which the greedy algorithm yields an approximation. For this general class of functions, we show a tight bound of $\frac{\alpha}{\gamma} \cdot \frac{e^\alpha}{e^\alpha - 1}$ on the approximation ratio of the greedy algorithm that tightly interpolates between bounds from the literature for functions of bounded submodularity ratio and for α-augmentable functions. In particular, as a by-product, we close a gap in [Math.Prog., 2020] by obtaining a tight lower bound for α-augmentable functions for all $\alpha \geq 1$. For weighted rank functions of independence systems, our tight bound becomes $\frac{\alpha}{\gamma}$, which recovers the known bound of $1/q$ for independence systems of rank quotient at least q.

Key words. greedy algorithm, approximation ratio, cardinality-constrained maximization, independence system, submodularity ratio, augmentability

MSC codes. 68W25, 90C27, 68Q25

1. Introduction. We consider cardinality-constrained maximization problems of the form

$$\begin{aligned} \max\ & f(X) \\ \text{s.t. } & |X| \leq k \\ & X \subseteq U, \end{aligned}$$

with a *monotone* objective function $f\colon 2^U \to \mathbb{R}_{\geq 0}$ over a finite ground set U. Additional constraints of the form $X \in \mathcal{X}$ can be modeled by the monotone objective $f'(X) := \max\{f(Y) | Y \in 2^X \cap \mathcal{X}\}$. In this way, every combinatorial, cardinality-constrained maximization problem with monotone objective can be captured, and we adopt this framework throughout the paper.[1] For example, the maximum weighted matching problem on a graph $G = (V, E)$ with edge weights $w\colon E \to \mathbb{R}_{\geq 0}$ yields the objective function $f(X \subseteq E) = \max\{\sum_{e \in M} w(e) | M \subseteq X, M \text{ is a matching in } G\}$.

We focus on the performance of the *greedy algorithm*. This algorithm iteratively produces a solution $S^{\mathrm{G}}_{f,k} = \{x_1, \ldots, x_k\}$ with

$$x_i \in \arg\max_{x \in U \setminus \{x_1, \ldots, x_{i-1}\}} f(\{x_1, \ldots, x_{i-1}\} \cup \{x\}),$$

for all $i \in [k] := \{1, \ldots, k\}$, i.e., it adds elements such that the increase in objective value is maximized in each step. The greedy algorithm is inherently incremental and may be regarded as the most natural approach for incrementally building up infrastructures that support changing active solutions (in the sense of the definition $f'(X)$ above). While this algorithm is widely used in practical applications, greedy solutions can be arbitrarily far away from optimal (e.g., for the knapsack problem).

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[1]Note that the objective function f may be computationally hard to evaluate. If we assume that the greedy algorithm has oracle access to f, it requires $O(|U|k)$ queries to the oracle.

A natural question in this context is, for which objective functions f the greedy algorithm gives a good solution. We are interested in characterizing these objective functions.

Note that we consider the *adaptive* greedy solution $S^{\mathrm{G}}_{f,k}$ as opposed to the *non-adaptive* greedy solution $\tilde{S}^{\mathrm{G}}_{f,k} := S^{\mathrm{G}}_{f,\min\{k,\bar{k}\}}$, where $\bar{k} \in [|U|]$ is the smallest cardinality such that $f(S^{\mathrm{G}}_{\bar{k}} \cup \{x\}) = f(S^{\mathrm{G}}_{\bar{k}})$ for all $x \in U \setminus S^{\mathrm{G}}_{\bar{k}}$. In other words, the non-adaptive greedy algorithm terminates as soon as it cannot improve the solution further. This non-adaptive variant of the greedy algorithm has often been considered in the early literature (e.g., [15, 16, 22, 23]). Note, that for submodular functions, i.e., functions with $f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y)$ for all $X, Y \subseteq U$, there is no difference between these two variants, and for our purposes both variants are interchangeable in the following sense.

Formally, we measure the quality of the greedy algorithm on a set of objectives $\mathcal{F}$ by the approximation ratio $\sup_{f\in\mathcal{F}} \max_{k\in[|U_f|]} f(S^*_{f,k})/f(X_{f,k})$, where U_f is the ground set of the function $f \in \mathcal{F}$, $S^*_{f,k} \in \arg\max_{X\subseteq U:|X|\leq k} f(X)$ denotes an optimum solution of cardinality at most k, and $X_{f,k} \in \{S^{\mathrm{G}}_{f,k}, \tilde{S}^{\mathrm{G}}_{f,k}\}$ refers to the (non-)adaptive greedy solution of cardinality k. We claim that the approximation ratios of both variants of the greedy algorithm coincide. To see this, observe that the non-adaptive setting is more restrictive, and that every lower bound instance in the non-adaptive setting can be made adaptive by introducing additional elements that add a vanishingly small but positive objective value when added to every solution. This implies that all our bounds on the approximation ratio of the (adaptive) greedy algorithm immediately apply to both variants.

From now on, we write $S^{\mathrm{G}}_k := S^{\mathrm{G}}_{f,k}$ and $S^*_k := S^*_{f,k}$, whenever f is clear from the context. In these terms, we are interested in characterizing the set of objectives for which the greedy algorithm has a bounded approximation ratio. Known examples include the objectives of maximum (weighted) (b-)matching, maximum (weighted) coverage, and many more [2, 3, 8, 18, 30], and we additionally introduce a multi-commodity flow problem (Section 2), where the greedy algorithm yields an approximation.

A well-known class of functions for which the greedy algorithm has a bounded approximation ratio of (exactly) $\frac{\mathrm{e}}{\mathrm{e}-1}$ are the monotone, submodular functions [22]. This class includes the maximum coverage problem, but fails to capture many other greedily approximable settings. See Figure 1 along with the following.

Das and Kempe [8] introduced the class of functions of bounded *submodularity ratio* as a generalization of submodular functions. Importantly, its definition depends on the greedy solutions for different cardinalities. We adapt and weaken the definition from [8] for consistency, by restricting ourselves to greedy solutions and by minimizing over all cardinalities.

Definition 1.1 ([8]). *The weak submodularity ratio of $f\colon 2^U \to \mathbb{R}_{\geq 0}$ is (using $\frac{0}{0} := 1$)*

$$\gamma(f) := \min_{X\in\{S^G_0,\dots,S^G_k\},Y\subseteq U\setminus X} \frac{\sum_{y\in Y}(f(X\cup\{y\}) - f(X))}{f(X\cup Y) - f(X)} \in [0,1].$$

Das and Kempe [8] showed an upper bound of $\frac{\mathrm{e}^\gamma}{\mathrm{e}^\gamma-1}$ on the approximation ratio of the greedy algorithm for the set of all monotone functions with submodularity ratio at least $\gamma > 0$, and Bian et al. [3] extended this to a tight bound that is additionally parameterized by the curvature of the objective. Since submodular functions have submodularity ratio 1, this bound generalizes the submodular bound. Crucially, it is

easy to verify that these results carry over to the set $\tilde{\mathcal{F}}_\gamma$ of all monotone functions with *weak* submodularity ratio at least $\gamma > 0$.[2]

Another generalization of submodularity was proposed by Bernstein et al. [2]. We extend the definition by a weakened variant in order to bring it more in line with Definition 1.1.

DEFINITION 1.2 ([2]). *The function $f\colon 2^U \to \mathbb{R}_{\geq 0}$ is (weakly) α-augmentable for $\alpha \geq 1$, if, for every $X \subseteq U$ ($X \in \{S_0^G, \dots, S_{\bar{k}}^G\}$) and $Y \subseteq U$ with $Y \not\subseteq X$, there exists an element $y \in Y \setminus X$ with*

$$f(X \cup \{y\}) - f(X) \geq \frac{f(X \cup Y) - \alpha f(X)}{|Y|}.$$

Bernstein et al. showed that the greedy algorithm has an approximation ratio of at most $\alpha \cdot \frac{e^\alpha}{e^\alpha - 1}$ on the set $\mathcal{F}_\alpha$ of monotone, α-augmentable functions, for $\alpha \geq 1$, and that this bound is tight for $\alpha \in \{1, 2\}$ and in the limit $\alpha \to \infty$. Since submodular functions are 1-augmentable, this bound again generalizes the submodular bound. The class of α-augmentable problems captures the objective of the maximum (weighted) α-dimensional matching problem, which is not submodular. In this paper, we introduce a natural α-commodity flow variant that is α-augmentable, and we prove a tight lower bound on the approximation ratio for all $\alpha \geq 1$.

Another well-known setting, besides submodularity, where the greedy algorithm has a bounded approximation ratio, are weighted rank functions of independence systems of bounded rank quotient [17]. An *independence system* is a tuple $(U, \mathcal{I} \subseteq 2^U)$, where $\mathcal{I}$ is closed under taking subsets and $\emptyset \in \mathcal{I}$. For a given weight function $w\colon U \to \mathbb{R}_{\geq 0}$, the *weighted rank function* of $(U, \mathcal{I})$ is given by $f(X) = \max\{\sum_{x \in Y} w(x) | Y \in \mathcal{I} \cap 2^X\}$. The *rank quotient* of an independence system $(U, \mathcal{I})$ is $q(U, \mathcal{I}) := \min_{X \subseteq U} \min_{B, B' \in \mathcal{B}(X)} |B|/|B'|$, where we set $\frac{0}{0} := 1$, and the set $\mathcal{B}(X)$ of all *bases* of some set $X \subseteq U$ is defined to be the set of inclusion-wise maximal subsets of $\mathcal{I} \cap 2^X$, i.e., $\mathcal{B}(X) := \{B \in \mathcal{I} \cap 2^X | \forall x \in X \setminus B\colon B \cup \{x\} \notin \mathcal{I}\}$. Jenkyns [15] and Korte and Hausmann [16] showed that the greedy algorithm has an approximation ratio of exactly $1/q$ on the set $\mathcal{F}_q$ of all weighted rank functions of independence systems with rank quotient at least $q > 0$.[3]

Our results. Our goal is to unify and to generalize the above classes of functions on which the greedy algorithm has a bounded approximation ratio. To this end, we first observe that each one of the classes $\tilde{\mathcal{F}}_\gamma$, $\mathcal{F}_\alpha$, and $\mathcal{F}_q$ uniquely captures greedily approximable objectives (cf. Figure 1 and Propositions 3.1, 3.2, 3.3, and 4.10). In particular, we construct a natural α-augmentable variant of multi-commodity flow that does not have bounded (weak) submodularity ratio (for $\alpha \in \mathbb{N} \setminus \{1\}$) and cannot be expressed as the maximization of a weighted rank function. Besides the α-dimensional matching problem, to our knowledge, the problem introduced in Section 2 is the only other natural α-augmentable problem to date.

PROPOSITION 1.3. *For every $\gamma, q \in (0, 1)$ and $\alpha \geq 1$, it holds that*

$$\tilde{\mathcal{F}}_\gamma \not\subseteq (\mathcal{F}_\alpha \cup \mathcal{F}_q) \quad \text{and} \quad \mathcal{F}_\alpha \not\subseteq (\tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_q) \quad \text{and} \quad \mathcal{F}_q \not\subseteq (\tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_\alpha).$$

This motivates the following definition to consolidate all three classes.

[2]Here and throughout we use the notation $\tilde{\mathcal{F}}$ as opposed to $\mathcal{F}$ to refer to a function class based on a *weak* definition.

[3]Note that we abuse notation, since, e.g., $\mathcal{F}_\alpha \neq \mathcal{F}_q$ for $\alpha = q = 1$. However, the set of functions we are referring to will always be clear by the naming of the indices.

Definition 1.4. *The function $f\colon 2^U \to \mathbb{R}_{\geq 0}$ is (weakly) γ-α-augmentable for $\gamma \in (0,1]$ and $\alpha \geq \gamma$ if, for all sets $X \subseteq U$ ($X \in \{S_0^G, ..., S_k^G\}$) and all $Y \subseteq U$ with $Y \not\subseteq X$, there exists $y \in Y$ with*

$$f(X \cup \{y\}) - f(X) \geq \frac{\gamma f(X \cup Y) - \alpha f(X)}{|Y|}.$$

Note that we need to consider the weak variant of this definition if we hope to encompass the class $\tilde{\mathcal{F}}_\gamma$, which enforces its defining property only for "greedy sets", however, any upper bound on the approximation ratio immediately carries over to the same bound in the stronger definition. Also note that γ-α-augmentability only requires $\alpha \geq \gamma$, unlike α-augmentability where $\alpha \geq 1$. This is in line with the definitions of α-augmentability where $\gamma = 1$ and of the submodularity ratio where $\alpha = \gamma$. We let $\tilde{\mathcal{F}}_{\gamma,\alpha}$ denote the set of all weakly γ-α-augmentable functions. The first part of our main result is that this set encompasses all functions in $\tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_\alpha \cup \mathcal{F}_q$ and captures additional functions (cf. Figure 1). Formally, we show the following (cf. Propositions 4.2 and 4.9).

Theorem 1.5. *For every $\gamma, q \in (0,1]$, every $\gamma' \in (0,1)$, every $\alpha \geq 1$, and every $\alpha' \geq \gamma'$, it holds that*

$$\tilde{\mathcal{F}}_{\gamma,\max\{\alpha,1/q\}} \supseteq \tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_\alpha \cup \mathcal{F}_q \qquad \text{and} \qquad \tilde{\mathcal{F}}_{\gamma',\alpha'} \not\subseteq \tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_\alpha \cup \mathcal{F}_q.$$

Note that α' and γ' in Theorem 1.5 do not depend on α, γ and q. The second part of our main result is a tight bound on the approximation ratio of the greedy algorithm on $\tilde{\mathcal{F}}_{\gamma,\alpha}$ (cf. Theorems 4.3 and 4.7).

Theorem 1.6. *The approximation ratio of the greedy algorithm on the class $\tilde{\mathcal{F}}_{\gamma,\alpha}$ of monotone, weakly γ-α-augmentable functions, with $\gamma \in (0,1]$ and $\alpha \geq \gamma$, is exactly*

$$\frac{\alpha}{\gamma} \cdot \frac{\mathrm{e}^\alpha}{\mathrm{e}^\alpha - 1}.$$

Importantly, this bound recovers exactly the known bound for functions of bounded submodularity ratio, since $\tilde{\mathcal{F}}_\gamma \subseteq \tilde{\mathcal{F}}_{\gamma,\gamma}$, as well as the known bound for α-augmentable functions, since $\mathcal{F}_\alpha \subseteq \tilde{\mathcal{F}}_{1,\alpha}$. In that sense, our new bound interpolates tightly between these two bounds and generalizes them. In addition, our tight lower bound for $\tilde{\mathcal{F}}_{1,\alpha}$ is obtained with an α-augmentable function. This means that, in particular, we are able to close the gap left in [2], by showing a tight lower bound for α-augmentable objectives, for all $\alpha \geq 1$ (cf. Propositions 4.6 and 4.8).

Corollary 1.7. *The approximation ratio of the greedy algorithm on the class $\mathcal{F}_\alpha$ of monotone, α-augmentable functions is exactly $\alpha \cdot \frac{\mathrm{e}^\alpha}{\mathrm{e}^\alpha - 1}$ for all $\alpha \geq 1$.*

Finally, we are also able to show a tight bound of α/γ for γ-α-augmentable, weighted rank functions on independence systems (cf. Propositions 4.11 and 4.12). Since $\mathcal{F}_q \subseteq \tilde{\mathcal{F}}_{1,1/q}$ (by Theorem 1.5), our bound recovers exactly the known bound of $1/q$ for the approximation ratio of the greedy algorithm when the rank quotient is bounded from below by $q > 0$. This means that the class of monotone, weakly γ-α-augmentable functions truly unifies and generalizes the three classes $\tilde{\mathcal{F}}_\gamma$, $\mathcal{F}_\alpha$, and $\mathcal{F}_q$ of greedily approximable functions (cf. Figure 1). Note that, in particular, the lower bound is tight already for α-augmentable functions, which implies a tight bound of α for the approximation ratio of the greedy algorithm on α-augmentable weighted rank functions.

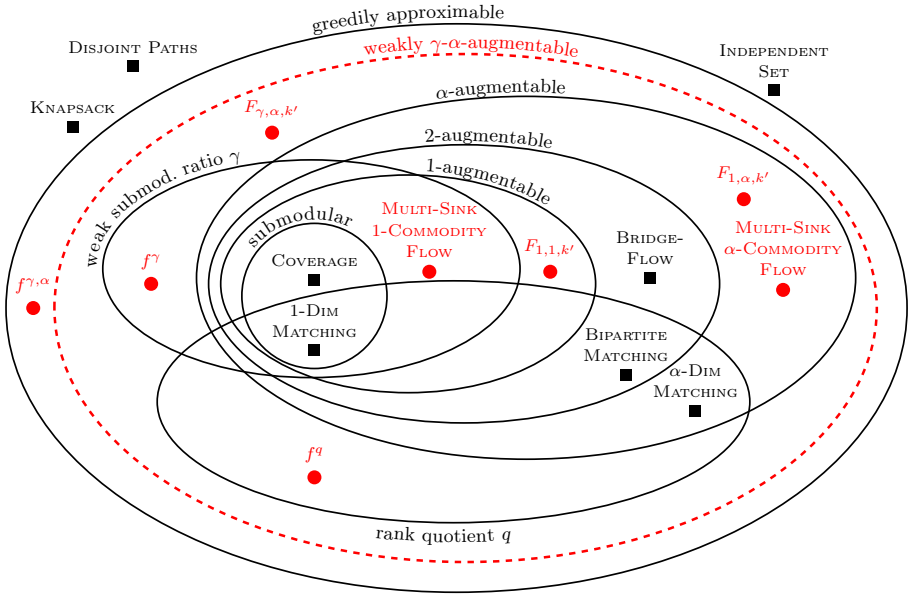

FIG. 1. *Relation of the different problem classes (nodes) and objective properties (ellipses). Anything that is contained within one ellipse has the property the ellipse stands for. Newly introduced classes and problems are marked in red with dashed lines and round nodes. The parameter k' is chosen sufficiently large, depending on γ and α.*

THEOREM 1.8. *Let $\mathcal{F}_{\text{IS}} := \bigcup_{q\in(0,1]} \mathcal{F}_q$ be the set of weighted rank functions on some independence system. The approximation ratio of the greedy algorithm on the class $\tilde{\mathcal{F}}_{\gamma,\alpha} \cap \mathcal{F}_{\text{IS}}$, with $\gamma \in (0,1]$ and $\alpha \geq \gamma$, is exactly $\frac{\alpha}{\gamma}$.*

Related Work. We can view our cardinality-constrained maximization framework as a special case of maximization over an independence system. In particular, the cardinality-constraint can be expressed as a uniform matroid constraint [17]. From that perspective, the most basic, non-trivial setting is the maximization of a linear (i.e., modular) objective over an independence system. Regarding the approximation ratio of the greedy algorithm, this classic setting is equivalent to the maximization of a weighted rank function, as considered in Theorem 1.8. This is easy to see by considering the non-adaptive variant of the greedy algorithm, and by observing that the greedy solution is guaranteed to remain feasible while the algorithm makes progress (cf. Lemma 4.1).

In that sense, the perfomance of the greedy algorithm for weighted rank function maximization has extensively been studied in the past. Rado [25] showed that the greedy algorithm is optimal for all weight functions if the underlying independence system is a matroid, and Edmonds [10] established the reverse implication. Jenkyns [15] extended this result by showing an upper bound of $1/q$ for the approximation ratio of the greedy algorithm on independence systems with rank quotient q, and Korte and Hausmann [16] gave a tight lower bound. Years later, Mestre [21] independently proved this tight bound for the subclass of k-extendible independence systems. Bouchet [4] gave a different generalization of the result by Rado and Edmonds by showing that the greedy algorithm remains optimal on symmetrical matroids.

Another prominent setting is the maximization of a submodular function over an independence system. Again, this includes cardinality-constrained maximization of a

submodular objective, which is equivalent to submodular maximization over a uniform matroid. Nemhauser, Wolsey, and Fisher [23] showed that the greedy algorithm has a tight approximation ratio of $\frac{\mathrm{e}}{\mathrm{e}-1}$ for maximizing a monotone, submodular function under a cardinality-constraint. Krause et al. [19] observed that the approximation ratio is unbounded when maximizing the minimum of two monotone, submodular functions. Non-monotone submodular maximization over a cardinality-constraint (and knapsack constraints) was considered by Lee et al. [20]. Feldman et al. [14] analyzed a variant of the continuous greedy algorithm [28] and showed an upper bound on its approximation ratio of $(1/e - o(1))^{-1}$. This bound for the non-monotone case with cardinality-constraint was later improved by Buchbinder et al. [5] and Ene and Nguyen [11] by further adapting the (continuous) greedy algorithm. For maximizing a submodular function subject to k-extentible system and k-systems constraints, Feldman et al. [12, 13] considered three variants of the greedy algorithm, a repeated greedy, a sample greedy and a simultaneous greedy. They were able to show approximation ratios of $k + O(1)$ for k-extendible system constraints and $k + O(\sqrt{k})$ for k-system constraints.

Maximization of a monotone, submodular function over a matroid was considered by Vondrák [28] and by Calinescu et al. [6], who showed that the continuous greedy algorithm has an approximation ratio of $\frac{\mathrm{e}}{\mathrm{e}-1}$ in this setting. Nemhauser, Wolsey, and Fisher [23], showed an upper bound of $p + 1$ for the regular greedy algorithm when maximizing over the intersection of p matroids. A generalization of this upper bound to the setting of maximizing subject to a p-system constraint was later proven by Calinescu et al. [6]. Conforti and Cornuejols [7] gave an upper bound of $p + c$ depending on the curvature c of the monotone submodular function – this interpolates between the submodular bound of [23] ($c = 1$) and the linear bound of [16] ($c = 0$). Vondrák [29] showed that the continuous greedy algorithm has an approximation ratio of at most $c\frac{\mathrm{e}^c}{\mathrm{e}^c-1}$ over an arbitrary matroid, and Sviridenko, Vondrák, and Ward [27] showed an improved upper bound of $\frac{\mathrm{e}}{\mathrm{e}-c}$ for the approximation ratio of a modified continuous greedy algorithm over a uniform matroid (i.e., a cardinality-constraint).

Other variants of the problem setting include the maximization of a monotone, submodular function over a knapsack constraint [26], and robust submodular maximization [1, 24].

1.1. Paper organization. This paper is structured as follows. In Section 2 we present an α-augmentable multi-commodity flow problem. We show that the greedy algorithm cannot achieve an approximation ratio smaller than the known upper bound of $\alpha \cdot \frac{\mathrm{e}^\alpha}{\mathrm{e}^\alpha-1}$ for this problem class for $\alpha \in \mathbb{N}$, i.e., we show Corollary 1.7 for $\alpha \in \mathbb{N}$. In Section 3 we prove most of Proposition 1.3, i.e., we show that of the function classes $\tilde{\mathcal{F}}_\gamma$, $\mathcal{F}_\alpha$ and $\mathcal{F}_q$, neither is contained within the other two. Lastly, in Section 4 we show Theorems 1.5, 1.6 and 1.8. I.e., we show that the class of γ-α-augmentable functions contains the three previously mentioned classes, and show a tight approximation ratios for the greedy algorithm on this class. Furthermore, we close the gaps left in the proofs of the previous sections.

2. The Multi-Sink α-Commodity Flow problem. In this section, we introduce a natural α-commodity flow problem that models, e.g., production processes where output is limited by availability of all components. The objective of this problem is (exactly) α-augmentable, but, for $\alpha \in \mathbb{N} \setminus \{1\}$, does not have a bounded (weak) submodularity ratio and cannot be expressed as a weighted rank function over an independence system. We will show that this problem also gives a tight lower bound for the approximation ratio of the greedy algorithm on α-augmentable functions, for

$\alpha \in \mathbb{N}$. We will extend this lower bound to all $\alpha \geq 1$ in Section 4.1, and thus close a gap left by [2].

DEFINITION 2.1. *For a directed graph $G = (V, E)$ with source $s \in V$, sinks $T \subseteq V$, and arc capacities $\mu\colon E \to \mathbb{R}_{\geq 0}$, we define an s-T-flow to be a function $\vartheta\colon E \to \mathbb{R}_{\geq 0}$ that satisfies*

$$\begin{aligned} \vartheta(e) \leq \mu(e) \quad & \forall e \in E && \textit{(capacity constraint)}, \\ \mathrm{ex}_\vartheta(v) = 0 \quad & \forall v \in V \setminus (\{s\} \cup T) && \textit{(flow conservation)}, \\ \mathrm{ex}_\vartheta(t) \geq 0 \quad & \forall t \in T && \textit{(T are sinks)}, \end{aligned}$$

where (using $\delta^+(v) := (\{v\} \times V) \cap E$, $\delta^-(v) := (V \times \{v\}) \cap E$) the excess of a vertex $v \in V$ is defined as

$$\mathrm{ex}_\vartheta(v) := \sum_{e \in \delta^-(v)} \vartheta(e) - \sum_{e \in \delta^+(v)} \vartheta(e).$$

We extend this notion to multi-commodity flows, where each commodity has an independent capacity function.

DEFINITION 2.2. *Let $\alpha \in \mathbb{N}$ and $G = (V, E)$ be a graph with $s \in V$ and $T \subseteq V$. Furthermore, let $\boldsymbol{\mu} = (\mu_i\colon E \to \mathbb{R}_{\geq 0})_{i \in [\alpha]}$ be capacity functions. A multicommodity-flow in G w.r.t. $\boldsymbol{\mu}$ is a tuple $\boldsymbol{\vartheta} = (\vartheta_1, ..., \vartheta_\alpha)$, where ϑ_i is an s-T-flow in G with respect to capacities μ_i. The minimum-excess of the sink vertex $t \in T$ in $\boldsymbol{\vartheta}$ is*

$$\mathrm{minex}_{\boldsymbol{\vartheta}}(t) := \min_{i \in [\alpha]} \mathrm{ex}_{\vartheta_i}(t).$$

For convenience, we let $\mu(u, v) := \mu((u, v))$, $\vartheta(u, v) := \vartheta((u, v))$, and we let $\mathrm{ex}_\vartheta(V') := \sum_{v \in V'} \mathrm{ex}_\vartheta(v)$ for $V' \subseteq V$, and $\mathrm{minex}_{\boldsymbol{\vartheta}}(T') := \sum_{t \in T'} \mathrm{minex}_{\boldsymbol{\vartheta}}(t)$ for $T' \subseteq T$ in the following.

An instance of the problem MULTI-SINK α-COMMODITY FLOW, for $\alpha \in \mathbb{N}$, is given by a tuple $(G, s, T, \boldsymbol{\mu})$, where $G = (V, E)$ is a directed graph, $s \in V$ is a source vertex, $T \subseteq V$ contains sink vertices, and $\boldsymbol{\mu} = (\mu_i\colon E \to \mathbb{R}_{\geq 0})_{i \in [\alpha]}$ are capacity functions. The problem is to find a subset of sinks $X \subseteq T$ with $|X| = k$ that maximizes the objective function

$$f(X) = \max_{\boldsymbol{\vartheta} \in \mathcal{M}_{G,\mu}} \mathrm{minex}_{\boldsymbol{\vartheta}}(X),$$

where $\mathcal{M}_{G,\mu}$ denotes the set of all multicommodity-flows in G w.r.t. capacities $\boldsymbol{\mu}$.

EXAMPLE 2.3. *For a prototypical application of MULTI-SINK α-COMMODITY FLOW, consider a factory where $k \in \mathbb{N}$ machines are to be built in a set T of potential locations. Each machine produces the same item and needs a number $\alpha \in \mathbb{N}$ of different resources. The output of a machine is limited by the resource it has available the least. All resources are delivered to the machines along different routes within the factory, e.g., some liquids might be transported via pipes, other resources might be transported on a conveyor belt or on pallets. The objective is to determine in which k locations the machines should be constructed in order to maximize overall production.*

THEOREM 2.4. *For every $\alpha \in \mathbb{N}$, the objective of MULTI-SINK α-COMMODITY FLOW is monotone and* α-augmentable.

Proof. Let $X \subseteq T$ and $t \in T \setminus X$. To prove monotonicity, fix some flow $\boldsymbol{\vartheta}$ with $\mathrm{minex}_{\boldsymbol{\vartheta}}(X) = f(X)$. By definition, $\mathrm{minex}_{\boldsymbol{\vartheta}}(X) \leq \mathrm{minex}_{\boldsymbol{\vartheta}}(X \cup \{t\}) \leq f(X \cup \{t\})$ holds and thus f is monotone.

To show α-augmentability, let (G, s, T, μ) be an instance of Multi-Sink α-Commodity Flow. Let $X, Y \subseteq T$ such that $Y' := Y \setminus X \neq \emptyset$. We show that there exists $y \in Y'$ with

$$f(X \cup \{y\}) - f(X) \geq \frac{f(X \cup Y') - \alpha f(X)}{|Y'|}.$$

This suffices because, with

$$\frac{f(X \cup Y') - \alpha f(X)}{|Y'|} = \frac{f(X \cup Y) - \alpha f(X)}{|Y'|} \geq \frac{f(X \cup Y) - \alpha f(X)}{|Y|},$$

α-augmentability of the problem follows.

Let $\boldsymbol{\vartheta}^{X \cup Y'} = (\vartheta_1^{X \cup Y'}, ..., \vartheta_\alpha^{X \cup Y'})$ be a multicommodity-flow in G that maximises the minimum-exess $\operatorname{minex}_{\boldsymbol{\vartheta}^{X \cup Y'}}(X \cup Y')$, i.e., $\operatorname{minex}_{\boldsymbol{\vartheta}^{X \cup Y'}}(X \cup Y') = f(X \cup Y')$, such that $\vartheta_i^{X \cup Y'}$ is a maximum s-$(X \cup Y')$-flow w.r.t. capacity μ_i for all $i \in [\alpha]$. Such a multicommodity-flow can be obtained by augmenting a flow that maximises $\operatorname{minex}_{\boldsymbol{\vartheta}^{X \cup Y'}}(X \cup Y')$, e.g., with the Edmonds-Karp algorithm (cf. [17]). Furthermore, we let $\boldsymbol{\vartheta}^X = (\vartheta_1^X, ..., \vartheta_\alpha^X)$ be a multicommodity-flow in G with $\operatorname{minex}_{\boldsymbol{\vartheta}^X}(X) = f(X)$, as well as $\operatorname{ex}_{\vartheta_i^X}(X) = f(X)$ and $\operatorname{ex}_{\vartheta_i^X}(T \setminus X) = 0$ for all $i \in [\alpha]$, i.e., $\boldsymbol{\vartheta}^X$ maximises the minimum-excess of the set X while the values of all flows ϑ_i^X are as small as possible. This multicommodity-flow can be obtained by reducing the flows of a multicommodity-flow that maximises $\operatorname{minex}_{\boldsymbol{\vartheta}^X}(X \cup Y')$ along paths of a path decomposition of the flow (cf. [17]). We define the function $g\colon X \to [\alpha]$, such that, for all $x \in X$, no flow $\tilde{\vartheta}$ w.r.t. capacity $\mu_{g(x)}$ exists with $\operatorname{ex}_{\tilde{\vartheta}}(x') \geq \operatorname{ex}_{\vartheta_{g(x)}^X}(x')$ for all $x' \in X \setminus \{x\}$ and with $\operatorname{ex}_{\tilde{\vartheta}}(x) > \operatorname{ex}_{\vartheta_{g(x)}^X}(x)$. This means that the flow $\vartheta_{g(x)}^X$ is one of the flows limiting the value of $\operatorname{minex}_{\boldsymbol{\vartheta}^X}(x)$. Let $g^{-1}(i) = \{x \in X \mid g(x) = i\}$ for all $i \in [\alpha]$ be the preimage of g. Obviously

$$\bigcup_{i=1}^{\alpha} g^{-1}(i) = X. \tag{2.1}$$

We add a super sink t to G and let $\tilde{G} = (\tilde{V}, \tilde{E})$ with $\tilde{V} := V \cup \{t\}$ and $\tilde{E} := E \cup \{(v, t) \mid v \in (X \cup Y')\}$ denote the resulting graph. Furthermore, we define the capacity functions $\tilde{\mu}_i\colon \tilde{E} \to \mathbb{R}$ for all $i \in [\alpha]$ such that, for $(u, v) \in \tilde{E}$,

$$\tilde{\mu}_i(u, v) := \begin{cases} \mu_i(u, v), & \text{if } (u, v) \in E, \\ \max\{\operatorname{ex}_{\vartheta_i^X}(u), \operatorname{ex}_{\vartheta_i^{X \cup Y'}}(u)\}, & \text{if } (u, v) \in X \times \{t\}, \\ \operatorname{ex}_{\vartheta_i^{X \cup Y'}}(u), & \text{if } (u, v) \in Y' \times \{t\}. \end{cases}$$

Now we extend the flow $\boldsymbol{\vartheta}^{X \cup Y'}$ to a flow $\tilde{\boldsymbol{\vartheta}}^{X \cup Y'}$ in $\tilde{G}$, such that, for all $i \in [\alpha]$ and $(u, v) \in \tilde{E}$,

$$\tilde{\vartheta}_i^{X \cup Y'}(u, v) := \begin{cases} \vartheta_i^{X \cup Y'}(u, v), & \text{if } (u, v) \in E, \\ \operatorname{ex}_{\vartheta_i^{X \cup Y'}}(u), & \text{else}, \end{cases}$$

holds, and analogously, we extend the flow $\boldsymbol{\vartheta}^X$ to a flow $\tilde{\boldsymbol{\vartheta}}^X$ in $\tilde{G}$. With this definition, $\tilde{\vartheta}_i^{X \cup Y'}$ is a maximum s-t-flow w.r.t. capacity $\tilde{\mu}_i$, because $\vartheta_i^{X \cup Y'}$ is a maximum s-$(X \cup Y')$-flow w.r.t. capacity μ_i.

For $i \in [\alpha]$, let $\tilde{\vartheta}_i$ be a maximum s-t-flow w.r.t. capacity $\tilde{\mu}_i$ in $\tilde{G}$ obtained from $\tilde{\vartheta}_i^X$ by using the Edmonds-Karp algorithm. Then its value is exactly $\mathrm{ex}_{\vartheta_i^{X\cup Y'}}(X \cup Y')$, as $\vartheta_i^{X\cup Y'}$ is a maximum s-$(X \cup Y')$-flow. We project $\tilde{\vartheta}_i$ onto a flow in G, i.e., we set $\vartheta_i := \tilde{\vartheta}_i|_E$ for $i \in [\alpha]$ and define $\boldsymbol{\vartheta} := (\vartheta_1, ..., \vartheta_\alpha)$. For all $x \in X$, by definition of $\boldsymbol{\vartheta}$, we have $\mathrm{ex}_{\vartheta_i}(x) \geq \mathrm{ex}_{\vartheta_i^X}(x)$, and thus, by definition of g,

$$\mathrm{ex}_{\vartheta_{g(x)}}(x) = \mathrm{ex}_{\vartheta^X_{g(x)}}(x). \tag{2.2}$$

Because $\tilde{\vartheta}_i$ is a maximum s-t-flow in $\tilde{G}$ w.r.t. capacity $\tilde{\mu}_i$, ϑ_i is a maximum s-$(X\cup Y')$-flow w.r.t. capacity μ_i in G. Since $\vartheta_i^{X\cup Y'}$ is also a maximum s-$(X \cup Y')$-flow w.r.t. capacity μ_i, we have

$$\mathrm{ex}_{\vartheta_i}(X \cup Y') = \mathrm{ex}_{\vartheta_i^{X\cup Y'}}(X \cup Y'). \tag{2.3}$$

For all $x \in X$, we know that the excess of x in ϑ_i is as large as the flow $\tilde{\vartheta}_i(x, t)$, i.e.,

$$\mathrm{ex}_{\vartheta_i}(x) = \tilde{\vartheta}_i(x,t) \leq \tilde{\mu}_i(x,t) = \max\{\mathrm{ex}_{\vartheta_i^X}(x), \mathrm{ex}_{\vartheta_i^{X\cup Y'}}(x)\} \leq \mathrm{ex}_{\vartheta_i^X}(x) + \mathrm{ex}_{\vartheta_i^{X\cup Y'}}(x). \tag{2.4}$$

By maximality of $\boldsymbol{\vartheta}^X$ and because $\mathrm{ex}_{\vartheta_i}(x) \geq \mathrm{ex}_{\vartheta_i^X}(x)$ for all $x \in X$, we have

$$\mathrm{minex}_{\boldsymbol{\vartheta}}(X) = \mathrm{minex}_{\boldsymbol{\vartheta}^X}(X) = f(X). \tag{2.5}$$

Since $X \cap Y' = \emptyset$, we obtain

$$\begin{aligned}
& \mathrm{ex}_{\vartheta_i^{X\cup Y'}}(Y') - \mathrm{ex}_{\vartheta_i}(Y') \\
= \;& \mathrm{ex}_{\vartheta_i^{X\cup Y'}}(X \cup Y') - \mathrm{ex}_{\vartheta_i}(X \cup Y') - \mathrm{ex}_{\vartheta_i^{X\cup Y'}}(X) + \mathrm{ex}_{\vartheta_i}(X) \\
\overset{(2.3)}{=} \;& \mathrm{ex}_{\vartheta_i}(X) - \mathrm{ex}_{\vartheta_i^{X\cup Y'}}(X) \\
= \;& \sum_{x\in X\setminus g^{-1}(i)} (\mathrm{ex}_{\vartheta_i}(x) - \mathrm{ex}_{\vartheta_i^{X\cup Y'}}(x)) + \sum_{x\in g^{-1}(i)} (\mathrm{ex}_{\vartheta_i}(x) - \mathrm{ex}_{\vartheta_i^{X\cup Y'}}(x)) \\
\overset{(2.4),(2.2)}{\leq} \;& \sum_{x\in X\setminus g^{-1}(i)} \mathrm{ex}_{\vartheta_i^X}(x) + \sum_{x\in g^{-1}(i)} (\mathrm{ex}_{\vartheta_i^X}(x) - \mathrm{ex}_{\vartheta_i^{X\cup Y'}}(x)) \\
= \;& f(X) - \sum_{x\in g^{-1}(i)} \mathrm{ex}_{\vartheta_i^{X\cup Y'}}(x),
\end{aligned} \tag{2.6}$$

where we used minimality of $\boldsymbol{\vartheta}^X$. Using this we can compute

$$\begin{aligned}
\operatorname{minex}_{\boldsymbol{\vartheta}^{X\cup Y'}}(Y') &= \sum_{y\in Y'} \min_{i\in[\alpha]}\{\operatorname{ex}_{\vartheta_i^{X\cup Y'}}(y)\} \\
&= \sum_{y\in Y'} \min_{i\in[\alpha]}\{\operatorname{ex}_{\vartheta_i}(y) + (\operatorname{ex}_{\vartheta_i^{X\cup Y'}}(y) - \operatorname{ex}_{\vartheta_i}(y))\} \\
&\leq \sum_{y\in Y'} \Big(\min_{i\in[\alpha]}\{\operatorname{ex}_{\vartheta_i}(y)\} + \sum_{i=1}^{\alpha}(\operatorname{ex}_{\vartheta_i^{X\cup Y'}}(y) - \operatorname{ex}_{\vartheta_i}(y))\Big) \\
&= \operatorname{minex}_{\boldsymbol{\vartheta}}(Y') + \sum_{i=1}^{\alpha}(\operatorname{ex}_{\vartheta_i^{X\cup Y'}}(Y') - \operatorname{ex}_{\vartheta_i}(Y')) \\
&\overset{(2.6)}{\leq} \operatorname{minex}_{\boldsymbol{\vartheta}}(Y') + \sum_{i=1}^{\alpha}\big(f(X) - \sum_{x\in g^{-1}(i)} \operatorname{ex}_{\vartheta_i^{X\cup Y'}}(x)\big) \\
&\overset{(2.1)}{=} \operatorname{minex}_{\boldsymbol{\vartheta}}(Y') + \alpha f(X) - \sum_{x\in X} \operatorname{ex}_{\vartheta_{g(x)}^{X\cup Y'}}(x). \qquad (2.7)
\end{aligned}$$

Finally, because of $X \cap Y' = \emptyset$, we get

$$\begin{aligned}
f(X\cup Y') &= \operatorname{minex}_{\boldsymbol{\vartheta}^{X\cup Y'}}(X) + \operatorname{minex}_{\boldsymbol{\vartheta}^{X\cup Y'}}(Y') \\
&= \sum_{x\in X}\big(\min_{i\in[\alpha]} \operatorname{ex}_{\vartheta_i^{X\cup Y'}}(x)\big) + \operatorname{minex}_{\boldsymbol{\vartheta}^{X\cup Y'}}(Y') \\
&\overset{(2.7)}{\leq} \sum_{x\in X} \operatorname{ex}_{\vartheta_{g(x)}^{X\cup Y'}}(x) + \operatorname{minex}_{\boldsymbol{\vartheta}}(Y') + \alpha f(X) - \sum_{x\in X} \operatorname{ex}_{\vartheta_{g(x)}^{X\cup Y'}}(x) \\
&= \sum_{y\in Y'} \operatorname{minex}_{\boldsymbol{\vartheta}}(y) + \alpha f(X),
\end{aligned}$$

which is equivalent to

$$\sum_{y\in Y'} \operatorname{minex}_{\boldsymbol{\vartheta}}(y) \geq f(X\cup Y') - \alpha f(X). \qquad (2.8)$$

Now, we show that $f(X\cup\{y\})-f(X) \geq \operatorname{minex}_{\boldsymbol{\vartheta}}(y)$ for all $y \in Y'$, which completes the proof,

$$\begin{aligned}
|Y'|\,\big(\max_{y\in Y'} f(X\cup\{y\}) - f(X)\big) &\geq \sum_{y\in Y'}\big(f(X\cup\{y\}) - f(X)\big) \\
&\geq \sum_{y\in Y'} \operatorname{minex}_{\boldsymbol{\vartheta}}(y) \overset{(2.8)}{\geq} f(X\cup Y') - \alpha f(X).
\end{aligned}$$

In order to show that $f(X\cup\{y\})-f(X) \geq \operatorname{minex}_{\boldsymbol{\vartheta}}(y)$ holds for all $y \in Y'$, let $y \in Y'$. Since $X \cap Y' = \emptyset$, we have

$$\operatorname{minex}_{\boldsymbol{\vartheta}}(X\cup\{y\}) = \operatorname{minex}_{\boldsymbol{\vartheta}}(X) + \operatorname{minex}_{\boldsymbol{\vartheta}}(y) \overset{(2.5)}{=} f(X) + \operatorname{minex}_{\boldsymbol{\vartheta}}(y).$$

Furthermore, we have $f(X\cup\{y\}) \geq \operatorname{minex}_{\boldsymbol{\vartheta}}(X\cup\{y\})$ because $\boldsymbol{\vartheta}$ is a multicommodity-flow in G. Combining these two insights yields $f(X\cup\{y\}) - f(X) \geq \operatorname{minex}_{\boldsymbol{\vartheta}}(y)$. Thus, we can conclude that f is α-augmentable. ∎

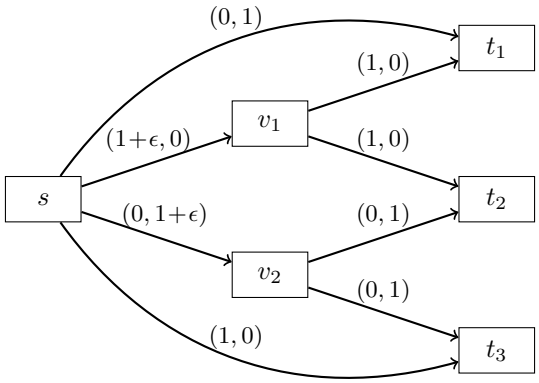

FIG. 2. *An instance of* MULTI-SINK α-COMMODITY FLOW *for* $\alpha = 2$ *where the objective has (weak) submodularity ratio 0 and cannot be modelled as weighted rank function of some independence system.*

PROPOSITION 2.5. *For every* $\gamma, q \in (0, 1)$*, and* $\alpha \in \mathbb{N}$ *with* $\alpha \geq 2$*, there exists an instance of* MULTI-SINK α-COMMODITY FLOW *where the objective is not in* $\tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_q$.

Proof. We will define such an instance of MULTI-SINK α-COMMODITY FLOW. Let

$$\begin{aligned}
T &:= \{t_1, t_2, t_3\},\\
V &:= \{s, v_1, v_2\} \cup T,\\
E &:= \{(s, v_1), (s, v_2), (s, t_1), (s, t_3), (v_1, t_1), (v_1, t_2), (v_2, t_2), (v_2, t_3)\},\\
G &:= (V, E),
\end{aligned}$$

and, with $0 < \epsilon < \frac{\gamma}{2}$, let

$$\mu \colon E \to \mathbb{R}^\alpha_{\geq 0}, \mu(e) = \begin{cases} (1+\epsilon, 0, 0, ..., 0), & \text{if } e = (s, v_1),\\ (0, 1+\epsilon, 1+\epsilon, ..., 1+\epsilon), & \text{if } e = (s, v_2),\\ (1, 0, 0, ..., 0), & \text{if } e \in \{(s, t_3), (v_1, t_1), (v_1, t_2)\},\\ (0, 1, 1, ..., 1), & \text{else.} \end{cases}$$

A diagram of the graph can be seen in Figure 2. With proper tie breaking (or by adding small extra capacities), the greedy algorithm picks the sink t_2 in the first iteration. Adding any other sink to this increases the objective value by ϵ, i.e., for all $t \in T$, we have $\sum_{t \in T}(f(S_1^{\mathrm{G}} \cup \{t\}) - f(S_1^{\mathrm{G}})) = 2\epsilon$. But since $f(S_1^{\mathrm{G}} \cup \{t_1, t_3\}) - f(S_1^{\mathrm{G}}) = 1$, the weak submodularity ratio of this problem is $\frac{2\epsilon}{1} < \gamma$.

If f could be modelled as the weighted rank function of some independence system, the corresponding weight function would have to satisfy $w(t_1) = w(t_2) = w(t_3) = 1$ because each sink alone has a minimum-excess of 1. Yet, we have $f(\{t_1, t_2\}) = 1 + \epsilon$. This cannot be possible if f is the weighted rank function of some independence system, as, in this case, we would have $f(\{t_1, t_2\}) \in \{0, 1, 2\}$, depending on which sets are independent. ∎

We will now construct a family of instances the MULTI-SINK α-COMMODITY FLOW problem to show a tight lower bound on the approximation ratio of the greedy algorithm for the class of α-augmentable objectives for $\alpha \in \mathbb{N}$.

For $\alpha = 2$, Multi-Sink α-Commodity Flow problem is equivalent to the BridgeFlow problem considered in [2]. We generalize the tight lower bound construction for BridgeFlow to arbitrary $\alpha \in \mathbb{N}$.

For $k \in \mathbb{N}$, $k \geq 2$ we let $x := \frac{k}{k-1}$. Now, we define the graphs $G_k = (V_k, E_k)$ (cf. Figure 3) via

$$\begin{aligned}
V_k &:= \{s, v_1, ..., v_{\alpha k}, t_1, ..., t_{2\alpha k}\},\\
E_k &:= \bigcup_{i=1}^{\alpha} \mathcal{E}_{k,i},\\
\mathcal{E}_{k,i} &:= E^1_{k,i} \cup E^\infty_{k,i} \cup \bigcup_{j=1}^{\alpha k} E_{k,i,j} \cup \bigcup_{j=1}^{\alpha k} E'_{k,i,j},\\
E^1_{k,i} &:= \{(s, t_{(\alpha+i-1)k+1}), ..., (s, t_{(\alpha+i)k})\},\\
E^\infty_{k,i} &:= \{(s, t_{\alpha k+1}), ..., (s, t_{2\alpha k})\} \setminus E^1_{k,i},\\
E_{k,i,j} &:= \{(s, v_j), (v_j, t_j)\} \forall j \in [\alpha k],\\
E'_{k,i,j} &:= \{(v_j, t_{(\alpha+i-1)k+1}), ..., (v_j, t_{(\alpha+i)k})\} \forall j \in [\alpha k],
\end{aligned}$$

capacity functions $\mu^k = (\mu^k_1, ..., \mu^k_\alpha)$ with $\mu^k_i \colon E^k \to \mathbb{R}$ for $i \in [\alpha]$ and

$$\mu^k_i(e) = \begin{cases} 1, & \text{if } e \in E^1_{k,i},\\ \infty, & \text{if } e \in E^\infty_{k,i},\\ x^{\alpha k - j + 1}, & \text{if } e \in E_{k,i,j} \text{ for some } j \in [\alpha k],\\ \frac{1}{k} x^{\alpha k - j + 1}, & \text{if } e \in E'_{k,i,j} \text{ for some } j \in [\alpha k],\\ 0, & \text{else.} \end{cases}$$

Note that only the arcs in $\mathcal{E}_{k,i}$ allow a flow of commodity i. We define s to be the source vertex and $T := \{t_1, ..., t_{2\alpha k}\}$ to be the set of sink vertices.

In the following proof we will need the following observation: Using $x = \frac{k}{k-1}$ and with $n \in \mathbb{N}$ the equation

$$\begin{aligned}
1 + \frac{1}{k}\sum_{j=1}^{n} x^j &= 1 + \frac{1}{k}\Big(\frac{x^{n+1}-1}{x-1} - 1\Big) = 1 + \frac{1}{k}\Big(\frac{\left(\frac{k}{k-1}\right)^{n+1} - 1}{\left(\frac{k}{k-1}\right) - 1} - 1\Big)\\
&= 1 + \frac{1}{k}\Big((k-1)\Big(\Big(\frac{k}{k-1}\Big)^{n+1} - 1\Big) - 1\Big)\\
&= 1 + \Big(\frac{k}{k-1}\Big)^n - \frac{1}{k}((k-1)+1) = \Big(\frac{k}{k-1}\Big)^n = x^n \qquad (2.9)
\end{aligned}$$

holds.

We will now show in which order the greedy algorithm picks the vertices from the set T. We assume that the tie-breaking works out in our favor. This can be achieved by introducing small offsets to the capacities. For better readability we omit this here.

Lemma 2.6. *Let $\alpha, k \in \mathbb{N}$. In iteration $\ell \in [\alpha k]$, the greedy algorithm picks sink vertex t_ℓ. Furthermore, a multicommodity-flow $\boldsymbol{\vartheta} = (\vartheta_1, ..., \vartheta_\alpha)$ with maximum minimum-excess of the vertices $\{t_1, ..., t_\ell\}$ for $\ell \in [\alpha k]$ always satisfies $\vartheta_i(e) = x^{\alpha k - j + 1}$ for all $e \in E_{k,i,j}$ with $i \in [\alpha]$ and $j \in [\ell]$, i.e., all arcs in $E_{k,i,j}$ are fully saturated for all $i \in [\alpha]$ and $j \in [\ell]$.*

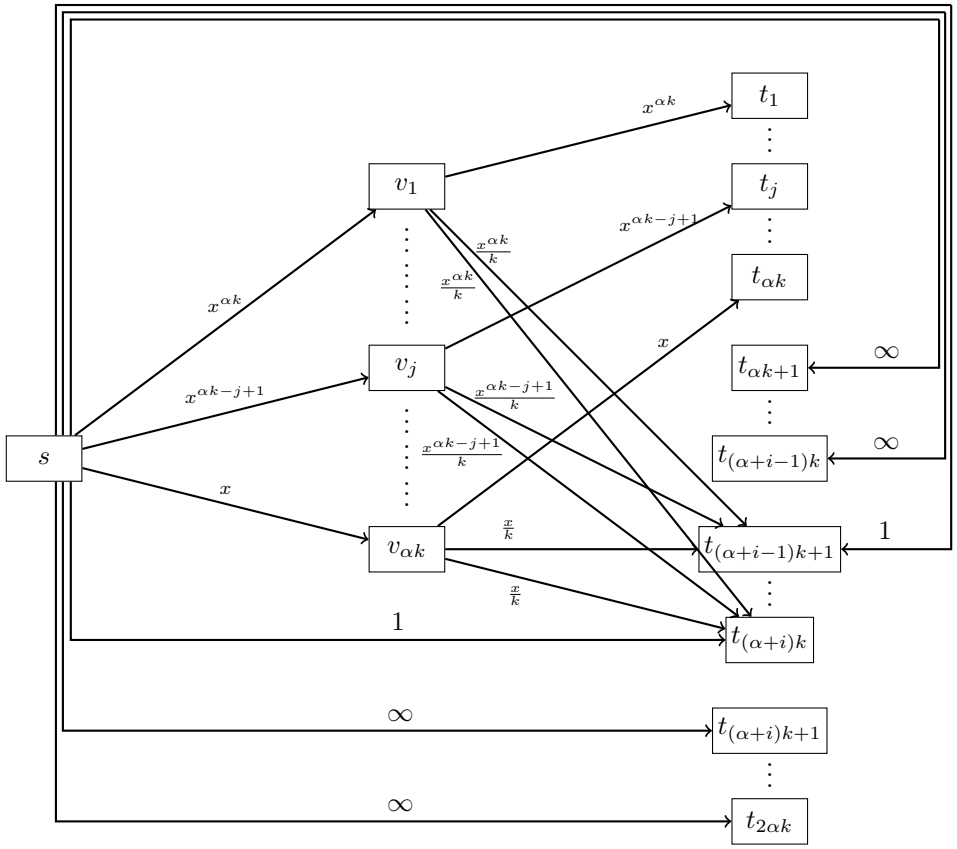

FIG. 3. *The graph G_k only with edges from $\mathcal{E}_{k,i}$ and capacities μ_i^k.*

Proof. We will prove the statement by induction. In iteration $\ell = 1$, the gain of picking vertex t_j with $j \in [\alpha k]$ is $x^{\alpha k-j+1}$, because for all $i \in [\alpha]$ we can have a flow of value $x^{\alpha k-j+1}$ of commodity i from s via v_j and the edges in $E_{k,i,j}$ to t_j and no more flows to t_j are possible, since the only incoming arc to t_j, which allows a flow of commodity i, is the arc $(v_j, t_j) \in E_{k,i,j}$. The gain of picking vertex t_j with $j \in \{\alpha k + 1, ..., 2\alpha k\}$ is the minimum of all commodities flowing to t_j and there is only one commodity which does not allow an unbounded flow to t_j, because for $i \in [\alpha] \setminus \{\lceil \frac{j-\alpha}{k} \rceil\}$ there is an arc from s_i to t_j in $E'_{k,i,j}$ with infinite capacity for commodity i. The maximum flow of the commodity with a finite flow to t_j is

$$1 + \frac{1}{k} \sum_{j=1}^{\alpha k} x^j \overset{(2.9)}{=} x^{\alpha k},$$

and, thus, with proper tie-breaking, the greedy algorithm chooses vertex t_1. For $i \in [\alpha]$, the only incoming path that allows a flow of commodity i from s to t_1 is along the edges in $E_{k,i,1}$, so they have to be fully saturated by a multicommodity-flow with maximum minimum-excess.

Now suppose the statement is true for some $\ell \in [\alpha k - 1]$, i.e., the greedy algorithm has picked edges $t_1, ..., t_\ell$ and a multicommodity-flow with maximum minimum-excess of the vertices $\{t_1, ..., t_\ell\}$ fully saturates all arcs in $E_{k,i,j}$ for all $i \in [\alpha]$ and $j \in [\ell]$.

Then the gain of picking vertex t_j for $j \in \{\ell+1, ..., \alpha k\}$ is still $x^{\alpha k - j + 1}$, because all s-t_j-paths for $i \in [\alpha]$ do not carry flow that contributes to the maximum minimum-excess. The gain of picking vertex t_j for $j \in \{\alpha k + 1, ..., 2\alpha k\}$ is still the minimum of all commodities flowing to t_j, and again there is only one commodity which does not allow an unbounded flow to t_j. Because all incoming flow at vertices $v_1, ..., v_\ell$ already saturates all incoming arcs, there is no flow of this commodity via a vertex in $\{v_1, ..., v_\ell\}$ to t_j possible without reducing the minimum-excess of another sink vertex by the same amount. Thus, the maximal flow of this commodity to t_j is

$$1 + \frac{1}{k} \sum_{j=1}^{\alpha k - \ell} x^j \overset{(2.9)}{=} x^{\alpha k - \ell},$$

so, with proper tie-breaking, the greedy algorithm picks vertex $t_{\ell+1}$ next. For $i \in [\alpha]$, the only incoming path that allows a flow of commodity i from s to t_j for $j \in [\ell]$ is along the edges in $E_{k,i,j}$, so they have to be fully saturated by a multicommodity-flow with maximum minimum-excess. ∎

With this, we obtain a lower bound for the approximation ratio of the greedy algorithm on $\mathcal{F}_\alpha$ for $\alpha \in \mathbb{N}$ that tightly matches the upper bound of [2], i.e., we obtain Corollary 1.7 for $\alpha \in \mathbb{N}$. In particular, it follows that the objective of Multi-Sink α-Commodity Flow is not β-augmentable for any $\beta < \alpha$. We will generalize the lower bound to all $\alpha \geq 1$ in Section 4.1.

Theorem 2.7. *For $\alpha \in \mathbb{N}$, the greedy algorithm has an approximation ratio of at least $\alpha \frac{\mathrm{e}^\alpha}{\mathrm{e}^\alpha - 1}$ for Multi-Sink α-Commodity Flow.*

Proof. By Lemma 2.6, the greedy algorithm picks the sinks $t_1, ..., t_{\alpha k}$ in the first αk iterations and the objective increases by $x^{\alpha k - j + 1}$ when sink vertex t_j is picked and thus the minimum-excess of the greedy solution is

$$f(S_k^{\mathrm{G}}) = \sum_{j=1}^{\alpha k} x^j \overset{(2.9)}{=} k(x^{\alpha k} - 1).$$

We compare this to the solution that picks the vertices $t_{\alpha k+1}, ..., t_{2\alpha k}$ (which is, in fact, an optimal solution for cardinality αk). Increasing the flow to one of these vertices does not reduce the flow to the others, so the minimum-excess of any of these vertices is

$$1 + \frac{1}{k} \sum_{j=1}^{\alpha k} x^j \overset{(2.9)}{=} x^{\alpha k},$$

and their total minimum-excess thus is $\alpha k x^{\alpha k}$. Using this and $x = \frac{k}{k-1}$, we calculate the ratio between this solution and the greedy solution and get

$$\frac{\alpha k x^{\alpha k}}{k(x^{\alpha k} - 1)} = \frac{\alpha x^{\alpha k}}{x^{\alpha k} - 1} = \frac{\alpha \left(\frac{k}{k-1}\right)^{\alpha k}}{\left(\frac{k}{k-1}\right)^{\alpha k} - 1} = \frac{\alpha \left(\left(\frac{k}{k-1}\right)^k\right)^\alpha}{\left(\left(\frac{k}{k-1}\right)^k\right)^\alpha - 1}.$$

Using the identity $\lim_{k\to\infty}(k/(k-1))^k = \mathrm{e}$, we obtain the limit

$$\lim_{k\to\infty} \frac{\alpha \left(\left(\frac{k}{k-1}\right)^k\right)^\alpha}{\left(\left(\frac{k}{k-1}\right)^k\right)^\alpha - 1} = \alpha \frac{\mathrm{e}^\alpha}{\mathrm{e}^\alpha - 1}.$$

∎

3. Separating Function Classes. In this section we will prove the second and third part of Proposition 1.3 for all $\alpha \geq 1$, as well as the first part of Proposition 1.3 for $\alpha \in \mathbb{N} \setminus \{1\}$. The case $\alpha \geq 1$ will be addressed in Section 4.1.

We start with the first part, i.e., we separate $\mathcal{F}_\alpha$ for $\alpha \in \mathbb{N} \setminus \{1\}$. This follows immediately from Theorem 2.4 and Proposition 2.5.

PROPOSITION 3.1. *For every $\gamma, q \in (0,1)$, and $\alpha \in \mathbb{N}$ with $\alpha \geq 2$, it holds that $\mathcal{F}_\alpha \nsubseteq (\tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_q)$.*

Proof. Let $\gamma, q \in (0,1)$, and $\alpha \in \mathbb{N}$ with $\alpha \geq 2$. By Theorem 2.4, every objective of an instance of MULTI-SINK α-COMMODITY FLOW is α-augmentable. By Proposition 2.5, there exists an instance of MULTI-SINK α-COMMODITY FLOW, where the objective is not in $\tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_q$. Combining this yields the desired result. ∎

We proceed to show the second and third part of Proposition 1.3 (for all $\alpha \geq 1$).

PROPOSITION 3.2. *For every $\gamma, q \in (0,1)$, $\alpha \geq 1$, it holds that $\tilde{\mathcal{F}}_\gamma \nsubseteq (\mathcal{F}_\alpha \cup \mathcal{F}_q)$.*

Proof. Consider the set $U = \{a, b\}$ and the objective function

$$f^\gamma \colon 2^U \to \mathbb{R}_{\geq 0}, f^\gamma(X) = \begin{cases} |X|, & \text{if } |X| \leq 1, \\ \frac{2}{\gamma}, & \text{else.} \end{cases}$$

If f^γ could be modelled as the weighted rank function of an independence system $(U, \mathcal{I})$, then we would have $U \in \mathcal{I}$ because $f(U) > f(X)$ for all $X \subsetneq U$. Then $\mathcal{I} = 2^U$ and f^γ would be linear which is not true. Thus f^γ cannot be modelled as the weighted rank function of an independence system, and $f^\gamma \notin \mathcal{F}_q$.

Furthermore, $f^\gamma \notin \mathcal{F}_\alpha$. To see this, consider $X = \emptyset$ and $Y = \{a, b\}$. Then we have $f^\gamma(X \cup \{y\}) - f^\gamma(X) = 1$ for all $y \in Y$, and we have $\frac{f^\gamma(X \cup Y) - \alpha f^\gamma(X)}{|Y|} = \frac{1}{\gamma}$. Since $\gamma < 1$, the problem is not α-augmentable.

Now, let $X, Y \subseteq U$ with $X \cap Y = \emptyset$. For $Y = \emptyset$ the ratio in the definition of the weak submodularity ratio is $\frac{0}{0} = 1$. Thus, assume $|Y| \geq 1$. If $X = \emptyset$, we have

$$\frac{\sum_{y \in Y} f^\gamma(X \cup \{y\}) - f^\gamma(X)}{f^\gamma(X \cup Y) - f^\gamma(X)} = \frac{|Y|}{f^\gamma(Y)} \in \{1, \gamma\}.$$

Otherwise, if $|X| = 1$, then $|Y| = 1$ and the ratio in the definition of the (weak) submodularity ratio is 1. In both cases, the ratio is at least γ, thus the (weak) submodularity ratio of this problem is γ, and $f^\gamma \in \tilde{\mathcal{F}}_\gamma$. ∎

PROPOSITION 3.3. *For every $\gamma, q \in (0,1)$, $\alpha \geq 1$, it holds that $\mathcal{F}_q \nsubseteq (\tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_\alpha)$.*

Proof. We fix $m, n \in \mathbb{N}$ with $q \leq \frac{m}{n} < 1$ and $\alpha \geq 1$. Let

$$\begin{aligned} A &:= \{a_1, ..., a_{\lceil \alpha \rceil n}\}, \\ B &:= \{b_1, ..., b_{\lceil \alpha \rceil n}\}, \\ C &:= \{c\} \\ U &:= A \cup B \cup C, \\ \mathcal{I} &:= 2^A \cup 2^B \cup \{X \subset U \mid |X| \leq \lceil \alpha \rceil m\}. \end{aligned}$$

We consider the independence system $(U, \mathcal{I})$ and the weight function $w \colon U \to \mathbb{R}_{\geq 0}$ defined by

$$w(e) = \begin{cases} 1, & e \in A, \\ \lceil \alpha \rceil (n - m) + 1, & \text{else.} \end{cases}$$

The weighted rank function f is given by

$$f^q\colon U \to \mathbb{R}_{\geq 0}, f^q(X) = \max\{w(Y) \mid Y \in 2^X \cap \mathcal{I}\}.$$

Obviously we have $q(U,\mathcal{I}) = \frac{m}{n}$, i.e., $f^q \in \mathcal{F}_q$.

For $X = A$, $Y = B$ and $y \in Y$, we calculate

$$\begin{aligned} f^q(X) &= \lceil\alpha\rceil n,\\ f^q(X \cup \{y\}) &= \max\{\lceil\alpha\rceil n, \lceil\alpha\rceil(n-m)+1+(\lceil\alpha\rceil m-1)\} = \lceil\alpha\rceil n,\\ f^q(X \cup Y) &= \lceil\alpha\rceil n(\lceil\alpha\rceil(n-m)+1). \end{aligned}$$

Suppose, f^q was α-augmentable. Then

$$f^q(X \cup \{y\}) - f^q(X) \geq \frac{f^q(X \cup Y) - \alpha f^q(X)}{|Y|},$$

i.e.,

$$\lceil\alpha\rceil n - \lceil\alpha\rceil n \geq \frac{\lceil\alpha\rceil n(\lceil\alpha\rceil(n-m)+1) - \alpha\lceil\alpha\rceil n}{\lceil\alpha\rceil n},$$

which is equivalent to

$$\alpha \geq \lceil\alpha\rceil(n-m)+1.$$

Since $n > m$, this is a contradiction, i.e., $f^q \notin \mathcal{F}_\alpha$.

Now, with $X = \{c, b_1, ..., b_{\lceil\alpha\rceil m-1}\}$ and $Y = B \setminus X = \{b_{\lceil\alpha\rceil m}, ..., b_{\lceil\alpha\rceil n}\}$, we have

$$\frac{\sum_{y\in Y} f^q(X \cup \{y\}) - f^q(X)}{f^q(X \cup Y) - f^q(X)} = 0.$$

Thus, and because the set X can be the greedy solution $S_{\bar{k}}^{\mathrm{G}}$, the weak submodularity ratio of this problem is $\gamma(f^q) = 0$, i.e., $f^q \notin \tilde{\mathcal{F}}_\gamma$. ∎

4. γ-α-Augmentability. In this section, we argue that the class $\tilde{\mathcal{F}}_{\gamma,\alpha}$ of weakly γ-α-augmentable functions unifies and generalizes the classes $\tilde{\mathcal{F}}_\gamma$, $\mathcal{F}_\alpha$, and $\mathcal{F}_q$. We start by proving the first half of Theorem 1.5. The second half will be shown in Section 4.1, together with lower bounds for the approximation ratio of the greedy algorithm.

We need the following simple lemma.

LEMMA 4.1. *Let $(U,\mathcal{I})$ be an independence system with weight function $w\colon U \to \mathbb{R}_{\geq 0}$ and weighted rank function f. Furthermore, let $k \in [\bar{k}]$ and $x \in U \setminus S_k^G$ with $w(x) > 0$. Then, the following are equivalent:*

(i) $S_k^G \cup \{x\} \in \mathcal{I}$

(ii) $f(S_k^G \cup \{x\}) - f(S_k^G) = w(x)$

(iii) $f(S_k^G \cup \{x\}) - f(S_k^G) > 0$

Proof. "*(i)* ⇒ *(ii)*": By definition of f as a weighted rank function and because $S_k^{\mathrm{G}} \cup \{x\} \in \mathcal{I}$, we have

$$f(S_k^{\mathrm{G}} \cup \{x\}) - f(S_k^{\mathrm{G}}) = \sum_{x' \in S_k^{\mathrm{G}} \cup \{x\}} w(x') - \sum_{x' \in S_k^{\mathrm{G}}} w(x') = w(x).$$

"*(ii)* ⇒ *(iii)*": This follows immediately from the fact that $w(x) > 0$.

"*(iii) ⇒ (i)*": Let $x \in U \setminus S_k^{\mathrm{G}}$ with $f(S_k^{\mathrm{G}} \cup \{x\}) - f(S_k^{\mathrm{G}}) > 0$. Suppose there is some $s' \in S_k^{\mathrm{G}}$ with $w(x) > w(s')$. This means that x was considered by the greedy algorithm before and not added to the solution, i.e., $\{s \in S_k^{\mathrm{G}} \mid w(s) \geq w(x)\} \cup \{x\} \notin \mathcal{I}$. The fact that $f(S_k^{\mathrm{G}} \cup \{x\}) - f(S_k^{\mathrm{G}}) > 0$ implies that there is $\emptyset \neq S \subseteq S_k^{\mathrm{G}}$ with $S_k^{\mathrm{G}} \setminus S \cup \{x\} \in \mathcal{I}$ and $w(S) < w(x)$. The last inequality implies that $\{s \in S \mid w(s) \geq w(x)\} = \emptyset$, which means that $\{s \in S_k^{\mathrm{G}} \mid w(s) \geq w(x)\} \subseteq S_k^{\mathrm{G}} \setminus S$. But then $\{s \in S_k^{\mathrm{G}} \mid w(s) \geq w(x)\} \cup \{x\} \in \mathcal{I}$, which is a contradiction. Therefore, we have $w(x) \leq w(s)$ for all $s \in S_k^{\mathrm{G}}$. If we would have $S_k^{\mathrm{G}} \cup \{x\} \notin \mathcal{I}$, then the equality $f(S_k^{\mathrm{G}} \cup \{x\}) - f(S_k^{\mathrm{G}}) = 0$ would hold because every element in S_k^{G} has a greater weight than x and because $S_k^{\mathrm{G}} \in \mathcal{I}$. Thus, the statement holds. ∎

Since (weak) γ-α-augmentability implies (weak) γ'-α'-augmentability for all $\gamma \geq \gamma'$ and $\alpha \leq \alpha'$, the following proposition implies the first part of Theorem 1.5.

PROPOSITION 4.2. *For every $\gamma, q \in (0, 1]$, and every $\alpha \geq 1$, it holds that*

$$\tilde{\mathcal{F}}_{1,\alpha} \supseteq \mathcal{F}_\alpha \qquad \text{and} \qquad \tilde{\mathcal{F}}_{\gamma,\gamma} \supseteq \tilde{\mathcal{F}}_\gamma \qquad \text{and} \qquad \tilde{\mathcal{F}}_{\gamma,\gamma/q} \supseteq \mathcal{F}_q.$$

Proof. If $f \in \mathcal{F}_\alpha$, then, for all $X, Y \subseteq U$, and, in particular $X \in \{S_0^{\mathrm{G}}, ..., S_k^{\mathrm{G}}\}$, there exists $y \in Y$ with

$$f(X \cup \{y\}) - f(X) \geq \frac{1 \cdot f(X \cup Y) - \alpha f(X)}{|Y|},$$

which means that $f \in \tilde{\mathcal{F}}_{1,\alpha}$.

For the second part of the proof, let $f \in \tilde{\mathcal{F}}_\gamma$, $X \in \{S_0^{\mathrm{G}}, ..., S_k^{\mathrm{G}}\}$ and $Y \subseteq U \setminus X$. Furthermore, let $y^* \in \arg\max_{y \in Y} f(X \cup \{y\})$. Then, we have

$$\begin{aligned} |Y|(f(X \cup \{y^*\}) - f(X)) &\geq \sum_{y \in Y} (f(X \cup \{y\}) - f(X)) \\ &\geq \gamma(f) f(X \cup Y) - \gamma(f) f(X), \end{aligned}$$

where the second inequality follows from the definition of the weak submodularity ratio. Since $\gamma(f) \geq \gamma$, this means that f is weakly γ-γ-augmentable, i.e., $f \in \tilde{\mathcal{F}}_{\gamma,\gamma}$.

For the last part of the proof, let $f \in \mathcal{F}_q$ be the weighted rank function of an independence system $(U, \mathcal{I})$, and let $w \colon U \to \mathbb{R}_{\geq 0}$ be the associated weight function. Furthermore, let $k \in [\bar{k}]$ and $Y \subseteq U$. We prove that, for every $\gamma \in (0, 1]$, there exists $y \in Y$ with

$$f(S_k^{\mathrm{G}} \cup \{y\}) - f(S_k^{\mathrm{G}}) \geq \frac{\gamma f(S_k^{\mathrm{G}} \cup Y) - \frac{\gamma}{q(U,\mathcal{I})} f(S_k^{\mathrm{G}})}{|Y|}.$$

If $f(S_k^{\mathrm{G}} \cup Y) - \frac{1}{q(U,\mathcal{I})} f(S_k^{\mathrm{G}}) < 0$, the inequality holds by monotonicity of f. Thus, assume from now on that

$$f(S_k^{\mathrm{G}} \cup Y) - \frac{1}{q(U,\mathcal{I})} f(S_k^{\mathrm{G}}) \geq 0. \tag{4.1}$$

Let $S' \subseteq S_k^{\mathrm{G}}$ and $Y' \subseteq Y$ with $S' \cup Y' \in \mathcal{I}$ and $f(S_k^{\mathrm{G}} \cup Y) = w(S' \cup Y')$. Furthermore, let $y^* := \arg\max_{y \in Y'} f(S_k^{\mathrm{G}} \cup \{y\})$. We define

$$\tilde{Y} := \begin{cases} \{y \in Y' \mid w(y) > w(y^*)\}, & \text{if } f(S_k^{\mathrm{G}} \cup \{y^*\}) > f(S_k^{\mathrm{G}}), \\ Y', & \text{if } f(S_k^{\mathrm{G}} \cup \{y^*\}) = f(S_k^{\mathrm{G}}), \end{cases}$$

and we define the independence system $(\tilde{U}, \tilde{\mathcal{I}})$ with

$$\begin{aligned}\tilde{U} &:= S_k^{\mathrm{G}} \cup \tilde{Y},\\ \tilde{\mathcal{I}} &:= 2^{S_k^{\mathrm{G}}} \cup 2^{S' \cup \tilde{Y}}.\end{aligned}$$

We have $\tilde{U} \subseteq U$ and $\tilde{\mathcal{I}} \subseteq \mathcal{I}$ and thus $q(\tilde{U}, \tilde{\mathcal{I}}) \geq q(U, \mathcal{I})$. The greedy solution for the maximization problem on the independence system $(\tilde{U}, \tilde{\mathcal{I}})$ is S_k^{G}. Let $S^* \subseteq \tilde{U}$ be the optimal solution. Then, as shown in [15, 16], we have

$$\begin{aligned} f(S_k^{\mathrm{G}}) \geq& q(\tilde{U}, \tilde{\mathcal{I}}) f(S^*) \geq q(\tilde{U}, \tilde{\mathcal{I}}) f(S' \cup \tilde{Y})\\ \geq& q(U, \mathcal{I}) f(S' \cup \tilde{Y}) = q(U, \mathcal{I}) w(S' \cup \tilde{Y}). \end{aligned} \tag{4.2}$$

If $f(S_k^{\mathrm{G}} \cup \{y^*\}) > f(S_k^{\mathrm{G}})$, Lemma 4.1 yields $f(S_k^{\mathrm{G}} \cup \{y^*\}) - f(S_k^{\mathrm{G}}) = w(y^*)$, and if $f(S_k^{\mathrm{G}} \cup \{y^*\}) = f(S_k^{\mathrm{G}})$, by definition of $\tilde{Y}$, we have $|Y' \setminus \tilde{Y}| = 0$. Using this and the definition of $\tilde{Y}$, we get

$$\begin{aligned} |Y|(f(S_k^{\mathrm{G}} \cup \{y^*\}) - f(S_k^{\mathrm{G}})) &\geq |Y' \setminus \tilde{Y}| w(y^*)\\ &\geq w(Y' \setminus \tilde{Y})\\ &\overset{(4.2)}{\geq} w(Y' \setminus \tilde{Y}) + w(S' \cup \tilde{Y}) - \frac{1}{q(U, \mathcal{I})} f(S_k^{\mathrm{G}})\\ &= w(S' \cup Y') - \frac{1}{q(U, \mathcal{I})} f(S_k^{\mathrm{G}})\\ &= f(S_k^{\mathrm{G}} \cup Y) - \frac{1}{q(U, \mathcal{I})} f(S_k^{\mathrm{G}})\\ &\overset{(4.1), 1 \geq \gamma}{\geq} \gamma f(S_k^{\mathrm{G}} \cup Y) - \frac{\gamma}{q(U, \mathcal{I})} f(S_k^{\mathrm{G}}). \end{aligned}$$

Since $q(U, \mathcal{I}) \geq q$, this yields weak γ-$\frac{\gamma}{q}$-augmentability, i.e., $f \in \tilde{\mathcal{F}}_{\gamma, \gamma/q}$. ∎

Having shown that $\tilde{\mathcal{F}}_{\gamma,\alpha}$ subsumes the other three classes of functions, we now prove the upper bound on the approximation ratio in Theorem 1.6 for this class. Observe that the upper bound trivially carries over to the class of monotone, γ-α-augmentable (not weakly) functions.

THEOREM 4.3. *The approximation ratio of the greedy algorithm on the class $\tilde{\mathcal{F}}_{\gamma,\alpha}$ of monotone, weakly γ-α-augmentable functions, with $\gamma \in (0, 1]$ and $\alpha \geq \gamma$, is at most*

$$\frac{\alpha}{\gamma} \cdot \frac{\mathrm{e}^{\alpha}}{\mathrm{e}^{\alpha} - 1}.$$

Proof. Let $f \in \tilde{\mathcal{F}}_{\gamma,\alpha}$. First we consider the case $k > \bar{k}$, i.e., the case that the greedy algorithm stops early because the value of the solution cannot be increased by adding any element. Because f is weakly γ-α-augmentable and there is no element $u \in U$ with $f(S_{\bar{k}} \cup \{u\}) - f(S_{\bar{k}}) > 0$, we have, for all $x \in S_{\bar{k}}^*$,

$$0 = |S_{\bar{k}}^*|(f(S_{\bar{k}}^{\mathrm{G}} \cup \{x\}) - f(S_{\bar{k}}^{\mathrm{G}})) \geq \gamma f(S_{\bar{k}}^{\mathrm{G}} \cup S_{\bar{k}}^*) - \alpha f(S_{\bar{k}}^{\mathrm{G}}) \geq \gamma f(S_{\bar{k}}^*) - \alpha f(S_{\bar{k}}^{\mathrm{G}}),$$

i.e., $f(S_{\bar{k}}^{\mathrm{G}}) \geq \frac{\gamma}{\alpha} f(S_{\bar{k}}^*) > \frac{\gamma}{\alpha} \cdot \frac{\mathrm{e}^{\alpha} - 1}{\mathrm{e}^{\alpha}} f(S_{\bar{k}}^*)$.

Now consider the case that $k \leq \bar{k}$. For ease of notation, we define the gain of the greedy algorithm in iteration j to be $\delta_j := f(S_j^{\mathrm{G}}) - f(S_{j-1}^{\mathrm{G}})$ for all $j \in [k]$. Let $i \in [k]$ and

$$x^* := \arg\max_{x \in S_k^*} f(S_{i-1}^{\mathrm{G}} \cup \{x\}) - f(S_{i-1}^{\mathrm{G}}).$$

We have

$$\delta_i \geq f(S^{\mathrm{G}}_{i-1} \cup \{x^*\}) - f(S^{\mathrm{G}}_{i-1}) \geq \frac{\gamma f(S^{\mathrm{G}}_{i-1} \cup S^*_k) - \alpha f(S^{\mathrm{G}}_{i-1})}{|S^*_k|}$$

$$\geq \frac{\gamma}{k} f(S^*_k) - \frac{\alpha}{k} f(S^{\mathrm{G}}_{i-1}) = \frac{\gamma}{k} f(S^*_k) - \frac{\alpha}{k} \sum_{j=1}^{i-1} \delta_j - \frac{\alpha}{k} f(\emptyset). \tag{4.3}$$

We prove by induction that, for all $\ell \in \{0, ..., k\}$, we have

$$f(S^*_k) - \frac{\alpha}{\gamma} \sum_{j=1}^{\ell} \delta_j - \frac{\alpha}{\gamma} f(\emptyset) \leq f(S^*_k)(1 - \frac{\alpha}{k})^{\ell}. \tag{4.4}$$

For $\ell = 0$ the equation obviously holds. Now suppose that (4.4) holds for some $\ell \in \{0, ..., k-1\}$. Then, for $\ell + 1$, we have

$$\begin{aligned}
& f(S^*_k) - \frac{\alpha}{\gamma} \sum_{j=1}^{\ell+1} \delta_j - \frac{\alpha}{\gamma} f(\emptyset) \\
= \;& f(S^*_k) - \frac{\alpha}{\gamma} \sum_{j=1}^{\ell} \delta_j - \frac{\alpha}{\gamma} \delta_{\ell+1} - \frac{\alpha}{\gamma} f(\emptyset) \\
\overset{(4.3)}{\leq} \;& f(S^*_k) - \frac{\alpha}{\gamma} \sum_{j=1}^{\ell} \delta_j - \frac{\alpha}{\gamma} \Big(\frac{\gamma}{k} f(S^*_k) - \frac{\alpha}{k} \sum_{j=1}^{\ell} \delta_j - \frac{\alpha}{k} f(\emptyset) \Big) - \frac{\alpha}{\gamma} f(\emptyset) \\
= \;& \Big(f(S^*_k) - \frac{\alpha}{\gamma} \sum_{j=1}^{\ell} \delta_j - \frac{\alpha}{\gamma} f(\emptyset) \Big) (1 - \frac{\alpha}{k}) \overset{(4.4)}{\leq} f(S^*_k)(1 - \frac{\alpha}{k})^{\ell+1},
\end{aligned}$$

and (4.4) continues to hold. Because of $1 + x \leq \mathrm{e}^x$ for $x \in \mathbb{R}$, we have

$$f(S^*_k) - \frac{\alpha}{\gamma} \sum_{j=1}^{\ell} \delta_j - \frac{\alpha}{\gamma} f(\emptyset) \overset{(4.4)}{\leq} f(S^*_k)(1 - \frac{\alpha}{k})^{\ell} \leq \mathrm{e}^{-\frac{\alpha}{k}\ell} f(S^*_k).$$

Rearranging this for $\ell = k$ and using the fact that $f(S^{\mathrm{G}}_k) = \sum_{j=1}^{k} \delta_j + f(\emptyset)$, yields

$$f(S^{\mathrm{G}}_k) \geq \frac{\gamma}{\alpha} \cdot \frac{\mathrm{e}^{\alpha} - 1}{\mathrm{e}^{\alpha}} f(S^*_k). \qquad \square$$

4.1. A Critical Function. To obtain the tight lower bound of Theorem 1.6 for weakly γ-α-augmentable problems and to separate this class from $\tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_\alpha \cup \mathcal{F}_q$, we introduce a function that is inspired by a construction in [3] for the submodularity ratio.

We fix $\gamma \in (0, 1]$ and $\alpha \geq \gamma$. Let $k \in \mathbb{N}$ with $k > \alpha$, and let $A = \{a_1, ..., a_k\}$ and $B = \{b_1, ..., b_k\}$ be disjoint sets. We set $U = A \cup B$, define $\xi_i := \frac{1}{k}(\frac{k-\alpha}{k})^{i-1}$ and let $h(x) := \frac{\gamma^{-1}-1}{k-1} x^2 + \frac{k-\gamma^{-1}}{k-1} x$. For our purpose, the important facts about h are $h(0) = 0$, $h(1) = 1$, $h(k) = \frac{k}{\gamma}$ and that h is convex and non-decreasing on $[0, k]$). With this in mind, we define the function $F_{\gamma,\alpha,k} \colon 2^U \to \mathbb{R}_{\geq 0}$ by

$$F_{\gamma,\alpha,k}(X) = \max_{X' \subseteq X} \Big\{ \frac{h(|\{b_1\} \cap X'| \cdot |B \cap X'|)}{k} \Big(1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in A \cap X'}} \xi_i \Big) + \sum_{\substack{i \in [k]: \\ a_i \in A \cap X'}} \xi_i \Big\}$$

If $h(|\{b_1\} \cap X| \cdot |B \cap X|) > \frac{k}{\alpha}$, we have

$$F_{\gamma,\alpha,k}(X) = \frac{h(|\{b_1\} \cap X| \cdot |B \cap X|)}{k},$$

and otherwise, if $h(|\{b_1\} \cap X| \cdot |B \cap X|) \leq \frac{k}{\alpha}$, we have

$$F_{\gamma,\alpha,k}(X) = \frac{h(|\{b_1\} \cap X| \cdot |B \cap X|)}{k}\Big(1 - \alpha \sum_{\substack{i \in [k]: \\ a_i \in A \cap X}} \xi_i\Big) + \sum_{\substack{i \in [k]: \\ a_i \in A \cap X}} \xi_i.$$

We observe that, for $X \subseteq B$, convexity of h, $h(0) = 0$, $h(k) = k/\gamma$ and $|X| \leq |B| = k$ imply that

$$h(|\{b_1\} \cap X| \cdot |X|) \leq \frac{|\{b_1\} \cap X| \cdot |X|}{\gamma}, \tag{4.5}$$

and, for $\ell \in \{0, ..., k\}$, we have

$$\sum_{i=1}^{\ell} \xi_i = \sum_{i=1}^{\ell} \frac{1}{k}\Big(\frac{k-\alpha}{k}\Big)^{i-1} = \frac{1}{k} \cdot \frac{1 - \big(\frac{k-\alpha}{k}\big)^{\ell}}{1 - \frac{k-\alpha}{k}} = \frac{1 - \big(\frac{k-\alpha}{k}\big)^{\ell}}{\alpha}. \tag{4.6}$$

We show that our modification of the function introduced in [3] retains the same structure in regard to greedy solutions.

Proposition 4.4. *For $i \in [k]$, the greedy algorithm picks the element a_i in iteration i, and, for $i \in [2k] \setminus [k]$, the greedy algorithm picks the element b_{i-k} in iteration i.*

Proof. First, we consider the case $i \in [k]$. Suppose that in iteration i, the initial solution is $\{a_1, ..., a_{i-1}\}$, where $\{a_1, ..., a_0\} = \emptyset$, with objective value $\sum_{\ell=1}^{i-1} \xi_\ell$. Adding an element from $\{b_2, ..., b_k\}$ does not increase the objective value because $\{b_1\} \cap \{b\} = \emptyset$ for all $b \in \{b_2, ..., b_k\}$. For $j \in \{i, ..., k\}$, adding a_j increases the objective value by $\xi_j = \frac{1}{k}(\frac{k-\alpha}{k})^{j-1}$. Since $k > \alpha$, we have $\xi_i \geq \xi_j$ for $j \geq i$. Adding the element b_1 to the solution $\{a_1, ..., a_{i-1}\}$ increases the objective value by

$$\frac{1}{k}\Big(1 - \alpha \sum_{\ell=1}^{i-1} \xi_\ell\Big) \overset{(4.6)}{=} \frac{1}{k}\Big(1 - \alpha \frac{1 - \big(\frac{k-\alpha}{k}\big)^{i-1}}{\alpha}\Big) = \frac{1}{k}\Big(\frac{k-\alpha}{k}\Big)^{i-1}.$$

Thus, with proper tie breaking, the greedy algorithm picks the element a_i in iteration i for $i \in [k]$.

Now, we consider the case that $i \in \{k+1, ..., 2k\}$. For $i = k+1$, adding an element from $\{b_2, ..., b_k\}$ does not increase the objective value, while adding b_1 increases it by $\frac{1}{k}(\frac{k-\alpha}{k})^k$. Thus, in iteration $k+1$, the element b_1 is added to the solution. For $i \geq k+2$, adding any element from $B \setminus S^{\mathrm{G}}_{i-1}$ to the greedy solution S^{G}_{i-1} increases the function value by the same amount. Therefore, with proper tie breaking, the greedy algorithm picks the element b_{i-k} in iteration i for $i \in \{k+1, ..., 2k\}$. ∎

With this, we can show that $F_{\gamma,\alpha,k}$ is weakly γ-α-augmentable.

Lemma 4.5. *For every $\gamma \in (0, 1]$, every $\alpha \geq \gamma$, and every $k \in \mathbb{N}$ with $k > \alpha$, it holds that $F_{\gamma,\alpha,k} \in \tilde{\mathcal{F}}_{\gamma,\alpha}$.*

Proof. The monotonicity of $F_{\gamma,\alpha,k}$ immediately follows from the maximum in the definition. To prove weak γ-α-augmentability, let $X \in \{S_0^{\mathrm{G}}, ..., S_{2k}^{\mathrm{G}}\}$ and $Y \subseteq U$. We define $Y' := Y \setminus X$. For better readability, we will write $F := F_{\gamma,\alpha,k}$.

First, consider the case that $X \subseteq A$. Then $F(X) = \sum_{i\in[k]:a_i\in X} \xi_i$ because $h(0) = 0$. Thus and because $h(1) = 1$, for all $y \in Y'$, we have

$$F(X \cup \{y\}) - F(X) = \begin{cases} \xi_i, & \text{if } y = a_i \in (A \cap Y'), \\ \frac{1}{k}(1 - \alpha \sum_{i\in[k]:a_i\in X} \xi_i), & \text{if } y = b_1, \\ 0, & \text{else,} \end{cases}$$

i.e.,

$$\begin{aligned} & |Y'|\big(\max_{y\in Y'} F(X \cup \{y\}) - F(X)\big) \\ \geq\ & \Big(\sum_{y\in A\cap Y'} \big(F(X \cup \{y\}) - F(X)\big)\Big) + \\ & |B \cap Y'|\big(\max_{y\in B\cap Y'} F(X \cup \{y\}) - F(X)\big) \\ =\ & \Big(\sum_{\substack{i\in[k]:\\ a_i\in A\cap Y'}} \xi_i\Big) + |\{b_1\} \cap Y'| \cdot |B \cap Y'| \frac{1}{k}\Big(1 - \alpha \sum_{\substack{i\in[k]:\\ a_i\in X}} \xi_i\Big). \end{aligned} \tag{4.7}$$

If $h(|\{b_1\}\cap Y'|\cdot|B\cap Y'|) \leq \frac{k}{\alpha}$, we use the fact that $F(X) = \sum_{i\in[k]:a_i\in X} \xi_i$ to calculate

$$\begin{aligned} & \gamma F(X \cup Y) - \alpha F(X) \\ \overset{B\cap X=\emptyset}{=}\ & \gamma\Big(\frac{h(|\{b_1\} \cap Y'| \cdot |B \cap Y'|)}{k}\Big(1 - \alpha \sum_{\substack{i\in[k]:\\ a_i\in X\cup(A\cap Y')}} \xi_i\Big) \\ & + \sum_{\substack{i\in[k]:\\ a_i\in X\cup(A\cap Y')}} \xi_i\Big) - \alpha \sum_{\substack{i\in[k]:\\ a_i\in X}} \xi_i \\ =\ & \Big[\frac{\gamma}{k} h(|\{b_1\} \cap Y'| \cdot |B \cap Y'|)\Big(1 - \alpha \sum_{\substack{i\in[k]:\\ a_i\in X}} \xi_i\Big)\Big] \\ & + \Big[\gamma\big(1 - \frac{\alpha}{k} h(|\{b_1\} \cap Y'| \cdot |B \cap Y'|)\big)\Big(\sum_{\substack{i\in[k]:\\ a_i\in A\cap Y'}} \xi_i\Big)\Big] + \Big[(\gamma - \alpha) \sum_{\substack{i\in[k]:\\ a_i\in X}} \xi_i\Big] \\ \leq\ & \Big[\frac{1}{k}|\{b_1\} \cap Y'| \cdot |B \cap Y'|\Big(1 - \alpha \sum_{\substack{i\in[k]:\\ a_i\in X}} \xi_i\Big)\Big] + \Big[\sum_{\substack{i\in[k]:\\ a_i\in A\cap Y'}} \xi_i\Big] + [0]. \end{aligned} \tag{4.8}$$

The first part of the last inequality follows from (4.5). The second part of the inequality follows from the fact that $\gamma \in (0, 1]$ and, for $x \geq 0$, we have $\frac{\alpha}{k} h(x) \geq 0$. The last part follows from the fact that $\gamma \leq \alpha$. Combining equations (4.7) and (4.8) together with the fact that $Y' \subseteq Y$ yields weak γ-α-augmentability.

Otherwise, if $h(|\{b_1\}\cap Y'|\cdot|B\cap Y'|)>\frac{k}{\alpha}$, we have

$$\begin{aligned}\gamma F(X\cup Y)-\alpha F(X) &= \gamma\frac{h(|\{b_1\}\cap Y'|\cdot|B\cap Y'|)}{k}-\alpha\sum_{\substack{i\in[k]:\\ a_i\in X}}\xi_i\\ &\overset{(4.5)}{\leq} \frac{1}{k}|\{b_1\}\cap Y'|\cdot|B\cap Y'|-\alpha\sum_{\substack{i\in[k]:\\ a_i\in X}}\xi_i\\ &\overset{|B|=k}{\leq} \frac{1}{k}|\{b_1\}\cap Y'|\cdot|B\cap Y'|\Big(1-\alpha\sum_{\substack{i\in[k]:\\ a_i\in X}}\xi_i\Big)\\ &\overset{(4.7)}{\leq} |Y'|\big(\max_{y\in Y'}F(X\cup\{y\})-F(X)\big),\end{aligned}$$

which yields γ-α-augmentability also in this case.

Now, consider the case that $X\not\subseteq A$. Then, by Proposition 4.4, we have $X=A\cup\{b_1,...,b_i\}$ for some $i\in[k]$. If $i=k$, i.e., $X=U$, we have $\gamma F(X\cup Y)-\alpha F(X)=(\gamma-\alpha)F(U)\leq 0$ and we are done. Thus, assume that $i<k$. Because h is convex and non-decreasing on $[0,k]$, we have

$$\frac{h(i+|Y'|)-h(i)}{|Y'|}\overset{|Y'|+i\leq|B|}{\leq}\frac{h(|B|)-h(i)}{|B|-i}. \tag{4.9}$$

With

$$H(i):=(k-i)\frac{h(i+1)-h(i)}{h(k)-h(i)},$$

we have

$$H'(i)=(k-1)\frac{2-3\gamma+\gamma^2}{(k-1+i-\gamma i)^2}\geq 0,$$

which yields

$$H(i)\geq H(0)=k\frac{1-0}{\frac{k}{\gamma}-0}=\gamma. \tag{4.10}$$

Combining this with (4.9), we obtain

$$\frac{|Y'|\big(h(i+1)-h(i)\big)}{h(i+|Y'|)-h(i)}\overset{(4.9)}{\geq}\frac{(|B|-i)\big(h(i+1)-h(i)\big)}{h(|B|)-h(i)}\overset{|B|=k}{=}H(i)\overset{(4.10)}{\geq}\gamma. \tag{4.11}$$

If $h(i+|Y'|)\leq\frac{k}{\alpha}$, because h is increasing for positive values, we have

$h(i) \leq h(i+1) \leq h(i+|Y'|) \leq \frac{k}{\alpha}$. Thus, for every $y \in Y'$, we have

$$\begin{aligned}
|Y|\big(F(X\cup\{y\})-F(X)\big) &\geq |Y'|\frac{h(i+1)-h(i)}{k}\Big(1-\alpha\sum_{j=1}^{k}\xi_j\Big)\\
&\overset{(4.11)}{\geq} \frac{\gamma\big(h(i+|Y'|)-h(i)\big)}{k}\Big(1-\alpha\sum_{j=1}^{k}\xi_j\Big)\\
&\overset{\gamma\leq\alpha}{\geq} \Big(\gamma\frac{h(i+|Y'|)}{k}-\alpha\frac{h(i)}{k}\Big)\Big(1-\alpha\sum_{j=1}^{k}\xi_j\Big) \qquad (4.12)\\
&\overset{\gamma\leq\alpha}{\geq} \gamma\Big[\frac{h(i+|Y'|)}{k}\Big(1-\alpha\sum_{j=1}^{k}\xi_j\Big)+\sum_{j=1}^{k}\xi_j\Big]\\
&\quad -\alpha\Big[\frac{h(i)}{k}\Big(1-\alpha\sum_{j=1}^{k}\xi_j\Big)+\sum_{j=1}^{k}\xi_j\Big]\\
&= \gamma F(X\cup Y)-\alpha F(X).
\end{aligned}$$

If $h(i) \leq h(i+1) \leq \frac{k}{\alpha} < h(i+|Y'|)$, then, for every $y \in Y'$, we have

$$\begin{aligned}
&|Y|\big(F(X\cup\{y\})-F(X)\big)\\
\overset{(4.12)}{\geq}\ &\Big(\gamma\frac{h(i+|Y'|)}{k}-\alpha\frac{h(i)}{k}\Big)\Big(1-\alpha\sum_{j=1}^{k}\xi_j\Big)\\
=\ &\gamma\frac{h(i+|Y'|)}{k}-\alpha\frac{h(i)}{k}\Big(1-\alpha\sum_{j=1}^{k}\xi_j\Big)-\frac{\gamma}{k}h(i+|Y'|)\alpha\sum_{j=1}^{k}\xi_j\\
\overset{h(i+|Y'|)\leq h(k)=\frac{k}{\gamma}}{\geq}\ &\gamma\frac{h(i+|Y'|)}{k}-\alpha\Big[\frac{h(i)}{k}\Big(1-\alpha\sum_{j=1}^{k}\xi_j\Big)+\sum_{j=1}^{k}\xi_j\Big]\\
=\ &\gamma F(X\cup Y)-\alpha F(X). \qquad (4.13)
\end{aligned}$$

If $h(i) \leq \frac{k}{\alpha} < h(i+1) \leq h(i+|Y'|)$, then

$$\frac{\alpha}{k}h(i+1) > \frac{\alpha}{k}\cdot\frac{k}{\alpha} = 1, \qquad (4.14)$$

which implies that, for every $y \in Y'$, we have

$$F(X\cup\{y\}) = \frac{h(i+1)}{k} \overset{(4.14)}{>} \frac{h(i+1)}{k}\Big(1-\alpha\sum_{j=1}^{k}\xi_j\Big)+\sum_{j=1}^{k}\xi_j. \qquad (4.15)$$

This implies

$$\begin{aligned}
|Y|\big(F(X\cup\{y\})-F(X)\big) &\overset{(4.15)}{\geq} |Y'|\frac{h(i+1)-h(i)}{k}\Big(1-\alpha\sum_{j=1}^{k}\xi_j\Big)\\
&\overset{(4.11)}{\geq} \frac{\gamma\big(h(i+|Y'|)-h(i)\big)}{k}\Big(1-\alpha\sum_{j=1}^{k}\xi_j\Big)\\
&\geq \frac{\gamma h(i+|Y'|)-\alpha h(i)}{k}\Big(1-\alpha\sum_{j=1}^{k}\xi_j\Big)\\
&\overset{(4.13)}{\geq} \gamma F(X\cup Y)-\alpha F(X).
\end{aligned}$$

If $\frac{k}{\alpha} < h(i) \leq h(i+1) \leq h(i+|Y'|)$, then, for every $y \in Y'$, we have

$$\begin{aligned}
|Y|\big(F(X\cup\{y\})-F(X)\big) &= |Y'|\frac{h(i+1)-h(i)}{k}\\
&\overset{(4.11)}{\geq} \frac{\gamma\big(h(i+|Y'|)-h(i)\big)}{k}\\
&\geq \gamma\frac{h(i+|Y'|)}{k}-\alpha\frac{h(i)}{k}\\
&= \gamma F(X\cup Y)-\alpha F(X),
\end{aligned}$$

i.e., also in all of these cases, F is γ-α-augmentable. ∎

It is straightforward to bound the approximation ratio of the greedy algorithm for $F_{\gamma,\alpha,k}$.

Proposition 4.6. *The approximation ratio of the greedy algorithm for maximizing the function $F_{\gamma,\alpha,k}$, with $\gamma \in (0,1]$, $\alpha \geq \gamma$ and $k \in \mathbb{N}$ with $k > \alpha$, is at least*

$$\frac{\alpha}{\gamma}\frac{1}{1-(1-\frac{\alpha}{k})^k}.$$

Proof. We compare the objective values of the greedy solution S_k^{G} of size k and the solution B, which also has size k. By Proposition 4.4, we have $S_k^{\mathrm{G}} = A$, and thus

$$F(S_k^{\mathrm{G}}) = F(A) = \sum_{i=1}^{k}\xi_i \overset{(4.6)}{=} \frac{1-\left(\frac{k-\alpha}{k}\right)^k}{\alpha}$$

and

$$F(B) = \frac{h(k)}{k} = \frac{\frac{k}{\gamma}}{k} = \frac{1}{\gamma}.$$

Thus, the greedy algorithm has an approximation ratio of at least

$$\frac{F(S_k^*)}{F(S_k^{\mathrm{G}})} \geq \frac{F(B)}{F(S_k^{\mathrm{G}})} = \frac{\alpha}{\gamma}\cdot\frac{1}{1-\left(\frac{k-\alpha}{k}\right)^k}.$$

∎

Theorem 4.7. *The approximation ratio of the greedy algorithm on the class $\tilde{\mathcal{F}}_{\gamma,\alpha}$ of monotone, weakly γ-α-augmentable functions, with $\gamma \in (0,1]$ and $\alpha \geq \gamma$, is at least*

$$\frac{\alpha}{\gamma}\cdot\frac{\mathrm{e}^{\alpha}}{\mathrm{e}^{\alpha}-1}.$$

Proof. By Lemma 4.5, $F_{\gamma,\alpha,k} \in \tilde{\mathcal{F}}_{\gamma,\alpha}$, and, by Proposition 4.6, the greedy algorithm has an approximation ratio of at least $\frac{\alpha}{\gamma}\frac{1}{1-(1-\frac{\alpha}{k})^k}$ for maximizing $F_{\gamma,\alpha,k}$. The general lower bound follows, since

$$\lim_{k\to\infty} \frac{1}{1-\left(\frac{k-\alpha}{k}\right)^k} = \frac{1}{1-\mathrm{e}^{-\alpha}} = \frac{\mathrm{e}^{\alpha}}{\mathrm{e}^{\alpha}-1}.$$

∎

It even turns out that, for $\gamma = 1$, the function $F_{\gamma,\alpha,k}$ is α-augmentable. This allows to carry the lower bound over to the class $\mathcal{F}_{\alpha}$.

PROPOSITION 4.8. *For every $\alpha \geq 1$, and every $k \in \mathbb{N}$ with $k \geq \alpha$, it holds that $F_{1,\alpha,k} \in \mathcal{F}_{\alpha}$.*

Proof. By Lemma 4.5, $F_{1,\alpha,k}$ is monotone. Thus, it suffices to prove that the function is α-augmentable. For better readability, we write $F := F_{1,\alpha,k}$. Observe that, if $\gamma = 1$, we have $h(x) = x$ for all $x \in \mathbb{R}$. Let $X, Y \subseteq U$ and $Y' := Y \setminus X$.

If $|\{b_1\} \cap (X \cup Y)| \cdot |B \cap (X \cup Y)| \leq \frac{k}{\alpha}$, for $y \in Y'$, we have

$$F(X \cup \{y\}) - F(X) = \begin{cases} \big(1 - \frac{|\{b_1\}\cap X|\cdot|B\cap X|}{k}\alpha\big)\xi_i, & \text{if } y = a_i \in A \cap Y', \\ \frac{|\{b_1\}\cap(X\cup\{y\})|\cdot|B\cap(X\cup\{y\})|-|\{b_1\}\cap X|\cdot|B\cap X|}{k} & \\ \quad\cdot\big(1-\alpha\sum_{i\in[k]:a_i\in A\cap X}\xi_i\big), & \text{if } y \in B \cap Y'. \end{cases} \tag{4.16}$$

This yields

$$\begin{aligned}
& F(X\cup Y) - \alpha F(X) \\
= \; & \Big(\frac{|\{b_1\}\cap(X\cup Y')|\cdot|B\cap(X\cup Y')|}{k}\Big(1-\alpha\sum_{\substack{i\in[k]:\\ a_i\in A\cap(X\cup Y')}}\xi_i\Big) + \sum_{\substack{i\in[k]:\\ a_i\in A\cap(X\cup Y')}}\xi_i\Big) \\
& -\alpha\Big(\frac{|\{b_1\}\cap X|\cdot|B\cap X|}{k}\Big(1-\alpha\sum_{\substack{i\in[k]:\\ a_i\in A\cap X}}\xi_i\Big) + \sum_{\substack{i\in[k]:\\ a_i\in A\cap X}}\xi_i\Big) \\
= \; & \Big[\frac{|\{b_1\}\cap(X\cup Y')|\cdot|B\cap(X\cup Y')| - \alpha|\{b_1\}\cap X|\cdot|B\cap X|}{k}\Big(1-\alpha\sum_{\substack{i\in[k]:\\ a_i\in A\cap X}}\xi_i\Big)\Big] \\
& +\Big[\Big(1-\frac{|\{b_1\}\cap(X\cup Y')|\cdot|B\cap(X\cup Y')|}{k}\alpha\Big)\sum_{\substack{i\in[k]:\\ a_i\in A\cap Y'}}\xi_i\Big] + \Big[(1-\alpha)\sum_{\substack{i\in[k]:\\ a_i\in A\cap X}}\xi_i\Big] \\
\leq \; & \Big[|B\cap Y'|\max_{y\in B\cap Y'}\Big\{\frac{|\{b_1\}\cap(X\cup\{y\})|\cdot|B\cap(X\cup\{y\})| - |\{b_1\}\cap X|\cdot|B\cap X|}{k} \\
& \big(1-\alpha\sum_{\substack{i\in[k]:\\ a_i\in A\cap X}}\xi_i\big)\Big\}\Big] + \Big[\Big(1-\frac{|\{b_1\}\cap X|\cdot|B\cap X|}{k}\alpha\Big)\sum_{\substack{i\in[k]:\\ a_i\in A\cap Y'}}\xi_i\Big] + [0] \\
\overset{(4.16)}{=} \; & |B\cap Y'|\big(\max_{y\in B\cap Y'}F(X\cup\{y\}) - F(X)\big) + \sum_{y\in A\cap Y'}(F(X\cup\{y\}) - F(X)) \\
\leq \; & |Y|\big(\max_{y\in Y}F(X\cup\{y\}) - F(X)\big).
\end{aligned}$$

This establishes α-augmentability if $|\{b_1\} \cap (X \cup Y)| \cdot |B \cap (X \cup Y)| \leq \frac{k}{\alpha}$.

Consider the case that $|\{b_1\}\cap X|\cdot|B\cap X|\leq\frac{k}{\alpha}<|\{b_1\}\cap(X\cup Y)|\cdot|B\cap(X\cup Y)|$. If, for $y\in B\cap Y'$, we have $|\{b_1\}\cap(X\cup\{y\})|\cdot|B\cap(X\cup\{y\})|\leq\frac{k}{\alpha}$, then

$$\begin{aligned}&F(X\cup\{y\})-F(X)\\ \overset{(4.16)}{=}\ &\frac{|\{b_1\}\cap(X\cup\{y\})|\cdot|B\cap(X\cup\{y\})|-|\{b_1\}\cap X|\cdot|B\cap X|}{k}\big(1-\alpha\sum_{i\in[k]:a_i\in A\cap X}\xi_i\big)\end{aligned}$$

and if

$$|\{b_1\}\cap(X\cup\{y\})|\cdot|B\cap(X\cup\{y\})|>\frac{k}{\alpha}\tag{4.17}$$

for $y\in B\cap Y'$, we have

$$\begin{aligned}&F(X\cup\{y\})-F(X)\\ =\ &\frac{|B\cap(X\cup\{y\})|}{k}-\frac{|\{b_1\}\cap X|\cdot|B\cap X|}{k}\big(1-\alpha\sum_{i\in[k]:a_i\in A\cap X}\xi_i\big)-\sum_{i\in[k]:a_i\in A\cap X}\xi_i\\ \overset{(4.17)}{\geq}\ &\frac{|\{b_1\}\cap(X\cup\{y\})|\cdot|B\cap(X\cup\{y\})|-|\{b_1\}\cap X|\cdot|B\cap X|}{k}\big(1-\alpha\sum_{i\in[k]:a_i\in A\cap X}\xi_i\big).\end{aligned}$$

This means that, in either case, for $y\in B\cap Y'$, we have

$$\begin{aligned}&F(X\cup\{y\})-F(X)\\ \geq\ &\frac{|\{b_1\}\cap(X\cup\{y\})|\cdot|B\cap(X\cup\{y\})|-|\{b_1\}\cap X|\cdot|B\cap X|}{k}\\ &\cdot\big(1-\alpha\sum_{\substack{i\in[k]:\\ a_i\in A\cap X}}\xi_i\big).\end{aligned}\tag{4.18}$$

Since we consider $|\{b_1\}\cap X|\cdot|B\cap X|\leq\frac{k}{\alpha}<|\{b_1\}\cap(X\cup Y)|\cdot|B\cap(X\cup Y)|$, we have

$$Y'\cap B\neq\emptyset\tag{4.19}$$

and

$$b_1\in X\cup Y=X\cup Y'.\tag{4.20}$$

This yields

$$
\begin{aligned}
& F(X\cup Y)-\alpha F(X)\\
= \;& \frac{|B\cap(X\cup Y')|}{k}-\alpha\Big(\frac{|\{b_1\}\cap X|\cdot|B\cap X|}{k}\Big(1-\alpha\sum_{\substack{i\in[k]:\\ a_i\in A\cap X}}\xi_i\Big)+\sum_{\substack{i\in[k]:\\ a_i\in A\cap X}}\xi_i\Big)\\
= \;& \Big[\frac{|B\cap(X\cup Y')|}{k}-\alpha\sum_{\substack{i\in[k]:\\ a_i\in A\cap X}}\xi_i\Big]-\Big[\alpha\frac{|\{b_1\}\cap X|\cdot|B\cap X|}{k}\Big(1-\alpha\sum_{\substack{i\in[k]:\\ a_i\in A\cap X}}\xi_i\Big)\Big]\\
\overset{\alpha\geq 1}{\leq} \;& \Big[\frac{|B\cap(X\cup Y')|}{k}-\alpha\sum_{\substack{i\in[k]:\\ a_i\in A\cap X}}\xi_i\Big]-\Big[\frac{|\{b_1\}\cap X|\cdot|B\cap X|}{k}\Big(1-\alpha\sum_{\substack{i\in[k]:\\ a_i\in A\cap X}}\xi_i\Big)\Big]\\
= \;& \Big[\frac{|B\cap(X\cup Y')|}{k}-\alpha\sum_{\substack{i\in[k]:\\ a_i\in A\cap X}}\xi_i\Big]\\
& -\Big[(|B\cap Y'|-(|B\cap Y'|-1))\frac{|\{b_1\}\cap X|\cdot|B\cap X|}{k}\Big(1-\alpha\sum_{\substack{i\in[k]:\\ a_i\in A\cap X}}\xi_i\Big)\Big]\\
\overset{|B|=k,|\{b_1\}\cap X|\leq 1}{\leq} \;& \Big[\frac{|B\cap(X\cup Y')|}{k}\big(1-\alpha\sum_{\substack{i\in[k]:\\ a_i\in A\cap X}}\xi_i\big)\Big]\\
& -\Big[\frac{|B\cap Y'|\cdot|\{b_1\}\cap X|\cdot|B\cap X|-(|B\cap Y'|-1)|B\cap X|}{k}\Big(1-\alpha\sum_{\substack{i\in[k]:\\ a_i\in A\cap X}}\xi_i\Big)\Big]\\
= \;& |B\cap Y'|\frac{|B\cap X|+1-|\{b_1\}\cap X|\cdot|B\cap X|}{k}\big(1-\alpha\sum_{\substack{i\in[k]:\\ a_i\in A\cap X}}\xi_i\big)\\
\overset{(4.19),(4.20)}{=} \;& |B\cap Y'|\max_{y\in B\cap Y'}\Big\{\frac{|\{b_1\}\cap(X\cup\{y\})|\cdot|B\cap(X\cup\{y\})|-|\{b_1\}\cap X|\cdot|B\cap X|}{k}\\
& \cdot\big(1-\alpha\sum_{\substack{i\in[k]:\\ a_i\in A\cap X}}\xi_i\big)\Big\}\\
\overset{(4.18)}{\leq} \;& |B\cap Y'|\big(\max_{y\in B\cap Y'}F(X\cup\{y\})-F(X)\big)
\end{aligned}
$$

Thus, if $|\{b_1\}\cap X|\cdot|B\cap X|\leq\frac{k}{\alpha}<|\{b_1\}\cap(X\cup Y)|\cdot|B\cap(X\cup Y)|$, the function F is α-augmentable.

If $\frac{k}{\alpha}<|\{b_1\}\cap X|\cdot|B\cap X|$, for $y\in Y'$, we have

$$
F(X\cup\{y\})-F(X)=\begin{cases}0 & \text{if } y=a_i\in A\cap Y',\\ \frac{|B\cap(X\cup\{y\})|-|B\cap X|}{k} & \text{if } y\in B\cap Y',\end{cases}
\tag{4.21}
$$

which yields

$$\begin{aligned}
F(X\cup Y)-\alpha F(X) &= \frac{|B\cap(X\cup Y')|}{k}-\alpha\frac{|B\cap X|}{k}\\
&\overset{\alpha\geq 1}{\leq} \frac{|B\cap(X\cup Y')|-|B\cap X|}{k}\\
&= \frac{|B\cap Y'|}{k}\\
&= |B\cap Y'|\max_{y\in B\cap Y'}\frac{|B\cap(X\cup\{y\})|-|B\cap X|}{k}\\
&\leq |Y|\max_{y\in B\cap Y'}\frac{|B\cap(X\cup\{y\})|-|B\cap X|}{k}\\
&\overset{(4.21)}{=} |Y|\big(\max_{y\in Y}F(X\cup\{y\})-F(X)\big).
\end{aligned}$$

This establishes α-augmentability if $\frac{k}{\alpha}<|\{b_1\}\cap X|\cdot|B\cap X|$. ∎

Together with Proposition 4.6, this extends the lower bound of Theorem 2.7 to all $\alpha\geq 1$ and thus proves Corollary 1.7.

With this, we can prove the second part of Theorem 1.5.

Proposition 4.9. *For every $\gamma'\in(0,1)$, $\alpha'\geq\gamma'$, $\alpha\geq 1$ and $k\in\mathbb{N}$ with $k>\alpha'$, it holds that $F_{\gamma',\alpha',k}\notin\mathcal{F}_\alpha$. For every $\gamma,\gamma',q\in(0,1]$ and $\alpha'\geq\gamma'$, there exists $k'\in\mathbb{N}$ with $k'>\alpha$ such that $F_{\gamma',\alpha',k'}\notin\tilde{\mathcal{F}}_\gamma\cup\mathcal{F}_q$.*

Proof. For the first part, let $\gamma'\in(0,1)$, $\alpha'\geq\gamma'$ and $k\in\mathbb{N}$ with $k>\alpha'$. Furthermore, let $X=\emptyset$ and $Y=B$. For $y\in Y$, we have

$$F_{\gamma',\alpha',k}(X\cup\{y\})-F_{\gamma',\alpha',k}(X)=F_{\gamma',\alpha',k}(\{y\})\leq F_{\gamma',\alpha',k}(\{b_1\})=\frac{1}{k}$$

and, for any $\alpha\geq 1$, we have

$$\frac{F_{\gamma',\alpha',k}(X\cup Y)-\alpha F_{\gamma',\alpha',k}(X)}{|Y|}=\frac{F_{\gamma',\alpha',k}(B)}{k}=\frac{1}{k\gamma'}>\frac{1}{k}$$

because $\gamma'<1$. Thus, $F_{\gamma',\alpha',k}$ is not α-augmentable for any $\alpha\geq 1$.

For the second part, let $\gamma'\in(0,1]$, $\alpha'\geq\gamma'$ and $k\in\mathbb{N}$ with $k>\alpha'$. Furthermore, let $X=A=S_k^{\mathrm{G}}$ and $Y=B$. For $y\in Y$, we have

$$F_{\gamma',\alpha',k}(X\cup\{y\})-F_{\gamma',\alpha',k}(X)=\begin{cases}\frac{1}{k}\big(1-\alpha'\sum_{i=1}^k\xi_i\big)=\frac{1}{k}\big(\frac{k-\alpha'}{k}\big)^k & \text{if } y=b_1,\\ 0, & \text{else,}\end{cases}$$

and

$$F_{\gamma',\alpha',k}(X\cup Y)-F_{\gamma',\alpha',k}(X)=F_{\gamma',\alpha',k}(B)-F_{\gamma',\alpha',k}(A)=\frac{1}{\gamma'}-\frac{1}{\alpha'}\big(1-\big(\frac{k-\alpha'}{k}\big)^k\big).$$

For $k\to\infty$, the (weak) submodularity ratio gets arbitrarily close to 0 because

$$\lim_{k\to\infty}\frac{\sum_{y\in Y}\big(F_{\gamma',\alpha',k}(X\cup\{y\})-F_{\gamma',\alpha',k}(X)\big)}{F_{\gamma',\alpha',k}(X\cup Y)-F_{\gamma',\alpha',k}(X)}=\lim_{k\to\infty}\frac{\frac{1}{k}\big(\frac{k-\alpha'}{k}\big)^k}{\frac{1}{\gamma'}-\frac{1}{\alpha'}\big(1-\big(\frac{k-\alpha'}{k}\big)^k\big)}=0,$$

i.e., for $k = k'$ large enough, $F_{\gamma',\alpha',k'} \notin \tilde{\mathcal{F}}_\gamma$. It remains to show that $F_{\gamma',\alpha',k'} \nsubseteq \mathcal{F}_q$. If $F_{\gamma',\alpha',k'} \in \mathcal{F}_q$ would hold, then there would be some independence system with weight function w such that $F_{\gamma',\alpha',k'}$ was the associated weighted rank function. The fact that $F_{\gamma',\alpha',k'}(\{b_2\}) = 0$ implies that b_2 must have weight 0 or $\{b_2\}$ is not independent, and the fact that $F_{\gamma',\alpha',k'}(\{b_1, b_2\}) - F_{\gamma',\alpha',k'}(\{b_1\}) = \frac{h(2)-h(1)}{k} > 0$ implies that b_2 must have a weight greater 0 and that $\{b_2\}$ has to be independent, which contradict each other. Thus, $F_{\gamma',\alpha',k'}$ cannot be modelled as the weighted rank function of an independence system, i.e., $F_{\gamma',\alpha',k'} \notin \mathcal{F}_q$. ∎

Finally, we can extend Proposition 3.1 to all $\alpha \geq 1$ by combining the fact that, by Proposition 4.8, for $\alpha \geq 1$, $\{F_{1,\alpha,k} \mid k \in \mathbb{N}, k > \alpha\} \subseteq \mathcal{F}_\alpha$ and the fact that, by Proposition 4.9, for every $\gamma, q \in (0, 1]$, $\{F_{1,\alpha,k} \mid k \in \mathbb{N}, k > \alpha\} \nsubseteq \tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_q$.

PROPOSITION 4.10. *For every $\gamma, q \in (0, 1]$, $\alpha \geq 1$, it holds that $\mathcal{F}_\alpha \nsubseteq (\tilde{\mathcal{F}}_\gamma \cup \mathcal{F}_q)$.*

4.2. γ-α-Augmentability on Independence Systems. To tightly capture the class $\mathcal{F}_q$ of weighted rank functions on independence systems, we show a stronger bound for the approximation ratio of the greedy algorithm on monotone, (weakly) γ-α-augmentable functions. In particular, it was already shown in [2] that the objective function of α-DIMENSIONAL MATCHING is (exactly) α-augmentable, while the greedy algorithm yields an approximation ratio of α, which beats the upper bound of $\alpha \cdot \frac{e^\alpha}{e^\alpha - 1}$ for this case. We show that this can be explained by the fact that α-DIMENSIONAL MATCHING can be represented via a weighted rank function over an independence system. We first show the upper bound of Theorem 1.8.

PROPOSITION 4.11. *Let $\mathcal{F}_{\text{IS}} := \bigcup_{q \in (0,1]} \mathcal{F}_q$ be the set of weighted rank functions on some independence system. The approximation ratio of the greedy algorithm on the class $\tilde{\mathcal{F}}_{\gamma,\alpha} \cap \mathcal{F}_{\text{IS}}$ is at most $\frac{\alpha}{\gamma}$, for every $\gamma \in (0, 1]$ and $\alpha \geq \gamma$.*

Proof. Let $f \in \tilde{\mathcal{F}}_{\gamma,\alpha} \cap \mathcal{F}_{\text{IS}}$, and let $w \colon U \to \mathbb{R}_{\geq 0}$ be the weight function that induces f. We use induction over k. For $k = 0$, the statement holds obviously. Now suppose, the statement holds for some $k \in [|U| - 1]$. If $f(S_k^{\text{G}}) \geq \frac{\gamma}{\alpha} f(S_{k+1}^*)$, then, by monotonicity of f, we have $f(S_{k+1}^{\text{G}}) \geq f(S_k^{\text{G}}) \geq \frac{\gamma}{\alpha} f(S_{k+1}^*)$. Otherwise, the weak γ-α-augmentability of f guarantees the existence of $x \in S_{k+1}^*$ with

$$f(S_k^{\text{G}} \cup \{x\}) - f(S_k^{\text{G}}) \geq \frac{\gamma f(S_k^{\text{G}} \cup S_{k+1}^*) - \alpha f(S_k^{\text{G}})}{|S_{k+1}^*|} \geq \frac{\gamma f(S_{k+1}^*) - \alpha f(S_k^{\text{G}})}{k+1} > 0.$$

By Lemma 4.1, this is equivalent to $f(S_k^{\text{G}} \cup \{x\}) = f(S_k^{\text{G}}) + w(x)$. We conclude

$$\begin{aligned} f(S_{k+1}^{\text{G}}) &\geq f(S_k^{\text{G}} \cup \{x\}) \\ &= f(S_k^{\text{G}}) + w(x) \\ &\overset{\text{ind}}{\geq} \frac{\gamma}{\alpha} f(S_k^*) + w(x) \\ &\geq \frac{\gamma}{\alpha} f(S_{k+1}^* \setminus \{x\}) + w(x) \\ &\overset{\alpha \geq \gamma}{\geq} \frac{\gamma}{\alpha} f(S_{k+1}^* \setminus \{x\}) + \frac{\gamma}{\alpha} w(x) \\ &\geq \frac{\gamma}{\alpha} f(S_{k+1}^*), \end{aligned}$$

i.e., the greedy algorithm has an approximation ratio of at most $\frac{\alpha}{\gamma}$. ∎

The lower bound of Theorem 1.8 follows directly from the well-known tight bound of $1/q$ for $\mathcal{F}_q$.

PROPOSITION 4.12. *Let $\mathcal{F}_{\mathrm{IS}} := \bigcup_{q\in(0,1]} \mathcal{F}_q$ be the set of weighted rank functions on some independence system. The approximation ratio of the greedy algorithm on the class $\tilde{\mathcal{F}}_{\gamma,\alpha} \cap \mathcal{F}_{\mathrm{IS}}$ is at least $\frac{\alpha}{\gamma}$, for every $\gamma \in (0,1]$ and $\alpha \geq \gamma$.*

Proof. Let $\gamma \in (0,1]$, $\alpha \geq \gamma$ and $q \in [\frac{\gamma}{\alpha}, 1] \cap \mathbb{Q}$. In [15] it was shown that the approximation ratio of the greedy algorithm on the set $\mathcal{F}_q$ is exactly $1/q$. By definition of $\mathcal{F}_{\mathrm{IS}}$, we have $\mathcal{F}_q \subseteq \mathcal{F}_{\mathrm{IS}}$, and, by Proposition 4.2, $\mathcal{F}_q \subseteq \tilde{\mathcal{F}}_{\gamma,\gamma/q} \subseteq \tilde{\mathcal{F}}_{\gamma,\alpha}$ holds, where we use the fact that $\frac{\gamma}{q} \leq \frac{\gamma}{\gamma/\alpha} = \alpha$. Thus, we can conclude that the approximation ratio of the greedy algorithm on the class $\tilde{\mathcal{F}}_{\gamma,\alpha} \cap \mathcal{F}_{\mathrm{IS}}$ is at least $1/q$, and since q can be chosen arbitrarily close to $\frac{\gamma}{\alpha}$, the statement follows. ∎

It can be shown that the lower bound of Proposition 4.12 already holds for γ-α-augmentable functions, i.e., in the non-weak subclass of $\tilde{\mathcal{F}}_{\gamma,\alpha}$. It follows that the tight bound of Theorem 1.8 carries over to this, in some sense more natural, class of functions. Since every α-augmentable function is 1-α-augmentable, and vice-versa, we additionally obtain the following. Note that this tightly captures the perfomance of the greedy algorithm for the α-DIMENSIONAL MATCHING problem, which can be represented as the maximization of an α-augmentable weighted rank function over an independence system [2].

COROLLARY 4.13. *The approximation ratio of the greedy algorithm on the class $\mathcal{F}_\alpha \cap \mathcal{F}_{\mathrm{IS}}$, with $\alpha \geq 1$, is exactly α.*

5. Outlook. The vision guiding our work is to precisely characterize the set of cardinality-constrained maximization problems for which the greedy algorithm yields an approximation, and to tightly bound the corresponding approximation ratio.

In this paper, we have made progress towards this goal by unifying and generalizing important classes of greedily approximable maximization problems, and by providing tight bounds on the approximation ratio for the resulting generalized class of problems. While this brings us closer to a full characterization, there are still settings that are not captured by (weak) γ-α-augmentability.

PROPOSITION 5.1. *For $\gamma \in (0,1]$ and $\alpha \geq \gamma$, there exists a monotone function $f^{\gamma,\alpha}$ that is not weakly γ-α-augmentable, and for which the greedy algorithm computes an optimum solution.*

Proof. Let U be any ground set of size $|U| > \frac{1}{\gamma}$ and consider the objective function $f^{\gamma,\alpha}\colon 2^U \to \mathbb{R}_{\geq 0}$ with

$$f^{\gamma,\alpha}(X) = |X|^2.$$

For all $X, Y \subseteq U$ with $|Y| > \frac{1}{\gamma}(2|X| + 1 + \alpha|X|^2)$ $(*)$, e.g., $X = \emptyset$ and $|Y| = \lfloor \frac{1}{\gamma} \rfloor + 1$, we have

$$\begin{aligned} |Y|(f^{\gamma,\alpha}(X \cup \{y\}) - f^{\gamma,\alpha}(X)) &= |Y|(2|X| + 1) \\ &\overset{(*)}{<} \gamma|Y|^2 - \alpha|Y||X|^2 \\ &\leq \gamma|X \cup Y|^2 - \alpha|X|^2 \\ &= \gamma f^{\gamma,\alpha}(X \cup Y) - \alpha f^{\gamma,\alpha}(X), \end{aligned}$$

i.e., $f^{\gamma,\alpha}$ is not weakly γ-α-augmentable. Yet, picking elements in any order is obviously optimal. Thus, there exists a problem that is not γ-α-augmentable, but where the greedy algorithm performs optimally. ∎

Remark 5.2. Objective functions as in the proof of Proposition 5.1 arise for example in the context of incremental maximum flows on a complete bipartite graph $G = (U \cup V, E)$ where we want to incrementally grow subsets of U and of V such that the flow from one of the subsets to the other (i.e., the cut size) is maximized.

We leave it as an open problem to find a natural generalization of weak γ-α-augmentability that captures a larger set of greedily approximable objectives. The challenge is to find a meaningful generalization in terms of a natural definition that does not directly depend on the behavior of the greedy algorithm, but rather enforces some structural property of the objective function. In that sense, the dependency of weak γ-α-augmentability on the greedy solutions $S_0^{\mathrm{G}}, \ldots, S_k^{\mathrm{G}}$ is a significant flaw. Note that we needed to introduce this dependency in order to encompass settings with bounded (weak) submodularity ratios, since the definition of the latter depends on the greedy solutions as well. Importantly, our upper bound on the approximation ratio of the greedy algorithm carries over to the stronger notion of γ-α-augmentability that requires the defining property to hold for *all* sets X, and not just the greedy solutions. Our tight lower bound does not immediately translate to this, more restrictive, definition, and it remains an open problem to construct a tight lower bound in this setting as well.

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