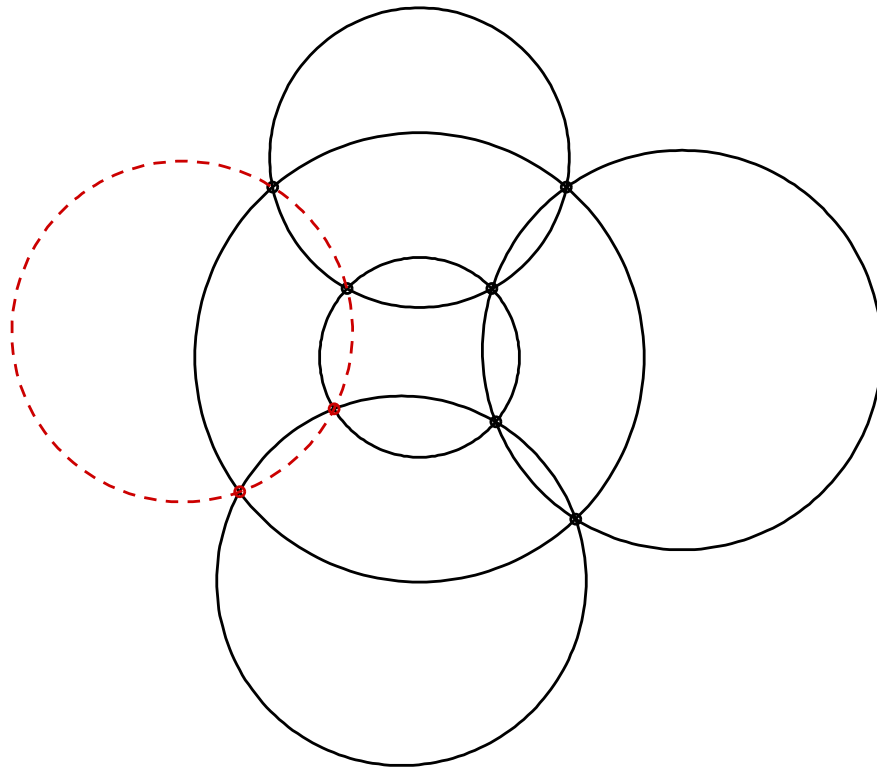


Planar Circle Geometries

an Introduction to Moebius-, Laguerre- and
Minkowski-planes



Erich Hartmann

Department of Mathematics
Darmstadt University of Technology

Contents

0	INTRODUCTION	7
1	RESULTS ON AFFINE AND PROJECTIVE GEOMETRY	11
1.1	Affine planes	11
1.1.1	The axioms of an affine plane	11
1.1.2	Collineations of an affine plane	12
1.1.3	Desarguesian affine planes	13
1.1.4	Pappian affine planes	14
1.2	Projective planes	15
1.2.1	The axioms of a projective plane	15
1.2.2	Collineations of a projective plane	15
1.2.3	Desarguesian projective planes	16
1.2.4	Homogeneous coordinates of a desarguesian plane	17
1.2.5	Collineations of a desarguesian projective plane	17
1.2.6	Pappian projective planes	17
1.2.7	The groups $P\Gamma L(2, K)$, $PGL(2, K)$ and $PSL(2, K)$	18
1.2.8	Basic 1–dimensional projective configurations	19
2	OVALS AND CONICS	23
2.1	Oval, parabolic and hyperbolic curve	23
2.2	The ovals \mathfrak{c}_1 and \mathfrak{c}_2	24
2.3	Properties of the ovals \mathfrak{c}_1 and \mathfrak{c}_2	26
2.4	Oval conics and their properties	27
2.5	Oval conics in affine planes	39
2.6	Finite ovals	40
2.7	Further examples of ovals	43
2.7.1	Translation–ovals	43
2.7.2	Moufang–ovals	44
2.7.3	Ovals in Moulton–planes	45
2.7.4	Ovals in planes over nearfields	45
2.7.5	Ovals in finite planes over quasifields	45
2.7.6	Real ovals on convex functions	46

3	MOEBIUS–PLANES	47
3.1	The classical real Moebius–plane	47
3.2	The axioms of a Moebius–plane	48
3.3	Miquelian Moebius–planes	50
3.3.1	The incidence structure $\mathfrak{M}(K, q)$	50
3.3.2	Representation of $\mathfrak{M}(K, q)$ over E	51
3.3.3	Automorphisms of $\mathfrak{M}(K, q)$, $\mathfrak{M}(K, q)$ is Moebius–plane	52
3.3.4	Cycle reflections	53
3.3.5	Angles in affine plane $\mathfrak{A}(K, q)$	55
3.3.6	Theorem of MIQUEL in $\mathfrak{M}(K, q)$	56
3.3.7	The sphere–model of a miquelian Moebius–plane	59
3.3.8	Isomorphic miquelian Moebius–planes	60
3.4	Ovoidal Moebius–planes	61
3.4.1	Plane model of an ovoidal Moebius–plane	63
3.4.2	Examples of ovoidal Moebius–planes	64
3.5	Non ovoidal Moebius–planes	64
3.6	Final remark	65
4	LAGUERRE–PLANES	67
4.1	The classical real Laguerre–plane	67
4.2	The axioms of a Laguerre–plane	68
4.3	Miquelian Laguerre–planes	70
4.3.1	The incidence structure $\mathfrak{L}(K)$	70
4.3.2	Automorphisms of $\mathfrak{L}(K)$, $\mathfrak{L}(K)$ is a Laguerre–plane	70
4.3.3	Parabolic measure for angles in $\mathfrak{A}(K)$	72
4.3.4	Theorem of MIQUEL in $\mathfrak{L}(K)$	72
4.4	Ovoidal Laguerre–planes	74
4.4.1	Definition of an ovoidal Laguerre–plane	74
4.4.2	The plane model of an ovoidal Laguerre–plane	75
4.4.3	Examples of ovoidal Laguerre–planes	77
4.4.4	Automorphisms of an ovoidal Laguerre–plane $\mathfrak{L}(K, f)$	77
4.4.5	The bundle theorem for Laguerre–planes	78
4.5	Non ovoidal Laguerre–planes	79
4.6	Automorphisms of Laguerre–planes	80
4.6.1	Transitivity properties of ovoidal Laguerre–planes	83
5	MINKOWSKI–PLANES	87
5.1	The classical real Minkowski–plane	87
5.2	The axioms of a Minkowski–plane	88
5.3	Miquelian Minkowski–planes	91
5.3.1	The incidence structure $\mathfrak{M}(K)$	91
5.3.2	Automorphisms of $\mathfrak{M}(K)$. $\mathfrak{M}(K)$ is a Minkowski–plane	92
5.3.3	Hyperbolic measure for angles in $\mathfrak{M}(K)$	93

5.3.4	Theorem of MIQUEL in $\mathfrak{M}(K)$	93
5.3.5	Cycle reflections and the Theorem of Miquel	95
5.3.6	The hyperboloid model of a miquelian Minkowski-plane	97
5.3.7	The bundle theorem for Minkowski-planes	98
5.4	Minkowski-planes over TITS-nearfields	98
5.4.1	The incidence structure $\mathfrak{M}(K, f)$	98
5.4.2	Automorphisms of $\mathfrak{M}(K, f)$. $\mathfrak{M}(K, f)$ is a Minkowski-plane . . .	99
5.5	Minkowski-planes and sets of permutations	100
5.5.1	Description of a Minkowski-plane by permutation sets	100
5.5.2	Minkowski-planes over permutation groups	101
5.5.3	Symmetry at a cycle. The rectangle axiom	103
5.5.4	The symmetry axiom	106
5.5.5	Minkowski-planes of even order	110
5.6	Further Examples of Minkowski-planes	111
5.6.1	A method for the generation of finite Minkowski-planes	111
5.6.2	Finite examples which do not fulfill (G)	112
5.6.3	Real examples, which do not fulfill (G)	113
5.7	Automorphisms of Minkowski-planes	114
6	Appendix: Quadrics	117
6.1	Quadratic forms	117
6.2	Definition and properties of a quadric	118
6.3	Quadratic sets	122
6.4	Final remarks	123
7	Appendix: Nearfields	125
7.1	Definition of a nearfield and some rules	125
7.2	Examples of nearfields	126
7.3	Planar nearfields	126
7.4	Nearfields and sharply 2-transitive permutationgroups	126
7.5	TITS-nearfields	127

Chapter 0

INTRODUCTION

Because of the parallel relation on the set of lines an *affine plane* is not a homogeneous geometric structure (some lines intersect, others not). This inhomogeneity can be omitted by extending the affine plane to its *projective completion* (Within a projective plane any pair of lines intersect). For *desarguesian* planes the formal inhomogeneity of the projective completion can be abolished by using the 3d-model (1-dim subspaces are points, 2-dim are lines).

Now we start from the real euclidean plane and merge the set of lines together with the set of circles to a set of blocks. This construction results in a rather inhomogeneous incidence structure: two points determine one line and a whole pencil of circles. The trick embedding this incidence structure into a homogeneous one is based on the following idea: Add to the point set the new point ∞ , which must lie on every line. Now any block is determined by exactly 3 points. This new homogeneous geometry is called classical **inversive geometry** or **Moebius-plane**.

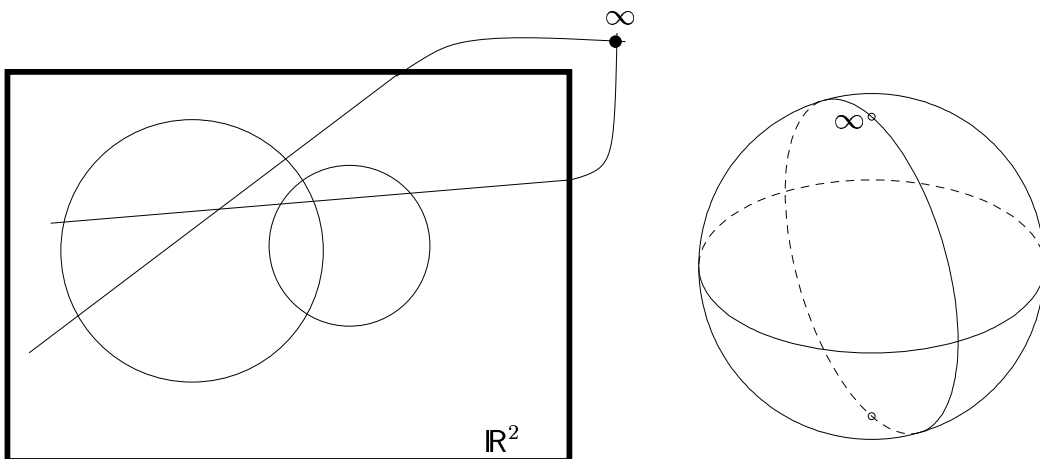


Figure 1: 2d- and 3d-model of a Moebius-plane

The still existing inhomogeneity of the description (lines, circles, new point) can be abol-

ished by using a 3d-model. From a stereographic projection we learn: the classical Moebius-plane is isomorphic to the geometry of plane sections (circles) on a sphere in euclidean 3-space.

Analogously to the (axiomatic) projective plane one calls an incidence structure, which exhibits essentially the same incidence properties, an (axiomatic) **Moebius-plane** (see Chapter 3). Expectedly there are a lot of Moebius-planes which are different from the classical one.

If we start again from \mathbb{R}^2 and take the curves with equations $y = ax^2 + bx + c$ (parabolas and lines) as blocks, the following homogenization is effective: Add to the curve $y = ax^2 + bx + c$ the new point (∞, a) . Hence the set of points is $(\mathbb{R} \cup \infty) \times \mathbb{R}$. This geometry of parabolas is called **classical Laguerre-plane**. (Originally it was designed as the geometry of the oriented lines and circles, see [BE'73]. Both geometries are isomorphic.)

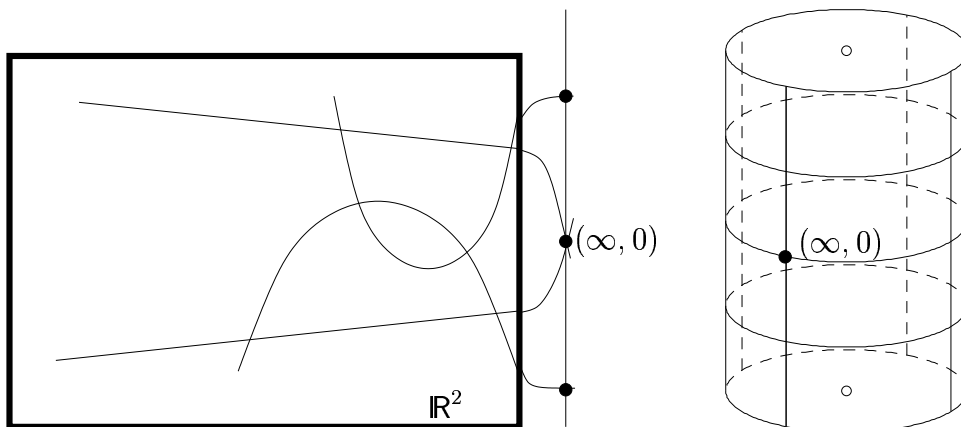


Figure 2: 2d- and 3d-model of a Laguerre-plane

As for the Moebius-plane there exists a 3d-model: the geometry of the elliptic plane sections on an orthogonal cylinder (in \mathbb{R}^3), see Chapter 4. An abstraction leads (analogously to the Moebius-plane) to the axiomatic **Laguerre-plane**.

At least, if we start from \mathbb{R}^2 and merge the lines $y = mx + d$, $m \neq 0$ with the hyperbolas $y = \frac{a}{x-b} + c$, $a \neq 0$ in order to get the set of blocks the following idea homogenizes the incidence structure: Add to any line the point (∞, ∞) and to any hyperbola $y = \frac{a}{x-b} + c$, $a \neq 0$ the two points $(b, \infty), (\infty, c)$. Hence the point set is $(\mathbb{R} \cup \infty)^2$. This geometry of the hyperbolas is called the **classical Minkowski-plane**.

Analogously to the classical Moebius- and Laguerre-planes there exists a 3d-model: The classical Minkowski-plane is isomorphic to the geometry of plane sections of a hy-

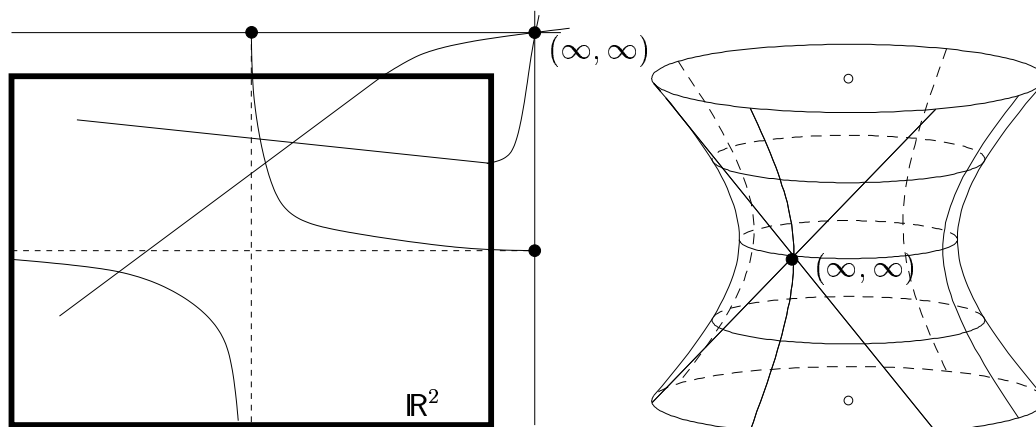


Figure 3: 2d- and 3d-model of a Minkowski-plane

perboloid of one sheet (non degenerated quadric of index 2) in 3-dimensional projective space (see Chapter 5). Similar to the first two cases we get the (axiomatic) **Minkowski-plane**.

Because of the essential role of the circle (considered as *the* non degenerate conic in a projective plane) and the plane description of the original models the three types of geometries are subsumed to **plane circle geometries**.

The prominent classes of the plane circle geometries are built on (commutative) fields with the aid of conics. Therefore a chapter on *oval conics* (Section 2.4) is included into this lecture notes. In order to give support for the understanding of the 3d-models there is added an appendix on *quadrics* (Chapter 6). The appendix on *nearfields* (7) is necessary for the understanding of a wide class of Minkowski-planes.

The lecture notes on hand arose from lectures held at the Department of Mathematics of Darmstadt University of Technology. Because the author's field of interest changed during the past years this lecture notes is not a report on cutting edge results. Its intention is to introduce interesting readers into the subject of circle geometries.

Darmstadt, Oktober 2004