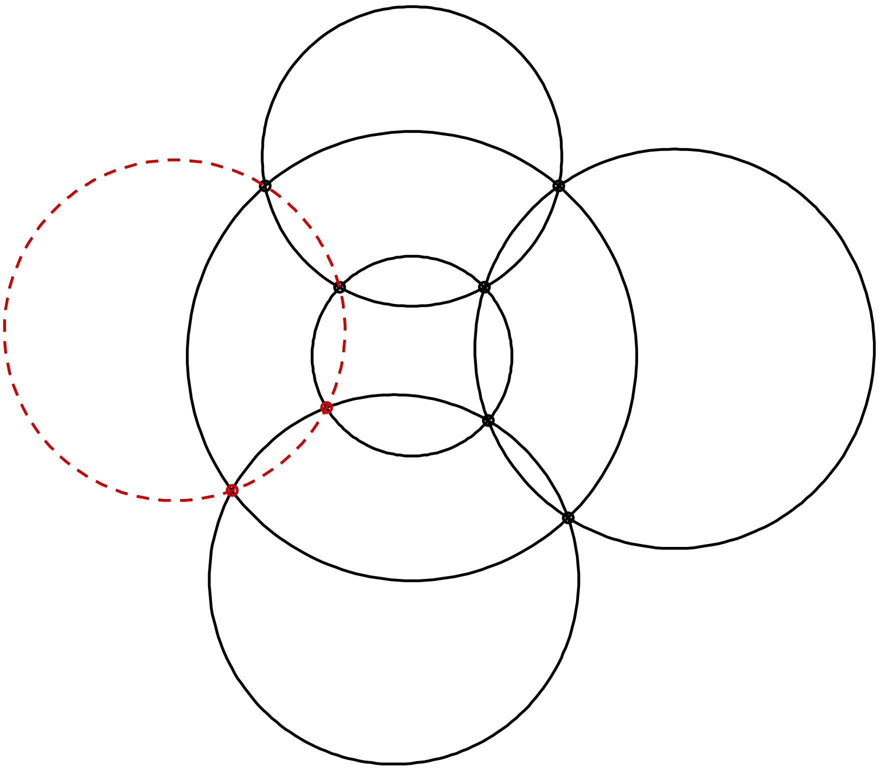
Planar Circle Geometries

an Introduction to Moebius–, Laguerre– and Minkowski–planes



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Chapter 0

INTRODUCTION

Because of the parallel relation on the set of lines an *affine plane* is not a homogeneous geometric structure (some lines intersect, others not). This inhomogenity can be omitted by extending the affine plane to its *projective completion* (Within a projective plane any pair of lines intersect). For *desarguesian* planes the formal inhomogenity of the projective completion can be abolished by using the 3d-model (1-dim subspaces are points, 2-dim are lines).

Now we start from the real euclidean plane and merge the set of lines together with the set of circles to a set of blocks. This construction results in a rather inhomogeneous incidence structure: two points determine one line and a whole pencil of circles. The trick embedding this incidence structure into a homogeneous one is based on the following idea: Add to the point set the new point ∞, which must lie on every line. Now any block is determined by exactly 3 points. This new homogeneous geometry is called classical **inversive geometry** or **Moebius–plane**.

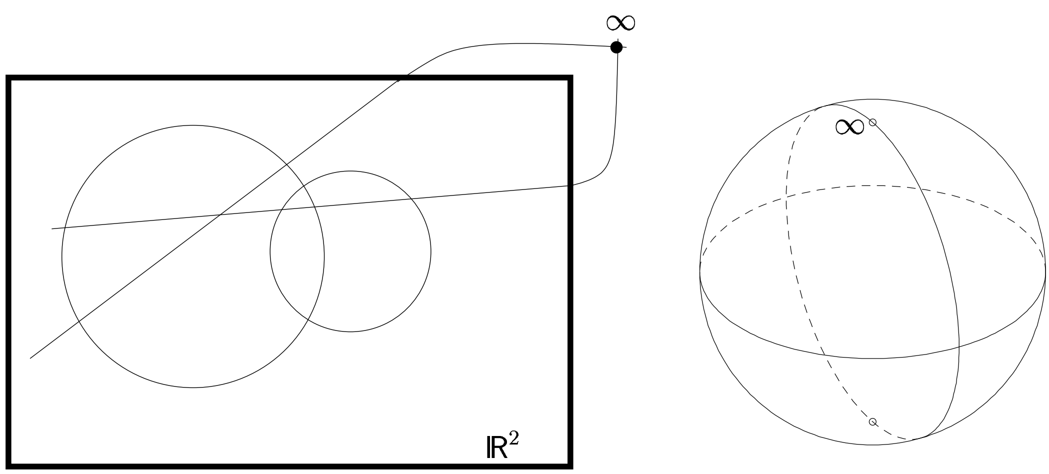

Figure 1: 2d– and 3d–model of a Moebius–plane

The still existing inhomogenity of the description (lines, circles, new point) can be abolished by using a 3d-model. From a stereographic projection we learn: the classical Moebius–plane is isomorphic to the geometry of plane sections (circles) on a sphere in euclidean 3-space.

Analogously to the (axiomatic) projective plane one calls an incidence structure, which exhibits essentially the same incidence properties, an (axiomatic) **Moebius–plane** (see Chapter 3). Expectedly there are a lot of Moebius–planes which are different from the classical one.

If we start again from $\mathbb{R}^2$ and take the curves with equations $y = ax^2 + bx + c$ (parabolas and lines) as blocks, the following homogenization is effective: Add to the curve $y = ax^2 + bx + c$ the new point (∞, a). Hence the set of points is $(\mathbb{R} \cup \infty) \times \mathbb{R}$. This geometry of parabolas is called **classical Laguerre–plane**.(Originally it was designed as the geometry of the oriented lines and circles, see [BE'73]. Both geometries are isomorphic.)

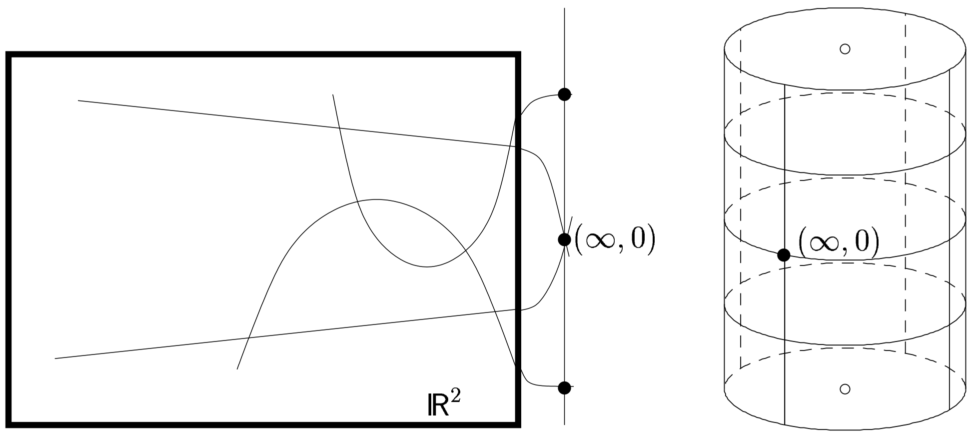

Figure 2: 2d– and 3d–model of a Laguerre–plane

As for the Moebius–plane there exists a 3d-model: the geometry of the elliptic plane sections on an orthogonal cylinder (in $\mathbb{R}^3$), see Chapter 4. An abstraction leads (analogously to the Moebius–plane) to the axiomatic **Laguerre–plane**.

At least, if we start from $\mathbb{R}^2$ and merge the lines $y = mx + d, m \neq 0$ with the hyperbolas $y = \frac{a}{x-b} + c, a \neq 0$ in order to get the set of blocks the following idea homogenizes the incidence structure: Add to any line the point (∞, ∞) and to any hyperbola $y = \frac{a}{x-b} + c, a \neq 0$ the two points $(b, \infty), (\infty, c)$. Hence the point set is $(\mathbb{R} \cup \infty)^2$. This geometry of the hyperbolas is called the **classical Minkowski–plane**.

Analogously to the classical Moebius– and Laguerre–planes there exists a 3d-model: The classical Minkowski–plane is isomorphic to the geometry of plane sections of a hy-

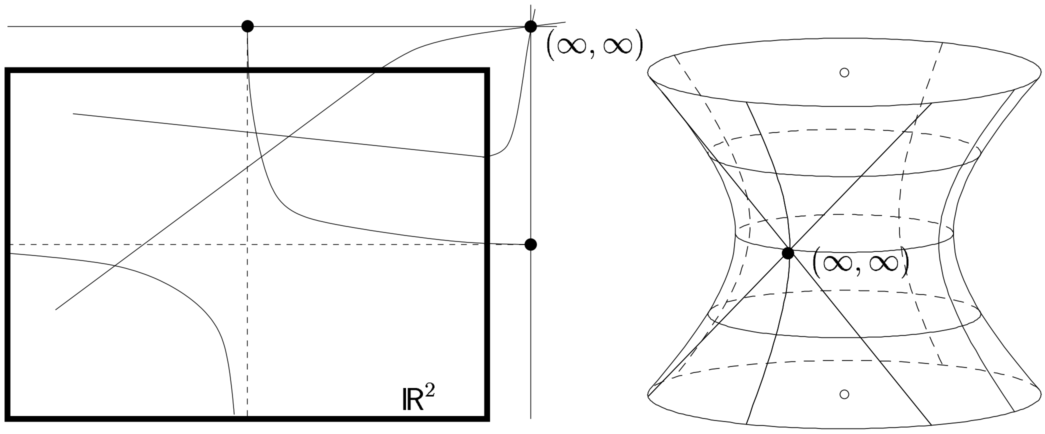

Figure 3: 2d– and 3d–model of a Minkowski–plane

perboloid of one sheet (non degenerated quadric of index 2) in 3-dimensional projective space (see Chapter 5). Similar to the first two cases we get the (axiomatic) **Minkowski–plane**.

Because of the essential role of the circle (considered as *the* non degenerate conic in a projective plane) and the plane description of the original models the three types of geometries are subsumed to **plane circle geometries**.

The prominent classes of the plane circle geometries are built on (commutative) fields with the aid of conics. Therefore a chapter on *oval conics* (Section 2.4) is included into this lecture notes. In order to give support for the understanding of the 3d–models there is added an appendix on *quadrics* (Chapter 6). The appendix on *nearfields* (7) is necessary for the understanding of a wide class of Minkowski–planes.

The lecture notes on hand arose from lectures held at the Department of Mathematics of Darmstadt University of Technology. Because the author's field of interest changed during the past years this lecture notes is not a report on cutting edge results. Its intention is to introduce interesting readers into the subject of circle geometries.

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