ELLIPTIC OPERATORS WITH UNBOUNDED DRIFT COEFFICIENTS ON DOMAINS

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Abstract. We show that second-order elliptic differential operators, whose coefficients of the first order terms are growing not more than $|x|\log(|x|)$ at infinity, equipped with Dirichlet boundary conditions, are generators of consistent, positive C_0-semigroups on $L^p(\Omega)$, $1 < p < \infty$, for every open set $\Omega \subseteq \mathbb{R}^d$. Furthermore, when the domain is enlarged, the semigroups increase monotonously in the sense of positive operators, as in the case of the Dirichlet Laplacian.

1. Introduction

In this paper we study the differential operator

$$\begin{aligned}\mathcal{A}u(x) &= \mathcal{A}_0u(x) + \mathcal{L}u(x)\\ &= \operatorname{div}\big(a(x)\nabla u(x)\big) + b(x)\cdot\nabla u(x), \quad x \in \Omega.\end{aligned}$$

Here $\Omega \subseteq \mathbb{R}^d$ is an open set and the functions $a \in C_b^1(\Omega;\mathbb{R}^{d\times d})$ and $b \in W^{1,\infty}_{\text{loc}}(\Omega;\mathbb{R}^d)$ satisfy the following hypotheses for some constants $\alpha > 0$ and $c \geq 0$.

(A1) $\xi^T a(x)\xi \geq \alpha|\xi|^2$ for all $\xi \in \mathbb{R}^d$ and almost all $x \in \Omega$,
(B1) $|b(x)| \leq c\big(1 \vee |x|\log(|x|)\big)$ for almost all $x \in \Omega$,
(B2) $\operatorname{div} b$ is bounded from below.

We show that suitable realisations of these operators generate positive, consistent C_0-semigroups on $L^p(\Omega)$ for $1 < p < \infty$ and that all these semigroups are dominated in the sense of positive operators by the corresponding semigroups on $\mathbb{R}^d$. Furthermore, a gradient estimate for the resolvent in $L^2(\Omega)$ is deduced.

In [HW05] we achieved this in the special case of Ornstein-Uhlenbeck operators where $\mathcal{A}_0$ has constant coefficients and $b(x) = Bx$ for some matrix $B \in \mathbb{R}^{d\times d}$. In this note we show how these results can be generalised to the above setting. Since part of the arguments stay the same, we do not always give self-contained proofs, but sometimes refer to [HW05]. Also for references on earlier results on Ornstein-Uhlenbeck operators we refer to this article.

In the recent article [AMP06] W. Arendt, G. Metafune and D. Pallara study Schrödinger operators with unbounded drift coefficients. In contrast to this note, their focus lies on the interplay between the potential and the drift coefficient and on conditions that guarantee Gaussian estimates for the semigroups. Setting the potential equal to zero, some of our results can, at least implicitely, also be found in this article. Note, that although it is only formulated for the case of $\Omega = \mathbb{R}^d$, their proofs of the generation results carry over to general open sets Ω, cf. [AMP06, Remark 5.8].

An alternative approach to operators with superlinearly growing drift coefficients on the whole space $\mathbb{R}^d$ was used by J. Prüss, A. Rhandi and R. Schnaubelt in [PRS05]. There a generation result is shown and the domain of the operator is determined. Furthermore, the authors show by an example for $d = 1$ that, imposing

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a domain that contains at least $C_b(\mathbb{R}) \cap C^2(\mathbb{R})$, no generation result can be expected for some drift coefficient b that grows like $|x|^{1+\varepsilon}$ at infinity even for arbitrarily small $\varepsilon > 0$.

Thus our growth assumption (B1) is a somehow natural restriction, even if it may be slightly weakened (cf. Remark 3).

Comparing our results in the special case $\Omega = \mathbb{R}^d$ with the results in [PRS05], we weaken the hypotheses on the coefficients, allowing non-bounded divergence of b and eliminating a condition that links b to the first derivatives of a, for the price of a less exact description of the domain of the operator.

If Ω is not the whole space, the description of the domain of the operator seems to be an open problem, even for regular boundaries.

2. Main result

We define the realisation of $\mathcal{A}$ on $L^2(\Omega)$ by

$$D(A_{\Omega,2}) = H_0^1(\Omega) \cap \{u \in H^2_{\mathrm{loc}}(\Omega) : \mathcal{A}u \in L^2(\Omega)\}, \qquad A_{\Omega,2}u = \mathcal{A}u$$

and we set for $1 < p < \infty$

$$\omega_p := \operatorname*{essinf}_{x \in \Omega} \frac{\operatorname{div} b(x)}{p}$$

which exists by (B2).

Now we can formulate our main result.

Theorem 1. *Let $\Omega \subseteq \mathbb{R}^d$ be open and (A1), (B1) and (B2) be satisfied. Then the operator $A_{\Omega,2}$ generates a positive C_0-semigroup $(T_{\Omega,2}(t))_{t\ge 0}$ on $L^2(\Omega)$ with $\|T_{\Omega,2}(t)\| \le \mathrm{e}^{-\omega_2 t}$ and for every $\lambda > 0$ we have the estimate*

$$\|\nabla R(\lambda, A_{\Omega,2} + \omega_2)\|_{\mathcal{L}(L^2(\Omega))} \le \frac{C}{\sqrt{\lambda}}.$$

Moreover, there exists a family of consistent positive C_0-semigroups $(T_{\Omega,p}(t))_{t\ge 0}$ on $L^p(\Omega)$ for $1 < p < \infty$ with $\|T_{\Omega,p}(t)\| \le \mathrm{e}^{-\omega_p t}$ and for every $\lambda > -\omega_p$ and every $t \ge 0$ we have the domination properties

$$|T_{\Omega,p}(t)f| \le T_{\mathbb{R}^d,p}(t)|\tilde{f}|, \qquad f \in L^p(\Omega),$$
$$|R(\lambda, A_{\Omega,p})f| \le R(\lambda, A_{\mathbb{R}^d,p})|\tilde{f}|, \quad f \in L^p(\Omega),$$

where $\tilde{f}$ denotes the extension of f by 0 and $A_{\Omega,p}$ is the generator of $(T_{\Omega,p}(t))_{t\ge 0}$.

We give some comments on the strategy of the proof. Proceeding as in [HW05], we first show that $A_{\Omega,2}$ is dissipative for arbitrary open sets Ω (Proposition 2) and show surjectivity of $\lambda - A_{\Omega,2}$ for some positive λ by approximating the domain from the interior with bounded regular open subsets Ω_n. On all these subsets the coefficients of our operator are bounded, so we get a sequence of solutions u_n on Ω_n, which can be shown to contain a subsequence that converges weakly to a solution of the elliptic problem on Ω (Proposition 4). In fact, the whole sequence converges even strongly, as can be seen from the proof of [AMP06, Theorem 3.1], but this is not needed in the following.

In a second step we show, also analogously to [HW05], that the semigroups $(T_{\Omega,2}(t))_{t\ge 0}$ generated by $A_{\Omega,2}$ are positive and growing in the sense of positive operators when the domain is enlarged, thus giving us the domination result of Theorem 1 for $p = 2$ (Proposition 5).

In contrast to the case of Ornstein-Uhlenbeck operators, it is not known whether this semigroup on the whole space is associated to a kernel that gives rise to a consistent family of operators in $L^p(\mathbb{R}^d)$ for $1 < p < \infty$, so it is not obvious how to extend the semigroups on $L^2(\Omega)$ to $L^p(\Omega)$ consistently.

To circumvent this problem we show dissipativity of $A_{\Omega,p}$ for $p \neq 2$ on $\mathbb{R}^d$ and on regular domains (Proposition 2). Using this, we adapt the proof of surjectivity described above for $L^2(\Omega)$ to the special case of $L^p(\mathbb{R}^d)$, $1 < p < \infty$, providing semigroups on $L^p(\mathbb{R}^d)$, that are consistent with that on $L^2(\mathbb{R}^d)$ (Proposition 6). This finally allows to extend our semigroups consistently by domination for arbitrary domains Ω.

3. Dissipativity of $A_{\Omega,p}$

In order to show dissipativity for all $1 < p < \infty$, we need the integration by parts formula

$$\int_\Omega \varphi\overline{u}|u|^{p-2}\nabla u = -\int_\Omega \varphi|u|^{p-2}u\nabla\overline{u} - \frac{p-2}{2}\int_\Omega \varphi|u|^{p-2}(\overline{u}\nabla u + u\nabla\overline{u}) - \int_\Omega \nabla\varphi|u|^p, \tag{1}$$

that is valid for every Lipschitz domain $\Omega \subseteq \mathbb{R}^d$ and for every $1 < p < \infty$, every $u \in W_0^{1,p}(\Omega)$ and every $\varphi \in W^{1,\infty}(\Omega)$. The proof follows the lines of [MS04, Theorem 3.1] and can be found in [Woo05, Proposition 5.2.2.], where it is formulated for $\varphi \in C_c^\infty(\mathbb{R}^d)$. It easily carries over to our more general functions φ.

Note, that for $p = 2$, this formula reduces to

$$\int_\Omega \varphi\overline{u}\nabla u = -\int_\Omega \varphi u\nabla\overline{u} - \int_\Omega \nabla\varphi|u|^2, \tag{2}$$

which is easily verified by partial integration and valid for arbitrary open sets $\Omega \subseteq \mathbb{R}^d$.

Proposition 2. *Let $p = 2$ and $\Omega \subseteq \mathbb{R}^d$ be open or let $p \in (1,\infty)$ and $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz domain. Then the operator A given by*

$$D(A) = W_0^{1,p}(\Omega) \cap \{f \in W^{2,p}_{\mathrm{loc}}(\Omega) : \mathcal{A}f \in L^p(\Omega)\}, \qquad Au = \mathcal{A}u + \omega_p u$$

is dissipative.

Proof. Choose a cut-off function $\eta \in C_c^\infty(\mathbb{R})$ with $0 \leq \eta(t) \leq 1$ for all $t \in \mathbb{R}$, $\eta(t) = 1$ for $t \in [-1,1]$ and $\eta(t) = 0$ for $|t| \geq 2$. In order to balance the superlinear growth of b we now put in contrast to [HW05] $\eta_m(x) = \eta(\log(|x|)/m)$ for $m \in \mathbb{N}$ and $x \in \mathbb{R}^d$. Then for all $m \in \mathbb{N}$ and $u \in D(A)$ we have by (1) or (2), respectively

$$\begin{aligned}
&\Re\left(\int_\Omega \mathcal{L}u\overline{u}|u|^{p-2}\eta_m\right) \\
=&\Re\left(-\int_\Omega ub\nabla\overline{u}|u|^{p-2}\eta_m - \frac{p-2}{2}\int_\Omega |u|^{p-2}(\overline{u}b\nabla u + ub\nabla\overline{u})\eta_m - \int_\Omega \operatorname{div}(\eta_m b)|u|^p\right).
\end{aligned}$$

Since b is real-valued we have $\Re(ub\nabla\overline{u}) = \Re(\overline{u}b\nabla u)$, so we get

$$=\Re\left(-(p-1)\int_\Omega \mathcal{L}u\overline{u}|u|^{p-2}\eta_m\right) - \int_\Omega \operatorname{div} b\,|u|^p\eta_m - \int_\Omega b\nabla\eta_m|u|^p,$$

which finally yields

$$\Re\left(\int_\Omega \mathcal{L}u\overline{u}|u|^{p-2}\eta_m\right) = \frac{1}{p}\left(-\int_\Omega \operatorname{div} b\,|u|^p\eta_m - \int_\Omega b\nabla\eta_m|u|^p\right).$$

By dissipativity of $\mathcal{A}_0$ and the definition of ω_p this implies

$$\begin{aligned}
\Re\left(\int_\Omega Au\overline{u}|u|^{p-2}\eta_m\right) &\leq \int_\Omega \left(\omega_p - \frac{\operatorname{div} b}{p}\right)|u|^p\eta_m - \frac{1}{p}\int_\Omega b\nabla\eta_m|u|^p \\
&\leq -\frac{1}{p}\int_\Omega b\nabla\eta_m|u|^p.
\end{aligned}$$

Now, we want to let $m \to \infty$. For the left hand side this is easy due to $\mathcal{A}u \in L^p(\Omega)$ and the boundedness of η_m. Concerning the integral on the right hand side, we note that

$$\nabla \eta_m(x) = \eta' \left(\frac{\log(|x|)}{m} \right) \frac{1}{m|x|} \frac{x}{|x|}.$$

Since $\eta'(\log(|x|)/m) = 0$, whenever $\log(|x|) \notin [m, 2m]$, the integrand converges pointwise to 0 as $m \to \infty$. Furthermore,

$$|\nabla \eta_m(x)| \le \|\eta'\|_\infty \frac{1}{m|x|} \mathbf{1}_{\{\log(|x|)/2 \le m \le \log(|x|)\}} \le C \frac{1}{1 \vee |x| \log(|x|)/2}$$

and (B1) imply

$$\left| b(x) \cdot \nabla \eta_m(x) |u(x)|^p \right| \le C \frac{1 \vee |x| \log(|x|)}{1 \vee |x| \log(|x|)/2} |u(x)|^p \le C |u(x)|^p.$$

So by Lebesgue's Theorem we end up with

$$\int_\Omega A u \overline{u} |u|^{p-2} \le -\frac{1}{p} \lim_{m \to \infty} \int_\Omega b \nabla \eta_m |u|^p = 0.$$

□

Remark 3. By putting $\log^{[1]}(|x|) = \log(|x|)$ and $\log^{[n+1]}(|x|) = \log(\log^{[n]}(|x|))$ for $n \ge 1$ and setting

$$\eta_m(x) = \eta \left(\frac{\log^{[n]}(|x|)}{m} \right)$$

for some $n \in \mathbb{N}$ in the above proof, we may also deal with a slightly stronger growth of the coefficient b. In fact, for this choice of η_m, we have

$$\nabla \eta_m(x) = \eta' \left(\frac{\log^{[n]}(|x|)}{m} \right) \frac{1}{|x| \log(|x|) \log^{[2]}(|x|) \cdot \ldots \cdot \log^{[n-1]}(|x|)} \frac{x}{m|x|}.$$

As above $\nabla \eta_m(x) = 0$ whenever $\log^{[n]}(|x|) \notin [m, 2m]$, consequently

$$|\nabla \eta_m(x)| \le \frac{C}{1 \vee |x| \log(|x|) \log^{[2]}(|x|) \cdot \ldots \cdot \log^{[n]}(|x|)}$$

in this case, and hypotheses (B1) can be weakened to

(B1′) $|b(x)| \le c\big(1 \vee |x| \log(|x|) \log^{[2]}(|x|) \cdot \ldots \cdot \log^{[n]}(|x|)\big)$ for some $n \in \mathbb{N}$ and for almost all $x \in \Omega$.

4. The case $p = 2$

The next goal is a generation result for the operator $A_{\Omega,2}$. In view of Proposition 2, it suffices to show the following statement.

Proposition 4. *For every $\lambda > 0$ the operator $\lambda - \omega_2 - A_{\Omega,2}$ is surjective.*

Proof. Let $\lambda > 0$ and $f \in L^2(\Omega)$ be given. Proceeding as in [HW05], by [DL90, II.4, Lemma 1] we choose a sequence $(\Omega_n)_{n \in \mathbb{N}}$ of bounded open subsets of Ω with smooth boundaries, such that $\Omega_n \subseteq \Omega_{n+1}$ for every $n \in \mathbb{N}$ and $\bigcup_{n=1}^\infty \Omega_n = \Omega$. Note, that in [DL90] the sets Ω_n are constructed in such a way that every Ω_n has only a finite number of connected components, so we get unique solutions $u_n \in H^2(\Omega_n) \cap H_0^1(\Omega_n)$ of

$$\begin{cases} \lambda u_n - \omega_2 u_n - A_{\Omega_n,2} u_n &= f|_{\Omega_n} \text{ in } \Omega_n, \\ u_n &= 0 \quad \text{ on } \partial \Omega_n. \end{cases}$$

Since $A_{\Omega_n,2} + \omega_2$ is dissipative by Proposition 2 we get the estimate

$$\|u_n\|_{L^2(\Omega_n)} \le \frac{1}{\lambda} \|f\|_{L^2(\Omega)} \tag{3}$$

independently from $n \in \mathbb{N}$.

In order to bound $\|\nabla u_n\|_{L^2(\Omega_n)}$ analogously to [HW05, Lemma 3.6], we suppose by the same argument as therein, u_n to be real-valued and we estimate by hypotheses (A1)

$$\|\nabla u_n\|^2_{L^2(\Omega_n)} \leq C \int_{\Omega_n} \nabla u_n a \nabla u_n = -C \int_{\Omega_n} \mathcal{A}_0 u_n u_n.$$

Following the proof of [HW05, Lemma 3.6] this implies

$$\begin{aligned}\|\nabla u_n\|^2_{L^2(\Omega_n)} &\leq C \left[\int_{\Omega_n} (\lambda - \omega_2 - A_{\Omega_n,2}) u_n u_n - \lambda \int_{\Omega_n} u_n^2 - \int_{\Omega_n} \left(\frac{\operatorname{div} b}{2} - \omega_2 \right) u_n^2 \right] \\ &\leq C \big(\|f\|_{L^2(\Omega)} \|u_n\|_{L^2(\Omega_n)} + \lambda \|u_n\|^2_{L^2(\Omega_n)} \big) \leq \frac{C}{\lambda} \|f\|^2_{L^2(\Omega)}\end{aligned}$$

by (3). Thus we have shown

$$\|\nabla u_n\|_{L^2(\Omega_n)} \leq \frac{C}{\sqrt{\lambda}} \|f\|_{L^2(\Omega)} \tag{4}$$

independently from $n \in \mathbb{N}$. Consequently, there is a subsequence of $(u_n)_{n\in\mathbb{N}}$ in $H^1_0(\Omega)$ that converges weakly to some u. As in the proof of [HW05, Proposition 3.7] one sees, that $u \in D(A_{\Omega,2})$ and $(\lambda - \omega_2 - A_{\Omega,2})u = f$. □

Therefore, by the Lumer-Phillips Theorem $A_{\Omega,2}$ generates a C_0-semigroup on $L^2(\Omega)$, that will be denoted by $(T_{\Omega,2}(t))_{t\geq 0}$ in the following.

Proposition 5. *The semigroup $(T_{\Omega,2}(t))_{t\geq 0}$ is positive and if $\Omega_1, \Omega_2 \subseteq \mathbb{R}^d$ are domains with $\Omega_1 \subseteq \Omega_2$, then for every $\lambda > -\omega_2$, every $t \geq 0$ and every function $f \in L^2(\Omega_1)$*

(a) $|T_{\Omega_1,2}(t) f| \leq T_{\Omega_2,2}(t)|\tilde{f}|$,
(b) $|R(\lambda, A_{\Omega_1,2}) f| \leq R(\lambda, A_{\Omega_2,2})|\tilde{f}|$,

where $\tilde{f}$ denotes the trivial extension of f to Ω_2.

Proof. The proofs in [HW05, Section 4] can be copied almost completely, replacing again, as we did in the proof of dissipativity, the linear scaling in the cut-off functions η_m by a logarithmic one. Only at the end of Lemma 4.1 we have to give an extra argument:

Let $u \in D(A_{\Omega,2})$ and $v \in H^1(\Omega) \cap \{f \in H^2_{\text{loc}}(\Omega) : \mathcal{A} f \in L^2(\Omega)\}$ be real-valued functions with $v \geq 0$, that fulfill $(\lambda - \mathcal{A})u \leq (\lambda - \mathcal{A})v$ a.e. for some $\lambda > -\omega_2$. Then we get as in the proof of [HW05, Lemma 4.1] the inequality

$$\begin{aligned}&\int_\Omega \left[\nabla (u-v)^+\right]^T a \nabla (u-v)^+ \eta_m + \int_\Omega [\nabla \eta_m]^T a \nabla (u-v)^+ (u-v)^+ \\ &\quad + \int_\Omega \left(\frac{\operatorname{div} b}{2} + \lambda \right) \left[(u-v)^+\right]^2 \eta_m + \frac{1}{2} \int_\Omega b \nabla \eta_m \left[(u-v)^+\right]^2 \leq 0,\end{aligned}$$

where η_m is defined as in the proof of Proposition 2. By positivity of the third integral and letting $m \to \infty$, this yields

$$\int_\Omega \left[\nabla (u-v)^+\right]^T a \nabla (u-v)^+ \leq 0.$$

Thus by the ellipticity condition (A1) we have $\nabla (u-v)^+ = 0$. As $(u-v)^+$ is known to be in $H^1_0(\Omega)$, this yields $u \leq v$ a.e. □

5. Proof of Theorem 1

For $p = 2$ the proof of Theorem 1 is contained in the last section. Note that the estimate for the gradient of the resolvent follows directly from (4), since the resolvent is the weak limit of a subsequence of $(u_n)_{n\in\mathbb{N}}$.

In order to prove the second part of Theorem 1 we begin with the case $\Omega = \mathbb{R}^d$. For $1 < p < \infty$ we define the realisation

$$D(A_{\mathbb{R}^d,p}) = \{f \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^d) : \mathcal{A}u \in L^p(\mathbb{R}^d)\}, \qquad A_{\mathbb{R}^d,p}u = \mathcal{A}u.$$

Proposition 6. *The operator $A_{\mathbb{R}^d,p}$ generates a C_0-semigroup $(T_{\mathbb{R}^d,p}(t))_{t\geq 0}$ on $L^p(\mathbb{R}^d)$ with $\|T_{\mathbb{R}^d,p}(t)\| \leq \mathrm{e}^{-\omega_p t}$ that is consistent with $(T_{\mathbb{R}^d,2}(t))_{t\geq 0}$.*

Proof. By Proposition 2 the operator $A := A_{\mathbb{R}^d,p} + \omega_p$ is dissipative. In order to show surjectivity of $\lambda - A$ for some $\lambda > 0$, we approximate solutions in $\mathbb{R}^d$ again by solutions in bounded, regular domains Ω_n (e.g. balls) as in the proof of Proposition 4. Since the coefficients of $\mathcal{A}$ are bounded on Ω_n for every $n \in \mathbb{N}$ we again get unique solutions $u_n \in W^{2,p}(\Omega_n) \cap W^{1,p}_0(\Omega_n)$ of

$$\begin{cases} \lambda u_n - \mathcal{A}u_n &= f|_{\Omega_n} \text{ in } \Omega_n, \\ u_n &= 0 \quad \text{ on } \partial\Omega_n, \end{cases}$$

that by Proposition 2 obey the estimate

$$\|u_n\|_{L^p(\Omega_n)} \leq \frac{1}{\lambda}\|f\|_{L^p(\Omega)}$$

independently from $n \in \mathbb{N}$. Extending all these solutions to $\mathbb{R}^d$ by zero, we get a bounded sequence $(u_n)_{n\in\mathbb{N}}$ in $L^p(\mathbb{R}^d)$, from which we may extract a weakly convergent subsequence with weak limit $u \in L^p(\mathbb{R}^d)$. Redoing the proof of [HW05, Proposition 3.7] in the L^p-setting we see, that $u \in D(A_{\mathbb{R}^d,p})$ and $(\lambda - A)u = f$. Thus A generates a contraction semigroup and the rescaled semigroup $(T_{\mathbb{R}^d,p}(t))_{t\geq 0}$ satisfies the stated norm estimate.

Finally we have to prove consistency. We will do this for the resolvents of $A_{\mathbb{R}^d,p}$ and $A_{\mathbb{R}^d,2}$ in the case of large λ. Then consistency of the semigroups follows by

$$T_{\mathbb{R}^d,2}(t)f = \lim_{n\to\infty}\left[\frac{n}{t}R\left(\frac{n}{t}, A_{\mathbb{R}^d,2}\right)f\right]^n = \lim_{n\to\infty}\left[\frac{n}{t}R\left(\frac{n}{t}, A_{\mathbb{R}^d,p}\right)f\right]^n = T_{\mathbb{R}^d,p}(t)f$$

for all $f \in L^2(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$. Now let $\lambda > \max\{-\omega_p, -\omega_2\}$ and $f \in L^2(\mathbb{R}^d)\cap L^p(\mathbb{R}^d)$ be given. Since the coefficients of $\mathcal{A}$ are bounded on Ω_n for every $n \in \mathbb{N}$, we have consistency of $R(\lambda, A_{\Omega_n,p})$ and $R(\lambda, A_{\Omega_n,2})$. Thus

$$u_n = R(\lambda, A_{\Omega_n,p})f|_{\Omega_n} = R(\lambda, A_{\Omega_n,2})f|_{\Omega_n} \in L^2(\Omega_n) \cap L^p(\Omega_n)$$

for every $n \in \mathbb{N}$. By the above construction, there is a weakly convergent subsequence of $(u_n)_{n\in\mathbb{N}}$ in $L^p(\mathbb{R}^d)$ with weak limit $u = R(\lambda, A_{\mathbb{R}^d,p})f$. Note that by dissipativity this weak limit is independent of the chosen subsequence. On the other hand, by the proof of Proposition 4, we may extract from this subsequence another subsequence, that converges weakly in $L^2(\mathbb{R}^d)$ to $R(\lambda, A_{\mathbb{R}^d,2})f$. Denoting this final subsequence again by $(u_n)_{n\in\mathbb{N}}$ we obtain for every $\varphi \in C^\infty_c(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} R(\lambda, A_{\mathbb{R}^d,p})f\varphi = \lim_{n\to\infty}\int_{\mathbb{R}^d} u_n\varphi = \int_{\mathbb{R}^d} R(\lambda, A_{\mathbb{R}^d,2})f\varphi.$$

The desired consistency of the resolvents follows. □

Now, let $\Omega \subseteq \mathbb{R}^d$ be open and $f \in L^p(\Omega) \cap L^2(\Omega)$. Then for every $t \geq 0$ we have by Proposition 5 and consistency

$$\|T_{\Omega,2}(t)f\|_{L^p(\Omega)} \leq \|T_{\mathbb{R}^d,2}(t)|\tilde{f}|\|_{L^p(\mathbb{R}^d)} = \|T_{\mathbb{R}^d,p}(t)|\tilde{f}|\|_{L^p(\mathbb{R}^d)} \leq \mathrm{e}^{-\omega_p t}\|f\|_{L^p(\Omega)}.$$

Thus as in [HW05, Section 2] we may extend the operators $T_{\Omega,2}(t)$ continuously to operators $T_{\Omega,p}(t)$ on $L^p(\Omega)$, obtaining C_0-semigroups, that are consistent with $(T_{\Omega,2}(t))_{t\geq 0}$ by construction and fulfill $\|T_{\Omega,p}(t)\| \leq \mathrm{e}^{-\omega_p t}$ by the above calculation.

Furthermore, we have for $t \geq 0$ and $f \in L^2(\Omega) \cap L^p(\Omega)$

$$T_{\mathbb{R}^d,p}(t)|\tilde{f}| - |T_{\Omega,p}(t)f| = T_{\mathbb{R}^d,2}(t)|\tilde{f}| - |T_{\Omega,2}(t)f| \geq 0$$

and since the cone of all non-negative functions is closed in $L^p(\Omega)$ we have

$$|T_{\Omega,p}(t)f| \leq T_{\mathbb{R}^d,p}(t)|\tilde{f}|$$

for every $f \in L^p(\Omega)$ and $t \geq 0$. By the same closedness argument we get positivity of $(T_{\Omega,p}(t))_{t\geq 0}$ from

$$T_{\Omega,p}(t)f = T_{\Omega,2}(t)f \geq 0$$

for positive functions $f \in L^p(\Omega) \cap L^2(\Omega)$.

Finally, for every $\lambda > -\omega_p$ and every $f \in L^p(\Omega)$ we have

$$|R(\lambda, A_{\Omega,p})f| = \left| \int_0^\infty \mathrm{e}^{-\lambda t} T_{\Omega,p}(t)f \,\mathrm{d}t \right| \leq \int_0^\infty \mathrm{e}^{-\lambda t} T_{\mathbb{R}^d,p}(t)|\tilde{f}| \,\mathrm{d}t = R(\lambda, A_{\mathbb{R}^d,p})|\tilde{f}|,$$

completing the proof of Theorem 1.

Remark 7. It is worthwhile to observe that the semigroups generated by $A_{\Omega,p}$ constructed in the proof of Theorem 1 coincide with the corresponding semigroups from [AMP06] (whose generators will be denoted by $\tilde{A}_{\Omega,p}$ in the following). Since both families are consistent it is enough to show this for the resolvents in the case $p = 2$.

Let $f \in L^2(\Omega)$ be given. In [AMP06] $R(\lambda, \tilde{A}_{\Omega,2})f$ is obtained as the strong limit of $R(\lambda, \tilde{A}_{\Omega_n,2})f$. Now Ω_n is bounded and we have $R(\lambda, A_{\Omega_n,2}) = R(\lambda, \tilde{A}_{\Omega_n,2})$. So every subsequence of this sequence converges weakly to the strong limit, which implies $R(\lambda, A_{\Omega,2})f = R(\lambda, \tilde{A}_{\Omega,2})f$.

It is also of interest, that the strong convergence of the resolvents of $A_{\Omega_n,p}$ implies a convergence result for the corresponding semigroups, see [AMP06, Proposition 3.6.].

References

[AMP06] W. Arendt, G. Metafune, and D. Pallara. Schrödinger operators with unbounded drift. *J. Operator Theory*, 55(1):185–211, 2006.

[DL90] R. Dautray and J.-L. Lions. *Mathematical analysis and numerical methods for science and technology. Vol. 1.* Springer-Verlag, Berlin, 1990.

[HW05] R. Haller-Dintelmann and J. Wiedl. Kolmogorov kernel estimates for the Ornstein-Uhlenbeck operator. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)*, 4:729–748, 2005.

[MS04] G. Metafune and C. Spina. An integration by parts formula in Sobolev spaces. *Ulmer Seminare*, 9:254–262, 2004.

[PRS05] J. Prüss, A. Rhandi, and R. Schnaubelt. The domain of elliptic operators on $L^p(\mathbb{R}^d)$ with unbounded drift coefficients. To appear in Houston J. Math., Preprint, 2005.

[Woo05] I. Wood. *Elliptic and Parabolic Problems in Non-Smooth Domains.* PhD thesis, Technische Universität Darmstadt, 2005.

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