Expansions in generalized eigenfunctions of the weighted Laplacian on star-shaped networks

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In memory of Günter Lumer

Abstract. We are interested in evolution phenomena on star-shaped networks composed of n semi-infinite branches which are connected at their origins. Using spectral theory we construct the equivalent of the Fourier transform, which diagonalizes the weighted Laplacian on the n-star. It is designed for the construction of explicit solution formulas to various evolution equations such as the heat, wave or the Klein-Gordon equation with different leading coefficients on the branches.

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1. Introduction

We study the foundations for the understanding of evolution phenomena on star-shaped networks composed of n semi-infinite branches which are connected at their origins. To this end, we construct the equivalent of the Fourier transform which diagonalizes the weighted Laplacian on the n-star, using spectral theory. This allows us to formulate a functional calculus for the weighted Laplacian, designed to construct explicit solution formulas to various evolution equations such as the heat, wave or the Klein-Gordon equation with different leading coefficients on the branches. The model of the n-star should lead to a comprehension of the phenomena happening locally in time and space near the ramification nodes of more complicated networks. The investigation of evolution equations on networks

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starts with G. Lumer [17] and subsequent papers. See [1, 4, 9] and the references mentioned therein.

Let $N_1, \dots, N_n$ be n disjoint copies of $(0; +\infty)$ ($n \in \mathbb{N}$, $n \geq 2$) and $c_k > 0$, for $k \in \{1, \dots, n\}$. A vector $(u_1, \dots, u_n)$ of functions $u_k : \overline{N_k} \to \mathbb{C}$ is said to satisfy the transmission conditions

$$(T_0), \text{ if } u_i(0) = u_k(0) \text{ for all } (i,k) \in \{1, \dots, n\}^2,$$

$$(T_1), \text{ if } \sum_{k=1}^{n} c_k^2 \partial_x u_k(0^+) = 0.$$

A vector $(u_k)_{k=1,\dots,n}$ satisfying (T_0) can also be viewed as a function on $N := \bigcup_{k=1}^n \overline{N_k}$, where the n boundary points corresponding to $0 \in \overline{N_k}$ are identified. This domain is called a star-shaped network or n-star with the branches $N_1, \dots, N_n$.

In this paper, we study the weighted Laplacian submitted to (T_0) and (T_1):

$$\begin{cases} D(A) := \left\{ (u_k) \in \prod_{k=1}^{n} H^2(N_k) \mid (u_k) \text{ satisfies } (T_0) \text{ and } (T_1) \right\}, \\ A(u_k) := (-c_k^2 \cdot \partial_x^2 u_k)_{k=1,\dots,n}. \end{cases}$$

This operator can be inserted for example in the abstract wave equation

$$\begin{cases} \ddot{u}(t) + Au(t) = 0, \\ u(0) = u_0, \ \dot{u}(0) = v_0, \end{cases}$$

which means in concrete terms:

$$\begin{cases} [\partial_t^2 - c_k^2 \partial_x^2] u_k(t,x) = 0, & \forall\, k \in \{1, \dots, n\}, \\ u_i(t,0) = u_k(t,0), & \forall\, (i,k) \in \{1, \dots, n\}^2, \\ \sum_{k=1}^{n} c_k^2 \partial_x u_k(t, 0^+) = 0, & \\ u_k(0,x) = u_k^0(x), & \forall\, k \in \{1, \dots, n\}, \\ \partial_t u_k(0,x) = v_k^0(x), & \forall\, k \in \{1, \dots, n\} \end{cases}$$

for $x, t \geq 0$, where $u_0 = (u_k^0)_{k=1,\dots,n}$, $v_0 = (v_k^0)_{k=1,\dots,n}$ and $u(t) = (u_k(t, \cdot))_{k=1,\dots,n}$.

The operator A is self-adjoint, its spectrum is $[0; +\infty)$ and has multiplicity n (in the sense of ordered spectral representations, see Definition XII.3.15, p. 1216 of [14]). The analytical core of this paper is a representation of the kernel of the resolvent of A in terms of a special choice of a family of n generalized eigenfunctions parametrized by $\lambda \in [0; +\infty)$.

After having proved a limiting absorption principle for the resolvent, we insert A in Stone's formula to obtain a representation of the resolution of the identity of A in terms of the generalized eigenfunctions. This classical procedure (see for example [3]) should lead to an expansion formula for functions in $H = \prod_{k=1}^n L^2(N_k)$ in terms of the family of generalized eigenfunctions.

We observe that the transition from the formula for the resolution of the identity to an expansion formula involving a generalized Fourier transform, which diagonalizes A, is not straightforward in the case of the n-star. This comes from the fact that the resolvent kernel, which is defined on $N \times N$, changes its structure when crossing the n diagonals of $N_k \times N_k$, $k = 1, \ldots, n$. These diagonals cut $N \times N$ into n connected pieces in accordance with the structure of the resolvent. Our special choice of the generalized eigenfunctions allows us to recombine the inner integral of the formula for the resolution of the identity across the diagonals of $N_k \times N_k$ to an integral over all of N, furnishing the desired generalized Fourier transformation V as well as its left inverse Z. It is not obvious, whether this recombination is possible for all choices of generalized eigenfunctions, although theoretical results imply that an expansion in generalized eigenfunctions always exists [11, 19]. Now, V can be extended to an isometry on H, which diagonalizes A, and an explicit functional calculus for A can be given. We plan to give explicit expressions for the solutions of evolution equations like the weighted wave, heat and Klein-Gordon equations on the n-star and to derive results on their qualitative behaviour in a subsequent paper.

Such expressions can be obtained (at least formally) also from representations of the resolution of the identity which are not recombined to Fourier-type transformations. But these expressions would be sums of terms with very poor regularity although their sum, representing the solution, is regular (like a decomposition of a C^∞-function by multiplying it with characteristic functions on sub-domains). These artificial singularities are totally undesirable for any kind of investigations. They occur for example in [13], a pioneering paper of theoretical physics explaining the phenomenon of advanced transmission of dispersive wave packets crossing a potential barrier. The authors obtain a solution formula using Laplace tranform in time, but which splits up into irregular terms. They do not attempt to prove that their formula represents a solution of the original problem, which should be possible only in some very weak sense. But this (artificial) lack of regularity permits only to study the advanced transmission phenomenon for gaussian wave packets using a highly special method.

In [7], the authors study the similar phenomenon of delayed reflection occurring at semi-infinite barriers. They construct an expansion in generalized eigenfunctions and thus avoid those artificial singularities. This expansion is used to define wave packets in frequency bands adapted to the transmission conditions. Thus it is possible to study the dependence of propagation patterns, in particular the delayed reflection, on the main frequency of the wave packets. In [8] it is pointed out using similar methods, that classical causality is valid for nonlinear dispersive waves hitting a semi-infinite barrier. In [6] a solution formula for the Klein-Gordon equation on the n-star but with one finite branch with an end with prescribed excitation is presented using Laplace transform in time. This result is not comparable with the present paper, because it does not concern an initial value problem.

There remains an unsatisfactory point in the present paper: our Fourier-type transformation V is not a spectral representation of A in the classical sense although it diagonalizes this operator: the natural norm on the range of V making V an isometry, as in the theorem of Plancherel, is not just a weighted L^2-norm on some measure space. This is due to the fact that the back transformation Z has a different expression on each branch, and this is caused by the ramification of the domain.

It is not clear to us how one could find a family of generalized eigenfunctions leading to a spectral representation of A. The existing general literature on expansions in generalized eigenfunctions ([11, 19, 20] for example) does not seem to be helpful for this kind of problem: their constructions start from an abstractly given spectral representation. But in concrete cases you do not have an explicit formula for it at the beginning.

In [10] the relation of the eigenvalues of the Laplacian in a $L^\infty-$setting on infinite, locally finite networks to the adjacency operator of the network is studied. The question of the completeness of the corresponding eigenfunctions, viewed as generalized eigenfunctions in an L^2-setting, could be asked. The n-star we consider is a particular case of the geometry studied by J. von Below and the completeness of the eigenfunctions is established in a way. In a recent paper ([15]), the authors consider general networks with semi-infinite ends. They give a construction to compute some generalized eigenfunctions from the coefficients of the transmission conditions (scattering matrix). The eigenvalues of the associated Laplacian are the poles of the scattering matrix and their asymptotic behaviour is studied. But no attempt is made to show the completeness of a given family of generalized eigenfunctions. Spectral theory for the Laplacian on finite networks has been studied since the 1980ies for example by J.P. Roth, J.v. Below, S. Nicaise, F. Ali Mehmeti (see [1]).

Natural perspectives for our expansion result are investigations on the qualitative behaviour of solutions of evolution equations on the n-star. For the weighted heat equation on the n-star, our expansion permits to prove Gaussian estimates (this feature shall be treated in a subsequent paper). For bounded networks and variable coefficients this has already been proved by D. Mugnolo ([18]) using different methods. In [16] the transport operator is considered on finite networks. The connection between the spectrum of the adjacency matrix of the network and the (discrete) spectrum of the transport operator is established. By adding semi-infinite branches to the finite network, continuous parts of the spectrum and generalized eigenfunctions might appear.

Many results have been obtained in spectral theory for elliptic operators on various types of unbounded domains in $\mathbb{R}^n$. Using the existing results on stratified bands [12] for example, one could reduce the spectral analysis of the Laplacian on networks of bands locally near the nodes to the case of the n-star. Time asymptotics for the associated evolution equations have also been studied extensively. For the Klein-Gordon equation on the n-star we conjecture that the maximum of the absolute value of the solutions decays as $t^{-1/2}$ when t tends to infinity as on the real line. For two branches with potential step this has been already proved using generalized eigenfunctions in [2]. An example for a three-dimensional coupled domain with singularities is treated in [5]. See also the other literature mentioned therein and in [3].

2. Data and functional analytic framework

Let us introduce some notation which will be used throughout the rest of the paper:

- **Domain and functions**: Let $N_1, \ldots, N_n$ be n disjoint sets identified with $(0; +\infty)$ ($n \in \mathbb{N}$, $n \geq 2$) and put $N := \bigcup_{k=1}^{n} \overline{N_k}$. Furthermore, we write $[a,b]_{N_k}$ for the interval $[a,b]$ in the branch N_k. For the notation of functions two viewpoints are used:
  - functions f on the object N taking their values in $\mathbb{R}$ and f_k is then the restriction of f to N_k.
  - n-tuples of functions on the branches N_k; then sometimes we write $f = (f_1, \ldots, f_n)$.
- **Transmission conditions**:

$$(T_0)\text{: } (u_k)_{k=1,\ldots,n} \in \prod_{k=1}^{n} C^0(\overline{N_k}) \text{ satisfies } u_i(0) = u_k(0),\ \forall\ (i,k) \in \{1, \ldots, n\}^2.$$

$$(T_1)\text{: } (u_k)_{k=1,\ldots,n} \in \prod_{k=1}^{n} C^1(\overline{N_k}) \text{ satisfies } \sum_{k=1}^{n} c_k^2 \cdot \partial_x u_k(0^+) = 0.$$

- **Definition of the operator**: Define the real Hilbert space

$$H = \prod_{k=1}^{n} L^2(N_k) \text{ with scalar product } ((u_k),(v_k))_H = \sum_{k=1}^{n} (u_k, v_k)_{L^2(N_k)}$$

  and the operator $A : D(A) \longrightarrow H$ by

$$\begin{cases} D(A) = \left\{ (u_k) \in \prod_{k=1}^{n} H^2(N_k) \mid (u_k) \text{ satisfies } (T_0) \text{ and } (T_1) \right\}, \\ A(u_k) = (A_k u_k)_{k=1,\ldots,n} = (-c_k^2 \cdot \partial_x^2 u_k)_{k=1,\ldots,n}. \end{cases}$$

  Note that, if $c_k = 1$ for every $k \in \{1, \ldots, n\}$, A is the Laplacian in the sense of the existing literature.
- **Notation for the resolvent**: The resolvent of an operator T is denoted by R, i.e. $R(z,T) = (zI - T)^{-1}$ for $z \in \rho(T)$.

Proposition 2.1 (spectrum of A). *The operator $A : D(A) \to H$ defined above is self-adjoint and satisfies $\sigma(A) = [0; +\infty)$.*

Proof. Simple adaptation of the proof of Lemma 1.1.5 in [3]. □

3. Expansion in generalized eigenfunctions

The aim of this section is to find an explicit expression for the kernel of the resolvent of the operator A on the star-shaped network defined in the previous section.

Definition 3.1 (generalized eigenfunction). Let $\lambda \in \mathbb{C}$ be fixed. An element $f \in \prod_{k=1}^{n} C^\infty(\overline{N_k})$ is called generalized eigenfunction of A if it satisfies (T_0), (T_1) and the formal differential expression $Af = \lambda f$.

Proposition 3.2 (an expression of the resolvent). *Let $\lambda \in \mathbb{C}$ be fixed. Let* $\mathrm{Im}(\lambda) \neq 0$ *and e_1^λ, e_2^λ be generalized eigenfunctions of A such that the wronskian $w_{1,2}^\lambda(x)$ satisfies for every x in N*

$$w_{1,2}^\lambda(x) = \det W(e_1^\lambda(x), e_2^\lambda(x)) = e_1^\lambda(x) \cdot (e_2^\lambda)'(x) - (e_1^\lambda)'(x) \cdot e_2^\lambda(x) \neq 0.$$

If for some $k \in \{1, \dots, n\}$ we have $e_1^\lambda|_{N_m} \in H^2(N_m)$ for all $m \neq k$ and $e_2^\lambda|_{N_k} \in H^2(N_k)$, then we have for any $f \in H$, $\lambda \in \mathbb{C}$ and $x \in N_k$

$$[R(\lambda, A)f](x) = \frac{1}{c_k^2 (w_{1,2}^\lambda)(x)} \cdot \left[\int_{[x;+\infty)_{N_k}} e_1^\lambda(x) e_2^\lambda(x') f(x')\, dx' \right. \tag{1}$$
$$\left. + \int_{N \setminus [x;+\infty)_{N_k}} e_2^\lambda(x) e_1^\lambda(x') f(x')\, dx' \right].$$

Note that by integral over N, we mean the sum of the integrals over N_k, $k = 1, \dots, n$.

Proof. The arguments are the same as in the proof of Theorem 1.3.4 of [3] (see also [2]) and the calculations are analogous. The integration by parts is replaced here by the Green formula for the star-shaped network that is given in the next lemma. □

Lemma 3.3 (Green's formula on the star-shaped network with n semi-infinite branches). *Denote by $V_{a_1,\dots,a_n}$ the subset of the network N defined by*

$$V_{a_1,\dots,a_n} = \{x \in N \mid x \in [0; a_k),\ \textit{where } k \textit{ is the index such that } x \in \overline{N_k}\}.$$

Then $u, v \in D(A)$ implies

$$\int_{V_{a_1,\dots,a_n}} u''(x)v(x)\, dx = \int_{V_{a_1,\dots,a_n}} u(x)v''(x)\, dx - \sum_{k=1}^{n} u(a_k)v'(a_k) + \sum_{k=1}^{n} u'(a_k)v(a_k).$$

Proof. Two successive integrations by parts are used and since both u and v belong to $D(A)$, they both satisfy the transmission conditions (T_0) and (T_1). So

$$\sum_{k=1}^{n} u_k(0)v_k'(0) = u_1(0) \sum_{k=1}^{n} v_k'(0) = 0.$$

Idem for $\sum_{k=1}^{n} u_k'(0)v_k(0)$. □

Definition 3.4 (generalized eigenfunctions of A). For $j \in \{1, \ldots, n\}$ let

$$s_j := -c_j^{-1} \cdot \sum_{l \neq j} c_l, \qquad d_{1,j} := (1 + s_j)/2 \qquad \text{and} \qquad d_{2,j} := (1 - s_j)/2.$$

The complex square root is chosen in such a way that $\sqrt{r \cdot e^{i\phi}} = \sqrt{r} e^{i\phi/2}$ with $r > 0$ and $\phi \in [-\pi; \pi)$. For $\lambda \in \mathbb{C}$ and $j, k \in \{1, \ldots, n\}$, $F_\lambda^{\pm,j} : N \to \mathbb{C}$ is defined for $x \in \overline{N_k}$ by $F_\lambda^{\pm,j}(x) := F_{\lambda,k}^{\pm,j}(x)$ with

$$\begin{cases} F_{\lambda,j}^{\pm,j}(x) = d_{1,j} \cdot \exp(\pm i c_j^{-1} \sqrt{\lambda} x) + d_{2,j} \cdot \exp(\mp i c_j^{-1} \sqrt{\lambda} x), & \\ F_{\lambda,k}^{\pm,j}(x) = \exp(\pm i c_k^{-1} \sqrt{\lambda} x), & \text{for } k \neq j. \end{cases}$$

Remark 3.5.
- $F_\lambda^{\pm,j}$ satisfies the transmission conditions (T_0) and (T_1).
- Formally it holds $AF_\lambda^{\pm,j} = \lambda F_\lambda^{\pm,j}$.
- Clearly $F_\lambda^{\pm,j}$ does not belong to H, thus it is not a classical eigenfunction.
- For $\mathrm{Im}(\lambda) \neq 0$, the function $F_{\lambda,k}^{\pm,j}$, where the $+$-sign (respectively $-$-sign) is chosen if $\mathrm{Im}(\lambda) > 0$ (respectively $\mathrm{Im}(\lambda) < 0$), belongs to $H^2(N_k)$ for $k \neq j$. This feature is used in the formula for the resolvent of A.

Definition 3.6 (kernel of the resolvent). For any $\lambda \in \mathbb{C}$, $j \in \{1, \ldots, n\}$ and $x \in \overline{N_j}$ we define

$$K(x, x', \lambda) = \begin{cases} \dfrac{1}{w(\lambda)} F_{\lambda,j}^{\pm,j}(x) F_{\lambda,j}^{\pm,j+1}(x'), \text{ for } x' \in \overline{N_j},\ x' > x, \\ \dfrac{1}{w(\lambda)} F_{\lambda,j}^{\pm,j+1}(x) F_\lambda^{\pm,j}(x'), \text{ for } x' \in \overline{N_k}, k \neq j \text{ or } x' \in \overline{N_j},\ x' < x, \end{cases}$$

where $w(\lambda) = \pm i\sqrt{\lambda} \cdot \sum_{j=1}^n c_j$. In the whole formula $+$ (respectively $-$) is chosen if $\mathrm{Im}(\lambda) > 0$ (respectively $\mathrm{Im}(\lambda) \leq 0$).

Here the index j is to be understood modulo n, that is to say, if $j = n$, then $j + 1 = 1$.

Note that in particular, if $c_j = c$ for all $j \in \{1, \ldots, n\}$, then $w(\lambda) = \pm i n \sqrt{\lambda}$, for all $j \in \{1, \ldots, n\}$.

Theorem 3.7 (expansion of the resolvent in the family $\{F_\lambda^{\pm,j}, j = 1, \ldots, n\}$). *Let $f \in H$. Then, for $x \in N$ and $\lambda \in \rho(A)$*

$$[R(\lambda, A)f](x) = \int_N K(x, x', \lambda) f(x')\, dx'.$$

Proof. In (1), the generalized eigenfunction e_1^λ can be chosen to be $F_\lambda^{\pm,j}$. Then e_2^λ can be $F_\lambda^{\pm,l}$ with any $l \neq j$ so we have chosen $j + 1$ to fix the formula. The choice has been done so that the integrands lie in $L^1(0, +\infty)$ (cf. the last item in Remark 3.5). □

4. Application of Stone's formula and limiting absorption principle

Let us first recall Stone's formula (see Theorem XII.2.11 in [14]).

Theorem 4.1 (Stone's formula). *Let E be the resolution of the identity of a linear unbounded self-adjoint operator $T : D(T) \to H$ in a Hilbert space H (i.e. $E(a,b) = \mathbf{1}_{(a,b)}(A)$ for $(a,b) \in \mathbb{R}^2$, $a < b$). Then, in the strong operator topology*

$$h(T)E((a,b)) = \lim_{\delta \to 0^+} \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} h(\lambda)[R(\lambda - \epsilon i, T) - R(\lambda + \epsilon i, T)]\, d\lambda$$

for all $(a,b) \in \mathbb{R}^2, a < b$ and for any continuous scalar function h defined on the real line.

To apply this formula we need to study the behaviour of the resolvent $R(\lambda, A)$ for λ approaching the spectrum of A.

Theorem 4.2 (limiting absorption principle for A). *For any $(x,x') \in N^2$ and $(\lambda, \epsilon) \in (\mathbb{R}^+)^2$, it holds with s_j, d_j as defined in Definition 3.4:*

1. $\lim_{\epsilon \to 0} K(x,x',\lambda - i\epsilon) = K(x,x',\lambda)$,
2. $|K(x,x',\lambda - i\epsilon)| \le M \cdot (\sqrt{\lambda})^{-1}$ *with*

$$M = \max_{j \in \{1,\dots,n\}} \left[\max(1; |d_{1,j}| + |d_{2,j}|) \cdot (\sum_{j=1}^{n} c_j)^{-1} \right].$$

Proof. 1. The complex square root is, by definition, continuous on $\{z \in \mathbb{C} \mid \operatorname{Im}(z) \le 0\}$ (cf. Definition 3.4), hence the continuity of $K(x,x',\lambda)$ at real positive numbers λ. (Note that x,x' are fixed parameters in this context.)

2. In concrete terms, the kernel is for $\operatorname{Im}(\mu) \le 0$ and $x \in \overline{N_j}$

$$K(x,x',\mu) = \frac{1}{w(\mu)} \begin{cases} e^{-i\sqrt{\mu}(c_j^{-1}x + c_k^{-1}x')}, & x' \in \overline{N_k},\ k \ne j, \\ d_{2,j} e^{-i\sqrt{\mu} c_j^{-1}(x-x')} + d_{1,j} e^{-i\sqrt{\mu} c_j^{-1}(x+x')}, & x' \in \overline{N_j},\ x' < x, \\ d_{2,j} e^{-i\sqrt{\mu} c_j^{-1}(x'-x)} + d_{1,j} e^{-i\sqrt{\mu} c_j^{-1}(x+x')}, & x' \in \overline{N_j},\ x' > x. \end{cases}$$

Now

$$\left| \frac{1}{w(\mu)} \right| = \left(\sum_{j=1}^{n} c_j \sqrt{|\lambda - i\epsilon|} \right)^{-1} = \left(\sum_{j=1}^{n} c_j \right)^{-1} (\lambda^2 + \epsilon^2)^{-1/4}$$
$$\le \left(\sum_{j=1}^{n} c_j \right)^{-1} \lambda^{-1/2}$$

for $\mu = \lambda - i\epsilon$, $\lambda > 0$, $\epsilon \ge 0$. Moreover, if $x' < x$,

$$\left| e^{-i(\sqrt{\lambda - i\epsilon}) c_j^{-1}(x-x')} \right| = e^{\operatorname{Im}(\sqrt{\lambda - i\epsilon}) c_j^{-1}(x-x')} \le 1,$$

since $\operatorname{sgn}(\operatorname{Im}(\sqrt{\lambda - i\epsilon})) = \operatorname{sgn}(\operatorname{Im}(\lambda - i\epsilon))$ (cf. Lemma 2.5.1 of [3], see also [2]). Idem for the other exponential terms. Hence the above estimate. □

Remark 4.3. Note that, in particular, if $c_j = c$, $j = 1, \ldots, n$, then $M = c(n-1)/n$.

Lemma 4.4. *For $(x, x') \in N^2$ and $\lambda \in \mathbb{C}$, it holds $K(x, x', \overline{\lambda}) = \overline{K(x, x', \lambda)}$.*

Proof. The choice of the branch cut of the complex square root has been made such that $\sqrt{\overline{\lambda}} = \overline{\sqrt{\lambda}}$ for all $\lambda \in \mathbb{C}$.

This implies $\overline{e^{i\sqrt{\lambda}x}} = e^{\overline{i\sqrt{\lambda}x}} = e^{-i\sqrt{\overline{\lambda}}x}$ for all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$. Thus it holds

$$\overline{F_\lambda^{+,j}(x)} = F_{\overline{\lambda}}^{-,j}(x) \quad \text{and} \quad \overline{F_\lambda^{-,j}(x)} = F_{\overline{\lambda}}^{+,j}(x)$$

for all $\lambda \in \mathbb{C}$, $x \in N$ and $j \in \{1, \ldots, n\}$. In the same way we have $\overline{w(\lambda)} = -w(\overline{\lambda})$. Observe, that switching from λ to $\overline{\lambda}$ the sign of the imaginary part is changing, so in the definition of $K(x, x', \lambda)$ we have to take the other sign whenever there is a $\pm$-sign in the formula. This gives the assertion. □

Proposition 4.5 (rewriting of the resolution of the identity of A). *Take $f \in H = \prod_{j=1}^n L^2(N_j)$, vanishing almost everywhere outside a compact set $B \subset N$ and let $-\infty < a < b < +\infty$. Then, for $x \in N$*

$$(E(a,b)f)(x) = \operatorname{Re}\left\{ \frac{1}{\pi} \int_a^b \sum_{j=1}^n \sigma_j(\lambda, x) \cdot F_\lambda^{-,j+1}(x) \left(\int_N f(x') \cdot F_\lambda^{-,j}(x')\, dx' \right) d\lambda \right\},$$

where E is the resolution of the identity of A (cf. Theorem 4.1) and

$$\sigma_j(\lambda, x) := \frac{1}{\sqrt{\lambda}} \sigma_j(x), \text{ where } \sigma_j(x) := \mathbf{1}_{\overline{N_j}}(x) \cdot \frac{1}{C} \text{ for } j \in \{1, \ldots, n\}.$$

Here $C = (\sum_k c_k)$ and the index j is to be understood modulo n, that is to say, if $j = n$, then $j + 1 = 1$.

Note that in particular if $c_j = c$ for all $j \in \{1, \ldots, n\}$, then $C = nc$, for all $j \in \{1, \ldots, n\}$.

Proof. The proof is analogous to that of Lemma 1.3.13 of [3] (see also [2]).

Let in addition $g \in H$ be vanishing outside B. Then

$$\begin{aligned}
&(E(a,b)f, g)_H \\
&= \left(\lim_{\delta \to 0+} \lim_{\varepsilon \to 0+} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} [R(\lambda - \varepsilon i, A) - R(\lambda + \varepsilon i, A)]\, d\lambda\, f, g \right)_H &(2)\\
&= \lim_{\delta \to 0+} \lim_{\varepsilon \to 0+} \frac{1}{2\pi i} \left(\int_{a+\delta}^{b-\delta} [R(\lambda - \varepsilon i, A) - R(\lambda + \varepsilon i, A)]\, d\lambda\, f, g \right)_H &(3)
\end{aligned}$$

$$= \lim_{\delta\to 0^+}\lim_{\varepsilon\to 0^+}\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}([R(\lambda-\varepsilon i,A)-R(\lambda+\varepsilon i,A)]f,g)_H\,d\lambda \tag{4}$$

$$= \lim_{\delta\to 0^+}\lim_{\varepsilon\to 0^+}\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\int_N f(x')[K(\cdot,x',\lambda-i\varepsilon)-K(\cdot,x',\lambda+i\varepsilon)]\,dx',g(\cdot)\right)_H d\lambda \tag{5}$$

$$= \lim_{\delta\to 0^+}\lim_{\varepsilon\to 0^+}\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\int_N f(x')[K(\cdot,x',\lambda-i\varepsilon)-\overline{K(\cdot,x',\lambda-i\varepsilon)}]\,dx',g(\cdot)\right)_H d\lambda \tag{6}$$

$$= \lim_{\delta\to 0^+}\lim_{\varepsilon\to 0^+}\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta}\left(\int_N f(x')\,2i\,\mathrm{Im}(K(\cdot,x',\lambda-i\varepsilon))\,dx',g(\cdot)\right)_H d\lambda \tag{7}$$

$$= \lim_{\delta\to 0^+}\frac{1}{\pi}\int_{a+\delta}^{b-\delta}\left(\int_N f(x')[\lim_{\varepsilon\to 0^+}\mathrm{Im}(K(\cdot,x',\lambda-i\varepsilon))]\,dx',g(\cdot)\right)_H d\lambda \tag{8}$$

$$= \left(\frac{1}{\pi}\int_a^b\int_N f(x')\mathrm{Im}(K(\cdot,x',\lambda-i0))\,dx'\,d\lambda,g(\cdot)\right)_H \tag{9}$$

$$= \int_N\frac{1}{\pi}\int_a^b\Big[\int_N f(x')\mathrm{Im}\Big[\frac{1}{\sqrt{\lambda}}\sum_{j=1}^n\frac{\mathbf{1}_{\overline{N_j}}(x)}{-iC}\Big(\mathbf{1}_{\{x'\in\overline{N_j},x'>x\}}(x')F_\lambda^{-,j}(x)F_\lambda^{-,j+1}(x') \tag{10}$$
$$+\mathbf{1}_{N\setminus\{x'\in\overline{N_j},x'>x\}}(x')F_\lambda^{-,j+1}(x)F_\lambda^{-,j}(x')\Big)\Big]dx'\Big]\,d\lambda\,g(x)\,dx$$

$$= \int_N\frac{1}{\pi}\int_a^b\Big[\int_N f(x')\frac{1}{\sqrt{\lambda}}\sum_{j=1}^n\mathbf{1}_{\overline{N_j}}(x)\mathrm{Re}\Big[\frac{1}{C}\big(\mathbf{1}_{\{x'\in\overline{N_j},x'>x\}}(x')F_\lambda^{-,j}(x)F_\lambda^{-,j+1}(x') \tag{11}$$
$$+\mathbf{1}_{N\setminus\{x'\in\overline{N_j},x'>x\}}(x')F_\lambda^{-,j+1}(x)F_\lambda^{-,j}(x')\big)\Big]dx'\Big]\,d\lambda\,g(x)\,dx$$

$$= \int_N\frac{1}{\pi}\int_a^b\Big[\int_N f(x')\frac{1}{C\sqrt{\lambda}}\sum_{j=1}^n\mathbf{1}_{\overline{N_j}}(x)\mathrm{Re}\Big[F_\lambda^{-,j+1}(x)F_\lambda^{-,j}(x')\Big]\,dx'\Big]\,d\lambda\,g(x)\,dx \tag{12}$$

$$= \int_N\mathrm{Re}\Big[\frac{1}{\pi}\int_a^b\frac{1}{C\sqrt{\lambda}}\sum_{j=1}^n\mathbf{1}_{\overline{N_j}}(x)F_\lambda^{-,j+1}(x)\Big(\int_N f(x')F_\lambda^{-,j}(x')\,dx'\Big)\Big]\,d\lambda\,g(x)\,dx.$$

Here, the justifications for the equalities are the following:

(2): Stone's formula (Theorem 4.1) applied with $h(\lambda)\equiv 1$.

(3): After applying the operator valued integral to f, the two limits are in H. So they commute with the scalar product in H.
(4): $(\cdot f, g)_H$ is a continuous linear form on $\mathcal{L}(H)$, and can therefore be commuted with the vector-valued integration.
(5): Theorem 3.7.
(6): Lemma 4.4.
(7): $z - \overline{z} = 2i \cdot \operatorname{Im} z \ \forall z \in \mathbb{C}$.
(8): Dominated convergence. Note that $\operatorname{supp} f$, $\operatorname{supp} g$ and $[a, b]$ are compact and use the limiting absorption principle (Theorem 4.2).
(9): Fubini.
(10): Definition 3.6.
(11): $\operatorname{Im}(z) = \operatorname{Re}(z/i)$ for all $z \in \mathbb{C}$. Note that, if $\lambda \in \mathbb{R}^-$, then $\lambda \in \rho(A)$ and thus the integrand in Stone's formula is zero.
(12): Note that

$$\begin{cases} (F_{\lambda,j}^{-,j})(x)(F_{\lambda,j}^{-,j+1})(x') = d_{2,j}e^{-ic_j^{-1}\sqrt{\lambda}(x-x')} + d_{1,j}e^{-ic_j^{-1}\sqrt{\lambda}(x+x')}, \\ (F_{\lambda,j}^{-,j+1})(x)(F_{\lambda,j}^{-,j})(x') = d_{2,j}e^{-ic_j^{-1}\sqrt{\lambda}(x'-x)} + d_{1,j}e^{-ic_j^{-1}\sqrt{\lambda}(x+x')}. \end{cases}$$

Since $e^{-ic_j^{-1}\sqrt{\lambda}(x-x')}$ and $e^{-ic_j^{-1}\sqrt{\lambda}(x'-x)}$ are conjugated for real λ, both expressions have the same real part. Thus the integrals on $\{x' \in \overline{N_j}, x' > x\}$ and its complement $N \setminus \{x' \in \overline{N_j}, x' > x\}$ recombine to a single integral on N. The formula of the theorem follows.

The assertion follows, because g was arbitrary with compact support. □

5. A Plancherel-type formula and a functional calculus for the operator

Now we use the explicit formula for the resolution of the identity of the operator A obtained in Proposition 4.5 to prove a Plancherel-type formula. As in [3] (see also [2]), we define the Fourier-type transformation V associated with the system of generalized eigenfunctions $\{F_\lambda^{-,j} \mid \lambda \in [0; +\infty), j \in \{1, \ldots, n\}\}$ on regular functions using Proposition 4.5.

The main difficulty here is that the coefficient $\sigma_j(x)$ appearing in Proposition 4.5 depends on $x \in N$: it is different on each branch of the star, unlike the situation in [2] and [3]. Thus $\sigma_j(x)$ does not commute with V and therefore the scalar product making the range of V a Hilbert space and V an isometry cannot be directly defined as in [2] and [3], but must be transferred from H via V. This introduces some additional technicalities. Apart from this we follow the lines of [2] and [3].

Definition 5.1. 1. For $f \in L^1(N)$ define $V_j f : [0; +\infty) \to \mathbb{R}$ by

$$V_j f(\lambda) = \int_N f(x) \cdot F_\lambda^{-,j}(x)\, dx,\ j = 1, \ldots, n$$

and $Vf : [0;+\infty) \to \mathbb{C}^n$ by $Vf = (V_j f)_{j\in\{1,\dots,n\}}$.

2. Let σ be defined as in Proposition 4.5 and $\chi \in C^\infty(\mathbb{R})$ be such that $\chi \equiv 0$ on $(-\infty, 1)$ and $\chi \equiv 1$ on $(2, +\infty)$. For $K_j \in C^\infty((0, +\infty), \mathbb{C})$ such that $\chi K_j \in \mathcal{S}(\mathbb{R})$, for $j \in \{1, \dots, n\}$ define $Z(K) : N \to \mathbb{R}$ by

$$Z(K_1, \dots, K_n)(x) = \frac{1}{\pi}\mathrm{Re}\left\{\int_0^{+\infty} \frac{1}{\sqrt{\lambda}} \sum_{j=1}^{n} \sigma_j(x) K_j(\lambda) F_\lambda^{-,j+1}(x)\, d\lambda\right\},\ x \in N.$$

Note that the integral on the right-hand side is absolutely convergent because $\lambda \mapsto 1/\sqrt{\lambda}$ is L^1_{loc}, K_j is continuous and rapidly decreasing at $+\infty$ and $|F_\lambda^{-,k}(x)| \le Const$, for all $\lambda \in (0; +\infty)$, $x \in N$, $k \in \{1, \dots, n\}$.

Remark 5.2. Unlike W in [3], Z is not injective: an easy computation shows that $Z(K) = 0$ is equivalent to

$$\mathrm{Re}\left[\mathcal{F}(K_j(\cdot^2) \cdot \mathbf{1}_{[0;+\infty)}(\cdot))(x)\right] = 0,\ \forall\, x \in N_j,\ \forall\, j \in \{1, \dots, n\},$$

where $\mathcal{F}$ denotes the Fourier transform. And there exist non-vanishing functions K_j satisfying this equation.

Lemma 5.3 (asymptotic behaviour of $V_j f$). *Consider $f \in \prod_{k=1}^n \mathcal{D}(N_k)$. Then $V_j f \in C^0([0; +\infty)) \cap C^\infty((0; +\infty))$ and $\chi V_j f \in \mathcal{S}(\mathbb{R})$ for any $j \in \{1, \dots, n\}$ with χ as in Definition* 5.1.

Proof. For $\lambda \in [0; +\infty)$ and $j \in \{1, \dots, n\}$, it holds

$$V_j f(\lambda) = \int_N f(x) F_\lambda^{-,j}(x)\, dx = \sum_{k=1}^{n} \int_{N_k} f_k(x) F_{\lambda,k}^{-,j}(x)\, dx.$$

Due to the definition of $F_{\lambda,k}^{-,j}$ and due to the fact that f_k is a test function having its support in $(0; +\infty)$, each term of the right-hand side is the Fourier transform of a test function and thus C^∞ and rapidly decreasing in λ. □

Proposition 5.4 (left inverse of V). *For $f \in H$ with f vanishing almost everywhere outside a compact set $B \subset N$ and $-\infty < a < b < +\infty$, it holds*

1. *$E(a, b)f = Z\mathbf{1}_{(a,b)}Vf$,* [1]
2. *$f = \lim\limits_{\substack{a\to-\infty\\ b\to+\infty}} E(a, b)f = ZVf$, if $f \in \prod_{k=1}^n \mathcal{D}(N_k)$,*
3. *V is injective on $\prod_{k=1}^n \mathcal{D}(N_k)$.*

Proof. 1. Follows directly from Proposition 4.5, using Definition 5.1.

2. Let us fix $f \in \prod_{k=1}^n \mathcal{D}(N_k)$. Lemma 5.3 implies that there exists $M_1(f) \ge 0$, such that

$$|V_j f(\lambda)| \le \frac{M_1(f)}{1 + \lambda^2},\ \forall\, \lambda > 0,\ j \in \{1, \dots, n\}.$$

[1]This formula is well defined using the expression for Z as defined in 5.1 in spite of the discontinuities introduced by the characteristic function

Clearly there exists $M_2 \geq 0$, such that

$$\left|\frac{1}{\sqrt{\lambda}}\sum_{j=1}^{n}\sigma_j(x)F_\lambda^{-,j+1}(x)\right| \leq \frac{M_2}{\sqrt{\lambda}}, \ \forall\ \lambda > 0,\ x \in N.$$

Thus the Theorem of Lebesgue implies that

$$\frac{1}{\pi}\mathrm{Re}\Big\{\int_0^{+\infty}\frac{1}{\sqrt{\lambda}}\mathbf{1}_{(a,b)}(\lambda)\sum_{j=1}^{n}\sigma_j(x)V_jf(\lambda)F_\lambda^{-,j+1}(x)\ d\lambda\Big\}$$

converges for $a \longrightarrow -\infty$ and $b \longrightarrow +\infty$ and almost every $x \in N$ towards the same expression with $\mathbf{1}_{(a,b)}$ replaced by 1.

3. Direct consequence of 2. □

Now we shall introduce a structure on the range of V which shall be later on identified as a scalar product.

Theorem 5.5 (Plancherel-type formula). *Let σ be defined as in the end of Proposition 4.5 and χ as in Definition 5.1. Let $f \in \prod_{k=1}^{n}\mathcal{D}(N_k)$ and $G = (G_1,\dots,G_n) \in (C^\infty(0;+\infty))^n \cap (C^0[0;+\infty))^n$ such that $\chi G_l \in \mathcal{S}(\mathbb{R})$ for $l \in \{1,\dots,n\}$. Define*

$$< Vf, G >_{\sigma,V} = \frac{1}{\pi}\mathrm{Re}\Big\{\sum_{j=1}^{n}\int_0^{+\infty}\frac{1}{\sqrt{\lambda}}V_{j+1}(\sigma_j(\cdot)f(\cdot))(\lambda)G_j(\lambda)\ d\lambda\Big\}$$

Then the integrals on the right-hand side are absolutely convergent and it holds $< Vf, G >_{\sigma,V} = (f, Z(G))_H$.

Proof. For $\lambda \in (0;+\infty)$, it holds

$$\left|\frac{1}{\sqrt{\lambda}}V_{j+1}(\sigma_j(\cdot)f(\cdot))(\lambda)\right| = \left|\frac{1}{\sqrt{\lambda}}\int_N \sigma_j(x)f(x)F_\lambda^{-,j+1}(x)\ dx\right| \leq C\frac{1}{\sqrt{\lambda}}\int_N |f(x)|\ dx. \tag{13}$$

Together with the fact that G_j is rapidly decreasing and continuous for any $j \in \{1,\dots,n\}$, the latter estimate ensures the absolute convergence of the integrals.

Estimate (13) also allows the application of the theorem of Fubini:

$$\begin{aligned}< Vf, G >_{\sigma,V} &= \frac{1}{\pi}\mathrm{Re}\Big\{\sum_{j=1}^{n}\int_0^{+\infty}\frac{1}{\sqrt{\lambda}}\Big(\int_N \sigma_j(x)f(x)F_\lambda^{-,j+1}(x)\ dx\Big)G_j(\lambda)\ d\lambda\Big\}\\ &= \frac{1}{\pi}\mathrm{Re}\Big\{\sum_{j=1}^{n}\int_N\frac{1}{\sqrt{\lambda}}\Big(\int_0^{+\infty}\sigma_j(x)F_\lambda^{-,j+1}(x)G_j(\lambda)\ d\lambda\Big)f(x)\ dx\Big\}\\ &= \int_N Z(G)(x)f(x)\ dx = (f, Z(G))_H.\end{aligned}$$

□

This Plancherel formula can now be combined with the fact that Z is the left inverse of V to prove that $< \cdot,\cdot >_{\sigma,V}$ is a scalar product and that V is an isometry.

Corollary 5.6. 1. *Let $(F,G) \in (V(\prod_{k=1}^{n}\mathcal{D}(N_k)))^2$ and $(f,g) \in (\mathcal{D}(N_k))^2$, such that $F = Vf$ and $G = Vg$. Then $< F, G >_{\sigma,V} = < Vf, Vg >_{\sigma,V} = (f,g)_H$.*

2. $< \cdot, \cdot >_{\sigma,V}$ *is a scalar product on* $V(\prod_{k=1}^{n} \mathcal{D}(N_k))$.
3. *Let* $L^2_{\sigma,V}$ *be the completion of* $V(\prod_{k=1}^{n} \mathcal{D}(N_k))$ *with respect to* $< \cdot, \cdot >_{\sigma,V}$. *We denote the extended scalar product by the latter bracket as well. Thus* $(L^2_{\sigma,V}, < \cdot, \cdot >_{\sigma,V})$ *is a Hilbert space.*
4. $V : \prod_{k=1}^{n} \mathcal{D}(N_k) \longrightarrow V(\prod_{k=1}^{n} \mathcal{D}(N_k))$ *extends to a surjective isometry* $\tilde{V} : H \longrightarrow L^2_{\sigma,V}$.
5. $Z = V^{-1} : V(\prod_{k=1}^{n} \mathcal{D}(N_k)) \longrightarrow \prod_{k=1}^{n} \mathcal{D}(N_k)$ *extends to a surjective isometry* $\tilde{Z} : L^2_{\sigma,V} \longrightarrow H$. *Thus* $\tilde{Z} = \tilde{V}^{-1}$.

Proof. 1. Lemma 5.3 implies that Vg is rapidly decreasing and thus Theorem 5.5 is applicable:

$$< F, G >_{\sigma,V} = < Vf, Vg >_{\sigma,V} = (f, Z(Vg))_H = (f, g)_H.$$

The last equality comes from Proposition 5.4.

2. $V : \prod_{k=1}^{n} \mathcal{D}(N_k) \longrightarrow \mathrm{Ran} V$ is linear and bijective (for the injectivity see Part 3 of Proposition 5.4). Thus $< \cdot, \cdot >_{\sigma,V}$ inherits the property of being a scalar product from $(\cdot, \cdot)_H$.
3. Clear by construction.
4. Clear by construction.
5. Theorem 5.5 implies $< Vf, G >_{\sigma,V} = (f, Z(G))_H$ for all $f \in \prod_{k=1}^{n} \mathcal{D}(N_k)$ and $G \in V(\prod_{k=1}^{n} \mathcal{D}(N_k))$. Thus it follows from 1.

$$\begin{aligned} |(f, Z(G))_H| &= | < Vf, G >_{\sigma,V} | \leq \|G\|_{\sigma,V} \|Vf\|_{\sigma,V} \\ &= \|G\|_{\sigma,V} \|f\|_H. \end{aligned} \tag{14}$$

Due to the denseness of $\prod_{k=1}^{n} \mathcal{D}(N_k)$ in H, inequality (14) is valid for all $f \in H$. Thus

$$\|Z(G)\|_H \leq \|G\|_{\sigma,V}.$$

Therefore Z extends by density-continuity to a continuous operator $\tilde{Z}$ on $L^2_{\sigma,V}$. □

Theorem 5.7. *Let* $h \in C(\mathbb{R})$ *and* $f \in H$, *such that* $\lambda \mapsto (h(\lambda)/\sqrt{\lambda})\tilde{V} f(\lambda)$ *is absolutely integrable on* $[0; +\infty)$. *Then we have for* $x \in N$

$$h(A)f(x) = \frac{1}{\pi} \mathrm{Re}\Big\{ \int_0^{+\infty} \frac{h(\lambda)}{\sqrt{\lambda}} \sum_{j=1}^{n} \sigma_j(x) V_j f(\lambda) F_\lambda^{-,j+1}(x)\, d\lambda \Big\}. \tag{15}$$

Proof. The same proof as in Proposition 4.5, but this time using Stone's formula (Theorem 4.1) with arbitrary $h \in C(\mathbb{R})$, yields

$$h(A)E(a,b)f(x) = \frac{1}{\pi} \mathrm{Re}\Big\{ \int_0^{+\infty} \frac{h(\lambda)}{\sqrt{\lambda}} \mathbf{1}_{(a,b)}(\lambda) \sum_{j=1}^{n} \sigma_j(x) V_j f(\lambda) F_\lambda^{-,j+1}(x)\, d\lambda \Big\}.$$

Now, the assertion follows from dominated convergence and the fact that $E(a,b)$ commutes with $h(A)$ and tends to the identity if $a \to -\infty$ and $b \to \infty$. □

Remark 5.8. 1. Formally (15) reads like

$$h(A)f = \tilde{Z} M_h \tilde{V} f, \tag{16}$$

where $(M_h K)(\lambda) := h(\lambda) K(\lambda)$. It should be investigated, if under the hypotheses of Theorem 5.7 we have $M_h \tilde{V} f \in L^2_{\sigma,V}$, and thus (16)) is rigorously valid.

2. Using Theorem 5.7, we can represent solutions of evolution equations involving A (heat, wave, Klein-Gordon, ...) in view of obtaining qualitative informations like decay properties in time on the n-star. It remains the open problem of describing the relation of the belonging of f to $D(A^s)$ and the decay of $\tilde{V} f$ at infinity. This is important, because for example $f \in D(A)$ ensures the twice differentiability of $u(t) = \cos(\sqrt{A}t)f$ and thus the validity of the abstract wave equation $\ddot{u}(t) + Au(t) = 0$.

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