IRREDUCIBILITY AND MIXED BOUNDARY CONDITIONS

ROBERT HALLER-DINTELMANN, MATTHIAS HIEBER, AND JOACHIM REHBERG

In memoriam Helmut H. Schaefer

Abstract. In this paper we consider positive semigroups on $L^p(\Omega)$ generated by elliptic operators A subject to mixed Dirichlet-Neumann boundary conditions on non-smooth domains Ω. We show in particular that these semigroups as well as those generated by multiplicative perturbations bA of A are irreducible, provided $b \in L^\infty(\Omega)$ is real and satisfies $b \geq \delta$ for some $\delta > 0$.

1. Introduction

The theory of *positive operators* and *positive semigroups* has many applications to evolutionary problems, see e.g. [14], [4], [6], [2]. In this note we consider operators related with diffusion equations in divergence form with mixed Dirichlet-Neumann boundary conditions on non-smooth domains $\Omega \subset \mathbb{R}^n$. It is known that the realizations A_p of the associated elliptic second order differential operators in $L^p(\Omega)$ generate positive semigroups on $L^p(\Omega)$ for all $p \in (1, \infty)$; see e.g. [8]. Note that formally the operators A_p may be expressed as $(\nabla \cdot \mu \nabla)$, where $\mu \in L^\infty(\Omega; M_{n\times n})$ denotes the diffusion coefficient.

Motivated by certain quasilinear problems, see e.g. [10], we are interested in multiplicative perturbations of A_p by real-valued L^∞-functions b bounded away from 0. These operators are formally given by $b(\nabla \cdot \mu \nabla)$.

It is an interesting question to ask whether the semigroup e^{tA_p} is irreducible and further whether bA_p generates a positive or even irreducible semigroup e^{tbA_p} on $L^p(\Omega)$. It is the aim of this paper to give an affirmative answer to this question. Observe that the semigroup property in the case where the boundary of Ω consists of two *seperate* components with either Dirichlet or Neumann conditions was already considered by Amann [1] in the situation of smooth boundaries.

In our situation, the underlying domains are Lipschitz domains and the boundary condition is *mixed*. By mixed Dirichlet-Neumann boundary condition we mean roughly speaking the following: there is an open set $\Gamma_N \subset \partial\Omega$ such that the elements of the domain of the operator have vanishing trace on $\partial\Omega \setminus \Gamma_N$ and a vanishing conormal derivative on Γ_N (in the distributional sense).

2. Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and denote by $\Gamma_N \subset \partial\Omega$ an open subset of $\partial\Omega$. For $1 < p < \infty$ we define $W^{1,p}_{\Gamma_N}(\Omega)$ as the closure of

$$\{\psi|_\Omega : \psi \in C_c^\infty(\mathbb{R}^n),\ \operatorname{supp}\psi \cap (\partial\Omega \setminus \Gamma_N) = \emptyset\}$$

in the Sobolev space $W^{1,p}(\Omega)$. If $p = 2$, we write $H^1(\Omega)$ or $H^1_{\Gamma_N}(\Omega)$ instead of $W^{1,2}(\Omega)$ or $W^{1,2}_{\Gamma_N}(\Omega)$. Of course, if $\Gamma_N = \emptyset$, then $W^{1,p}_{\Gamma_N}(\Omega) = W^{1,p}_0(\Omega)$.

2000 *Mathematics Subject Classification.* Primary 35B45; Secondary 35B65, 35K10.

Key words and phrases. Irreducibility, mixed Dirichlet-Neumann conditions.

Consider $\mu \in L^\infty(\Omega; M_{n\times n})$, where $M_{n\times n}$ denotes the set of all real, symmetric $n\times n$ matrices. Suppose that additionally

$$\inf_{x\in\Omega}\inf_{|\varsigma|=1}\mu(x)\varsigma\cdot\varsigma>0.$$

For a closed subspace $V\subset H^1(\Omega)$ such that $H^1_0(\Omega)\subset V$ we define the form $a: V\times V\to\mathbb{R}$ by

$$a(u,v):=-\int_\Omega \mu\nabla u\cdot\nabla v\,\mathrm{d}x,\quad u,v\in V.$$

The form induces a continuous mapping $\mathcal{A}: V\to V'$ such that

$$a(u,v)=\langle \mathcal{A}u,v\rangle,\quad u,v\in V.$$

Here, for $v\in L^2(\Omega)$, $f_v(u):=\langle v,u\rangle_{L^2}$ defines an element $f_v\in V'$ and $v\mapsto f_v: L^2(\Omega)\to V'$ defines a continuous injection. In the following, we identify v with f_v. We then define the operator A as

$$\begin{aligned}D(A)&:=\{u\in V:\exists f\in L^2(\Omega),\ a(u,\phi)=\langle f,\phi\rangle\ \forall\phi\in V\},\\ Au&:=f.\end{aligned}$$

It is well known that A generates an analytic semigroup on $L^2(\Omega)$ which is positivity preserving.

The following result gives sufficient conditions on the subspace V, such that e^{tA} satisfies an upper Gaussian bound.

Proposition 2.1. *[Arendt, terElst [3]] Assume that V is a closed subspace of $H^1(\Omega)$ satisfying*

a) $H^1_0(\Omega)\subseteq V$,
b) *V has the L^1-H^1 extension property,*
c) *$u\in V$ implies $|u|$, $\inf(|u|,1)\in V$,*
d) *$u\in V$, $v\in H^1(\Omega)$, $|v|\le u$ implies $v\in V$.*

Then e^{tA} satisfies an upper Gaussian estimate, i.e.

$$(\mathrm{e}^{tA}f)(x)=\int_\Omega K_t(x,y)f(y)\,\mathrm{d}y,\qquad x\in\Omega,\ f\in L^2(\Omega)\tag{2.1}$$

for some measurable function $K_t:\Omega\times\Omega\to\mathbb{R}_+$ and there exist constants $M,a>0$ and $\omega\in\mathbb{R}$ such that

$$0\le K_t(x,y)\le\frac{M}{t^{n/2}}\,\mathrm{e}^{\frac{-a|x-y|^2}{t}}\,\mathrm{e}^{\omega t},\quad t>0,\ a.a.\ x,y\in\Omega.\tag{2.2}$$

Let $H^1_{\Gamma_N}(\Omega)$ be defined as above. It was shown in [10] that $V:=H^1_{\Gamma_N}(\Omega)$ satisfies the assumptions a) – d) of Proposition (2.1).

This means that e^{tA} satisfies an upper Gaussian bound. Thus e^{tA} extends to an analytic semigroup on $L^p(\Omega)$ for all $1<p<\infty$ (see e.g. [4] or [3, Theorem 5.3]). We will denote the generator of the semigroup on $L^p(\Omega)$ by A_p. Furthermore, these semigroups are even contractive for all these p. This can be seen by combining Proposition 4.11 and Theorem 4.28 in [12].

We now turn to multiplicative perturbations of A. Let $b\in L^\infty(\Omega)$ be real-valued with $b\ge\delta$ for some $\delta>0$. Then we define the operator bA on $L^2(\Omega)$ with $D(bA)=D(A)$ and $bAu(x)=b(x)Au(x)$ for all $u\in D(A)$. It is remarkable that this multiplicative perturbation preserves the Gaussian bound. This essentially follows from a result due to Duong and Ouhabaz [5].

Proposition 2.2. *Let $b\in L^\infty(\Omega;\mathbb{R})$ and $b\ge\delta>0$. Then bA generates an analytic C_0-semigroup on $L^2(\Omega)$ given by kernel operators. Moreover, the associated kernels $p_t(x,y)$ satisfy an upper Gaussian bound of the form*

$$|p_t(x,y)|\le\frac{C}{t^{n/2}}\,\mathrm{e}^{-c\frac{|x-y|^2}{t}}\,\mathrm{e}^{\gamma t},\quad t>0,\ a.a.\ x,y\in\Omega,$$

for some constants $c,C>0$ and $\gamma\in\mathbb{R}$.

Proof. The fact that bA generates an analytic C_0-semigroup follows from [5, Proposition 2.1].

Let $\gamma \geq \omega\|b\|_\infty$. Then $\gamma/b - \omega \geq 0$, so $\mathrm{e}^{t(A-\omega)}$ and $\mathrm{e}^{-t(\gamma/b-\omega)}$ are contraction semigroups. Thus, by the Trotter product formula (see e.g. [6, Corollary III.5.8]) we have for every $f \in L^2(\Omega)$, using the positivity of $\mathrm{e}^{t(A-\omega)}$,

$$\begin{aligned} |\,\mathrm{e}^{t(A-\omega-(\gamma/b-\omega))}\,f| &= \lim_{n\to\infty} \left| \left(\mathrm{e}^{t/n(A-\omega)}\,\mathrm{e}^{-t/n(\gamma/b-\omega)}\right)^n f \right| \\ &\leq \lim_{n\to\infty} \mathrm{e}^{t/n(A-\omega)} \left| \left(\mathrm{e}^{t/n(A-\omega)}\,\mathrm{e}^{-t/n(\gamma/b-\omega)}\right)^{n-1} f \right| \\ &\leq \cdots \leq \lim_{n\to\infty} \left(\mathrm{e}^{t/n(A-\omega)}\right)^n |f| = \mathrm{e}^{t(A-\omega)}\,|f|. \end{aligned}$$

This implies that the semigroup $\mathrm{e}^{t(A-\gamma/b)} = \mathrm{e}^{t(A-\omega-(\gamma/b-\omega))}$ has the same Gaussian estimate as $\mathrm{e}^{t(A-\omega)}$, that is (2.2) without the factor $\mathrm{e}^{\omega t}$. So we may now apply [5, Theorem 2.2] to this semigroup and get a Gaussian bound without exponential factor for the semigroup $\mathrm{e}^{t(bA-\gamma)}$, which yields the assertion. □

Next, we remind the reader that the concept of irreducibility has undergone many developments in the last decades. In the following, we follow the approach and definition introduced by H. H. Schaefer [14].

Definition 2.3. Let X be a Banach lattice. A linear supspace Y of X is called an *ideal*, if for every choice of $f \in Y$ and $g \in X$ with $|g| \leq |f|$, we have $g \in Y$.

A C_0-semigroup $T(t)$ on X is called *irreducible*, if there is no $T(t)$-invariant ideal except $\{0\}$ and X.

Remark 2.4. It is interesting to note that for positive semigroups $T(t)$ on Banach lattices of the form $X = L^p(\Omega)$, $1 \leq p < \infty$, which may be represented as kernel operators, the following characterization of irreducibility in terms of the underlying kernels is known:

Let $T(t)$ be a C_0-semigroup on $L^p(\Omega)$ and assume that $T(t)$ is given by kernels as in (2.1). Then $T(t)$ is irreducible if and only if for any two measurable subsets M and N of Ω with positive and finite measure, such that $M \cap N$ is a nullset, there exists $t > 0$ with

$$\int_M \int_N K_t(x,y)\,\mathrm{d}x\,\mathrm{d}y > 0.$$

For a proof, see [11, Example C-III.3.4].

3. Main Result and Proof

We are now in the position to state the main result of this paper.

Theorem 3.1. *Let $1 < p < \infty$ and let $b \in L^\infty(\Omega;\mathbb{R})$ such that $b \geq \delta$ for some $\delta > 0$. Then the semigroup e^{tbA_p} is positive, irreducible and compact.*

The main idea of the proof of irreducibility of e^{tbA_p} is to show, that the resolvent of bA is *positivity improving*. This means, that for some $\lambda > s(A)$ and all $f \in L^p(\Omega) \setminus \{0\}$ with $f \geq 0$ almost everywhere the function $R(\lambda, A)f$ is strictly positive, i.e. $R(\lambda, A)f > 0$ almost everywhere in Ω. Then irreducibility follows by a contradiction argument using Remark 2.4.

Due to the relation

$$(\lambda - bA)^{-1} = (W_\lambda - A)^{-1} b^{-1} \tag{3.1}$$

with $W_\lambda := \frac{\lambda}{b}$, this is obtained whenever the resolvent of $A - W_\lambda$ is positivity improving. Thus, it is our first aim to show this.

In order to do so, we cite the following minimum principle from [7, Section 8.7].

Proposition 3.2. *Let $V \in L^\infty(\Omega)$ be positive and let $u \in H^1(\Omega)$ satisfy $(A - V)u \le 0$ in Ω. If for some open ball B with $\overline{B} \subset \Omega$ we have*

$$\inf_B u = \inf_\Omega u \le 0,$$

then u is constant in Ω.

In the following, let $W \in L^\infty(\Omega)$ satisfy $W \ge \varepsilon$ for some $\varepsilon > 0$. Since $A - W$ is self-adjoint and has compact resolvent, the spectrum of this operator consists only of eigenvalues and since $s(A) \le 0$ and $W \ge \varepsilon$, we have $\sigma(A-W) \subseteq (-\infty, \lambda_1]$, where $\lambda_1 < 0$ denotes the largest eigenvalue. Furthermore we have

$$\lambda_1 = \sup\big\{\langle (A - W)\eta, \eta\rangle : \eta \in H^1_{\Gamma_N}(\Omega),\ \|\eta\|_2 = 1\big\} \tag{3.2}$$

and the supremum is attained for every eigenfunction of $A - W$ associated to λ_1.

Lemma 3.3. *If $\psi \ne 0$ is a positive eigenfunction of $A - W$ belonging to λ_1, then $\psi > 0$ in Ω.*

Proof. If $\inf_\Omega \psi > 0$ there is nothing to prove, so we concentrate on the case where $\inf_\Omega \psi = 0$. Thus $(A-W)\psi = \lambda_1\psi \le 0$, so if we assume that ψ does not admit a strictly positive lower bound on a ball B with $\overline{B} \subseteq \Omega$, then ψ has to be constant on Ω in view of Proposition 3.2. But this can be true, since the infimum of ψ on Ω is zero, only provided $\psi = 0$. □

Lemma 3.4. *If $\psi \ne 0$ is a real eigenfunction of $A - W$ belonging to λ_1, then $\psi > 0$ on Ω or $\psi < 0$ on Ω. Any other eigenfunction belonging to λ_1 is a (scalar) multiple of ψ.*

Proof. Without loss of generality we may normalize our eigenfunction to $\|\psi\|_2 = 1$. Let $\psi = \psi_+ - \psi_-$ be the usual decomposition of ψ in a positive and negative part. If $\psi_- = 0$ one may directly apply Lemma 3.3 and obtains the first assertion. The same is true, if $\psi_+ = 0$, arguing via $-\psi$. Let us now suppose $\psi_+ \ne 0 \ne \psi_-$. First we will show that the function $\tilde{\psi}_+ := \frac{\psi_+}{\|\psi_+\|_2}$ also maximizes the expression in (3.2). In order to do so, consider functions $\eta = \alpha\psi_+ - \beta\psi_-$, with the additional conditions $\alpha \in [0, \frac{1}{\|\psi_+\|_2}]$ and $\|\eta\|_2 = 1$. In view of $\psi_- \ne 0$ it is clear that $\|\psi_+\|_2 < 1$.

Obviously, the normalizing condition $\|\eta\|_2 = 1$ is equivalent to

$$\beta^2 = \frac{1 - \alpha^2\|\psi_+\|_2^2}{\|\psi_-\|_2^2}. \tag{3.3}$$

By an explicit calculation and taking into account (3.3) we obtain

$$\begin{aligned}
\langle (A - W)\eta, \eta\rangle &= -\alpha^2\Big(\int_\Omega \langle \mu\nabla\psi_+, \nabla\psi_+\rangle + \int_\Omega W\psi_+^2\Big) - \beta^2\Big(\int_\Omega \langle \mu\nabla\psi_-, \nabla\psi_-\rangle + \int_\Omega W\psi_-^2\Big)\\
&= -\alpha^2\Big(\int_\Omega \langle \mu\nabla\psi_+, \nabla\psi_+\rangle + \int_\Omega W\psi_+^2\Big)\\
&\qquad - \frac{1 - \alpha^2\|\psi_+\|_2^2}{\|\psi_-\|_2^2}\Big(\int_\Omega \langle \mu\nabla\psi_-, \nabla\psi_-\rangle + \int_\Omega W\psi_-^2\Big)\\
&= -\Big[\int_\Omega \langle \mu\nabla\psi_+, \nabla\psi_+\rangle + \int_\Omega W\psi_+^2 - \frac{\|\psi_+\|_2^2}{\|\psi_-\|_2^2}\Big(\int_\Omega \langle \mu\nabla\psi_-, \nabla\psi_-\rangle + \int_\Omega W\psi_-^2\Big)\Big]\alpha^2\\
&\qquad - \frac{1}{\|\psi_-\|_2^2}\Big(\int_\Omega \langle \mu\nabla\psi_-, \nabla\psi_-\rangle + \int_\Omega W\psi_-^2\Big).
\end{aligned}$$

This means that the prefactor of α^2 must be zero, since otherwise the expression, seen as a function in α, is strictly decreasing or increasing (depending on the sign of the prefactor) and

$\alpha = 1$ cannot maximize (3.2) on the interval $[0, \frac{1}{\|\psi_+\|_2}]$. This implies

$$\frac{\int_\Omega \langle \mu \nabla \psi_-, \nabla \psi_- \rangle + \int_\Omega W \psi_-^2}{\|\psi_-\|_2^2} = \frac{\int_\Omega \langle \mu \nabla \psi_+, \nabla \psi_+ \rangle + \int_\Omega W \psi_+^2}{\|\psi_+\|_2^2}$$

and, setting $\alpha = \beta = 1$ (observe that $\eta = \psi$ in this case),

$$\begin{aligned} \langle (A - W)\psi, \psi \rangle &= -\frac{1}{\|\psi_-\|_2^2} \left(\int_\Omega \langle \mu \nabla \psi_-, \nabla \psi_- \rangle + \int_\Omega W \psi_-^2 \right) \\ &= \langle (A - W)\tilde{\psi}_+, \tilde{\psi}_+ \rangle. \end{aligned}$$

Thus, the normalized function $\tilde{\psi}_+$ also maximizes (3.2).

Since $\tilde{\psi}_+$ is an element of the form domain, and hence from the domain of $(A - W)^{1/2}$, the series

$$\sum_m |\langle \tilde{\psi}_+, u_m \rangle|^2 \lambda_m$$

converges. Here the λ_m's denote the eigenvalues of $A - W$ and $\{u_m\}_{m \in \mathbb{N}}$ is an orthonormal eigenbasis of $A - W$. Hence, we have

$$\sum_m |\langle \tilde{\psi}_+, u_m \rangle|^2 \lambda_1 = \lambda_1 = \langle (A - W)\tilde{\psi}_+, \tilde{\psi}_+ \rangle = \sum_m |\langle \tilde{\psi}_+, u_m \rangle|^2 \lambda_m.$$

Note, that the sums on both sides consist only of non-positive summands. Consequently, all terms $\langle \tilde{\psi}_+, u_m \rangle$ have to vanish if the corresponding λ_m is larger than λ_1. Thus, $\tilde{\psi}_+$ must belong to the eigenspace of λ_1. Now Lemma 3.3 shows that $\psi_+ > 0$ in Ω and ψ_- must vanish, in contradiction to our above supposition.

Assume now that ψ and φ are two linearly independent eigenfunctions belonging to λ_1. Then we have $(A - W - \lambda_1)\psi = (A - W - \lambda_1)\varphi = 0$, so by [7, Theorem 8.22] ψ and φ are locally Hölder continuous in Ω. Thus, by linear independence, there are points $x, y \in \Omega$ such that the system

$$\psi(x)r + \varphi(x)s = -1, \quad \psi(y)r + \varphi(y)s = 1$$

has a solution (in r, s). But this implies that the function $r\psi + s\varphi$ is also an eigenfunction to the eigenvalue λ_1 that takes positive and negative values. This, however, is impossible by the preceding considerations. □

Lemma 3.5. *$R(\lambda, bA)$ is positivity improving for all $\lambda > 0$.*

Proof. Putting $W_\lambda := \frac{\lambda}{b}$, one first notices $W_\lambda > \varepsilon_\lambda > 0$ almost everywhere. Hence, for every $\lambda > 0$ Lemma 3.3 and Lemma 3.4 apply for $W = W_\lambda$. Thus the largest eigenvalue of $A - W_\lambda$ is simple and there is a corresponding eigenfunction which is strictly positive. Furthermore, this operator is self-adjoint, bounded from above and the semigroup $\mathrm{e}^{t(A - W_\lambda)}$ is positive. To see this last point, observe that e^{tA} and e^{-tW_λ} are both positive contraction semigroups, so again by the Trotter product formula, we obtain

$$\mathrm{e}^{t(A - W_\lambda)} f = \lim_{n \to \infty} \left(\mathrm{e}^{\frac{t}{n} A} \, \mathrm{e}^{-\frac{t}{n} W_\lambda} \right)^n f,$$

which yields the claimed positivity.

Thus, we may apply [13, Theorem XIII.44], which states that $R(0, A - W_\lambda)$ is positivity improving. According to (3.1) this also holds for $(\lambda - bA)^{-1}$. □

Proof of Theorem 3.1. Since e^{tbA} is analytic, it maps $L^2(\Omega)$ into $D(A)$. The latter is contained in the form domain and thus in $H^1(\Omega)$ and so e^{tbA} is compact by the compact embedding of $H^1(\Omega)$ in $L^2(\Omega)$. As e^{tbA_p}, $1 < p < \infty$, is a consistent family of semigroups, the compactness in $L^p(\Omega)$ for $p \in [2, \infty)$ follows by complex interpolation via [4, Theorem 1.6.1] and for $p \in (1, 2)$ by duality. This proves the last assertion.

Regarding positivity, observe that it suffices to handle the case $p = 2$ due to consistency and to show that $R(\lambda, bA)$ is a positive operator for large λ. In order to do so, set again $W_\lambda := \lambda/b$. Then, we already saw in the proof of Lemma 3.5, that $\mathrm{e}^{t(A-W_\lambda)}$ is positive. Hence, the same is true for $R(0,(A-W_\lambda)) = (W_\lambda - A)^{-1}$. Using once more (3.1), we see that $R(\lambda, bA)$ is positive.

We now turn to prove irreducibility. By Proposition 2.2, the semigroup e^{tbA_p} is given by kernels $p_t \in L^\infty(\Omega \times \Omega)$ for $t > 0$. Thus in view of the criterion given in Remark 2.4, it suffices to argue in $L^2(\Omega)$.

Also, the upper Gaussian bound implies by [9, Theorem 2.2] that the resolvent of bA at the point 1, i.e. $(1 - bA)^{-1}$ is a regular integral operator with kernel

$$K^R(x,y) = \int_0^\infty \mathrm{e}^{-t}\, p_t(x,y)\;\mathrm{d}t.$$

Due to the positivity of e^{tbA} this kernel is positive. Furthermore, for every choice of measurable sets $M, N \subset \Omega$ with positive measure it satisfies

$$\int_M \int_N K^R(x,y)\;\mathrm{d}x\;\mathrm{d}y = \int_N \big(R(1,bA)\mathbf{1}_M\big)(x)\;\mathrm{d}x > 0, \tag{3.4}$$

since the characteristic function of M is positive and not identically zero. Hence, $R(1,bA)\mathbf{1}_M$ is strictly positive by Lemma 3.5.

Finally, suppose that the semigroup e^{tbA} is not irreducible. By Remark 2.4 this means that there exist measurable sets $M_0, N_0 \subset \Omega$ with positive measure such that for all $t > 0$ we have

$$\int_{M_0} \int_{N_0} p_t(x,y)\;\mathrm{d}x\;\mathrm{d}y = 0.$$

Note that this expression cannot be negative, since $p_t \geq 0$. But this implies immediately

$$\int_{M_0} \int_{N_0} K^R(x,y)\;\mathrm{d}x\;\mathrm{d}y = \int_0^\infty \mathrm{e}^{-t} \int_{M_0} \int_{N_0} p_t(x,y)\;\mathrm{d}x\;\mathrm{d}y = 0,$$

in contradiction to (3.4). Hence e^{tbA} must be irreducible. $\square$

References

[1] H. Amann, *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems*, in: Function spaces, differential operators and nonlinear analysis, H.-J. Schmeisser (ed.) et al., 9–126, Teubner-Texte Math., vol. 133, Stuttgart, 1993.

[2] W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Monographs Math., vol. 96, Birkhäuser, 2001.

[3] W. Arendt, A. F. M. terElst, *Gaussian estimates for second order elliptic operators with boundary conditions*, J. Oper. Th. **38** (1997), no. 1, 87–130.

[4] E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, vol. 92, Cambridge University Press, Cambridge, 1990.

[5] X. T. Duong, E. M. Ouhabaz, *Complex multiplicative perturbations of elliptic operators: heat kernel bounds and holomorphic functional calculus*, Differential Integral Equations **12** (1999), no. 3, 395–418.

[6] K. J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, vol. 194, Springer, New York-Berlin-Heidelberg, 2000.

[7] D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Grundlehren Math. Wiss., vol. 224, Springer, Berlin-Heidelberg-New York, 1977.

[8] J. A. Griepentrog, H.-C. Kaiser, J. Rehberg, *Heat kernel and resolvent properties for second order elliptic differential operators with general boundary conditions on L^p*, Adv. Math. Sci. Appl. **11** (2001), no. 1, 87–112.

[9] M. Hieber, *Gaussian estimates and holomorphy of semigroups on L^p spaces*, J. London Math. Soc. (2) **54** (1996), no. 1, 148–160.

[10] M. Hieber, J. Rehberg, *Quasilinear parabolic systems with mixed boundary conditions*, WIAS Preprint 1194 (2006), submitted.

[11] R. Nagel (ed.), *One-Parameter Semigroups of Positive Operators*, Lect. Notes in Math., vol. 1184, Springer, 1986.

[12] E. M. Ouhabaz, *Analysis of Heat Equations on Domains*, London Math. Soc. Monographs, vol. 31, Princeton University Press, 2005.

[13] M. Reed, B. Simon, *Methods of Modern Mathematical Physics, IV*, Academic Press, New York-London, 1978.

[14] H. H. Schaefer, *Banach Lattices and Positive Operators*, Grundlehren Math. Wiss., vol. 215, Springer, 1974.

Technische Universität Darmstadt, Fachbereich Mathematik, Schlossgartenstr. 7, 64289 Darmstadt, Germany

E-mail address: haller@mathematik.tu-darmstadt.de

Technische Universität Darmstadt, Fachbereich Mathematik, Schlossgartenstr. 7, 64289 Darmstadt, Germany

E-mail address: hieber@mathematik.tu-darmstadt.de

Weierstrass-Institut für Angewandte Analysis und Stochastik (WIAS), Mohrenstr. 39, 10117 Berlin, Germany

E-mail address: rehberg@wias-berlin.de