IRREDUCIBILITY AND MIXED BOUNDARY CONDITIONS

ROBERT HALLER-DINTELmann, MATTHIAS HIEBER, AND JOACHIM REHERG

In memoriam Helmut H. Schaefer

Abstract. In this paper we consider positive semigroups on $L^p(\Omega)$ generated by elliptic operators $A$ subject to mixed Dirichlet-Neumann boundary conditions on non-smooth domains $\Omega$. We show in particular that these semigroups as well as those generated by multiplicative perturbations $bA$ of $A$ are irreducible, provided $b \in L^\infty(\Omega)$ is real and satisfies $b \geq \delta$ for some $\delta > 0$.

1. Introduction

The theory of positive operators and positive semigroups has many applications to evolutionary problems, see e.g. [14], [4], [6], [2]. In this note we consider operators related with diffusion equations in divergence form with mixed Dirichlet-Neumann boundary conditions on non-smooth domains $\Omega \subset \mathbb{R}^n$. It is known that the realizations $A_p$ of the associated elliptic second order differential operators in $L^p(\Omega)$ generate positive semigroups on $L^p(\Omega)$ for all $p \in (1, \infty)$; see e.g. [8]. Note that formally the operators $A_p$ may be expressed as $(\nabla \cdot \mu \nabla)$, where $\mu \in L^\infty(\Omega; M_{n \times n})$ denotes the diffusion coefficient.

Motivated by certain quasilinear problems, see e.g. [10], we are interested in multiplicative perturbations of $A_p$ by real-valued $L^\infty$-functions $b$ bounded away from 0. These operators are formally given by $b(\nabla \cdot \mu \nabla)$.

It is an interesting question to ask whether the semigroup $e^{tA_p}$ is irreducible and further whether $bA_p$ generates a positive or even irreducible semigroup $e^{tbA_p}$ on $L^p(\Omega)$. It is the aim of this paper to give an affirmative answer to this question. Observe that the semigroup property in the case where the boundary of $\Omega$ consists of two separate components with either Dirichlet or Neumann conditions was already considered by Amann [1] in the situation of smooth boundaries.

In our situation, the underlying domains are Lipschitz domains and the boundary condition is mixed. By mixed Dirichlet-Neumann boundary condition we mean roughly speaking the following: there is an open set $\Gamma_N \subset \partial \Omega$ such that the elements of the domain of the operator have vanishing trace on $\partial \Omega \setminus \Gamma_N$ and a vanishing conormal derivative on $\Gamma_N$ (in the distributional sense).

2. Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and denote by $\Gamma_N \subset \partial \Omega$ an open subset of $\partial \Omega$. For $1 < p < \infty$ we define $W^{1,p}_{\Gamma_N}(\Omega)$ as the closure of
\[ \{ \psi|_{\Omega} : \psi \in C^\infty_c(\mathbb{R}^n), \supp \psi \cap (\partial \Omega \setminus \Gamma_N) = \emptyset \} \]
in the Sobolev space $W^{1,p}(\Omega)$. If $p = 2$, we write $H^1(\Omega)$ or $H^1_{\Gamma_N}(\Omega)$ instead of $W^{1,2}(\Omega)$ or $W^{1,2}_{\Gamma_N}(\Omega)$. Of course, if $\Gamma_N = \emptyset$, then $W^{1,p}_{\Gamma_N}(\Omega) = W^{1,p}(\Omega)$.

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Consider $\mu \in L^\infty(\Omega; M_{n \times n})$, where $M_{n \times n}$ denotes the set of all real, symmetric $n \times n$ matrices. Suppose that additionally

$$\inf_{x \in \Omega} \inf_{|k|=1} \mu(x) \zeta \cdot \zeta > 0.$$ 

For a closed subspace $V \subset H^1(\Omega)$ such that $H^1_0(\Omega) \subset V$ we define the form $a: V \times V \to \mathbb{R}$ by

$$a(u, v) := -\int_\Omega \mu \nabla u \cdot \nabla v \, dx, \quad u, v \in V.$$ 

The form induces a continuous mapping $A: V \to V'$ such that

$$a(u, v) = \langle Au, v \rangle, \quad u, v \in V.$$ 

Here, for $v \in L^2(\Omega)$, $f_v(u) := \langle v, u \rangle_{L^2}$ defines an element $f_v \in V'$ and $v \mapsto f_v : L^2(\Omega) \to V'$ defines a continuous injection. In the following, we identify $v$ with $f_v$. We then define the operator $A$ as

$$D(A) := \{ u \in V : \exists f \in L^2(\Omega), a(u, \phi) = \langle f, \phi \rangle \forall \phi \in V \},$$

$$Au := f.$$ 

It is well known that $A$ generates an analytic semigroup on $L^2(\Omega)$ which is positivity preserving.

The following result gives sufficient conditions on the subspace $V$, such that $e^{tA}$ satisfies an upper Gaussian bound.

**Proposition 2.1.** [Amendt, terElst [3]] Assume that $V$ is a closed subspace of $H^1(\Omega)$ satisfying

a) $H^1_0(\Omega) \subset V$,

b) $V$ has the $L^1$-$H^1$ extension property,

c) $u \in V$ implies $|u|, \inf(|u|, 1) \in V$,

d) $u \in V, v \in H^1(\Omega), |v| \leq u$ implies $v \in V$.

Then $e^{tA}$ satisfies an upper Gaussian estimate, i.e.

$$e^{tA}f(x) = \int_\Omega K_t(x, y)f(y) \, dy, \quad x \in \Omega, f \in L^2(\Omega)$$

for some measurable function $K_t : \Omega \times \Omega \to \mathbb{R}_+$ and there exist constants $M, c > 0$ and $\omega \in \mathbb{R}$ such that

$$0 \leq K_t(x, y) \leq M e^{\omega t} e^{\omega t} = e^{-\frac{t}{4} + \frac{\omega}{2}}, \quad t > 0, \text{ a.a. } x, y \in \Omega.$$ 

Let $H^1_{V, \infty}(\Omega)$ be defined as above. It was shown in [10] that $V := H^1_{V, \infty}(\Omega)$ satisfies the assumptions a) - d) of Proposition (2.1).

This means that $e^{tA}$ satisfies an upper Gaussian bound. Thus $e^{tA}$ extends to an analytic semigroup on $L^p(\Omega)$ for all $1 < p < \infty$ (see e.g. [4] or [3, Theorem 5.3]). We will denote the generator of the semigroup on $L^p(\Omega)$ by $A_p$. Furthermore, these semigroups are even contractive for all these $p$. This can be seen by combining Proposition 4.11 and Theorem 4.28 in [12].

We now turn to multiplicative perturbations of $A$. Let $b \in L^\infty(\Omega)$ be real-valued with $b \geq \delta$ for some $\delta > 0$. Then we define the operator $bA$ on $L^2(\Omega)$ with $D(bA) = D(A)$ and $bA u(x) = b(x)Au(x)$ for all $u \in D(A)$. It is remarkable that this multiplicative perturbation preserves the Gaussian bound. This essentially follows from a result due to Duong and Ouhabaz [5].

**Proposition 2.2.** Let $b \in L^\infty(\Omega; \mathbb{R})$ and $b \geq \delta > 0$. Then $bA$ generates an analytic $C_0$-semigroup on $L^2(\Omega)$ given by kernel operators. Moreover, the associated kernels $p_t(x, y)$ satisfy an upper Gaussian bound of the form

$$|p_t(x, y)| \leq \frac{C}{\pi^{n/2}} e^{-\frac{|x-y|^2}{C}} e^{\gamma t}, \quad t > 0, \text{ a.a. } x, y \in \Omega,$$

for some constants $c, C > 0$ and $\gamma \in \mathbb{R}$. 
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Proof. The fact that $bA$ generates an analytic $C_0$-semigroup follows from [5, Proposition 2.1]. Let $\gamma \geq \omega \|b\|_\infty$. Then $\gamma/b - \omega \geq 0$, so $e^{t(A-\omega)}$ and $e^{-t(\gamma/b - \omega)}$ are contraction semigroups. Thus, by the Trotter product formula (see e.g. [6, Corollary III.5.8]) we have for every $f \in L^2(\Omega)$, using the positivity of $e^{t(A-\omega)}$,

$$|e^{t(A-\omega-(\gamma/b - \omega))} f| = \lim_{n \to \infty} \left| \left(e^{t/n(A-\omega)} e^{-t/n(\gamma/b - \omega)} \right)^n f \right| \leq \lim_{n \to \infty} e^{t/n(A-\omega)} \left| \left(e^{t/n(A-\omega)} e^{-t/n(\gamma/b - \omega)} \right)^{n-1} f \right| \leq \cdots \leq \lim_{n \to \infty} \left( e^{t/n(A-\omega)} \right)^n |f| = e^{t(A-\omega)} |f|.
$$

This implies that the semigroup $e^{t(A-\gamma/b)} = e^{t(A-\omega-(\gamma/b - \omega))}$ has the same Gaussian estimate as $e^{t(A-\omega)}$, that is (2.2) without the factor $e^{at}$. So we may now apply [5, Theorem 2.2] to this semigroup and get a Gaussian bound without exponential factor for the semigroup $e^{t(bA-\gamma)}$, which yields the assertion.

Next, we remind the reader that the concept of irreducibility has undergone many developments in the last decades. In the following, we follow the approach and definition introduced by H. H. Schaefer [14].

**Definition 2.3.** Let $X$ be a Banach lattice. A linear subspace $Y$ of $X$ is called an ideal, if for every choice of $f \in Y$ and $g \in X$ with $|g| \leq |f|$, we have $g \in Y$.

A $C_0$-semigroup $T(t)$ on $X$ is called irreducible, if there is no $T(t)$-invariant ideal except $\{0\}$ and $X$.

**Remark 2.4.** It is interesting to note that for positive semigroups $T(t)$ on Banach lattices of the form $X = L^p(\Omega)$, $1 \leq p < \infty$, which may be represented as kernel operators, the following characterization of irreducibility in terms of the underlying kernels is known:

Let $T(t)$ be a $C_0$-semigroup on $L^p(\Omega)$ and assume that $T(t)$ is given by kernels as in (2.1). Then $T(t)$ is irreducible if and only if for any two measurable subsets $M$ and $N$ of $\Omega$ with positive and finite measure, such that $M \cap N$ is a nullset, there exists $t > 0$ with

$$\int_M \int_N K_t(x,y) \, dx \, dy > 0.$$

For a proof, see [11, Example C-III.3.4].

3. **MAIN RESULT AND PROOF**

We are now in the position to state the main result of this paper.

**Theorem 3.1.** Let $1 < p < \infty$ and let $b \in L^\infty(\Omega; \mathbb{R})$ such that $b \geq \delta$ for some $\delta > 0$. Then the semigroup $e^{t(bA)}$ is positive, irreducible and compact.

The main idea of the proof of irreducibility of $e^{t(bA)}$ is to show, that the resolvent of $bA$ is positivity improving. This means, that for some $\lambda > s(A)$ and all $f \in L^p(\Omega) \setminus \{0\}$ with $f \geq 0$ almost everywhere the function $R(\lambda, A)f$ is strictly positive, i.e. $R(\lambda, A)f > 0$ almost everywhere in $\Omega$. Then irreducibility follows by a contradiction argument using Remark 2.4.

Due to the relation

$$\lambda - bA = (W_\lambda - A)^{-1} b^{-1}$$

with $W_\lambda := \frac{\lambda}{b}$, this is obtained whenever the resolvent of $A - W_\lambda$ is positivity improving. Thus, it is our first aim to show this.

In order to do so, we cite the following minimum principle from [7, Section 8.7].
Proposition 3.2. Let \( V \in L^\infty(\Omega) \) be positive and let \( u \in H^1(\Omega) \) satisfy \( (A - V)u \leq 0 \) in \( \Omega \). If for some open ball \( B \) with \( \overline{B} \subset \Omega \) we have

\[
\inf_B u = \inf_{\Omega} u \leq 0,
\]
then \( u \) is constant in \( \Omega \).

In the following, let \( W \in L^\infty(\Omega) \) satisfy \( W \geq \varepsilon \) for some \( \varepsilon > 0 \). Since \( A - W \) is self-adjoint and has compact resolvent, the spectrum of this operator consists only of eigenvalues and since \( s(A) \leq 0 \) and \( W \geq \varepsilon \), we have \( \sigma(A - W) \subseteq (-\infty, \lambda_1] \), where \( \lambda_1 < 0 \) denotes the largest eigenvalue. Furthermore we have

\[
\lambda_1 = \sup \{ \langle (A - W)\eta, \eta \rangle : \eta \in H^1_0(\Omega), \|\eta\|_2 = 1 \}
\]
and the supremum is attained for every eigenfunction of \( A - W \) associated to \( \lambda_1 \).

Lemma 3.3. If \( \psi \neq 0 \) is a positive eigenfunction of \( A - W \) belonging to \( \lambda_1 \), then \( \psi > 0 \) in \( \Omega \).

Proof. If \( \inf_{\Omega} \psi > 0 \) there is nothing to prove, so we concentrate on the case where \( \inf_{\Omega} \psi = 0 \). Thus \( (A - W)\psi = \lambda_1 \psi \leq 0 \), so if we assume that \( \psi \) does not admit a strictly positive lower bound on a ball \( B \) with \( \overline{B} \subset \Omega \), then \( \psi \) has to be constant on \( \Omega \) in view of Proposition 3.2. But this can be true, since the infimum of \( \psi \) on \( \Omega \) is zero, only provided \( \psi = 0 \).

Lemma 3.4. If \( \psi \neq 0 \) is a real eigenfunction of \( A - W \) belonging to \( \lambda_1 \), then \( \psi > 0 \) on \( \Omega \) or \( \psi < 0 \) on \( \Omega \). Any other eigenfunction belonging to \( \lambda_1 \) is a (scalar) multiple of \( \psi \).

Proof. Without loss of generality we may normalize our eigenfunction to \( \|\psi\|_2 = 1 \). Let \( \psi = \psi_+ - \psi_- \) be the usual decomposition of \( \psi \) in a positive and negative part. If \( \psi = 0 \) one may directly apply Lemma 3.3 and obtains the first assertion. The same is true, if \( \psi_+ = 0 \), arguing via \( -\psi \). Let us now suppose \( \psi_+ \neq 0 \neq \psi_- \). First we will show that the function \( \tilde{\psi}_+ := \frac{\psi_+}{\|\psi_+\|_2} \) also maximizes the expression in (3.2). In order to do so, consider functions \( \eta = \alpha \psi_+ - \beta \psi_- \), with the additional conditions \( \alpha \in [0, \frac{1}{\|\psi_+\|_2}] \) and \( \|\eta\|_2 = 1 \). In view of \( \psi_- \neq 0 \) it is clear that \( \|\psi_+\|_2 < 1 \).

Obviously, the normalizing condition \( \|\eta\|_2 = 1 \) is equivalent to

\[
(3.3) \quad \beta^2 = \frac{1 - \alpha^2 \|\psi_+\|_2^2}{\|\psi_-\|_2^2}.
\]

By an explicit calculation and taking into account (3.3) we obtain

\[
\langle (A - W)\eta, \eta \rangle = -\alpha^2 \left( \int_\Omega (\mu \nabla \psi_+ \cdot \nabla \psi_+) + \int_\Omega W \psi_+^2 \right) - \beta^2 \left( \int_\Omega (\mu \nabla \psi_- \cdot \nabla \psi_-) + \int_\Omega W \psi_-^2 \right)
\]
\[
= -\alpha^2 \left( \int_\Omega (\mu \nabla \psi_+ \cdot \nabla \psi_+) + \int_\Omega W \psi_+^2 \right)
\]
\[
- \frac{1 - \alpha^2 \|\psi_+\|_2^2}{\|\psi_-\|_2^2} \left( \int_\Omega (\mu \nabla \psi_- \cdot \nabla \psi_-) + \int_\Omega W \psi_-^2 \right)
\]
\[
= - \left[ \int_\Omega (\mu \nabla \psi_+ \cdot \nabla \psi_+) + \int_\Omega W \psi_+^2 - \|\psi_+\|_2^2 \left( \int_\Omega (\mu \nabla \psi_- \cdot \nabla \psi_-) + \int_\Omega W \psi_-^2 \right) \right] \alpha^2
\]
\[
- \frac{1}{\|\psi_-\|_2^2} \left( \int_\Omega (\mu \nabla \psi_- \cdot \nabla \psi_-) + \int_\Omega W \psi_-^2 \right).
\]

This means that the prefactor of \( \alpha^2 \) must be zero, since otherwise the expression, seen as a function in \( \alpha \), is strictly decreasing or increasing (depending on the sign of the prefactor) and
\( \alpha = 1 \) cannot maximize (3.2) on the interval \([0, 1]\). This implies
\[
\frac{\int_{\Omega}(\mu \nabla \psi_{-} \cdot \nabla \psi_{-}) + \int_{\Omega} W \psi_{-}^2}{\|\psi_{-}\|_{L^2}^2} = \frac{\int_{\Omega}(\mu \nabla \psi_{+} \cdot \nabla \psi_{+}) + \int_{\Omega} W \psi_{+}^2}{\|\psi_{+}\|_{L^2}^2}
\]
and, setting \( \alpha = \beta = 1 \) (observe that \( \eta = \psi \) in this case),
\[
\langle (A - W)\psi, \psi \rangle = \frac{1}{\|\psi_{-}\|_{L^2}^2}\left(\int_{\Omega}(\mu \nabla \psi_{-} \cdot \nabla \psi_{-}) + \int_{\Omega} W \psi_{-}^2\right) = \langle (A - W)\tilde{\psi}_{+}, \tilde{\psi}_{+} \rangle.
\]
Thus, the normalized function \( \tilde{\psi}_{+} \) also maximizes (3.2).

Since \( \tilde{\psi}_{+} \) is an element of the form domain, and hence from the domain of \((A - W)^{1/2}\), the series
\[
\sum_{m} |(\tilde{\psi}_{+}, u_m)|^2 \lambda_m
\]
converges. Here the \( \lambda_m \)'s denote the eigenvalues of \( A - W \) and \( \{u_m\}_{m \in \mathbb{N}} \) is an orthonormal eigenbasis of \( A - W \). Hence, we have
\[
\sum_{m} |(\tilde{\psi}_{+}, u_m)|^2 \lambda_1 = \lambda_1 = \langle (A - W)\tilde{\psi}_{+}, \tilde{\psi}_{+} \rangle = \sum_{m} |(\tilde{\psi}_{+}, u_m)|^2 \lambda_m.
\]
Note, that the sums on both sides consist only of non-positive summands. Consequently, all terms \( \langle \tilde{\psi}_{+}, u_m \rangle \) have to vanish if the corresponding \( \lambda_m \) is larger than \( \lambda_1 \). Thus, \( \tilde{\psi}_{+} \) must belong to the eigenspace of \( \lambda_1 \). Now Lemma 3.3 shows that \( \psi_{+} > 0 \) in \( \Omega \) and \( \psi_{-} \) must vanish, in contradiction to our above supposition.

Assume now that \( \psi \) and \( \varphi \) are two linearly independent eigenfunctions belonging to \( \lambda_1 \). Then we have \((A - W - \lambda_1)\psi = (A - W - \lambda_1)\varphi = 0\), so by [7, Theorem 8.22] \( \psi \) and \( \varphi \) are locally Hölder continuous in \( \Omega \). Thus, by linear independence, there are points \( x, y \in \Omega \) such that the system
\[
\psi(x)r + \varphi(x)s = -1, \quad \psi(y)r + \varphi(y)s = 1
\]
has a solution \((r, s)\). But this implies that the function \( r\psi + s\varphi \) is also an eigenfunction to the eigenvalue \( \lambda_1 \) that takes positive and negative values. This, however, is impossible by the preceding considerations. \( \square \)

**Lemma 3.5.** \( R(\lambda, bA) \) is positivity improving for all \( \lambda > 0 \).

**Proof.** Putting \( W_\lambda := \frac{1}{\lambda} \), one first notices \( W_\lambda > \varepsilon_\lambda > 0 \) almost everywhere. Hence, for every \( \lambda > 0 \) Lemma 3.3 and Lemma 3.4 apply for \( W = W_\lambda \). Thus the largest eigenvalue of \( A - W_\lambda \) is simple and there is a corresponding eigenfunction which is strictly positive. Furthermore, this operator is self-adjoint, bounded from above and the semigroup \( e^{t(A-W_\lambda)} \) is positive. To see this last point, observe that \( e^{tA} \) and \( e^{-tW_\lambda} \) are both positive contraction semigroups, so again by the Trotter product formula, we obtain
\[
e^{t(A-W_\lambda)} f = \lim_{n \to \infty} (e^{tA} e^{-tW_\lambda})^n f,
\]
which yields the claimed positivity.

Thus, we may apply [13, Theorem XIII.44], which states that \( R(0, A - W_\lambda) \) is positivity improving. According to (3.1) this also holds for \((\lambda - bA)^{-1}\). \( \square \)

**Proof of Theorem 3.1.** Since \( e^{tA} \) is analytic, it maps \( L^2(\Omega) \) into \( D(A) \). The latter is contained in the form domain and thus in \( H^1(\Omega) \) and so \( e^{tA} \) is compact by the compact embedding of \( H^1(\Omega) \) in \( L^2(\Omega) \). As \( e^{tA_{b,p}} \), \( 1 < p < \infty \), is a consistent family of semigroups, the compactness in \( L^p(\Omega) \) for \( p \in [2, \infty) \) follows by complex interpolation via [4, Theorem 1.6.1] and for \( p \in (1, 2) \) by duality. This proves the last assertion. \( \square \)
Regarding positivity, observe that it suffices to handle the case \( p = 2 \) due to consistency and to show that \( R(\lambda, bA) \) is a positive operator for large \( \lambda \). In order to do so, set again \( W_\lambda := \lambda/b \).

Then, we already saw in the proof of Lemma 3.5, that \( e^{t(A-W_\lambda)} \) is positive. Hence, the same is true for \( R(0, (A - W_\lambda)) = (W_\lambda - A)^{-1} \). Using once more (3.1), we see that \( R(\lambda, bA) \) is positive.

We now turn to prove irreducibility. By Proposition 2.2, the semigroup \( e^{tA} \) is given by kernels \( p_t \in L^\infty(\Omega \times \Omega) \) for \( t > 0 \). Thus in view of the criterion given in Remark 2.4, it suffices to argue in \( L^2(\Omega) \).

Also, the upper Gaussian bound implies by [9, Theorem 2.2] that the resolvent of \( bA \) at the point 1, i.e. \((1 - bA)^{-1}\) is a regular integral operator with kernel

\[
K^R(x,y) = \int_0^\infty e^{-t} p_t(x,y) \, dt.
\]

Due to the positivity of \( e^{tA} \) this kernel is positive. Furthermore, for every choice of measurable sets \( M, N \subset \Omega \) with positive measure it satisfies

\[
(3.4) \quad \int_M \int_N K^R(x,y) \, dx \, dy = \int_N \left( R(1,bA)1_M \right)(x) \, dx > 0,
\]

since the characteristic function of \( M \) is positive and not identically zero. Hence, \( R(1,bA)1_M \) is strictly positive by Lemma 3.5.

Finally, suppose that the semigroup \( e^{tA} \) is not irreducible. By Remark 2.4 this means that there exist measurable sets \( M_0, N_0 \subset \Omega \) with positive measure such that for all \( t > 0 \) we have

\[
\int_{M_0} \int_{N_0} p_t(x,y) \, dx \, dy = 0.
\]

Note that this expression cannot be negative, since \( p_t \geq 0 \). But this implies immediately

\[
\int_{M_0} \int_{N_0} K^R(x,y) \, dx \, dy = \int_0^\infty \int_{M_0} \int_{N_0} p_t(x,y) \, dx \, dy = 0,
\]

in contradiction to (3.4). Hence \( e^{tA} \) must be irreducible.  

\[\square\]

\section*{References}


Technische Universität Darmstadt, Fachbereich Mathematik, Schloßgartenstr. 7, 64289 Darmstadt, Germany
E-mail address: haller@mathematik.tu-darmstadt.de

Technische Universität Darmstadt, Fachbereich Mathematik, Schloßgartenstr. 7, 64289 Darmstadt, Germany
E-mail address: hieber@mathematik.tu-darmstadt.de

Weierstrass-Institut für Angewandte Analysis und Stochastik (WIAS), Mohrenstr. 39, 10117 Berlin, Germany
E-mail address: rehberg@wias-berlin.de