$L^p - L^q$-ESTIMATES FOR PARABOLIC SYSTEMS IN NON-DIVERGENCE FORM WITH VMO-COEFFICIENTS

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ABSTRACT. Consider a parabolic $N \times N$-system of order $m$ on $\mathbb{R}^n$ with top-order coefficients $a_\alpha \in \text{VMO} \cap L^\infty$. Let $1 < p, q < \infty$ and let $\omega$ be a Muckenhoupt weight. It is proved that systems of this kind possess a unique solution $u$ satisfying

$$\|u\|_{L^q(J; L^p(\mathbb{R}^n)^N)} + \|Au\|_{L^q(J; L^p(\mathbb{R}^n)^N)} \leq C \|f\|_{L^q(J; L^p(\mathbb{R}^n)^N)},$$

where $Au = \sum_{|\alpha| \leq m} a_\alpha D^\alpha u$ and $J = [0, \infty)$. In particular, closing $\omega = 1$, the realization of $A$ in $L^p(\mathbb{R}^n)^N$ has maximal $L^p - L^q$-regularity.

1. Introduction

$L^p - L^q$-estimates for parabolic differential equations are of particular interest since, combined with interpolation theory, they provide a powerful tool for many nonlinear problems. Whereas $L^p - L^q$-regularity properties for a single second-order equation in divergence form are fairly well understood (see e.g. the recent monograph of Auscher [Aus04]), the situation is far from clear for systems in divergence form, for higher order operators or for equations (or systems) having a non-divergence structure.

In this paper we consider higher order parabolic $N \times N$-systems on $\mathbb{R}^n$ in non-divergence form, i.e. systems of the form

$$u_t - \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u = f,$$

where the coefficients $a_\alpha$ for $|\alpha| = m$ belong to the class VMO$(\mathbb{R}^n; C^{N \times N})$. Here VMO$(\mathbb{R}^n; C^{N \times N})$ denotes the Serapion space of all functions on $\mathbb{R}^n$ with values in $C^{N \times N}$ having vanishing mean oscillation. Our aim is to prove that equation (1.1) admits a unique solution which satisfies an estimate of the form

$$\|u\|_{L^q(J; L^p(\mathbb{R}^n)^N)} + \|Au\|_{L^q(J; L^p(\mathbb{R}^n)^N)} \leq C \|f\|_{L^q(J; L^p(\mathbb{R}^n)^N)},$$

where $1 < p, q < \infty$, $Au = \sum_{|\alpha| \leq m} a_\alpha D^\alpha u$, $J = [0, \infty)$ and $\omega \in A_p$ is a Muckenhoupt weight.

Note that in particular the Sobolev space $W^\alpha, r/\theta(\mathbb{R}^n)$ for $\theta \in (0, 1]$ is contained in VMO$(\mathbb{R}^n)$; thus our approach allows to treat parabolic systems with not necessarily continuous coefficients.

The elliptic problem $Au = f$ for a single differential operator $A$ of order $2m$ in non-divergence form subject to general boundary conditions was solved in the $L^p$-setting in a classical paper by Agmon, Douglis and Nirenberg [ADN59], provided the top-order coefficients of $A$ are bounded and uniformly continuous. The corresponding result for parabolic equations (and even systems) was proved recently by Denk, Hieber and Prüss in [DHP03]. For previous results dealing with the case of $\mathbb{R}^n$, second order operators with particular boundary conditions or with additional regularity assumptions on the coefficients, we refer to [LSU68], [Duo90], [PS93], [AHS94], [Ama95], [HP97] and [DS97].

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1
The study of linear and quasilinear elliptic equations with VMO-coefficients started with the pioneering work of Chinzena, Frasca and Longo [CFL91], [CFL93]. They proved $W^{2,p}$-estimates for the solution $u$ of the Dirichlet problem associated to
\[ \sum_{i,j=1}^{n} a_{ij} D_i D_j u = f, \]
provided $1 < p < \infty$, $f \in L^p$ and $a_{ij} \in \text{VMO} \cap L^\infty$. Their proof was based on parameter-dependent Calderón-Zygmund theory. The techniques were later extended by Palagachev, Di Fazio, Muggeri and Softova [Pa95], [DP96], [MP98], [MP99] to quasilinear elliptic equations and to the oblique derivative problem.

The corresponding result for certain elliptic systems was proved recently by Ragusa [Rag02].

The first result for a scalar, second order parabolic equation is due to Bramanti and Cerutti [BC93]. They extended the technique developed in [CFL93] to the parabolic situation. The oblique derivative problem in the parabolic situation was solved more recently by Softova [So00]. Using wavelet techniques, Duong and Yan [DY02] proved that the operators associated to these equations admit a bounded $H^\infty$-calculus on $L^p(\mathbb{R}^n)$; see also previous work by Angeletti, Mazet and Tchamitchian [AMT97]. Combining their result with the Dore-Venni theorem one obtains also estimates of the form (1.2) for second order operators. For a result on elliptic boundary value problems with coefficients in VMO see Guidetti [Gui02].

We already mentioned that $L^p - L^q$-estimates for parabolic systems in non-divergence form have been considered so far only for coefficients which are bounded and uniformly continuous (see [DHP03]). The approach there was to localize and to consider perturbations which are small in $L^\infty$. Of course, this method needs uniformly continuous coefficients.

Our approach to parabolic systems with VMO-coefficients is very different. It is based on a particular representation of the highest order derivatives of the solution $u$ of $\lambda u + A u = f$. This representation allows to estimate $D^\nu u$ for $|\nu| = m$ by parameter depending Calderón-Zygmund theory and commutator techniques. More precisely, we are aiming for a weighted a priori estimate of the form
\[ \sum_{|\nu| \leq m} ||\lambda|^{-|\nu|} D^\nu u||_{p,\omega} \leq C||\lambda + A|| u||_{p,\omega}, \]
where $\omega \in A_p$ is a Muckenhoupt weight, $u \in W^{m,p}_\omega(\mathbb{R}^n)$, $\lambda \in C$ belongs to a suitable sector of the complex plane and the coefficients of $A$ are small in BMO.

The reason for introducing Muckenhoupt weights in this context is the following, combining the characterization theorem of maximal $L^p$-regularity due to Weis [Wei00] with results due to Rubio de Francia [Rub00] one sees that estimate (1.2) is implied by weighted estimates of the form
\[ ||(\lambda + A)^{-1} f||_{p,\omega} \leq C|| f||_{p,\omega}, \quad \omega \in A_p, \quad f \in L^p(\mathbb{R}^n)^N, \]
where $A$ is the realization of $A$ given by $A u = \sum_{|\nu| \leq m} a_{\nu} D^\nu u$ in $L^p_\omega(\mathbb{R}^n)^N$, $\lambda$ lies in a suitable sector of the complex plane and the constant $C$ is allowed to depend on the $A_p$-constant of the weight $\omega$ only. For details see the following section and [HHH03]. This method was already successfully used in [HHH03] for scalar parabolic equations.

The problem of determining the “minimal” regularity assumption on the top-order coefficients $a_{ij}$ of $A$ such that (1.1) admits a unique solution satisfying (1.2) is still far from being solved. It is quite surprising that the answer also depends on the space dimension $n$. Indeed, it is shown in [HW04] that for scalar second order elliptic operators with $L^\infty$-coefficients the estimate (1.2) holds true provided $n = 2$, $1 < q < \infty$ and $p \in (1,2]$ is close to 2. This is no longer true for $n > 2$, even for $p = 2$, as the following example due to Talenti [Tal65] shows: let $n > 2$ and let $\Omega$ be the unit ball in $\mathbb{R}^n$. Set
\[ A = \sum_{i,j=1}^{n} \left( \frac{x_i x_j}{|x|^2} (1 - cn) + \delta_{ij} \right) D_{ij} := \sum_{i,j=1}^{n} a_{ij} D_{ij}. \]
for $0 < c < \frac{n-2}{n(n-1)}$. Then $A : H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)$ is not an isomorphism. We refer also to Mauger, Palagachev and Softova [MPS00] and the references therein. Observe that in the above case $a_{ij} \in W^{1,n-\varepsilon}(\Omega)$ for all $\varepsilon > 0$. Thus parabolic equations associated to operators with coefficients $a_{ij} \in W^{1,n-\varepsilon}(\Omega)$ for $n \geq 3$ cannot fulfill an estimate of the form (1.2) in general. On the other hand, if $a_{ij} \in W^{1,n+\varepsilon}(\Omega)$, then by Morrey’s theorem $a_{ij} \in C^\alpha(\overline{\Omega})$ and estimate (1.2) is well known in this case; see e.g. [HP97] or [DDHPV04]. Since $W^{1,n}(\mathbb{R}^n)$ is a subspace of VMO$(\mathbb{R}^n)$, our regularity assumptions allow in particular to treat the limiting case above.

This paper is organized as follows. After collecting certain results on Muckenhoupt weights and fundamental solutions of parabolic systems in Section 2, we state our main result in Section 3. Section 4 deals with singular integrals, parameter depending Calderón-Zygmund kernels and commutator estimates in weighted $L^p$-spaces. These results are fundamental for the a priori estimate given in Section 5. Finally, in Section 6 we give a proof of our main result.

2. Preliminaries

We start this section with the definition of the Muckenhoupt class $A_p$ for $1 < p < \infty$. More precisely, a function $0 \leq \omega \in L^{1,\infty}_{\text{loc}}(\mathbb{R}^n)$ is called an $A_p$-weight in the sense of Muckenhoupt, if there is a constant $C > 0$ such that

$$\left( \frac{1}{|Q|} \int_Q \omega \, dx \right)^{1/p} \leq C,$$

for all cubes $Q \subseteq \mathbb{R}^n$ with sides parallel to the axes. The smallest such $C$ is called the $A_p$-constant of $\omega$. We call a constant $C = C(\omega)$ to be $A_p$-consistent if it depends on the $A_p$-constant of $\omega$ only. For operators acting in weighted $L^p$-spaces with a weight belonging to $A_p$ the following extrapolation theorem (see [GR85, Theorem IV.5.19] or [Ste93, V.6.17]) is true. Let $1 < p, q < \infty$ and $T$ be a family of operators such that for all $\omega \in A_p$ there exists a constant $C$, depending only on the $A_p$-constant of $\omega$, such that

$$\|Tf\|_{L^q} \leq C\|f\|_{L^p}, \quad T \in T.$$

Then it follows that the same inequality, with $p$ replaced by $q$, holds for all $\omega \in A_q$. This extrapolation theorem allows to give the following sufficient criterion for maximal $L^p - L^q$-regularity, and so (1.2), on $L^p$-spaces, see [HHH03]. The proof combines a result due to Garcia-Cuervo and Rubio de Francia [GR85, Theorem V.6.4] with a recent Fourier multiplier theorem due to Weis [Wei01].

**Proposition 2.1** ([HHH03]). Let $1 < p, q < \infty$ and assume that $A$ is a sectorial operator in $L^p(\mathbb{R}^n)^N$ of angle $\varphi < \frac{\pi}{2}$. Suppose that a weighted estimate of the form

$$\|s(is + A)^{-1}f\|_{L^q} \leq C\|f\|_{L^p}, \quad \omega \in A_p, \quad f \in L^p(\mathbb{R}^n)^N,$$

holds, where the constant $C$ depends only on the $A_p$-constant of the weight $\omega$. Then there exists a constant $M > 0$, such that

$$\|Au\|_{L^q(J; L^p(\mathbb{R}^n)^N)} + \|A\|_{L^q(J; L^p(\mathbb{R}^n)^N)} \leq M\|f\|_{L^q(J; L^p(\mathbb{R}^n)^N)}.$$

In the following, we consider systems of differential operators of the form $A = \sum_{|\alpha| \leq m} a_\alpha(x)D^\alpha$, where $D = -i(\partial_1, \ldots, \partial_n)$ and $a_\alpha \in L^\infty(\mathbb{R}^n; \mathbb{C}^N \otimes \mathbb{C}^N)$. We will assume that $A$ is $(M, \theta)$-elliptic; this means that there exist constants $\theta \in [0, \pi)$ and $M > 0$, such that the principal part $A_\#(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x)\xi^\alpha$ of the symbol of $A$ satisfies the following conditions:

$$\sigma(A_\#(x, \xi)) \subseteq \Sigma_\theta$$

and

$$\|A_\#(x, \xi)^{-1}\| \leq M$$

for all $\xi \in \mathbb{R}^n$, $|\xi| = 1$,

for almost all $x \in \mathbb{R}^n$. The set of all such $x$ will be denoted by $E$. Here $\Sigma_\theta$ denotes the sector in the complex plane defined by $\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \theta\}$ and the spectrum of an $N \times N$-matrix $M$ is denoted by $\sigma(M)$.
Let $A_{p,\omega}$ be the realization of the differential operator $A$ in $L^p_w(\mathbb{R}^n)^N$, that is defined by
\begin{equation}
A_{p,\omega} u := Au, \\
D(A_{p,\omega}) := W^m_p(\mathbb{R}^n)^N.
\end{equation}

For the time being, we freeze the elliptic operator $A$ in $x_0 \in E$ and consider the fundamental solution $\gamma^{x_0}_{\lambda}$ of $\lambda + A_{x_0}$ for $\lambda \in \Sigma_{x_0}$, where $\phi \in (0,\pi)$. Then by [DHP03] we have the following pointwise estimates for the fundamental solution $\gamma^{x_0}_{\lambda}$:
\begin{equation}
\|D^\nu \gamma^{x_0}_{\lambda}(x)\| \leq C_{\phi,k}|\lambda|^{-\frac{2\nu}{n} - 1}p^{n}_{m,k}(c_{\phi}|\lambda|^{-\frac{2}{n}}|x|), \quad x \in \mathbb{R}^n, \; k := |\nu|,
\end{equation}
where
\begin{align*}
p^n_{m,k}(r) &= \int_0^{\infty} \frac{r^{n-2}}{(1+s)^{m-\frac{n}{2}+1}}e^{-r(1+s)} \, ds, \quad n \geq 2, \\
p^m_{m,k}(r) &= \int_0^{\infty} \frac{1}{(1+s)^{m-\frac{n}{2}}e^{-r(1+s)}} \, ds.
\end{align*}
This means that $D^\nu \gamma^{x_0}_{\lambda}$ is integrable for all $\nu$ with $|\nu| < m$.

We next consider the spaces of functions with bounded mean oscillation $\text{BMO}(\mathbb{R}^n)$ and vanishing mean oscillation $\text{VMO}(\mathbb{R}^n)$, respectively. For this it is useful to introduce the following notation. For $f \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{C}^{N \times N})$ and $G \subseteq \mathbb{R}^n$ open and bounded we write
\[ f_G := \frac{1}{|G|} \int_G f(x) \, dx. \]
for the mean value of $f$ over $G$. We then say that a function $f \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{C}^{N \times N})$ has bounded mean oscillation, or $f \in \text{BMO}(\mathbb{R}^n; \mathbb{C}^{N \times N})$, if
\[ \|f\|_* := \sup_{B \subseteq \mathcal{B}_r} \frac{1}{|B|} \int_B |f(x) - f_B| \, dx = \sup_{B \subseteq \mathcal{B}_r} \|f - f_B\|_B < \infty, \]
where $\mathcal{B}_r$ denotes the set of all balls in $\mathbb{R}^n$.

For given $r > 0$ we write $\mathcal{B}_r$ for the set of all balls in $\mathbb{R}^n$ with radius less than $r$. We then further define the VMO-modulus $\eta_f$ of a function $f \in \text{BMO}(\mathbb{R}^n; \mathbb{C}^{N \times N})$ via
\[ \eta_f(r) = \sup_{B \subseteq \mathcal{B}_r} \frac{1}{|B|} \int_B |f(x) - f_B| \, dx, \quad r > 0. \]
and we say that $f$ has vanishing mean oscillation, or $f \in \text{VMO}(\mathbb{R}^n; \mathbb{C}^{N \times N})$, if
\[ \lim_{r \to 0+} \eta_f(r) = 0. \]
If $N = 1$ we will shortly write $\text{BMO}(\mathbb{R}^n)$ or $\text{VMO}(\mathbb{R}^n)$ for $\text{BMO}(\mathbb{R}^n; \mathbb{C}^{1 \times 1})$ or $\text{VMO}(\mathbb{R}^n; \mathbb{C}^{1 \times 1})$, respectively. Notation that a function $f = (f_{j,k})_{j,k=1}^N$ belongs to $\text{BMO}(\mathbb{R}^n; \mathbb{C}^{N \times N})$ or $\text{VMO}(\mathbb{R}^n; \mathbb{C}^{N \times N})$, if and only if every component $f_{j,k}$ of $f$ belongs to $\text{BMO}(\mathbb{R}^n)$ or $\text{VMO}(\mathbb{R}^n)$, respectively.

Next we introduce the sharp function $f^\#$ of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ by
\[ f^\#(x) := \sup_{B \subseteq \mathcal{B}_2, B \ni x} \frac{1}{|B|} \int_B |f(y) - f_B| \, dy. \]

Then the following holds.

**Lemma 2.2.** Let $1 < p < \infty$ and $\omega \in A_p$. Then there exists an $A_p$-consistent constant $C$ such that
\[ \|f\|_{p,\omega} \leq C ||f^\#||_{p,\omega}, \quad f \in L^p_w(\mathbb{R}^n). \]

For a proof of this fact see e.g. Proposition 5.4 in [HH03].

The following inequality will be often useful to have.
\begin{equation}
\int_I |f(x) - f| \, dx \leq 2 \int_I |f(x) - c| \, dx,
\end{equation}
for any constant $c \in \mathbb{R}$ and any measurable set $I \subseteq \mathbb{R}^n$ with positive measure.

Furthermore, we recall the well known John-Nirenberg inequality (see e.g. [Ste93, IV.1.3]). Denote by $I$ a cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes and let $1 < p < \infty$. Then

$$
\left( \int_I |f(x) - f_I|^p \, dx \right)^{\frac{1}{p}} \leq C_p \|f\|_*.
$$

Moreover, let $2^j I, j \in \mathbb{N}$, be the cube with the same center as $I$ but with sidelength $2^j d$, where $d$ is the sidelength of $I$. Then

$$
\left( \int_{2^j I} |f(x) - f_I|^p \, dx \right)^{\frac{1}{p}} \leq C_p (j + 1) \|f\|_*.
$$

We note that (2.6) follows from (2.5) by induction.

3. Main Results

We are now in the position to state the main result of this paper.

**Theorem 3.1.** Let $n \geq 2, 1 < p, q < \infty$, $\omega \in A_p, \phi_0 \in (0, \pi), \phi > \phi_0$ and $M > 0$. Assume that $A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ is an $(M, \phi_0)$-elliptic operator in $L^p_\omega(\mathbb{R}^n)^N$ with coefficients $a_\alpha$ satisfying

- $a_\alpha \in L^\infty(\mathbb{R}^n; \mathbb{C}^{m \times N}) \cap \text{VMO}(\mathbb{R}^n; \mathbb{C}^{m \times N})$ for $|\alpha| = m$,
- $a_\alpha \in L^\infty(\mathbb{R}^n; \mathbb{C}^{m \times N})$ for $|\alpha| < m$.

Then there are $A_p$-consistent constants $\lambda_0, C > 0$, such that

$$
\|(\lambda + \lambda_0 + A_{p, \omega})^{-1}\|_{L(L^p_\omega(\mathbb{R}^n)^N)} \leq \frac{C}{|\lambda|}, \quad \lambda \in \Sigma_{\pi - \phi}.
$$

The following corollary follows immediately from Proposition 2.1 and Theorem 3.1.

**Corollary 3.2.** Let $1 < p, q < \infty$. Assume that $\phi_0 < \frac{\pi}{2}$. Then there exist constants $M, \mu \geq 0$ such that

$$
\|u\|_{L^q(J; L^p_\omega(\mathbb{R}^n)^N)} + \|\mu + A_{p, \omega} u\|_{L^q(J; L^p_\omega(\mathbb{R}^n)^N)} \leq M \|f\|_{L^q(J; L^p_\omega(\mathbb{R}^n)^N)},
$$

In particular, $-(\mu + A_{p, \omega})$ generates an analytic semigroup on $L^p_\omega(\mathbb{R}^n)^N$.

**Remark 3.3.** Obviously, choosing $\omega = 1$, the above assertions hold also in the unweighted space $L^p(\mathbb{R}^n)^N$.

4. Singular Integrals and Commutators

In this section we consider integral operators of Calderón-Zygmund type and related commutators with BMO-functions. More precisely, we call a function $K \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^{N \times N})$ a Calderón-Zygmund kernel, if for a constant $C > 0$ the following three conditions are satisfied:

a) $\|FK\|_\infty \leq C$,

b) $|K(x)| \leq C|x|^{-n}$ for all $x \in \mathbb{R}^n \setminus \{0\}$,

c) $|K(x - y) - K(x)| \leq C|y||x|^{-(n+1)}$ for all $x, y \in \mathbb{R}^n$, where $|x| > 2|y| > 0$.

Here we denote by $\mathcal{F}$ the Fourier transform.

**Remark 4.1.** It is clear that the entries $K_{j, k}$ of $K = (K_{j, k})_{j, k=1}^N$ are scalar Calderón-Zygmund kernels if and only if $K$ is a Calderón-Zygmund kernel.

The following result on scalar-valued kernels was proved in [HH03, Propositions 5.1 and 5.4]. It extends in particular a classical commutator result due to Coifman, Rochberg and Weiss [CRW76] to the weighted situation.
Proposition 4.2. Let $1 < p < \infty$, $\omega \in A_p$, $a \in \text{BMO}(\mathbb{R}^n)$ and $k \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ be a (scalar-valued) Calderón-Zygmund kernel. Then
\[ Tf := k \ast f \quad \text{and} \quad [T, a]f := T(af) - aTf, \quad f \in \mathcal{S}(\mathbb{R}^n), \]
define bounded linear operators on $L^p_{\omega}(\mathbb{R}^n)$ and there is an $A_p$-consistent constant $C$ such that
\[ \|Tf\|_{p,\omega} \leq C\|f\|_{p,\omega} \quad \text{and} \quad \|[T, a]f\|_{p,\omega} \leq C\|a\|_\infty \|f\|_{p,\omega}. \]

We next give a result analog to Proposition 4.2 but now for kernels having two variables. We assume that the kernel is homogeneous in the second variable. By $S^{n-1}$ we denote the unit sphere in $\mathbb{R}^n$.

Proposition 4.3. Let $1 < p < \infty$, $\omega \in A_p$ and $a \in \text{BMO}(\mathbb{R}^n)$. Further let $k : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{C}$ be measurable, such that
a) the function $k(x, \cdot)$ is homogeneous of degree $-n$ for a.e. $x \in \mathbb{R}^n$,
b) $\int_{S^{n-1}} k(x, y) \, dy = 0$, 
c) $\|D^\alpha_y k(x, y)\|_{L^\infty(\mathbb{R}^n \times S^{n-1})} \leq M, \quad |\alpha| \leq 2n$.
Consider the operators $T$ and $[T, a]$ given by
\[ Tf(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} k(x, x - y) f(y) \, dy = \text{p.v.} \int \kappa(x, x - y) f(y) \, dy, \]
\[ [T, a]f(x) = \text{p.v.} \int k(x, x - y)(a(x) - a(y)) f(y) \, dy. \]
Then $T$ and $[T, a]$ are bounded on $L^p_{\omega}(\mathbb{R}^n)$ and there is an $A_p$-consistent constant $C$ such that
\[ \|Tf\|_{p,\omega} \leq C\|f\|_{p,\omega}, \quad \|[T, a]f\|_{p,\omega} \leq C\|a\|_\infty \|f\|_{p,\omega}. \]
The proof of this proposition is given in [HH03, Proposition 5.5] and uses a representation of $|y|^\alpha k(x, y)$ by spherical harmonics. As we need this method in the following, we shortly describe it in the next proposition.

Proposition 4.4. Let $k$ be as in Proposition 4.3. Then for $j \in \mathbb{N}$ there exist $b_j \in L^\infty(\mathbb{R}^n)$ and $Y_j \in L^\infty(\mathbb{R}^n \setminus \{0\})$ such that
a) $k(x, y) = \sum_{j=1}^\infty b_j(x) Y_j(y)/|y|^n$, 
b) $Y_j(y)/|y|^n$ is a Calderón-Zygmund kernel, 
c) the norms of the associated operators on $L^p_{\omega}(\mathbb{R}^n)$ are uniformly bounded in $j \in \mathbb{N}$, 
d) $\sum_{j=1}^\infty \|b_j\|_\infty < \infty$.

The following result is a consequence of (2.3), see e.g. [HH03, Lemma 6.1].

Proposition 4.5. Let $B$ be an $(M, \phi_0)$-elliptic differential operator, homogenous of degree $m$, with constant coefficients. Further, let $\phi > \phi_0$ and let $\gamma_\lambda$ be the fundamental solution of $\lambda + B$ for $\lambda \in \Sigma_{\phi, -\phi}$. Then, for every $\lambda \in \Sigma_{\phi, -\phi}$ and $|\nu| \leq m$, the functions
\[ \lambda^{1-|\nu|/m} D^\nu \gamma_\lambda \]
are Calderón-Zygmund kernels. Furthermore, the constant $C$, that appears in the definition of Calderón-Zygmund kernels, can be chosen independently of $\lambda$.

Remark 4.6. In particular, the kernels $\gamma_\lambda^{\phi_0}$, $x_0 \in E$, fulfill the conditions of this proposition. Note that the inequalities in (2.3) do not depend on $x_0$. Therefore, the constant $C$ can also be chosen independently of $x_0 \in E$.

The next result ensures the existence of a certain integral kernel.
Lemma 4.7. Let \( h \in C^\infty(\mathbb{R}^n \setminus \{0\} ; C^{N \times N}) \) be homogeneous of degree 0 and \( T \in L^2(L^2(\mathbb{R}^n)^N) \) be defined as \( Tf = F^{-1}hFf \). Then there exists a function \( k \in C^\infty(\mathbb{R}^n \setminus \{0\} ; C^{N \times N}) \) which is homogeneous of degree \(-n\) and satisfies \( \int_{S^{n-1}} k(x) \, dx = 0 \). Furthermore,
\[
Tf = c \cdot f + \lim_{r \to 0} \int_{|y| > r} k(y) f(-y) \, dy, \quad f \in L^2(\mathbb{R}^n)^N,
\]
where \( c = \int_{S^{n-1}} h(x) \, dx \) and
\[
||D^2 k||_{L^\infty(S^{n-1})} \leq C \sum_{|\beta| \leq r} ||D^\beta h||_{L^\infty(S^{n-1})}
\]
for some \( r \in \mathbb{N} \) depending only on \( n \) and \( |\alpha| \).

Proof. The proof follows easily from the scalar-valued case given in [HH03]. In fact, set \( h = (h_{j,\ell})_{j,\ell=1}^N \). Then \( h_{j,\ell} \in C^\infty(\mathbb{R}^n \setminus \{0\}) \) for all \( j, \ell \in \{1, \ldots, N\} \). Furthermore, for \( x \neq 0 \) and \( \lambda \in \mathbb{R} \) we have
\[
h_{j,\ell}(\lambda x) = e_j^T h(\lambda x) e_\ell = e_j^T h(x) e_\ell = h_{j,\ell}(x),
\]
which implies the homogeneity of each component \( h_{j,\ell} \). By [HH03, Lemma 6.2] we obtain scalar kernels \( k_{j,\ell} \in C^\infty(\mathbb{R}^n \setminus \{0\}) \) and \( c_{j,\ell} \) with the desired properties. Now \( k := (k_{j,\ell})_{j,\ell=1}^N \) and \( c := (c_{j,\ell})_{j,\ell=1}^N \) is the function and the constant we were looking for.

5. A Priori Estimates

The main result of this section is the following a priori estimate. We denote by \( \mathcal{A} \) always a homogeneous, \((M, \phi_0)\)-elliptic differential operator of order \( m \), where \( \phi_0 \in (0, \pi) \). We also assume that \( a_\alpha \in L^\infty(\mathbb{R}^n; C^{N \times N}) \) for all \( \alpha \) with \( \vert \alpha \vert = m \).

Theorem 5.1. Let \( 1 < p < \infty, \omega \in A_p \) and \( \phi > \phi_0 \). Then there exist \( A_\omega \)-consistent constants \( C, \eta > 0 \) and \( \lambda_0 \geq 0 \), such that for all \( u \in W^{m,p}(\mathbb{R}^n)^N \) and all \( \lambda \in \Sigma_{\pi-\phi} \) with \( \vert \lambda \vert > \lambda_0 \)
\[
\sum_{\vert \alpha \vert \leq m} ||\lambda^{\frac{m-m_0}{m}} D^\alpha u||_{p,\omega} \leq C(||(\lambda + A)u||_{p,\omega},
\]
provided \( ||a_\alpha|| \leq \eta \) for all \( \vert \alpha \vert = m \).

The proof of the above theorem is based on a representation of \( D^\nu u \) in terms of kernels \( \gamma_\lambda^{\nu_0} \) and operators \( \lambda + A_{x_\nu} \). More precisely, we have the following representation formula.

Lemma 5.2. Let \( u \in D(\mathbb{R}^n)^N \) and \( x \in E \). If \( \vert \nu \vert < m \), then
\[
D^\nu u(x) = \int_{\mathbb{R}^n} D^\nu \gamma_\lambda^{\nu_0}(x-y)(\lambda + A_x)u(y) \, dy.
\]
Moreover, for \( \vert \nu \vert = m \)
\[
D^\nu u(x) = \nu \cdot \int_{\mathbb{R}^n} K_\nu(x,x-y)(\lambda + A_x)u(y) \, dy + c_\nu(x)(\lambda + A)u(x)
\]
\[
- \int_{\mathbb{R}^n} \lambda \gamma_\lambda^{\nu_0}(x-y) \left( \nu \cdot \int_{\mathbb{R}^n} K_\nu(x,y-z)(\lambda + A_x)u(z) \, dz \right) dy
\]
\[
- c_\nu(x) \int_{\mathbb{R}^n} \lambda \gamma_\lambda^{\nu_0}(x-y)(\lambda + A_x)u(y) \, dy.
\]
Here \( c_\nu \in L^\infty(\mathbb{R}^n) \) and \( K_\nu \) is a matrix-valued integral kernel, whose entries fulfill the assumptions of Proposition 4.3.
Proof. For every $x_0 \in E$ we have $F u = (\lambda + A\#(x_0, \xi))^{-1} F(\lambda + A\#) u$. The first part of the assertion follows easily by setting $x_0 = x$ since by the estimates (2.3) $D^v \gamma_{\lambda}^{x_0}$ is in $L^1(\mathbb{R}^n)^N$ for every $|v| < m$.

Hence we turn to the case where $|v| = m$ and see that
\[ F D^v u(\xi) = \xi^v (\lambda + A\#(x_0, \xi))^{-1} (F(\lambda + A\#) u)(\xi) \]
\[ = \xi^v (A\#(x_0, \xi))^{-1} (F(\lambda + A\#) u)(\xi) + \left[ \xi^v (A\#(x_0, \xi))^{-1} - \xi^v (A\#(x_0, \xi))^{-1} \right] (F(\lambda + A\#) u)(\xi). \]

We may now replace $(\lambda + A\#(x_0, \xi))^{-1} - (A\#(x_0, \xi))^{-1}$ by $-\lambda (\lambda + A\#(x_0, \xi))^{-1}(A\#(x_0, \xi))^{-1}$, since the two matrices commute. Hence we obtain
\[ F D^v u(\xi) = \xi^v (A\#(x_0, \xi))^{-1} (F(\lambda + A\#) u)(\xi) - \lambda (\lambda + A\#(x_0, \xi))^{-1} \xi^v (A\#(x_0, \xi))^{-1} (F(\lambda + A\#) u)(\xi). \]

As $|v| = m$ and $A$ is an $(M, \phi_0)$-elliptic operator, the symbol $h_{x_0}(\xi) := \xi^v (A\#(x_0, \xi))^{-1}$ is a smooth homogeneous function of degree 0.

By Lemma 4.7 there exist kernels $K_v(x_0, \cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\}; C^{N \times N})$, that are homogeneous of degree $-n$ with $\int_{S^{n-1}} K_v(x_0, y) \, dy = 0$. Furthermore, the ellipticity of $A$ yields the estimate $\|D^\beta h_{x_0}\|_{L^\infty(S^{n-1})} \leq C_k$ uniformly in $x_0 \in E$ and for all $|\beta| \leq k$. This implies, that the derivatives of $K_v(x_0, \cdot)$ are bounded on the unit sphere independently of $x_0 \in E$. By this we finally obtain, that the entries of $K_v$ fulfill the assumptions of Proposition 4.3.

For the constants $c_v(x_0)$, which we obtain from Lemma 4.7, ellipticity of $A$ yields
\[ |c_v(x_0)| \leq \int_{S^{n-1}} |\xi^v (A\#(x_0, \xi))^{-1}| \, d\xi \leq \int_{S^{n-1}} M \, d\xi \leq CM. \]

Thus $c_v \in L^\infty(\mathbb{R}^n)$.

Inserting these kernels and the constants $c_v$ into the above equation and applying $F^{-1}$ we get
\[ D^v u(x) = \text{p.v.} \int_{\mathbb{R}^n} K_v(x_0, x - y)(\lambda + A\#) u(y) \, dy + c_v(x_0)(\lambda + A\#) u(x) \]
\[ - \int_{\mathbb{R}^n} \lambda \gamma_{\lambda}^{x_0}(x - y) \left( \text{p.v.} \int_{\mathbb{R}^n} K_v(x_0, y - z)(\lambda + A\#) u(z) \, dz \right) \, dy \]
\[ - c_v(x_0) \int_{\mathbb{R}^n} \gamma_{\lambda}^{x_0}(x - y)(\lambda + A\#) u(y) \, dy \]
for every $x_0 \in E$. The assertion follows again by setting $x_0 = x$. \qed

Before proving Theorem 5.1 some comments on our notation are in order. The entries of the matrices $a_v (x)$ and $\gamma_{\lambda}^{x_0}$ are denoted by $a_{j,k}^{v}(x)$ and $g_{j,k}^{x_0}$ for $j,k = 1, \ldots, N$, respectively. Furthermore, we write $u_j, j = 1, \ldots, N$, for the components of $u$.

By Lemma 5.2 we get for $|v| < m$
\[ \lambda^{1-\frac{|v|}{m}} D^v u(x) = \int_{\mathbb{R}^n} \lambda^{1-\frac{|v|}{m}} D^v \gamma_{\lambda}^{x_0}(x - y)(\lambda + A\#) u(y) \, dy \]
\[ = \int_{\mathbb{R}^n} \lambda^{1-\frac{|v|}{m}} D^v \gamma_{\lambda}^{x_0}(x - y)(\lambda + A) u(y) \, dy \]
\[ + \int_{\mathbb{R}^n} \lambda^{1-\frac{|v|}{m}} D^v \gamma_{\lambda}^{x_0}(x - y)(A\# - A) u(y) \, dy. \]
Looking at one component one obtains
\[
\lambda^{1-\frac{m}{2}} D^\nu u_j(x) = \sum_{k=1}^{N} \int_{\mathbb{R}^n} \lambda^{1-\frac{m}{2}} D^\nu g_{j,k}^{\lambda,\nu} (x-y) \left( (\lambda + A) u(y) \right)_k \, dy \\
+ \sum_{\ell=1}^{N} \sum_{k=1}^{N} \int_{\mathbb{R}^n} \lambda^{1-\frac{m}{2}} D^\nu g_{j,k}^{\lambda,\nu} (x-y) \sum_{|\alpha|\leq m} (a_{k,\ell}^{(\alpha)}(x) - a_{k,\ell}^{(\alpha)}(y)) D^\alpha u_\ell(y) \, dy \\
= \sum_{k=1}^{N} (T^{3,\lambda,\nu}_{j,k} ((\lambda + A) u)_k)(x) + \sum_{\ell=1}^{N} \sum_{k=1}^{N} \sum_{|\alpha|\leq m} ([T^{3,\lambda,\nu}_{j,k}, a_{k,\ell}^{(\alpha)}] D^\alpha u_\ell)(x),
\]
where
\[
T^{3,\lambda,\nu}_{j,k}(f)(x) = \int_{\mathbb{R}^n} \lambda^{1-\frac{m}{2}} D^\nu g_{j,k}^{\lambda,\nu} (x-y) f(y) \, dy,
\]
\[
[T^{3,\lambda,\nu}_{j,k}, a] f(x) = \int_{\mathbb{R}^n} \lambda^{1-\frac{m}{2}} D^\nu g_{j,k}^{\lambda,\nu} (x-y) (a(x) - a(y)) f(y) \, dy
\]
for $j, k = 1, \ldots, N$, $|\nu| < m$, $\lambda \in \Sigma_{n-\phi}$ and $a \in \text{BMO}(\mathbb{R}^n)$. By the same calculations, we derive from the representation formula in Lemma 5.2 for $|\nu| = m$
\[
D^\nu u(x) = \text{p.v.} \int_{\mathbb{R}^n} K_\nu(x, x-y) (\lambda + A) u(y) \, dy + \text{p.v.} \int_{\mathbb{R}^n} K_\nu(x, x-y) (A_x - A) u(y) \, dy \\
+ c_\nu (x)(\lambda + A) u(x) - \int_{\mathbb{R}^n} \lambda \gamma_\nu^\lambda (x-y) \left( \text{p.v.} \int_{\mathbb{R}^n} K_\nu(x, x-y) (\lambda + A) u(z) \, dz \right) \, dy \\
- \int_{\mathbb{R}^n} \lambda \gamma_\nu^\lambda (x-y) \left( \text{p.v.} \int_{\mathbb{R}^n} K_\nu(x, x-y) (A_x - A) u(z) \, dz \right) \, dy \\
- c_\nu (x) \int_{\mathbb{R}^n} \lambda \gamma_\nu^\lambda (x-y) (\lambda + A) u(y) \, dy - c_\nu (x) \int_{\mathbb{R}^n} \lambda \gamma_\nu^\lambda (x-y) (A_x - A) u(y) \, dy.
\]
For the $j$-th component this means
\[
D^\nu u_j(x) = \sum_{k=1}^{N} \text{p.v.} \int_{\mathbb{R}^n} K_{j,k}^\nu(x, x-y) (\lambda + A) u(y)_k \, dy \\
+ \sum_{|\alpha|\leq m} \sum_{\ell=1}^{N} \text{p.v.} \int_{\mathbb{R}^n} K_{j,k}^\nu(x, x-y) (a_{k,\ell}^{(\alpha)}(x) - a_{k,\ell}^{(\alpha)}(y)) D^\alpha u_\ell(y) \, dy \\
+ \sum_{k=1}^{N} c_{j,k}^\nu((\lambda + A) u(x))_k \\
- \sum_{\ell=1}^{N} \sum_{k=1}^{N} \text{p.v.} \int_{\mathbb{R}^n} K_{k,\ell}^\nu(x, y-z) (\lambda + A) u(z)_\ell \, dz \, dy \\
- \sum_{|\alpha|\leq m} \sum_{r=1}^{N} \sum_{\ell=1}^{N} \sum_{k=1}^{N} \text{p.v.} \int_{\mathbb{R}^n} \lambda g_{j,k}^{\lambda,\nu} (x-y) \, dz \, dy \\
\cdot \text{p.v.} \int_{\mathbb{R}^n} K_{k,\ell}^\nu(x, y-z) (a_{k,\ell}^{(\alpha)}(x) - a_{k,\ell}^{(\alpha)}(z)) D^\alpha u_\ell(z) \, dz \, dy \\
- \sum_{\ell=1}^{N} \sum_{k=1}^{N} c_{j,k}^\nu(x) \int_{\mathbb{R}^n} \lambda g_{k,\ell}^{\lambda,\nu} (x-y) (\lambda + A) u(y)_\ell \, dy \\
- \sum_{|\alpha|\leq m} \sum_{r=1}^{N} \sum_{\ell=1}^{N} \sum_{k=1}^{N} c_{j,k}^\nu(x) \int_{\mathbb{R}^n} \lambda g_{k,\ell}^{\lambda,\nu} (x-y) (a_{k,\ell}^{(\alpha)}(x) - a_{k,\ell}^{(\alpha)}(y)) D^\alpha u_\ell(y) \, dy,
where $K^\nu_{j,k}(x, x-y)$ and $c^\nu_{j,k}(x)$ are the entries of the matrices $K_\nu(x, x-y)$ and $c_\nu(x)$ respectively.

In order to get a formal grip on this formula we introduce the following operators. Let $j, k, \ell \in \{1, \ldots, N\}, |\nu| = m, \lambda \in \Sigma_{\pi-\phi}$ and let $a \in \text{BMO}(\mathbb{R}^n)$ and $f \in L^p_\text{c}(\mathbb{R}^n)$. Then we define

$$T^1_{j,k,\nu}(x) = \text{p.v.} \int_{\mathbb{R}^n} K^\nu_{j,k}(x, x-y)f(y) \, dy,$$

$$[T^1_{j,k,\nu}, a]f(x) = \text{p.v.} \int_{\mathbb{R}^n} K^\nu_{j,k}(x, x-y)(a(x) - a(y))f(y) \, dy,$$

$$T^2_{j,k,\nu,\ell}(x, y) = \int_{\mathbb{R}^n} g^\nu_{j,k}(x, y) \text{p.v.} \int_{\mathbb{R}^n} K^\nu_{k,\ell}(y, z)f(z) \, dz \, dy,$$

$$[T^2_{j,k,\nu,\ell}, a](x, y) = \int_{\mathbb{R}^n} g^\nu_{j,k}(x, y) \text{p.v.} \int_{\mathbb{R}^n} K^\nu_{k,\ell}(y, z)(a(x) - a(z))f(z) \, dz \, dy.$$

Using this notation we finally write

$$D^\nu u_j(x) = \sum_{k=1}^N T^1_{j,k,\nu}((\lambda + A)u)_k(x) + \sum_{|\alpha| = m} \sum_{\ell=1}^N \sum_{k=1}^N \left[ T^1_{j,k,\nu}, a_{k,\ell}^{(\alpha)} \right] D^\alpha u_{k\ell}(x)$$

$$+ \sum_{k=1}^N c^\nu_{j,k}(x)((\lambda + A)u(x))_k - \sum_{\ell=1}^N \sum_{k=1}^N T^2_{j,k,\nu,\ell}((\lambda + A)u)_{k\ell}(x)$$

$$- \sum_{|\alpha| = m} \sum_{r=1}^N \sum_{k=1}^N \sum_{l=1}^N \left[ T^2_{j,k,\nu,\ell}, a_{k,\ell}^{(\alpha)} \right] D^\alpha u_r(x) - \sum_{|\alpha| = m} \sum_{r=1}^N \sum_{k=1}^N c^\nu_{j,k}(x) \left[ T^3_{j,k,\nu,\ell}^{3,\lambda,0}(\lambda + A)u)_{k\ell}(x) \right]$$

(5.2)

Note, that the operator $[T^2_{j,k,\nu,\ell}, a]$ is not a commutator. As we can handle it by the same methods as the commutators, we however use this notation.

We now prove the following mapping properties of the above operators.

**Lemma 5.3.** Let $j, k, \ell \in \{1, \ldots, N\}, |\nu| = m, |\mu| < m, \lambda \in \Sigma_{\pi-\phi}, a \in L^\infty(\mathbb{R}^n), 1 < p < \infty, \omega \in A_p$ and $f \in L^p_\text{c}(\mathbb{R}^n)$. Then

$$||T^1_{j,k,\nu}f||_{p,\omega} \leq C ||f||_{p,\omega},$$

$$||[T^1_{j,k,\nu}, a]f||_{p,\omega} \leq C ||a||_\infty ||f||_{p,\omega},$$

$$||T^2_{j,k,\nu,\ell}f||_{p,\omega} \leq C ||f||_{p,\omega},$$

$$||[T^2_{j,k,\nu,\ell}, a]f||_{p,\omega} \leq CA_*(1 + ||a||_\infty)||f||_{p,\omega},$$

$$||T^3_{j,k,\nu,\ell}f||_{p,\omega} \leq C ||f||_{p,\omega},$$

$$||[T^3_{j,k,\nu,\ell}, a]f||_{p,\omega} \leq CA_*(1 + ||a||_\infty)||f||_{p,\omega},$$

where

$$A_* := \max \left\{ ||a||_\infty, \max_{|\alpha| = m} \max_{r=1}^N ||a_{r,s}^{(\beta)}||_* \right\}.$$

**Proof.** By Lemma 5.2 the kernels $K^\nu_{j,k}$ fulfill the hypotheses of Proposition 4.3. Thus the assertion concerning the $T^1$-operators follows.

We turn to the operators containing $T^3$. Using (2.3), we have

$$||D^\mu \gamma^\nu(z)|| \leq C\lambda^{2p\frac{|z|}{m-1}} P_{m-1}(c\lambda^{\frac{1}{m}}|z|)$$

for all $|\mu| < m, \lambda \in \Sigma_{\pi-\phi}$ and $x, z \in \mathbb{R}^n$. Consequently, we have the same estimate for the entries $g^\nu_{j,k}$ and the boundedness of $T^3_{j,k,\nu,\ell}$ follows as in [HH03, Proposition 6.5].

Dealing with the commutator of $T^3_{j,k,\nu,\ell}$ with BMO-functions is much harder. We first show a point-wise estimate for the sharp function of $[T^3_{j,k,\nu,\ell}, a]f$. 


Let $I$ be an axis-parallel cube in $\mathbb{R}^n$ and $x \in I$. Then

$$\left[ T_{j,k}^{3,\lambda,\mu}, a \right] f(z) = \left( a(z) - a_I \right) T_{j,k}^{3,\lambda,\mu} f(z) - T_{j,k}^{3,\lambda,\mu} \left( (a - a_I) 1_{2I} f(z) \right) - T_{j,k}^{3,\lambda,\mu} \left( (a - a_I) 1_{(2I)^c} f(z) \right) =: A(z) + B(z) + C(z).$$

We estimate $A^\#(x)$, $B^\#(x)$, and $C^\#(x)$ separately. For $A$ and $B$ we choose $c = 0$ in (2.4) and look at $\int_I |A(z)| \, dz$ and $\int_I |B(z)| \, dz$.

The Hölder inequality yields for all $1 < r < \infty$

$$\int_I |A(z)| \, dz \leq C \left( \int_I |a(z) - a_I|^r \, dz \right)^{\frac{1}{r}} \left( \int_I |T_{j,k}^{3,\lambda,\mu} f(z)|^r \, dz \right)^{\frac{1}{r}}.$$

For the first factor we use the John Nirenberg inequality (2.5), and we estimate the second one by the supremum over all cubes that contain $x$, and thus by the maximal operator. This yields

$$\int_I |A(z)| \, dz \leq C ||a||_*(M|T_{j,k}^{3,\lambda,\mu} f|)^{\frac{1}{r}}(x).$$

For estimating $B^\#$, we first apply Jensen’s inequality:

$$\int_I |B(z)| \, dz \leq \left( \int_I |T_{j,k}^{3,\lambda,\mu} \left( (a - a_I) 1_{2I} f(z) \right)|^q \, dz \right)^{\frac{1}{q}} \leq \left( \frac{1}{|I|} \right)^{\frac{1}{q}} ||T_{j,k}^{3,\lambda,\mu} \left( (a - a_I) 1_{2I} f(z) \right)||_q,$$

where $1 < q < \infty$. Now we use the boundedness of $T_{j,k}^{3,\lambda,\mu}$ in $L^q(\mathbb{R}^n)$ and then again Hölder’s inequality as above. Doing so, we get for every $u \in (1, \infty)$ and every $1 < r = qu < \infty$:

$$\int_I |B(z)| \, dz \leq C \left( \frac{1}{|I|} \int_I |a(z) - a_I|^{qu} \, dz \right)^{\frac{1}{qu}} \left( \frac{1}{|I|} \int_I |f(z)|^{qu} \, dz \right)^{\frac{1}{qu}} \leq C ||a||_*(M|f|)^{\frac{1}{r}}(x),$$

where we again estimated the first factor by the John Nirenberg inequality and the second one by the maximal operator.

The third part $C^\#(x)$ is more involved. Here we have to use the structure of our kernels and we cannot choose $c = 0$ in (2.4). We define the averaged operator

$$A_I := \sum_{|a| = m} \int_I a(x) \, dx D^n_a,$$

which is a homogeneous differential operator with constant coefficients. Thus there exist suitable constants $\lambda_0, K \geq 0$, such that the operator $\lambda_0 + A_I$ is $(K, \phi_0/2)$-elliptic. By Proposition 4.5 we see, that $(\lambda + A_I)^{-1}$ is given by a kernel $K_{I, \lambda}$ for all $\lambda \in \lambda_0 + \Sigma_{\pi - \phi_0/2}$. Choosing $\lambda_0 \geq \lambda_0$ so big, that $\Sigma_{\pi - \phi_0} \setminus B(0, \lambda_0) \subseteq \lambda_0 + \Sigma_{\pi - \phi_0/2}$, we get for all $\phi > \rho_0$ and $\lambda \in \Sigma_{\pi - \phi}$ satisfying $|\lambda| > \lambda_0$ the estimate

$$||D^\beta K_{I, \lambda}(x)|| \leq C|\lambda|^{\frac{n-1}{n-|\beta|}} \int_{\mathbb{R}^n} \left| \sum_{|\beta| = m} c_\beta |\lambda|^{\frac{|\beta|}{m}} \right|,$$

as in (2.3). Thus for all these $\lambda$ and for all $|\beta| \leq m$ the function $\lambda^{1-|\beta|/m} D^\beta K_{I, \lambda}$ is a Calderón-Zygmund kernel and the associated operators are bounded in $L^p(\mathbb{R}^n)$ uniformly in $\lambda \in \Sigma_{\pi - \phi} \setminus B(0, \lambda_0)$.

With this preparations in hand, we now estimate $C^\#(x)$. To this end we choose the constant $c$ in (2.4) as

$$c := \int_{\mathbb{R}^n} \lambda^{1-\frac{1}{m}} D^\mu K_{I, \lambda}^{3,\mu}(x_I - y)(a(y) - a_I) 1_{(2I)^c}(y) f(y) \, dy,$$
where \( x_I \) is the center of the cube \( I \). Setting \( g(y) := (a(y) - a_I)1_{(2^I)^c}(y) f(y) \) we obtain

\[
\int_I |C(z) - c| \, dz = \int_I \left| \int_{\mathbb{R}^n} \lambda_1^{-1/2} D^\mu \left( g_{j,k}^{\lambda,\gamma}(z - y) - K_{j,k}^{\lambda,\gamma}(x_I - y) \right) g(y) \, dy \right| \, dz
\]

\[
= \int_I \left| \int_{\mathbb{R}^n} \lambda_1^{-1/2} D^\mu \left( g_{j,k}^{\lambda,\gamma}(z - y) - K_{j,k}^{\lambda,\gamma}(z - y) \right) g(y) \, dy \right| \, dz
\]

\[
+ \int_I \left| \int_{\mathbb{R}^n} \lambda_1^{-1/2} D^\mu \left( K_{j,k}^{\lambda,\gamma}(z - y) - K_{j,k}^{\lambda,\gamma}(x_I - y) \right) g(y) \, dy \right| \, dz
\]

\[
=: I_1 + I_2.
\]

Therefore we need information about the differences \( g_{j,k}^{\lambda,\gamma} - K_{j,k}^{\lambda,\gamma} \) and \( K_{j,k}^{\lambda,\gamma}(z - \cdot) - K_{j,k}^{\lambda,\gamma}(x_I - \cdot) \). To handle the first difference we calculate for the corresponding matrix-valued kernels \( \gamma_\lambda^\gamma \) and \( K_{\lambda}^\lambda \) and for every \( \varphi \in \mathcal{D}(\mathbb{R}^n)^N \)

\[
(\gamma_\lambda^\gamma - K_{\lambda}^\lambda) * \varphi = \mathcal{F}^{-1} \left( (\lambda + A_{\#}(z, \xi))^{-1} - (\lambda + A_{\#}((\xi))^{-1} \right) \mathcal{F} \varphi
\]

\[
= \mathcal{F}^{-1} (\lambda + A_{\#}(z, \xi))^{-1} (A_{\#}(\xi) - A_{\#}(z, \xi)) (\lambda + A_{\#}(\xi))^{-1} \mathcal{F} \varphi
\]

\[
= \sum_{|\beta|=m} \mathcal{F}^{-1} (\lambda + A_{\#}(z, \xi))^{-1} ((a_\beta)_I - a_\beta(z)) \xi_\beta (\lambda + A_{\#}(\xi))^{-1} \mathcal{F} \varphi
\]

\[
= \sum_{|\beta|=m} \gamma_\lambda^\gamma [((a_\beta)_I - a_\beta(z)) D^\beta K_{\lambda}^\lambda] * \varphi.
\]

For one component this identity means, that for every \( f \in \mathcal{D}(\mathbb{R}^n) \)

\[
\left[ \lambda_1^{-1/2} D^\mu (g_{j,k}^{\lambda,\gamma} - K_{j,k}^{\lambda,\gamma}) \right] * f(z)
\]

\[
= \sum_{|\beta|=m} \sum_{|\beta|=m} \sum_{|\beta|=m} ((a^{(\beta)}_{\ell,\ell})_I - a^{(\beta)}_{\ell,\ell}(z)) \int_{\mathbb{R}^n} \lambda_1^{-1/2} D^\mu g^{\lambda,\gamma}_{j,k}(z - y) \int_{\mathbb{R}^n} D^\beta K_{n,k}^{\lambda,\gamma}(y - u) f(u) \, du \, dy
\]

\[
=: \sum_{|\beta|=m} \sum_{|\beta|=m} \sum_{|\beta|=m} ((a^{(\beta)}_{\ell,\ell})_I - a^{(\beta)}_{\ell,\ell}(z)) S^{\beta,\lambda,\mu}_{j,k,\ell,\ell} f(z),
\]

where \( S^{\beta,\lambda,\mu}_{j,k,\ell,\ell} \) is a bounded operator on \( L^2(\mathbb{R}^n) \), as it is a composition of two convolutions with Calderón-Zygmund kernels. Propositions 4.2 and 4.5 even allow us to estimate the norm of this operator \( A_{\#} \)-consistently and independently of \( \lambda \).

Using this to estimate \( I_1 \), we get

\[
I_1 \leq \sum_{|\beta|=m} \sum_{|\beta|=m} \sum_{|\beta|=m} \left[ \int_I \left| (a^{(\beta)}_{\ell,\ell})_I - a^{(\beta)}_{\ell,\ell}(z) \right| \left| S^{\beta,\lambda,\mu}_{j,k,\ell,\ell} (a 1_{2^I}) f \right|(z) \, dz \right.
\]

\[
+ \left. \int_I \left| (a^{(\beta)}_{\ell,\ell})_I - a^{(\beta)}_{\ell,\ell}(z) \right| \left| S^{\beta,\lambda,\mu}_{j,k,\ell,\ell} (a 1_{2^I}) f \right|(z) \, dz \right] =: I_{11} + I_{12}.
\]

In order to treat \( I_{11} \) we write \( 1_{(2^I)^c} f = f - 1_{2^I} f \). The triangle inequality then yields for every \( 1 < r < \infty \)

\[
I_{11} \leq \sum_{|\beta|=m} \sum_{|\beta|=m} \sum_{|\beta|=m} \left[ \int_I \left| (a^{(\beta)}_{\ell,\ell})_I - a^{(\beta)}_{\ell,\ell}(z) \right| \left| S^{\beta,\lambda,\mu}_{j,k,\ell,\ell} (a f) \right|(z) \, dz \right.
\]

\[
+ \left. \int_I \left| (a^{(\beta)}_{\ell,\ell})_I - a^{(\beta)}_{\ell,\ell}(z) \right| \left| S^{\beta,\lambda,\mu}_{j,k,\ell,\ell} (a 1_{2^I}) f \right|(z) \, dz \right]
\]

\[
\leq C \sum_{|\beta|=m} \sum_{|\beta|=m} \sum_{|\beta|=m} \left( \left| (a^{(\beta)}_{\ell,\ell})_I - a^{(\beta)}_{\ell,\ell}(z) \right| \left| (M[S^{\beta,\lambda,\mu}_{j,k,\ell,\ell} (a f)]^r)^{1/2}(z) \right| + \|a\|_{\infty} (M[f]^{1/2})(z) \right),
\]
in the same way as we estimated \( \int |A(z)| \, dz \) and \( \int |B(z)| \, dz \). Analogously

\[
I_2 \leq C \sum_{|\beta|=m} \sum_{\ell=1}^{N} \|a(\beta)\|_\ast \left\| (M|S^3_{j,k,\ell,\alpha}f|)^{\frac{1}{2}}(z) + \|a\|_{\infty} (M|f|)^{\frac{1}{2}}(z) \right\|
\]

For \( I_2 \) we use that every entry of \( \lambda^{1-|m|/m} D^\mu K_{j,\lambda} \) is a Calderón-Zygmund kernel:

\[
I_2 = \int_I \left( \int_{\mathbb{R}^n} \lambda^{1-|m|/m} D^\mu \left( K_{j,k}^I(z-y) - K_{j,k}^I(x_I-y) \right) g(y) \, dy \right) \, dz
\]
\[
\leq C \int_I \left( \int_{\mathbb{R}^n \setminus \{x_I\}} \lambda^{1-|m|/m} D^\mu \left( K_{j,k}^I(z-y) - \lambda^{1-|m|/m} D^\mu K_{j,k}^I(x_I-y) \right) |a(y) - a_I|^r \, dy \right) \, dz
\]
\[
\leq C \int_I \left( \int_{\mathbb{R}^n \setminus \{x_I\}} \left| \frac{x_I - z}{|x_I - y|^{n+1}} \right| |a(y) - a_I|^r \, dy \right) \, dz
\]

This estimate is possible, as \( |x_I - y| > 2|x_I - z| \) is valid. Applying the Hölder inequality

\[
I_2 \leq C \int_I \left( \int_{\mathbb{R}^n \setminus \{x_I\}} \left| \frac{x_I - z}{|x_I - y|^{n+1}} \right| |a(y) - a_I|^r \, dy \right)^{\frac{1}{r}} \left( \int_{\mathbb{R}^n \setminus \{x_I\}} \left| \frac{x_I - z}{|x_I - y|^{n+1}} |f(y)|^r \, dy \right) \, dz
\]

follows for every \( 1 < r < \infty \). Denoting the side-length of \( I \) by \( d \), we estimate the first factor:

\[
\int_{\mathbb{R}^n \setminus \{x_I\}} \left| \frac{x_I - z}{|x_I - y|^{n+1}} \right| |a(y) - a_I|^r \, dy
\]
\[
= \sum_{\ell=2}^{\infty} \int_{2^{\ell}I \setminus 2^{\ell-1}I} \left| \frac{x_I - z}{|x_I - y|^{n+1}} \right| |a(y) - a_I|^r \, dy \leq \sum_{\ell=2}^{\infty} \frac{d}{(2^{\ell-2}d)^{n+1}} \int_{2^{\ell}I} |a(y) - a_I|^r \, dy
\]
\[
= 4^{n+1} \sum_{\ell=2}^{\infty} 2^{-\ell} \int_{2^{\ell}I} |a(y) - a_I|^r \, dy \leq C \sum_{\ell=2}^{\infty} 2^{-\ell} (\ell + 1)^r \|a\|_{\ast}^r \leq C \|a\|_{\ast}^r
\]

by (2.6) and since \( (2^{\ell-2}d)^{n+1} = 2^{\ell-4(\ell-1)}d^{2\ell I} \).

Analogously we get for the second factor

\[
\int_{\mathbb{R}^n \setminus \{x_I\}} \left| \frac{x_I - z}{|x_I - y|^{n+1}} |f(y)|^r \, dy \leq C \sum_{\ell=2}^{\infty} 2^{-\ell} \int_{2^{\ell}I} |f(y)|^r \, dy \leq CM|f|^r(x).
\]

Altogether we obtain for \( I_2 \)

\[
I_2 \leq C \|a\|_{\ast} (M|f|^r)^{\frac{1}{2}}(x).
\]

Combining this with the estimate for \( I_1 \) we see that

\[
\int_I |C(z) - c| \, dz \leq C \left[ \|a\|_{\ast} (M|f|^r)^{\frac{1}{2}}(x) + \sum_{|\beta|=m} \sum_{\ell=1}^{N} \sum_{s=1}^{N} \|a(\beta)\|_\ast \left( (M|S^3_{j,k,\ell,\alpha}f|)^{\frac{1}{2}}(x) + \|a\|_{\infty} (M|f|)^{\frac{1}{2}}(x) \right) \right].
\]
Collecting the results for $A$, $B$ and $C$, we have for every $1 < r < \infty$
\[
\left( T^{3,\lambda,\mu}_{j,k}, a \right) f \right) \leq A^\#(x) + B^\#(x) + C^\#(x)
\leq C||a||_\infty \left( ||M||T^{3,\lambda,\mu}_{j,k}f||_r \right)^{1/r}(x) + \left( ||M||^{\lambda,\mu}_{j,k}a \right) f \right) \leq C||a||_\infty \left( ||M||S^{3,\lambda,\mu}_{j,k,a}f||_r \right)^{1/r}(x) + \left( ||M||^{\lambda,\mu}_{j,k}a \right) f \right) \leq C||a||_\infty \left( ||M||S^{3,\lambda,\mu}_{j,k,a}f||_r \right)^{1/r}(x) + \left( ||M||^{\lambda,\mu}_{j,k}a \right) f \right)
\]
As in the third step of the proof of Proposition 5.4 in [HH03] there exists an $r > 1$, such that
\[
||\left( ||M||f\right)||^{1/r}_{p,\omega} \leq C||f||^{1/r}_{p,\omega}, \quad f \in L^p_\omega(\mathbb{R}^n),
\]
with an $A_{r'}$-consistent constant $C$. This and Lemma 2.2 now yield
\[
\left\| T^{3,\lambda,\mu}_{j,k}, a \right\| \leq C \left\| \left( T^{3,\lambda,\mu}_{j,k}, a \right) f \right\|^{1/r}_{p,\omega}
\leq CA \left\| T^{3,\lambda,\mu}_{j,k}f \right\|_{P,\omega} + \left\| f \right\|_{P,\omega}
\leq \left( \sum_{|\beta| = m} \sum_{\ell=1}^{N} \left\| T^{3,\lambda,\mu}_{j,k} S^{2,\lambda,\mu}_{j,k,a} f \right\|_{P,\omega} + \left\| f \right\|_{P,\omega} \right)^2 \leq CA \left( \left\| f \right\|_{P,\omega} + \left\| a \right\|_{\infty} \right) \left\| f \right\|_{P,\omega},
\]
where $C$ is an $A_{r'}$-consistent constant.

We finally show the boundedness of the operators $T^{2,\lambda,\nu}_{j,k}$ and $T^{2,\lambda,\nu}_{j,k,a}$. For this purpose, we use Proposition 4.4 to expand the kernel of the principal value integral in the definition of $T^{2,\lambda,\nu}_{j,k}$. This yields
\[
T^{2,\lambda,\nu}_{j,k} f(x) = \sum_{s=1}^{\infty} b^{s,k}_{j,\nu}(x) f(x) \quad \text{p.v.} \quad \int_{\mathbb{R}^n} \frac{Y_{\nu,k}(y)}{|y|^n} f(z) \, dz \, dy,
\]
where $b^{s,k}_{j,\nu}$ is a Calderón-Zygmund operator on $L^p_\omega(\mathbb{R}^n)$, whose norm can be bounded independently of $s$. Hence
\[
\left\| T^{2,\lambda,\nu}_{j,k} f \right\|_{P,\omega} \leq \left\| \sum_{s=1}^{\infty} b^{s,k}_{j,\nu} \left( T^{3,\lambda,0} R^{\nu,k}_{j,k} f \right) \right\|_{P,\omega}
\leq \left\| \sum_{s=1}^{\infty} \left| b^{s,k}_{j,\nu} \right| \right\|_{P,\omega} \left\| T^{3,\lambda,0} R^{\nu,k}_{j,k} f \right\|_{P,\omega} \leq C \sum_{s=1}^{\infty} \left| b^{s,k}_{j,\nu} \right| \left\| f \right\|_{P,\omega} \leq C \left| f \right|_{P,\omega}
\]
with an $A_{r'}$-consistent constant $C$.

In order to bound the operator $T^{2,\lambda,\nu}_{j,k,a}$, we write
\[
T^{2,\lambda,\nu}_{j,k,a} f(x) = \int_{\mathbb{R}^n} \lambda^{\nu,k}_{j,\nu}(x-y) f(z) \, dz \, dy
= \int_{\mathbb{R}^n} \lambda^{\nu,k}_{j,\nu}(x-y) f(z) \, dz \, dy
+ \int_{\mathbb{R}^n} \lambda^{\nu,k}_{j,\nu}(x-y) f(z) \, dz \, dy.
\]
Applying once more Proposition 4.4 to expand the kernel \( K_{\nu}^{\nu} \), we get
\[
[T_{j,k}^{2,\lambda}, q] f(x) = \lim_{s \to 0} \sum_{k=1}^{\infty} b_{s}^{\nu,k,\ell}(x) \left( [T_{j,k}^{3,\lambda,0}, q] R_{s}^{\nu,k,\ell} f(x) + \sum_{s=0}^{\infty} \left([R_{s}^{\nu,k,\ell}, q] f(x)\right) \right).
\]
As we have already bounded the \( T^{3} \)-operators, we finally conclude with the help of Propositions 4.3 and 4.4
\[
\left\| T_{j,k}^{2,\nu}, q \right\|_{p,\infty} \leq C \sum_{s=0}^{\infty} \left\| b_{s}^{\nu,k,\ell} \right\|_{p,\infty} (A_{s}(1 + ||q||_{p,\infty}||f||_{p,\infty}) + ||a||_{p,\infty}||f||_{p,\infty})
\]
\[
\leq C A_{s}(1 + ||q||_{p,\infty}||f||_{p,\infty}). \quad \square
\]

**Proof of Theorem 5.1:** For every \(|\nu| < m\) and all \( j = 1, \ldots, N \) we have by (5.1)
\[
\lambda^{1-\nu} D^{\nu} u_{j}(x) = \sum_{k=1}^{N} (T_{j,k}^{3,\nu}((\lambda + A)u)_{k})(x) + \sum_{\ell=1}^{N} \sum_{k=1}^{N} \left( [T_{j,k}^{3,\nu}, q_{k,\ell}] D^{\nu} u_{\ell}(x) \right).
\]
By the preceding lemma we see that
\[
\left\| \lambda^{1-\nu} D^{\nu} u_{j} \right\|_{p,\infty} \leq C \left( \sum_{k=1}^{N} \left\| ((\lambda + A)u)_{k} \right\|_{p,\infty} + \alpha_{*} \alpha_{\infty} \sum_{\ell=1}^{N} \sum_{k=1}^{N} \left\| D^{\nu} u_{\ell} \right\|_{p,\infty} \right),
\]
where
\[
\alpha_{*} := \max_{[a-m] k, \ell=1} \left\| q_{k,\ell}^{(a)} \right\|_{p,\infty} \quad \text{and} \quad \alpha_{\infty} := 1 + \max_{[a-m] k, \ell=1} \left\| q_{k,\ell}^{(a)} \right\|_{p,\infty}.
\]
Analogously we get by (5.2) for all \(|\nu| = m\) and all \( j = 1, \ldots, N \)
\[
\left\| D^{\nu} u_{j} \right\|_{p,\infty} \leq \left( \sum_{k=1}^{N} \left\| ((\lambda + A)u)_{k} \right\|_{p,\infty} + \alpha_{*} \alpha_{\infty} \sum_{k=1}^{N} \sum_{[a-m] k} \left\| D^{\nu} u_{k} \right\|_{p,\infty} \right)
\]
\[
+ \sum_{k=1}^{N} \left\| ((\lambda + A)u)_{k} \right\|_{p,\infty} + \alpha_{*} \alpha_{\infty} \sum_{k=1}^{N} \sum_{[a-m] k} \left\| D^{\nu} u_{k} \right\|_{p,\infty} \right).
\]
Combining these two estimates, we conclude that
\[
\sum_{|\nu| \leq m} \left\| \lambda^{1-\nu} D^{\nu} u_{j} \right\|_{p,\infty} \leq C \sum_{j=1}^{N} \sum_{|\nu| \leq m} \left\| \lambda^{1-\nu} D^{\nu} u_{j} \right\|_{p,\infty}
\]
\[
\leq C \left( \sum_{k=1}^{N} \left\| ((\lambda + A)u)_{k} \right\|_{p,\infty} + \alpha_{*} \alpha_{\infty} \sum_{k=1}^{N} \sum_{[a-m] k} \left\| D^{\nu} u_{k} \right\|_{p,\infty} \right)
\]
With an \( A_{p} \)-consistent constant \( C \). If \( \eta < (C_{A_{\infty}})^{-1} \), we have \( C_{A_{\infty}} \leq C_{\eta A_{\infty}} < 1 \). Thus we may bring the second term on the left hand side to the right hand side and therefore finish the proof. \( \square \)
6. Proof of the Main Result

Due to the a priori estimate given in the previous section, it remains to prove that the operator \( \lambda + A \) from \( W^{m,p}_\omega (\mathbb{R}^n) \) onto \( L^p_\omega (\mathbb{R}^n)^N \) is surjective. This follows from the continuity method in the same way as in [HH03]. We just refer to Theorem 6.7 in [HH03]. We thus have the following result.

**Proposition 6.1.** Let \( n \geq 2 \), \( 1 < p < \infty \), \( \phi_0 \in (0, \pi) \), \( \phi > \phi_0 \), \( \omega \in A_p \). Let \( A \) be as above. Then there are \( A_p \)-consistent constants \( \eta > 0 \) and \( \lambda_0 \geq 0 \), such that for every \( f \in L^p_\omega (\mathbb{R}^n)^N \) and every \( \lambda \in \Sigma_{\pi - \phi} \) there exists a unique function \( u \in W^{m,p}_\omega (\mathbb{R}^n)^N \) such that
\[
(\lambda + \lambda_0 + A)u = f,
\]
whenever \( \max_{|x| = r} \| a_x \| \leq \eta \).

**Corollary 6.2.** Under the assumptions of Proposition 6.1, there exist \( A_p \)-consistent constants \( C, \lambda \geq 0 \) such that
\[
\| (\lambda + \lambda_0 + A_{p,\omega})^{-1} \|_{p,\omega} \leq \frac{C}{|\lambda|} \quad \lambda \in \Sigma_{\pi - \phi}.
\]
If \( \phi_0 < \pi/2 \), then \( -(\lambda_0 + A_{p,\omega}) \) generates an analytic semigroup on \( L^p_\omega (\mathbb{R}^n) \). Moreover, \( \lambda_0 + A_{p,\omega} \) satisfies the estimate (1.2).

**Proof.** Theorem 5.1 implies that \( \lambda + \lambda_0 + A_{p,\omega} \) is one to one as well as the resolvent estimate. Surjectivity of \( \lambda + \lambda_0 + A_{p,\omega} \) follows from Proposition 6.1. The \( L^p-L^q \)-estimate is now a consequence of Proposition 2.1. \( \square \)

In order to apply a localization procedure to differential operators with VMO coefficients, we need the following lemma.

**Lemma 6.3.** Let \( f \in \text{VMO}(\mathbb{R}^n; \mathbb{C}^{N \times N}) \). Then for every \( \varepsilon > 0 \), there is a radius \( r = r(\varepsilon, f) > 0 \), such that for all \( x_0 \in \mathbb{R}^n \), there exists a function \( g \in \text{VMO}(\mathbb{R}^n; \mathbb{C}^{N \times N}) \) with \( \| g \|_\ast \leq \varepsilon \) and \( f(x) = g(x) \) for all \( x \in B(x_0, r) \).

**Proof.** The corresponding result for \( N = 1 \) is in [HH03, Lemma 3.2]. By this, for every entry \( f_{j,k} \), \( j, k = 1, \ldots, N \), of \( f \) we get a function \( g_{j,k} \) having the above properties. Setting \( g = (g_{j,k})_{j,k=1}^N \) yields the result. \( \square \)

**Proof of Theorem 3.1.** In order to apply the usual localization procedure, we have to define local operators \( A_j \), \( j \in N \), that are equal to \( A \) on balls \( B(x_j, r) \subseteq \mathbb{R}^n \), but whose coefficients have BMO norm smaller than \( \eta \), where \( \eta \) is the constant from Theorem 5.1.

In order to do so, we choose for every \( |a| = m \) a radius \( r_\alpha > 0 \) given by Lemma 6.3 with \( \varepsilon = \eta \). We set \( r := \min_{|a| = m} r_\alpha \). By Lemma 6.3 there exist functions \( a_{\alpha, j} \in \text{BMO}(\mathbb{R}^n; \mathbb{C}^{N \times N}) \) with \( \| a_{\alpha, j} \|_\ast \leq \eta \) and \( a_{\alpha, j} = a_\alpha \) on \( B(x_j, r) \). Taking \( a_{\alpha, j} \) as coefficients for suitable differential operators \( A_j \) the result follows by the localization method used in [AHS94] and [HH03]. \( \square \)

**References**


$L^p - L^q$-ESTIMATES FOR PARABOLIC SYSTEMS WITH VMO-COEFFICIENTS


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