DIRECT FINITENESS OF REPRESENTABLE REGULAR $*$-RINGS

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Dedicated to the memory of Susan M. Roddy

Abstract. For $*$-regular rings R of endomorphisms of inner product spaces we provide a condition on r sufficient for r being a unit. In contrast to the claim made in earlier versions of this note it remains open whether this condition can be used to establish direct finiteness of R.

0. Erratum

A long standing problem of Handelman's (cf. Problem 48 in [5]) asks whether all $*$-regular rings are directly finite. In [10, Thm.3.1] and, in more general context, in [11, Thm.4] direct finiteness has been claimed for representable $*$-regular rings, i.e. for regular subrings R of endomorphisms of inner product spaces (V_F, Φ) where a proper involution on R is given by $r \mapsto r^*$, the adjoint of r.

Though, what has been shown is only the following: Let $\mathcal{F}$ denote the set of finite dimensional subspaces U of V_F. Then r is a unit of R provided that $(*)$: For each $U \in \mathcal{F}$ there are $W \supseteq U$ in $\mathcal{F}$ and a unit α of $\mathsf{End}(W_F)$ such that $\alpha|U = r|U$ and $\alpha^*|U = r^*|U$.

In [10] it has been tacitly assumed that $(*)$ holds if $sr = 1$ for some $s \in R$. No proof has been given. Thus, the following claims remain without proof: Thm. 3.1 in [10], Thm. 4 and Cor. 6 in [11], Fact 5 and Thm. 7 in [12]. Lemma 5 in [11] is wrong: $(*)$ fails for the shift matrix r in the $*$-ring of row and column finite matrices over $\mathbb{R}$ (which is not regular, as this author learned from Fred Wehrung [17, Ex.8.19], cf. [13]) embedded (as ring) into $\mathsf{End}(V_{\mathbb{R}})$, $\dim V_{\mathbb{R}} = \aleph_0$.

To the best of this authors knowledge, the problem of direct finiteness remains open for the class of $*$-regular rings as well as for its subclass of representables and, more specially, of semi-artinians.

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1. Introduction

A $*$-ring, that is ring with involution, is called *finite* if $rr^* = 1$ implies $r^*r = 1$. This is a basic notion in the classification of von Neumann algebras; in particular, as shown by Murray and von Neumann, a finite von Neumann algebra admits a finite $*$-ring of quotients. This ring is also $*$-regular and *directly finite*, that is $rs = 1$ implies $sr = 1$. As Ara and Menal [1] have shown, any $*$-regular ring is at least finite, while direct finiteness remains an open question, as stated by Handelman [5, Problem 48]. The present note gives for $*$-regular rings R of endomorphisms of inner product spaces a condition in terms of the action of r on finite dimensional subspaces sufficient for r being a unit.

For $*$-rings, there is a natural and well established concept of [faithful] representation in a vector space V_F endowed with a non-degenerate orthosymmetric sesquilinear form: an embedding into the $*$-ring $\mathsf{End}^*(V_F)$ of those endomorphisms of V_F which admit an adjoint. Famous examples are due to Gel'fand-Naimark-Segal (C^*-algebras in Hilbert space) and Kaplansky (primitive $*$-rings with a minimal right ideal). For $*$-regular rings of classical quotients of finite Rickart C^*-algebras existence of representations has been established in [8], jointly with M. Semenova. N. Niemann [15, 6] has shown that a subdirectly irreducible $*$-regular ring is representable if and only its ortholattice lattice of principal right ideals is representable within the ortholattice of closed subspaces of some V_F.

According to joint work with Susan M. Roddy [7], representability of modular ortholattices is equivalent to membership in a variety generated by finite height members. Using ideas from Tyukavkin [16], the analogue for $*$-regular rings was obtained by F. Micol [14]. Here, we rely on the presentation given in [9]: A regular $*$-ring can be represented within V_F, respectively some ultrapower thereof, if and only if it can be obtained via formation of ultraproducts, regular $*$-subrings, and homomorphic images from the class of the $\mathsf{End}^*(U_F)$, U_F ranging over finite dimensional non-degenerate subspaces of V_F. In the setting of [9] we derive a condition on r sufficient for r being a unit in R. It remains open whether this can be used to establish direct finiteness.

Thanks are due to Ken Goodearl for the hint to reference [4].

2. Preliminaries

When mentioning rings, we always mean associative rings R with unit 1_R, considered as a constant. In any ring, if a has a *left inverse* x, that is $xa = 1$, then a is *left cancellable*, that is $ay = az$ implies $y = z$. R is *directly finite* if, for all $r, s \in R$, $sr = 1$ implies $rs = 1$. In

such ring, a has a left inverse if and only if a is a unit (and $x = a^{-1}$). The endomorphism ring $\mathsf{End}(V_F)$ of a vector space is directly finite if and only if $\dim V_F < \omega$. A $*$-*ring* is a ring endowed with an involution $r \mapsto r^*$; an element e of such a ring is a *projection*, if $e = e^2 = e^*$.

A ring R is [*von Neumann*] *regular* if for any $a \in R$, there is an element $x \in R$ such that $axa = a$; such an element is called a *quasi-inverse* of a. If, for all a, x can be chosen a unit, then R is *unit regular*. Examples of such are the $\mathsf{End}(V_F)$, $\dim V_F < \omega$. A detailed discussion of direct finiteness in regular rings is given in Goodearl [5]. A regular $*$-ring is $*$-*regular* if $xx^* = 0$ only for $x = 0$.

In a regular ring, any left cancellable a has a left inverse; indeed $axa = a$ implies $xa = 1$. If $xa = 1$ and $aua = a$ with a unit u then $a = u^{-1}$ and $x = a^{-1}$. It follows

Fact 1. *In a directly finite regular ring every left cancellable element is a unit – similarly on the right. Every unit regular ring is directly finite.*

Fact 2. *If R is a regular subring of the ring R' and $r \in R$ a unit in R' then r is a unit in R.*

We recall some basic concepts and facts from [9] (here, Λ can be taken the $*$-ring of integers). In the sequel, F will be a division ring endowed with an involution and V_F a [right] F-vector space of $\dim V_F > 1$ endowed with a non-degenerate sesquilinear form $\langle . \mid . \rangle$ which is *orthosymmetric*, that is $\langle v \mid w \rangle = 0$ iff $\langle w \mid v \rangle = 0$. Such space will be called *pre-hermitean* and denoted by V_F, too. Within such space, any endomorphism φ has at most one adjoint φ^*; and these φ form a subring of $\mathsf{End}(V_F)$ which is a $*$-ring $\mathsf{End}^*(V_F)$ under the involution $\varphi \mapsto \varphi^*$. For $\dim V_F < \omega$, $\mathsf{End}^*(V_F)$ contains all of $\mathsf{End}(V_F)$. A [faithful] *representation* of a $*$-ring R is an embedding of R into some $\mathsf{End}^*(V_F)$.

Consider a linear subspace U of V_F, $1 < \dim U_F < \omega$. With the induced sesquilinear form, U_F is pre-hermitean if and only if $V = U \oplus U^\perp$; in particular, there is a projection $\pi_U \in \mathsf{End}^*(V_F)$ such that $U = \mathsf{im}\,\pi_U$ and such that the inclusion map $\varepsilon_U : U \to V$ is the adjoint of π_U (here, considered as a map $V \to U$). We write in this case $U \in \mathbb{O}(V_F)$ and say that U is a *finite-dimensional orthogonal summand*. A crucial fact is that V_F is the directed union of the U_F, $U \in \mathbb{O}(V_F)$. Let $C(F)$ denote the center of F and, for $U \in \mathbb{O}(V_F)$,

$$\begin{aligned} B_U &= \{\varepsilon_U \varphi \pi_U + \lambda\,\mathsf{id}_V \mid \varphi \in \mathsf{End}^*(U_F),\ \lambda \in C(F)\} \\ &= \{\psi \in \mathsf{End}(V_F) \mid \psi(U) \subseteq U \ \&\ \exists \lambda \in C(F)\, \psi|U^\perp = \lambda\,\mathsf{id}_{U^\perp}\} \end{aligned}$$

(φ and ψ are related via $\varphi(v) = \psi(v) + \lambda v$). Thus, B_U is a $*$-subring of $\mathsf{End}^*(V_F)$ and embeds into $\mathsf{End}^*(W_F)$ for any $W \in \mathbb{O}(V_F)$, $U \subset W \neq U$.

In particular, B_U is directly finite. Moreover, B_U is unit regular; indeed, $\chi \in B_U$ is a unit quasi-inverse of $\psi \in B_U$ if $\chi|U$ is one of $\psi|U$ and $\chi|U^\perp = (\psi|U^\perp)^{-1}$ (considering these as endomorphisms of U and $U^\perp$, respectively). We put

$$\begin{aligned} J(V_F) &= \{\varphi \in \mathsf{End}^*(V_F) \mid \dim \mathsf{im}\, \varphi < \omega\} \\ \hat{J}(V_F) &= \{\varphi + \lambda\, \mathsf{id}_V \mid \varphi \in J(V_F),\, \lambda \in C(F)\} \end{aligned}$$

According to [9, Proposition 4.4] $\hat{J}(V_F)$ is a $*$-subring of $\mathsf{End}^*(V_F)$ and $J(V_F)$ is an ideal of $\mathsf{End}^*(V_F)$ closed under the involution. Also, the following holds.

(∗) For any finite $\Phi \subseteq J(V_F)$ there is $U \in \mathbb{O}(V_F)$ such that $\varphi = \pi_U\varphi = \varphi\pi_U$ for all $\varphi \in \Phi$.

Thus, $\hat{J}(V_F)$ is the directed union of the B_U, $U \in \mathbb{O}(V_F)$, whence unit-regular.

Lemma 3. *Every regular $*$-subring R of $\mathsf{End}^*(V_F)$ extends to a regular $*$-subring $\hat{R}$ of $\mathsf{End}^*(V_F)$ containing $\hat{J}(V_F)$ and such that $J(V_F)$ is an ideal of $\hat{R}$.*

Proof. $\{\lambda\varphi \mid \varphi \in R,\, \lambda \in C(F)\}$ is a regular $*$-subring R' of $\mathsf{End}^*(V_F)$ and [9, Proposition 4.5] applies to R'. □

Recall that a [faithful] representation of a $*$-ring R within a pre-hermitian space V_F is an embedding $\varepsilon\colon R \to \mathsf{End}^*(V_F)$. It is convenient to consider representations as unitary R-F-bimodules ${}_RV_F$ (where the action of R is given as $rv = \varepsilon(r)(v)$) with sesquilinear form on V_F; that is, a 3-sorted structure with sorts V, R, and F. Considering a $*$-subring A of R we may add a fourth sort, A, and the embedding map. to obtain $({}_RV_F; A)$. Any elementary extension $({}_{\tilde{R}}\tilde{V}_{\tilde{F}}; \tilde{A})$ is again such a structure, that is, a representation of $\tilde{R}$ and a $*$-ring $\tilde{A}$ which may be considered as $*$-subring of $\tilde{R}$. It is a *modestly saturated* extension if, for each set $\Sigma(\bar{x})$ of first order formulas in finitely many [sorted] variables and with parameters from $({}_RV_F; A)$, one has $({}_{\tilde{R}}\tilde{V}_{\tilde{F}}; \tilde{A}) \models \exists\bar{x}.\Sigma(\bar{x})$, provided that $\Sigma(\bar{x})$ is *finitely realized* in $({}_RV_F; A)$, that is $({}_RV_F; A) \models \exists\bar{x}.\Psi(\bar{x})$ for every finite subset $\Psi(\bar{x})$ of $\Sigma(\bar{x})$. Such extension always exists, cf. [2, Corollary 4.3.1.4].

3. A sufficient condition for invertibility

Proposition 4. *Let R be a regular $*$-ring represented within the pre-hermitean space V_F. Then $r \in R$ is a unit in R provided that the following holds*

(**) *For each* $U \in \mathbb{O}(V_F)$ *there are* $W \in \mathbb{O}(V_F)$ *and a unit* α *of* $\mathsf{End}(W_F)$ *such that* $\alpha|U = r|U$ *and* $\alpha^*|U = r^*|U$.

Proof. We recall the relevant steps of the proof of [9, Theorem 10.1]. Given a representation ${}_RV_F$ of the regular $*$-ring R, we may assume that $\dim V_F \geqslant \omega$. In view of Proposition 3 and Fact 2, we also may assume that R is a $*$-subring of $\mathsf{End}^*(V_F)$ containing $A = \hat{J}(V_F)$ and having ideal $J(V_F)$. Choose ${}_{\tilde{R}}\tilde{V}_{\tilde{F}}; \tilde{A})$ a modestly saturated elementary extension of $({}_RV_F; A)$.

Let J_0 denote the set of projections in $J(V_F)$. For $a \in \tilde{A}$ and $r \in R$, we put $a \sim r$ if $ae = re$ and $a^*e = r^*e$ for all $e \in J_0$. According to Claims 1–4 in the proof of [9, Theorem 10.1], $S = \{a \in \hat{A} \mid a \sim r \text{ for some } r \in R\}$ is a regular $*$-subring of $\hat{A}$ and there is a surjective homomorphism $g : S \to R$ such that $g(a) = r$ if and only if $a \sim r$. Being an elementary extension of A, $\tilde{A}$ is directly finite and so is its subring S.

Now, consider a finite set $E \subseteq J_0$. According to $(*)$, there is $e \in J_0$ such that $ef = f$ and $er^*f = r^*f$ for all $f \in E$. Take $a = re$ and observe that $af = ref = rf$ and $a^*f = er^*f = r^*f$ for all $f \in E$ (actually, this follows by the proof of Linells's Theorem [4]).

Thus, by $(**)$ the set

$$\Sigma(x) = \big\{[xe = re] \;\&.\; [x^*e = r^*e] \& [\exists y.\, yx = 1 = xy] \mid e \in J_0\big\}$$

of formulas with variables x, y of type A is finitely realized in $({}_{\tilde{R}}\tilde{V}_{\tilde{F}}; \tilde{A})$. By saturation, there are $a, b \in \tilde{A}$ $ba = 1 = ba$ and $a \sim r$, whence $a \in S$ and $g(a) = r$. By Fact 2, a is a unit in S. Moreover, a is left cancellable in $\tilde{R}$ whence in the subring S. Hence, $r = g(a)$ is a unit of R. □

Problem 1. *Is* $(**)$ *also necessary for* r *being a unit in the represented* $*$*-regular ring* R*?*

An example of a simple regular $*$-ring which is not finite is obtained as follows: Let V_F a vector space of countably infinite dimension, and $R = \mathsf{End}(V_F)/J(V_F)$. Of course, R is not directly finite. Define the involution on the direct product $R \times R^{op}$ by exchange: $(r, s)^* = (s, r)$ to obtain the $*$-ring S. Now, if $rs = 1$ but $sr \neq 1$ then $xx^* = 1$ but $x^*x \neq 1$ in S for $x = (r, s)$.

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