Lattice Drawings and Morphisms

Vincent Duquenne

Equipe Combinatoire et Optimisation,
CNRS & Université Pierre et Marie Curie,
175 rue du Chevaleret, 75013 Paris, France
duquenne@math.jussieu.fr

Abstract. Let L → H be a lattice homomorphism and let a "readable" drawing of H be given. It is natural to make use of it to try getting a clear(er) drawing of L. Hence, the following question is explored: How the knowledge of the congruence lattice Con(L) of L can help in getting "better" drawings of L? This will be done by proposing rank shelling procedures of (M(Con(L),≤) and will be illustrated with examples coming either from math. or social sciences.

Keywords: Lattice drawing, lattice homomorphism, congruence lattice.

1 Introduction

Since the very beginning of FCA ("Formal Concept Analysis" see [20], [16]) and more generally of the use of *lattices* (and *orders*) in Data Analysis (see [6], [10] and the applications quoted therein), the question of getting "good" / "readable" lattice drawings has been crucial. Whenever one sustains the claim that lattices are instrumental to decipher structures (*associations*, *implications*) extracted from data, better get understandable structures. Otherwise their promotion will be … hazardous.

A main evident difficulty, the lattice size can explode exponentially (regarding its input). A second one comes from the arbitrariness of "observed" lattices, so that the methods which are adapted for highly structured lattices encountered in math. cannot be of help. Here, we are dealing with arbitrary finite lattices. Besides this, defining precisely what could be a "good" drawing is not that easy, and will never receive a unique answer. It depends on the data nature and size, on its degree of structuring, and these questions are certainly context sensitive (depending on scientific contexts etc).

Now, this need for lattice drawings has been addressed specifically and this in two directions. Recently in the way of *Automated Lattice Drawing* (see [14]) by using *attraction / repulsion forces*, but all along the development of FCA, by using much more structural approaches based on several *lattice decompositions* (see [23], some of them already present in [20]). This structural attitude has been followed in developing our program GLAD (see [9]) in which *nested line diagrams* ([16] p. 75) and *gluing decompositions* ([22], [16] p. 195) have been implemented since the mid-eighties. This is also the direction followed here, with the proposal of new iterated *shelling decompositions* which are dependent on the global structure of the *congruence lattice*.

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2 Congruence Emergency Toolkit

From classical sources (for Lattice Theory see [2], [17] and FCA see [16]), we will shortly extract a minimal set of notions and properties that are required in the sequel.

Homomorphisms are "structure preserving maps between algebraic structures". Hence for lattices, they should preserve the two operations and the order relation. For two lattices $(L_1,\leq,\vee,\wedge)$ and $(L_2,\leq,\vee,\wedge)$, $\varphi:L_1 \to L_2$ is a (*lattice*) *homorphism* iff

$\phi(x\wedge y)=\phi(x)\wedge\phi(y)$ and $\phi(x\vee y)=\phi(x)\vee\phi(y)$ hold for all $x,y\in L_1$.

Through $x\leq y \Leftrightarrow x=x\wedge y\Rightarrow\phi(x)=\phi(x)\wedge\phi(y)\Rightarrow \phi(x)\leq\phi(y)$, ϕ must actually be *isotone*.

A lattice homomorphism $\varphi:L_1 \to L_2$ quotients L_1 by a *congruence* Φ defined by $x\Phi y$ iff $\phi(x)=\phi(y)$ for $x,y\in L_1$. Φ is an equivalence on L_1 that respects the *substitution properties*: $x\Phi y$ and $u\Phi v$ implies $x\wedge u\Phi y\wedge v$ and $x\vee u\Phi y\vee v$, for $x,y,u,v\in L_1$. Now let $x\Phi y$, $x\Phi x$ and $x\Phi y \Rightarrow x=x\wedge x\Phi x\wedge y$, and similarly $x\Phi x\vee y$ holds, so that $x\Phi y$ implies $x\wedge y\Phi x\vee y$, and the Φ-classes are *convex intervals* of L_1. x/Φ denotes the class of x.

Surjective homomorphisms and congruence relations express the two sides of a single phenomenon -depending on which side of the arrow the focus is- which is made precise through the so-called *homomorphism theorems* (see [2] and [17] p. 16): for a surjective homomorphism $\varphi:L_1 \to L_2$, the *quotient lattice* $L/\Phi:=(x/\Phi \mid x\in L)$ is isomorphic to L_2 ; of particular importance here as it gives to drawings a "coherence of inheritance": for a congruence Θ on a lattice L, the congruence lattice of $L/\Theta:=(x/\Theta \mid x\in L)$ is isomorphic with the interval $[\Theta,1]$ of the congruence lattice of L.

Let Con(L) be the congruence lattice of a lattice L, ordered by set inclusion (as subsets of $L\times L$). Con(L) is a 01-sublattice of the partition lattice Part(L). Moreover (see [15]) Con(L) is *distributive* (quite a strong property among other algebras).

When Con(L)={0,1} L is called *simple*. When Con(L) has a single atom, its zero element (identity) cannot be expressed as the meet of others, in which case L is called *subdirectly irreducible*. Otherwise, when there are several atoms $\Theta_j(j\in J)$ in Con(L), thanks to distributivity, the identity can be expressed as its *minimal expression in meet-irreducible elements* (meet of all the meet-irreducible congruences that are *perspective* to the $\Theta_j(j\in J)$ in Con(L)). In this case, L is called *subdirectly reducible* and is a *subdirect product* of the irreducible *factors* $L/\Theta_j(j\in J)$, as it can be identified as a *sublattice* of their product which projects surjectively onto the factors $L/\Theta_j(j\in J)$.

Some notations will be needed for dealing concretely with congruences. Let L be a lattice, J:=J(L) and M:=M(L) its sets of *join-* and *meet-irreducible* elements, and $(J,M,\leq)$ be its *standard context* (table). For $j\in J$ and $m\in M$, $j\uparrow m$ means that m is maximal not above j, dually $m\downarrow j$ that j is minimal not below m, in which case they are *weakly perspective*. A pair of elements in $J\cup M$ is *weakly projective* if there exists a path of alternating weak perspectivities between them (ex: $j_1\uparrow m_1\downarrow j_2\uparrow m_2\downarrow j_3\uparrow m_3$...).

How to calculate Con(L)? (see [21] p. 280, [14] and http://www.latdraw.org/ for a program). For $m\in M$ let $m/\approx$ be its *weakly projective closure* in $J\cup M$. $(M(\mathrm{Con}(L)),\leq)$ is isomorphic to the ordered set (by dual inclusion) of all weakly projective closure of $m\in M$ (in words, this is just the consequence of "heredity of not collapsing").

Finally, congruences on a lattice L are represented into the Hasse diagram of L (representing its *cover relation* obtained by *transitive reduction* of $(L,\leq)$, while the latter is reconstructed by *transitive closure*). Lattice homomorphism corresponds to homomorphism of the covering relation, hence their importance here for diagrams.

3 On Frattini Congruences

As a motivating example let consider the *permutohedron* Perm(4), that is the set of all *permutations* on {1,2,3,4} rooted at 1234 (see Fig. 1), and ordered by *transposing* adjacent elements (other orders are sometimes considered). The structure of Perm(n) was extensively studied (see [12], with subsequent papers in literature on *weak orders* and *multinomial lattices*) up to characterizing iteratively its *standard context* (J(Perm(n)),M(Perm(n)),≤) out of (J(Perm(n-1)),M(Perm(n-1)),≤) through a simple copy-and-paste procedure, which could also iteratively characterize the *arrow-graph* expressing the *weak-perspectivities* between its join- and meet-irreducible elements.

This permitted the construction of the congruence lattice Con(Perm(4)) (see Fig. 2) through a characterization of its ordered set of meet-irreducible elements / congruences (M(Con(Perm(4))),≤), thanks to distributivity. From its structure it could be derived that it has 2**(n-2)=4 minimal elements (see [12], p. 80) so that Perm(4) is a subdirect product with as many hence four irreducible factors. In Fig. 2, these minimal meet-irreducible congruences are labelled by "I","K","M", and "O". Consequently, Perm(4) has four subdirectly irreducible factors that are defined by the *order filter* that they generate in (M(Con(Perm(4))),≤), namely:

IEJCFL, MENCFL, KGJCFL and OGNCFL, respectively.

CLAIM 1. The notion of subdirect product is clear and neat (but requires training since one must characterize which is the sublattice of the product see [21]). The uniqueness of … "subdirect product of irreducible factors" provides some reassuring canonicity. But despite this, in such situations as above where the factors of Perm(4) share a lot of attributes ("C","F","L" globally and more pairwise), they will not help in obtaining "understandable" drawings of Perm(4), since the product of factors becomes so big compared with Perm(4) (4×6=24 / 11 meet-irreducibles in Perm(4)).

Here we will call *Frattini congruence* of a lattice the intersection of its *coatomistic* (lower covers of the unit) congruences, just like a *Frattini sublattice* (see [18]) of a lattice is the intersection of its maximal proper sublattices (beware that in literature another definition involving congruences on sublattices was also considered).

The interest of our Frattini congruences will be better understood if we first examine under which conditions on L its congruence lattice Con(L) is Boolean, that is –being already distributive- when its Frattini congruence equals identity in Part(L). This is the case: iff the weak-projectivity relation on J(L)∪M(L) is symmetric, iff the arrow-graph of weak-perspectivities on J(L)∪M(L) is the disjoint union of strongly connected components, iff L is a subdirect product of simple factors. These lattices are called *weakly-modular* and comprise *modular*, hence *distributive*, and *relatively complemented* lattices which do have Boolean congruence lattices.

These above conditions express that for such a lattice L, M(L) and J(L) are mutually splitted into blocks of pairwise symmetrically weakly-projective elements, and that otherwise any pair of elements belonging to two distinct blocks are not weakly-projective (nor –perspective). Here is a harsh contrast, between what we will call *full symmetric wp-dependence* (within the blocks) and *full wp-independence*.

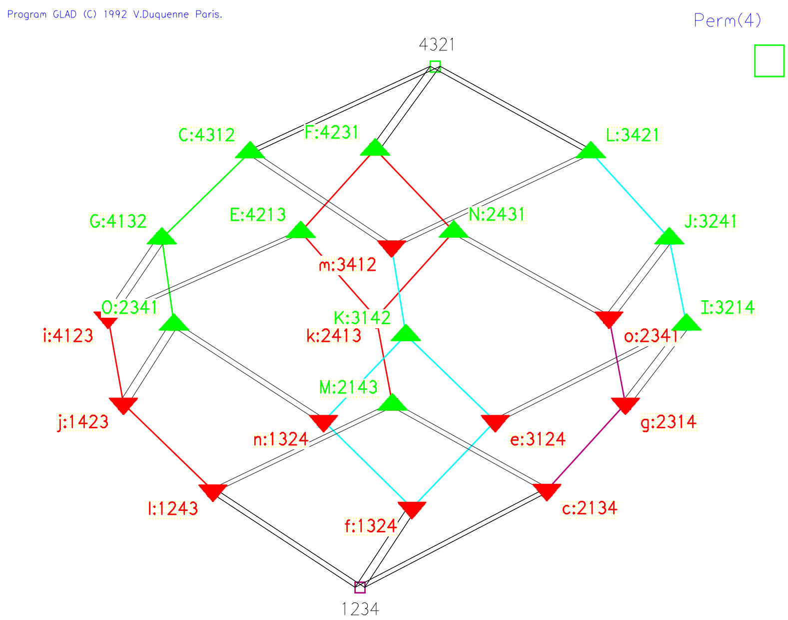

Fig. 1. Decomposition of the permutohedron Perm(4) by its Frattini / Glivenko congruence (congruence classes are colored with single lines, while they are connected with double lines)

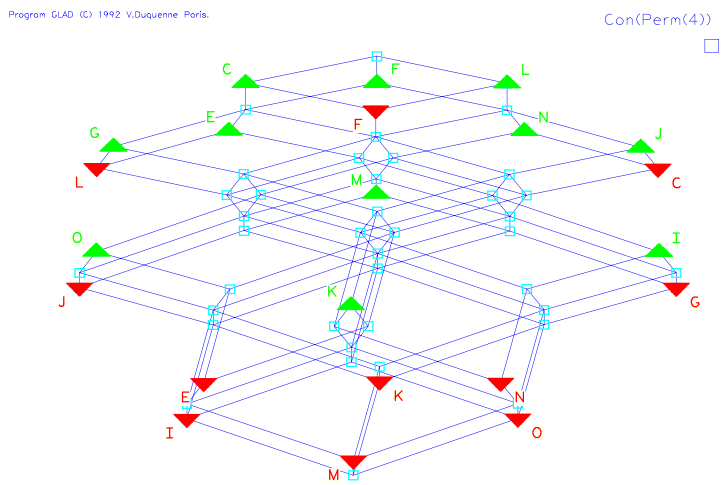

Fig. 2. The congruence lattice of the permutohedron Perm(4) was initially (see [12]) calculated and derived through characterizing of the set of its meet-irreducible elements: (M(Perm(4)),≤)

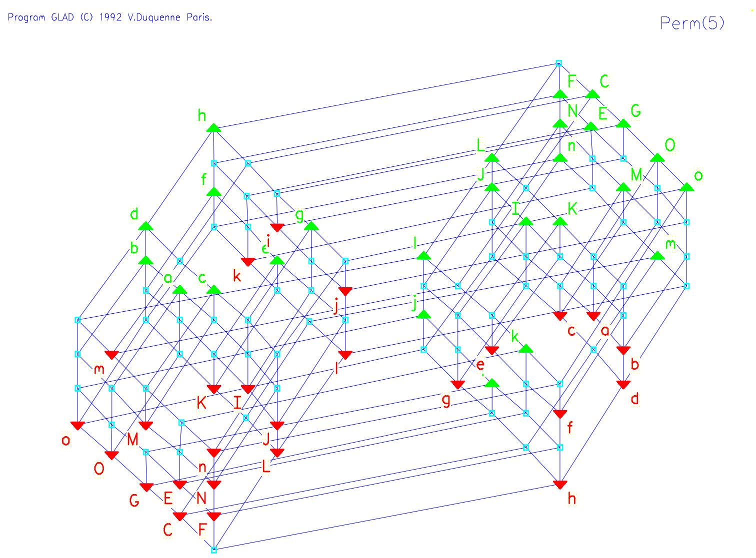

Fig. 3. The permutohedron Perm(5) (order-) embedded into a product of chains: 5x4x3x2

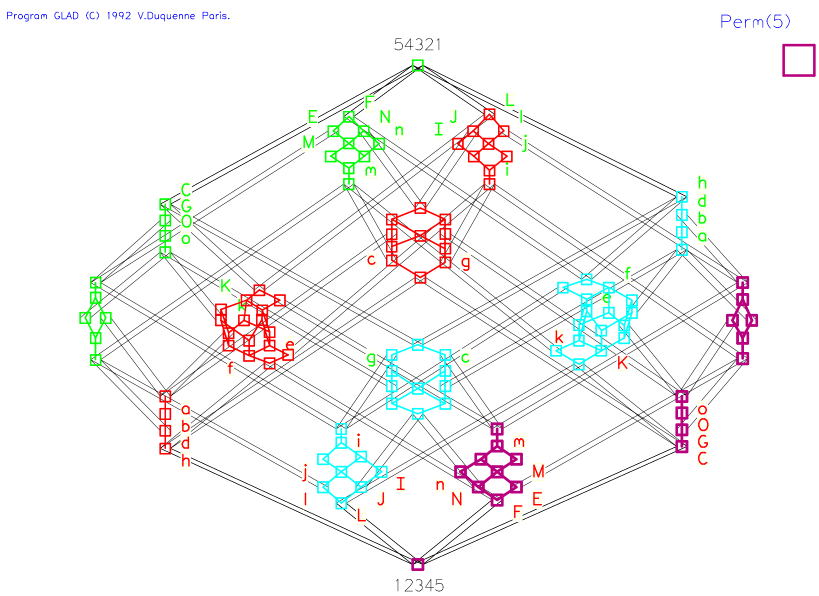

Fig. 4. The permutohedron Perm(5) quotiented by its Frattini / Glivenko congruence

CLAIM 2. Considering the Frattini congruence Φ_L of an arbitrary L will similarly identify those meet / join-irreducible elements –think of objects / attributes in data analysis- that are fully wp-dependent / independent. Moreover Considering L/Φ_L , and drawing L according to an "understandable" drawing of L/Φ_L would clarify L.

Coming back to our motivating example Perm(4), let denote by Φ its Frattini congruence (also known as *Glivenko congruence* for such *complemented* lattice and interpreted as "sharing the same complement"). Φ(Perm(4)) is a Boolean lattice 2**3. Fig. 1 provides quite a clarified drawing of Perm(4)/Φ along this Boolean lattice (compare with [14] p. 127 which is very similar in shape although it was obtained through a completely different à la *spring embedder* method). Similarly, Fig. 3 provides an (order) embedding of Perm(5) into a product of chains (compare with [16] p. 56), while Fig. 4 gives a more symmetric view of Perm(5) through the decomposition by its Frattini congruence into a Boolean 2**4 quotient lattice.

It may seem reasonable that this method can clarify lattices coming from math, just by respecting the symmetries like the left-right symmetry of the rooted Perm(n). But dealing with arbitrary lattices coming out of FCA & data analysis, a real question is: Will such decompositions by Frattini congruences clarify lattices coming from data?

A first illustration coming from social networks is given in Fig. 5. The original study (see [19]) focus on global structures of overlapping relations among Brazilian youth organizations and their projects, during the so-called "1992 impeachment movement". The data is a small 29×8 *context* (binary relation / matrix). Let us call for short L its concept (/ Galois) lattice. Its congruence lattice Con(L) (see Fig. 6) shows that L is subdirectly irreducible, and that its unique atom is equal to its Frattini congruence Φ_L. The quotient lattice L/Φ_L is isomorphic to a 2×3 product of chains, while interestingly the five remaining attributes that are collapsed by Φ_L belong to the top class: the complex structure of wp-dependency among these attributes is simply unfolded downwards by the three wp-independent attributes "C","F","G" into a quite simple distributive product of chains. Of course, it would be interesting that the specialists could interpret these facts regarding their knowledge on associations.

A second application comes from the pedagogy of mathematics (see [4]) and is represented in Fig. 8. The data was previously analyzed through *implications* and *gluing decompositions* (see [11]). The congruence lattice (see Fig. 7) is just a bit more complex, with the grafting a three element chain below the Frattini congruence. Here again as before L/Φ_L is a distributive lattice generating a clear split between wp-independent / dependent attributes and simplifying drastically the drawing.

A third example comes from a study on partial Down syndrome in genetics (see [5] and [13]), where the data describes young patients in terms of triplication of genetic bands in Chromosome 21. The quotient lattice L/Φ_L is again distributive, and interestingly the attributes which are collapsed by Φ_L belong to the top class that corresponds to the center of Chromosome 21, which is unfolded at left / right, with a bifurcation for the latter, generating the three dimensional quotient lattice.

Hence, Frattini congruences Φ_L help in clarifying these three small examples, but they share that Φ_L are not far away from the identity in their congruence lattices, and all quotient lattices are distributive. What can be done with more complex examples?

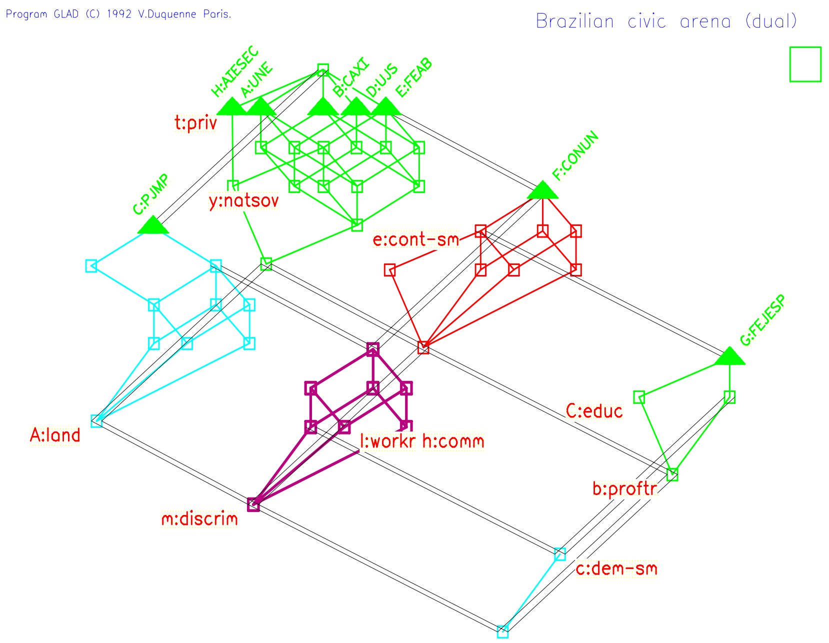

Fig. 5. A lattice coming from a study (see [19]) on Brazilian youth (organizations $\times$ projects). Its Frattini congruence quotients the lattice onto a simple distributive 2×3 product of chains, by collapsing together five pairwise weakly-projective attributes within the top class

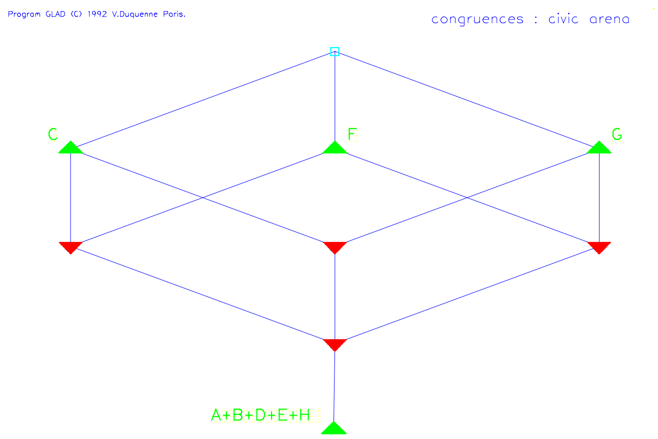

Fig. 6. Its congruence lattice splits three "independent" / five weakly-projective attributes

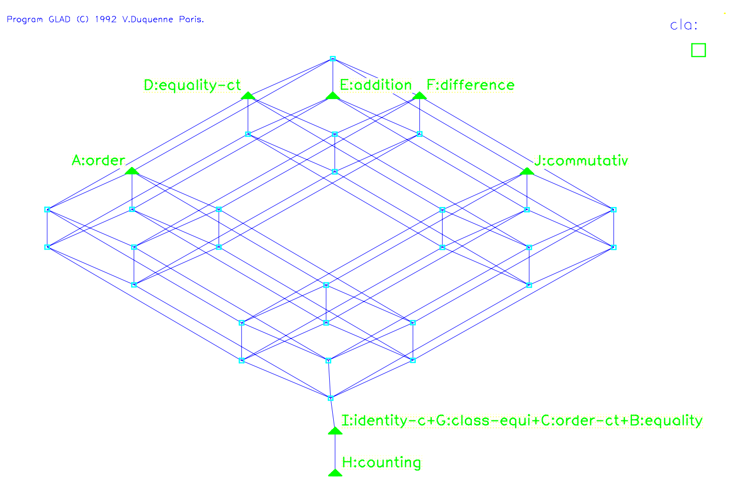

Fig. 7. A congruence lattice from a study ([4,11]) on math. teaching. The data describes whether children master properties of natural numbers and operations (addition, counting…). The aim is to evaluate implications between them, and locate the children in the lattice.

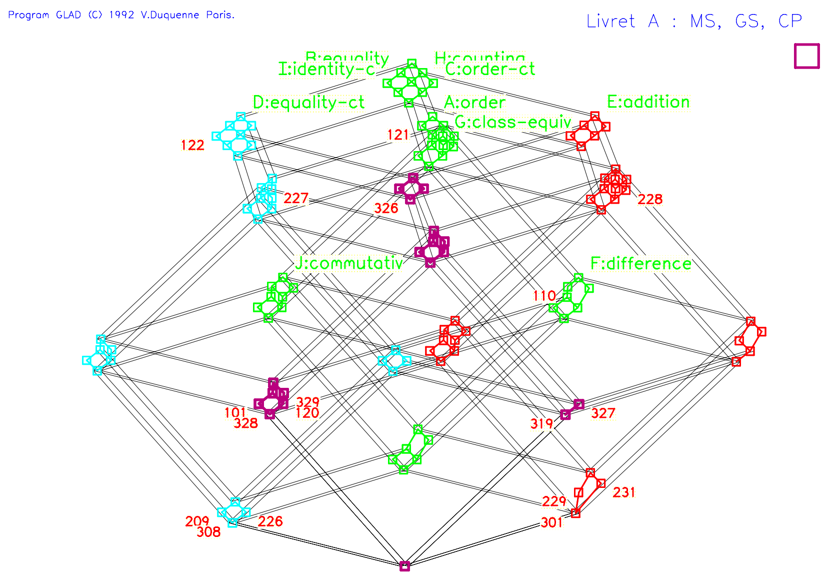

Fig. 8. The Frattini congruence generates a similar split between attributes and quotients into a distributive lattice that therefore reveals a simple hierarchy between "independent" attributes

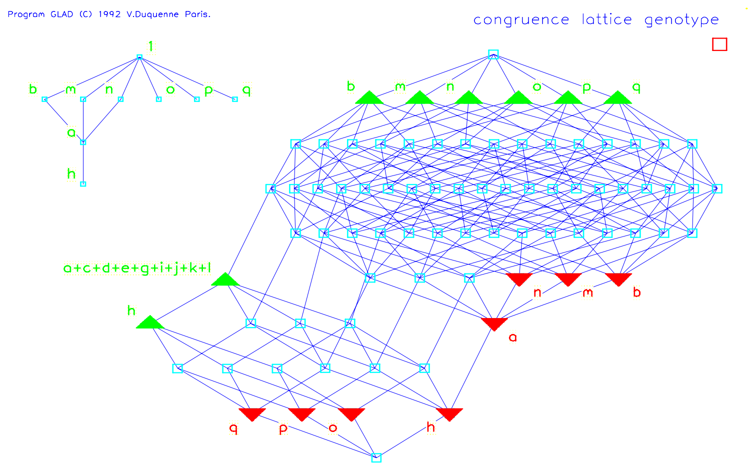

Fig. 9. A congruence lattice coming from genetics, with its meet irreducible elements. The data describes patients in terms of triplication of genetic bands in Chromosome 21 (see [5,13]).

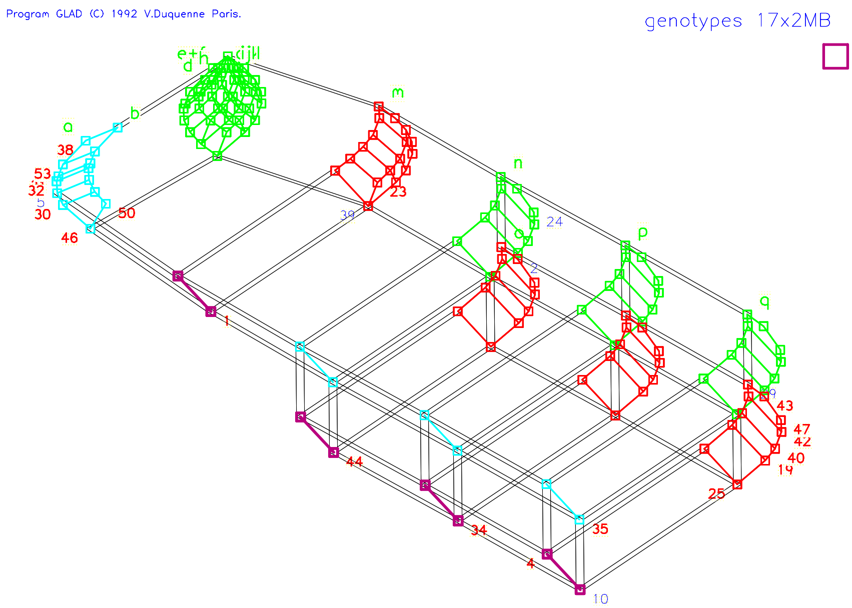

Fig. 10. Its Frattini congruence collapses the left “a” & center “cd…kl” of chromosone 21 within the top class while remaining bands unfold & project the lattice on a distributive factor

4 Nested Congruences and Diagrams

Our motivating example will this time come from math. and even closer from "FCA on math". In his paper on "subdirect product decomposition of concept lattices" (see [21]) R. Wille gave a nice drawing from an example describing homomorphisms of partial algebras by abstract properties. It was imitated in Fig. 11. It could be said that the drawing of this lattice –say L- is clear, constructed under the control of $(M(L),\leq)$ by applying the parallel law of *additive line diagrams* (see [16] p. 76).

This lattice was certainly chosen because it is a subdirect product of four irreducible factors which are not too complex, and is reconstructed by a kind of fusion (see [21] p. 229) of their *scaffoldings* which are some kind of distinguished generating subsets (for the notion of *partial sub-semilattice*). We bettered this by showing that it is also a same kind of fusion of the (*minimal* for this kind of construction through partial subsemilattices) factor *meet-cores* (see [7] , [8] Th. 4 p. 137).

CLAIM 3. If these (re)constructions of a subdirect product L are clear on an abstract level and can be programmed, they don't help for clarifying the drawing of L because the product of the factors is generally big, and these fusion procedures are complex (see [7] p. 229-233) since carrying all the projections of L onto the factors.

To try going a bit further, Con(L) and $(M(Con(L)),\leq)$ have been represented in Fig. 12. $(M(Con(L)),\leq)$ has four minimal elements, as expected, ("j","g","h","f"). The Frattini congruence Φ_L generates a quotient lattice L/Φ_L (see Fig. 13) which is Boolean and meet-generated along maximal elements of $M(Con(L)),\leq)$: "a","b","d". Now, seven attributes are left, so that in such situation where Con(L) is far away from being Boolean, we propose to reiterate the process along a "shelling procedure".

For M:=M(L), let $(M, \geqq)$ be the preorder induced on M by the *weakly projective closure*: $m_1 \geqq m_2 \Leftrightarrow m_1/\approx \supseteq m_2/\approx$. The equivalence classes of this preorder are ordered isomorphically to $(M(Con(L)),\geq)$. Let us call in short $M_R:=\{M_r \ / \ r=1,2,\ldots\}$ *shelling partition* of M that is obtained by a "*rank down shelling*" of $(M(Con(L)),\geq)$:

$$M_1=\max(M, \geqq),\ M_2=\max(M \setminus M_1, \geqq),\ldots,M_r=\max(M \setminus \cup\{M_s/s=1,r\text{-}1\}, \geqq).$$

CLAIM 4. The "*rank down shelling*" of $(M(Con(L)),\geq)$ procedure leads to define:
- *A canonical representation of L*, by drawings which refine a drawing of the Frattini quotient L/Φ_L by refining the classes iteratively along the shelling partition M_R.
- *A canonical presentation of the "standard context"* $(J(L),M(L),\leq)$ of L (see Fig. 15) with the *arrow graph* of weak perspectivities, which is quotiented into strongly connected components by the map $M(L) \to M(Con(L))$. The elements $m\in M$ are ordered from left to right along the shelling partition M_R, and J(L) and M(L) are permuted so that the strongly connected components be connected into the context.
The claim is that these (re)presentations put what is clear (*full wp-independence / dependence*…) in front line, and nest the remaining complexity in a canonical order.

Such a "canonical" drawing is represented in Fig. 13. To get an even more readable drawing, we erased in Fig. 14 the nested boxes –as is often done for "nested line diagrams"-, and distinguish / reconstruct the three shelling levels by using three different distances between attributes $m\in M$ and their unique upper covers m^* in L.

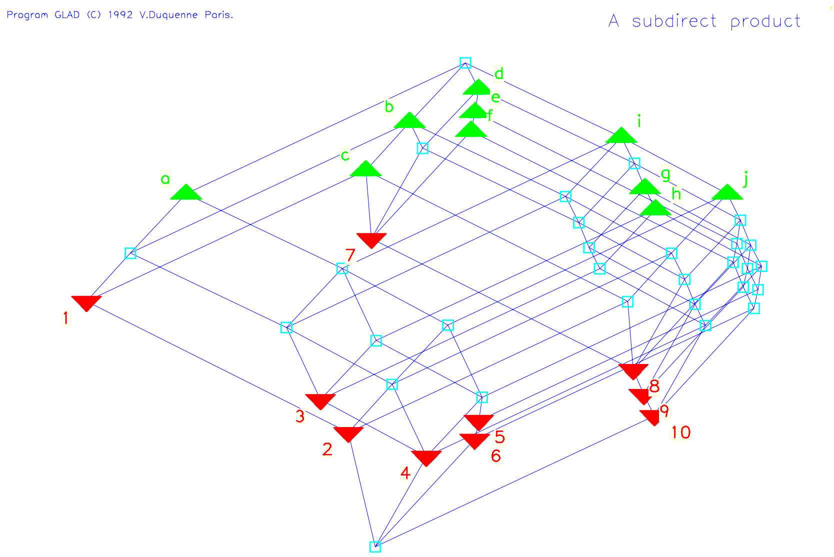

Fig. 11. A subdirectly reducible lattice L coming from mathematics (description of homomorphisms on partial algebra see [21], from which the lattice drawing has been borrowed)

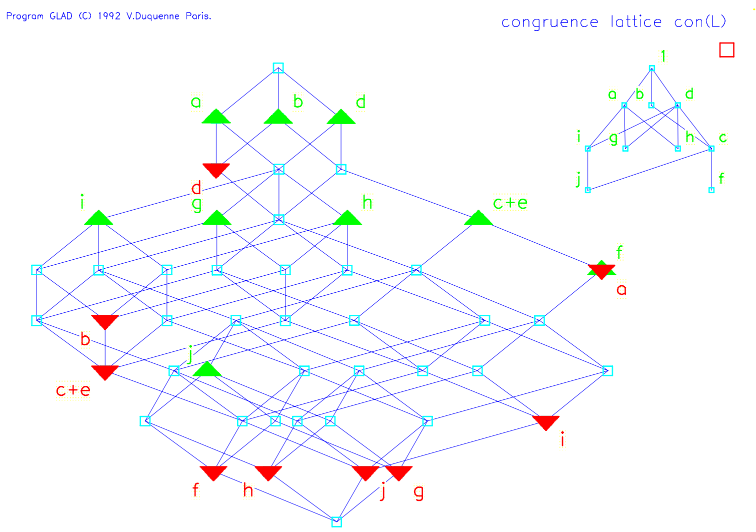

Fig. 12. The order (M(Con(L)),≤) has four minimal elements so that L is the subdirect product of four irreducible factors (capturing respectively all attributes above “j”, “g”, “h” and “f”)

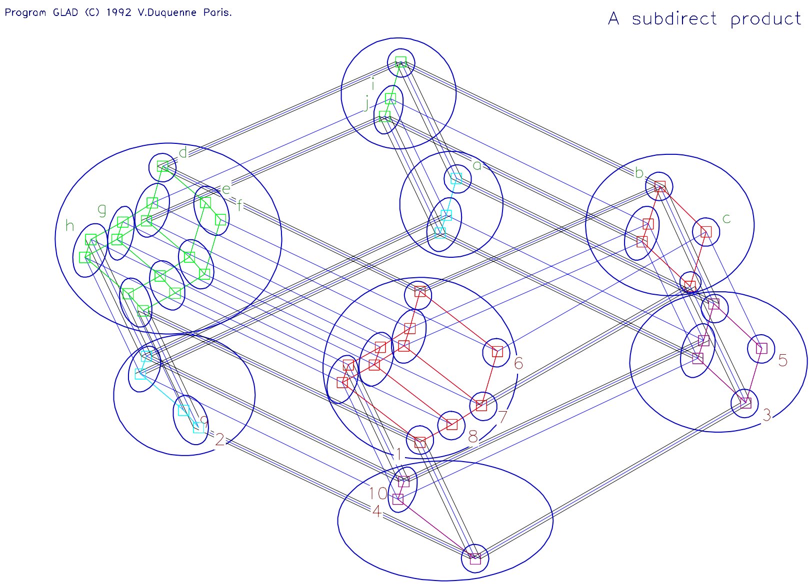

Fig. 13. A canonical presentation of L by "*rank down shelling*" of (M(Con(L)),≥): the Frattini congruence Φ_L generates a quotient lattice L/Φ_L which is Boolean and meet-generated along: "a","b","d" (first level of boxes). The L/Φ_L-classes are refined the same way at the next level.

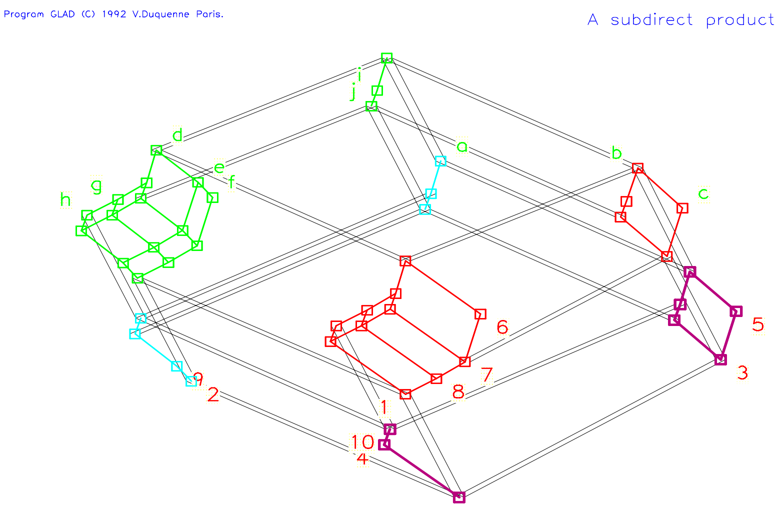

Fig. 14. The same canonical presentation of L by "*rank down shelling*" of (M(Con(L)),≥), where the drawing is simplified by removing boxes and lines as often done for nested line diagrams. The three shelling levels can be easily identified by using three different distances.

↕	x	x	x	x	x	x	x	x	X		1
x	↕	x	↓	x	x	x	X	x	X	J_1	2
x	x	↕	x	↓	x	↓	↓	↓	X		3
x	x	x	↕	↕	x	x	X	↓	X		4
x	x	↑	x		↕				↓		5
↑	x	x	x	x	↕			x	↓	J_2	6
↑	x	x	x	x	x	↕		x	X		7
↑	x	x	x	x	x	x	↕	x	X		8
x	↑	x		x	x	x	x	↕	X		9
x	x	x	↑	↑	x	x	x		↕	J_3	10
	M_1				M_2				M_3		
a	b	d	c	e	i	g	h	f	j		

Fig. 15. A canonical presentation of the "standard context" (J(L),M(L),≤) with the arrow graph along the *rank down shelling* of (M(Con(L)),≥), which defines the partition (M_1,M_2,M_3). J(L) and M(L) have been permuted so that the graph strongly connected components be connected. There are obviously no up-arrows (down-arrows) above (left of) the subtables J_rxM_r (r=1,2,3).

5 On the "Importance of Morphisms"

This question has been advocated for our present and pragmatic concerns regarding lattice drawing clarification, but it was raised initially at the general level of philosophy of mathematics. There, many authors (see [1] p. 209) have agreed that:

"From Dedekind, through Noether, and the work of Eilenberg and Mac Lane, the fact has clearly emerged that mathematical structure is determined by a system of objects *and their mappings*, rather than by any specific features of mathematical objects viewed in isolation. To a great degree, the structural approach in modern mathematics is characterized by increased attention to (system of) *mappings*, and the idea that mathematical objects are determined by their *'admissible' transformations*."

Moreover, as attested by G. Birkhoff (see [3, p. 773], among S. Mac Lane, N. Bourbaki etc), in half of a century, this shift developed widely with enthusiasm to reformulate algebra and even to call for rethinking the foundations of mathematics:

"The tidal wave generated by enthusiasm about abstract algebra had wider repercussions. (…) The "universal" approach to algebra, which I had initiated in the 1930's and 1940's stressing the role of lattices, was developed much further in two important books by Cohn and Grätzer. In a parallel development, Lawvere (1965) proposed "The category of categories as a foundation for mathematics," beginning with the statement: '*In the mathematical development of recent decades one sees clearly the rise of the conviction that the relevant properties of mathematical objects are those which can be stated in terms of their abstract structure rather than in terms of the elements which the objects were thought to be made of. The question thus naturally arises whether one can give a foundation for mathematics (…) in which classes and membership in classes do not play any role*.' "

Now, this abstract movement could have been wild and dominating in the 70's. At the end of a lecture, I remember a great professor asking "why don't you embed this into Category Theory?", which has had terrified me for twenty years, up to relaxing only after seeing it as leading in a "list of best stupid questions in Mathematics".

Here, more modestly, it has been advocated and illustrated that *congruence relations* and *lattices* -although abstract- can be instrumental in taming the complexity of lattice drawings. This has been done through a series of claims and examples.

A first motivating example has been the permutohedron Perm(n), on which the *Frattini congruence* defines a Boolean lattice of classes, each of which can be interpreted as putting together lattice elements *sharing the same complements*. This decomposition gives new views / perspectives of Perm(n). To get such interpretable congruence that gives new insights of a lattice should be most useful for data analysis.

Hence, three examples from social sciences and genetics demonstrate that the Frattini congruence can help in extracting –hopefully big- distributive factors that are helpful in simplifying lattice drawings thanks to the nice properties of distributivity.

For more complex examples for which congruence lattices depart from being Boolean, we introduced a procedure of *rank down shelling* of (M(Con(L)),≥), that decomposes congruence classes iteratively along a chain of nested congruences. The kind of drawing clarification that can be obtained is illustrated on an example coming from math. The general idea behind this procedure is to put what is clear, the *contrast* between full weakly projective *dependence* / in*dependence* in front line, and to nest the remaining complexity in a *linear order* of such embedded contrasts, which is determined by the *congruence lattice global structure*: all the *morphisms* are collectively used to shape an appropriate … *morphology* to the lattice. This hierarchy of contrasts is very close to the "hierarchy of contradictions" often met in philosophy: this follows a balanced claim for more "concreteness & abstraction, nothing less!".

6 Added in Proofs

Congruences can be used for formalizing dualities like *identification* / *distinction* via the *collapsing* / *splitting* of lattice elements -under the substitution properties and both operation respects. Hence, the above *rank down shelling* procedure gives *priority to distinction* (splitting). But there are situations where identification (collapsing) should prevail…This leads to consider a dual *up shelling* procedure scanning through the order (M(Con(L)),≥) –as well as the induced preorder (M, ≧)- from bottom to top.

The resulting drawing is represented in Fig. 15, based on the up shelling partition of M(L): M_1=ghfj, M_2=cei, M_3=abd. As compared with Fig. 13, the "boxes" should now be more easily read from the smaller (inside) to the bigger (outside). If the two drawings are not that different (in this particular case), here the priority is to collapse pairs "having no other consequences" for collapsing, by considering them as similar. The shelling procedure provides a canonical ordering of such similarities. Comparing the two logic of *distinction* / *identification* should now be done on appropriate data...

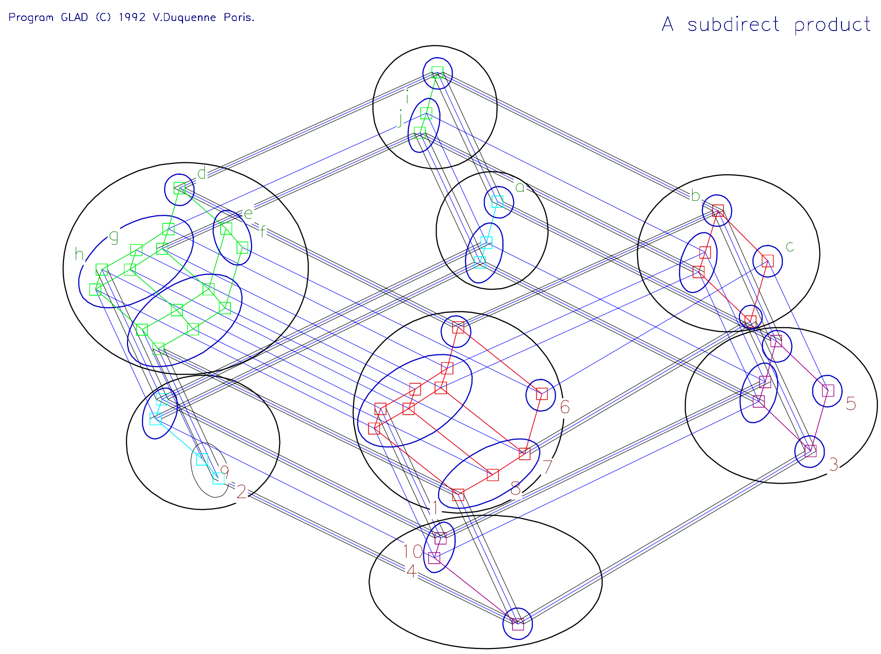

Fig. 15. Another canonical presentation of L by "*rank up shelling*" of (M(Con(L)),≥). Here the *shelling partition* is: M_1=min(M, ≧), M_2=min(M \ M_1, ≧),…,M_r=min(M \ ∪{M_s/s=1,r-1}, ≧).

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Thanks are also due to the referees for their remarks and comments. In particular a proposal has been to drop out our assumption of homomorphism *surjectivity*, with the following motivation: for an homomorphism ϕ:L → H and a "nice" drawing of H, it is possible that removing the elements of H \ ϕ(L) makes the drawing becoming "ugly". This extension would give more flexibility and should be explored … in the future.

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