RINGS OF QUOTIENTS AND PROJECTION LATTICES OF FINITE AW*-ALGEBRAS: FAITHFUL REPRESENTATION AND EQUATIONAL THEORY

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Abstract. We show that the maximal ring of quotients of a finite AW^*-algebra A (which is also a $*$-regular ring and a $\mathbb{C}$-algebra according to Berberian [4]) admits a faithful representation in the $*$-$\mathbb{C}$-algebra of endomorphisms of a closed subspace of an elementary extension of a Hilbert space. We also show that any existence variety generated by regular subalgebras of such $*$-$\mathbb{C}$-algebras is either generated by a single matrix $\mathbb{C}$-algebra $\mathbb{C}^{n\times n}$ or by all $\mathbb{C}$-algebras $\mathbb{C}^{n\times n}$, $n<\omega$. The latter class $\mathcal{N}_{\mathbb{C}}$ has a decidable equational theory when considered as consisting of 2-sorted structures with quasi-inverse. Using results of [31] it is shown that the associated projection ortholattices form a variety whose equational theory has decision problem in PSPACE. The latter result is transferred to the $*$-rings in $\mathcal{N}_{\mathbb{C}}$ endowed with the Rickart relative inversion.

1. Introduction

Murray and von Neumann [48] constructed for any finite von Neumann algebra factor A an extension to a $*$-regular ring of unbounded operators affiliated with A. This $*$-ring has the same projections as A. For a recent exposition, see Reich [53, 54]. In an abstract setting, this result was extended to finite AW^*-algebras by Berberian [4], who constructed for a finite AW^*-algebra A a $*$-regular extension $Q_B(A)$ of A with the same projections. It was shown by Hafner [20] and Pyle [52] that the resulting ring is the maximal ring of quotients $Q_M(A)$ of A. Using a different approach, Handelman [22] constructed for any finite Rickart C^*-algebra A an extension to a $*$-regular subring of $Q_M(A)$ again having the same projections as A. Finally, it was shown by Ara and Menal [1] that the ring constructed by Handelman is the classical ring of quotients $Q_C(A)$ of A.

The purpose of the present paper is twofold: to find faithful representations of such $*$-rings within endomorphism rings of inner product spaces with (partial) involution given by adjunction (as defined in §7) and to study the equational theory of such $*$-rings and their projection ortholattices.

It is shown in §17 that representation exists for $*$-regular rings of generalized operators, a concept modeled after the approach of Murray, von Neumann, and Berberian. Applying the GNS-construction, this yields a faithful representation for the quotient ring $Q_M(A)$ of a finite AW^*-algebra A, see Theorem 57 in §19. The representation is within a closed subspace of an elementary extension of a Hilbert

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space and yields a representation of the projection lattice of A (a concept defined in §9). This extends the corresponding result from [26] for finite von Neumann algebra factors.

Classes of $*$-regular rings representable in inner product spaces have been studied by Micol [47] (based on Tyukavkin [57]). In particular, it has been shown that, for an axiomatic class $\mathcal{S}$ of inner product spaces, the $*$-regular rings having a subdirect decomposition into factors representable in spaces from $\mathcal{S}$ form an *existence variety*, that is a class closed under direct products, regular $*$-subrings, and homomorphic images. Moreover, this existence variety is generated by the endomorphism rings of finite dimensional members from $\mathcal{S}$. This is reviewed in §4, §7, and §8.

Here, our main concern is $*$-regular algebras representable over the field of reals $\mathbb{R}$ and over the field of complex numbers $\mathbb{C}$ in particular. In order to obtain existence varieties, we have to pass to the corresponding axiomatic classes of $*$-fields; that is, *real closed ordered fields* and *complexifications* of those (§10-11).

The existence variety $\mathcal{N}_{\mathbb{C}}$ of $*$-$\mathbb{C}$-algebras having a faithful representation over some elementary extension of $\mathbb{C}$ is generated by the matrix algebras $\mathbb{C}^{n\times n}$, $n < \omega$, with canonical involution (§13). Moreover, using the relationship between von Neumann n-frames and n-distributivity (§14) and the fact that in a finite Rickart C^*-algebra perspectivity is equivalent to the existence of a partial isometry, we can show in §20 that any existence variety generated by finite AW^*-algebras is either equal $\mathcal{N}_{\mathbb{C}}$ or generated by a single $*$-$\mathbb{C}$-algebra $\mathbb{C}^{n\times n}$ and an analogous result holds for the associated variety of projection lattices.

The above and decidability of $\mathbb{R}$ resp. $\mathbb{C}$ imply that the equational theory of the class $\mathrm{L}_{\mathcal{N}}$ of projection ortholattices of members from $\mathcal{N}_{\mathbb{C}}$ is decidable. With a suitable choice of quasi-inverse, we will also prove decidability of the equational theory of $\mathcal{N}_{\mathbb{C}}$, considering scalars as constants or as elements of a second sort. The fact that frames can be forced via terms and that the same applies to elements in their coordinate domains (§15) leads to equivalence (via term interpretations) of the decision problems for the equational theories of $\mathrm{L}_{\mathcal{N}}$ and $\mathcal{N}^q$, the variety generated by members of $\mathcal{N}_{\mathbb{C}}$ considered as $*$-rings with Rickart relative inverse (§16).

For matrix $*$-algebras $\mathbb{R}^{d\times d}$ and $\mathbb{C}^{d\times d}$ and their projection ortholattices 'weak satisfiability', i.e. $t(\overline{x}) > 0$ for some $\overline{x}$, for terms $t(\overline{x})$ have been studied in [31]. In particular, it was shown that for indefinite d, the weak satisfiability problem is in PSPACE. Since the decision problem for identities in $\mathrm{L}_{\mathcal{N}}$ is the complement of weak satisfiability in indefinite dimension, and since term interpretations are computable in polynomial time, both above mentioned decision problems are in PSPACE, too.

§14-16 require only §1-2 as prerequisites, taking Theorems 36(i) and 37(ii) as definitions, where $F_n \in \{\mathbb{R}, \mathbb{C}\}$. §17 and §19 require only §2, §6, and §7 and basic definitions from §18.

Related results concerning tautologies (in the sense of Łukasiewicz) of MV-algebras associated with certain AF C^*-algebras are due to Mundici [46] —in particular coNP-completeness of the tautological theory of the free AF C^*-algebra.

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2. $*$-REGULAR RINGS

We refer to [50, 5, 7, 40, 18] for more details. We call a ring with involution a $*$-*ring*. Throughout this paper, let Λ be a commutative ring with involution $\lambda \mapsto \overline{\lambda}$. A $*$-Λ-*algebra* is an associative (left) Λ-algebra R with unit 1_R and involution * such that $(\lambda r)^* = \overline{\lambda} r^*$ for all $r \in R$. Unless it is stated otherwise, we consider $*$-Λ-algebras in the signature $\sigma_\Lambda = \{+, \cdot, -, ^*, 0, 1\} \cup \{\lambda\cdot \mid \lambda \in \Lambda\}$, while we consider $*$-rings in the signature $\sigma = \{+, \cdot, -, ^*, 0, 1\}$. In particular, a *subring* is always closed under involution. An element e of a $*$-ring R is a *projection*, if $e = e^2 = e^*$. Let $\mathbb{L}(R)$ denote the set of all projections of R.

An ring R is [*von Neumann*] *regular* if for any $a \in R$, there is a *quasi-inverse* $x \in R$ such that $axa = a$. A $*$-ring R is $*$-*regular* if it is regular and $aa^* = 0$ implies $a = 0$ for all $a \in R$. Equivalently, for any $a \in R$, there is a (unique) projection $e \in \mathbb{L}(R)$ such that $aR = eR$ (alternatively: for any a there is $e \in \mathbb{L}(R)$ with $Ra = Re$). Those are called *left* and *right* projections of a and are denoted by $LP(a)$ and $RP(a)$ respectively. By $\mathcal{R}_\Lambda$, we denote the class of all $*$-regular $*$-Λ-algebras.

Lemma 1. *If R is a $*$-ring and $e \in \mathbb{L}(R)$, then the following statements hold.*

1. *The* corner *eRe is a $*$-ring.*
2. *eRe is a homomorphic image of $S = eRe + (1-e)R(1-e)$ which is a subring of R.*
3. *If R is a $*$-regular ring, then eRe and S are also $*$-regular rings.*

Proof. See [36, page 30] and [7, Chapter 2]. The claim about S follows from the fact that $S \cong eRe \times (1-e)R(1-e)$. □

Lemma 2. [7, 2.7] *Let R be a $*$-regular ring, let $a, b \in R$, and let e, f be projections such that $aR = eR$ and $Ra = Rf$. Then*

1. *there is a unique $b \in fR$ such that $ab = e$;*
2. *moreover, $b \in fRe$, $ba = f$, $aba = a$, and $bab = b$.*

Within the notation of Lemma 2, $b \in R$ from (i) is called the *Rickart relative inverse* of a. It also follows from Lemma 2(ii) that if b is the Rickart relative inverse of a, then a is the Rickart relative inverse of b.

For any $*$-Λ-algebra R, the Λ-algebra $R^{n\times n}$ of all $(n \times n)$-matrices endowed with the *canonical* involution

$$(r_{ij})^*_{i,j\leqslant n} = (r^*_{ji})_{i,j\leqslant n}$$

is a $*$-Λ-algebra, which is regular according to [18, Lemma 1.6] if R is regular. But $R^{n\times n}$ fails to be $*$-regular, in general.

Idempotents $e_{ij}, (1 \leq i, j \leq n)$ form an *n-system of $*$-matrix units* in the $*$-Λ-algebra R if $e^*_{ij} = e_{ji}$ for all i, j and

$$e_{ij}e_{kl} = \begin{cases} e_{il}, & \text{if } j = k; \\ 0, & \text{if } j \neq k. \end{cases}$$

This system is *nontrivial* if some $e_{ij} \neq 0$. For the following, see [5, §16 Proposition 1].

Lemma 3. *For any $n > 0$, $\Lambda^{n\times n}$ is a $*$-Λ-subalgebra of some corner of any $*$-Λ-algebra R containing a nontrivial n-system of $*$-matrix units.*

3. Modular ortholattices and projection lattices

We refer to [8, 9]. An algebraic structure $\langle L; \cdot, +, ', 0, 1\rangle$ is an *ortholattice* if the structure $\langle L; \cdot, +, 0, 1\rangle$ is a bounded lattice satisfying the condition

$$a \leqslant b \quad \text{iff} \quad b' \leqslant a'.$$

It is a *modular ortholattice* (shortly, an MOL) if, in addition, $a \geqslant c$ implies $a(b+c) = ab+c$. An element a of an MOL L resp. an interval $[0, a]$ has *height* n (which is also called the *dimension* of a and denoted by $\dim a$), if $[0, a]$ has $(n+1)$-element maximal chains. Every interval $[v, u]$ of an MOL is an MOL with *induced* orthocomplement $x \mapsto x'u + v$. In particular, $[0, u]$ is a *sectional sub-MOL* of L. Intervals $[0, u]$ and $[u', 1]$ are anti-isomorphic as MOLs and they are homomorphic images of the sub-ortholattice $[0, u] \cup [u', 1]$.

Lemma 4. *Let $s = s(\overline{x})$ and $t = t(\overline{x})$ be ortholattice terms. Then for any $\overline{c}$ in an MOL, one has $s(\overline{c}) = t(\overline{c})$ if and only if $1 = u(\overline{c})$, where $u = (s + s't')(t + s't')$, if and only if $0 = v(\overline{c})$ where $v = s(st)' + t(st)'$.*

Proof. Observe that for $b \leq a$ in an MOL, one has $a = b$ if and only $b + a' = 1$, if and only if $ab' = 0$. □

For any $*$-ring R, the set $\mathbb{L}(R)$ of all projections of R can be partially ordered by

$$e \leqslant f \quad \text{iff} \quad ef = e.$$

If R is $*$-regular then $\mathbb{L}(R)$ is an MOL isomorphic to the lattice $\overline{L}(R_R)$ of all principal right ideals of R, matching a principal right ideal with its unique generating projection, cf. [37, 50, 40]. The orthocomplement in $\mathbb{L}(R)$ is given by $e' = 1 - e$. Infimum and supremum of $e, f \in \mathbb{L}(R)$ will be denoted by $e \cap f$ and $e \vee f$ in order to avoid a possible confusion with ring operations. For any $e \in \mathbb{L}(R)$, $\mathbb{L}(eRe)$ is the interval $[0, e]$ in $\mathbb{L}(R)$. If $e \leq f'$ then e and f are *orthogonal* to each other. Equivalently, $ef^* = 0$.

Proposition 5. *The operations of MOLs $\mathbb{L}(R)$, R any $*$-regular ring, can be expressed uniformly by terms in the signature $\sigma \cup \{q\}$, where q is an operation q of quasi-inversion, satisfying $q(a^*a)^* = q(a^*a)$ for all a in R. In particular this applies to Rickart relative inversion.*

Proof. Cf. [47, Proposition 1.2]. Define $e = p(x) = xq(x^*x)x^* \in xR$. Clearly, $e^* = e$. For $f = e - e^2$, one computes $ff^* = 0$, whence $f = 0$. Moreover, $x = ex$ since $x^*(x - ex) = 0$. Thus, $p(x) = xq(x^*x)x^*$ is a projection such that $xR = p(x)R$.

Now, for projections $a, b \in R$, according to [15, §3.2] one obtains the lattice operations of $\mathbb{L}(R)$ as follows:

$$a \vee b = p\big(a + (b - ab)u\big); \quad a \cap b = p\big(b - bu(b - ab)\big), \text{ where } u = q(b - ab).$$

Considering the Rickart relative inverse b of a, we have by Lemma 2 that $fR = a^*R$, $Re = Ra^*$, $b^* \in eRf \subseteq eR$, and $a^*b^* = f$, i.e. b^* is the Rickart relative inverse of a^*. □

Proposition 6. *For any $*$-regular ring R, there is a canonical isomorphism between the congruence lattice of R and the one of $\mathbb{L}(R)$. These lattices are distributive.*

Proof. This is well known, see e.g. [30, Proposition 5] for the case of regular rings and complemented modular lattices. Now, for $*$-regular R, one has $I = I^*$ for each ideal of R since $a^* \in Re \subseteq I$ for any $a \in I$ and $e \in \mathbb{L}(R)$ with $aR = eR$. On the other hand, any lattice congruence θ of an MOL is an ortholattice congruence since $b \leq a$ and $a\,\theta\,b$ imply $ab'\,\theta\,bb' = 0$ and $b' = ab' + a'\,\theta\,0 + a' = a'$. □

Lemma 7. *For a finite height MOL L and an ortholattice term $t(\overline{x})$, if there is $\overline{a}$ in L such that $t(\overline{a}) \neq 0$, then there are $u \in L$ such that $\dim u \leqslant |t|$ and $\overline{b}$ in $[0, u]$ such that $t(\overline{b}) \neq 0$ in the sectional sub-MOL $[0, u]$.*

Here, $|t|$ denotes the number of occurrences of variables in the term t.

Proof. Follows from [26, Lemma 2.2], see also [31, Theorem 24b]. □

4. Existence varieties of $*$-regular algebras

For a class $\mathcal{K}$ of algebraic structures of a fixed similarity type, by $\mathsf{H}(\mathcal{K})$, $\mathsf{S}(\mathcal{K})$, $\mathsf{P}(\mathcal{K})$, $\mathsf{P}_s(\mathcal{K})$, $\mathsf{P}_\omega(\mathcal{K})$, and $\mathsf{P}_u(\mathcal{K})$, we denote the class of all homomorphic images, of structures isomorphic to substructures, direct products, subdirect products, direct products of finitely many factors, and ultraproducts of members of $\mathcal{K}$, respectively. For more details, we refer to [12, 19, 43]. A *variety* is a class definable by *identities*; that is, first-order sentences of the form

$$\forall \overline{x}\ A_0(\overline{x})\ \&\ \ldots\ \&\ A_{n-1}(\overline{x}),$$

where $A_i(\overline{x})$, $i < n$, are atomic formulas. Equivalently, a variety is a class closed under operators H, S, and P. The smallest variety containing a class $\mathcal{K}$ is $\mathsf{V}(\mathcal{K}) = \mathsf{HSP}(\mathcal{K})$.

An algebraic structure A is *subdirectly irreducible* if it has a least non-trivial congruence. In particular, if A is subdirectly irreducible and $A \in \mathsf{SP}(\mathcal{K})$, then $A \in \mathsf{S}(\mathcal{K})$. By Birkhoff's Theorem, any algebraic structure is a subdirect product of its subdirectly irreducible homomorphic images. For $*$-regular rings [$*$-regular algebras, respectively] congruence relations stay in one-to-one correspondence with ideals, since all ideals are closed under involution, see Proposition 6.

It is obvious that the class $\mathcal{R}_\Lambda$ of all $*$-regular $*$-Λ-algebras is closed under operators P, P_u, and $\mathsf{S}_\exists$. Moreover, $\mathsf{H}(\mathcal{R}_\Lambda) \subseteq \mathcal{R}_\Lambda$ by [50, Theorem II.4.5].

For a class $\mathcal{K} \subseteq \mathcal{R}_\Lambda$, the class $\mathsf{S}_\exists(\mathcal{K})$ [$\mathsf{P}_{s\exists}(\mathcal{K})$] consists of those members from $\mathsf{S}(\mathcal{K})$ [from $\mathsf{P}_\mathsf{s}(\mathcal{K})$, respectively] which are regular as rings.

A subclass $\mathcal{K} \subseteq \mathcal{R}_\Lambda$ which is closed under operators H, $\mathsf{S}_\exists$, and , P is called an $\exists$-*variety*. The proof of [30, Proposition 10] obviously extends to any class of algebraic structures of a given similarity type and having regular ring reducts. Therefore, we have the following

Proposition 8. *For any $\mathcal{K} \subseteq \mathcal{R}_\Lambda$ the following statements hold.*

(1) *The smallest $\exists$-variety $\mathsf{V}_\exists(\mathcal{K})$ containing $\mathcal{K}$ is $\mathsf{HS}_\exists\mathsf{P}(\mathcal{K})$.*
(2) *If $A \in \mathsf{V}_\exists(\mathcal{K})$ is subdirectly irreducible, then $A \in \mathsf{HS}_\exists\mathsf{P}_u(\mathcal{K})$.*
(3) *$\mathsf{V}_\exists(\mathcal{K})$ is generated by its finitely generated subdirectly irreducible members.*

Lemma 9. *A class $\mathcal{K} \subseteq \mathcal{R}_\Lambda$ is an $\exists$-variety if and only if it is closed under H, $\mathsf{S}_\exists$, P_u, and P_ω.*

Proof. Consider $R_i \in \mathcal{K}$, $i \in I$, and their expansions $R_i^\alpha \in \mathcal{V}^\alpha$, where $\alpha(x, y)$ is the formula $xyx = x$, cf. §5. The direct product $\prod_{i\in I} R_i^\alpha$ belongs to $\mathsf{SP}_u\mathsf{P}_\omega(R_i^\alpha \mid i \in I)$ according to [19, Corollary 2.3.4]. Hence the σ_Λ-reduct R of $\prod_{i\in I} R_i^\alpha$ belongs to $\mathsf{S}_\exists\mathsf{P}_u\mathsf{P}_\omega(R_i \mid i \in I)$. Thus the desired conclusion follows from Proposition 8(i). □

Given a class $\mathcal{K} \subseteq \mathcal{R}_\Lambda$, we define $\mathsf{M}_n(\mathcal{K}) = \big(R^{n\times n} \mid R \in \mathcal{K}\big)$.

Lemma 10. *Let $R \in \mathcal{R}_\Lambda$ and let $n > 0$. Then $R^{m\times m} \in \mathsf{HS}(R^{n\times n})$ for any $m \leqslant n$. If $R^{n\times n}$ is $*$-regular, then $R^{m\times m}$ is also $*$-regular.*

Proof. Choose e as a diagonal matrix whose diagonal contains 1_R m times, and the rest elements are 0_R. Then the corner $eR^{n\times n}e$ is isomorphic to $R^{m\times m}$. The claim follows with Lemma 1. □

Lemma 11. *$\mathbb{R}^{n\times n} \in \mathsf{S}_\exists(\mathbb{C}^{n\times n})$ and $\mathbb{C}^{n\times n} \in \mathsf{S}_\exists(\mathbb{R}^{2n\times 2n})$ for any $n > 0$.*

Proof. The first claim is obvious, while the second follows by mapping $a + bi$ to the matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in \mathbb{R}^{2\times 2}$$

and matrices from $\mathbb{C}^{n\times n}$ to corresponding block matrices in $\mathbb{R}^{2n\times 2n}$. □

Lemma 12. *Let $\mathcal{K}$ be a class of $*$-regular $*$-Λ-algebras and let $R \in \mathcal{K}$. Then $\mathbb{L}(S) \in \mathsf{S}(R)$ for $S \in \mathsf{S}_\exists(R)$, $\mathbb{L}(S) \in \mathsf{O}(R)$ for $S \in \mathsf{O}(R)$, $\mathsf{O} \in \{\mathsf{H}, \mathsf{P}\}$, and*

$$\mathsf{V}\big(\mathbb{L}(R) \mid R \in \mathcal{K}\big) = \mathsf{V}\big(\mathbb{L}(R) \mid R \in \mathsf{V}_\exists(\mathcal{K})\big).$$

In particular, $\mathbb{L}(R^{m\times m}) \in \mathsf{HS}\big(\mathbb{L}(R^{n\times n})\big)$ for any $m \leqslant n < \omega$. Moreover, $J = J^*$ and the map

$$\mathbb{L}(R) \to \mathbb{L}(R/J), \quad e \mapsto e + J$$

is a homomorphism for any (two-sided) ideal $J \subseteq R$.

Proof. Follows from [30, Lemma 30] and Lemma 10. The last statement is [47, Theorem 1.4]. □

5. Equational definition of quasi-inverse

Consider formulas $\alpha(x, y)$ in the first-order language of $*$-Λ-algebras with free variables x, y such that sentences

$$\forall x \exists y\ \alpha(x, y);$$
$$\forall x \forall y\ \alpha(x, y) \ \longrightarrow\ xyx = x$$

hold in $\mathcal{R}_\Lambda$. Let q be a new unary operation symbol. For a class $\mathcal{K} \subseteq \mathcal{R}_\Lambda$, let $\mathcal{K}^\alpha$ denote the class of all algebras $R \in \mathcal{K}$ endowed with a new unary operation q (also denoted by R^α) such that

$$R^\alpha \models \forall x\ \alpha(x, q(x)).$$

We put $\sigma_\Lambda^\alpha = \sigma_\Lambda \cup \{q\}$.

Lemma 13. $\mathsf{V}(\mathcal{K}^\alpha) \subseteq \big(\mathsf{V}_\exists(\mathcal{K})\big)^\alpha$ *for any* $\mathcal{K} \subseteq \mathcal{R}_\Lambda$.

Proof. Since $\mathcal{K} \subseteq \mathsf{V}_\exists(\mathcal{K})$, we have $\mathcal{K}^\alpha \subseteq \big(\mathsf{V}_\exists(\mathcal{K})\big)^\alpha$. Moreover, if $A = \prod_{i\in I} A_i$ and $A_i \in \mathcal{K}^\alpha$ for all $i \in I$, then the σ_Λ-reduct A_i' of A_i belongs to $\mathcal{K}$, whence $A' \in \mathsf{P}(\mathcal{K})$ and therefore $A \in \big(\mathsf{P}(\mathcal{K})\big)^\alpha$ due to the fact that $A_i \models \forall x\ \alpha(x, q(x))$ for all $i \in I$ implies that $A \models \forall x\ \alpha(x, q(x))$. Hence $\mathsf{P}(\mathcal{K}^\alpha) \subseteq \big(\mathsf{P}(\mathcal{K})\big)^\alpha$.

Further, if B is a substructure of $A \in \big(\mathsf{P}(\mathcal{K})\big)^\alpha$, then its σ_Λ-reduct B' is a substructure of $A' \in \mathsf{P}(\mathcal{K})$ and $B \models \forall x\ \alpha(x, q(x))$, whence $B' \in \mathsf{S}_\exists\mathsf{P}(\mathcal{K})$. Thus $B \in \big(\mathsf{S}_\exists\mathsf{P}(\mathcal{K})\big)^\alpha$.

Finally, if C is a homomorphic image of $B \in \big(\mathsf{S}_\exists\mathsf{P}(\mathcal{K})\big)^\alpha$, then $C \models \forall x\ \alpha(x, q(x))$. Now, its σ_Λ-reduct C' is a homomorphic image of the σ_Λ-reduct $B' \in \mathsf{S}_\exists\mathsf{P}(\mathcal{K})$, whence $C' \in \mathsf{HS}_\exists\mathsf{P}(\mathcal{K})$. This implies that $C \in \big(\mathsf{HS}_\exists\mathsf{P}(\mathcal{K})\big)^\alpha = \big(\mathsf{V}_\exists(\mathcal{K})\big)^\alpha$ and $\mathsf{V}(\mathcal{K}^\alpha) = \mathsf{HSP}(\mathcal{K}^\alpha) \subseteq \big(\mathsf{V}_\exists(\mathcal{K})\big)^\alpha$. □

We call $\alpha(x, y)$ an *equational definition of quasi-inverse*, if $\mathsf{V}(\mathcal{K}^\alpha) = \big(\mathsf{V}_\exists(\mathcal{K})\big)^\alpha$ for any $\mathcal{K} \subseteq \mathcal{R}_\Lambda$.

Lemma 14. *Let $\alpha(x, y)$ be either the formula $xyx = x$ or an existentially quantified conjunction of atomic formulas such that sentences*

$$\forall x \exists y\ \alpha(x, y);$$
$$\forall xy\ \alpha(x, y) \longrightarrow xyx = x;$$
$$\forall xyz\ \alpha(x, y)\ \&\ \alpha(x, z) \longrightarrow y = z$$

hold in $\mathcal{R}_\Lambda$. Then $\alpha(x, y)$ is an equational definition of quasi-inverse. In particular, the Rickart relative inverse arises from an equational definition of quasi-inverse.

Proof. If $\alpha(x, y)$ is the formula $xyx = x$, then the required statement follows from [30, Proposition 10(i)]. We consider the second case. For any $R \in \mathcal{R}_\Lambda$, the structure R^α is uniquely defined in this case. Let $\mathcal{K} \subseteq \mathcal{R}_\Lambda$, let $\varphi\colon B \to C$ be a homomorphism such that $B \in \mathcal{K}$, and let $b \in B$. Since $B \models \alpha(b, q(b))$, $\alpha(x, y)$ is an existential formula, and φ is a homomorphism, we conclude that $C \models \alpha\big(\varphi(b), \varphi(q(b))\big)$. As we also have $C \models \alpha\big(\varphi(b), q(\varphi(b))\big)$, we get by uniqueness that $\varphi(q(b)) = q(\varphi(b))$, whence φ is a homomorphism from B^α into C^α. This implies that $\big(\mathsf{H}(\mathcal{K})\big)^\alpha \subseteq \mathsf{H}(\mathcal{K}^\alpha)$. Uniqueness of expansions yields that $\big(\mathsf{S}_\exists(\mathcal{K})\big)^\alpha \subseteq \mathsf{S}(\mathcal{K}^\alpha)$ and $\big(\mathsf{P}(\mathcal{K})\big)^\alpha \subseteq \mathsf{P}(\mathcal{K}^\alpha)$, from where we get that $\big(\mathsf{V}_\exists(\mathcal{K})\big)^\alpha \subseteq \mathsf{V}(\mathcal{K}^\alpha)$ for any $\mathcal{K} \subseteq \mathcal{R}_\Lambda$. Now the desired statement follows with Lemma 13. To the Rickart relative inverse Lemma 2 applies. □

If scalars are to be considered as variables too, then the following concept is appropriate. Given an equational definition α of quasi-inverse in $\mathcal{R}_\Lambda$, a 2-*sorted conditional identity* is a pair $\langle \varphi, \psi \rangle$, where $\varphi(\overline{\xi})$ is a quantifier-free formula in the

language of ordered $*$-fields or their complexifications (see §10) and $\psi(\overline{\xi}, \overline{x})$ is a formula which is obtained from an atomic formula of signature $\sigma^{\alpha}_{\Lambda}$ by replacing unary operation symbols $\lambda\cdot$, $\lambda \in \Lambda$, with variables ξ. The 2-*sorted conditional identity*

$$\langle \varphi(\xi_1, \ldots, \xi_n), \psi(\xi_1, \ldots, \xi_n, x_1, \ldots, x_m) \rangle$$

holds in $R \in \mathcal{R}^{\alpha}_{\Lambda}$, if $R \models \psi(\lambda_1, \ldots, \lambda_n, a_1, \ldots, a_m)$ for all $a_1, \ldots, a_m \in R$ and all $\lambda_1, \ldots, \lambda_n \in \Lambda$ such that $\Lambda \models \varphi(\lambda_1, \ldots, \lambda_n)$.

6. Inner product spaces

An *inner product space* is a (right) vector space V_F over a division $*$-ring F endowed with an *anisotropic $*$-hermitean form* $\Phi\colon V_F \times V_F \to F$, that is for all $u, v, w \in V$ and all $\lambda \in F$, the following holds:

$$\begin{aligned}
&\Phi(u, v + w) = \Phi(u, v) + \Phi(u, w);\\
&\Phi(u, v\lambda) = \Phi(u, v)\lambda :\\
&\overline{\Phi(u, v)} = \Phi(v, u);\\
&\Phi(v, v) = 0 \text{ implies } v = 0.
\end{aligned}$$

We write simply V sometimes, if the underlying division $*$-ring F is clear from the context. We are mainly concerned with the case of commutative F, where we may write $\lambda v = v\lambda$. For $U \subseteq V_F$, we define

$$U^{\perp} = \{x \in V \mid \Phi(u, x) = 0 \text{ for all } u \in U\}.$$

A subset $U \subseteq V_F$ is *closed*, if $U = U^{\perp\perp}$. Given an endomorphism φ of V_F, we say that φ^* is the *adjoint* of φ *with respect to* Φ, if

$$\Phi\big(\varphi(x), y\big) = \Phi\big(x, \varphi^*(y)\big)$$

for all $x, y \in V$. We consider inner product spaces as 2-sorted structures with sorts V and F. For an inner product space (V_F, Φ), where F is a $*$-Λ-algebra, let $\mathsf{End}_{\Lambda}(V_F)$ denote the Λ-algebra of all endomorphisms of V_F, where $(\lambda\varphi)(v) = \varphi(v)(\lambda 1_F)$. Let $\mathsf{End}_{\Lambda}(V_F, \Phi)$ be the subalgebra of $\mathsf{End}_{\Lambda}(V_F)$ consisting of all endomorphisms having an adjoint with respect to the form Φ. As $\mathsf{End}_{\Lambda}(V_F, \Phi)$ is closed under * it will be considered as a $*$-Λ-algebra. If $\Lambda = \mathbb{Z}$, then we write simply $\mathsf{End}(V_F)$ and $\mathsf{End}(V_F, \Phi)$. The following fact is well known.

Proposition 15. $\mathsf{End}(V_F, \Phi)$ *is a $*$-regular ring for any inner product space (V_F, Φ). The Rickart relative inverse of φ is equal to $\chi \oplus \psi$, where $\chi\colon (\mathsf{im}\,\varphi)^{\perp} \to 0$ and $\psi\colon \mathsf{im}\,\varphi \to U = (\ker\varphi)^{\perp}$ is the inverse of $\varphi|_U\colon U \to \mathsf{im}\,\varphi$.*

Proof. Observe that $LP(\varphi)$ is the orthogonal projection onto $\mathsf{im}\,\varphi = (\ker\varphi^*)^{\perp}$ and $RP(\varphi)$ is the orthogonal projection onto $(\ker\varphi)^{\perp} = \mathsf{im}\,\varphi^*$. □

For an inner product space (V_F, Φ), let $\mathrm{Sub}(V_F, \Phi)$ denote the lattice of all subspaces of V_F, with $0 = \{0\}$, $1 = V$, and the map $X \mapsto X^{\perp}$. Observe that $X \subseteq Y^{\perp}$ if and only if $Y \subseteq X^{\perp}$ for all X, $Y \subseteq V$; that is, the map $X \mapsto X^{\perp}$ is self-adjoint. Moreover, $X \cap X^{\perp} = 0$ for all $X \subseteq V$. Thus $\mathrm{Sub}(V_F, \Phi)$ is an MOL (and of height

n) if and only if $n = \dim V_F < \infty$. In this case, any endomorphism has an adjoint and one has the isomorphism

$$\varphi\colon \mathbb{L}\big(\mathsf{End}(V_F, \Phi)\big) \to \mathrm{Sub}(V_F, \Phi), \quad e \mapsto e(V).$$

Moreover, for a subspace U of V_F, the MOL $\mathrm{Sub}(U_F, \Phi|_U)$ is the interval $[0, U]$ of $\mathrm{Sub}(V_F, \Phi)$ and it is isomorphic to $\mathbb{L}\big(e\,\mathsf{End}(V_F, \Phi)e\big)$, where e is the orthogonal projection onto U.

7. Representations of $*$-algebras

A *representation* of a $*$-Λ-algebra R is a σ_Λ-homomorphism $\varepsilon\colon R \to \mathsf{End}_\Lambda(V_F, \Phi)$. In particular, $\varepsilon(\lambda \cdot 1_R) = \lambda\,\mathsf{id}_V$, whence $\varepsilon(\lambda \cdot r) = \lambda\varepsilon(r)$ for all $\lambda \in \Lambda$ and all $r \in R$. Our main concern will be *faithful* representations, that is, one-to-one representations.

In the sense of Mal'cev axiomatic correspondence [42], a representation of a $*$-Λ-algebra $\varepsilon\colon R \to \mathsf{End}_\Lambda(V_F, \Phi)$ can be viewed as a 3-sorted structure $(R, V_F, \Phi, \varepsilon)$ with sorts R, V_F, and F and with the map ε captured by the ternary relation (also denoted by ε)

$$\{(a, v, w) \mid w = \varepsilon(a)(v)\} \subseteq R \times V_F \times V_F.$$

Corollary 16. *Let $\varepsilon\colon R \to \mathsf{End}_\Lambda(V_F, \Phi)$ be a (faithful) representation considered as a 3-sorted structure. If $\varepsilon'\colon R' \to \mathsf{End}_\Lambda(V'_{F'}, \Phi')$ is elementary equivalent to ε, then ε' is a (faithful) representation of R'.*

The following statement elaborates on Micol [47, Theorem 3.8].

Proposition 17. *If a regular $*$-Λ-algebra R admits a faithful representation $\varepsilon\colon R \to \mathsf{End}_\Lambda(V_F, \Phi)$, then for any homomorphic image S of R, there are an elementary extension $(V'_{F'}, \Phi')$ of (V_F, Φ), a closed subspace U of $(V'_{F'}, \Phi')$ invariant under the induced action of R, and a faithful representation of S in $\mathsf{End}_\Lambda(U_{F'}, \Phi'|_U)$.*

The proof of Proposition 17 relies on the following concept of saturation: Given a first order structure A, we enrich the original signature of A with the set of constants $\{c_a \mid a \in A\}$ (which are also called *parameters*) with the natural interpretation of each constant symbol in A. Then we consider a set $\Sigma = \Sigma(x_1, \dots, x_n)$ of first order formulas with free variables among $x_1, \dots x_n$ in this enriched signature. The set Σ is said to be *realized* in A, if there are $a_1, \dots, a_n$ in A such that $A \models \sigma(a_1, \dots, a_n)$ for all $\sigma \in \Sigma$. Call the set Σ *finitely realized* in A, if each of its finite subsets is realized in A. Finally, call A *ω-saturated* over a subset $B \subseteq A$, if every set Σ of formulas with parameters which is finitely realized in A and contains only finitely many constants not in B is also realized in A.

Proposition 18. *Every first order structure B has an elementary extension A which is ω-saturated over B.*

Proof. According to [33, Corollary 10.2.2], one may choose A as an elementary extension of B which is κ-saturated for some infinite cardinal $\kappa > |B|$. □

Proof. (Proposition 17) Let $f\colon R \to S$ be an onto homomorphism and let I be the kernel of f. Consider an elementary extension $(R', V'_{F'}, \Phi', \varepsilon')$ of the 3-sorted structure $(R, V_F, \Phi, \varepsilon)$ which is ω-saturated over I. Then $\varepsilon'\colon R' \to \mathsf{End}_\Lambda(V'_{F'}, \Phi')$ is a faithful representation according to Corollary 16. Consider the subspace

$$U = \{v \in V' \mid \varepsilon'(a)(v) = 0 \text{ for all } a \in I\}$$

of V' with the induced inner product $\Phi'|_U$. Now, for any $a \in I$, there is a projection $e \in I$ with $Ra = Re$. We obtain therefore that

$$U = \bigcap_{a\in I} \ker \varepsilon'(a) = \bigcap_{e\in I\cap \mathbb{L}(R)} \ker \varepsilon'(e) = \Big(\sum_{e\in I\cap\mathbb{L}(R)} \operatorname{im} \varepsilon'(1-e)\Big)^\perp$$

is a closed subspace of V'. Let $r \in R$ and let $v \in U$. Then one has $ar \in I$ for all $a \in I$, whence

$$\varepsilon'(a)\big(\varepsilon'(r)(v)\big) = \varepsilon'(ar)(v) = 0 \quad \text{for all} \quad a \in I.$$

Therefore, $\varepsilon'(r)(v) \in U$. Thus, one obtains representations

$$\varepsilon'_U\colon R \to \mathsf{End}_\Lambda(U_{F'}, \Phi'|_U), \qquad \varepsilon'_U(r) = \varepsilon'(r)|_U;$$
$$\xi\colon S \to \mathsf{End}_\Lambda(U_{F'}, \Phi'|_U), \qquad \xi(s) = \varepsilon'_U\big(f^{-1}(s)\big).$$

It remains to show that ξ is a faithful representation. Let $a \notin I$ and let $b_0, \dots, b_{n-1} \in I$. Since I is an ideal and R is regular, there is $b \in R$ such that $Rb = \sum_{i<n} Rb_i \subseteq I$. Again, by regularity of R, we conclude that $b \in Rb \subseteq I$ and that $a \in Ra \nsubseteq Rb$.

Suppose that for all $x \in V'$, $\varepsilon'(b)(x) = 0$ implies that $\varepsilon'(a)(x) = 0$. Consider any $c \in R$ such that $bc = 0$. For any $y \in V'$ and $x = \varepsilon'(c)(y)$, we have $\varepsilon'(b)(x) = 0$, whence $\varepsilon'(a)(x) = 0$. Thus, $\varepsilon'(ac)(y) = \varepsilon'(a)\big(\varepsilon'(c)(y)\big) = 0$ for all $y \in V'$ and $ac = 0$ since ε' is one-to-one. Now, choose $c = 1 - e$, where $e \in \mathbb{L}(R)$ and $Re = Rb$. Then $bc = 0$ whence $a = ae \in Rb$, a contradiction. Thus there is $x \in V'$ such that $\varepsilon'(b)(x) = 0$ but $\varepsilon'(a)(x) \neq 0$. Since $Rb_i \subseteq Rb$ for all $i < n$, there are $c_0, \dots, c_{n-1} \in R$ such that $b_i = c_i b$ for all $i < n$, whence

$$\varepsilon'(b_i)(x) = \varepsilon'(c_i b)(x) = \varepsilon'(c_i)\varepsilon'(b)(x) = 0.$$

By saturation, we get that for any $a \notin I$ there is $x \in V'$ such that $\varepsilon'(a)(x) \neq 0$ and $\varepsilon'(b)(x) = 0$ for all $b \in I$, which means that $x \in U$ and $\varepsilon'(a) \neq 0$. □

8. Representable classes

Given a class $\mathcal{F}_\Lambda$ of $*$-Λ-algebras which are division rings and a class $\mathcal{S}$ of inner product spaces over division rings from $\mathcal{F}_\Lambda$, we denote by $\mathcal{A}_{\mathcal{S},\Lambda}$ the class of all $*$-Λ-algebras admitting a faithful representation in some inner product space in $\mathcal{S}$. Let $\mathcal{R}_{\mathcal{S},\Lambda}$ denote the class of $*$-regular members of $\mathcal{A}_{\mathcal{S},\Lambda}$; that is, $\mathcal{R}_{\mathcal{S},\Lambda} = \mathcal{R}_\Lambda \cap \mathcal{A}_{\mathcal{S},\Lambda}$.

Remark 19. If Λ' is a subring of Λ, then $\mathcal{F}_{\Lambda'}$ consists of $*$-Λ'-algebras which are $\sigma_{\Lambda'}$-reducts of $*$-Λ-algebras from $\mathcal{F}_\Lambda$.

Theorem 20. *If $\mathcal{S}$ and $\mathcal{F}_\Lambda$ are axiomatic classes, then the following statements hold.*

(1) *$\mathcal{A}_{\mathcal{S},\Lambda}$ is a universal class, while $\mathcal{R}_{\mathcal{S},\Lambda}$ is an axiomatic class.*

(2) *If the class $\mathcal{F}_\Lambda$ is recursively axiomatizable, then both classes $\mathcal{A}_{\mathcal{S},\Lambda}$ and $\mathcal{R}_{\mathcal{S},\Lambda}$ are also recursively axiomatizable.*
(3) *Given $R \in \mathcal{R}_\Lambda$, one has $R \in \mathcal{A}_{\mathcal{S},\Lambda}$ $[R \in \mathcal{R}_{\mathcal{S},\Lambda}]$ if and only if for any countable elementary substructure Λ' of Λ and for any countable elementary $*$-Λ'-subalgebra R' of R, one has $R' \in \mathcal{A}_{\mathcal{S},\Lambda'}$ $[R' \in \mathcal{R}_{\mathcal{S},\Lambda'}$, respectively$]$.*

Proof. Obviously, $\mathcal{A}_{\mathcal{S},\Lambda}$ is closed under substructures. If $\mathcal{S}$ and $\mathcal{F}_\Lambda$ are axiomatic classes, then for fixed Λ and F ranging over $\mathcal{F}_\Lambda$, the class $\mathcal{A}^3_{\mathcal{S},\Lambda}$ of faithful representations, viewed as 3-sorted structures which arise from members of $\mathcal{A}_{\mathcal{S},\Lambda}$, forms an axiomatic class too. In particular, it is closed under ultraproducts, which immediately implies that the class $\mathcal{A}_{\mathcal{S},\Lambda}$ is also closed under ultraproducts. Thus $\mathcal{A}_{\mathcal{S},\Lambda}$ is a universal class according to [13, Theorem 3.2.2], see also [43, Corollary IV.8.3.4], whence (i) follows.

Let Γ denote a set of first order axioms for $\mathcal{A}^3_{\mathcal{S},\Lambda}$ and let Σ denote the set of all universal sentences in the signature of $*$-Λ-algebras which are consequences of Γ. By Gödel's Completeness Theorem, Σ is recursively enumerable if Γ is recursive. By Craig's trick [33, Exercise 6.1.3], Σ is also recursive, whence (ii) follows.

To prove (iii), let $\{X_i \mid i \in I\}$ denote the set of all finite subreducts of R. Then for any $i \in I$, there is a countable elementary subalgebra Λ_i of Λ such that X_i can be extended to a countable $*$-Λ_i-algebra R_i which is an elementary $*$-Λ_i-subalgebra of the σ_{Λ_i}-reduct of R. According to our assumption, $R_i \in \mathcal{A}_{\mathcal{S},\Lambda_i}$. By Mal'cev's Theorem [43, Theorem IV.8.3.2], this provides a faithful representation of R in an ultraproduct V of vector spaces V_i over $F_i \in \mathcal{F}_{\Lambda_i}$. In particular, V is a vector space over the corresponding ultraproduct F of division rings F_i, $i \in I$, while F is a $*$-algebra over the corresponding ultraproduct of rings Λ_i, $i \in I$. Since the latter ultraproduct contains Λ as an elementary subring, F is a $*$-Λ-algebra. Moreover, since $\mathcal{F}_\Lambda$ is an axiomatic class, $F \in \mathcal{F}_\Lambda$, whence $R \in \mathcal{A}_{\mathcal{S},\Lambda}$.

Considering the case of regular algebras, in (i)-(iii) we have to consider R in the signature $\sigma_\Lambda \cup \{q\}$, where $q(x)$ is some operation of quasi-inverse. Then the argument repeats the one above. □

Theorem 21. [Tyukavkin [57], Micol [47]] *Let $\mathcal{F}_\Lambda$ be an axiomatic class, let $\mathcal{S}$ be a universal class, and let $\mathcal{K} = \{\mathsf{End}(V_F, \Phi) \mid (V_F, \Phi) \in \mathcal{S},\ \dim V < \infty\}$.*

(1) *$\mathcal{R}_{\mathcal{S},\Lambda}$ is closed under H, $\mathsf{S}_\exists$, and P_u.*
(2) *For any subdirectly irreducible algebra $R \in \mathcal{R}_\Lambda$, one has $R \in \mathcal{R}_{\mathcal{S},\Lambda}$ if and only if $R \in \mathsf{HS}_\exists\mathsf{P}_u(\mathcal{K})$.*
(3) $\mathsf{P}_{s\exists}(\mathcal{R}_{\mathcal{S},\Lambda}) = \mathsf{V}_\exists(\mathcal{K})$.

Proof. (i) The class $\mathcal{R}_{\mathcal{S},\Lambda}$ is closed under P_u according to Theorem 20(i) and it is obviously closed under $\mathsf{S}_\exists$. Moreover, $\mathcal{R}_{\mathcal{S},\Lambda}$ is closed under H by Proposition 17.

Statements (ii) and (iii) are equivalent in view of Proposition 8(ii), thus it suffices to prove (iii). According to (i), $\mathsf{V}_\exists(\mathcal{K}) \subseteq \mathsf{P}_{s\exists}(\mathcal{R}_{\mathcal{S},\Lambda})$. Therefore by Proposition 8(iii) and Theorem 20(iii), it remains to show that $R \in \mathsf{V}_\exists(\mathcal{K})$ for any countable $R \in \mathcal{R}_{\mathcal{S},\Lambda}$. Following Tyukavkin [57] and Micol [47, Theorem 3.5], R has a faithful representation over some $(V_F, \Phi) \in \mathcal{S}$, where V_F has a countable basis, and R can

be identified with a ring of row and column finite matrices over F. The latter ring belongs to $\mathsf{V}_\exists(\mathcal{K})$ by [57] and [47, Theorem 3.15]. □

9. Representations of MOLs

A *representation* of an MOL L in an inner product space (V_F, Φ) is an $(0,1)$-lattice homomorphism $\varepsilon\colon L \to \mathrm{Sub}(V_F, \Phi)$ such that $\varepsilon(x') = \varepsilon(x)^\perp$ for all $x \in L$. Observe that, by modularity, $\varepsilon(x) = \varepsilon(x)^{\perp\perp}$ for all $x \in L$. A representation ε is *faithful*, if it is one-to-one.

As in Section 7, a representation of an MOL $\varepsilon\colon L \to \mathrm{Sub}(V_F, \Phi)$ can be viewed as a 3-sorted structure with sorts L, V_F, and F and with the map ε being captured by the binary relation (denoted by ε, again)

$$\{(a, v) \mid v \in \varepsilon(a)\} \subseteq L \times V_F.$$

The proof of [26, Corollary 3.4] given for unitary spaces generalizes to the setting of Section 7 in an obvious way to provide the following

Proposition 22. *If ε is a faithful representation of a $*$-regular ring R in an inner product space (V_F, Φ), then $e \mapsto \mathsf{im}\,\varepsilon(e)$ defines a faithful representation of the MOL $\mathbb{L}(R)$ in (V_F, Φ).*

The following statement is a version of [27, Theorem 1.1], cf. also [24, Theorem 13.1].

Proposition 23. *Let (V_F, Φ) be an inner product space, let $\varepsilon\colon L \to \mathrm{Sub}(V_F, \Phi,]vep)$ be a faithful representation of an MOL L, and let $M \in \mathsf{H}(L)$. Then there is an elementary extension $(\tilde{V}_{\tilde{F}}, \tilde{\Phi})$ of (V_F, Φ), there is a closed subspace U of $(\tilde{V}_{\tilde{F}}, \tilde{\Phi})$, and there is a faithful representation $\xi\colon M \to \mathrm{Sub}(U_{\tilde{F}}, \tilde{\Phi}|_U)$.*

Proof. Let $\varphi\colon L \to M$ be an onto ortholattice homomorphism, and let $I = \{x \in L \mid \varphi(x) = 0\}$. Then I is an ideal of L, and for $x \le y$ in L one has $\varphi(x) = \varphi(y)$ if and only if $x'y \in I$ (indeed, I is a 'neutral' ideal and $y = x + xy'$). Consider an elementary extension $(\tilde{L}, \tilde{V}_{\tilde{F}}, \tilde{\Phi}, \tilde{\varepsilon})$ of $(L, V_F, \Phi, \varepsilon)$ which is ω-saturated over I. Then $\tilde{\varepsilon}\colon \tilde{L} \to \mathrm{Sub}(\tilde{V}_{\tilde{F}}, \tilde{\Phi})$ is a faithful representation according to the lattice analogue of Corollary 16. Consider the closed subspace

$$U = \Big(\bigcup_{a \in I} \tilde{\varepsilon}(a)\Big)^\perp$$

of $\tilde{V}$ with the induced inner product $\tilde{\Phi}|_U$. We put

$$\xi\big(\varphi(x)\big) = \tilde{\varepsilon}(x) \cap U.$$

It is straightforward that for any $x \le y$ in L with $\varphi(x) = \varphi(y)$, one has

$$\tilde{\varepsilon}(y) \cap U = \big(\tilde{\varepsilon}(x) + \tilde{\varepsilon}(x'y)\big) \cap U = \tilde{\varepsilon}(x) \cap U,$$

as $\tilde{\varepsilon}(x'y) \subseteq U^\perp$. Thus ξ is well-defined. The arguments from [27, Section 4.2] (cf. also [24, Claim 13.4]) show that ξ preserves joins: observe that $v \in U$ if and only if $v \in \tilde{\varepsilon}(b)$ for all b belonging to the 'neutral' filter $F = \{a \in L \mid a' \in I\}$. Obviously, ξ preserves all the other operations and constants, whence it is a representation.

We show that ξ is faithful. Let $a \notin I$ and let $b_0, \dots, b_{n-1} \in I$. Since I is an ideal, $b = b_0 + \ldots + b_{n-1} \in I$ and $a \not\leq b$. Suppose that $\tilde{\varepsilon}(b)^\perp \subseteq \tilde{\varepsilon}(a)^\perp$. Then $\tilde{\varepsilon}(a) = \tilde{\varepsilon}(a)^{\perp\perp} \subseteq \tilde{\varepsilon}(b)^{\perp\perp} = \tilde{\varepsilon}(b)$, whence $a \leq b$ as $\tilde{\varepsilon}$ is an embedding. This implies that $a \in I$, a contradiction. Thus there is $v \in \tilde{V}$ such that $v \in \tilde{\varepsilon}(b_i)^\perp$ for all $i < n$ but $v \notin \tilde{\varepsilon}(a)^\perp$. By saturation, we get that for any $a \notin I$ there is $v \in \tilde{V}$ such that $v \notin \tilde{\varepsilon}(a)^\perp$ and $v \in \tilde{\varepsilon}(b)^\perp$ for all $b \in I$, which means that $v \in U$ and that $\xi\big(\varphi(a)\big) \neq 0$. Since M is complemented, this implies that ξ is faithful. □

Given a class $\mathcal{F}$ of division $*$-rings and a class $\mathcal{S}$ of inner product spaces over members of $\mathcal{F}$, we define $\mathcal{L}_\mathcal{S}$ as the class of all MOLs having a faithful representation in some member of $\mathcal{S}$. From Proposition 23 we get immediately

Corollary 24. *For any class $\mathcal{F}$ of division $*$-rings and any class $\mathcal{S}$ of inner product spaces over $\mathcal{F}$, the following statements hold.*

(1) *The class $\mathcal{L}_\mathcal{S}$ is closed under S.*
(2) *If $\mathcal{F}$ is an axiomatic class and $\mathcal{S}$ is a universal class, then the class $\mathcal{L}_\mathcal{S}$ is closed under H.*
(3) *If $\mathcal{S}$ is a universal class then $\mathsf{P}_s(\mathcal{L}_\mathcal{S}) = \mathsf{V}\big(\mathrm{Sub}(V_F, \Phi) \mid (V_F, \Phi) \in \mathcal{S},\ \dim V < \infty\big)$.*

Proof. (i) is obvious and (ii) follows from Proposition 23. To prove (iii), consider an MOL L having a faithful representation over some $(V_F, \Phi) \in \mathcal{S}$. By [27, Proposition 3.12] (cf. also [24, Theorem 2.1]), there is an atomic extension $\hat{L}$ of L which also has a faithful representation over (V_F, Φ). Hence

$$L \in \mathsf{V}\big(\mathrm{Sub}(U_F, \Phi|_U) \mid U \in \mathrm{Sub}(V_F, \Phi),\ \dim U < \infty\big)$$

by [28, Proposition 1.2]. Therefore,

$$\mathsf{P}_s(\mathcal{L}_\mathcal{S}) \subseteq \mathsf{V}\big(\mathrm{Sub}(V_F, \Phi) \mid (V_F, \Phi) \in \mathcal{S},\ \dim V < \infty\big).$$

Since the class $\mathcal{L}_\mathcal{S}$ is closed under substructures and homomorphic images by (i)-(ii), the class $\mathsf{P}_s(\mathcal{L}_\mathcal{S})$ is a variety, whence the reverse inclusion follows. □

The following will not be used later.

Theorem 25. [Niemann [51]] *Let $\mathcal{F}$ be an axiomatic class, let $\mathcal{S}$ be a universal class, and let R be a subdirectly irreducible $*$-regular ring. Then $R \in \mathcal{R}_{\mathcal{S},\mathbb{Z}}$ if and only if $\mathbb{L}(R) \in \mathcal{L}_\mathcal{S}$.*

Proof. If $R \in \mathcal{R}_{\mathcal{S},\mathbb{Z}}$ then $\mathbb{L}(R) \in \mathcal{L}_\mathcal{S}$ by Proposition 22. Conversely, consider a subdirectly irreducible R $*$-ring with a minimal ideal I and with a given faithful representation of $\mathbb{L}(R)$ within $\mathcal{S}$. Let $e \in I$ be a projection. Then the corner eRe is a simple $*$-regular ring with unit e, cf. [51, Lemma 4.5.18]. Now $eRe \in \mathsf{HS}_\exists(R)$ by Lemma 1, whence $\mathbb{L}(eRe) \in \mathsf{HS}(\mathbb{L}(R))$ by Lemma 12. It follows from Corollary 24(i)-(ii) that $\mathbb{L}(eRe)$ has a faithful representation in $\mathcal{S}$. This replaces the doubtful hypothesis of [51, Theorem 1.6.9] that faithful representations exist for all subdirectly irreducible Arguesian MOLs. Based on results on frames in representable MOLs, Niemann shows in the proof of [51, Corollary 4.5.12] that eRe has a faithful representation in $\mathcal{S}$. The ideal I is a simple $*$-regular ring (without unit) which has a

faithful representation in $\mathsf{P}_u(\mathcal{S}) = \mathcal{S}$ according to [51, Proposition 4.5.13]. It follows with [51, Lemma 4.5.15] and [51, Proposition 4.5.16] that R has a representation in $\mathcal{S}$. □

10. Real closed fields and complexifications

Let Λ be the ring of integers $\mathbb{Z}$ or a real closed ordered field. By $\mathtt{RC}_\Lambda$, we denote the class of all $*$-Λ-algebras F which are real closed ordered fields endowed with the identity involution. We consider algebras from $\mathtt{RC}_\Lambda$ in the $*$-Λ-algebra signature σ_Λ (cf. §2) endowed with a binary relation $<$; we denote this signature by $\sigma_\Lambda^<$.

Given an ordered field F_0 with the identity involution, the *complexification* of F_0 is a $*$-field extension $F_0(\iota) = \{a + b\iota \mid a, b \in F_0\}$ of F_0 with $\iota^2 = -1$ and involution defined by $\overline{(a + b\iota)} = a - b\iota$ for all $a, b \in F_0$. Observe that $(a+b\iota)\overline{(a + b\iota)} = a^2+b^2 > 0$ for all $a, b \in F_0$.

If $\Lambda_0 = \mathbb{Z}$ or Λ_0 is a real closed ordered field with identity involution and $\Lambda = \Lambda_0(\iota)$ is its complexification, let $\mathtt{RC}_\Lambda$ denote the class of all complexifications of algebras $F_0 \in \mathtt{RC}_{\Lambda_0}$; that is, the class of $*$-Λ-algebras F considered in the signature $\sigma_\Lambda \cup \{i, P, <\}$, where i is a constant symbol such that $i = \iota 1$, $F = F_0(\iota)$, and P is a unary relation symbol which stands for the subfield $F_0 \in \mathtt{RC}_{\Lambda_0}$ ordered by $<$. For any $\Lambda_0 \subseteq \mathbb{R}$, we have $\mathbb{R} \in \mathtt{RC}_{\Lambda_0}$ and $\mathbb{C} \in \mathtt{RC}_{\Lambda_0(\iota)}$. Observe that both Λ_0 and $\Lambda_0(\iota)$ are recursive, if Λ_0 consists of the algebraic reals.

In the sequel, Λ always denotes either the ring of integers $\mathbb{Z}$, or a real closed ordered field, or complexification of one of those, while $\mathtt{RC}_\Lambda$ will always denote one of the classes just defined.

We say that a class $\mathcal{K}$ of structures enjoys the *joint embedding property*, if for any $F_0, F_1 \in \mathcal{K}$, there is $F \in \mathcal{K}$ such that both F_0 and F_1 embed into F.

Proposition 26. (1) *The class* $\mathtt{RC}_\Lambda$ *is first-order axiomatizable.*

(2) *The* $*$-Λ-*algebra* Λ *is an elementary substructure of each member of* $\mathtt{RC}_\Lambda$*. In particular, the first-order theory* T *of* $\mathtt{RC}_\Lambda$ *is complete.*

(3) *If* Λ *is recursive, then* $\mathtt{RC}_\Lambda$ *is recursively axiomatizable and the first-order theory of* $\mathtt{RC}_\Lambda$ *is decidable.*

(4) $\mathtt{RC}_\Lambda$ *has the joint embedding property.*

Proof. The class $\mathtt{RC}_\mathbb{Z}$ is the well known class of all real closed ordered fields which satisfies all the claims (i)-(iv) and admits quantifier elimination, cf. Marker [44, Corollaries 3.3.6, 3.3.15].

Now, suppose that Λ is a real closed field. Adding the atomic diagram of Λ to the axioms of $\mathtt{RC}_\mathbb{Z}$, one obtains an axiomatization T_0 of T. Thus (i) holds.

Besides that, $\mathtt{RC}_\Lambda$ admits quantifier elimination too. Indeed, given a formula φ, replace any occurrence of a term λt (in other words, the term $\lambda(t)$ considering λ a unary operation symbol) by $x_\lambda \cdot t$, where x_λ is a new free variable. For the resulting formula ψ of signature $\sigma^<$, there is a quantifier free formula χ having the same free variables as ψ such that ψ and χ are equivalent in the theory of real closed ordered fields. Replace in χ any occurrence of x_λ by $\lambda 1$ to obtain a quantifier free formula of signature $\sigma_\Lambda^<$ having the same free variables as φ and equivalent to φ within the class $\mathtt{RC}_\Lambda$, as $(\lambda 1) \cdot t = \lambda t$ for any term t.

Observe that the field Λ may be identified with the subfield $\{\lambda 1 \mid \lambda \in \Lambda\}$ of any $F \in \mathtt{RC}_\Lambda$. Since Λ is real closed, this identification is also an order embedding: $\lambda > 0$ implies $\lambda = \mu^2$ for some $\mu \in \Lambda$, whence $\lambda 1 = (\mu 1)^2 > 0$. Thus considered as a member of $\mathtt{RC}_\Lambda$, Λ is an elementary substructure of any member of $\mathtt{RC}_\Lambda$. In view of quantifier elimination and [44, Propositions 3.1.14, 3.1.15], this proves (ii).

If Λ is recursive, T_0 is also recursive, and decidability of T follows from completeness and the Completeness Theorem. Thus (iii) holds.

Finally, according to [33, Theorem 8.4.1], the set $T_\forall$ of universal sentences belonging to T possesses the amalgamation property. In particular, given F_0, $F_1 \in \mathtt{RC}_\Lambda$, there is $F \models T_\forall$ such that F_i embeds into F for any $i < 2$. In particular, F is an ordered integral domain. According to [44, Corollary 3.3.12], F embeds (as an ordered integral domain) into an ordered field F' which, in turn, embeds into its real closure $\overline{F}$. Since $\lambda a = (\lambda 1) \cdot a$ for all $a \in \overline{F}$ and since $\lambda 1 \in F$, the embedding of F into $\overline{F}$ is also a $\sigma^<_\Lambda$-embedding, which proves (iv).

The case $\mathtt{RC}_{\Lambda(\iota)}$ follows using an obvious 2-dimensional interpretation $\mathtt{RC}_{\Lambda(\iota)}$ in $\mathtt{RC}_\Lambda$, cf. [33, §5.1]. The proof is now complete. □

The following proposition summarizes some well known facts.

Proposition 27. *Let F, $F_1 \in \mathtt{RC}_\Lambda$, let $R = F^{n\times n}$, $n > 0$, and let (V_F, Φ) be an n-dimensional vector space with scalar product Φ.*

(1) *$R \in \mathcal{R}_\Lambda$ and $R \cong \mathsf{End}(V_F, \Phi)$.*
(2) *For any projection $e \in R$, one has $eRe \cong F^{m\times m}$, where $m = \dim e$ in $\mathbb{L}(R)$.*
(3) *$*$-Λ-algebras $F^{n\times n}$ and $F_1^{n\times n}$ have the same first order theory.*
(4) *$\mathsf{V}_\exists(F^{n\times n}) = \mathsf{V}_\exists(F_1^{n\times n})$.*

Proof. The first part of (i) refers to the well known fact that AA^* is positive definite for any $A \in F^{n\times n}$ if $F = \mathbb{R}$ or $\mathbb{C}$. By elementary equivalence which follows from Proposition 26(ii), this extends to any $F \in \mathtt{RC}_\Lambda$. Moreover, with the canonical scalar Φ-product on F^n, we have $F^{n\times n} \cong \mathsf{End}(F_F^n, \Phi)$. Now, suppose that we are given two n-dimensional inner product spaces (V_F, Φ) and (W_F, Ψ). Since F is quadratically closed, we may choose an orthonormal basis in each of those and match these bases so that $(V_F, \Phi) \cong (W_F, \Psi)$. (ii) follows using the fact that projections correspond to orthogonal projections in (V_F, Φ).

To prove (iii), we use the concept of interpretation, cf. Hodges [33, Chapter 5]. Let $F_0 = F$. Clearly, $F_i^{n\times n}$, has an $(n \times n)$-dimensional interpretation in F_i for all $i < 2$. Furthermore, according to Proposition 26(ii), F_0 and F_1 are elementary equivalent, whence $F_0^{n\times n}$ and $F_1^{n\times n}$ are also elementary equivalent. (iv) follows from (iii) and the fact that existence varieties are axiomatic classes. □

Proposition 28. *Let $F \in \mathtt{RC}_\Lambda$.*

(1) *If Λ is recursive, then there is an algorithm which on input n and sentence φ in the language of $*$-Λ-algebras decides whether φ is valid in $F^{n\times n}$.*
(2) *There is an algorithm which on input n and sentence φ in the language of ortholattices decides whether φ is valid in $\mathbb{L}(F^{n\times n})$.*

Proof. (i) In view of Proposition 26(ii)-(iii), there is a decision algorithm for the theory of F. Given φ and n, translate φ according to the $(n \times n)$-dimensional interpretation into a sentence in the language of F and apply this algorithm to the translation. (ii) follows due to Lemma 5 (with $\Lambda = \mathbb{Z}$). □

11. Existence varieties of representables

In what follows, we consider *positive definite* inner product spaces over $F \in \mathtt{RC}_\Lambda$; that is, inner product spaces (V_F, Φ) such that $\Phi(v, v) > 0$ for all $0 \neq v \in V$. By $\mathcal{N}_\Lambda$ and $\mathrm{L}_{\mathcal{N}_\Lambda}$ we denote the classes of all $*$-regular $*$-Λ-algebras resp. MOLs admitting a faithful representation in some positive inner product space over some $F \in \mathtt{RC}_\Lambda$.

Theorem 29. *The class $\mathcal{N}_\Lambda$ is an existence variety, while $\mathrm{L}_{\mathcal{N}_\Lambda}$ is a variety. If Λ is recursive and $\alpha(x, y)$ is an equational definition of quasi-inverse, then the variety $\mathcal{N}_\Lambda^\alpha$ is recursively axiomatizable.*

Proof. Given positive faithful representations $\varepsilon_i \colon R_i \to \mathsf{End}_\Lambda(V_{iF_i}, \Phi_i)$, $i < 2$, we may assume by Proposition 26(iv) that F_0, F_1 have a joint extension F. This yields faithful representations $\eta_i \colon R \to \mathsf{End}_\Lambda(W_i, \hat{\Phi}_i)$, $i < 2$, where $W_i = F \otimes_{F_i} V_i$ and Φ_i lifts to $\hat{\Phi}_i$ on W_i canonically. The orthogonal sum provides a faithful representation

$$\eta_1 \times \eta_2 \colon R_1 \times R_2 \to \mathsf{End}_\Lambda\big((W_0, \Phi_0) \oplus^\perp (W_1, \Phi_1)\big).$$

Thus, the class $\mathcal{N}_\Lambda$ is closed under P_ω. An analogous reasoning applies to MOLs. Closure of $\mathcal{N}_\Lambda$ under $\mathsf{S}_\exists$ and P_u is follows from Theorem 20(i) and closure under H is granted by Proposition 17. It follows by Lemma 9 that $\mathcal{N}_\Lambda$ is an $\exists$-variety. Corollary 24 implies that $\mathrm{L}_{\mathcal{N}_\Lambda}$ is a variety. If Λ_0 is recursive, then recursive axiomatizability follows from Theorem 20(ii) and Proposition 26(iii). □

Lemma 30. *Let Λ be a real closed field or $\Lambda = \mathbb{Z}$ and let $F \in \mathtt{RC}_\Lambda$. A $*$-$\Lambda(\iota)$-algebra R admits a positive faithful representation over $F(\iota) \in \mathtt{RC}_{\Lambda(\iota)}$ if and only if R, considered as a $*$-Λ-algebra, admits a positive faithful representation over F.*

Proof. Necessity follows when considering an $F(\iota)$-vector space V as an F-vector space and replacing Φ by its real part. Conversely, given a positive faithful representation of the $*$-Λ-algebra R in a vector space (V_F, Φ_0) over F, consider V as a faithful R-module and form the R-module $F(\iota) \otimes_F V$ with Φ given by

$$\Phi\big((a + b\iota)v, (c + d\iota)w\big) = \Phi_0(av, cw) + \Phi_0(bv, dw) + \Phi_0(av, dw)\iota - \Phi_0(bv, cw)\iota$$

to obtain a faithful representation over $F(\iota)$. □

Corollary 31. *Let Λ be a real closed field or $\Lambda = \mathbb{Z}$ and let R be a $*$-$\Lambda(i)$-algebra. Then $R \in \mathcal{N}_{\Lambda(i)}$ if and only the σ_Λ-reduct of R belongs to $\mathcal{N}_\Lambda$.*

12. Generators and decidability

Theorem 32. *Let $F \in \mathtt{RC}_\Lambda$, let N be an infinite set of positive integers, and let $\alpha(x, y)$ be an equational definition of quasi-inverse in $\mathcal{R}_\Lambda$.*

1. $\mathcal{N}_\Lambda = \mathsf{V}_\exists\big(F^{n\times n} \mid n \in N\big)$.
2. $R \in \mathsf{HS}_\exists\mathsf{P}_u\big(F^{n\times n} \mid n < \omega\big)$ *for any subdirectly irreducible algebra $R \in \mathcal{N}_\Lambda$.*

(3) $\mathcal{N}_\Lambda^\alpha = \mathsf{V}\big((F^{n\times n})^\alpha \mid n \in N\big)$.
(4) *If Λ is recursive, then $\mathcal{N}_\Lambda^\alpha$ has a decidable equational theory.*
(5) $R^{n\times n} \in \mathcal{N}_\Lambda$ *for any* $R \in \mathcal{N}_\Lambda$ *and any* $n < \omega$.

Proof. By Lemma 10, Theorem 21(iii), and Theorem 29, we have $\mathcal{N}_\Lambda = \mathsf{V}_\exists\big(F^{n\times n} \mid F \in \mathtt{RC}_\Lambda,\ n \in N\big)$. According to Proposition 27(iv), for a fixed $n < \omega$ and for any F_0, $F_1 \in \mathtt{RC}_\Lambda$, the $*$-Λ-algebras $F_0^{n\times n}$ and $F_1^{n\times n}$ generate the same $\exists$-variety, whence (i) follows. Statement (ii) follows with (i) and Proposition 8(ii), and statement (iii) follows from (i) by the definition of an equational definition of quasi-inverse. Statement (iv) follows from Theorem 29 and the Completeness Theorem.

Concerning (v), observe that $(F^{m\times m})^{n\times n} \cong F^{mn\times mn}$ also as $*$-rings due to the fact that matrix rings are endowed with the canonical involution. Moreover, one easily verifies that $\mathsf{M}_n\mathsf{O}(\mathcal{K}) \subseteq \mathsf{OM}_n(\mathcal{K})$ for any operator $\mathsf{O} \in \{\mathsf{H}, \mathsf{S}_\exists, \mathsf{P}\}$. □

Proposition 33. *If $\Lambda \in \mathtt{RC}_{\Lambda_1}$, then $\mathcal{N}_\Lambda^\alpha$ and $\mathcal{N}_{\Lambda_1}^\alpha$ have the same theory of 2-sorted conditional identities.*

Proof. Let $\langle \varphi(\xi_1, \ldots, \xi_n), \psi(\xi_1, \ldots, \xi_n, x_1, \ldots, x_m)\rangle$ be a 2-sorted conditional identity. To prove that $\mathcal{N}_\Lambda^\alpha \models \langle \varphi, \psi\rangle$ implies that $\mathcal{N}_{\Lambda_1}^\alpha \models \langle \varphi, \psi\rangle$, one simply considers algebras from $\mathcal{N}_{\Lambda_1}^\alpha$ just as $*$-Λ_1-algebras.

Conversely, assume that $\mathcal{N}_\Lambda^\alpha \not\models \langle \varphi, \psi\rangle$. This means that there are $R \in \mathcal{N}_\Lambda^\alpha$, $\lambda_1, \ldots, \lambda_n \in \Lambda$, and $a_1, \ldots, a_m \in R$ such that $\Lambda \models \varphi(\lambda_1, \ldots, \lambda_n)$, but $R \not\models \psi(\lambda_1, \ldots, \lambda_n, a_1, \ldots, a_m)$. Keeping $\lambda_1, \ldots, \lambda_n$ fixed, we may assume by Theorem 32(iii) that $R = \Lambda^{n\times n}$ and we consider it as a 2-sorted structure. Then according to Proposition 26(i), as a 2-sorted structure, $\Lambda_1^{n\times n}$ is an elementary substructure of (the 2-sorted structure) $\Lambda^{n\times n}$. Hence there are $\mu_1, \ldots, \mu_n \in \Lambda_1$ and $b_1, \ldots, b_m \in \Lambda_1^{n\times n}$ such that $\Lambda_1 \models \varphi(\mu_1, \ldots, \mu_n)$, but $\Lambda_1^{n\times n} \not\models \psi(\mu_1, \ldots, \mu_n, b_1, \ldots, b_m)$, whence $\mathcal{N}_{\Lambda_1}^\alpha \not\models \langle \varphi, \psi\rangle$. □

Corollary 34. *For any equational definition α of quasi-inverse, the theory of 2-sorted conditional equations of $\mathcal{N}_\Lambda^\alpha$ is decidable.*

Proof. Let Λ_1 be the set of algebraic reals [all algebraic numbers, respectively]. The desired conclusion follows from Theorem 32(iv) and Proposition 33. □

13. The von Neumann variety

We write $F \in \mathtt{RC}$ if F is a real closed ordered field with identity involution or complexification of such a field (with conjugation as involution). Thus F can be considered a member of $\mathtt{RC}_\mathbb{Z}$ or $\mathtt{RC}_{\mathbb{Z}(\iota)}$ respectively.

Let $\mathcal{N}$ resp. $\mathrm{L}_\mathcal{N}$ denote the class of all $*$-regular rings resp. ortholattices having a faithful representation over some $F \in \mathtt{RC}$. For a class $\mathcal{Q} \subseteq \mathcal{R}_\mathbb{Z}$, let $\mathcal{Q}^q$ denote the class of members of $\mathcal{Q}$, endowed with Rickart relative inversion; its signature is denoted by σ^q. $\mathcal{N}^q$ shall be called the *von Neumann variety*.

Lemma 35. $\mathcal{N} = \mathsf{V}_\exists\big(F_n^{n\times n} \mid n \in N\big)$ *for any infinite set N of natural numbers and any $F_n \in \mathtt{RC}$, $n \in N$.*

Proof. Let $\mathcal{N}_r$ and $\mathcal{N}_c$, respectively, denote the classes of those members of $\mathcal{N}$ which have a faithful representation over some $F \in \mathtt{RC}_{\mathbb{Z}}$ resp. $F \in \mathtt{RC}_{\mathbb{Z}(\iota)}$. By Theorem 32(i), one gets $\mathcal{N}_r = \mathsf{V}_{\exists}\big(\mathbb{R}^{n\times n} \mid n \in N\big)$ and $\mathcal{N}_c = \mathsf{V}_{\exists}\big(\mathbb{C}^{n\times n} \mid n \in N\big)$. By Lemma 11, $\mathcal{N}_r = \mathcal{N}_c = \mathcal{N}$ and thus (ii) holds in view of Lemma 10 and Proposition 27(iv). □

Theorem 36. (1) $\mathcal{N} = \mathsf{V}_{\exists}\big(\mathbb{C}^{n\times n} \mid n < \omega\big) = \mathsf{V}_{\exists}\big(\mathbb{R}^{n\times n} \mid n < \omega\big)$.

(2) *The equational theory of $\mathcal{N}^\alpha$ is decidable for any equational definition α of quasi-inverse. In particular, this applies to $\mathcal{N}^q$.*

(3) $R \in \mathsf{HS}_{\exists}\mathsf{P}_u\big(\mathbb{C}^{n\times n} \mid n < \omega\big)$ *for any subdirectly irreducible algebra $R \in \mathcal{N}$.*

(4) $R^{n\times n} \in \mathcal{N}$ *for any $R \in \mathcal{N}$ and any $n < \omega$.*

Proof. Follows from Lemma 35 and Theorem 32(i), (iv), (ii), and (v). □

Theorem 37. (1) $\mathrm{L}_{\mathcal{N}}$ *is a recursively axiomatizable MOL variety.*

(2) *For any infinite set N of natural numbers and any $F_n \in \mathtt{RC}$, $n \in N$,*

$$\mathrm{L}_{\mathcal{N}} = \mathsf{V}\big(\mathbb{L}(R) \mid R \in \mathcal{N}\big) = \mathsf{V}\big(\mathbb{L}(F_n^{n\times n}) \mid n \in N\big).$$

(3) *Given any $F \in \mathtt{RC}$, an ortholattice identity $s = t$ is valid in $\mathrm{L}_{\mathcal{N}}$ if and only if it is valid in $F^{d\times d}$, where $d = 3|s| + 3|t|$.*

(4) *The equational theory of $\mathrm{L}_{\mathcal{N}}$ is decidable.*

(5) *For any $*$-regular ring R, $R \in \mathcal{N}$ if and only if $\mathbb{L}(R) \in \mathrm{L}_{\mathcal{N}}$.*

Proof. By Corollary 24(iii), Lemma 12, and Lemma 35, one has $\mathrm{L}_{\mathcal{N}} \subseteq \mathsf{V}\big(\mathbb{L}(F_n^{n\times n}) \mid F_n \in \mathtt{RC},\ n \in N\big)$. By Lemma 12, $\mathsf{V}\big(\mathbb{L}(R) \mid R \in \mathcal{N}\big)$. This proves (ii) and allows to consider in (i) representations over real closed ordered fields only and to conclude with Theorem 29 that $\mathrm{L}_{\mathcal{N}} = \mathrm{L}_{\mathcal{N}_{\mathbb{Z}}}$ is a recursively axiomatizable variety. (iii) follows from Lemmas 4 and 7, considering the term $s(st)' + t(st)'$. (iv) follows from (i) and Proposition 28, see also [26, Theorem 5.1]. Theorem 25 proves (v) for subdirectly irreducible R. The claim follows from Lemma 12, since $\mathcal{N}$ is closed under $\mathsf{P}_{s\exists}$ and $\mathrm{L}_{\mathcal{N}}$ is closed under P_s. □

14. Frames and n-distributivity in MOLs

A system $\overline{a} = (a_{ij} \mid 1 \leqslant i, j \leqslant n)$, $n > 0$, of elements of a modular ortholattice L is an *n-frame*, if the following conditions are satisfied, where $a_i = a_{ii}$:

(1) $a_{ij} = a_{ji}$ for all $i \neq j$;

(2) $a_i \sum_{k\neq i} a_k = a_i a_{ij} = a_j a_{ij} = 0_L < a_i$ for all distinct $i, j \leqslant n$;

(3) $a_i + a_j = a_i + a_{ij}$ for all distinct $i, j \leqslant n$;

(4) $a_{ij} = (a_{ik} + a_{kj})(a_i + a_j)$ for all pairwise distinct $i, j, k \leqslant n$.

It suffices to have a_i, $1 \leqslant i \leqslant n$, and a_{1i}, $1 \leqslant i \leqslant n$, satisfying (ii)-(iii) above. By modularity, there are unique a_{ij}, $i, j \leqslant n$, completing these to an n-frame.

An n-frame $\overline{a}$ is *orthogonal* if $a_i \leq a_j'$ for all $i \neq j$; $\overline{a}$ is *spanning* if $\sum_{k\leqslant n} a_k = 1_L$; $\overline{a}$ is *trivial* if $a_{ij} = 0$ for all i, j.

For any orthogonal basis $v_1, \ldots, v_n$ of an inner product space (V_F, Φ), there is an associated spanning orthogonal n-frame of $\mathrm{Sub}(V_F, \Phi)$ given by

$$a_i = v_i F, \quad a_{ij} = (v_i - v_j)F.$$

For $F \in \mathtt{RC}$ and the canonical basis of F_F^n, this is referred to as the *canonical n-frame* of $\mathbb{L}(F_F^n, \Phi) \cong \mathbb{L}(F^{n\times n})$.

Lemma 38. *Let $n > 1$.*

(1) *For any n-frame $\overline{a}$ in a modular lattice, $\dim a_i = \dim a_j$ for all i, j. Moreover, if $a_i > 0$ for some i then the map $(i, j) \mapsto a_{ij}$ for $i \leqslant j$ is injective and $\dim \sum_{i\in I} a_i = \sum_{i\in I} \dim a_i$ for any $I \subseteq \{1, \dots, n\}$.*
(2) *Let L be an MOL and $\dim L = n$. Then every non-trivial n-frame is spanning. Moreover, L contains a spanning orthogonal n-frame if and only if L is subdirectly irreducible.*

Proof. (i) $a_1, \dots, a_n$ are independent and a_i is perspective to a_j via a_{ij} for $i \neq j$, cf. [8, IV.6]. (ii) The first claim and 'only if' in the second from (i) follow immediately. Conversely, any MOL of height n has n orthogonal atoms: choose a_1 and proceed with the MOL $[0, a_1']$. If L is subdirectly irreducible, then all atoms are perspective, cf. [8, IV.7]. Choose a_{1i} for $i > 1$ witnessing that a_1 is perspective to a_i and put $a_{ij} = (a_i + a_j)(a_{1i} + a_{1j})$. □

An alternative approach to frames is due to Huhn [34]: define an $(n-1)$-*diamond* in a modular (ortho)lattice L as a system $\overline{d} = (d_i \mid 0 \leqslant i \leqslant n)$ such that

$$d_j \sum_{1\leqslant i\leqslant n,\, i\neq j} d_i = 0 \quad \text{for all } j = 1, \dots, n;$$

$$d_0 + d_j = \sum_{i=1}^{n} d_i, \quad d_0 d_j = 0 \quad \text{for all } j = 1, \dots, n.$$

and call it *spanning* if $\sum_{i=1}^n d_i = 1$ and *orthogonal* if $d_i \leq d_j'$ for all distinct $1 \leqslant i, j \leqslant n$. Both concepts are equivalent in the following sense.

Lemma 39. *If $\overline{d}$ is an $(n-1)$-diamond in an MOL, then $\overline{a}$ with*

$$a_i = d_i,\ 1 \leqslant i \leqslant n, \quad a_{ij} = d_0(a_i + a_j) \text{ for } i \neq j$$

is an n-frame. Conversely, if $\overline{a}$ is an n-frame then

$$d_i = a_i,\ 1 \leqslant i \leqslant n, \quad d_0 = \sum_{i=2}^{n} a_{1i} \text{ for } i \neq j$$

defines an $(n-1)$-diamond. In both cases, $\overline{a}$ is spanning resp. orthogonal if and only if $\overline{d}$ is so.

Proof. This is well known and shown by easy calculations using modularity. E.g. given an $(n-1)$-diamond and pairwise distinct $i, j, k \geqslant 1$, one gets $\big(d_0(a_i + a_j) + d_0(a_j + a_k)\big)(a_i + a_k) = (d_0(a_i + a_j) + a_j + a_k)d_0(a_i + a_k) = \big((d_0 + a_j)(a_i + a_j) + a_k\big)d_0(a_i + a_k) = (a_i + a_j + a_k)d_0(a_i + a_k) = d_0(a_i + a_k)$. Conversely, given an n-frame, one has e.g. for $j > 1$ $d_0 d_j = d_0 \prod_{k\neq j} \sum_{l\neq k} a_l = \sum_{j=2}^n a_{1j} \sum_{l\neq j} a_l = \sum_{j=2}^n a_{1j} a_1 = 0$. □

According to Huhn [34] a lattice is *n-distributive* if it satisfies the identity

$$x + \prod_{i=0}^{n} y_i = \prod_{j=1}^{n} (x + \prod_{i=0,\ i\neq j}^{n} y_i).$$

For modular lattices, this identity is equivalent to its dual [35].

Theorem 40. [45] *For each $n > 1$, there are ortholattice terms $d_i(\overline{x})$, $0 \leqslant i \leqslant n$, with variables $\overline{x} = (x_i \mid 0 \leqslant i \leqslant n)$ such that $(d_i(\overline{c}) \mid 0 \leqslant i \leqslant n)$ is an orthogonal $(n-1)$-diamond for any MOL L and any substitution $\overline{c}$ in L that fails the $(n-1)$-distributive law and such that $d_i(\overline{c}) = c_i$, $0 \leqslant i \leqslant n$, if $\overline{c}$ is an orthogonal $(n-1)$-diamond. Moreover, L is $(n-1)$-distributive if and only if it contains no nontrivial orthogonal n-frame.*

Proof. We provide a proof more accessible than [45]. Clearly, any $(n-1)$-diamond induces a failure of the $(n-1)$-distributive law. Conversely, consider fixed but arbitrary elements $x_i = c_i$, $0 \leqslant i \leqslant n$, in an MOL which fail the $(n-1)$-distributive law and put

$$z_j = x_j \prod_{1\leqslant i\leqslant n\, i\neq j} x_i'.$$

Then $z_j' \geq x_j' \geq z_k$ for all distinct $j, k \geqslant 1$. Following the approach of Huhn [34], put

$$w_0 = x_0; \quad y_j = \sum_{1\leqslant i\leqslant n\ i\neq j} z_i; \quad u_0 = \prod_{i=1}^{n} (x_0 + z_i)$$

and observe that

$$\prod_{i=1}^{n} y_i = 0, \qquad \prod_{1\leqslant i\leqslant n\, i\neq j} y_i = z_j.$$

Define

$$v = \sum_{i=1}^{n} w_0 z_i; \quad b_i = (z_i + v)u_0,\ i > 0; \quad u = \sum_{i=1}^{n} b_i; \quad b_0 = uw_0.$$

According to the proof of [34, Satz 2.1], $u_0 \geq v$ and $\overline{b} = (b_i \mid 0 \leqslant i \leqslant n)$ is a spanning $(n-1)$-diamond in the interval sublattice $[v, u]$ and it is nontrivial, if $x_0, \ldots x_n$ do not satisfy the $(n-1)$-distributive law. Now

$$z_j u = z_j\big(\sum_{i=1}^{n} z_i u_0 + v\big) = z_j\big(\sum_{i=1}^{n} (z_i u_0 + z_i w_0)\big) = z_j u_0 + z_j w_0 + X,$$

where

$$X \leq z_j \sum_{1\leqslant i\leqslant n,\ i\neq j} z_i \leq z_j z_j' = 0.$$

It follows that

$$z_j u + v = z_j u_0 + z_j w_0 + v = z_j u_0 + v = b_j.$$

On the interval $[v, u]$, the induced orthocomplement is given by $x \mapsto x^\dagger = x'u + v$. In particular,

$$b_j^\dagger = z_j'u + v \geq z_k u + v = b_k \quad \text{for } j \neq k,\ j, k > 0.$$

Thus, $\overline{b}$ is an orthogonal $(n-1)$-diamond in $[v, u]$. With $t = uv'$ we get that $x \mapsto xt$ is an ortholattice isomorphism of $[v, u]$ onto $[0, t]$, each with the induced orthocomplement, taking $\overline{b}$ to the spanning $(n-1)$-diamond $\overline{d}$ in $[0, t]$ where $d_i = tb_i$. But then $\overline{d}$ is orthogonal in L too. If $\overline{c}$ is an orthogonal $(n-1)$-diamond, then $\overline{d} = \overline{c}$, obviously. □

Proposition 41. *A subdirectly irreducible MOL L is n-distributive if any only if $\dim L \leq n$. In particular, $\mathbb{L}(F^{m\times m})$, $F \in \mathtt{RC}$, is n-distributive if and only if $m \leq n$.*

Proof. Cf. [45, Theorem 1.3]. As a lattice, L embeds into the subspace lattice M of a projective space P with $M \in \mathsf{V}(L)$. Then M is the direct product of the subspace lattices of the irreducible components of P; since L is subdirectly irreducible we may assume that P is irreducible. But then $\dim L \leqslant \dim M \leq n$ by Lemma 38 if L is n-distributive. The converse follows from Lemma 38 and Theorem 40. □

15. Coordinates

The *coordinate domain* $R_{12\overline{a}}$ associated with an n-frame $\overline{a}$ in a lattice L is defined as $\{x \in L \mid xa_2 = 0,\ x + a_2 = a_1 + a_2\}$.

Proposition 42. *Let $n \geqslant 3$ and let $\overline{z} = (z_{ij} \mid 1 \leqslant i, j \leqslant n)$ be a list of variables. There are terms $x \oplus_{\overline{z}} y$, $x \otimes_{\overline{z}} y$, $\ominus_{\overline{z}} x$, $x^{\circledast_{\overline{z}}}$, $\hat{0}_{\overline{z}}$, $\hat{1}_{\overline{z}}$, $\hat{q}_{\overline{z}}(x)$ in the signature of MOLs such that for any $F \in \mathtt{RC}$, for any k, and $R = F^{k\times k}$, if $\overline{a}$ is a spanning orthogonal n-frame in $\mathbb{L}(R)$ with $e_1 = a_1$, then $R_{12\overline{a}}$ with the operations $x \oplus_{\overline{a}} y$, $x \otimes_{\overline{a}} y$, $\ominus_{\overline{a}} x$, $x^{\circledast_{\overline{a}}}$, $\hat{0}_{\overline{a}}$, $\hat{1}_{\overline{a}}$, $\hat{q}_{\overline{a}}(x)$ is isomorphic to the $*$-ring e_1Re_1 with Rickart relative inverse.*

Proof. This is more generally true for MOLs with $n \geqslant 4$ resp. for Arguesian MOLs with $n \geqslant 3$ and implicitly contained in proofs of (unique) coordinatization — we refer to [9], [50], and [40, XII Satz 3.1]. As stated, the proposition can be proved by elementary Linear Algebra cf. [51, §4.4-5] and [25]. In view of Proposition 27, we consider R as the endomorphism ring of $V_F = F_F^k$ with the involution given as adjunction with respect to the scalar product Φ and $L = \mathbb{L}(R) \cong \mathrm{Sub}(V_F, \Phi)$. Then $V_i = a_i$ yields an orthogonal decomposition $V = V_1 \oplus^\perp \ldots \oplus^\perp V_n$. Clearly, $\mathsf{End}(V_1, \Phi|_{V_1}) \cong e_1Re_1$. Furthermore, for $i \neq j$, the set $\mathsf{Hom}(V_i, V_j)$ is mapped bijectively onto

$$R_{ij} = \{x \in L \mid x + a_j = a_i + a_j,\ xa_j = 0\}$$

by the map

$$\phi \mapsto \Gamma(\phi) = \{x - \phi(x) \mid x \in V_i\}$$

using the (negative) graph of ϕ. Observe that

$$\Gamma(\psi \circ \phi) = (a_i + a_k)(\Gamma(\psi) + \Gamma(\phi)) \quad \text{for } \phi \in \mathsf{Hom}(V_i, V_j),\ \psi \in \mathsf{Hom}(V_j, V_k).$$

In particular for each $i \neq j$, there is a linear isomorphism $\varepsilon_{ij}\colon V_i \to V_j$ such that $a_{ij} = \{x - \varepsilon_{ij}(x) \mid x \in V_i\}$ and due to (iv) in the definition of frames, one has $\varepsilon_{jk} \circ \varepsilon_{ij} = \varepsilon_{ik}$. Moreover, $\varepsilon_{ji} = \varepsilon_{ij}^{-1}$. Thus

$$\pi_{ijk}(x) = (x + a_{jk})(a_i + a_k)$$

is a bijection from R_{ij} onto R_{ik} such that

$$\pi_{ijk}\Gamma(\phi) = \Gamma(\varepsilon_{jk} \circ \phi) \quad \text{for } \phi \in \mathsf{Hom}(V_i, V_j)$$

and also a bijection from R_{ji} onto R_{ki} such that

$$\pi_{ijk}\Gamma(\phi) = \Gamma(\phi \circ \varepsilon_{kj}) \quad \text{for } \phi \in \mathsf{Hom}(V_j, V_i).$$

It follows easily that

$$\iota\colon \mathsf{End}(V_1) \to R_{12}, \quad \iota\colon \phi \mapsto \Gamma(\varepsilon_{12} \circ \phi)$$

is a bijection and a ring isomorphism if R_{12} is endowed with the following operations:

$$\begin{aligned}
x \oplus y &= (a_1 + a_2)\big[(x + a_3)(a_{13} + a_2) + \pi_{123}(y)\big];\\
x \otimes y &= (a_1 + a_2)\big[\pi_{213}(x) + \pi_{213}(y)\big];\\
\ominus x &= (a_1 + a_2)\big[a_3 + (a_{23} + a_1)\big(a_2 + \pi_{213}(x)\big)\big];\\
\hat{0} &= a_1;\\
\hat{1} &= a_{12}.
\end{aligned}$$

More generally, for pairwise distinct i, j, k put

$$\ominus_{ij} x = (a_i + a_j)\big[a_k + (a_{jk} + a_i)\big(a_j + \pi_{jik}(x)\big)\big]$$

to obtain

$$\Gamma(-\phi) = \ominus_{ij}\Gamma(\phi) \quad \text{for } \phi \in \mathsf{Hom}(V_i, V_j).$$

Concerning the involution, observe that for $i \neq j$, $\psi \in \mathsf{Hom}(V_j, V_i)$ is the adjoint of $\phi \in \mathsf{Hom}(V_i, V_j)$ if and only if

$$\Gamma(\phi) \perp \Gamma(-\psi).$$

In particular,

$$\Gamma(\varepsilon_{ij}^*) = (\ominus_{ij} a_{ij})'(a_i + a_j)$$

and

$$\pi_{ijk}^*(x) = (x + \Gamma(\varepsilon_{jk}^*))(a_i + a_k)$$

is a bijection from R_{ki} onto R_{ji} such that

$$\pi_{ijk}^*\Gamma(\phi) = \Gamma(\phi \circ \varepsilon_{kj}^*) \quad \text{for } \phi \in \mathsf{Hom}(V_k, V_i).$$

Also $\varepsilon_{ji} \circ \varepsilon_{ij} = \mathsf{id}_{V_i}$ for all i, j implies $\varepsilon_{ij}^* \circ \varepsilon_{ji}^* = \mathsf{id}_{V_j}$ for all i, j. Now, by the above observation, for any $\phi \in \mathsf{End}(V_1)$,

$$\Gamma(\phi^* \circ \varepsilon_{12}^*) = \Gamma((\varepsilon_{12} \circ \phi)^*) = (\ominus\Gamma(\varepsilon_{12} \circ \phi))'(a_1 + a_2) \in R_{21},$$

whence

$$\Gamma(\varepsilon_{13}\circ(\phi^*\circ\varepsilon_{12}^*))=\pi_{123}(\Gamma(\phi^*\circ\varepsilon_{12}^*))\in R_{23};$$
$$\Gamma(\varepsilon_{13}\circ\phi^*)=\Gamma\big((\varepsilon_{13}\circ(\phi^*\circ\varepsilon_{12}^*))\circ\varepsilon_{21}^*\big)=\pi_{312}^*\Gamma(\varepsilon_{13}\circ(\phi^*\circ\varepsilon_{12}^*))\in R_{13};$$
$$\Gamma(\varepsilon_{12}\circ\phi^*)=\Gamma(\varepsilon_{32}\circ\varepsilon_{13}\circ\phi^*)=\pi_{132}\Gamma(\varepsilon_{13}\circ\phi^*)\in R_{12}.$$

Thus, defining

$$x^{\circledast}=\pi_{132}\pi_{312}^*\pi_{123}(\ominus x)'(a_1+a_2),$$

we achieve that $\iota(\phi^*)=(\iota(\phi))^{\circledast}$, i.e. ι is a $*$-ring isomorphism.

Finally, for $\phi\in\mathsf{Hom}(V_1,V_2)$ and $x=\Gamma(\phi)$, the kernel, the image, and the graph of the relation inverse to ϕ are obtained as

$$\mathsf{ker}\,x=xa_1;\quad \mathsf{im}\,x=(x+a_1)a_2;\quad x^{-1}=\pi_{231}\pi_{213}\pi_{123}(x).$$

Thus, in view of Proposition 15, with

$$\hat q(x)=(\mathsf{im}\,x)'a_2+x^{-1}(a_2+(\mathsf{ker}\,x)'a_1)$$

we have $\iota(q(\phi))=\hat q(\iota(\phi))$. □

Lemma 43. *Given variables $\overline{x}=(x_i\mid 0\leqslant i\leqslant 3)$, there are ortholattice terms $a_{ij}(\overline{x})$ and, for any new variable y, a term $r(y,\overline{x})$ such that for any substitution $\overline{b}$ in an MOL, L, $\overline{a}=(a_{ij}(\overline{b})\mid 1\leqslant i,j\leqslant 3)$ forms an orthogonal 3-frame and, furthermore, for any substitution c for y, one has $r(c,\overline{b})\in R_{12\overline{a}}$.*

Proof. According to Lemma 39 and Theorem 40, the terms

$$a_i(\overline{x})=d_i(\overline{x}),\quad a_{ij}(\overline{x})=d_0(\overline{x})(d_i(\overline{x})+d_j(\overline{x}))$$

evaluate to an orthogonal 3-frame under any substitution $\overline{b}$ for $\overline{x}$ which fails 3-distributivity. Now, consider the ortholattice term

$$l(x,z_1,z_2,z_3)=\Big(x'+x\big(x'+z_1+z_3\big)\Big)\big(x+x'(z_1+z_3)\big)\big(z_1+z_2\big).$$

Observe that $\hat x=\big(x'+x(xz_2)'\big)\big(x+(x+z_2)'\big)$ is a complement of x in the interval $[x(xz_2)',x+(x+z_2)']$, whereas $x(xz_2)'$ is a complement of xz_2 in the interval $[0,x]$ and $x+(x+z_2)'$ is a complement of $x+z_2$ in the interval $[x,1]$. Therefore, $\hat x$ is a complement of z_2 and $\hat x(z_1+z_2)$ is a complement of z_2 in the interval $[0,z_1+z_2]$. Thus for a given spanning orthogonal 3-frame $\overline{a}=(a_{ij}\mid 1\leqslant i,j\leqslant 3)$, one has $l(x,a_1,a_2,a_3)=\hat x(a_1+a_2)$, whence $l(x,a_1,a_2,a_3)\in R_{12\overline{a}}$. One easily verifies that $l(x,a_1,a_2,a_3)=x$ if $x\in R_{12\overline{a}}$. Using the above term for the 3-frame we put

$$r(y,\overline{x})=l\Big(y\big(a_1(\overline{x})+a_2(\overline{x})+a_3(\overline{x})\big),a_1(\overline{x}),a_2(\overline{x}),a_3(\overline{x})\Big).$$

□

16. Equivalence of equational theories of $\mathcal{N}^q$ and $\mathrm{L}_{\mathcal{N}}$

A *term interpretation* Ψ for the equational theory of one class $\mathcal{A}_1$ of structures of signature σ_1 into that of another, $\mathcal{A}_2$, of signature σ_2 fixes a list $\overline{z}$ of new auxiliary variables, associates with each each k-ary operation symbol $f \in \sigma_1$ a σ_2-term $\hat{f}(y_1, \ldots, y_k, \overline{z})$, and then associates with any σ_1-term $t(\overline{x}$ the σ_2-term $\Psi(t(\overline{x}) = \hat{t}(\overline{x}, \overline{z})$ according to the recursive definition $\Psi(x) = x$,

$$\Psi(f(t_1, \ldots, t_k)) = \hat{f}(\Psi(t_1), \ldots, \Psi(t_k), \overline{z})$$

such that an identity $s = t$ is valid in $\mathcal{A}_1$ if and only if the identity $\Psi(s) = \Psi(t)$ is valid in $\mathcal{A}_2$. An *equivalence* of the equational theories of $\mathcal{A}_1$ and $\mathcal{A}_2$ *via term interpretations* is given by a pair of term interpretations Ψ_1 in one direction, Ψ_2 in the other. Recall that σ^q denotes the signature of $*$-rings with additional operation symbol q for the Rickart relative inverse.

Lemma 44. *There is a constant c_1 and a term interpretation Ψ from terms t in signature σ^q to ortholattice terms $\hat{t} = \Psi(t)$ such that $|\hat{t}| \leqslant c_1|t|$ and such that the following holds for any $*$-regular ring $S = F^{k\times k}$, $F \in \mathtt{RC}$, with Rickart relative inverse q: the identity $\hat{s} = \hat{t}$ is valid in $\mathbb{L}(S)$ if and only if the identity $s = t$ is valid in e_1Se_1 for any orthogonal 3-frame $\overline{a}$ in $\mathbb{L}(S)$ and $e_1 = a_1$.*

Proof. Given MOL-variables $\overline{z} = (z_i \mid 0 \leqslant i \leqslant 3)$, we choose the terms $a_{ij}(\overline{z})$ according to Lemma 43. Now, for any k-ary operation symbol $f \in \sigma^q$, we put

$$\hat{f}(y_1, \ldots, y_k, \overline{z}) = f^{\#}(r(y_1, \overline{a}(\overline{z})), \ldots, r(y_k, \overline{a}(\overline{z})), \overline{a}(\overline{z}))$$

with $r(y, \overline{x})$ as in Lemma 43 and $f^{\#}(y_1, \ldots, y_k, \overline{z})$ as in Proposition 42. This defines Ψ. Now, consider any substitution $\overline{b} \in \mathbb{L}(S)$ for $\overline{z}$. Then one has the spanning orthogonal 3-frame $a_i(\overline{b}), a_{ij}(\overline{b})$ of the interval $L = [0, e]$ of $\mathbb{L}(S)$, where $e = \sum_{i=1}^{3} a_i(\overline{b})$. Applying Proposition 42 to $R = eSe$ (with $\mathbb{L}(R) = L$), one has $L \models \forall \overline{x}\ \hat{s}(\overline{x}, \overline{b}) = \hat{t}(\overline{x}, \overline{b})$ if and only if $e_1Se_1 = e_1Re_1 \models \forall \overline{x}\ s(\overline{x}) = t(\overline{x})$, where $e_1 = a_1(\overline{b})$. □

Theorem 45. *Let $F \in \mathtt{RC}$.*

1. *There is a constant c_2 such that for any identity $s = t$ in signature σ^q, if $s = t$ fails in some $F^{n\times n}$ then it fails in $F^{m\times m}$ for some $m \leqslant c_2(|s| + |t|)$ as well as for $m = c_2(|s| + |t|)$.*
2. *The equational theories of $\mathcal{N}^q$ and $\mathrm{L}_{\mathcal{N}}$ are equivalent via term interpretations.*

Proof. Apply the term interpretation of Lemma 44. (i) If $s = t$ fails in some $F^{n\times n}$, then $\hat{s} = \hat{t}$ fails in $\mathbb{L}(F^{3n\times 3n}) \cong \mathbb{L}\big((F^{n\times n})^{3\times 3}\big)$ as witnessed by the canonical 3-frame $\overline{a}$ and $F^{n\times n} \cong e_1(F^{3n\times 3n})e_1$. By Theorem 37(iii), $\hat{s} = \hat{t}$ fails in $\mathbb{L}(F^{k\times k})$ for some $k \leqslant 3(|\hat{s}| + |\hat{t}|)$. Again by Lemma 44, $s = t$ fails in some $e(F^{k\times k})e \cong F^{m\times m}$, cf. Proposition 27(ii). Thus, $c_2 = 3c_1$ will do. The claim for $m = c_2(|s| + |t|)$ follows by Lemma 10.

(ii) Lemma 5 provides a term interpretation from ortholattice terms u to σ^q-terms $\tilde{u}$ such that, for any $n > 0$, $u = 1$ holds in $\mathbb{L}(F^{n\times n})$ if and only if $\tilde{u} = 1$ holds in $F^{n\times n}$. Apply Lemma 4. Conversely, Lemma 44 provides a term interpretation $t \mapsto \hat{t}$

such that $\hat{s} = \hat{t}$ holds in $\mathbb{L}(F^{k\times k})$ for all k if and only if $s = t$ holds in $F^{m\times m}$ for all m. □

Corollary 46. *The decision problems for the equational theories of $\mathcal{N}^q$ and $\mathrm{L}_{\mathcal{N}}$ are polynomial time equivalent and belong to the complexity class PSPACE.*

Proof. Clearly, in any suitable encoding of terms, given a term interpretation Ψ, the output $\Psi(T)$ can be computed from the input t in time polynomial in the length of the input t. Thus by Theorem 45 the two problems are polynomial time equivalent. On the other hand, the complement of validity of identities $t = 0$ in $\mathrm{L}_{\mathcal{N}}$, the 'weak satisfiability' of t in the class $\{\mathbb{L}(\mathbb{C}^{d\times d}) \mid d < \omega\}$ (i.e. in "indefinite dimension") has been shown to be in PSPACE in [31, Theorem 25] cf. [3, 32]. Since PSPACE is closed under complements and by Lemma 4, validity of arbitrary identities in $\mathrm{L}_{\mathcal{N}}$ is in PSPACE too. □

17. Generalized rings of operators

The following is motivated by the construction, due to Murray and von Neumann [48], of the $*$-regular ring of unbounded operators affiliated with a finite von Neumann algebra factor and its generalization to AW^*-algebras by Berberian [4].

For a vector space V_F, by $\mathsf{PEnd}(V_F)$ we denote the set of all pairs (φ, U), where U is a linear subspace of V_F and $\varphi\colon U \to V$ is a linear map. We consider this set endowed with partial operations $(\varphi, U)+(\psi, W) = (\varphi+\psi,\, U\cap W)$, $(\varphi, U)\circ(\psi, W) = (\varphi \circ \psi,\, W \cap \psi^{-1}(U))$, and $\lambda(\varphi, U) = (\lambda\varphi, U)$, $\lambda \in F$. Since $U = \mathsf{dom}\,\varphi$ is uniquely determined by φ we may use φ to denote the pair (φ, U). We also write $\varphi\psi$ instead of $\varphi \circ \psi$.

Given a $*$-Λ-algebra F which is a division ring, an inner product space (V_F, Φ), and a downward directed set $\mathcal{D}$ of linear subspaces of V_F, consider an algebraic structure $\langle R; \oplus, \odot, \ominus, \underline{\lambda}\ (\lambda \in \Lambda), {}^{\circledast}, \underline{0}, \underline{1}\rangle$ with a binary relation $\approx$ in the signature of $*$-Λ-algebras σ_Λ which acts as a set of partial linear maps on V_F; that is, for each $\varphi \in R$ there is a linear subspace $\mathsf{dom}\,\varphi$ of V_F such that the map $x \mapsto \varphi(x)$, $x \in \mathsf{dom}\,\varphi$, is F-linear. Thus, we associate with $\varphi \in R$ a member of $\mathsf{PEnd}(V_F)$, also denoted by φ. This association is well defined but it may not be one-to-one. We shortly write R for this structure. We define $\mathcal{D}_\varphi = \{D \in \mathcal{D} \mid D \subseteq \mathsf{dom}\,\varphi \cap \mathsf{dom}\,\varphi^{\circledast}\}$, and assume that the following holds for all φ, $\psi \in R$:

(1) $D \subseteq \mathsf{dom}\,\varphi$ for some $D \in \mathcal{D}$;
(2) For any $D \in \mathcal{D}$, there is $D' \in \mathcal{D}$ such that $D' \subseteq \varphi^{-1}(D)$;
(3) $\mathsf{dom}\,\varphi \oplus \psi \in \mathcal{D}_\varphi \cap \mathcal{D}_\psi$ and $(\varphi \oplus \psi)(v) = \varphi(v) + \psi(v)$ for all $v \in \mathsf{dom}\,\varphi \oplus \psi$;
(4) $\underline{0}(v) = 0$ and $\underline{1}(v) = v$ for all $v \in V$;
(5) $\mathsf{dom}\,\varphi\odot\psi \subseteq \psi^{-1}(\mathsf{dom}\,\varphi)\cap\mathsf{dom}\,\psi\cap\mathsf{dom}\,\varphi^{\circledast}\cap(\varphi^{\circledast})^{-1}(\mathsf{dom}\,\psi^{\circledast})$ and $(\varphi\odot\psi)(v) = \varphi(\psi(v))$ for all $v \in \mathsf{dom}\,\varphi \odot \psi$;
(6) $\mathsf{dom}\,\underline{\lambda}\varphi \in \mathcal{D}_\varphi$ and $(\underline{\lambda}\varphi)(v) = \lambda(\varphi(v))$ for all $\lambda \in \Lambda$ and $v \in \mathsf{dom}\,\underline{\lambda}\varphi$;
(7) $\Phi(\varphi(v), w) = \Phi(v, \varphi^{\circledast}(w))$ for all $v \in \mathsf{dom}\,\varphi$ and $w \in \mathsf{dom}\,\varphi^{\circledast}$;
(8) $\approx$ is a congruence relation on R, where $\varphi \approx \psi$ if and only if $\varphi|_D = \psi|_D$ and $\varphi^{\circledast}|_D = \psi^{\circledast}|_D$ for some $D \in \mathcal{D}_\varphi \cap \mathcal{D}_\psi$.

Denoting the congruence class of φ by $[\varphi]$, call $R/_{\approx}$ a $*$-Λ-algebra of *generalized operators* on (V_F, Φ) if, in addition, $R/_{\approx}$ is a $*$-ring.

Theorem 47. *Any $*$-regular $*$-Λ-algebra $R/_{\approx}$ of generalized operators on (V_F, Φ) admits a faithful representation $\varepsilon\colon R/_{\approx} \to \mathsf{End}_\Lambda(U_{F'}, \Phi'|_U)$ for some elementary extension $(V'_{F'}, F')$ of (V_F, F) and a closed subspace U of $(V'_{F'}, \Phi')$.*

Proof. Let L_0 denote the lattice of finite dimensional linear subspaces of V_F. For any $U \in L_0$, let $\pi_U\colon V \to U$ denote the orthogonal projection onto U. We put

$$\mathcal{J} = \{\pi_U \mid U \in L_0\};$$
$$\mathcal{J}_D = \{\pi_U \in \mathcal{J} \mid U \subseteq D\},\ D \in \mathcal{D}.$$

For any $\pi \in \mathcal{J}$, $T_\pi = \pi\,\mathsf{End}_\Lambda(V_F, \Phi)\pi$ can be viewed as a regular $*$-Λ-subalgebra (without unit) of $\mathsf{End}_\Lambda(V_F, \Phi)$ according to Lemma 1. We put

$$T = \{T_\pi \mid \pi \in \mathcal{J}\} \cup \{\mathsf{id}_V\}.$$

Then T is a regular $*$-Λ-subalgebra of $\mathsf{End}_\Lambda(V_F, \Phi)$. Consider

$$\langle V_F; F_\Lambda; \mathsf{PEnd}(V_F); \mathcal{D}; L_0; \mathcal{J}; R\rangle$$

as a 7-sorted structure with $\mathsf{PEnd}(V_F)$ acting on V_F as a set of partial maps, with a unary predicate T on $\mathsf{PEnd}(V_F)$, with a binary operation $\Phi\colon V_F \times V_F \to F$, and with a binary relation $\mathsf{dom} = \{(\varphi, v) \mid v \in \mathsf{dom}\,\varphi\} \subseteq \mathsf{PEnd}(V_F) \times V_F$. Choose an elementary extension

$$(\hat{V}, \hat{F}_\Lambda, \widehat{\mathsf{PEnd}(V_F)}, \hat{\mathcal{D}}, \hat{L}_0, \hat{\mathcal{J}}, \hat{R})$$

with predicate $\hat{T}$ and operation $\hat{\Phi}$ which is ω-saturated over $\mathcal{J}$. Then following Mal'cev, this extension can be understood in terms of the algebra of generalized operators on $(\hat{V}_{\hat{F}}, \hat{\Phi})$. Also $\hat{T}$ may be considered as a regular $*$-Λ-subalgebra of $\mathsf{End}(\hat{V}_{\hat{F}}, \hat{\Phi})$ and $\hat{L}_0$ as an atomic modular sublattice of the lattice of closed subspaces of $(\hat{V}_{\hat{F}}, \hat{\Phi})$. Denoting the embedding in the form $x \mapsto \hat{x}$, we have $\mathsf{dom}\,\hat{\varphi} = \widehat{\mathsf{dom}\,\varphi}$ for all $\varphi \in \mathsf{PEnd}(V_F)$ and $\varphi \mapsto \hat{\varphi}$ is an embedding of $\mathsf{PEnd}(V_F)$ into $\mathsf{PEnd}(\hat{V}_{\hat{F}})$. Besides that for any $U \in L_0$, $\hat{U}$ is a finite dimensional subspace of $\hat{V}_{\hat{F}}$ with orthogonal projection $\hat{\pi}$, where $\pi = \pi_U$. Moreover in this case, $\hat{\pi}\hat{T}\hat{\pi} = \hat{\pi}\,\mathsf{End}(\hat{V}_{\hat{F}})\hat{\pi} \cong \mathsf{End}(\hat{U}_{\hat{F}})$. Observe that for any $\sigma, \tau \in \mathsf{End}(\hat{V}_{\hat{F}})$, one has

$$\sigma|_{\hat{U}} = \tau|_{\hat{U}} \text{ if and only if } \sigma\hat{\pi} = \tau\hat{\pi}.$$

Put for $\sigma \in \hat{T}$ and $\varphi \in R$:

$$\sigma \sim \varphi \ \Leftrightarrow\ \exists D \in \mathcal{D}_\varphi\ \forall \pi \in \mathcal{J}_D\ \sigma\hat{\pi} = \hat{\varphi}\hat{\pi} \text{ and } \sigma^*\hat{\pi} = \widehat{\varphi^{\circledast}}\hat{\pi}.$$

We claim that

$$S = \{\sigma \in \hat{T} \mid \sigma \sim \varphi \text{ for some } \varphi \in R\}$$

is a regular $*$-Λ-subalgebra of $\hat{T}$ and that the map

$$G\colon S \to R/_{\approx};\quad \sigma \mapsto [\varphi] \ \text{ where } \ \sigma \sim \varphi$$

is a $*$-Λ-algebra homomorphism from S onto $R/_{\approx}$. Then since T is represented in (V_F, Φ), $\hat{T}$ is represented in $(\hat{V}_{\hat{F}}, \hat{\Phi})$ by Corollary 16. Therefore, its substructure S is also represented in $(\hat{V}_{\hat{F}}, \hat{\Phi})$. Then the claim of Theorem follows with Proposition 17.

So, we prove now that S is a $*$-Λ-subalgebra of $\hat{T}$ and G is a $*$-Λ-algebra homomorphism. Observe that by (viii), $\sigma \sim \varphi \approx \psi$ implies $\sigma \sim \psi$ and $\sigma \sim \varphi$, $\sigma \sim \psi$ imply $\varphi \approx \psi$ for any $\sigma \in S$ and any $\varphi, \psi \in R$. Thus, the map G is well-defined and does not depend on the representative φ of $[\varphi]$. Moreover, it is obvious that $\sigma \sim \varphi$ implies $\sigma^* \sim \varphi^{\circledast}$ for any $\sigma \in S$ and any $\varphi \in R$. Therefore, S is closed under involution, and G preserves involution.

Let $\sigma \sim \varphi$ and $\tau \sim \psi$ be witnessed by D_φ and D_ψ. Then there is $D' \in \mathcal{D}$ such that $D' \subseteq D_\varphi \cap D_\psi$. According to (ii), there is $D \in \mathcal{D}_{\psi \odot \varphi}$ such that $D \subseteq D' \cap \varphi^{-1}(D') \cap (\psi^{\circledast})^{-1}(D')$. Let $\pi \in \mathcal{J}_D$. Then $\pi(V) \subseteq D \subseteq D' \subseteq D_\varphi$, while $\varphi\pi(V) \subseteq \varphi(D) \subseteq D' \subseteq D_\psi$. Similarly, $\pi(V) \subseteq D_\psi$, while $\psi^{\circledast}\pi(V) \subseteq D_\varphi$. Therefore, one has $\tau\sigma\hat{\pi} = \tau\hat{\varphi}\hat{\pi} = \hat{\psi}\hat{\varphi}\hat{\pi} = \widehat{\psi\varphi}\hat{\pi} = \widehat{(\psi \odot \varphi)}\hat{\pi}$ by (v). Similarly, $(\tau\sigma)^*\hat{\pi} = \sigma^*\tau^*\hat{\pi} = \widehat{(\varphi^{\circledast} \odot \psi^{\circledast})}\hat{\pi} = \widehat{(\psi \odot \varphi)^{\circledast}}\hat{\pi}$. This implies that $\tau\sigma \sim \psi \odot \varphi$. Simpler calculations yield $\tau + \sigma \sim \psi \oplus \varphi$ with the help of any $D \in \mathcal{D}_{\varphi \oplus \psi}$ such that $D \subseteq D_\varphi \cap D_\psi$ and $\lambda\sigma \sim \underline{\lambda}\varphi$ via any $D \in \mathcal{D}_{\underline{\lambda}\varphi}$ such that $D \subseteq D_\varphi$. Of course, $0 \sim \underline{0}$ and $1 \sim \underline{1}$. Thus S is a subalgebra of $\hat{T}$ and G is a homomorphism.

We prove surjectivity of G using saturation. It suffices to show that given $\varphi \in R$, $D \in \mathcal{D}_\varphi$ and $\pi_1, \dots, \pi_n \in \mathcal{J}_D$ there is $\sigma \in \hat{T}$ such that $\sigma\hat{\pi}_i = \hat{\varphi}\hat{\pi}_i$ and $\sigma^*\hat{\pi}_i = \widehat{\varphi^{\circledast}}\hat{\pi}_i$ for all $i \in \{1, \dots, n\}$. Put

$$U = \sum_{i=1}^{n} \mathsf{im}\, \pi_i,\ \pi = \pi_U,\ W = U + \varphi(U) + \varphi^{\circledast}(U),\ \text{and}\ \rho = \pi_W.$$

Now put $W_0 = W_1 = \hat{U}$, $\varphi_0 = \hat{\varphi}|_{W_0}$, and $\varphi_1 = \widehat{\varphi^{\circledast}}|_{W_1}$. Taking into consideration (vii), we may apply Lemma 48 below to W_0, W_1 and φ_0, φ_1 to get σ_0 satisfying all the conclusions of Lemma 48. Then for $\sigma = \sigma_0\hat{\rho}$, we have $\sigma = \hat{\rho}\sigma\hat{\rho} \in \hat{T}$. Moreover, $\sigma^* = \hat{\rho}\sigma^*\hat{\rho} \in \hat{T}$. By construction, $\sigma\hat{\pi} = \hat{\varphi}\hat{\pi}$ and $\sigma^*\hat{\pi} = \widehat{\varphi^{\circledast}}\hat{\pi}$. It follows that $\sigma\hat{\pi}_i = \hat{\varphi}\hat{\pi}_i$ and $\sigma^*\hat{\pi}_i = \widehat{\varphi^{\circledast}}\hat{\pi}_i$ for all $i \in \{1, \dots, n\}$, as required.

Since $\mathsf{im}\, G = R/_{\approx}$ is regular, to prove regularity of S, it suffices to show that $\ker G$ is regular according to [18, Lemma 1.3]. The latter statement is immediate by Lemma 49 below. □

Lemma 48. *Let (W, Φ) be a finite dimensional inner product space. Consider subspaces W_i and linear maps $\varphi_i \colon W_i \to W$, $i < 2$, such that $\Phi\big(x_0, \varphi_1(x_1)\big) = \Phi\big(\varphi_0(x_0), x_1\big)$ for all $x_i \in W_i$. Then there is $\sigma \in \mathsf{End}(W)$ such that $\sigma|_{W_0} = \varphi_0$ and $\sigma^*|_{W_1} = \varphi_1$.*

Proof. Choose bases $v_1, \dots, v_n$ and $w_1, \dots, w_n$ of W extending bases $v_1, \dots, v_k$ of W_0 and $w_1, \dots, w_l$ of W_1 respectively. There are unique $\sigma, \tau \in \mathsf{End}(W)$ such that

$$\Phi\big(\sigma(v_i), w_j\big) = \Phi\big(v_i, \tau(w_j)\big) = \begin{cases} \Phi\big(v_i, \varphi_2(w_j)\big) & \text{for } j \leqslant l; \\ \Phi\big(\varphi_1(v_i), w_j\big) & \text{for } i \leqslant k; \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\tau = \sigma^*$, $\sigma|_{W_0} = \varphi_0$, and $\sigma^*|_{W_1} = \tau|_{W_1} = \varphi_1$. □

Lemma 49. *Let R be a $*$-regular ring and let $a \in R$. Then there is $b \in R$ such that $aba = a$ and for any projection $p \in R$, $ap = a^*p = 0$ implies $bp = b^*p = 0$.*

Proof. Let $e = LP(a)$, let $f = RP(a)$, and let b be the Rickart relative inverse of a. Then $b \in fRe$ by Lemma 2(ii). Equalities $ap = a^*p = 0$ imply that $p(1-e) = 1-e$ and $(1-f)p = 1-f$, whence $ep = fp = 0$ and $bp = bep = 0$, $b^*p = b^*fp = 0$. □

Corollary 50. *Let $F \in \mathtt{RC}_\Lambda$ and let $S = R/_\approx$ be a $*$-regular $*$-Λ-algebra of generalized operators over a positive definite inner product space (V_F, Φ). Then $S \in \mathcal{N}_\Lambda$.*

18. Finite Rickart C^*-algebras and their rings of quotients

A $*$-ring R is *finite* if $xx^* = 1$ implies $x^*x = 1$ for all $x \in R$. R is *Rickart* if for any $a \in R$, there is a projection e generating the right annihilator of any element of R, that is $\{x \in R \mid ax = 0\} = eR$ for all $a \in R$. It follows that $aa^* = 0$ implies $a = 0$ cf. [7, 1.10]. Any $*$-regular ring is a finite Rickart $*$-ring [1]. R is a *Baer $*$-ring* if for any $S \subseteq R$, there is a projection e such that $\{x \in R \mid sx = 0 \text{ for all } s \in S\} = eR$. This definition is equivalent to the one in Kaplansky [36], see also Berberian [5, Exercise 7.1]. Of course, any Baer $*$-ring is Rickart.

A normed $*$-$\mathbb{C}$-algebra A which is complete with respect to the norm is a *Banach algebra*; A is a *C^*-algebra* if, in addition, $\|x^*x\| = \|x\|^2$ for all $x \in A$. A C^*-algebra which is a Rickart resp. Baer $*$-ring is called a Rickart C^*-algebra resp. an *AW^*-algebra*. Every von Neumann algebra is an AW^*-algebra, see [5, Proposition 4.9].

For a ring A, let $Q_M(A)$ denote the *maximal* (or *complete*) *ring of* [*right*] *quotients* of A, cf. Lambek [38, §4.3]. Let $Q_C(A)$ denote the *classical ring of* (*right*) *quotients* of a ring A, cf. Lambek [38, §4.6]. If $Q_C(A)$ exists, it can be considered as the subring of $Q_M(A)$ consisting of elements of the form ab^{-1}, where $a, b \in A$ and b is neither right nor left zero divisor.

Theorem 51. *For any finite AW^*-algebra A, there is a $*$-regular $*$-$\mathbb{C}$-algebra extension $Q_B(A)$ of A having $Q_M(A)$ as ring reduct such that $\mathbb{L}\big(Q_B(A)\big) = \mathbb{L}(A)$. Any finite Rickart C^*-algebra A admits a $*$-regular $*$-$\mathbb{C}$-algebra extension $Q_H(A)$ of A having $Q_C(A)$ as ring reduct such that $\mathbb{L}\big(Q_H(A)\big) = \mathbb{L}(A)$. For a finite AW^*-algebra A, $Q_B(A)$ embeds into $Q_H(A)$ as a $*$-$\mathbb{C}$-algebra.*

Proof. For any finite AW^*-algebra A, Berberian [4] has constructed $Q_B(A)$ which was later shown by Hafner [20] and Pyle [52] to have $Q_M(A)$ as ring reduct. Handelman [22, Theorem 2.1] has constructed $Q_H(A)$, which has $Q_C(0)$ as ring reduct according to Ara and Menal [1, Theorem 3.1]. □

Proposition 52. *For any finite Rickart C^*-algebra A, the set $\mathbb{L}(A)$ is an MOL ordered by $e \leq f$. The orthocomplement is given by $e' = 1 - e$. As MOLs, $\mathbb{L}(A) = \mathbb{L}\big(Q_H(A)\big)$ and, if A is an AW^*-algebra, then $\mathbb{L}(A) = \mathbb{L}\big(Q_B(A)\big)$.*

Proof. $\mathbb{L}(Q_H(A))$ is an MOL since $Q_H(A)$ is $*$-regular. Modularity of $\mathbb{L}(A)$ also follows from [22, Corollary 1.1]. □

An MOL L is *continuous* if $\sup X$ and $\inf X$ exist for any $X \subseteq L$ and $a \sup X = \sup\{ax \mid x \in X\}$ for any upward directed set $X \subseteq L$ and $a + \inf X = \inf\{a + x \mid x \in X\}$ for any downward directed set $X \subseteq L$.

Theorem 53. *Let A be a subdirectly irreducible finite Rickart C^*-algebra. Then A is a simple AW^*-algebra, $\mathbb{L}(A)$ a simple continuous MOL, and $Q_H(A) = Q_B(A)$ is a simple $*$-ring.*

Proof. A is a simple finite AW^*-algebra by [23, Theorem 3.1] and [22, Theorem 4.3]. Thus, by [36, Theorem 6.5] $\mathbb{L}(A)$ is a continuous MOL. By the proof of (3) in [22, Theorem 4.3] $Q_H(A)$ is simple. By Proposition 6 $\mathbb{L}(A)$ is simple too. [21, Proposition 2.10 and Theorem 2.3] imply that $Q_H(A) = Q_B(A)$. □

Theorem 54. *For any finite Rickart C^*-algebra A and a natural number $n > 0$, the following conditions are equivalent.*

(1) *There is a nontrivial n-system of $*$-matrix units of A.*
(2) *$\mathbb{C}^{n\times n}$ is a $*$-$\mathbb{C}$-subalgebra of some corner of A.*
(3) *$\mathbb{L}(\mathbb{C}^{n\times n})$ is a sectional sub-MOL of $\mathbb{L}(A)$.*
(4) *$\mathbb{L}(A)$ is not $(n-1)$-distributive.*

Proof. (iii) and (iv) are equivalent by Theorem 40. If (iv)holds, then by Theorem 40, there is a nontrivial orthogonal n-frame $\overline{a}$ in $\mathbb{L}(A)$, in particular $a_i \cap a_j = 0$ for $i \neq j$. By Handelman [22, Corollary 1.1], there are 'partial isometries' $w_i \in A$ such that $w_i^* w_i = a_1$ and $w_i w_i^* = a_i$. Then $w_i a_1 = w_i$ and $a_i w_i^* = w_i^*$ by [7, 6.2]. Thus, one obtains a nontrivial n-system of $*$-matrix units in A (cf. [49, Lemma 2.6.2]

$$e_{ii} = a_{ii}, \quad e_{ij} = w_i w_j^*.$$

Then (i) implies (ii) by Lemma 3 and (ii) implies (iii) by Lemma 12. □

Corollary 55. *Any variety $\mathcal{V}$ generated by the projection MOLs of some class of finite Rickart C^*-algebras is either n-distributive for some $n > 0$ or contains $\mathbb{L}(\mathbb{C}^{n\times n})$ for all $n > 0$. In particular, if $\mathcal{V}$ is not 2-distributive, then it contains a simple MOL of height 3.*

The latter verifies a conjecture of Gunter Bruns [11] for this particular kind of MOL-varieties. Though, the approach fails in general: there are subdirectly irreducible MOLs of infinite height which are generated by an orthogonal 3-frame [27, Corollary 4.16].

19. Representation of the regular ring of a finite AW^*-algebra

In this section, let A be a finite AW^*-algebra. We refer to [5, 4, 6]. A *strongly dense domain* (SDD, for short) is an ascending sequence $(e_n \mid n < \omega)$ of projections in A such that $\sup e_n = 1_A$. Very often, we denote an SDD $(e_n \mid n < \omega)$ by $\mathbf{e}$. If $\mathbf{f} = (f_n \mid n < \omega)$ is another SDD, then $\mathbf{e} \cap \mathbf{f} = (e_n \cap f_n \mid n < \omega)$ is also an SDD according to [5, Lemma 47.1].

An *operator with closure* (OWC, for short) is a sequence $(\langle x_n, e_n\rangle \mid n < \omega)$ of pairs, where $(e_n \mid n < \omega)$ is an SDD and $x_n \in A$, $n < \omega$, are such that

$$x_n e_m = x_m e_m \quad \text{and} \quad x_n^* e_m = x_m^* e_m \quad \text{for all} \quad m < n.$$

Again, very often, we denote an OWC $(\langle x_n, e_n\rangle \mid n < \omega)$ just by $\langle \mathbf{x}, \mathbf{e}\rangle$. For a projection $e \in A$ and an element $x \in A$, let $x^{-1}(e)$ denote the projection $g \in A$ such that gA is the right annihilator of the element $(1-e)x$; in particular $xg = exg$.

On the set $R(A)$ of all OWCs of A, operations are defined as follows, cf. [4, §2] or [5, Lemmas 47.3, 47.6]:

$$\langle \mathbf{x}, \mathbf{e}\rangle \oplus \langle \mathbf{y}, \mathbf{f}\rangle = \langle \mathbf{x}+\mathbf{y}, \mathbf{e}\cap\mathbf{f}\rangle,$$
$$\text{where } (x+y)_n = x_n + y_n \text{ and } (e\cap f)_n = e_n \cap f_n;$$
$$\langle \mathbf{x}, \mathbf{e}\rangle \otimes \langle \mathbf{y}, \mathbf{f}\rangle = \langle \mathbf{xy}, \mathbf{g}\rangle,$$
$$\text{where } (xy)_n = x_n y_n \text{ and } g_n = e_n \cap y_n^{-1}(e_n) \cap f_n \cap (x_n^*)^{-1}(f_n);$$
$$\underline{\lambda}\langle \mathbf{x}, \mathbf{e}\rangle = \langle \lambda\mathbf{x}, \mathbf{e}\rangle, \text{ where } (\lambda x)_n = \lambda x_n;$$
$$\langle \mathbf{x}, \mathbf{e}\rangle^{\circledast} = \langle \mathbf{x}^*, \mathbf{e}\rangle, \text{ where } (x^*)_n = x_n^*;$$
$$\underline{0} = \langle \mathbf{0}, \mathbf{1}\rangle, \text{ where } 0_n = 0_A \text{ and } 1_n = 1_A;$$
$$\underline{1} = \langle \mathbf{1}, \mathbf{1}\rangle.$$

According to [5, Lemma 47.1], $\mathbf{e}\cap\mathbf{f}$ is an SDD, while according to [5, Lemma 47.3], $\langle \mathbf{x}+\mathbf{y}, \mathbf{e}\cap\mathbf{f}\rangle$ is an OWC. We note also that $\mathbf{g}$ above is an SDD according to [5, Lemma 47.5], while $\langle \mathbf{x}, \mathbf{e}\rangle \otimes \langle \mathbf{y}, \mathbf{f}\rangle = \langle \mathbf{xy}, \mathbf{g}\rangle$ is an OWC according to [5, Lemma 47.6].

On the set $R(A)$ of all OWCs, one can define a binary relation $\equiv$ by

$$\langle \mathbf{x}, \mathbf{e}\rangle \equiv \langle \mathbf{y}, \mathbf{f}\rangle \quad \Leftrightarrow \quad \text{there is an SDD } \mathbf{g} \text{ such that}$$
$$x_n g_n = y_n g_n \text{ and } x_n^* g_n = y_n^* g_n \text{ for all } n < \omega.$$

According to [5, Lemma 7], $\equiv$ is an equivalence relation on $R(A)$. The equivalence class of $\langle \mathbf{x}, \mathbf{e}\rangle$ is denoted by $[\mathbf{x}, \mathbf{e}]$.

According to [5, Theorem 47.1] (see also [4, §2]), $\equiv$ is a congruence relation on $R(A)$. Moreover, one can identify

$$Q_B(A) = \{[\mathbf{x}, \mathbf{e}] \mid \langle \mathbf{x}, \mathbf{e}\rangle \text{ is an OWC in } A\},$$

so $Q_B(A)$ becomes a $*$-$\mathbb{C}$-algebra. Besides that, the map $x \mapsto \mathbf{x}$, where $x_n = x$ for all $n < \omega$, defines an embedding of A into $Q_B(A)$. Furthermore, $Q_B(A)$ is a $*$-regular ring according to [5, Theorem 52.1] (see also [4, Theorem 2.1 and Corollary 7.1]).

Theorem 56. *For a finite AW*-algebra A and a faithful representation $\varepsilon\colon A \to \mathsf{End}_{\mathbb{C}}(V, \Phi)$, $Q_B(A)$ is a $*$-regular $*$-$\mathbb{C}$-algebra of generalized operators on (V, Φ).*

Proof. For $x, y \in A$ and a projection $e \in A$, one has

$$xe = ye \quad \Leftrightarrow \quad \varepsilon(x)(v) = \varepsilon(y)(v) \text{ for all } v \in \mathsf{im}\,\varepsilon(e).$$

Furthermore, for an SDD $\mathbf{e}$ in A, we put $D(\mathbf{e}) = \bigcup_{n<\omega} \mathsf{im}\,\varepsilon(e_n)$; then $D(\mathbf{e}) \subseteq V_F$. We put also

$$\mathcal{D} = \{D(\mathbf{e}) \mid \mathbf{e} \text{ is an SDD in } A\}.$$

Then the set $\mathcal{D}$ is downward directed by [5, Lemma 47.1]. For an OWC $\langle \mathbf{x}, \mathbf{e}\rangle \in R(A)$, we define a partial endomorphism $(\mathbf{x}, \mathbf{e})$ on V_F by

$$\mathsf{dom}(\mathbf{x}, \mathbf{e}) = D(\mathbf{e}), \quad (\mathbf{x}, \mathbf{e})(v) = \varepsilon(x_n)(v) \text{ for } v \in \mathsf{im}\,\varepsilon(e_n) \subseteq D(\mathbf{e}).$$

Let $v \in \mathsf{im}\,\varepsilon(e_m)$; without loss of generality, we may assume that $m < n$. Then $\varepsilon(e_n)(v) = v = \varepsilon(e_m)(v)$, whence according to the definition of an OWC and the above,

$$\begin{aligned}\varepsilon(x_m)(v) = \varepsilon(x_m)\varepsilon(e_m)(v) = \varepsilon(x_m e_m)(v) = \varepsilon(x_n e_m)(v) = \\ = \varepsilon(x_n)\varepsilon(e_m)(v) = \varepsilon(x_n)(v).\end{aligned}$$

Therefore, $(\mathbf{x}, \mathbf{e})$ is well-defined.

We prove now that the set $\{(\mathbf{x}, \mathbf{e}) \mid \langle \mathbf{x}, \mathbf{e}\rangle \in R(A)\}$ with the operations defined as above satisfies conditions (i)-(viii) of §17. First of all, conditions (i) and (vi) are obviously satisfied. Condition (iv) is also trivial as $\mathsf{dom}\,\underline{0} = \mathsf{dom}\,\underline{1} = \mathsf{im}\,\varepsilon(1_A) = \mathsf{im}\,\mathsf{id}_V = V$.

For (ii), let $\langle \mathbf{x}, \mathbf{e}\rangle$ be an OWC and let $D \in \mathcal{D}$. Then there is an SDD $\mathbf{f}$ such that $D = D(\mathbf{f})$. We put $g_n = e_n \cap x_n^{-1}(f_n)$; then $\mathbf{g}$ is an SDD according to [5, Lemma 47.5]. If $v \in \mathsf{im}\,\varepsilon(g_n) \subseteq \mathsf{im}\,\varepsilon(e_n)$, then $\varepsilon(g_n)(v) = v$, whence

$$\begin{aligned}(\mathbf{x}, \mathbf{e})(v) = \varepsilon(x_n)(v) = \varepsilon(x_n)\varepsilon(g_n)(v) = \varepsilon(x_n g_n)(v) = \varepsilon(f_n x_n g_n)(v) = \\ = \varepsilon(f_n)\varepsilon(x_n g_n)(v) \in \mathsf{im}\,\varepsilon(f_n) \subseteq D,\end{aligned}$$

as $(1 - f_n)x_n g_n = 0$ implies $x_n g_n = f_n x_n g_n$. Therefore, $(\mathbf{x}, \mathbf{e})\big(D(\mathbf{g})\big) \subseteq D$, and (ii) holds.

To prove (iii), we assume that $\langle \mathbf{x}, \mathbf{e}\rangle$ and $\langle \mathbf{y}, \mathbf{f}\rangle$ are OWCs and we put $\varphi = (\mathbf{x}, \mathbf{e})$ and $\psi = (\mathbf{y}, \mathbf{f})$. Therefore, $\mathsf{dom}(\varphi \oplus \psi) = D(\mathbf{e} \cap \mathbf{f}) \subseteq D(\mathbf{e}) \cap D(\mathbf{f}) \in \mathcal{D}_\varphi \cap \mathcal{D}_\psi$, and for all $n < \omega$ and all $v \in \mathsf{im}\,\varepsilon(e_n \cap f_n)$, one has

$$(\varphi \oplus \psi)(v) = \varepsilon(x_n + y_n)(v) = \varepsilon(x_n)(v) + \varepsilon(y_n)(v) = \varphi(v) + \psi(v).$$

To prove (v), let again $\varphi = (\mathbf{x}, \mathbf{e})$ and $\psi = (\mathbf{y}, \mathbf{f})$. If $g_n = e_n \cap y_n^{-1}(e_n) \cap f_n \cap (x_n^*)^{-1}(f_n)$, then

$$\mathsf{im}\,\varepsilon(g_n) \subseteq \mathsf{im}\,\varepsilon(e_n) \cap \mathsf{im}\,\varepsilon\big(f_n \cap y_n^{-1}(e_n)\big) \cap \mathsf{im}\,\varepsilon(f_n) \cap \mathsf{im}\,\varepsilon\big(e_n \cap (x_n^*)^{-1}(f_n)\big).$$

According to the definition of dom and the argument for (ii), one has

$$\begin{aligned}&\mathsf{im}\,\varepsilon(e_n) \subseteq \mathsf{dom}\,\varphi = \mathsf{dom}\,\varphi^{\circledast};\\ &\mathsf{im}\,\varepsilon(f_n) \subseteq \mathsf{dom}\,\psi;\\ &\mathsf{im}\,\varepsilon\big(f_n \cap y_n^{-1}(e_n)\big) \subseteq \psi^{-1}\big(\mathsf{dom}\,\varphi\big);\\ &\mathsf{im}\,\varepsilon\big(e_n \cap (x_n^*)^{-1}(f_n)\big) \subseteq (\varphi^{\circledast})^{-1}\big(\mathsf{dom}\,\psi\big) = (\varphi^{\circledast})^{-1}\big(\mathsf{dom}\,\psi^{\circledast}\big).\end{aligned}$$

We get therefore:

$$\mathsf{dom}(\varphi \otimes \psi) = D(\mathbf{g}) \subseteq \mathsf{dom}\,\varphi^{\circledast} \cap \psi^{-1}\big(\mathsf{dom}\,\varphi\big) \cap \mathsf{dom}\,\psi \cap (\varphi^{\circledast})^{-1}\big(\mathsf{dom}\,\psi^{\circledast}\big),$$

and for $v \in \mathsf{im}\,\varepsilon(g_n) \subseteq \mathsf{im}\,\varepsilon(f_n)$, we also get

$$(\varphi \otimes \psi)(v) = \varepsilon(x_n y_n)(v) = \varepsilon(x_n)\varepsilon(y_n)(v) = \varphi\big(\psi(v)\big).$$

To prove (vii), let $\varphi = (\mathbf{x}, \mathbf{e})$, let $v \in \mathsf{dom}\,\varphi$, and let $w \in \mathsf{dom}\,\varphi^{\circledast} = \mathsf{dom}\,\varphi$. Since $\mathbf{e}$ is an SDD, there is $n < \omega$ such that $v, w \in \mathsf{im}\,\varepsilon(e_n)$. Then taking into consideration

that ε is a faithful representation of A, we get

$$\begin{aligned}\Phi\big(\varphi(v), w\big) &= \Phi\big(\varepsilon(x_n)(v), w\big) = \Phi\big(v, \varepsilon(x_n)^*(w)\big) = \\ &= \Phi\big(v, \varepsilon(x_n^*)(w)\big) = \Phi\big(v, \varphi^{\circledast}(w)\big).\end{aligned}$$

Finally to prove (viii), it suffices to show that for any OWCs $\langle \mathbf{x}, \mathbf{e}\rangle$, $\langle \mathbf{y}, \mathbf{f}\rangle \in R(A)$,

$$\langle \mathbf{x}, \mathbf{e}\rangle \equiv \langle \mathbf{y}, \mathbf{f}\rangle \quad \text{if and only if} \quad (\mathbf{x}, \mathbf{e}) \approx (\mathbf{y}, \mathbf{f}).$$

Indeed, let $\langle \mathbf{x}, \mathbf{e}\rangle \equiv \langle \mathbf{y}, \mathbf{f}\rangle$ be witnessed by an SDD $\mathbf{g}$. Then $\mathbf{h} = \mathbf{e} \cap \mathbf{f} \cap \mathbf{g}$ is an SDD by [5, Lemma 47.1], and for any $n < \omega$ and any $v \in \mathsf{im}\,\varepsilon(h_n) \subseteq \mathsf{im}\,\varepsilon(e_n) \cap \mathsf{im}\,\varepsilon(f_n) \cap \mathsf{im}\,\varepsilon(g_n)$, we have

$$\begin{aligned}(\mathbf{x}, \mathbf{e})(v) &= \varepsilon(x_n)(v) = \varepsilon(x_n)\varepsilon(g_n)(v) = \varepsilon(x_n g_n)(v) = \varepsilon(y_n g_n)(v) = \\ &= \varepsilon(y_n)\varepsilon(g_n)(v) = \varepsilon(y_n)(v) = (\mathbf{y}, \mathbf{f})(v); \\ (\mathbf{x}, \mathbf{e})^{\circledast}(v) &= \varepsilon(x_n^*)(v) = \varepsilon(x_n^*)\varepsilon(g_n)(v) = \varepsilon(x_n^* g_n)(v) = \varepsilon(y_n^* g_n)(v) = \\ &= \varepsilon(y_n^*)\varepsilon(g_n)(v) = \varepsilon(y_n^*)(v) = (\mathbf{y}, \mathbf{f})^{\circledast}(v).\end{aligned}$$

Therefore, $(\mathbf{x}, \mathbf{e}) \approx (\mathbf{y}, \mathbf{f})$ is witnessed by $D(\mathbf{h}) \in \mathcal{D}$. Conversely, let $(\mathbf{x}, \mathbf{e})|_{D(\mathbf{g})} = (\mathbf{y}, \mathbf{f})|_{D(\mathbf{g})}$ and $(\mathbf{x}, \mathbf{e})^{\circledast}|_{D(\mathbf{g})} = (\mathbf{y}, \mathbf{f})^{\circledast}|_{D(\mathbf{g})}$ for OWCs $\langle \mathbf{x}, \mathbf{e}\rangle$, $\langle \mathbf{y}, \mathbf{f}\rangle$ and an SDD $\mathbf{g}$. Again, we consider the SDD $\mathbf{h} = \mathbf{e} \cap \mathbf{f} \cap \mathbf{g}$ and take $n < \omega$ and $v \in \mathsf{im}\,\varepsilon(h_n) \subseteq \mathsf{im}\,\varepsilon(e_n) \cap \mathsf{im}\,\varepsilon(f_n) \cap \mathsf{im}\,\varepsilon(g_n)$. Then we get

$$\begin{aligned}\varepsilon(x_n)(v) = (\mathbf{x}, \mathbf{e})(v) = (\mathbf{y}, \mathbf{f})(v) = \varepsilon(y_n)(v), \\ \varepsilon(x_n^*)(v) = (\mathbf{x}, \mathbf{e})^{\circledast}(v) = (\mathbf{y}, \mathbf{f})^{\circledast}(v) = \varepsilon(y_n^*)(v),\end{aligned}$$

whence $x_n h_n = y_n h_n$ and $x_n^* h_n = y_n^* h_n$. Therefore, $\langle \mathbf{x}, \mathbf{e}\rangle \equiv \langle \mathbf{y}, \mathbf{f}\rangle$ is witnessed by $\mathbf{h}$, and the proof is complete. □

Theorem 57. *For any finite AW*-algebra A, the $*$-regular $*$-$\mathbb{C}$-algebra $Q_B(A)$ and the projection lattice $\mathbb{L}(A)$ admit faithful representations in a closed subspace of some ultrapower of a Hilbert space.*

Proof. First, let A be a finite AW^*-algebra. Due to the GNS-construction, cf. [55, Theorem 1.16.6], there is a Hilbert space (V, Φ) and a faithful representation ε of A in the algebra of bounded operators of (V, Φ). Application of Theorem 47 and Theorem 56 proves the claim for algebras. With Propositions 52 and 22, the claim for MOLs follows. □

Proposition 58. *If A is a finite Rickart C^*-algebra such that the projection lattice $\mathbb{L}(A)$ satisfies the ascending chain condition, then A is isomorphic as a C^*-algebra to $\prod_{i=1}^{k} \mathbb{C}^{n_i \times n_i}$, where $\dim \mathbb{L}(A) = n = \sum_{i=1}^{k} n_i$. Moreover, $A \cong Q_H(A) \cong Q_B(A)$.*

Proof. According to [41, Lemma 1.3], A is an AW^*-algebra, cf. also [7, Proposition 1.21]. Thus for any SDD $\mathbf{e}$ in A, there is $n < \omega$ such that $e_m = 1$ for all $m \geqslant n$. Due to the definition of OWCs we obtain a well defined map

$$\varphi\colon R(A) \to A, \quad \varphi\langle \mathbf{x}, \mathbf{e}\rangle = x_m \;\Leftrightarrow\; e_m = 1.$$

The map φ is onto as witnessed by the OWC $\langle \mathbf{x}, \mathbf{e}\rangle$, where $e_n = 1$ and $x_n = x \in A$ for all $n < \omega$. Moreover, one easily sees that φ is a homomorphism of $R(A)$ onto

A. Finally, $\langle \mathbf{x}, \mathbf{e}\rangle \equiv \langle \mathbf{y}, \mathbf{f}\rangle$ if and only if $\varphi\langle \mathbf{x}, \mathbf{e}\rangle = \varphi\langle \mathbf{y}, \mathbf{f}\rangle$. Therefore, the induced map $\varphi/_{\equiv}$ is an isomorphism of $Q_B(A)$ onto A with $(\varphi/_{\equiv})^{-1}$ being the canonical embedding of A into $Q_B(A)$. Thus $A \cong Q_H(A) \cong Q_B(A)$, canonically.

In particular, A is an artinian regular ring, and by the Wedderburn-Artin theorem, $A \cong \prod_{i=1}^{k} D_i^{n_i \times n_i}$, where $D_i = e_i A e_i$ is a division ring, cf. [38, Proposition 3.5.2 and Proposition 3.4.6]. But D_i is also a $\mathbb{C}$-algebra inheriting the norm from A. By the Mazur-Gelfand theorem [17, 2], it follows that $D_i \cong \mathbb{C}$. Thus, A is a finite-dimensional $\mathbb{C}$-algebra and the claim follows with [56, Theorem I.11.2]. □

20. The von Neumann variety revisited

Theorem 59. *Let $\mathcal{C}$ consist of $*$-regular rings such that for any $R \in \mathcal{C}$ there is a finite AW^*-algebra A such that $R \in \mathsf{S}_\exists\big(Q_B(A)\big)$. Put $\mathcal{Q} = \mathsf{V}_\exists(\mathcal{C})$ and $\mathcal{L} = \mathsf{V}\big(\mathbb{L}(R) \mid A \in \mathcal{C}\big)$. Then*

$$\mathcal{Q} \subseteq \mathcal{N} = \mathsf{V}_\exists\big(\mathbb{C}^{n\times n} \mid n < \omega\big), \quad \mathcal{Q}^q \subseteq \mathcal{N}^q = \mathsf{V}\big(\mathbb{C}^{n\times n} \mid n < \omega\big)$$

and

$$\mathcal{L} \subseteq \mathrm{L}_{\mathcal{N}} = \mathsf{V}\big(\mathbb{L}(\mathbb{C}^{n\times n}) \mid n < \omega\big).$$

Moreover, the following conditions are equivalent for any $n > 0$:

(i) $\mathcal{Q} = \mathsf{V}_\exists(\mathbb{C}^{n\times n})$; (ii) $\mathcal{Q} \neq \mathcal{N}$;
(iii) $\mathcal{Q}^q = \mathsf{V}(\mathbb{C}^{n\times n})$; (iv) $\mathcal{Q}^q \neq \mathcal{N}^q$;
(v) $\mathcal{L} = \mathsf{V}\big(\mathbb{L}(\mathbb{C}^{n\times n})\big)$; (vi) $\mathcal{L} \neq \mathrm{L}_{\mathcal{N}}$;
(vii) $\mathcal{L}$ *is $(n+1)$-distributive but not n-distributive.*

Analogous results hold for $\mathcal{N}_{\mathbb{C}}$ when considering $*$-$\mathbb{C}$-algebras.

Proof. The first claims follow from Theorems 57, 32, and 37 and Proposition 22. The equivalence of statements (i)-(vii) is then immediate by Theorem 54 and Proposition 41. □

Call a finite Rickart C^*-algebra A a *type* II_1 algebra if it is subdirectly irreducible and $\mathbb{L}(A)$ not n-distributive for any n. By Theorem 53, A is an AW^*-algebra, $Q_H(A) = Q_B(A)$ simple, and $\mathbb{L}(A)$ is simple.

Corollary 60. (1) *Any finite von Neumann algebra factor A of type II_1 is a type II_1 finite Rickart C^*-algebra.*
(2) $\mathcal{N} = \mathsf{V}_\exists\big(Q(A)\big)$ *for any type II_1 finite Rickart C^*-algebra.*
(3) $\mathrm{L}_{\mathcal{N}} = \mathsf{V}\big(\mathbb{L}(A)\big)$ *for any type II_1 finite Rickart C^*-algebra.*
(4) $Q(A_2) \in \mathsf{HS}_\exists\mathsf{P}_u\big(Q(A_1)\big)$ *for any finite Rickart C^*-algebras A_i, $i < 2$, of type II_1.*

Analogous results hold for $\mathcal{N}_{\mathbb{C}}$ when considering $*$-$\mathbb{C}$-algebras.

Proof. (i) follows from Theorem 53. On the other hand, if A is a type II_1 von Neumann algebra factor then $Q_H(A)$ is an indecomposable (cf. [40, VI Satz 3.5]) continuous regular ring which is even simple, see e.g. [18, Corollary 13.26]. Now, let M be a maximal ideal of A and let J be the ideal of $Q_H(A)$ generated by the projections of M. Again by the proof of (3) in [22, Theorem 4.3] J is a maximal ideal of $Q_H(A)$ whence $J = 0$ and from [58, Lemma 2.1] it follows that $M = J$.

Thus, A is simple. Finally, by [49, Theorem XIII] and Theorem 54 $\mathbb{L}(A)$ is not n-distributive for any n. This proves (ii). (iii) and (iv) follow from (ii), Lemma 12, and Proposition 8(ii). □

An analogue of Theorem 59 would hold for $Q_H(A)$, A is a finite Rickart C^*-algebra, if the following problem had a positive answer.

Problem 61. Is it true that $\mathbb{L}(A) \in \mathrm{L}_{\mathcal{N}}$ for any finite Rickart C^*-algebra A?

Indeed, by Proposition 22 and Theorem 25 this is equivalent to $Q_H(A) \in \mathcal{N}$. The subdirect decomposition of A into finite AW^*-factors and the embedding of A into a finite AW^*-algebra given by Handelman, Higgs, and Lawrence [23, Theorem 3.1] might be considered as a kind of positive evidence. Though, even representability in any inner product space is still an open problem.

References

[1] P. Ara, P. Menal, On regular rings with involution, Arch. Math. 42 (1984) 126-130.
[2] V.A. Belfi, R.S. Doran, Norm and spectral characterizations in Banach algebras, L'Enseignement Mathématique 26 (1980) 103-130.
[3] M. Ben-Or, D. Kozen, and J. Reif, The complexity of elementary algebra and geometry, 16th annual ACM-SIGACT symposium on the theory of computing (Washington, D.C., 1984). J. Comput. System Sci. 32 (1986) 251-264.
[4] S.K. Berberian, The regular ring of a finite AW^*-algebra, Ann. Math. 65 (1957) 224-240.
[5] S.K. Berberian, Baer $*$-Rings, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen 195, Springer-Verlag, Berlin–Heidelberg–New York, 1972.
[6] S.K. Berberian, The regular ring of a finite Baer $*$-ring, J. Algebra 33 (1972) 35-65.
[7] S.K. Berberian, Baer Rings and Baer $*$-Rings, Austin, 1988; http://www.ma.utexas.edu/mp_arc/c/03/03-181.pdf
[8] G. Birkhoff, Lattice Theory, third ed., American Mathematical Society, Providence, 1967.
[9] G. Birkhoff, J. von Neumann, The logic of quantum mechanics, Ann. Math. 37 (1936) 823-843.
[10] M. Bodirsky, H. Chen, Qualitative temporal and spatial reasoning revisited, J. Logic Comput. 19 (2009) 1359-1383.
[11] G. Bruns, Varieties of modular ortholattices, Houston J. Math. 9 (1983) 1-7.
[12] S. Burris, H.P. Sankappanavar, A Course in Universal Algebra, Springer, Berlin, 1981; www.math.uwaterloo.ca/ snburris/htdocs/ualg.html
[13] C.C. Chang, H.J. Keisler, Model Theory, third ed., Amsterdam, 1990.
[14] J. Dixmier, Les C^*-algèbres et Leurs Représentations, Deuxième édition, Paris, Gauthier-Villars, 1969.
[15] K.D. Fryer, I. Halperin, Coordinates in geometry, Trans. Roy. Soc. Canada, Sect III (3) 48 (1954) 11-26.
[16] L. Giudici, http://www.nohay.net/mat
[17] I.M. Gelfand, Normierte Ringe, Mat. Sbornik 9 (1941) 3-24.
[18] K.R. Goodearl, Von Neumann Regular Rings, second ed., Krieger, Malabar, 1991.
[19] V.A. Gorbunov, Algebraic Theory of Quasivarieties, Plenum, New York, 1998.
[20] I. Hafner, The regular ring and the maximal ring of quotients of a finite Baer $*$-ring, Michigan Math. J. 21 (1974) 153-160.
[21] D. Handelman, Coordinatization applied to Baer $*$-rings, Trans. Amer. Math. Soc. 235 (1978) 1-34.
[22] D. Handelman, Finite Rickart C^*-algebras and their properties, Studies in analysis, in: Adv. Math. Suppl. Stud. 4, Academic Press, New York–London, 1979, pp. 171-196.

[23] D. Handelman, D. Higgs, J. Lawrence, Directed abelian groups, countably continuous rings, and Rickart C^*-algebras, J. London Math. Soc. (2) 21 (1980) 193-202.
[24] C. Herrmann, Complemented modular lattices with involution and orthogonal geometry, Algebra Universalis 61 (2009) 339-364.
[25] C. Herrmann, Generators for complemented modular lattices and the von-Neumann-Jónsson coordinatization theorems, Algebra Universalis 63 (2010) 45-64.
[26] C. Herrmann, On the equational theory of projection lattices of finite von Neumann factors, J. Symb. Logic 75 (2010) 1102-1110.
[27] C. Herrmann, M.S. Roddy, Proatomic modular ortholattices. Representation and equational theory, Note di Matematica e Fisica 10 (1999) 55-88.
[28] C. Herrmann, M.S. Roddy, A note on the equational theory of modular ortholattices, Algebra Universalis 44 (2000) 165-168.
[29] C. Herrmann, F. Micol, M.S. Roddy, On n-distributive modular ortholattices, Algebra Universalis 53 (2005) 143-147.
[30] C. Herrmann, M. Semenova, Existence varieties of regular rings and complemented modular lattices, J. Algebra 314 (2007) 235-251.
[31] C. Herrmann, M. Ziegler, Computational complexity of quantum satisfiability, to appear in LICS in 2011.
[32] C. Herrmann, M. Ziegler, Computational complexity of quantum satisfiability, Full paper, in preparation.
[33] W. Hodges, Model Theory, Encyclopedia of Mathematics and Its Applications 42, Cambridge University Press, 1993.
[34] A.P. Huhn, Schwach distributive Verbände I, Acta Sci. Math. 33 (1972) 297-305.
[35] A.P. Huhn, Schwach distributive Verbände, Acta Fac. Rer. Nat. Univ. Comeniae (1971).
[36] I. Kaplansky, Projections in Banach algebras, Ann. Math. 63 (1951) 235-249.
[37] I. Kaplansky, Any orthocomplemented complete modular lattice is a continuous geometry, Ann. Math. 67 (1955) 524-541.
[38] J. Lambek, Lectures on Rings and Modules, Blaisdell, Waltham–Toronto–London, 1966.
[39] J. van Leeuwen, Handbook of Theoretical Computer Science, vol. A., Elsevier, Amsterdam, 1990.
[40] F. Maeda, Kontinuierliche Geometrien, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen 45, Springer, Berlin, 1958.
[41] F. Maeda, On a ring whose principal right ideals form a lattice, J. Sci. Hiroshima Univ. Ser. A 24 (1960) 509-525.
[42] A.I. Mal'cev, A correspondence between rings and groups, in: The Metamathematics of Algebraic Systems, Amsterdam, 1971.
[43] A.I. Mal'cev, Algebraic Systems, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen 192, Springer, Berlin, 1973.
[44] D. Marker, Model Theory: An Introduction, Graduate Texts in Mathematics 217, Springer, New York, 2002.
[45] R. Mayet, M.S. Roddy, n-Distributivity in ortholattices, in: Contributions to General Algebra **5**, Johannes Heyn, Klagenfurt, 1987, pp. 285-294.
[46] D. Mundici, The Turing complexity of AF C^*-algebras with lattice-ordered K_0. Computation theory and logic, In: Lecture Notes in Comput. Sci. 270, Springer, Berlin, 1987, pp. 256-264.
[47] F. Micol, On Representability of $*$-Regular Rings and Modular Ortholattices, Ph.D. thesis, Technische Universität Darmstadt, 2003;
http://elib.tu-darmstadt.de/diss/000303/diss.pdf
[48] F.J. Murray, J. von Neumann, On rings of operators, Ann. Math. 37 (1936) 116-229.
[49] F.J. Murray, J. von Neumann, On rings of operators IV, Ann. Math. 44 (1943) 761-808.
[50] J. von Neumann, Continuous Geometry, Princeton University Press, Princeton, New Jersey, 1960.

[51] N. Niemann, On representability of $*$-regular rings in endomorphism rings of vector spaces, to appear.
[52] E.S. Pyle, The regular ring and the maximal ring of quotients of a finite Baer $*$-ring, Trans. Amer. Math. Soc. 203 (1975) 201-213.
[53] H. Reich, Group von Neumann Algebras and Related Algebras, Dissertation, Universität Göttingen, 1999; http://www.math.uni-duesseldorf.de/∼reich/publ/diss/diss.dvi
[54] H. Reich, On the K- and L-theory of the algebra of operators affiliated to a finite von Neumann algebra, K-Theory 24 (2001) 303-326.
[55] S. Sakai, C^*-Algebras and W^*-Algebras, Ergebnisse der Mathematik und Ihrer Grenzgebiete 60, Springer-Verlag, Berlin–Heidelberg–New York, 1971.
[56] M. Takesaki, Theory of Operator Algebras I, Springer, New York, 1979.
[57] D.V. Tyukavkin, Regular rings with involution, Vestnik Moskovskogo Universiteta. Matematika 39 (1984) 29-32 (Russian).
[58] F.B. Wright, A reduction for algebras of finite type, Ann. Math. 60 (3) (1954) 560-570.

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