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BKNOTES - 4/97

The 11 sections and one appendix drafts enclosed were intended for a book to be titled "Lattices and Categories of Modules".

Although some informal checking was done as the drafts were written, these were not regarded as finished drafts. So, full checking of all statements from beginning to end was not done. Accordingly, the reader is warned that there may be errors in the text.

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Chapter I. Mathematical Theories Associated with Rings and Modules.

§1. Introduction.

This book is about rings and modules, but it is not primarily concerned with the elements, operations, and structure of particular rings or modules. Most of the results are expressed in terms of the following theories:

modular lattices,

abelian categories,

additive relation categories,

endomorphism algebras of additive relation categories, *without* constants,

endomorphism algebras of additive relation categories, *with* constants.

These five diverse theories have a common overlap, which will be displayed and developed here as a unified theory.

We will be concerned with associative rings with unit 1, and all modules will satisfy $1x = x$. Given a (left) module M over a fixed ring R, consider the lattices Su(M) of submodules of M and Con(M) of congruences of M. For any M, Su(M) and Con(M) are isomorphic complete and algebraic lattices which satisfy the modularity law. Special cases of lattices of the form Su(M) motivated the original development of modular lattice theory. For example, Su(M) is a projective geometry (in lattice form) for R a division ring and M a vector space.

Current module theory makes heavy use of abelian category methods, since R-**Mod** is an abelian category if R is a ring. Note particularly the importance of the well known embedding theorem for small abelian categories: any such category has an exact embedding into the category of **Z**-modules (abelian groups) and homomorphisms. So, a small abelian category can be regarded as an abstraction of a subcategory of the category of abelian groups and homomorphisms, such that the subcategory admits kernels and cokernels of homomorphisms and direct sums of pairs of objects.

Like abelian category theory, the theory of additive relation categories introduced by S. MacLane [1C] and D. Puppe [1B] is partly motivated by the study of modules over a fixed ring R with unit. For R-modules M and N, a relation $u \subseteq M \times N$ is called *R-linear* if u is a submodule of the direct sum $M \oplus N$. Elements u of $Su(M \oplus N)$ are in one-one correspondence with R-linear isomorphisms $\bar{u}:M_1/M_0 \longrightarrow N_1/N_0$ such that $M_0 = u^{-1}[0] \subseteq u^{-1}[N] = M_1$ in Su(M) and $N_0 = u[0] \subseteq u[M] = N_1$, where $\bar{u}(a + M_0) = b + N_0$ iff $\langle a,b \rangle \in u$. Graphs of R-linear homomorphisms $M \longrightarrow N$ correspond exactly to those isomorphisms with $M_1 = M$ and $N_0 = 0$ (everywhere-defined and single-valued relations). One can define the category R-**Rel** of R-modules and R-linear relations, where the morphisms $M \longrightarrow N$ in R-**Rel** are the elements of $Su(M \oplus N)$, and the composite of $u:M \longrightarrow N$ and $v:N \longrightarrow P$ in R-**Rel** is the usual composition of relations:

$$uv = \{\langle a,c \rangle \in M \oplus P: (\exists b)(b \in N,\ \langle a,b \rangle \in u \text{ and } \langle b,c \rangle \in v)\}.$$

(We shall follow this *forward* notation uv for u followed by v, both for composition of relations and for composition of functions and functors in general.) Each morphism $u:M \longrightarrow N$ in R-**Rel** has a *converse* $u^{\#}:N \longrightarrow M$, given by:

$$u^{\#} = \{\langle b,a \rangle: \langle a,b \rangle \in u\}.$$

The diagonal relation is a unit for multiplication:

$$1_M = \{\langle a,a \rangle: a \in M\}.$$

The Hom sets $Su(M \oplus N)$ are $(\mathbf{0},\mathbf{I})$ modular lattices when ordered by inclusion, so that meet $u \wedge v$ is intersection, join $u \vee v$ is (set) sum of submodules, $\mathbf{I}_{MN} = M \oplus N$ and $\mathbf{0}_{MN} = \{0\} = \{\langle 0,0 \rangle\}$. Following P. Hilton [1A, §5], a weak additive structure on $Su(M \oplus N)$ is given, using:

$$u + v = \{\langle a,b+c \rangle: a \in M,\ b,c \in N,\ \langle a,b \rangle \in u,\ \langle a,c \rangle \in v\}$$

to define *relational sum*. Now, $M \oplus 0$ is an additive 0, but not every element of $Su(M \oplus N)$ has a negative for relational sum. However, there is a (generalized) *negative* operation satisfying $u + (-u) + u = u$, given by:

$$-u = \{\langle a,-b \rangle: \langle a,b \rangle \in u\}.$$

The categories of form R-**Rel**, provided with the structures described above, are the prototypes from which additive relation category theory was developed. The abelian category R-**Mod** is isomorphic to a subcategory of R-**Rel**.

The endomorphism algebras of R-**Rel** are just the sets $\mathrm{Su}(M \oplus M)$ provided with structures induced by R-**Rel**. It is convenient to introduce two forms of endomorphism algebras, with and without constants. The theory of type τ_A has just the composition, converse, meet, join, relational sum and negative operations, but no constants. The theory of type τ_B has all of the above operations plus the constants 1, 0, **O** and **I**. Computation with these algebras is similar to diagram-chasing for additive relation categories. Since operations are everywhere defined for the algebras, complications associated with partially defined operations can be avoided by using them.

For each ring R with unit, the associated R-modules yield an abelian category R-**Mod** and an additive relation category R-**Rel**. For the algebraic theories of modular lattices and of additive relation algebras, R-modules yield classes of models:

$$\mathfrak{M}_L(R) = \{\mathrm{Su}(M)\colon M \text{ an } R\text{-module}\},$$

$$\mathfrak{M}_A(R) = \{\mathrm{Su}(M \oplus M) \text{ as a } \tau_A\text{-algebra: } M \text{ an } R\text{-module}\}.$$

$$\mathfrak{M}_B(R) = \{\mathrm{Su}(M \oplus M) \text{ as a } \tau_B\text{-algebra: } M \text{ an } R\text{-module}\}.$$

The primitive elements of these theories (submodules, additive relations, homomorphisms viewed as function graphs) represent modules as a whole. The choice of the ring R has no effect on the syntax of terms and formulas in these *external* theories. However, such a formula may represent a true statement for some choices of R and a false statement for others.

For modular lattices and additive relation algebras, we will investigate *finitely-presented word problems*. In a typical such problem, a finite list of polynomial equations are assumed to be satisfied by a list of submodules of a module, and it is required to determine whether another given polynomial equation is necessarily satisfied by these submodules. Lattice polynomials are obtained by combining variables using binary meet and join symbols, as usual. For additive relation algebras, the polynomials combine variables (and possibly constants 0, 1, **O** and **I**) using unary converse and negative, and binary meet, join, composition and sum symbols. For additive relation algebras, the submodules must belong to some Su(M⊕M), which is used to compute the polynomials. Each such word problem can be represented by a *basic universal Horn formula* Γ, of form:

$$(\forall x_1, x_2, \ldots, x_m)((p_1 = q_1 \wedge p_2 = q_2 \wedge \ldots \wedge p_n = q_n) \Rightarrow (p_0 = q_0))$$

where $p_i = p_i(x_1, \ldots, x_m)$ and $q_i = q_i(x_1, \ldots, x_m)$ are all lattice polynomials (or are all additive relation polynomials of type τ_A or of type τ_B). When $n = 0$, we have a free word problem

$$(\forall x_1, x_2, \ldots, x_m)(p_0(x_1, \ldots, x_m) = q_0(x_1, \ldots, x_m)),$$

which is an algebraic *identity* if true. Here, Γ is *true for R-modules* if it is satisfied for all members of $\mathfrak{M}_L(R)$, $\mathfrak{M}_A(R)$ or $\mathfrak{M}_B(R)$, respectively.

For abelian category theory, the corresponding concerns are usually called *diagram-chasing* problems (Noether isomorphisms, connecting homomorphism, five lemma, etc.). Typically, a diagram of modules and linear maps is given, and certain commutativity, exactness, additivity and other relationships are assumed. Then specified diagram relationships are showed to be necessary consequences of the hypotheses. Another kind of diagram-chasing, using relation category structures, is available for additive relation category theory. The informal use of diagrams is an effective means for presenting and organizing multiple hypotheses and their consequences in category-based theories. Our analysis will be restricted to diagram problems involving finitely many elementary hypotheses, in the appropriate sense.

Word problems and diagram-chasing problems can be expressed as true/false determinations for certain propositions in the external theories of modules described above. To make this precise, we consider formulations of the first-order predicate calculus with equality for these five theories. For the algebraic cases (lattices and additive relation algebras with our without constants), standard methods are used. For abelian category logic and additive relation category logic, we use purely relational logics. These logics have no constants or primitive functions; the only terms are variables.

The atomic predicates represent various partially-defined operations and relationships appropriate for these categories. The *universal sentences* (that is, closed prenex formulas with only universal quantifiers) in all five logics will be showed to represent the same body of knowledge about modules over a fixed ring. In Chapter I, we develop the results needed to precisely state and prove this assertion. Translation functions are constructed which map universal basic Horn sentences in one theory to those of another. Given a ring R with unit, the translated sentence must be true for R-modules if and only if the original sentence was true.

In Chapter I, we also begin the study of *inclusions* and *equivalences* of our theories for different rings. We show that $S\mathfrak{M}_L(R)$ is a *quasivariety* (or *universal Horn class*) of lattices for any ring R. (As usual, $S\mathcal{C}$ denotes the class of isomorphic images of subalgebras of algebras in $\mathcal{C}$, where $\mathcal{C}$ is any class of algebras of a given type.) So, $S\mathfrak{M}_L(R)$ is precisely the class of lattices satisfying all the universal sentences that are satisfied in Su(M) for all R-modules M. This result leads to a proof that $S\mathfrak{M}_A(R)$ is the quasivariety of additive relation algebras generated by $\mathfrak{M}_A(R)$, and similarly for $S\mathfrak{M}_B(R)$ and $\mathfrak{M}_B(R)$. For rings R and S, we write $R \precsim S$ if R has *fewer* models than S, so that *more* universal sentences are true for R-modules than for S-modules. Using the translation functions, we can characterize relations $R \precsim S$ in *all five* of our theories. For example, $S\mathfrak{M}_A(R) \subseteq S\mathfrak{M}_A(S)$ is one such characterization. The most useful characterization is by abelian categories: $R \precsim S$ iff there exists an exact embedding functor R-**Mod**$\longrightarrow$S-**Mod**. We call the relation $R \precsim S$ *diagram inclusion*, and there is a corresponding *diagram equivalence* relation $R \sim S$, when $R \precsim S$ and $S \precsim R$.

In Chapter II, the *computability* and *unsolvability* of decision predicates is analyzed, with respect to the universal sentences described above. A finite presentation (two generators and one defining equation) of type τ_B is given and proved to be generally unsolvable. (That is, this presentation has a recursively unsolvable word problem with respect to all quasivarieties $S\mathfrak{M}_B(R)$, where R is a nontrivial ring.) Using the universal basic Horn sentence translation functions, similar R-unsolvability results follow for universal sentences of the other four logics. However, the *free* word problems for lattices and additive relation algebras (algebraic identities) are recursively decidable for many rings R. The analysis reveals the distinct *varieties* of additive relation algebras generated by classes $\mathfrak{M}_B(R)$, R a ring. Essentially the same analysis holds for $\mathfrak{M}_A(R)$, and for the varieties of modular lattices generated by the classes $\mathfrak{M}_L(R)$ (see Hutchinson and Czédli [1D]).

In Chapter III, we show that the study of general diagram equivalence can be reduced to the study of diagram equivalence of rings R and S having the same prime power characteristic p^k. We develop a system of criteria for this special case, based on certain *identifiable* ring ideals. Most of the results depend upon the many known methods for constructing exact functors between module categories (change of rings, projective, injective and flat modules, ring idempotents, etc.).

Axiomatization and representation problems are considered in Chapter IV. For any ring R, a constructive process is given in Chapter I which generates an infinite axiomatization of $S\mathfrak{M}_L(R)$ by universal basic Horn sentences for lattices. For many rings R, it is known that no finite such axiomatization of $S\mathfrak{M}_L(R)$, $S\mathfrak{M}_A(R)$ or $S\mathfrak{M}_B(R)$ is possible. For other nontrivial rings, the *finite basis* problem for these quasivarieties remains open. A four-way system of dualities is also studied, based on order duality for lattices (exchange meet and join) and category duality (reverse the arrows).

Some of our representation theorems have the form:

> Each additive relation algebra satisfying property Ξ is in $S\mathfrak{M}_A(R)$, and every X in $S\mathfrak{M}_A(R)$ is a subalgebra of an additive relation algebra satisfying Ξ.

Here, such Ξ will always be expressible as a closed prenex sentence of our first-order additive relation algebra logic, having both existential and universal quantifiers. Other representation theorems have the above form adapted for modular lattices. The proofs are based on abelian category embedding methods.

A substantial simplification of the results for rings and modules occurs if only division rings and vector spaces are considered. In Chapter V, we give the analysis that applies in this special case. There are also special results that involve lattice complementation, since Su(M) and Su(M⊕M) are complemented modular lattices if M is a vector space.

Readers unfamiliar with universal algebra and modular lattice theory will find some basic facts and terminology described in Appendix A. Elementary information about rings and modules has been collected in Appendix B. Basic information about categories, including abelian categories and additive relation categories, is in Appendix C. There are other appendices with additional analysis of material in the main text.

§2. Modular Lattices that are Representable by Modules.

In this section, we verify that the classes of lattices $S\mathfrak{M}_L(R)$ are quasivarieties, using a specialized logical technique. It also follows that only countable rings need be considered for analysis of these lattice classes.

2.1. Definitions and Properties. Let τ_L denote the algebraic type for lattices (binary meet and join): $\tau_L = \langle\wedge,\vee\rangle$, arities $\langle 2,2\rangle$.

Suppose R is a ring. A lattice L is *representable by an R-module* if there is a lattice embedding (that is, one-one τ_L-homomorphism) $L \longrightarrow Su(M)$ for some R-module M. Let $\mathcal{L}(R)$ denote the class of all lattice that are representable by R-modules. That is:

$$\mathcal{L}(R) = S\mathfrak{M}_L(R) = \mathbf{S}\{Su(M)\colon M \text{ in } R\text{-}\mathbf{Mod}\} = \mathbf{S}\{Con(M)\colon M \text{ in } R\text{-}\mathbf{Mod}\}.$$

2.1a. For any R, $\mathcal{L}(R)$ admits isomorphic images, sublattices, and arbitrary products of lattices including the trivial lattice. (If $L_j \longrightarrow Su(M_j)$ is a lattice embedding, M_j an R-module, for all j in an index set J, then there is an obvious lattice embedding of $\prod_{j\in J} L_j$ into $Su(\prod_{j\in J} M_j)$.)

In view of 2.1a and A?, it is sufficient to show that $\mathcal{L}(R)$ admits ultraproducts to conclude that $\mathcal{L}(R)$ is a quasivariety of lattices. This can be done by methods of model theory, as showed by B. Schein [2A] and by M. Makkai and G. McNulty [2B]. Makkai and McNulty prove a number of properties of the quasivarieties $\mathcal{L}(R)$, including several properties discussed below.

Our approach yields these results of [2B], and we also obtain a completeness theorem for our specialized logic. In [2C], a related method was used to generate an infinite axiomatization of $\mathcal{L}(R)$ by universal basic Horn sentences when R is commutative. There is also a close connection with the logical methods of [2D]. An alternative method has been considered by G. Czédli [2E].

Given a ring R and lattice L which is not representable by an R-module, we first show that there exists a finite sequence of certain elementary steps by which this fact can be proved. These sequences resemble formal proofs, in that they are generated by *axioms* (which can begin a sequence or be added at the end of any already constructed sequence) and *rules of inference* (by which sequences containing certain specified premisses can be extended by adding consequences of these rules). However, the sequence elements are not formulas, but are instead pairs belonging to a Cartesian product $R^{(B)} \times L$, where $R^{(B)}$ is an appropriate free R-module.

2.2. Definitions. Let R be a ring and L a lattice. Let $R^{(B)}$ denote the free R-module generated by a set of variables B, where B consists of pairwise distinct variables of two sorts: a variable $\mathbf{b}_x$ for each x in L, and a variable b_i for each $i \geq 1$. An ⟨L,R⟩-*sequence* is a finite sequence of elements of $R^{(B)} \times L$ which is constructed according to the axiom scheme 2.2a and rules of inference 2.2b through 2.2e below. Letting u,v denote arbitrary elements of $R^{(B)}$ and x,y arbitrary elements of L:

2.2a. $\langle \mathbf{b}_x, x \rangle$ is an axiom.

2.2b. Premisses $\langle u,x \rangle$ and $\langle u,y \rangle$ yield the consequence $\langle u, x \wedge y \rangle$.

2.2c. A premiss $\langle u,x \rangle$ and $r \in R$ yield the consequence $\langle ru, x \rangle$.

2.2d. Premisses $\langle u,x \rangle$ and $\langle v,y \rangle$ yield the consequence $\langle u+v, x \vee y \rangle$.

2.2e. From a premiss $\langle u, x \vee y \rangle$, an ⟨L,R⟩-sequence of n terms can be extended to $n+2$ terms by adding $\langle b_n, x \rangle$ and $\langle u - b_n, y \rangle$.

Adapting the usual statement, the set $\mathcal{W}(L,R)$ (or just $\mathcal{W}$) of all ⟨L,R⟩-sequences is defined as the smallest set of finite sequences on $R^{(B)} \times L$ such that (1) $\langle w_1 \rangle$ is in $\mathcal{W}$ if w_1 is an axiom 2.2a, (2) $\langle w_1, \ldots, w_{n+1} \rangle$ is in $\mathcal{W}$ if $\langle w_1, \ldots, w_n \rangle$ is in $\mathcal{W}$ and w_{n+1} is either an axiom 2.2a or a consequence of a rule 2.2b,c,d whose premisses belong to $\{w_1, \ldots, w_n\}$, and (3) $\langle w_1, \ldots, w_{n+2} \rangle$ is in $\mathcal{W}$ if $\langle w_1, w_2, \ldots, w_n \rangle$ is in $\mathcal{W}$ and w_{n+1}, w_{n+2} are consequences of an instance of 2.2e with premiss in $\{w_1, \ldots, w_n\}$. A pair $\langle \mathbf{b}_x, y \rangle$ is ⟨L,R⟩-*derivable* if it is the last term of some ⟨L,R⟩-sequence.

Roughly speaking, the pair $\langle u,x\rangle$ represents an assertion that a certain R-module element corresponding to u belongs to a certain submodule corresponding to x. The generator $\mathbf{b}_x$ is intended to represent an element in *general position* in the x-submodule, so that 2.2a is the *only* assumption made about $\mathbf{b}_x$. The rules of inference 2.2b,c,d are straightforward. Rule 2.2e is more complicated: it is intended to express the fact that any element u in $x \vee y$ is the sum of some element b_n in x and (necessarily) $u - b_n$ in y. Hence b_n implicitly has an existential quantifier. To avoid a conflict of variables b_i in construction of an $\langle L,R\rangle$-sequence involving several uses of 2.2e, we impose the restriction that b_n be used in applying 2.2e to an $\langle L,R\rangle$-sequence of n terms.

The following derived rule of inference is convenient for construction of $\langle L,R\rangle$-sequences.

2.3. Proposition. Suppose R is a ring, L is a lattice,

$\mathbf{w} = \langle w_1, w_2, \ldots, w_n\rangle$ is an $\langle L,R\rangle$-sequence,

$\langle u_j, x_j\rangle \in \{w_1, w_2, \ldots, w_n\}$ for $j = 1,2,\ldots,m$,

$y \geq x_1 \vee x_2 \vee \ldots \vee x_m$ in L, and $r_1, r_2, \ldots, r_m$ are in R.

Then $\mathbf{w}$ can be extended to an $\langle L,R\rangle$-sequence $\langle w_1, w_2, \ldots, w_{n+2m+2}\rangle$ such that

$$w_{n+2m+2} = \langle r_1u_1 + r_2u_2 + \ldots + r_mu_m,\ y\rangle.$$

Proof: Assuming the hypotheses, extend $\mathbf{w}$ by $\langle \mathbf{b}_y, y\rangle$ using 2.2a, $\langle 0,y\rangle$ using 2.2c, $\langle r_1u_1, x_1\rangle$ using 2.2c, $\langle 0 + r_1u_1,\ y \vee x_1\rangle = \langle r_1u_1, y\rangle$ using 2.2d, and then use 2.2c and 2.2d alternately. ∎

The next result is the appropriate completeness theorem for the specialized logic of $\langle L,R\rangle$-derivability.

2.4. Theorem. Let R be a ring and L a lattice. For x,y in L, the following are equivalent:

2.4a. $\langle \mathbf{b}_x, y\rangle$ is $\langle L,R\rangle$-derivable.

2.4b. If $h: L \longrightarrow Su(M)$ is a lattice homomorphism for some R-module M, then

$h(x) \subseteq h(y)$.

Proof: Assume the hypotheses, and suppose $h: L \longrightarrow Su(M)$ is a homomorphism for some M in R-**Mod**. Assuming 2.4a, there exists an $\langle L,R\rangle$-sequence $\mathbf{w} = \langle w_1, \ldots, w_n\rangle$ such that $w_n = \langle \mathbf{b}_x, y\rangle$. Let $w_i = \langle u_i, x_i\rangle$ for $i \leq n$, and assume that $v \in h(x)$. By induction, we define R-linear maps $g_i: R^{(B)} \longrightarrow M$ for $i = 1,2,\ldots,n$ such that $g_i(\mathbf{b}_x) = v$, $g_i(\mathbf{b}_z) = 0$ for $z \neq x$ in L, $g_i(b_j) = 0$ for all $j \geq i$, and $g_i(u_j) \in h(x_j)$ for all $j \leq i$. Clearly, g_1 has already been defined and has the required properties. Assume the induction hypothesis: g_i has the required properties, and that $\langle w_1, \ldots, w_i\rangle$ is an $\langle L,R\rangle$-sequence, for some $i < n$. If $u_{i+1} \neq b_i$, then w_{i+1} is either an axiom 2.2a or is obtained by one of the rules 2.2b,c,d. Then define $g_{i+1} = g_i$, and note that g_{i+1} has the required properties. For $u_{i+1} = b_i$, $\langle w_1, \ldots, w_i\rangle$ has been extended by rule 2.2e, so there exists $k \leq i$ such that:

$$w_k = \langle u_k, x_{i+1} \vee x_{i+2}\rangle, \quad w_{i+1} = \langle b_i, x_{i+1}\rangle, \quad w_{i+2} = \langle u_k - b_i, x_{i+2}\rangle.$$

By hypothesis $g_i(u_k) \in h(x_{i+1}) \vee h(x_{i+2})$, so there exists v_i in $h(x_{i+1})$ such that $g_i(u_k) - v_i$ is in $h(x_{i+2})$. Now define g_{i+1} by $g_{i+1}(b_i) = v_i$ and $g_{i+1}(b_j) = g_i(b_j)$ for $j \neq i$, and $g_{i+2} = g_{i+1}$. Since the variable b_i has coefficient 0 in all terms u_j, $j \leq i$, g_{i+1} and g_{i+2} have the required properties. This completes the induction, and we immediately obtain:

$$v = g_n(\mathbf{b}_x) = g_n(u_n) \in h(x_n) = h(y).$$

Therefore, $h(x) \subseteq h(y)$, which proves 2.4a $\Rightarrow$ 2.4b.

We have shown the *correctness* of our proofs: the derivability of $\langle \mathbf{b}_x, y\rangle$ implies the property 2.4b for homomorphisms of L into R-submodule lattices. The *completeness* of this method, that $\langle \mathbf{b}_x, y\rangle$ is $\langle L,R\rangle$-derivable when x and y in L satisfy 2.4b, is proved by a lengthy and complex argument. It has been separated from the main text (see Appendix D). ∎

2.5. Corollary. Let R be a ring and L a lattice. Then the following are equivalent:

2.5a. L is representable by an R-module, that is, $L \in \mathcal{L}(R)$.

2.5b. For all x,y in L, $\langle \mathbf{b}_x, y\rangle$ is $\langle L,R\rangle$-derivable $\Rightarrow$ $x \le y$.

Proof: Suppose $L \in \mathcal{L}(R)$, with $\lambda: L \longrightarrow Su(N)$ a lattice embedding for an R-module N. If $\langle \mathbf{b}_x, y\rangle$ is $\langle L,R\rangle$-derivable, then $\lambda(x) \subseteq \lambda(y)$ by 2.4, so $x \le y$ since λ is a lattice embedding. Therefore, 2.5a $\Rightarrow$ 2.5b.

Let $\theta = \bigcap \{\varphi \in Con(L): L/\varphi \in \mathcal{L}(R)\}$, so L/θ is in $\mathcal{L}(R)$ by 2.1a and A?. If 2.5a fails, then there are x,y in L with $x \nleq y$ but $\langle x,y\rangle \in \theta$. Now 2.4b is satisfied for x and y, since $h(x) = h(y)$ for any such h. So, $\langle \mathbf{b}_x, y\rangle$ is $\langle L,R\rangle$-derivable by 2.4, and 2.5b fails. Therefore, 2.5b $\Rightarrow$ 2.5a. ■

The following brief example shows that the standard argument that each Su(M) satisfies lattice modularity can be expressed using $\langle L,R\rangle$-sequences. A longer example of $\langle L,R\rangle$-sequence calculation is given in Appendix D.

2.6. Example. Suppose $L \in \mathcal{L}(R)$ for an arbitrary ring R. To prove L is modular, we show that for y_1, y_2, y_3 in L, $y_1 \le y_3$ implies:

$$(y_1 \vee y_2) \wedge y_3 = p \le q = y_1 \vee (y_2 \wedge y_3).$$

An $\langle L,R\rangle$-sequence is constructed as follows:

1. $\langle \mathbf{b}_p, p\rangle$ by 2.2a.

5. $\langle \mathbf{b}_p, y_1 \vee y_2\rangle$ by 2.3 using 1 (terms 2,3,4 omitted).

6. $\langle b_5, y_1\rangle$ by 2.2e using 5.

7. $\langle \mathbf{b}_p - b_5, y_2\rangle$ by 2.2e continued.

13. $\langle \mathbf{b}_p - b_5, y_3\rangle$ by 2.3 using 1 and 6 (terms 8-12 omitted).

14. $\langle \mathbf{b}_p - b_5, y_2 \wedge y_3\rangle$ by 2.2b for 7 and 13.

15. $\langle \mathbf{b}_p, y_1 \vee (y_2 \wedge y_3)\rangle$ by 2.2d for 6 and 14.

Since $\langle \mathbf{b}_p, q\rangle$ is $\langle L,R\rangle$-derivable if $y_1 \le y_3$, it follows by 2.5 that $p \le q$, hence L is modular.

By using 2.5, we can confirm that each $\mathcal{L}(R)$ is a quasivariety, and prove variants 2.8 and 2.10 of some useful results of Makkai and McNulty (see [2B, Thm. 3, p. 29, and Cor. 3, Cor. 6, p. 30]). Each $\langle L,R\rangle$-sequence construction can be abstracted as a collection of relations between variables representing ring elements plus a collection of relations between variables representing lattice elements.

2.7. Definitions. Let τ_R denote the algebraic type of rings with unit, with binary sum and product, unary negation, and constants 0 and 1:

$$\tau_R = \langle +,-,0,*,1\rangle, \quad \text{arities } \langle 2,1,0,2,0\rangle.$$

A *system of ring equations formula* is a sentence of first-order ring theory of the form below, for some $m,n \geq 1$:

$$(\exists x_1)(\exists x_2)\ldots(\exists x_m)((p_1=0)\wedge(p_2=0)\wedge\ldots\wedge(p_n=0)),$$

where each $p_i = p_i(x_1,x_2,\ldots,x_m)$, $i \leq n$, is a τ_R-polynomial.

If L is not in $\mathcal{L}(R)$, then by 2.5 there exist x and y in L such that $x \nleq y$ but $\langle \mathbf{b}_x, y\rangle$ is $\langle L,R\rangle$-derivable. Let **w** denote an $\langle L,R\rangle$-sequence with last term $\langle \mathbf{b}_x, y\rangle$. We can construct a system of ring equations Ψ and a basic universal Horn sentence Γ for lattices corresponding to **w**. More precisely, if Ψ is satisfied in a ring S and the hypotheses of Γ are satisfied in a lattice K, then we can define a $\langle K,S\rangle$-sequence with step by step construction following the same pattern as **w**. The conclusion of Γ corresponds to $x \leq y$, so that Γ is not satisfied in L. However, Γ is satisfied in every lattice in $\mathcal{L}(S)$ if Ψ is satisfied in S. In particular, Ψ is satisfied in R, and so Γ is satisfied in every lattice in $\mathcal{L}(R)$.

In example 2.6, we gave an $\langle L,R\rangle$-sequence $\langle w_1,w_2,\ldots,w_{15}\rangle$ involving elements $\mathbf{b}_p$, b_3 and b_5 of B. We could introduce ring variables r_{ip}, r_{i3} and r_{i5} and lattice variables x_i such that w_i is represented by $\langle u_i, x_i\rangle$ with $u_i = r_{ip}\mathbf{b}_p + r_{i3}b_3 + r_{i5}b_5$ for $i = 1,2,\ldots,15$. Since w_{14} was obtained by 2.2b for w_7 and w_{13}, we would include equations $r_{14,p} = r_{7,p} = r_{13,p}$, $r_{14,3} = r_{7,3} = r_{13,3}$ and $r_{14,5} = r_{7,5} = r_{13,5}$ in Ψ to force $u_{14} = u_7 = u_{13}$, and equation $x_{14} = x_7 \wedge x_{13}$ as a hypothesis of Γ. It is straightforward to develop such equations for all the rules of inference of 2.2; the details are given in Appendix D. The result stated below is then obtained.

2.8. Proposition. Suppose R is a ring and L is a lattice not in $\mathcal{L}(R)$. Then there exist a system of ring equations Ψ and a universal basic Horn sentence Γ for lattices such that Ψ is satisfied in R, Γ is not satisfied in L, and for any ring S, Ψ satisfied in S implies Γ is satisfied in every lattice in $\mathcal{L}(S)$.

The known fact that each $\mathcal{L}(R)$ is a quasivariety follows now.

2.9. Corollary. For each ring R, $\mathcal{L}(R)$ is a quasivariety of lattices, axiomatizable by a set of universal basic Horn sentences of first-order lattice theory.

Proof: Suppose L is in the quasivariety generated by $\mathcal{L}(R)$. If L is not in $\mathcal{L}(R)$, then there exists a basic universal Horn sentence Γ such that $\mathcal{L}(R) \models \Gamma$ but Γ is not satisfied in L, by 2.8. This is a contradiction, proving that $\mathcal{L}(R)$ is a quasivariety. ∎

The next result, from [2B], shows that only countable rings need be considered when studying the quasivarieties $\mathcal{L}(R)$.

2.10. Corollary. For each ring R, there exists a countable subring S of R such that $\mathcal{L}(R) = \mathcal{L}(S)$.

Proof: For any ring T, let $\mathcal{E}_T$ denote the set of formulas Ψ which are systems of ring equations satisfied in T. We can assume that such a Ψ is a sentence on some fixed countable alphabet, so that $\mathcal{E}_T$ is a countable set. Given R, each Ψ in $\mathcal{E}_R$ is satisfied in some finitely generated subring of R. So, there exists a countable subring S of R such that $\mathcal{E}_S = \mathcal{E}_R$. By 2.8, a lattice L is not in $\mathcal{L}(R)$ iff it is not in $\mathcal{L}(S)$. But then $\mathcal{L}(R) = \mathcal{L}(S)$. ∎