§3. Additive Relation Algebras.

In this section, we begin the axiomatic study of additive relation algebras. It is useful to introduce two varieties of additive relation algebra, with and without constants. The form with constants is similar to that described in [3A, 3B], and is closely related to the endomorphism algebras obtained from the additive relation category theories of S. MacLane, D. Puppe, P. Hilton and H.-B. Brinkmann [3C,3D,3E,3F,3G,3H]. The new form without constants is even more closely connected to category theory, as will be showed in later chapters. Some of the computations here are adapted from this previous work, but we will omit individual references in most cases.

3.1. Definitions. The algebraic types for additive relation algebras without and with constants are given as follows:

$\tau_A = \langle +,-,\cdot,\#,\wedge,\vee \rangle$, arities $\langle 2,1,2,1,2,2 \rangle$,

$\tau_B = \langle +,-,0,\cdot,1,\#,\wedge,\vee,\mathbf{0},\mathbf{I} \rangle$, arities $\langle 2,1,0,2,0,1,2,2,0,0 \rangle$.

The variety $\mathcal{V}_A$ consists of all τ_A-algebras satisfying 3.1a-h, and members of $\mathcal{V}_A$ are called *additive relation algebras*. The variety $\mathcal{V}_B$ consists of all τ_B-algebras satisfying 3.1a-i, and members of $\mathcal{V}_B$ are called *additive relation algebras with unit* (see 3.12).

3.1a. All modular lattice identities are satisfied by $\wedge$ and $\vee$. (Let $\leq$ denote the lattice order, and let $f \leq g$ denote the equation $f \wedge g = f$, in the usual way.)

3.1b. Sum is associative and commutative, and is monotonic in both arguments (that is, $(f \wedge g) + (h \wedge k) \leq f + h$).

3.1c. Multiplication is associative, and is monotonic in both arguments (that is, $(f \wedge g)(h \wedge k) \leq fh$).

3.1d. Converse (written $f^\#$) is an *involution* and *generalized inverse* for multiplication, that is:

$(f^\#)^\# = f$, $(fg)^\# = g^\# f^\#$ and $ff^\# f = f$.

It preserves meet and join:

$(f \wedge g)^{\#} = f^{\#} \wedge g^{\#}$ and $(f \vee g)^{\#} = f^{\#} \vee g^{\#}$.

3.1e. Negation is an *involution* and *generalized negative* for sum, that is:

$-(-f) = f$, $-(f+g) = (-f)+(-g)$ and $f+(-f)+f = f$.

Negation preserves meet and join:

$-(f \wedge g) = (-f) \wedge (-g)$ and $-(f \vee g) = (-f) \vee (-g)$.

It commutes with converses and multiplication:

$-(f^{\#}) = (-f)^{\#}$ and $-(fg) = (-f)g = f(-g)$.

3.1f. $f+(-f)$ is a *null* element, that is:

$(f+(-f))g(f+(-f)) = f+(-f)$.

It also satisfies:

$(f+(-f))^{\#}(f+(-f)) \le (f+g)^{\#}(f+g)$, and

$(f+(-f))(f+(-f))^{\#} \ge (f+g)(f+g)^{\#}$.

3.1g. $fg \wedge h \le ff^{\#}h$ and $hf^{\#}f \le gf \vee h$.

3.1h. $(f+g)h \ge fh+gh$ and $f(g+h) \le fg+fh$.

The additional axiom group for the constants in τ_B is:

3.1i. 1 is a unit ($1f = f1 = f$), $0 = 1+(-1)$, 0 is a zero ($f+0 = f$), $\mathbf{0} = 0^{\#}0$, $\mathbf{I} = 00^{\#}$ and $\mathbf{0} \le f \le \mathbf{I}$.

From the above, we see that $\mathcal{V}_A$ is axiomatized by 32 τ_A-identities and $\mathcal{V}_B$ by 40 τ_B-identities. As previously observed, τ_A is the reduct obtained from τ_B by omitting the constants (nullary operations). Note that the first five operations of τ_B are precisely the τ_R operations used for rings with unit. However, additive relation algebras are not additive groups for sum in general (unless B in $\mathcal{V}_B$ is trivial, $f+(-f) = 0$ is not satisfied everywhere in B), and we have only the half-distributivities 3.1h, not full distributivity as in ring theory. The last four operations of τ_B are the $(\mathbf{0},\mathbf{I})$ lattice operations, and B in $\mathcal{V}_B$ is indeed a modular lattice with smallest element $\mathbf{0}$ and largest element $\mathbf{I}$.

Most of the axioms are familiar from ring theory or lattice theory. The properties of relational sums were studied in [3C], [3D] and [3E]. Less familiar axioms such as 3.1f,g can be understood more readily as the theory is developed. In particular, axioms 3.1g are modifications (removing the constants) of the axioms $f\mathbf{I} \wedge g \le ff^{\#}g \le f\mathbf{0} \vee g$ of [3A], and are related to the axioms K2a and K2b of [3C] and [3D].

We have not closely studied the dependence relationships among these axioms. However, it is clear that only one of the identities involving $\wedge$ and $\vee$ in 3.1d is necessary, and similarly for 3.1e. We will eventually see that parts of 3.1i could be omitted also.

The τ_B-algebras obtained from modules by the operations described in §1 (that is, endomorphism algebras for R-**Rel**) are members of $\mathcal{V}_B$, and their reducts to type τ_A are members of $\mathcal{V}_A$:

3.2. Definition and Properties. For any ring R and R-module M, let Rel(M) denote Su(M⊕M) regarded as a τ_B-algebra, and let $Rel_*(M)$ denote Su(M⊕M) as a τ_A-algebra. For R a ring with unit, let

$$\mathcal{A}(R) = \mathbf{S}\mathcal{M}_A(R) = \mathbf{S}\{Rel_*(M)\colon M \text{ an R-module}\},$$

$$\mathcal{B}(R) = \mathbf{S}\mathcal{M}_B(R) = \mathbf{S}\{Rel(M)\colon M \text{ an R-module}\}.$$

We say that a τ_A-algebra A is *representable by an* R-*module* if it is in $\mathcal{A}(R)$; or equivalently if there exists a τ_A-monomorphism from A into some $Rel_*(M)$. The same terminology is used for members of $\mathcal{B}(R)$.

3.2a. Rel(M) is an additive relation algebra with unit, and $Rel_*(M)$ is an additive relation algebra. (Proof by direct verification of the defining properties is omitted.)

3.2b. For any ring R with unit, $\mathcal{A}(R) \subseteq \mathcal{V}_A$ and $\mathcal{B}(R) \subseteq \mathcal{V}_B$.

We will show in the next section that $\mathcal{A}(R)$ and $\mathcal{B}(R)$ are quasivarieties of algebras of types τ_A and τ_B, respectively.

Throughout the remainder of §3, we will assume that A is an arbitrary additive relation algebra, and that B is an arbitrary additive relation algebra with unit.

3.3. Proposition. If $p(x_1, x_2, \ldots, x_n)$ is any τ_A-polynomial and $f_i \le g_i$ in A for $i = 1,2,\ldots,n$, then $p(f_1, f_2, \ldots, f_n) \le p(g_1, g_2, \ldots, g_n)$. Similarly, τ_B-polynomials are monotonic in all arguments on B.

The proof is by induction on τ_A-polynomial or τ_B-polynomial length, using 3.1a,b,c,d,e.

3.4. Definitions and Properties. An element f of A (or of B) is called *symmetric* if $f = f^{\#}$, is called an *idempotent* if $f = ff$, and is called *null* if $fgf = f$ for all g.

3.4a. For all f in A, $ff^{\#}$ and $f^{\#}f$ are symmetric idempotents (3.1d). Also, e in A is a symmetric idempotent iff $e = ee^{\#}$ iff $e = e^{\#}e$ iff $e^{\#} = ee^{\#}$ iff $e^{\#} = e^{\#}e$.

3.4b. If z is null, then fz, zg and fzg are null for all f and g in A. If any of the four maps z, $z^{\#}$, $zz^{\#}$ and $z^{\#}z$ is null, then all four are null (3.1d).

3.4c. If y and z are null, then $yfz = yz$ for all f ($yfz = yfzyz = yz$). In particular, a null element is an idempotent ($z = zz^{\#}z = zz$).

3.4d. If z is null, then $z = -z = z + z$. (Clearly $-z = -(zz^{\#}z) = z(-z^{\#})z = z$ using 3.1e. So, $z + z + z = z + (-z) + z = z$. Then

$$z = z(z + z + z) \le zz + z(z + z) = z + z(zz + zz) \le z + z(z + z)z = z + z,$$

using 3.1h, 3.3 and 3.4c, so $z \le z + z \le z + z + z = z$.)

3.4e. If y and z are null, then $y \wedge z$, $y \vee z$ and $y + z$ are null. (We have $y \wedge z \le (y \wedge z)y^{\#}(y \wedge z) \le yy^{\#}y \wedge zy^{\#}z = y \wedge z$, so $y \wedge z$ is null using 3.4b. Similarly, $y \vee z$ is null. By 3.4d and 3.1b,e,f, $y + z = y + (-y) + z + (-z) = (y + z) + -(y + z)$ is null.)

3.4f. If $y \le f$ for y symmetric and null, then $yf + y = yf$. If $f \le z$ for z symmetric and null, then $fz + z = fz$. (For $h = yf + y$, we have $h = yf + yy \le yf + yf = yf$ since yf is null (3.4b,d). Also,

$$yf = yf(yf)^{\#}yf \le yfh^{\#}h \le yfh^{\#}yf + yfh^{\#}y = yf + y = h,$$

using 3.1d,f,h, since $yf = yf + (-yf)$. The second part is similar.)

3.4g. If $f \le g$, $ff^{\#} \ge gg^{\#}$ and $f^{\#}f \ge g^{\#}g$, then $f = g$ ($g = gg^{\#}g \le ff^{\#}g \le$

$fg^{\#}g \le ff^{\#}f = f$ by 3.1d and 3.3).

3.4h. Suppose e is in Rel(M) for some R-module M. Then e is a symmetric idempotent iff there exist $C \le B$ in Su(M) such that:

$$e = \{\langle a,b\rangle : a,b \in B,\ a-b \in C\}.$$

Furthermore, z in Rel(M) is null iff there exist B and C in Su(M) such that $z = B \oplus C$. Also, z is symmetric and null iff $z = B \oplus B$ for some B in Su(M). (Proof omitted.)

The theory of additive relation algebras has two simple duality principles, which we describe next.

3.5. Definition and Properties. Let A^{con}, called the *converse dual* of A, denote the τ_A-algebra with the same elements and τ_A-structure as A, except that for all u and v, uv in A^{con} equals vu in A and $u+v$ in A^{con} equals the *converse sum* $(u^{\#}+v^{\#})^{\#}$ in A. For B^{con}, the τ_A-operations are the same as for A^{con}, and the constants 1, 0, **O** and **I** of B^{con} are the elements 1, $0^{\#}$, **O** and **I** of B, respectively.

Let A^{ord}, the *order dual* of A, denote the τ_A-algebra with the same elements as A and τ_A-structure obtained from A by exchanging meet and join (lattice duality), replacing sum by converse sum, and keeping multiplication, converse, and negation as in A. For B^{ord}, the τ_A-operations are the same as for A^{ord}, and the constants 1, 0, **O** and **I** of B^{ord} are respectively equal to the elements 1, $0^{\#}$, and (exchanging) **I** and **O**, in B.

Let A^* denote $(A^{ord})^{con}$, the *order-converse dual* of A, which equals $(A^{con})^{ord}$. In A^*, lattice operations are exchanged, multiplication is reversed, and converse, sum and negation are the same as in A. For $B^* = (B^{ord})^{con} = (B^{con})^{ord}$, 1 and 0 are the same as in B, and **O** and **I** are exchanged.

3.5a. For C in $\mathcal{V}_A$ or $\mathcal{V}_B$, $C = (C^{con})^{con} = (C^{ord})^{ord} = (C^*)^*$.

3.5b. For C in $\mathcal{V}_A$ (respectively, $\mathcal{V}_B$), $u \longmapsto u^{\#}$ determines reciprocal τ_A-isomorphisms (respectively, τ_B-isomorphisms) $C^{con} \longrightarrow C$ and $C \longrightarrow C^{con}$.

3.5c. If C is in $\mathcal{V}_A$ (respectively, $\mathcal{V}_B$), then C^{con}, C^{ord} and C^* are in $\mathcal{V}_A$ (respectively, $\mathcal{V}_B$). (For C^{con}, use 3.5b. Verify 3.1a-h or 3.1a-i for C^* directly, and then use $C^{ord} = (C^*)^{con}$ by 3.5a.)

Symmetric null elements characterize singleton τ_A-subalgebras.

3.6. Proposition. An element z of A is symmetric and null iff $\{z\}$ is a τ_A-subalgebra of A.

Proof: If $\{z\}$ is a τ_A-subalgebra, then $z = z^\# = z + (-z)$, so z is symmetric and null (3.1f). Suppose z is symmetric and null. We have $z = z^\# = z \wedge z = z \vee z$, and $z = zz$ by 3.4c, and $z = -z = z + z$ by 3.4d. Therefore, $\{z\}$ is a τ_A-subalgebra of A. ■

3.7. Corollary. Suppose $p(x_1, x_2, \ldots, x_n)$ is a τ_A-polynomial, and z is symmetric and null in A. If $f_i \le z \le g_i$ for $i = 1, 2, \ldots, n$, then $p(f_1, f_2, \ldots, f_n) \le z \le p(g_1, g_2, \ldots, g_n)$.

Proof: By 3.3 and 3.6. ■

From 3.7, we see that the intervals $\{f: y \le f \le z\}$ are τ_A-subalgebras of A, for symmetric null y and z in A with $y \le z$. We next give a technical result, followed by introduction of some convenient notation, to prepare for the analysis of such intervals.

3.8. Proposition. If y is a symmetric null element of A such that $y \le f \wedge g$, then $y(f + g) = yf \vee yg$. If z is a symmetric null element of A such that $z \ge f \vee g$, then $(f + g)z = fz \wedge gz$.

Proof: Assume that y is symmetric null and $y \le f \wedge g$, and let h denote $yf \vee yg$. Using 3.1d,h, 3.4b,c,d, 3.3 and 3.7, we have

$$y(f + g) \le yf + yg = yyf + yyg \le yh + yh = yh \le hh^\# h = h.$$

Now $y(f + g) = y(ff^\# f + g) \ge y(yf + y) = yyf = yf$, using 3.7 and 3.4c,e. Since $y(f + g) \ge yg$ similarly, we have $h = yf \vee yg \le y(f + g) \le h$. The second result is dual. ■

Recall from §1 that f in Rel(M) corresponds to an isomorphism

$\bar{f}: C_1/C_0 \longrightarrow D_1/D_0$ for appropriate $C_0 \le C_1$ and $D_0 \le D_1$ in Su(M). We can define symmetric null elements corresponding to C_1, C_0, D_1 and D_0 in any additive relation algebra.

3.9. Definitions and Properties. For f in A, define:

$$\mathbf{p}(f) = (f+(-f))^{\#}(f+(-f)) \text{ and } \mathbf{q}(f) = (f+(-f))(f+(-f))^{\#}.$$

3.9a. For all f in A, $\mathbf{p}(f)$ and $\mathbf{q}(f)$ are null symmetric idempotents, by 3.1f and 3.4a,b,c. Also, $\mathbf{q}(f)\mathbf{p}(f) = f+(-f)$, so $f+\mathbf{q}(f)\mathbf{p}(f) = f$.

3.9b. $\mathbf{p}(f) \le (f+g)^{\#}(f+g)$ and $\mathbf{q}(f) \ge (f+g)(f+g)^{\#}$ (3.1f). So, $\mathbf{p}(f) \le f^{\#}f$ and $\mathbf{q}(f) \ge ff^{\#}$, taking $g = (-f)+f$ and using 3.1e. Therefore, $\mathbf{p}(f^{\#}) \le \mathbf{q}(f)$ and $\mathbf{p}(f) \le \mathbf{q}(f^{\#})$.

3.9c. If $f \le g$, then $\mathbf{p}(f) \le \mathbf{p}(g)$ and $\mathbf{q}(f) \le \mathbf{q}(g)$ (3.3).

3.9d. $\mathbf{p}(fg) \ge \mathbf{p}(g)$ and $\mathbf{q}(fg) \le \mathbf{q}(f)$. In particular, $\mathbf{p}(f) = \mathbf{p}(f^{\#}f)$ and $\mathbf{q}(f) = \mathbf{q}(ff^{\#})$. (For $h = g+(-g)$, we have

$$\mathbf{p}(fg) = (fg+(-fg))^{\#}(fg+(-fg)) \ge (fh)^{\#}fh = h^{\#}f^{\#}fh = h^{\#}h = \mathbf{p}(g),$$

using 3.1d,e,f,h and 3.4c. The second part is dual, and the remaining parts follow from $f = ff^{\#}f$.)

3.9e. If z is null, $\mathbf{p}(z) = z^{\#}z$ and $\mathbf{q}(z) = zz^{\#}$ (3.4d). In particular, $\mathbf{p}(\mathbf{p}(f)) = \mathbf{q}(\mathbf{p}(f)) = \mathbf{p}(f)$ and $\mathbf{p}(\mathbf{q}(f)) = \mathbf{q}(\mathbf{q}(f)) = \mathbf{q}(f)$ for all f (3.9a).

3.9f. Let e be a symmetric idempotent of A. Then $\mathbf{p}(e) \le e \le \mathbf{q}(e)$, $\mathbf{p}(e) = e\mathbf{p}(e) = \mathbf{p}(e)e$ and $\mathbf{q}(e) = e\mathbf{q}(e) = \mathbf{q}(e)e$. (We have $\mathbf{p}(e) \le e \le \mathbf{q}(e)$ by 3.9b, and $\mathbf{p}(e) = \mathbf{p}(e)(e+(-e)) = \mathbf{p}(e)e+\mathrm{p}(e)(-e) = \mathbf{p}(e)e$ using 3.4c,d, 3.9b, 3.8 and 3.1e. Then $\mathbf{p}(e) = e\mathbf{p}(e)$ by taking converses, and the other parts are dual.)

3.9g. Suppose $f \in \mathrm{Rel}_*(M)$ for some R-module M. Then

$$\mathbf{q}(f) = C_1 \oplus C_1 \text{ for } C_1 = \{v \in M\colon (\exists w)\ \langle v,w\rangle \in f\},$$

$$\mathbf{p}(f^{\#}) = C_0 \oplus C_0 \text{ for } C_0 = \{v \in M\colon \langle v,0\rangle \in f\},$$

$$\mathbf{q}(f^{\#}) = D_1 \oplus D_1 \text{ for } D_1 = \{v \in M\colon (\exists u)\ \langle u,v\rangle \in f\}, \text{ and}$$

$$\mathbf{p}(f) = D_0 \oplus D_0 \text{ for } D_0 = \{v \in M\colon \langle 0,v\rangle \in f\}.$$

(Proof omitted.)

3.10. Definition and Properties. Suppose y and z are symmetric null elements of A with $y \le z$. Then $[y,z]$ denotes $\{f\colon y \le f \le z\}$, which is called an *interval subalgebra* of A.

3.10a. $[y,z]$ is a τ_A-subalgebra of A (3.7).

3.10b. If $f \le g$ in $[y,z]$ such that $yf \ge yg$ and $fz \ge gz$, then $f = g$. (We have $g = fz \wedge g \le ff^{\#}g \le fg^{\#}g \le yg \vee f = f$, using 3.7, 3.1g, etc.)

3.10c. If $f \in [y,z]$, then $yf = y\mathbf{p}(f)$ and $fz = \mathbf{q}(f)z$. (We have $y\mathbf{p}(f) = y(f + (-f)) = yf + (-y)f = yf$ by 3.4c,d, 3.1e and 3.8, and the second part is similar.)

3.10d. If $f \in [y,z]$, then $f + (-f) = \mathbf{q}(f)\mathbf{p}(f) = fzyf$ (3.4c, 3.10c). So, $f = f + zy = f + fzy = f + zyf = f + fzyf$. (We have $f = f + fzyf$ by 3.1e, then apply 3.10b to $f + fzy \le f + zyf$ using 3.8, and finally note that $f + fzy \le f + zy \le f + zyf$ and $f + fzy \le f + fzyf \le f + zyf$.)

3.10e. If e is a symmetric idempotent of A and $f \in [y,z]$, then $ef = ey \vee (f \wedge ez)$ and $fe = ye \vee (f \wedge ze)$. (By lattice modularity,

$$ef \le (ey \vee f) \wedge ez = ey \vee (f \wedge ez) \le ef,$$

using 3.1g. The second part is obtained by taking converses.)

3.10f. Let e be a symmetric idempotent. Then $f = ef = fe$ for all f in $[y,z]$ iff $\mathbf{p}(e) \le y \le z \le \mathbf{q}(e)$. (If $y = eye$, then $\mathbf{p}(e) = \mathbf{p}(e)y\mathbf{p}(e) \le eye = y$, and similarly $z \le \mathbf{q}(e)$. If $f \le z \le \mathbf{q}(e)$, then $f = e\mathbf{q}(e) \wedge f \le ee^{\#}f = ef$ by 3.1g and 3.9f, and similarly $ef \le f$ if $f \ge \mathbf{p}(e)$.)

3.10g. If $f_1, f_2, \ldots, f_n$ are in A, then there exist symmetric null elements y and z of A such that $f_i \in [y,z]$ for $i = 1,2,\ldots,n$. (Let $y = \mathbf{p}(h \wedge h^{\#})$ for $h = f_1 \wedge f_2 \wedge \ldots \wedge f_n$. For $i \le n$, $y = yhy \le hh^{\#}hh^{\#}h = h \le f_i$. Let $z = \mathbf{q}(k \vee k^{\#})$ for $k = f_1 \vee f_2 \vee \ldots \vee f_n$, so $f_i \le z$ for $i \le n$ similarly.)

3.10h. If d and e are symmetric idempotents such that $ed = d$, then $d = (d \wedge e)d = d(d \wedge e) = (d \vee e)d = d(d \vee e)$. (Choose $[y,z]$ containing d and e by 3.10g, so $(d \wedge e)d = yd \vee (d \wedge e \wedge zd) = (yd \vee e) \wedge d = d$ by 3.10e and modularity, since $yd \vee e \ge ed^{\#}d = d$ by 3.1g. The remaining equations are obtained dually.)

3.10i. If d and e are symmetric idempotents, then de = d iff ed = d iff $\mathbf{p}(e) \le d \le \mathbf{q}(e)$ iff $\mathbf{p}(e) \le \mathbf{p}(d) \le \mathbf{q}(d) \le \mathbf{q}(e)$. (Note de = d iff ed = d by converses, then use 3.9d,f and 3.10f.)

3.10j. If y and z are symmetric, then $y \wedge z$ and $y \vee z$ are symmetric. If y and z are null, then $y \wedge z$, $y \vee z$ and $y + z$ are null. (The first part is by 3.1d. Suppose y,z are in the interval subalgebra [x,w] by 3.10g. Then $y \wedge z \le (y \wedge z)w(y \wedge z) \le ywy \wedge zwz = y \wedge z$ implies $y \wedge z$ is null by 3.4b. Dually, $y + z$ is null, and $y + z \le (y + z)w(y + z) \le (yw \wedge zw)(y + z) \le ywy + zwz = y + z$ by 3.8 and 3.1h, proving that $y + z$ is null.)

By 3.10g, any finitely-generated τ_A-subalgebra of A is contained in some interval subalgebra [y,z]. Now, a symmetric idempotent in A determines an interval subalgebra which is an additive relation algebra *with unit*.

3.11. Proposition. Suppose y and z are symmetric null elements of A with $y \le z$. Then the following are equivalent:

3.11a. There exist unique elements 1, 0, **O** and **I** of [y,z] such that [y,z] is an additive relation algebra with unit (a member of $\mathcal{V}_B$).

3.11b. There exists an element u of [y,z] such that f = uf = fu for all f in [y,z].

3.11c. There exists a symmetric idempotent d of A such that $y = \mathbf{p}(d)$ and $z = \mathbf{q}(d)$.

3.11d. There exists a symmetric idempotent e of A such that $y \ge \mathbf{p}(e)$ and $z \le \mathbf{q}(e)$.

Proof: Obviously 3.11a ⇒ 3.11b and 3.11c ⇒ 3.11d. Assuming 3.11b, we see that $u^\# = u^\# u$, so u is a symmetric idempotent (3.4a) and $\mathbf{p}(u) \le y \le z \le \mathbf{q}(u)$ by 3.10f. But $\mathbf{p}(u)$ and $\mathbf{q}(u)$ are in [y,z], so we have 3.11b ⇒ 3.11c.

Assume 3.11d, and note that f = ef = fe for f in [y,z] by 3.10f. Let $d = y \vee e$, so $d = d^\#$ and $d \le dd \le ddd = d$. Then d is a symmetric idempotent, and $\mathbf{q}(d) = \mathbf{q}(e)$ by 3.7 and 3.9c. Also,

$$y \le yd = (y\mathbf{p}(e) \vee y \vee e) \wedge y\mathbf{q}(e) = y \vee (e \wedge y\mathbf{q}(e)) \le y \vee ye = y$$

by 3.10e, modularity and 3.1g. So, $y = \mathbf{p}(y) = \mathbf{p}(yd) \geq \mathbf{p}(d) \geq \mathbf{p}(y)$, proving $\mathbf{p}(d) = y$. Then $f = fd = df$ for f in $[y,z]$ by 3.10f. A similar argument shows that $c = d \wedge z$ is a symmetric idempotent in $[y,z]$ such that $\mathbf{p}(c) = y$ and $\mathbf{q}(c) = z$. Now define $1 = c$, $0 = c + (-c) = \mathbf{q}(c)\mathbf{p}(c)$, $\mathbf{O} = \mathbf{p}(c)$ and $\mathbf{I} = \mathbf{q}(c)$. All the axioms of 3.1i are satisfied for $[y,z]$, by the above and 3.10d,f. So, $[y,z]$ is an additive relation algebra with unit. The uniqueness follows from the uniqueness of a multiplicative unit for $[y,z]$ and 3.1i. ∎

3.12. Corollary. An additive relation algebra A is the reduct to τ_A of an additive relation algebra with unit (a member of $\mathcal{V}_B$) iff it has a multiplicative unit u.

Proof: The forward implication follows from 3.1i. A multiplicative unit u is a symmetric idempotent ($u^\# = u^\# u$), so $A = [\mathbf{p}(u),\mathbf{q}(u)]$ by 3.10f,g. Then A is in $\mathcal{V}_B$ for uniquely determined 1, 0, $\mathbf{O}$ and $\mathbf{I}$ by 3.11. ∎

Given the multiplicative unit, we show next some of the additional elementary properties that can be obtained for B in $\mathcal{V}_B$.

3.13. Properties of Additive Relation Algebras with Unit.

3.13a. In B, $\mathbf{O}$ and $\mathbf{I}$ are symmetric and null (3.10g), and 1 is a symmetric idempotent ($1^\# = 1^\# 1$).

3.13b. For f in B, $f + (-f) = f\mathbf{I}\mathbf{O}f$ (3.10d), so $\mathbf{p}(f) = f^\#\mathbf{O}f$ and $\mathbf{q}(f) = f\mathbf{I}f^\#$ (3.13a).

3.13c. For f in B, $\mathbf{O}f^\# f = \mathbf{O}f = \mathbf{O}\mathbf{I} \wedge f$, $ff^\#\mathbf{O} = f\mathbf{O} = \mathbf{I}\mathbf{O} \wedge f$, $\mathbf{I}f^\# f = \mathbf{I}f = \mathbf{I}\mathbf{O} \vee f$ and $ff^\#\mathbf{I} = f\mathbf{I} = \mathbf{O}\mathbf{I} \vee f$ ($\mathbf{O}f \leq \mathbf{O}f^\# f \leq \mathbf{O}ff^\# f = \mathbf{O}f$ and $\mathbf{O}\mathbf{I} \wedge f \leq \mathbf{O}\mathbf{O}^\# f \leq \mathbf{O}f^\# f \leq \mathbf{O}\mathbf{I} \wedge ff^\# f = \mathbf{O}\mathbf{I} \wedge f$ by 3.1d,g and 3.3, etc.).

3.13d. The elements $\{\mathbf{O},\mathbf{O}\mathbf{I},\mathbf{I}\mathbf{O},1,\mathbf{I}\}$ form a $(\mathbf{O},\mathbf{I})$ sublattice of B with five elements and length two, unless B is trivial. (We have $\mathbf{O}\mathbf{I} \wedge 1 = \mathbf{O}1 = \mathbf{O} = 1\mathbf{O} = \mathbf{I}\mathbf{O} \wedge 1$ and $\mathbf{O}\mathbf{I} \wedge \mathbf{I}\mathbf{O} = \mathbf{O}\mathbf{I}\mathbf{O} = \mathbf{O}$. Dually, $\mathbf{O}\mathbf{I} \vee 1 = \mathbf{I}\mathbf{O} \vee 1 = \mathbf{O}\mathbf{I} \vee \mathbf{I}\mathbf{O} = \mathbf{I}$.)

3.13e. For f,g in B, $\mathbf{O}(f+g) = \mathbf{O}f \vee \mathbf{O}g$ and $(f+g)\mathbf{I} = f\mathbf{I} \wedge g\mathbf{I}$ (3.8).

3.13f. For f in B, $(-1)f = f(-1)$. Also, $\mathbf{O} = \mathbf{O}(-1) = (-1)\mathbf{O}$, $\mathbf{I} = \mathbf{I}(-1) = (-1)\mathbf{I}$, $(-1)(-1) = 1$ and $(-1)^\# = -1$. (The first part is by 3.1e,f,i. Also,

$\mathbf{0} = \mathbf{0I0} = \mathbf{0}(1+(-1)) = \mathbf{0}1+\mathbf{0}(-1)$, so $\mathbf{0}(-1) = \mathbf{0}$, etc. Finally, $(-1)(-1) = 1$ by 3.1e,f, and $(-1)^{\#} = -1$ by 3.1e.)

3.13g. If $f \le g$, $\mathbf{0}f \ge \mathbf{0}g$ and $f\mathbf{I} \ge g\mathbf{I}$ in B, then $f = g$ (3.10b).

3.13h. For e,f in B with e a symmetric idempotent, $ef = (e\mathbf{0} \vee f) \wedge e\mathbf{I}$ and $fe = (\mathbf{0}e \vee f) \wedge \mathbf{I}e$ (3.10e).

3.13i. If e is a symmetric idempotent, then $e = e(e \wedge 1) = (e \wedge 1)e = e(e \vee 1) = (e \vee 1)e$ (3.10h). Also, $1 \wedge e = 1 \wedge e\mathbf{I} = 1 \wedge \mathbf{I}e$ and $1 \vee e = 1 \vee e\mathbf{0} = 1 \vee \mathbf{0}e$ ($e\mathbf{I} \wedge 1 \le e$ by 3.1g, etc.). Finally, $\mathbf{0}(1 \vee e) = \mathbf{0}e$ and $(1 \wedge e)\mathbf{I} = e\mathbf{I}$ ($\mathbf{0}e \le \mathbf{0}(1 \vee e) \le \mathbf{0}e(1 \vee e) = \mathbf{0}e$ and dually).

3.13j. Suppose c,d,f are in B with $c \le 1 \le d$. Then c and d are symmetric idempotents, $cf = c\mathbf{I} \wedge f$, $fc = \mathbf{I}c \wedge f$, $df = d\mathbf{0} \vee f$ and $fd = \mathbf{0}d \vee f$. If $b,c \le 1$, then $bc = cb = b \wedge c$. If $d,e \ge 1$, then $de = ed = d \vee e$. (Note that $c = cc^{\#}c \ge cc^{\#}1 \ge c1^{\#}1 = c$, so $c = cc^{\#}$, and dually. Since $c\mathbf{0} \le \mathbf{0}$, etc., 3.13h leads to the next four equations. Finally, $b \wedge c = (b \wedge c)(b \wedge c) \le bc \le b \wedge c$, and dually.)

3.13k. The set S of symmetric null elements of B is a sublattice of B which is lattice isomorphic to each of the interval sublattices $[\mathbf{0},\mathbf{0I}]$, $[\mathbf{0},\mathbf{I0}]$, $[\mathbf{0},1]$, $[\mathbf{0I},\mathbf{I}]$, $[\mathbf{I0},\mathbf{I}]$ and $[1,\mathbf{I}]$ of B. (By 3.1d and 3.4e, S is a sublattice of B, and by 3.13c, $y \mapsto \mathbf{0}y$ is a lattice isomorphism from S into $[\mathbf{0},\mathbf{0I}]$ with reciprocal $z \mapsto z^{\#}z$. The six intervals above are projective intervals of B as a modular lattice by 3.13d, and so are lattice isomorphic by composites of transpose isomorphisms such as $z \mapsto 1 \vee z$ from $[\mathbf{0},\mathbf{0I}]$ into $[1,\mathbf{I}]$ and its reciprocal $e \mapsto \mathbf{0I} \wedge e = \mathbf{0}e$.)

We now return to consideration of additive relation algebras without unit. It is possible to make such an algebra, consisting only of null elements, from any modular lattice.

3.14. Definition and Properties. Suppose L is a modular lattice, and L^2 denotes the τ_A-algebra on $L \times L$ defined by:

$$\langle y,z\rangle + \langle w,x\rangle = \langle y \wedge w,\ z \vee x\rangle,$$

$-\langle y,z\rangle = \langle y,z\rangle$,

$\langle y,z\rangle\langle w,x\rangle = \langle y,x\rangle$,

$\langle y,z\rangle^{\#} = \langle z,y\rangle$,

$\langle y,z\rangle \wedge \langle w,x\rangle = \langle y \wedge w,\ z \wedge x\rangle$, and

$\langle y,z\rangle \vee \langle w,x\rangle = \langle y \vee w,\ z \vee x\rangle$.

3.14a. L^2 is an additive relation algebra such that every element of L^2 is null. An element $\langle y,z\rangle$ of L^2 is symmetric iff $y = z$. (Proof by direct calculation.)

3.15. Proposition. The subset $N = \{z\colon z \text{ is null}\}$ of A is a τ_A-subalgebra of A. The subset $S = \{y\colon y \text{ is symmetric and null}\}$ of A is a (modular) sublattice of A, and $\kappa\colon S^2 \longrightarrow N$ such that $\kappa(y,z) = yz$ is a τ_A-isomorphism.

Proof: Suppose y and z are null. By 3.4b,d,e, $-y$, $y^{\#}$, yz, $y \wedge z$, $y \vee z$ and $y \vee z$ are null. Therefore, N is a τ_A-subalgebra of A.

For y and z symmetric and null, $y \wedge z$ and $y \vee z$ are symmetric by 3.1d and are null by the above. So, S is a sublattice of A.

Defining $\lambda\colon N \longrightarrow S^2$ by $\lambda(z) = \langle zz^{\#}, z^{\#}z\rangle$, it is easily checked that κ and λ are reciprocal bijections which preserve order. Therefore, κ and λ are lattice isomorphisms. We observe that κ preserves negation by 3.4d, preserves products by 3.4c, and preserves converses by 3.1d.

To prove κ preserves sums, it suffices to show that $wx + yz = (w \wedge y)(z \vee x)$ for symmetric null w, x, y and z. By 3.10g, choose an interval subalgebra $[s,t]$ containing w, x, y and z. For $u = (w \wedge y)(x + z)$ and $v = (w + y)(x \vee z)$, we have

$$u \le wx + yz \le v \text{ and } u \le (w \wedge y)(x \vee z) \le v,$$

using 3.1h, 3.3 and 3.4d. Now $su = sx \vee sz \le s(x \vee z) = sv$ and $ut = (w \wedge y)t \le wt \wedge yt = vt$ by 3.8 and 3.4c. Applying 3.4c,g, we obtain $su = sv$ and $ut = vt$, hence $u = v$ by 3.10b. This proves that κ is a τ_A-isomorphism. ∎

By 3.15, we see that for symmetric null elements $y < z$ in A, there may not exist a symmetric idempotent e such that $y = \mathbf{p}(e)$ and $z = \mathbf{q}(e)$

(compare 3.11c). If A is in $\mathcal{V}_B$, however, such an e always exists by 3.11.

We now indicate the method by which category structures can be recovered from an additive relation algebra A. The objects of the category are the symmetric idempotents of A. Two kinds of morphisms are considered, one corresponding to additive relation categories and the other to abelian categories. This method is closely related to category constructions given in [3I] and by R. Vescan in [3J].

3.16. Definitions and Properties. For any symmetric idempotents c and d of A, let rel(c,d) denote the set

$$\{f \in A: cf = f = fd\},$$

and let hom(c,d) denote the subset

$$\{f \in A: cf = f = fd,\ ff^{\#} \geq c,\ f^{\#}f \leq d\}.$$

3.16a. f is in rel(c,d) iff $\mathbf{p}(c)\mathbf{p}(d) \leq f \leq \mathbf{q}(c)\mathbf{q}(d)$ iff:

$$\mathbf{p}(c) \leq \mathbf{p}(ff^{\#}) \leq \mathbf{q}(ff^{\#}) \leq \mathbf{q}(c) \text{ and } \mathbf{p}(d) \leq \mathbf{p}(f^{\#}f) \leq \mathbf{q}(f^{\#}f) \leq \mathbf{q}(d).$$

(Use 3.10i, plus $\mathbf{p}(c)\mathbf{p}(d) = \mathbf{p}(c)f\mathbf{p}(d) \leq ff^{\#}ff^{\#}f = f$ by 3.4c, etc.)

3.16b. f is in hom(c,d) iff: $\mathbf{p}(c) \leq \mathbf{p}(ff^{\#}) \leq \mathbf{q}(ff^{\#}) = \mathbf{q}(c)$ and $\mathbf{p}(d) = \mathbf{p}(f^{\#}f) \leq \mathbf{q}(f^{\#}f) \leq \mathbf{p}(d)$ (3.16a, 3.9c).

3.16c. For any symmetric idempotents c,d and e, $f \in$ rel(c,d) and $g \in$ rel(d,e) implies $fg \in$ rel(c,e), and $fg \in$ hom(c,e) when $f \in$ hom(c,d) and $g \in$ hom(d,e). Also, d in hom(d,d) is like a category unit for the object d: fd = f for f in rel(c,d) and dg = g for g in rel(d,e).

3.16d. There are no proper inclusions in hom sets: if $f \leq g$ in hom(c,d), then f = g (3.4g).

3.16e. For any symmetric idempotent d, rel(d,d) is an additive relation algebra with unit, where the constants 1, 0, **O** and **I** are d, $\mathbf{q}(d)\mathbf{p}(d)$, $\mathbf{p}(d)$ and $\mathbf{q}(d)$, respectively.

3.16f. For any symmetric idempotent d, hom(d,d) is closed for sum, product and negation, and under these operations it is a ring with unit d and zero $\mathbf{q}(d)\mathbf{p}(d)$.

3.16g. For any symmetric idempotents c and d of A, rel(c,d) is the interval sublattice $[\mathbf{p}(c)\mathbf{p}(d), \mathbf{q}(c)\mathbf{q}(d)]$ of A, and $f^{\#}$ is in rel(d,c) for each f in rel(c,d). Also, rel(c,d) is closed for sum and negation, and has the zero $\mathbf{q}(c)\mathbf{p}(d)$ for sum.

3.16h. For any symmetric idempotents c and d of A, hom(c,d) is an abelian group with zero $\mathbf{q}(c)\mathbf{p}(d)$. It is a left-R, right-S bimodule for R = hom(c,c) and S = hom(d,d). More generally, $(f+g)h = fh+gh$ and $g(h+k) = gh+gk$ for f,g in hom(c,d) and h,k in hom(d,e).

3.16i. Suppose c and d are symmetric idempotents in Rel(M) for some R-module M, and c and d correspond to subquotients $C = C_1/C_0$ and $D = D_1/D_0$ of M respectively, as in 3.4f. Then

$$\alpha_{cd}(f) = \{\langle a+C_0, b+D_0\rangle : \langle a,b\rangle \in f\}$$

defines a bijection α_{cd} from rel(c,d) onto $\mathrm{Su}(C \oplus D)$ such that for f,g in rel(c,d) we have:

$f \le g$ iff $\alpha_{cd}(f) \le \alpha_{cd}(g)$,

$\alpha_{cd}(f+g) = \alpha_{cd}(f)+\alpha_{cd}(g)$,

$\alpha_{cd}(-f) = -\alpha_{cd}(f)$,

$\alpha_{dc}(f^{\#}) = \alpha_{cd}(f)^{\#}$,

$\alpha_{cc}(1) = 1$, and

$\alpha_{ce}(gh) = \alpha_{cd}(g)\alpha_{de}(h)$ for h in rel(d,e).

In particular, α_{cd} is a $(\mathbf{0},\mathbf{I})$ lattice isomorphism preserving sum and negation between rel(c,d) and $\mathrm{Su}(C \oplus D)$, which induces by restriction of the domain and codomain an abelian group isomorphism between hom(c,d) and Hom(C,D) for R-**Mod**. Furthermore, α_{cc} is a τ_B-isomorphism between rel(c,c) and Rel(C), which induces by restriction of the domain and codomain a ring isomorphism preserving 1 between hom(c,c) and the ring of endomorphisms Hom(C,C).

From 3.16, we see how to recover from Rel(M) a category equivalent to the full subcategory of R-**Rel** (or of R-**Mod**) determined by the set of subquotients of M, and that an abstraction of this process can be used for an arbitrary additive relation algebra.