§4. Frames in Modular Lattices.

In the introduction, we asserted that universal Horn sentences could be translated from other external theories of modules into modular lattice theory. The techniques that are needed involve certain lattice configurations called *frames*, which were introduced by von Neumann in his development of continuous geometry [4A]. The use of lattice frames to construct rings from which modular lattice representations are obtained, as in [4A, Theorem 14.1, p. 208], is called *coordinatization*. Frames and coordinatization have been extensively used in recent years to obtain many results of modular lattice theory; for examples see [4B,4C,4D]. In this section, we describe some basic properties of frames and begin the contruction of the universal Horn sentence translation functions. First, consider the lattice $\mathbf{M}_3$, which is the fundamental building block used to construct a frame.

In the following, L will denote an arbitrary modular lattice.

4.1. Definitions and Properties. A quintuple $\langle a,x,y,z,b\rangle$ of elements of L *forms an* $\mathbf{M}_3$ *sublattice* if

$$a = x \wedge y = x \wedge z = y \wedge z \quad \text{and} \quad b = x \vee y = x \vee z = y \vee z.$$

That is, these elements generate the sublattice of L shown below:

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  b
 /|\
x y z
 \|/
  a
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We will also say that $\{x,y,z\}$ *generates* an $\mathbf{M}_3$, which *spans* the interval sublattice $[a,b] = \{w \in L\colon a \le w \le b\}$ of L, and *spans L* if $L = [a,b]$.

4.1a. An $\mathbf{M}_3$ sublattice is *simple*; proper homomorphic images of $\mathbf{M}_3$ are trivial.

4.1b. If $\{X,Y,Z\}$ generates an $\mathbf{M}_3$ spanning Su(M) for an R-module M, then there are R-linear isomorphisms $f\colon M \longrightarrow X \oplus Z$ and $g\colon X \longrightarrow Z$ such that f[Y] is the graph of g. Note the Noether isomorphisms X to M/Y to Z to M/X to Y to M/Z; g is the composite of X to M/Y to Z.

4.1c. If $\{X,Y,Z\}$ generates an $\mathbf{M}_3$ spanning $[U,V]$ in Su(M), then we can relativize to isomorphisms $f:V/U \longrightarrow X/U \oplus Z/U$ and $g:X/U \longrightarrow Z/U$, with $f[Y/U]$ the graph of g.

4.1d. If B is an additive relation algebra with unit, then $\{\mathbf{IO}, 1, \mathbf{OI}\}$ generates an $\mathbf{M}_3$ spanning B. (From 3.1g,i, $\mathbf{OI} \wedge 1 = \mathbf{OI} \wedge \mathbf{IO} = \mathbf{O}$, etc.) If e is an idempotent of an additive relation algebra A, then $\{\mathbf{q}(e)\mathbf{p}(e),\ e,\ \mathbf{p}(e)\mathbf{q}(e)\}$ generates an $\mathbf{M}_3$ spanning $[\mathbf{p}(e),\mathbf{q}(e)]$ (see 3.11).

Although x, y and z appear symmetrically in 4.1, it is often useful to regard an $\mathbf{M}_3$ sublattice as representing two disjoint isomorphic modules and an isomorphism graph, or even as elements **IO**, 1 and **OI** of an additive relation algebra with unit. For frames, this view is extended to n isomorphic modules, $n \geq 2$, which satisfy appropriate disjointness and isomorphism graph conditions.

4.2. Definitions. For $n \geq 2$, an n-*frame* in a modular lattice L is a system of elements a, b_i for $i \leq n$ and c_{ij} for $i \neq j$, $1 \leq i,j \leq n$, that either satisfies 4.2a,b,c below or is trivial.

4.2a. $(b_1 \vee b_2 \vee \ldots \vee b_i) \wedge b_{i+1} = a$ for $1 \leq i \leq n$.

4.2b. $\{b_i, c_{ij}, b_j\}$ generates an $\mathbf{M}_3$ spanning $[a,\ b_i \vee b_j]$ and $c_{ji} = c_{ij}$ for $1 \leq i < j \leq n$.

4.2c. $c_{ik} = (c_{ij} \vee c_{jk}) \wedge (b_i \vee b_k)$ for distinct i,j,k, $1 \leq i,j,k \leq n$.

(An n-frame is *trivial* if $a = b_i = c_{ij}$ for all $j \neq i$, $1 \leq i,j \leq n$.)

We say that an n-frame *spans* [a,b] for $b = b_1 \vee b_2 \vee \ldots \vee b_n$, since the sublattice of L generated by the frame elements contains a and b and is contained in the interval sublattice [a,b] of L. The elements $b_1, b_2, \ldots, b_n$ are called *independent* over a in L if they satisfy 4.2a.

4.3. Properties. A nontrivial 2-frame in L is essentially an $\mathbf{M}_3$ sublattice generated by $\{b_1, c_{12}, b_2\}$, since $a = b_1 \wedge b_2$ and $c_{21} = c_{12}$. For $n \geq 3$, the sublattice of L generated by an n-frame may be infinite.

4.3a. If M is an R-module and submodules $A = \mathbf{O}$, B_i and C_{ij} of M form an n-frame which spans Su(M), then there are R-linear isomorphisms

$$f:M \longrightarrow B_1 \oplus B_2 \oplus \ldots \oplus B_n$$

and $g_{ij}:B_i \longrightarrow B_j$ for all $j \neq i$, $1 \leq i,j \leq n$, such that $g_{ji} = g_{ij}^{-1}$ and $g_{ij}g_{jk} = g_{ik}$ for all distinct i, j and k, $1 \leq i,j,k \leq n$, and $f[C_{ij}]$ is the negative graph of g_{ij} regarded as a submodule of the n-fold direct sum; that is, $\langle v_1, v_2, \ldots, v_n \rangle$ is in $f[C_{ij}]$ iff $v_k = 0$ for all $k \leq n$ except $k = i$ and $k = j$, $v_i \in B_i$ and $v_j = -g_{ij}(v_i)$ in B_j.

4.3b. For M an R-module and $n \geq 3$, let $\kappa_i:M \longrightarrow M^{(n)}$ denote the i-th insertion map for $i \leq n$. Then there is a canonical n-frame spanning $Su(M^{(n)})$ given by $A = \mathbf{0}$, $B_i = \kappa_i[M]$ for $i \leq n$, and $C_{ij} = C_{ji} = (\kappa_i - \kappa_j)[M]$ for $1 \leq i < j \leq n$. If $M = R$ and $\{v_1, v_2, \ldots, v_n\}$ is a free generating set for $R^{(n)}$, then $B_i = Rv_i$ for $i \leq n$, and $C_{ij} = C_{ji} = R(v_i - v_j)$ for $1 \leq i < j \leq n$.

4.3c. For M an R-module and an n-frame A, B_i for $i \leq n$ and $C_{ij} = C_{ji}$ for $1 \leq i < j \leq n$ in Su(M) spanning [A,B] for $B = B_1 \oplus B_2 \oplus \ldots \oplus B_n$, there are R-linear isomorphisms

$$f:B/A \longrightarrow B_1/A \oplus B_2/A \oplus \ldots \oplus B_n/A$$

and $g_{ij}:B_i/A \longrightarrow B_j/A$ for all $j \neq i$, $1 \leq i,j \leq n$, such that $g_{ji} = g_{ij}^{-1}$ and $g_{ik} = g_{ij}g_{jk}$ for all distinct i, j, k between 1 and n. Here, $f[C_{ij}/A]$ is the negative graph of g_{ij} for $j \neq i$, $1 \leq i,j \leq n$.

The use of negative graphs is not necessary, but it simplifies certain formulas. Consider $C_{13} = (C_{12} \vee C_{23}) \wedge (B_1 \vee B_3)$, which corresponds to the composition formula $g_{13} = g_{12}g_{23}$, say for the canonical 4-frame of 4.3b. If negative graphs are used, then $C_{12} \vee C_{23}$ contains quadruples $\langle u, -u+v, -v, 0 \rangle$ for $u,v \in M$, and quadruples of the above form in $B_1 \vee B_3$ are just the negative graph quadruples $\langle u, 0, -u, 0 \rangle$ of C_{13}. If positive graphs are used for C_{12} and C_{23}, then $C_{12} \vee C_{23}$ contains the quadruples $\langle u, u+v, v, 0 \rangle$, again leading to the negative graph quadruples $\langle u, 0, -u, 0 \rangle$ in $B_1 \vee B_3$. The minus signs can be avoided by using $Con(M^{(4)})$ instead of $Su(M^{(4)})$, but this would require consideration of *octuples* of elements of M. We could also work with positive graphs at the *top* of the frame, using D_{12} and D_{23} consisting of quadruples $\langle u,u,w,y\rangle$ and $\langle z,v,v,x\rangle$ in $M^{(4)}$, so that $D_{12}\wedge D_{23}$ consists of elements $\langle u,u,u,y\rangle$ in $M^{(4)}$, and the join with $B_2\vee B_4$ contains quadruples $\langle u,t,u,y\rangle$ corresponding to D_{13}. This may be interpreted as a dual composition formula for D_{13} given by $(D_{12}\wedge D_{23})\vee(E_1\wedge E_3)$, where $E_1 = B_2\vee B_3\vee B_4$, $E_2 = B_1\vee B_3\vee B_4$ and $E_3 = B_1\vee B_2\vee B_4$, and D_{ij} is in $[E_i\wedge E_j, M^{(4)}]$ for $1\le i<j\le 3$. There is an obvious adaptation of this procedure to n-frames for other values of $n\ge 3$. Although these alternative formulations would also work, we will employ the usual negative graph representations hereafter.

There is another approach to n-frames using a configuration of $n+1$ symmetrical elements, called an n-*diamond*. (A set $D=\{d_1,d_2,\ldots,d_{n+1}\}$ which spans [a,b] in L is an n-diamond if any n-element subset of D is independent over a and has join b.) This terminology was introduced by Huhn [4G]; the simplified version above is discussed in Day [4H]. For our purposes, the formulation of 4.2 is more suitable.

The universal Horn sentence translation from additive relation algebras with unit to modular lattices is constructed by expressing τ_B-operations by lattice polynomials in n-frames, $n\ge 3$.

4.4. Definitions and Properties. Let a, b_i and c_{ij} for $j\ne i$, $1\le i,j\le n$, form a system H of elements of a τ_L-algebra K, $n\ge 3$. Relative to H, define τ_B-operations on K as below, for x and y in K:

$$x +_H y = ([(x\vee c_{13})\wedge(b_2\vee b_3)]\vee[(y\vee b_3)\wedge(b_2\vee c_{13})])\wedge(b_1\vee b_2),$$

$$-_H x = ([([(x\vee b_3)\wedge(b_1\vee c_{23})]\vee b_2)\wedge(b_1\vee b_3)]\vee c_{23})\wedge(b_1\vee b_2),$$

$$x\cdot_H y = ([(x\vee c_{23})\wedge(b_1\vee b_3)]\vee[(y\vee c_{13})\wedge(b_2\vee b_3)])\wedge(b_1\vee b_2),$$

$$x^{\#_H} = ([([(x\vee c_{13})\wedge(b_2\vee b_3)]\vee c_{12})\wedge(b_1\vee b_3)]\vee c_{23})\wedge(b_1\vee b_2),$$

$$x\vee_H y = x\vee y,\quad x\wedge_H y = x\wedge y,$$

$$0_H = b_1,\quad 1_H = c_{12},\quad \mathbf{0}_H = a \text{ and } \mathbf{I}_H = b_1\vee b_2.$$

Let K_H denote the τ_B-algebra on K provided with the above operations. Of course, K_H is not an additive relation algebra with unit.

If $h:K \longrightarrow L$ is a τ_L-homomorphism of τ_L-algebras, let h(H) denote the system of elements h(a), $h(b_i)$ and $h(c_{ij})$ for $j \neq i$, $1 \leq i,j \leq n$.

4.4a. If K is a lattice, then the interval sublattice $[\mathbf{0}_H, \mathbf{I}_H]$ is a τ_B-subalgebra of K_H.

4.4b. Suppose $h:K \longrightarrow L$ is a τ_L-homomorphism of τ_L-algebras. Then $h:K_H \longrightarrow L_{h(H)}$ is a τ_B-homomorphism. If K and L are lattices and H is an n-frame, then h(H) is an n-frame and h induces a τ_B-homomorphism $k:[\mathbf{0}_H, \mathbf{I}_H] \longrightarrow [\mathbf{0}_{h(H)}, \mathbf{I}_{h(H)}]$ by restriction of the domain and codomain.

For M an R-module and the canonical n-frame Z on $Su(M^{(n)})$ for $n \geq 3$ as in 4.3b, the interval τ_B-subalgebra $[\mathbf{0}_Z, \mathbf{I}_Z] = [A, B_1 \vee B_2]$ of 4.4a is essentially just Rel(M). Using negative graphs, let:

$$\hat{\mu}(f) = \{\kappa_1(a) - \kappa_2(b) : \langle a,b \rangle \in f\},$$

which determines a τ_B-monomorphism $\hat{\mu}:Rel(M) \longrightarrow Su(M^{(n)})$ with image $[\mathbf{0}_Z, \mathbf{I}_Z]$. For example, if f and g are in Rel(M) we have:

$$D_1 = (\hat{\mu}(f) \vee C_{13}) \wedge (B_2 \vee B_3) =$$
$$= \{\langle u+w, -v, -w, 0, \dots, 0 \rangle : \langle u,v \rangle \in f,\ u+w = 0\}$$
$$= \{\langle 0, -v, u, 0, \dots, 0 \rangle : \langle u,v \rangle \in f\}, \text{ and}$$

$$D_2 = (\hat{\mu}(g) \vee B_3) \wedge (B_2 \vee C_{13}) =$$
$$= \{\langle w, -x, -w, 0, \dots, 0 \rangle : \langle w,x \rangle \in g\}, \text{ so}$$

$$\hat{\mu}(f) +_Z \hat{\mu}(g) = (D_1 \vee D_2) \wedge (B_1 \vee B_2)$$
$$= \{\langle w, -v-x, u-w, 0, \dots, 0 \rangle : \langle u,v \rangle \in f,\ \langle w,x \rangle \in g,\ u-w = 0\}$$
$$= \{\langle u, -(v+x), 0, \dots, 0 \rangle : \langle u,v \rangle \in f,\ \langle u,x \rangle \in g\} = \hat{\mu}(f+g).$$

Similar arguments show that $\hat{\mu}$ preserves all the other τ_B-operations, and it is clear that $\hat{\mu}$ is one-one and has image $[\mathbf{0}_Z, \mathbf{I}_Z]$.

Using 4.3c, calculations of the above sort can be adapted to an arbitrary n-frame in Su(M), $n \geq 3$. We will omit the routine proof of the following general result below.

4.5. Proposition. Suppose A, B_i for $i \le n$ and $C_{ij} = C_{ji}$ for $1 \le i < j \le n$ form an n-frame H in Su(M) for M an R-module, $n \ge 3$. Let $\mu:\mathrm{Rel}(B_1/A) \longrightarrow \mathrm{Su}(M)_H$ be given by the negative graph insertion:

$$\mu(f) = \{u - g_{12}(v) : \langle u + A,\ v + A\rangle \in f\}.$$

Then μ is a one-one τ_B-homomorphism with image $[0_H, I_H]$. In particular, $[0_H, I_H]$ is in $\mathcal{B}(R)$.

Von Neumann defined the auxiliary ring associated with an n-frame, $n \ge 4$, to be the set of complements of b_2 relative to the interval $[a,\ b_1 \vee b_2]$; this set for 4.3b corresponds to the ring of endomorphisms of M, which can be regarded as the τ_R-subalgebra hom(1,1) of Rel(M). Our formulas for the ring operations $+_H$, $-_H$, 0_H, $\cdot_H$ and 1_H are essentially the same, but we apply these formulas to the entire interval $[a,\ b_1 \vee b_2]$ and add the other τ_B-operations, corresponding to the entire additive relation algebra Rel(M).

To translate universal Horn sentences, adapt the considerations above in terms of polynomials of types τ_B and τ_L.

4.6. Definitions. To set up a 3-frame, the first ten variables of $X = \{x_1, x_2, x_3, \ldots\}$ are relabelled, to obtain $Y = \langle x_1, x_2, \ldots, x_{10}\rangle =$

$$\langle a, b_1, b_2, b_3, c_{12}, c_{21}, c_{13}, c_{31}, c_{23}, c_{32}\rangle.$$

Also, denote x_{i+10} by y_i for all $i \ge 1$.

Recall that $P(X,\tau)$ denotes the τ-algebra of all τ-polynomials on X, for any algebraic type τ. Define $P(X,\tau_L)_Y$ to be a τ_B-algebra given by the formulas of 4.4, and let $\kappa: P(X,\tau_B) \longrightarrow P(X,\tau_L)_Y$ be the unique τ_B-homomorphism such that $\kappa(x_i) = y_i$ for all $i \ge 1$. It is clear that κ is a recursive function, if polynomials of type τ_L and τ_B are regarded as words on a suitable alphabet.

4.7. Definitions and Properties. Suppose Λ is a basic universal Horn sentence for additive relation algebras with unit, say

$$(\forall x_1, x_2, \ldots, x_m)((p_1 = q_1\ \&\ p_2 = q_2\ \&\ \ldots\ \&\ p_n = q_n) \Rightarrow p_0 = q_0)$$

where $p_i = p_i(x_1, x_2, \ldots, x_m)$ and $q_i = q_i(x_1, x_2, \ldots, x_m)$ in $P(X, \tau_B)$ for $0 \le i \le n$ and $x_1, x_2, \ldots, x_m$ in X. Define $\mathbf{T}_1(\Lambda)$, a basic universal Horn sentence for lattices, as:

$$(\forall x_1, x_2, \ldots, x_{m+10})((E_1 \ \& \ E_2 \ \& \ \ldots \ \& \ E_{d+2m+n}) \Rightarrow \mathbf{t}(p_0) = \mathbf{t}(q_0)),$$

with the equations $E_1, E_2, \ldots, E_{d+2m+n}$ given as follows:

4.7a. E_j for $j \le d$ are equations as in 4.2 specifying that a, b_i and c_{ij} form a 3-frame (possibly trivial). More precisely, a τ_L-homomorphism $h: P(X, \tau_L) \longrightarrow L$ is a frame assignment iff h satisfies these equations. (In 4.2, 29 such equations are given, but many are redundant.)

4.7b. E_{d+j} is the equation $a \le y_j$ and E_{d+m+j} is $y_j \le b_1 \vee b_2$, for $1 \le j \le m$.

4.7c. E_{d+2m+j} is the equation $\kappa(p_j) = \kappa(q_j)$ for $1 \le j \le n$.

We are now prepared for our first translation result.

4.8. Proposition. Suppose R is a ring with unit and Λ is a basic universal Horn sentence for additive relation algebras with unit (type τ_B). Then Λ holds in Rel(M) for all R-modules M iff $\mathbf{T}_1(\Lambda)$ holds in Su(M) for all R-modules M.

Proof: Assume the hypotheses, and suppose $\mathrm{Su}(N) \models \mathbf{T}_1(\Lambda)$ for all N in R-**Mod**. Let $g: P(X, \tau_B) \longrightarrow \mathrm{Rel}(M)$ be a τ_B-homomorphism such that $g(p_i) = g(q_i)$ for $i \le n$. Using 4.5, μ is a one-one τ_B-homomorphism from Rel(M) into $\mathrm{Su}(M^{(3)})_Z$ for Z the canonical 3-frame of 4.3b. Let $h: P(X, \tau_L) \longrightarrow \mathrm{Su}(M^{(3)})$ be the τ_L-homomorphism such that $h(a) = \mathbf{0}$, $h(b_i) = \kappa_i[M]$, $h(c_{ij}) = h(c_{ji}) = (\kappa_i - \kappa_j)[M]$ for $1 \le i < j \le 3$, and $h(y_i) = \mu(g(x_i))$ for $i \ge 1$. Since h is a frame assignment with $h(Y) = Z$, $h: P(X, \tau_L)_Y \longrightarrow \mathrm{Su}(M^{(3)})_Z$ is a τ_B-homomorphism by 4.4b. Then by the hypotheses, we have the commutative diagram of τ_B-homorphisms below:

$$\begin{array}{ccc} P(X, \tau_B) & \xrightarrow{\ \kappa\ } & P(X, \tau_L)_Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ \mathrm{Rel}(M) & \xrightarrow[\ \mu\ ]{} & \mathrm{Su}(M^{(3)})_Z \end{array}$$

Therefore, $g(p_i) = g(q_i)$ implies $h(\kappa(p_i)) = h(\kappa(q_i))$ for $i = 1, 2, \ldots, n$. Note that all of the hypotheses of $\mathbf{T}_1(\Lambda)$ are satisfied for h, and so $h(\kappa(p_0)) = h(\kappa(q_0))$. Since μ is one-one, we have $g(p_0) = g(q_0)$, and therefore $Rel(M) \models \Lambda$.

Now suppose that $Rel(N) \models \Lambda$ for all R-modules N. Let $h: P(X,\tau_L) \longrightarrow Su(M)$ be a lattice homomorphism satisfying all the hypotheses of $\mathbf{T}_1(\Lambda)$. Then $h: P(X,\tau_L)_Y \longrightarrow Su(M)_V$ is a τ_B-homomorphism by 4.4b, where $V = h(Y)$. By 4.7a, V is a 3-frame in Su(M). So, $[\mathbf{0}_V, \mathbf{I}_V]$ is in $\mathcal{B}(R)$ by 4.5, and note that $h(y_i)$ is in $[\mathbf{0}_V, \mathbf{I}_V]$ for $1 \le i \le m$ by 4.7b. Let $g: P(X,\tau_B) \longrightarrow [\mathbf{0}_V, \mathbf{I}_V]$ be any τ_B-homomorphism such that $g(x_i) = h(y_i)$ for $1 \le i \le m$. By construction, $\mu(g(p)) = h(\kappa(p))$ if $p = p(x_1, x_2, \ldots, x_m)$ in $P(X,\tau_B)$. Since μ is one-one, $g(p_i) = g(q_i)$ for $1 \le i \le n$ by 4.7c. Then $g(p_0) = g(q_0)$ using Λ, so $h(\kappa(p_0)) = h(\kappa(q_0))$. This proves that $Su(M) \models \mathbf{T}_1(\Lambda)$ for all R-modules M. ∎

Recall the classes $\mathcal{A}(R)$ and $\mathcal{B}(R)$ of additive relation algebras (without or with unit) which are representable by R-modules (3.2). Using frames, we can show that these classes are quasivarieties.

4.9. Proposition. If R is a ring with unit, then $\mathcal{A}(R)$ is a quasivariety of τ_A-algebras, and $\mathcal{B}(R)$ is a quasivariety of τ_B-algebras.

Proof: Suppose B is a τ_B-algebra not in $\mathcal{B}(R)$. Let C be a disjoint union $Y \cup B$, where Y contains ten 3-frame variables as in 4.6. We construct a commutative diagram of τ_B-homomorphisms as shown below.

$$\begin{array}{ccccc} P(B,\tau_B) & \xrightarrow{\ \kappa\ } & P(C,\tau_L)_Y & \xrightarrow{\ \eta\ } & L_Y \\ \big\downarrow g & & & & \big\downarrow \lambda \\ B & \xrightarrow[\ \mu\ ]{} & [\mathbf{0}_V, \mathbf{I}_V] & \xrightarrow[\ \iota\ ]{} & Su(M)_V \end{array}$$

Define g and κ by $g(f) = f = \kappa(f)$ for all f in B. Define L by a presentation $\mathcal{L}(R)\{C|W\}$, where $\mathcal{L}(R)$ is a lattice quasivariety and $W = W_1 \cup W_2 \cup W_3 \subseteq P(C,\tau_L)^2$ as given below. Let W_1 consist of d pairs corresponding to the equations of 4.7a, specifying that elements a, b_i and c_{ij} of Y form a 3-frame in L. Let W_2 consist of the set of $2|B|$ pairs specifying that $a \le f \le b_1 \vee b_2$ for all f in B, similar to 4.7b. Finally, let W_3 consist of all pairs $\langle \kappa(p), \kappa(q) \rangle$ such that $g(p) = g(q)$, for p and q in $P(B,\tau_B)$ (compare 4.7c). Let η be the canonical τ_L-homomorphism, so $h(c) = c$ for all c in C, and note that η is a τ_B-homomorphism by 4.4b. There exists a lattice embedding $\lambda:L\longrightarrow Su(M)$, since $L \in \mathcal{L}(R)$. From W_1, Y is a 3-frame in L, and so $V = \lambda(Y)$ is a 3-frame in $Su(M)$ and λ is a τ_B-homomorphism by 4.4b. Also, the inclusion ι is a τ_B-homomorphism by 4.4a. Now $\eta(\kappa(f))$ is in $[\mathbf{0}_Y,\mathbf{I}_Y]$ for all f in B by W_2, so $\lambda(\eta(\kappa(f)))$ is in $[\mathbf{0}_V,\mathbf{I}_V]$. Using W_3, we can define a unique τ_B-homomorphism μ such that $\mu(f) = \lambda(f)$ for all f in B, so $\kappa\eta\lambda = g\mu\iota$. Now μ is not one-one, since B is not in $\mathcal{B}(R)$ but $[\mathbf{0}_V,\mathbf{I}_V]$ is in $\mathcal{B}(R)$ by 4.5. So, there are distinct f_1 and f_2 in B such that $\mu(f_1) = \mu(f_2)$. Since λ is one-one, we have $\eta(f_1) = \eta(f_2)$ also.

By A?, there exists a finite subset $C_m = Y \cup \{f_1,f_2,\dots,f_m\}$ of C (containing f_1 and f_2 above) and a finite subset $W' \subseteq W \cap P(C_m,\tau_L)^2$ such that $\eta'(f_1) = \eta'(f_2)$ if $\eta':P(C_m,\tau_L)\longrightarrow L'$ is canonical for $L' = \mathcal{L}(R)\{C_m|W'\}$. Suppose $W_3 \cap W'$ consists of pairs $\langle p_j,q_j\rangle$ in $P(B,\tau_B)$, where

$$p_j = p_j(f_1,f_2,\dots,f_m) \text{ and } q_j = q_j(f_1,f_2,\dots,f_m) \text{ for } j \le n.$$

Let Λ be the universal Horn sentence of type τ_B with hypotheses

$$p_j(x_1,x_2,\dots,x_m) = q_j(x_1,x_2,\dots,x_m)$$

for $j \le n$, and conclusion $x_1 = x_2$. By the construction and A?, $\mathcal{L}(R) \models \mathbf{T}_1(\Lambda)$, and so $\mathcal{B}(R) \models \Lambda$ by 4.8. Clearly Λ is not satisfied in B, and it follows that $\mathcal{B}(R)$ is a quasivariety. The proof that $\mathcal{A}(R)$ is a quasivariety is similar. ∎

We next observe that *inclusion* functions can also serve as basic universal Horn sentence translation functions.

4.10. Proposition. If Λ is a universal Horn sentence of type τ_A, then $\mathcal{A}(R) \models \Lambda$ iff $\mathcal{B}(R) \models \Lambda$. If Ω is a universal Horn sentence for lattices (type τ_L), then $\mathcal{L}(R) \models \Omega$ iff $\mathcal{A}(R) \models \Omega$ iff $\mathcal{B}(R) \models \Omega$.

Proof: From 3.2, $\mathcal{A}(R) \models \Lambda$ iff $Rel_*(M) \models \Lambda$ for all R-modules M iff $Rel(M) \models \Lambda$ for all R-modules M iff $\mathcal{B}(R) \models \Lambda$. In particular, $\mathcal{A}(R) \models \Omega$ iff $\mathcal{B}(R) \models \Omega$. Since each $Su(M)$ is isomorphic to the sublattice of symmetric null elements of Rel(M) by 3.4g and 3.16, $\mathcal{L}(R) \models \Omega$ iff $Su(M) \models \Omega$ for all M iff $Rel(M) \models \Omega$ for all M iff $\mathcal{B}(R) \models \Omega$ iff $\mathcal{A}(R) \models \Omega$. ∎

At this point, we can already prove the algebraic parts of the unification theorems of §7.

4.11. Corollary. For rings R and S with unit, $\mathcal{L}(R) \subseteq \mathcal{L}(S)$ iff $\mathcal{A}(R) \subseteq \mathcal{A}(S)$ iff $\mathcal{B}(R) \subseteq \mathcal{B}(S)$. Therefore, $\mathcal{L}(R) = \mathcal{L}(S)$ iff $\mathcal{A}(R) = \mathcal{A}(S)$ iff $\mathcal{B}(R) = \mathcal{B}(S)$.

That is, R and S have the same lattices representable by modules iff they have the same additive relation algebras (without or with unit) representable by modules. By the quasivariety characterizations in 2.9 and 4.9, it is also equivalent to assert that the same universal Horn sentences are satisfied for $\mathcal{L}(R)$ and $\mathcal{L}(S)$, or $\mathcal{A}(R)$ and $\mathcal{A}(S)$, or $\mathcal{B}(R)$ and $\mathcal{B}(S)$.

To prove the equivalences displayed above, note first that if $\mathcal{L}(R) \subseteq \mathcal{L}(S)$ is false, there is a basic universal Horn sentence for lattices Ω such that $\mathcal{L}(S) \models \Omega$ but not $\mathcal{L}(R) \models \Omega$. Then $\mathcal{A}(S) \models \Omega$ but not $\mathcal{A}(R) \models \Omega$ by 4.10, so $\mathcal{A}(R) \subseteq \mathcal{A}(S)$ is false. Similarly, if $\mathcal{A}(R) \subseteq \mathcal{A}(S)$ is false, so is $\mathcal{B}(R) \subseteq \mathcal{B}(S)$. If $\mathcal{B}(R) \subseteq \mathcal{B}(S)$ is false, then there is a basic universal Horn sentence Λ of type τ_B such that $\mathcal{B}(S) \models \Lambda$ but not $\mathcal{B}(R) \models \Lambda$. But then $\mathcal{L}(S) \models \mathbf{T}_1(\Lambda)$ but not $\mathcal{L}(R) \models \mathbf{T}_1(\Lambda)$ by 4.8, so that $\mathcal{L}(R) \subseteq \mathcal{L}(S)$ is false also.