§5. Relation Categories and Functors.

In this section, we introduce an elementary axiomatization of additive relation category theory. Our approach partly follows the axiomatizations of MacLane [] and Puppe []. The differences are introduced to obtain technical advantages in the treatment of full subcategories and in comparing the categories with additive relation algebras.

Just as the abelian category R-**Mod** was regarded as a subcategory of R-**Rel** in §1, we identify an exact subcategory of proper morphisms which are formally analogous to R-linear maps, by the method of Puppe [,§?]. Also, the relationships between homomorphisms of additive relation algebras and structure-preserving functors of relation categories and exact subcategories are described.

5.1. Definition. Suppose a system $\mathcal{C}$ is provided with category structures (objects, morphisms with domain and codomain, composition, unit morphisms), a converse operation # which determines a morphism $f^{\#}:B\longrightarrow A$ corresponding to each morphism $f:A\longrightarrow B$ of $\mathcal{C}$, and for all objects A and B, $\mathcal{C}(A,B)$ is provided with binary operations $\wedge_{AB}$, $\vee_{AB}$ and $+_{AB}$ and a unary operation $-_{AB}$, and designated morphisms $\mathbf{O}_{AB}$, $\mathbf{I}_{AB}$ and 0_{AB} in $\mathcal{C}(A,B)$. If $\mathcal{C}$ satisfies conditions 5.1a-i below, it is called a *strongly exact relation category*. If $\mathcal{C}$ satisfies 5.1a-h, it is called an *almost strongly exact relation category*. (As usual, we will omit many subscripts. To avoid ambiguity, $\mathbf{O}$, $\mathbf{I}$ and 0 will abbreviate only equal subscript cases $\mathbf{O}_{AA}$, $\mathbf{I}_{BB}$, 0_{CC}, etc., not $\mathbf{O}_{AB}$, $\mathbf{I}_{AB}$ or 0_{AB} if $A \neq B$ is possible.)

5.1a. $\mathcal{C}$ is a category.

5.1b. For all A and B in $\mathcal{C}$, $\mathcal{C}(A,B)$ is a modular lattice under $\wedge_{AB}$ and $\vee_{AB}$, with smallest element $\mathbf{O}_{AB}$ and largest element $\mathbf{I}_{AB}$. (Let $\leq_{AB}$ denote the partial order induced by this lattice.)

5.1c. Sum $+_{AB}$ is commutative and associative on $\mathcal{C}(A,B)$, and $0_{AB} = \mathbf{I}\mathbf{O}_{AB} = \mathbf{I}_{AB}\mathbf{O}$ is a zero for addition: $f + 0_{AB} = f$ for $f:A\longrightarrow B$.

5.1d. For $f \leq g$ and $h \leq k$ in $\mathcal{C}(A,B)$, $f + h \leq g + k$. For $f \leq g$ in $\mathcal{C}(A,B)$ and

$h \le k$ in $\mathcal{C}(B,C)$, $fh \le gk$.

5.1e. The converse operation is an involution that preserves the lattice operations. That is, for $f,g:A\longrightarrow B$ and $h:B\longrightarrow C$ in $\mathcal{C}$, we have $f^{\#\#} = f$, $(gh)^{\#} = h^{\#}g^{\#}$, $(f \wedge g)^{\#} = f^{\#} \wedge g^{\#}$ and $(f \vee g)^{\#} = f^{\#} \vee g^{\#}$.

5.1f. For $f:A\longrightarrow B$ in $\mathcal{C}$, $-f = (-1_A)f = f(-1_B)$, and -1_A is a negative unit: $1_A + (-1_A) = 0_{AA}$.

5.1g. For $f,g:A\longrightarrow B$ in $\mathcal{C}$, $f\mathbf{I} \wedge g \le ff^{\#}g$ and $gf^{\#}f \le \mathbf{0}f \vee g$.

5.1h. For $f,g:A\longrightarrow B$ and $h,k:B\longrightarrow C$, $(f+g)h \ge fh + gh$ and $g(h+k) \le gh + gk$.

5.1i. If $e:A\longrightarrow A$ is in $\mathcal{C}$ such that $e = e^{\#} = ee$, then there exists $f:B\longrightarrow A$ for some B in $\mathcal{C}$ such that $ff^{\#} = 1_B$ and $f^{\#}f = e$.

In §1, we described such structures for R-**Rel**, the category of R-modules and additive relations between them. We omit the proof of the next result.

5.2. Proposition. For any ring R with unit, R-**Rel** is a strongly exact relation category. A full subcategory of R-**Rel** which admits subobjects and quotient objects is a strongly exact relation category. Any full subcategory of an almost strongly exact relation category is an almost strongly exact relation category. In particular, any full subcategory of R-**Rel** is an almost strongly exact relation category.

Obviously, the category definitions are closely related to our axioms for additive relation algebras (3.1).

5.3. Proposition. If $\mathcal{C}$ is an almost strongly exact relation category, then each endomorphism algebra $\mathcal{C}(A,A)$ is an additive relation algebra with unit. If B is an additive relation algebra with unit, then there is an almost strongly exact relation category $\mathcal{C}_B$ with one object X and $\mathcal{C}_B(X,X) = B$.

Proof: For $f:A\longrightarrow B$ in $\mathcal{C}$, $f = f\mathbf{I} \wedge f \le ff^{\#}f \le \mathbf{0}f \vee f = f$ using 5.1g. So, 3.1a,b,c,d,g,h can be shown to hold in $\mathcal{C}(A,A)$ for A in $\mathcal{C}$. Also,

$$f = (1_A + (-1_A) + 1_A)f \ge f + (-f) + f \ge f(1_B + (-1_B) + 1_B) = f,$$

by 5.1f,h. Taking $f = -1_A$, $-(-1_A) = 1_A$ follows, and hence $-(-f) = f$, using 5.1f. Then $-1_A = -1_A(-1_A)^{\#}-1_A = (-1_A)^{\#}$, and so $-(f^{\#}) = (-f)^{\#}$ by 5.1e,f.

Some computation then shows that $\mathcal{C}(A,A)$ satisfies 3.1e.

In $\mathcal{C}(A,A)$, we have by 5.1b,d,f,h that

$$\mathbf{0} \le \mathbf{0I0} = \mathbf{0}(1+(-1)) \le \mathbf{0}1+\mathbf{0}(-1) = \mathbf{0}+(-1)\mathbf{0} \le \mathbf{0}+\mathbf{I0} = \mathbf{0}.$$

It follows that $\mathbf{0}$ is null in $\mathcal{C}(A,A)$, and similarly $\mathbf{I}$ is null. Also, $\mathbf{0}$ and $\mathbf{I}$ are symmetric by applying 5.1b,e. Now $\mathbf{0}g = \mathbf{0}g+\mathbf{0}$ by the arguments proving 3.4b,c,d,e, so $\mathbf{0}g \le g+h$ using 5.1d, for $g,h:A\longrightarrow A$. Similarly, $g+h \le g\mathbf{I}$. Also $g\mathbf{I0} = g(1+(-1)) \le g+(-g)$, and so

$$g\mathbf{I0}g \le g\mathbf{I0}g^{\#}g \le \mathbf{0}g \vee g\mathbf{I0} \le g+(-g),$$

using 5.1d,g. Dual arguments show that $g\mathbf{I0}g = g+(-g)$, and it follows that $\mathcal{C}(A,A)$ satisfies 3.1f. Then $\mathcal{C}(A,A)$ is an additive relation algebra with unit by 3.12, proving the first part. We omit the calculations proving the second part. ∎

Much of the elementary theory of almost strongly exact relation categories can be adapted from similar results for additive relation algebras. Implicitly using 5.3, we can apply results of §3 below.

5.4. Definitions and Properties. For $f:A\longrightarrow B$ in $\mathcal{C}$, f is *null* if $fgf = f$ for all $g:B\longrightarrow A$ in $\mathcal{C}$. For $d:A\longrightarrow A$ in $\mathcal{C}$, d is *symmetric* if $d = d^{\#}$, and d is *idempotent* if $d = dd$. (Compare 3.4.)

5.4a. If $f:A\longrightarrow B$, then $f = ff^{\#}f = f+(-f)+f$ (see 5.3).

5.4b. If $f:A\longrightarrow B$, $x:B\longrightarrow C$ and $g:C\longrightarrow D$ such that x is null, then fxg is null. If $y:A\longrightarrow B$ and $z:C\longrightarrow D$ are null, then $yhz = ykz$ for all $h,k:B\longrightarrow C$. If $w:A\longrightarrow B$ is null, then $w = -w = w+w$. (See 3.4b,c,d, and note that $w = ww^{\#}w = w(w^{\#}w+w^{\#}w) \le w+w$ by 5.1h.)

5.4c. For all A and B, $\mathbf{0}_{AB}$ and $\mathbf{I}_{AB}$ are null, and are symmetric if $A = B$. For all A, B and C, $\mathbf{0}_{AC}\mathbf{0}_{CB} = \mathbf{0}_{AB}$, $\mathbf{I}_{AC}\mathbf{I}_{CB} = \mathbf{I}_{AB}$, $\mathbf{I}_{AC}\mathbf{0}_{CB} = 0_{AB}$ and $\mathbf{0}_{AC}\mathbf{I}_{CB} = 0_{AB}{}^{\#}$. (Note $0_{AB} = \mathbf{I0}_{AB} \le \mathbf{I0}_{AC}\mathbf{0}_{CB} \le \mathbf{I}_{AC}\mathbf{0}_{CB} \le \mathbf{I}_{AC}\mathbf{0}_{CA}\mathbf{0}_{AB} \le \mathbf{I0}_{AB}$, etc. Using 5.4b and 3.13a, $\mathbf{0}_{AB} = \mathbf{00}_{AB}$ and $\mathbf{I}_{AB} = \mathbf{II}_{AB}$ are null, and are symmetric if $A = B$.)

5.4d. For $f:A\longrightarrow B$, $\mathbf{0}f = \mathbf{0}_{AB}f^{\#}f = \mathbf{0}_{AB}\mathbf{I}\wedge f$ and $f\mathbf{I} = ff^{\#}\mathbf{I}_{AB} = \mathbf{0I}_{AB}\vee f$ since

$\mathbf{O}f \le \mathbf{O}_{AB}f^\#f \le \mathbf{O}ff^\#f = \mathbf{O}f$, $\mathbf{O}_{AB}\mathbf{I} \wedge f \le \mathbf{O}_{AB}\mathbf{O}_{AB}{}^\#f = \mathbf{O}f$, etc.)

5.4e. For all A in $\mathcal{C}$, $(-1_A)(-1_A) = 1_A$ and $(-1_A)^\# = -1_A$. Also, $\mathbf{O}(-1_A) = \mathbf{O} = (-1_A)\mathbf{O}$ and $\mathbf{I}(-1_A) = \mathbf{I} = (-1_A)\mathbf{I}$. (Note $(-1)^2 = -(-1) = 1$ as in 5.3. Then $-1 = (-1)(-1)^\#(-1) = (-1)^\#$. For the rest, use 5.1f and 5.4b,c.)

5.4f. Suppose $f:A \longrightarrow B$. If $d:A \longrightarrow A$ is a symmetric idempotent, then $df = (d\mathbf{O}_{AB} \vee f) \wedge d\mathbf{I}_{AB}$. If $e:B \longrightarrow B$ is a symmetric idempotent, then $fe = (\mathbf{O}_{AB}e \vee f) \wedge \mathbf{I}_{AB}e$. (Note $d\mathbf{I} \wedge ff^\# \le dff^\#$, so $df \ge (d\mathbf{I}_{AB}f^\# \wedge ff^\#)f \ge (d\mathbf{I}_{AB} \wedge f)f^\#f \ge hh^\#h = h$ for $h = d\mathbf{I}_{AB} \wedge f$. Prove $df \le d\mathbf{O}_{AB} \vee f$ similarly, then use modularity as in 3.10e. Take converses to obtain the second formula.)

5.4g. If $f,g:A \longrightarrow B$, then $\mathbf{O}(f+g) = \mathbf{O}f \vee \mathbf{O}g$ and $(f+g)\mathbf{I} = f\mathbf{I} \wedge g\mathbf{I}$. Since $\mathbf{O}_{BA}f + \mathbf{O} = \mathbf{O}_{BA}f$ by 3.4e, we have $\mathbf{O}f \le \mathbf{O}_{AB}(\mathbf{O}_{BA}f + \mathbf{O}_{BA}g) \le \mathbf{O}f \vee \mathbf{O}g \le f+g$ using 5.4c. Now $\mathbf{O}g \le f+g$ similarly, so $h \le \mathbf{O}(f+g)$ for $h = \mathbf{O}f \vee \mathbf{O}g$ using 5.4d. But $\mathbf{O}(f+g) \le \mathbf{O}f + \mathbf{O}g \le \mathbf{O}h + \mathbf{O}h = \mathbf{O}h \le h$ by 5.4b,c, proving the first part. The second part is dual.)

5.4h. If $f,g:A \longrightarrow B$ satisfy $f \le g$, $\mathbf{O}f \ge \mathbf{O}g$ and $f\mathbf{I} \ge g\mathbf{I}$, then $f = g$. (By 5.1g and the hypotheses, $g = f\mathbf{I} \wedge g \le ff^\#g \le fg^\#g \le \mathbf{O}g \vee f = f$.)

5.4i. Suppose A and B are R-modules. Then $w:A \longrightarrow B$ is null in R-**Rel** iff there exist submodules A_0 of A and B_0 of B such that $w = A_0 \oplus B_0$. (For symmetric idempotents and symmetric null elements, see 3.4h.)

We give some further elementary results here, going beyond the analysis of §3.

5.5. Properties of Almost Strongly Exact Relation Categories.

5.5a. Suppose $f,g:A \longrightarrow B$ with $d = 1_A \wedge ff^\# \wedge gg^\#$ and $e = 1_B \vee f^\#f \vee g^\#g$. Then $dfe\mathbf{I} = d\mathbf{I}_{AB} = dge\mathbf{I}$, $\mathbf{O}dfe = \mathbf{O}_{AB}e = \mathbf{O}dge$ and $f+g = d(f+g) = df+dg = (f+g)e = fe+ge = dfe+dge$. (Note that $d\mathbf{I}_{AB} \ge dfe\mathbf{I} \ge df\mathbf{I} = dff^\#\mathbf{I}_{AB} \ge dd\mathbf{I}_{AB} = d\mathbf{I}_{AB}$ using 5.4c and 3.13j. So, $dfe\mathbf{I} = d\mathbf{I}_{AB}$, and $dge\mathbf{I} = d\mathbf{I}_{AB}$ and $\mathbf{O}dfe = \mathbf{O}_{AB}e = \mathbf{O}dge$ similarly. For $h = f+g$,

$$1_A \wedge hh^\# \le 1 \wedge h\mathbf{I}\mathbf{I}_{BA} = 1 \wedge (f\mathbf{I} \wedge g\mathbf{I})\mathbf{I}_{BA} \le 1 \wedge ff^\#\mathbf{I} \wedge gg^\#\mathbf{I} = d,$$

using 5.4c,g and 3.13i. Then $h = (1 \wedge hh^\#)hh^\#h \le dh$ by 3.13i, and $he \le h$

similarly. So, $h \le dh \le df + dg \le dfe + dge \le fe + ge \le he \le h$, using 5.1d,h.)

5.5b. For $f,g:A \longrightarrow B$, $(f \wedge g)\mathbf{O} = f\mathbf{O} + g\mathbf{O}$ and $\mathbf{I}(f \vee g) = \mathbf{I}f + \mathbf{I}g$. (For $h = f\mathbf{O} \wedge g\mathbf{O}$ and $k = f\mathbf{O} + g\mathbf{O}$, $(f \wedge g)\mathbf{O} = h = h + h \le k = f\mathbf{OIO} + g\mathbf{OIO} \le k\mathbf{IO} = (f\mathbf{OI} \wedge g\mathbf{OI})\mathbf{O} \le h$, using 5.4b,c,d,g and 5.1h. The second part is dual.)

5.5c. If $f,g:A \longrightarrow B$ and $h,k:A \longrightarrow C$, then $(f+g)^\#(h+k) \le f^\# h \vee g^\# k$. If $f,g:B \longrightarrow A$ and $h,k:C \longrightarrow A$, then $fh^\# \wedge gk^\# \le (f+g)(h+k)^\#$. (Given $f,g:A \longrightarrow B$ and $h,k:A \longrightarrow C$, let $c = 1_A \wedge ff^\# \wedge gg^\#$ and $d = 1_A \wedge hh^\# \wedge kk^\#$. Then c and d are symmetric idempotents with $cd = dc = c \wedge d$ by 3.13j, and $c(f+g) = f+g$ and $d(h+k) = h+k$ by 5.5a. Let $f_0 = cdf$, $g_0 = cdg$, $h_0 = cdh$ and $k_0 = cdk$. Now

$$cd\mathbf{I} = cd(1 \wedge ff^\# \wedge gg^\#)\mathbf{I} \le cdff^\#\mathbf{I} \le f_0\mathbf{I}_{BA} = f_0 f_0^\#\mathbf{I} \le cd\mathbf{I},$$

and continuing we obtain $cd\mathbf{I} = f_0 f_0^\#\mathbf{I} = g_0 g_0^\#\mathbf{I} = h_0 h_0^\#\mathbf{I} = k_0 k_0^\#\mathbf{I}$. Let $s = (f_0 + g_0)^\#(h_0 + k_0)$, $t = f_0^\# h_0 \vee g_0^\# k_0$ and $e = (f_0 + g_0)^\#(f_0 + g_0)$. Note that $s \le et$ because by 5.4b,f and the above,

$$h_0 + k_0 \le f_0 f_0^\# h_0 + g_0 g_0^\# k_0 \le (f_0 + g_0)t.$$

Also $et \le t$ by 5.4f, since $e\mathbf{O}_{AC} = (f_0 + g_0)^\#\mathbf{O}_{BC} = (\mathbf{O}_{CB} f_0 \vee \mathbf{O}_{CB} g_0)^\# = f^\#\mathbf{O}_{BC} \vee g^\#\mathbf{O}_{BC} \le t$ using 5.4c,d,g. Then

$$(f+g)^\#(h+k) = [cd(f+g)]^\# cd(h+k) \le s \le et \le t \le f^\# h \vee g^\# k$$

using 5.1e,h, proving the first part. The second part is dual.)

5.5d. Suppose $f:A \longrightarrow B$ and $g,h:B \longrightarrow C$. If $\mathbf{O}f^\# f \le gg^\#$ or $\mathbf{O}f^\# f \le hh^\#$, then $f(g \wedge h) = fg \wedge fh$ and $f(g \vee h) = fg \vee fh$. If $k:C \longrightarrow D$ such that $g^\# g \le kk^\#\mathbf{I}$ or $h^\# h \le kk^\#\mathbf{I}$, then $(g \vee h)k = gk \vee hk$ and $(g+h)k = gk + hk$. (Given f,g,h and $\mathbf{O}f^\# f \le hh^\#$, let $e = 1_B \vee f^\# f$. Then $e\mathbf{O}_{BC} = f^\# f\mathbf{OO}_{BC} \le hh^\#\mathbf{O}_{BC} \le h$, by 3.13i and 5.4c. So, $eh = h$ using 5.4f and $e \ge 1$, and similarly $f = fe$. Then $e(g \wedge h) = e\mathbf{O}_{BC} \vee (g \wedge h) = eg \wedge h$ by modularity and 5.4f. For $r = fg \wedge fh$, $ff^\# r = r$ using 5.4f, so $f(g \wedge h) = fe(g \wedge h) = f(eg \wedge eh) \ge f(f^\# fg \wedge f^\# fh) \ge ff^\# r = r \ge f(g \wedge h)$. For $s = f(g+h)$ and $t = fg + fh$, $s \le t$ by 5.1h, and $s\mathbf{I} = f(g\mathbf{I} \wedge h\mathbf{I}) = fg\mathbf{I} \wedge fh\mathbf{I} = t\mathbf{I}$ by 5.4g and the above. Finally, $\mathbf{O}s = \mathbf{O}f(g+h)(1+\mathbf{IO})^\# \ge \mathbf{O}f(g \wedge h\mathbf{OI}) = \mathbf{O}fg \wedge \mathbf{O}fh\mathbf{OI} = \mathbf{O}fg$ by 5.5c and the above. Similarly $\mathbf{O}s \ge \mathbf{O}fh$, so $\mathbf{O}s \ge \mathbf{O}fg \vee \mathbf{O}fh = \mathbf{O}t$. By 5.4h, $s = t$. The remaining parts are similar or dual.)

5.5e. Suppose $f,g:A\longrightarrow B$. Then $\mathbf{O}(f\vee g) = \mathbf{OI}(f+(-g))$ and $(f\wedge g)\mathbf{I} = (f+(-g))\mathbf{OI}$. (For $h = \mathbf{OI}(f+(-g))$ and $k = f+g$, we have $h = (1+(-1))^{\#}(f+(-g)) \leq 1^{\#}f\vee(-1)^{\#}(-g) = k$ by 5.5c and 5.4e, so $h \leq \mathbf{O}k$ by 5.4d. Using 5.5d,

$$\mathbf{O}k \leq \mathbf{O}_{AB}g^{\#}(f\vee g) = \mathbf{O}_{AB}(g^{\#}f\vee g^{\#}g) \leq \mathbf{O}_{AB}(1\vee g^{\#}f\vee g^{\#}g) = \mathbf{O}_{AB}(1\vee g^{\#}f),$$

since $1\vee g^{\#}g = 1\vee g^{\#}g\mathbf{O} \leq 1\vee g^{\#}f$ by 3.13i. For $t = g^{\#}f+(-1)$ then,

$$\mathbf{O}k \leq \mathbf{O}_{AB}(1\vee(1+\mathbf{IO})^{\#}(1+t)) \leq \mathbf{O}_{AB}(1\vee 1^{\#}1\vee \mathbf{OI}t) \leq \mathbf{O}_{AB}\mathbf{I}t$$

by 5.5d since $\mathbf{O}(1\vee t^{\#}\mathbf{I}t) = \mathbf{OI}t$ by 3.13i and 3.4c. Now $t \leq t\mathbf{I} \leq g^{\#}f\mathbf{I} \leq g^{\#}\mathbf{I}_{AB} = g^{\#}g\mathbf{I}$ by 5.4d,g, so

$$\mathbf{I}_{AB}t \leq \mathbf{I}_{AB}g^{\#}gt \leq \mathbf{I}_{AB}(g^{\#}f+(-g^{\#}g)) = \mathbf{I}_{AB}g^{\#}(f+(-g)) \leq \mathbf{I}(f+(-g)),$$

using 5.4e,f and 5.5d. Then $\mathbf{O}k \leq h$ follows, and the rest similarly.)

Next, we consider functors preserving additive relation category structures.

5.6. Definitions and Properties. Suppose $\mathcal{C}$ and $\mathcal{D}$ are almost strongly exact relation categories. A *relation functor* $F:\mathcal{C}\longrightarrow\mathcal{D}$ is a functor such that for f,g in $\mathcal{C}(A,B)$, $F(f^{\#}) = (Ff)^{\#}$, $F(f\wedge g) = Ff\wedge Fg$, $F(f\vee g) = Ff\vee Fg$, $F(f+g) = Ff+Fg$ and $F(-f) = -Ff$, and for $C = F(A)$ and $D = F(B)$, $F(\mathbf{O}_{AB}) = \mathbf{O}_{CD}$, $F(\mathbf{I}_{AB}) = \mathbf{I}_{CD}$ and $F(0_{AB}) = 0_{CD}$.

5.6a. Composites of relation functors are relation functors, and the identity functor is a relation functor. Inclusion functors of full subcategories are relation functors.

5.6b. Suppose $F:\mathcal{C}\longrightarrow\mathcal{D}$ is a relation functor. Then F induces a τ_B-homomorphism $\mathcal{C}(A,A)\longrightarrow\mathcal{D}(F(A),F(A))$ for each A in $\mathcal{C}$. If F is an embedding functor, then F induces a one-one τ_B-homomorphism.

5.6c. A τ_B-homorphism $f:B\longrightarrow C$ of additive relation algebras with unit can be regarded as a relation functor $\mathcal{C}_B\longrightarrow\mathcal{C}_C$ (see 5.3).

The following result is adapted from [RARAM, 2.19, p. 73]. It gives a convenient test for relation functors.

5.7. Proposition. Suppose $F:\mathcal{C}\longrightarrow\mathcal{D}$ is a functor of almost strongly exact relation categories such that F preserves converses and order, that is, $F(f^\#) = F(f)^\#$ and $f \le g$ implies $F(f) \le F(g)$). Also, suppose that $F(f+g) \le F(f)+F(g)$ or $F(f+g) \ge F(f)+F(g)$ if $\mathbf{0}f = \mathbf{0}g$ and $f\mathbf{I} = g\mathbf{I}$, for all $f,g:A\longrightarrow B$ in $\mathcal{C}$. Then F is a relation functor.

Proof: Assume the hypotheses, and suppose $F(A) = C$ and $b = F(\mathbf{0}_{AA})$. By 5.1b,c,d and the hypotheses, $b+b \le b$ or $b+b \ge b$. Using 3.1e,f, $b+b \le b$ implies $b \le b+(-b) \le -b$, hence $b = b+(-b)$ is null. Similarly, b is null if $b+b \ge b$, so $b = b\mathbf{0}b \le \mathbf{0}$ because $b \le 1_C$. This proves $F(\mathbf{0}_{AA}) = \mathbf{0}_{CC}$ in all cases, and $F(\mathbf{I}_{AA}) = \mathbf{I}_{CC}$ dually. It follows that $F(\mathbf{0}_{AB}) = \mathbf{0}_{CD}$, $F(\mathbf{I}_{AB}) = \mathbf{I}_{CD}$ and $F(0_{AB}) = 0_{CD}$ for $D = F(B)$, using 5.4b,c and 5.1c.

Suppose $f,g:A\longrightarrow B$ and $c = 1_B \vee f^\# f$. By 3.13i and 5.4c,d, $\mathbf{0}f = \mathbf{0}_{AB}c$. So, $\mathbf{0}f \vee g = \mathbf{0}_{AB}c \vee g = gc$ by 5.4f, and

$$F(\mathbf{0}f \vee g) = F(g)F(c) = \mathbf{0}_{CD}F(c) \vee F(g) = \mathbf{0}F(f) \vee F(g).$$

Similarly, $F(f\mathbf{I} \wedge g) = F(f)\mathbf{I} \wedge F(g)$. By 5.4g and the above,

$$\mathbf{0}F(f+g) = F(\mathbf{0}f \vee \mathbf{0}g) = \mathbf{0}F(f) \vee \mathbf{0}F(g) = \mathbf{0}(F(f)+F(g)).$$

Similarly, $F(f+g)\mathbf{I} = (F(f)+F(g))\mathbf{I}$. If follows by 5.4h that $F(f+g) = F(f)+F(g)$ if $F(f+g) \le F(f)+F(g)$ or $F(f+g) \ge F(f)+F(g)$. By 5.5a, there exist $d \le 1_A$ and $e \ge 1_B$ such that $dfe\mathbf{I} = d\mathbf{I}_{AB} = dge\mathbf{I}$, $\mathbf{0}dfe = \mathbf{0}_{AB}e = \mathbf{0}dge$ and $f+g = df+dg = dfe+dge$. Using the final hypothesis of 5.7, $F(dfe+dge) = F(dfe)+F(dge)$. Since $e \ge 1$, $F(df+dg) = F(dfe)+F(dge) \ge F(df)+F(dg)$, so $F(df+dg) = F(df)+F(dg)$. Finally, $F(f+g) = F(df)+F(dg) \le F(f)+F(g)$, so $F(f+g) = F(f)+F(g)$.

Now $F(-1_A) = -1_{F(A)}$ using 5.1f and the above, and then $F(-f) = F(f)$ follows.

Clearly, $F(f \wedge g) \le F(f) \wedge F(g)$. Since $\mathbf{0}(f \wedge g) = (f^\#\mathbf{0} + g^\#\mathbf{0})^\#$ by 5.5b and 5.1c, we have $\mathbf{0}F(f \wedge g) = \mathbf{0}(F(f) \wedge F(g))$ by the above. Similarly, $(f \wedge g)\mathbf{I} = (f+(-g))\mathbf{0}\mathbf{I}$ by 5.5e, leading to $F(f \wedge g)\mathbf{I} = (F(f) \wedge F(g))\mathbf{I}$. Then $F(f \wedge g) = F(f) \wedge F(g)$ by 5.4h. Since $F(f \vee g) = F(f) \vee F(g)$ dually, F is a relation functor. ■

We now show that almost strongly exact relation categories can be characterized as full subcategories of strongly exact relation categories. Given an almost strongly exact relation category $\mathcal{C}$, we construct a strongly exact relation category $\tilde{\mathcal{C}}$ extending $\mathcal{C}$, where $\tilde{\mathcal{C}}$ is minimal up to equivalence of relation categories. Similar constructions are in [GH]; the first journal publication of such a construction is by R. Vescan [Ve]. Essentially, we generalize the use of symmetric idempotents and the sets rel(c,d) of 3.16.

5.8. Definitions. Suppose $\mathcal{C}$ is an almost strongly exact relation category. Define $\tilde{\mathcal{C}}$ as follows:

The objects of $\tilde{\mathcal{C}}$ are the symmetric idempotents $c:A\longrightarrow A$ of $\mathcal{C}$.

The morphisms $\tilde{\mathcal{C}}(c,d)$ for symmetric idempotents $c:A\longrightarrow A$ and $d:B\longrightarrow B$ of $\mathcal{C}$ are triples $\langle c,f,d\rangle$ such that $f:A\longrightarrow B$ and $cf = f = fd$ in $\mathcal{C}$.

The category structures and converses are given by:

$\langle c,f,d\rangle\langle d,g,e\rangle = \langle c,fg,e\rangle$ from $\tilde{\mathcal{C}}(c,d)\wedge\tilde{\mathcal{C}}(d,e)$ to $\tilde{\mathcal{C}}(c,e)$,

$\langle c,f,d\rangle^{\#} = \langle d,f^{\#},c\rangle$ from $\tilde{\mathcal{C}}(c,d)$ to $\tilde{\mathcal{C}}(d,c)$, and

$1_c = \langle c,c,c\rangle$ in $\tilde{\mathcal{C}}(c,c)$ for each object c.

The $(\mathbf{0},\mathbf{I})$ lattice operations are defined in $\tilde{\mathcal{C}}(c,d)$ by:

$\langle c,f,d\rangle \vee \langle c,g,d\rangle = \langle c,f\vee g,d\rangle$, $\langle c,f,d\rangle \wedge \langle c,g,d\rangle = \langle c,f\wedge g,d\rangle$,

$\mathbf{0}_{cd} = \langle c,c\mathbf{0}_{AB}d,d\rangle$ and $\mathbf{I}_{cd} = \langle c,c\mathbf{I}_{AB}d,d\rangle$.

Relational sum structures are defined in $\tilde{\mathcal{C}}(c,d)$ by:

$\langle c,f,d\rangle + \langle c,g,d\rangle = \langle c,f+g,d\rangle$, $-\langle c,f,d\rangle = \langle c,-f,d\rangle$ and

$0_{cd} = \langle c,c0_{AB}d,d\rangle$.

Define $H:\mathcal{C}\longrightarrow\tilde{\mathcal{C}}$ by $H(A) = 1_A$ and $H(f) = \langle 1_A,f,1_B\rangle$ for $f:A\longrightarrow B$ in $\mathcal{C}$.

5.9. Proposition. Suppose $\mathcal{C}$ is an almost strongly exact relation category. Then $\tilde{\mathcal{C}}$ is a strongly exact relation category, and H is an embedding relation functor which induces an isomorphism between $\mathcal{C}$ and the full subcategory of $\tilde{\mathcal{C}}$

determined by the class of objects $\{1_A\colon A \text{ in } \mathcal{C}\}$. If $F\colon C \longrightarrow D$ is a embedding relation functor from $\mathcal{C}$ into a strongly exact relation category $\mathcal{D}$, then there exists an embedding relation functor $\tilde{F}\colon \tilde{C} \longrightarrow D$ such that $F = H\tilde{F}$.

Proof: Clearly, the category structures and converses in $\tilde{\mathcal{C}}$ are well-defined. For $\langle c,f,d\rangle$ in $\tilde{\mathcal{C}}(c,d)$, $f = cfd$ implies $c\mathbf{0}_{AB}d \le f \le c\mathbf{I}_{AB}d$. Conversely, $c\mathbf{0}_{AB}d \le f \le c\mathbf{I}_{AB}d$ implies $cf = f = fd$ (because $c\mathbf{I} \ge ff^{\#}$ implies $cff^{\#} \ge ff^{\#}$ by 3.13h, and so on). So, the $(\mathbf{0},\mathbf{I})$ lattice structures for $\tilde{\mathcal{C}}(c,d)$ are well-defined. For $\langle c,f,d\rangle$ and $\langle c,g,d\rangle$ in $\tilde{\mathcal{C}}(c,d)$, sum is well defined because $c\mathbf{0}_{AB}d = c\mathbf{0}_{AB}d + c\mathbf{0}_{AB}d \le f + g$ by 5.4b and 5.1d, etc. Negation is well-defined using 5.1f, and 0_{cd} is well-defined.

Calculations show that 5.1a,b,d,e,g,h are satisfied in $\tilde{\mathcal{C}}$, as are commutativity and associativity of sum from 5.1c. Now $cf = f = fd$ implies $f = f + 0_{AB} \le f + 0_{AB}d \le (f + 0_{AB})d = fd = f$ by 5.1f,h for $\mathcal{C}$, and so $f = f + 0_{AB}d \ge f + c0_{AB}d \ge c(f + 0_{AB}d) = f$, proving 5.1c for $\tilde{\mathcal{C}}$.

Clearly $-\langle c,f,d\rangle = (-1_c)\langle c,f,d\rangle = \langle c,f,d\rangle(-1_d)$ from 5.1f. Also, $0_{cd} = \mathbf{I}\mathbf{0}_{cd} = \mathbf{I}_{cd}\mathbf{0}$ is easily seen ($c0_{AB}d = c\mathbf{I}\mathbf{0}_{AB}d = c\mathbf{I}cc\mathbf{I}\mathbf{0}_{AB}d$, etc.). To complete the verification of 5.1f for $\tilde{\mathcal{C}}$, note that $1_c + (-1_c) = 0_{cc}$ because $c + (-c) = c\mathbf{I}\mathbf{0}c$ by 3.13b.

Now suppose $e = \langle c,d,c\rangle$ is a symmetric idempotent of $\tilde{\mathcal{C}}$, so d is a symmetric idempotent of $\mathcal{C}$. Then $f = \langle d,d,c\rangle$ in $\tilde{\mathcal{C}}(d,c)$ satisfies $ff^{\#} = 1_d$ and $f^{\#}f = e$, proving 5.1i for $\tilde{\mathcal{C}}$. So, $\tilde{\mathcal{C}}$ is a strongly exact relation category.

Verification of the properties of H is routine. Assume $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ satisfies the hypotheses above. For each symmetric idempotent $c\colon A \longrightarrow A$ of $\mathcal{C}$, define $F(c)$ in $\mathcal{D}$ by using 5.1i to select some $h_c\colon F(c) \longrightarrow F(A)$ in $\mathcal{D}$ such that $h_c h_c^{\#} = 1_{F(c)}$ and $h_c^{\#}h_c = F(c)$. If $c = 1_A$, choose $F(c) = F(A)$ and $h_c = 1_{F(A)}$. To define $\tilde{F}$ for a morphism $\langle c,f,d\rangle$ in $\tilde{\mathcal{C}}(c,d)$, use $\tilde{F}\langle c,f,d\rangle = h_c F(f) h_d^{\#}$. Computation using 5.7 shows that $\tilde{F}$ is a well-defined embedding relation functor having the desired properties. ∎

We omit the proof of the following result, which is suggested by the results of 5.2.

5.10. Proposition. Suppose R is a ring and $\mathcal{C}$ is a full subcategory of R-**Rel** with inclusion functor F. Then $\tilde{F}$ can be chosen to be a relation functor equivalence from $\tilde{\mathcal{C}}$ into the full subcategory of R-**Rel** determined by the class of subquotients of objects of $\mathcal{C}$.

We now turn to consideration of proper morphisms and the subcategories associated with them.

5.11. Definition and Properties. A morphism $f:A\longrightarrow B$ in $\mathcal{C}$ is called *proper* if $ff^{\#} \geq 1_A$ and $f^{\#}f \leq 1_B$.

Suppose $f:A\longrightarrow B$ in $\mathcal{C}$, and $y:A\longrightarrow A$ and $z:B\longrightarrow B$ are symmetric null morphisms of $\mathcal{C}$. Then $f^{\#}yf$ is called the *image* of y under f, and $fzf^{\#}$ is called the *preimage* of z under f.

For $f:A\longrightarrow B$ and $g:B\longrightarrow C$ in $\mathcal{C}$, $\langle f,g\rangle$ is called *exact* if $f^{\#}\mathbf{I}f = g\mathbf{O}g^{\#}$ (that is, the image of $\mathbf{I}$ under f equals the preimage of $\mathbf{O}$ under g).

5.11a. For $f:A\longrightarrow B$, f is proper iff $\mathbf{O}f = \mathbf{O}_{AB}$ and $f\mathbf{I} = \mathbf{I}_{AB}$. (Note $ff^{\#} \geq 1_A$ implies $f\mathbf{I} \geq ff^{\#}\mathbf{I}_{AB} \geq \mathbf{I}_{AB}$, and $f\mathbf{I} = \mathbf{I}_{AB}$ implies $ff^{\#}\mathbf{I} = \mathbf{I}$, hence $1_A \leq ff^{\#}$ by 5.1g, etc.)

5.11b. If $f,g:A\longrightarrow B$ are proper and $f \leq g$, then $f = g$ (5.4h).

5.11c. For any A and B, 1_A, -1_A and 0_{AB} are proper. For proper $f,g:A\longrightarrow B$, $-f$ and $f+g$ are proper. If $h:B\longrightarrow C$ is also proper, then gh is proper. If $z:A\longrightarrow B$ is null and proper, then $z = 0_{AB}$. (Use 5.4b,c,e,g and 5.11a.)

5.11d. Images and preimages are symmetric and null. If $f:A\longrightarrow B$ and $g:B\longrightarrow C$, the image of $y:A\longrightarrow A$ under fg is the image of the image since $g^{\#}(f^{\#}yf)g = (fg)^{\#}yfg$, and similarly for preimages of $z:C\longrightarrow C$ under fg.

5.11e. If $\langle f,g\rangle$ is exact, then fg is null ($fg \leq ff^{\#}\mathbf{I}fg = fg\mathbf{O}g^{\#}g \leq fg$).

5.11f. In R-**Rel**, $f:A\longrightarrow B$ is proper iff it is the graph of an R-linear homomorphism. A pair $\langle f,g\rangle$ of proper maps is exact in R-**Rel** iff the corresponding R-linear homomorphisms are exact in R-**Mod**. If $h:A\longrightarrow B$ in R-**Rel** and $y:A\longrightarrow A$ is symmetric and null, so that $y = A_0 \oplus A_0$ for some submodule A_0 of A, then the image $fyf^{\#}$ equals $B_0 \oplus B_0$ for

$$B_0 = \{b \in B\text{: there exists a in } A_0 \text{ such that } \langle a,b \rangle \in f\},$$

and similarly for preimages.

5.12. Definition and Properties. If $\mathcal{C}$ is a strongly exact relation category, let $\mathbf{P}(\mathcal{C})$ denote the system of all objects and all proper morphisms of $\mathcal{C}$. If $F:\mathcal{C} \longrightarrow \mathcal{D}$ is a relation functor for some strongly exact relation category $\mathcal{D}$, let $\mathbf{P}(F)(A)$ denote $F(A)$ for A in $\mathcal{C}$, and $\mathbf{P}(F)(f)$ denote $F(f)$ if $f:A \longrightarrow B$ is proper in $\mathcal{C}$.

5.12a. $\mathbf{P}(\mathcal{C})$ is a subcategory of $\mathcal{C}$.

5.12b. $\mathbf{P}(F)(A)$ and $\mathbf{P}(F)(f)$ determine a functor $\mathbf{P}(F):\mathbf{P}(\mathcal{C}) \longrightarrow \mathbf{P}(\mathcal{D})$ (also see 5.14).

5.12c. For R a ring, R-**Mod** is isomorphic to $\mathbf{P}$(R-**Rel**), by the usual identification of R-linear homomorphisms with their graphs (5.11f).

5.13. Definitions and Properties. A category is *exact* if it has null morphisms, kernels, cokernels, normal monomorphisms and conormal epimorphisms, and every morphism factors as an epimorphism followed by a monomorphism. An *additive structure* on a category $\mathcal{A}$ is given by (additively written) abelian group structures on $\mathcal{A}(A,B)$ for all A and B in $\mathcal{A}$, such that composition distributes over sum on the left and right whenever the appropriate composites are defined. (This has been called a *preadditive* structure by some authors.) An *exact additive category* is an exact category with an additive structure. If $\mathcal{A}$ and $\mathcal{B}$ are exact additive categories, then a functor $F:\mathcal{A} \longrightarrow \mathcal{B}$ is *exact* if it preserves exact sequences and is *additive* if it preserves sums.

Of course, an abelian category has a unique additive structure for which it is an exact additive category, and an exact functor of abelian categories is additive (see C?).

5.14. Proposition. If $\mathcal{C}$ is a strongly exact relation category, then $\mathbf{P}(\mathcal{C})$ is an exact additive category. If $F:\mathcal{C} \longrightarrow \mathcal{D}$ is a relation functor of strongly exact relation categories, then $\mathbf{P}(F):\mathbf{P}(\mathcal{C})\longrightarrow\mathbf{P}(\mathcal{D})$ is an exact and additive functor. If F is an embedding relation functor, then $\mathbf{P}(F)$ is also an embedding functor.

Proof: Assuming $\mathcal{C}$ is nonempty, choose C in $\mathcal{C}$ and $z:X\longrightarrow C$ such that $zz^{\#} = 1_X$ and $z^{\#}z = \mathbf{O}$ by 5.1i. Since $\mathbf{O}$ is null and $1_X = z\mathbf{O}z^{\#}$, 1_X is null. By 5.11c, X is a zero object of $\mathbf{P}(\mathcal{C})$ with respect to morphisms 0_{AX} and 0_{XB}, and $0_{AB} = 0_{AX}0_{XB}$.

Given $f:A\longrightarrow B$ in $\mathbf{P}(\mathcal{C})$, use 5.1i to select $k:K\longrightarrow A$ such that $kk^{\#} = 1_K$ and $k^{\#}k = 1_A \wedge f\mathbf{O}f^{\#}$, and to select $h:C\longrightarrow B$ such that $hh^{\#} = 1_C$ and $h^{\#}h = 1_B \vee f^{\#}\mathbf{I}f$. Then k is a proper monomorphism and $h^{\#}$ is a proper epimorphism. Some calculation shows that k is a kernel of f and $h^{\#}$ is a cokernel of f in $\mathbf{P}(\mathcal{C})$. We can verify that $\langle f,g\rangle$ is exact in $\mathbf{P}(\mathcal{C})$ iff $f^{\#}\mathbf{I}f = g\mathbf{O}g^{\#}$ in $\mathcal{C}$. Routine computation then shows that monomorphisms are normal and epimorphisms are conormal in $\mathbf{P}(\mathcal{C})$. Again, choose $m:D\longrightarrow B$ such that $mm^{\#} = 1_D$ and $m^{\#}m = f^{\#}f$ by 5.1i. Then m is a proper monomorphism and $e = fm^{\#}$ is a proper epimorphism such that $f = em$ in $\mathbf{P}(\mathcal{C})$, proving that $\mathbf{P}(\mathcal{C})$ is an exact category.

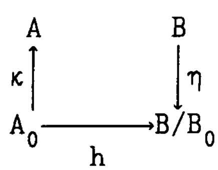
Since the sum of proper morphisms is proper by 5.11c, $\mathbf{P}(\mathcal{C})(A,B)$ is a commutative semigroup with zero 0_{AB} under $+_{AB}$ by 5.1c. Also, composition distributes over addition on the left and right in $\mathbf{P}(\mathcal{C})$ by 5.1h and 5.11b. Finally, $f+(-f)$ is null and proper, so $-f$ is a negative for f in $\mathbf{P}(\mathcal{C})(A,B)$ by 5.11c. Therefore, $\mathbf{P}(\mathcal{C})$ is an additive category.

Suppose $F:\mathcal{C}\longrightarrow\mathcal{D}$ is a relation functor of strongly exact relation categories, and $G = \mathbf{P}(F)$. Clearly G is an additive functor, and G preserves exactness by the characterization $f^{\#}\mathbf{I}f = g\mathbf{O}g^{\#}$ above. Finally, G is an embedding if F is an embedding. ■

Brinkmann and Puppe [BP] showed that a type of relation category $\mathbf{K}(\mathcal{A})$ could be constructed from any exact category $\mathcal{A}$. Their analysis shows that $\mathbf{P}(\mathbf{K}(\mathcal{A}))$ and $\mathcal{A}$ are isomorphic categories, and $\mathbf{K}(\mathbf{P}(\mathcal{C}))$ and $\mathcal{C}$ are isomorphic as relation categories, if $\mathcal{C}$ is such a relation category. Also, they construct a functor $\mathbf{K}(G):\mathbf{K}(\mathcal{A})\longrightarrow\mathbf{K}(\mathcal{B})$ preserving appropriate relation category structures from any exact functor $G:A\longrightarrow B$ of exact categories.

In Chapter IV, the abstract **P** and **K** constructions will be studied for strongly exact relation categories. In this section, we construct only the relation functor $\mathbf{K}(G):R\text{-}\mathbf{Rel}\longrightarrow S\text{-}\mathbf{Rel}$ from an exact functor $G:R\text{-}\mathbf{Mod}\longrightarrow S\text{-}\mathbf{Mod}$. The basis of the **K** construction is the observation that an additive relation $f:A\longrightarrow B$ in R-**Rel** can be uniquely identified with an R-linear homomorphism $h:A_0\longrightarrow B/B_0$ for suitable submodules A_0 of A and B_0 of B.

5.15. Definitions and Properties. For R a ring, identify R-**Mod** with $\mathbf{P}(R\text{-}\mathbf{Rel})$ as in 5.12c. For an additive relation $f:A\longrightarrow B$ in R-**Rel**, there exist A_0 in $\mathrm{Su}(A)$ and B_0 in $\mathrm{Su}(B)$ such that $f\mathbf{I}f^\# = A_0 \oplus A_0$ and $f^\#\mathbf{O}f = B_0 \oplus B_0$ (5.11f). Let $\kappa:A_0\longrightarrow A$ be the inclusion map and $\eta:B\longrightarrow B/B_0$ be the canonical quotient map. Define $h:A_0\longrightarrow B/B_0$ in R-**Rel** by $h = \kappa f\eta$. Some calculation shows that h is proper, with $h(a) = b+B_0$ iff $\langle a,b\rangle \in f$, and $f = \kappa^\# h\eta^\#$. The triple $\langle\kappa,h,\eta\rangle$, corresponding to the diagram



in R-**Mod**, is called the *standard representation* of f in R-**Mod**.

Let $G:R\text{-}\mathbf{Mod}\longrightarrow S\text{-}\mathbf{Mod}$ be an exact functor for rings R and S. For R-modules A, let $\mathbf{K}(G)(A) = G(A)$ in S-**Rel**. For $f:A\longrightarrow B$ in R-**Rel**, define $\mathbf{K}(G)(f):\mathbf{K}(G)(A)\longrightarrow\mathbf{K}(G)(B)$ in S-**Rel** by

$$\mathbf{K}(G)(f) = G(\kappa)^\# G(h)G(\eta)^\#,$$

where $\langle\kappa,h,\eta\rangle$ is the standard representation for f in R-**Mod**.

5.15a. If $f:A\longrightarrow B$ in R-**Rel**, then f is proper iff $\langle 1_A,f,1_B\rangle$ is the standard representation of f. (We identify $\eta:B\longrightarrow B/0$ with 1_B.)

5.15b. Suppose $f:A\longrightarrow B$ in R-**Rel** has standard representation $\langle\kappa,h,\eta\rangle$. If $f = gm^\#$ for g proper and m a monomorphism of R-**Mod**, then $\eta = 1_B$. If $f = e^\# g$ for g proper and e an epimorphism of R-**Mod**, then $\kappa = 1_A$.

To verify that $\mathbf{K}(G)$ is a relation functor, we first give some preparatory material. The material on exact squares below is adapted partly from standard abelian category theory (see [Fr, pp. 37-38]), and partly from analysis of additive relation categories (see [Hi, §3] and [BP, §5]). All these results are easily proved by elementary calculations for R-**Rel**.

5.16. Definition and Properties. Suppose fg = hk for the diagram below in R-**Mod**.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[k]{} & D \end{array}$$

Then $\langle f,g,h,k\rangle$ is called an *exact square* if $f^{\#}h = gk^{\#}$ in R-**Rel**. (Of course, $\langle h,k,f,g\rangle$ is then also an exact square by 5.1e.)

5.16a. If $\langle f,g,h,k\rangle$ is an exact square, then $A_0 = A_1 \vee A_2$ in Su(A) for A_0 = Ker fg = Ker hk, A_1 = Ker f and A_2 = Ker h.

5.16b. If $\langle f,g,h,k\rangle$ is an exact square, then $D_0 = D_1 \wedge D_2$ in Su(D) for D_0 = Im fg = Im hk, D_1 = Im g and D_2 = Im k.

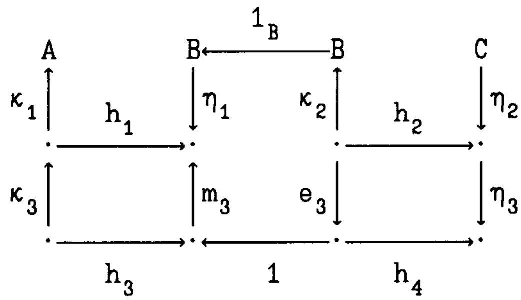
5.16c. If fg = hk for monomorphisms f and k, then $\langle f,g,h,k\rangle$ is an exact square iff f is a kernel for the composite g(coker k).

5.16d. If fg = hk for epimorphisms f and k, then $\langle f,g,h,k\rangle$ is an exact square iff k is a cokernel for (ker f)h.

5.17. Proposition. Suppose G:R-**Mod**⟶S-**Mod** is an exact (additive) functor for rings R and S. Then **K**(G):R-**Rel**⟶S-**Rel** is a relation functor, and **P**(**K**(G)) = G. If G is an exact embedding functor, then **K**(G) is also an embedding functor.

Proof: Assume the hypotheses, and let F denote **K**(G). By 5.15a, F(f) = G(f) if F is proper, and so $F(1_A) = 1_{F(A)}$ for A an R-module.

Suppose f_1:A⟶B and f_2:B⟶C in R-**Rel**. Assume that f_i has standard representation $\langle \kappa_i, h_i, \eta_i\rangle$, for i = 1,2. Factor $\kappa_2\eta_1$ as an epimorphism followed by monomorphism, say $\kappa_1\eta_2 = e_3m_3$. By 5.15b, we can obtain standard representations $\langle \kappa_3, h_3, 1\rangle$ for $\eta_1 m_3^{\#}$ and $\langle 1, h_4, \eta_3\rangle$ for $e_3^{\#}\eta_2$. This leads to the diagram of proper maps below.



Now $\kappa_3{}^{\#}h_3 = h_1m_3{}^{\#}$, and $\kappa_3h_1 \geq \kappa_3h_1m_3{}^{\#}m_3 = \kappa_3\kappa_3{}^{\#}h_3m_3 = h_3m_3$ implies $\kappa_3h_1 = h_3m_3$ by 5.4?, so $\langle\kappa_3, h_1, h_3, m_3\rangle$ is an exact square. Similarly, $\langle e_3, h_4, h_2, \eta_3\rangle$ is an exact square. We can suppose that $\eta_2\eta_3$ is a canonical map $C \longrightarrow C/C_0$ by using a Noether isomorphism, so that $\langle\kappa_3\kappa_1,\ h_3h_4,\ \eta_2\eta_3\rangle$ is the standard representation for f_1f_2. Since G is exact, $G(\eta_1)^{\#}G(\kappa_2)^{\#} = G(m_3)^{\#}G(e_3)^{\#}$ by 5.1e, $G(h_1)G(m_3)^{\#} = G(\kappa_3)^{\#}G(h_3)$ by 5.16c and $G(e_3)^{\#}G(h_2) = G(h_4)G(\eta_3)^{\#}$ by 5.16d, so:

$$F(f_1f_2) = G(\kappa_1)^{\#}G(\kappa_3)^{\#}G(h_3)G(h_4)G(\eta_3)^{\#}G(\eta_2)^{\#}$$
$$= G(\kappa_1)^{\#}G(h_1)G(\eta_1)^{\#}G(\kappa_2)^{\#}G(h_2)G(\eta_2)^{\#} = F(f_1)F(f_2),$$

by the diagram above. Therefore, F is a functor and **P**(F) = G.

Suppose $f\colon A \longrightarrow B$ in R-**Rel** has standard representation $\langle\kappa, g, \eta\rangle$, and let $g = em$ for an epimorphism e and a monomorphism m in R-**Mod**. Applying 5.15b, there are standard representations $\langle 1, h, \eta_0\rangle$ for $e^{\#}k$ and $\langle\kappa_0, k, 1\rangle$ for $hm^{\#}$. Since $\kappa_0{}^{\#}kh\eta_0{}^{\#} = \eta m^{\#}e^{\#}\kappa = (\kappa^{\#}em\eta^{\#})^{\#} = f^{\#}$, $\langle\kappa_0, kh, \eta_0\rangle$ is the standard representation for $f^{\#}$. By 5.1e and 5.16c,d again, we have:

$$F(f^{\#}) = G(\kappa_0)^{\#}G(hk)G(\eta_0)^{\#} = G(\eta)G(m)^{\#}G(e)^{\#}G(\kappa) =$$
$$= (G(\kappa)^{\#}G(em)G(\eta)^{\#})^{\#} = F(f)^{\#},$$

proving that F preserves converses.

Suppose that $\mathbf{O}f = \mathbf{O}g$ and $f\mathbf{I} = g\mathbf{I}$ for $f, g\colon A \longrightarrow B$ in R-**Rel**. Then there are κ, η, k and h such that $\langle\kappa, h, \eta\rangle$ and $\langle\kappa, k, \eta\rangle$ are the standard representations for f and g, respectively, and $\langle\kappa,\ h+k,\ \eta\rangle$ is the standard representation for $f+g$. Now $G(\kappa)^{\#}(G(h)+G(k)) = G(\kappa)^{\#}G(h)+G(\kappa)^{\#}G(k)$ because $G(\kappa)$ is a monomorphism, so

$$F(f+g) = G(\kappa)^{\#}(G(h)+G(k))G(\eta)^{\#} = (G(\kappa)^{\#}G(h)+G(\kappa)^{\#}G(k))G(\eta)^{\#}$$
$$\geq G(\kappa)^{\#}G(h)G(\eta)^{\#}+G(\kappa)^{\#}G(k)G(\eta)^{\#} = F(f)+F(g).$$

Suppose $f_1 \leq f_2$ for $f_1, f_2 : A \longrightarrow B$ in R-**Rel**. Let $\langle \kappa_i, h_i, \eta_i \rangle$ be the standard representation for f_i, with $h_i : A_i \longrightarrow B/B_i$, for i = 1,2. Clearly $A_1 \leq A_2$ and $B_1 \leq B_2$, so there exist $\kappa_{12} : A_1 \longrightarrow A_2$ and $\eta_{12} : B/B_1 \longrightarrow B/B_2$ such that $\kappa_1 = \kappa_{12}\kappa_2$, $\kappa_{12}h_2 = h_1\eta_{12}$ and $\eta_1\eta_{12} = \eta_2$. Since $G(\kappa_{12})$ is a monomorphism and $G(\kappa_1) = G(\kappa_{12})G(\kappa_2)$, we have $G(\kappa_1)^{\#}G(\kappa_{12}) \leq G(\kappa_2)^{\#}$, and similarly $G(\eta_1)^{\#} \leq G(\eta_{12})G(\eta_2)^{\#}$. Then

$$F(f_1) \leq G(\kappa_1)^{\#}G(h_1)G(\eta_{12})G(\eta_2)^{\#} = G(\kappa_1)^{\#}G(\kappa_{12})G(h_2)G(\eta_2)^{\#} \leq F(f_2).$$

Therefore, F preserves order. Then F is a relation functor by 5.7.

Suppose G is an embedding functor and $F(g_1) = F(g_2)$ for $g_1, g_2 : A \longrightarrow B$ in R-**Rel**. Let $f_1 = g_1 \wedge g_2$ and $f_2 = g_1 \vee g_2$, so $F(f_1) = F(f_2)$ also. Using the construction of the above paragraph, we see that $G(\kappa_{12})$ and $G(\eta_{12})$ must be isomorphisms. Since G is an exact embedding, κ_{12} and η_{12} must be isomorphisms, and so $f_1 = f_2$, hence $g_1 = g_2$. Therefore, F is an embedding functor. ∎

The results above lead to the next corollary, which is another step in our unification of external theories for modules.

5.18. Corollary. For rings R and S, the following are equivalent:

5.18a. There exists an embedding relation functor F:R-**Rel**⟶S-**Rel**.

5.18b. There exists an exact embedding functor G:R-**Mod**⟶S-**Mod**.

Proof: Use 5.14 with G = **P**(F) and 5.17 with F = **K**(G). ∎

By 5.6b, we see that 5.18a implies $\mathcal{B}(R) \subseteq \mathcal{B}(S)$. We will prove in §7 that $\mathcal{B}(R) \subseteq \mathcal{B}(S)$ implies 5.18b. So, these three conditions are equivalent to each other, to $\mathcal{L}(R) \subseteq \mathcal{L}(S)$, and to $\mathcal{A}(R) \subseteq \mathcal{A}(S)$, by 4.11.