§6. Category Logic and Horn Sentence Translations.

We now define two first-order logical languages for categories. By taking conjunctions of the atomic formulas of these languages, we can implicitly form category diagrams with variables representing objects and morphisms. Using the first language, commutativity and exactness relations can be imposed in exact additive categories, and **Z**-linear combinations of morphisms in each Hom set can be constructed indirectly. In the second language, the atomic predicates express diagram properties for the structures of almost strongly exact relation categories.

By defining our logical languages for categories, we determine the set $\mathcal{U}_{AC}$ of all basic universal Horn sentences for exact additive categories and the set $\mathcal{U}_{RC}$ of all basic universal Horn sentences for almost strongly exact relation categories. Given a ring R, let $\mathcal{U}_{AC}(R)$ denote the set of just those sentences in $\mathcal{U}_{AC}$ which are satisfied for R-**Mod**. Similarly, let $\mathcal{U}_{RC}(R)$ denote the subset of $\mathcal{U}_{RC}$ of sentences which are satisfied for R-**Rel**. Essentially, these formalizations allow certain diagram-chasing properties of R-**Mod** and R-**Rel** to be represented by formulas. Since sentences are finite, we are restricted to diagrams with finitely many elementary hypotheses.

In §4, universal Horn sentences for the algebraic types τ_L (lattices) τ_A (additive relation algebras) and τ_B (additive relation algebras with unit) were already discussed. Let $\mathcal{U}_L$, $\mathcal{U}_A$ and $\mathcal{U}_B$ denote the sets of all basic universal Horn sentences for each of these algebraic types. Here, the sentences satisfied for R-modules are determined by the algebraic quasivarieties. That is, $\mathcal{U}_L(R)$ denotes the subset of $\mathcal{U}_L$ consisting of exactly those basic universal Horn sentences which are satisfied in every member of $\mathcal{L}(R)$, and similarly for $\mathcal{U}_A(R)$ and $\mathcal{A}(R)$, and for $\mathcal{U}_B(R)$ and $\mathcal{B}(R)$.

Our objective in this section is to construct a closed loop of recursive Horn sentence translation functions

$$\mathfrak{U}_A \xrightarrow{T_0} \mathfrak{U}_B \xrightarrow{T_1} \mathfrak{U}_L \xrightarrow{T_2} \mathfrak{U}_{AC} \xrightarrow{T_3} \mathfrak{U}_{RC} \xrightarrow{T_4} \mathfrak{U}_A,$$

which are to be defined independently of the choice of any ring. For each nontrivial ring R, however, the set of basic universal Horn sentences satisfied for R will be preserved. That is, $\Gamma \in \mathfrak{U}_Y(R)$ iff $T_i(\Gamma) \in \mathfrak{U}_Z(R)$ for $i = 0,1,2,3,4$, substituting the appropriate labels for Y and Z in each case.

To explicitly construct recursive translation functions, we must precisely define our logical languages for categories. These languages are cumbersome, and not very useful for working with category diagrams directly. By examining the formalizations, however, we can see which diagram-chasing problems correspond to basic universal Horn sentences.

We begin with the logical language for exact additive categories.

6.1. Definitions. Let τ_{AC} denote the structure type having a unary predicate Object_{AC}, binary predicates equality (=) and Unit_{AC}, ternary predicates Morphism_{AC} and Zero_{AC}, a predicate Negative_{AC} of arity four, predicates Sum_{AC} and Exact_{AC} of arity five, and a predicate Composition_{AC} of arity six. Let $L(\tau_{AC})$ denote the language of a first-order predicate calculus with equality with atomic predicates of type τ_{AC}. Note that $L(\tau_{AC})$ is a purely relational first-order language, without any constants or functions. The only terms of $L(\tau_{AC})$ are variables. Fix a denumerably infinite set of variables $X = \{x_1, x_2, x_3, \ldots\}$ for $L(\tau_{AC})$. The only atomic formulas for $L(\tau_{AC})$ are constructed by providing an argument list of variables in X for one of the nine atomic predicates given above.

If $\mathfrak{A}$ is an exact additive category, then each variable of X may represent either an object or a morphism of $\mathfrak{A}$. (To avoid complications, we will assume that no element is both an object and a morphism of $\mathfrak{A}$. This is easily arranged for module categories.) A function

$$d: X \longrightarrow \text{Objects}(\mathfrak{A}) \cup \text{Morphisms}(\mathfrak{A}) \quad \text{(disjoint union)}$$

is interpreted as true for an atomic formula according to the specifications below. Let A, B, C, f, g, h, x and y denote arbitrary elements of X.

$x = y$	$d(x) = d(y)$ (any variables)
$\mathrm{Object}_{AC}(A)$	$d(A)$ is an object of $\mathcal{A}$
$\mathrm{Morphism}_{AC}(f,A,B)$	$d(f)$ is a morphism from $d(A)$ into $d(B)$ in $\mathcal{A}$
$\mathrm{Unit}_{AC}(f,A)$	$d(f) = 1_{d(A)}$ in $\mathcal{A}$
$\mathrm{Composition}_{AC}(f,g,h,A,B,C)$	$d(f) = d(g)d(h)$ for $d(g):d(A)\longrightarrow d(B)$ and $d(h):d(B)\longrightarrow d(C)$ in $\mathcal{A}$
$\mathrm{Sum}_{AC}(f,g,h,A,B)$	$d(f) = d(g)+d(h)$ for $d(g)$ and $d(h)$ from $d(A)$ into $d(B)$ in $\mathcal{A}$
$\mathrm{Negative}_{AC}(f,g,A,B)$	$d(f) = -d(g)$ for $d(g):d(A)\longrightarrow d(B)$ in $\mathcal{A}$
$\mathrm{Zero}_{AC}(f,A,B)$	$d(f)$ is the zero morphism from $d(A)$ into $d(B)$ in $\mathcal{A}$
$\mathrm{Exact}_{AC}(f,g,A,B,C)$	$\langle d(f),d(g)\rangle$ is an exact pair of morphisms for $d(f):d(A)\longrightarrow d(B)$ and $d(g):d(B)\longrightarrow d(C)$ in $\mathcal{A}$

Any argument labelled A, B or C above is intended to represent an object, and is called an *object* argument of the atomic predicate in question. Also, arguments labelled f, g or h above are intended to represent morphisms, and are called *morphism* arguments. Note that atomic predicates having one or more morphism arguments also have object arguments corresponding to the domain and codomain of each morphism argument. Specifically, each morphism argument x_i for an atomic formula has a corresponding *morphism triple* $\langle x_i, x_j, x_k\rangle$, where x_j and x_k represent the domain and codomain objects for x_i. For example, $\mathrm{Negative}_{AC}(x_3,x_7,x_1,x_4)$ has the object arguments x_1 and x_4, morphism argument x_3 with morphism triple $\langle x_3,x_1,x_4\rangle$, and morphism argument x_7 with morphism triple $\langle x_7,x_1,x_4\rangle$. If an interpretation d is true for this formula, then $d(x_3)$ and $d(x_7)$ must be morphisms with domain $d(x_1)$ and codomain $d(x_4)$.

If Γ is a sentence (closed formula) of the language $L(\tau_{AC})$, we write $\mathcal{A} \models \Gamma$ if Γ is satisfied in $\mathcal{A}$ according to the above interpretations. It is clear that a first-order theory for exact additive categories could be axiomatized using sentences of $L(\tau_{AC})$.

We now introduce our second logical language, appropriate for almost strongly exact relation categories.

6.2. Definitions. Let τ_{RC} denote the structure type having a unary predicate Object_{RC}, binary predicates equality (=) and Unit_{RC}, ternary predicates Morphism_{RC}, Zero_{RC}, Smallest_{RC}, Largest_{RC}, Leftproper_{RC} and Rightproper_{RC}, predicates Negative_{RC}, Converse_{RC} and Inclusion_{RC} of arity four, predicates Sum_{RC}, Meet_{RC}, Join_{RC} and Exact_{RC} of arity five, and a predicate Composition_{RC} of arity six. Let $L(\tau_{RC})$ denote the language of a first-order predicate calculus with equality with atomic predicates of type τ_{RC}.

Note that each atomic predicate in $L(\tau_{AC})$ has a corresponding atomic predicate in $L(\tau_{RC})$ (Object_{AC} corresponding to Object_{RC}, etc.). Like $L(\tau_{AC})$, $L(\tau_{RC})$ is a purely relational structure, with variables of X the only terms. Each atomic formula for $L(\tau_{RC})$ is obtained by providing variables of X as arguments for one of the seventeen atomic predicates.

If $\mathcal{C}$ is an almost strongly exact relation category, then each variable may represent either an object or a morphism of $\mathcal{C}$, but not both. Using the set of variables X, a function

$$d: X \longrightarrow \text{Objects}(\mathcal{C}) \cup \text{Morphisms}(\mathcal{C}) \quad \text{(disjoint union)}$$

is a true interpretation for each atomic formula of $L(\tau_{RC})$ according to the specifications below. Again, A, B, C, f, g, h, x and y are in X.

$x = y$	$d(x) = d(y)$ (any variables)
$\text{Object}_{RC}(A)$	$d(A)$ is an object of $\mathcal{C}$
$\text{Morphism}_{RC}(f,A,B)$	$d(f)$ is a morphism from $d(A)$ into $d(B)$ in $\mathcal{C}$
$\text{Unit}_{RC}(f,A)$	$d(f) = 1_{d(A)}$ in $\mathcal{C}$
$\text{Composition}_{RC}(f,g,h,A,B,C)$	$d(f) = d(g)d(h)$ for $d(g):d(A) \longrightarrow d(B)$

	and $d(h):d(B)\longrightarrow d(C)$ in $\mathcal{C}$
$\text{Converse}_{RC}(f,g,A,B)$	$d(f) = d(g)^{\#}$ for $d(g):d(A)\longrightarrow d(B)$ in $\mathcal{C}$
$\text{Inclusion}_{RC}(f,g,A,B)$	$d(f) \le d(g)$ for $d(f)$ and $d(g)$ from $d(A)$ to $d(B)$ in $\mathcal{C}$
$\text{Meet}_{RC}(f,g,h,A,B)$	$d(f) = d(g) \wedge d(h)$ for $d(g)$ and $d(h)$ from $d(A)$ into $d(B)$ in $\mathcal{C}$
$\text{Join}_{RC}(f,g,h,A,B)$	$d(f) = d(g) \vee d(h)$ for $d(g)$ and $d(h)$ from $d(A)$ into $d(B)$ in $\mathcal{C}$
$\text{Smallest}_{RC}(f,A,B)$	$d(f) = \mathbf{O}_{DE}$ for $D = d(A)$ and $E = d(B)$ in $\mathcal{C}$
$\text{Largest}_{RC}(f,A,B)$	$d(f) = \mathbf{I}_{DE}$ for $D = d(A)$ and $E = d(B)$ in $\mathcal{C}$
$\text{Sum}_{RC}(f,g,h,A,B)$	$d(f) = d(g) + d(h)$ (relational sum) for $d(g),d(h):d(A)\longrightarrow d(B)$ in $\mathcal{C}$
$\text{Negative}_{RC}(f,g,A,B)$	$d(f) = -d(g)$ (relational negative) for $d(g):d(A)\longrightarrow d(B)$ in $\mathcal{C}$
$\text{Zero}_{RC}(f,A,B)$	$d(f) = 0_{DE}$ for $D = d(A)$ and $E = d(B)$ in $\mathcal{C}$
$\text{Exact}_{AC}(f,g,A,B,C)$	$d(f):d(A)\longrightarrow d(B)$ and $d(g):d(B)\longrightarrow d(C)$ satisfy $d(f)^{\#}\mathbf{I}d(f) = d(g)\mathbf{O}d(g)^{\#}$ in $\mathcal{C}$
$\text{Leftproper}_{RC}(f,A,B)$	$d(f):d(A)\longrightarrow d(B)$ and $d(f)d(f)^{\#} \ge 1_{d(A)}$ in $\mathcal{C}$
$\text{Rightproper}_{RC}(f,A,B)$	$d(f):d(A)\longrightarrow d(B)$ and $d(f)^{\#}d(f) \le 1_{d(B)}$ in $\mathcal{C}$

Note that $d(f):d(A)\longrightarrow d(B)$ is a proper map of $\mathcal{C}$ iff $\text{Leftproper}_{RC}(f,A,B)$ and $\text{Rightproper}_{RC}(f,A,B)$ are satisfied for d.

As before, arguments labelled A, B or C above are called *object* arguments, and arguments labelled f, g or h are called *morphism* arguments. Again, each atomic predicate contains domain and codomain object arguments for each of its morphism arguments, so that each morphism argument has a *morphism triple* as in 6.1.

If Γ is a sentence for $L(\tau_{RC})$, then $\mathcal{C} \models \Gamma$ if Γ is satisfied in $\mathcal{C}$ for all the interpretations above. Again, a first-order theory for almost strongly exact relation categories could be formalized using sentences in the language $\mathbf{L}(\tau_{RC})$.

6.3. Definitions. For τ a structure type with associated first-order logical language $\mathbf{L}(\tau)$, we consider *basic universal Horn sentences* Γ having the special form

$$(\forall x_1, x_2, \ldots, x_n)(W),$$

where W is called the *open formula* for Γ. Here, W must either be an atomic formula P_0, or an implication $Q \Rightarrow P_0$ for Q an (open) *conjunctive* formula $P_1 \wedge P_2 \wedge \ldots \wedge P_m$ $(m \geq 1)$, where $P_i = P_i(x_1, x_2, \ldots, x_n)$ is an atomic formula of $\mathbf{L}(\tau)$ for $i = 0, 1, \ldots, m$. (By convention, take $m = 0$ when W equals P_0.)

As before, let each of $\mathcal{U}_{AC}$, $\mathcal{U}_{RC}$, $\mathcal{U}_L$, $\mathcal{U}_A$ and $\mathcal{U}_B$ denote the set of basic universal Horn sentences for $\mathbf{L}(\tau)$, where τ is τ_{AC}, τ_{RC}, τ_L, τ_A or τ_B, respectively.

Given a ring R with unit, we can restate previous definitions as follows:

$$\mathcal{U}_{AC}(R) = \{\Gamma \in \mathcal{U}_{AC} : R\text{-}\mathbf{Mod} \models \Gamma\},$$

$$\mathcal{U}_{RC}(R) = \{\Gamma \in \mathcal{U}_{RC} : R\text{-}\mathbf{Rel} \models \Gamma\},$$

$$\mathcal{U}_L(R) = \{\Gamma \in \mathcal{U}_L : \mathcal{L}(R) \models \Gamma\},$$

$$\mathcal{U}_A(R) = \{\Gamma \in \mathcal{U}_A : \mathcal{A}(R) \models \Gamma\},$$

$$\mathcal{U}_B(R) = \{\Gamma \in \mathcal{U}_B : \mathcal{B}(R) \models \Gamma\}.$$

That is, each set above contains the basic universal Horn sentences which are satisfied for the corresponding external theory of R-modules.

We now state our translation theorem for basic universal Horn sentences.

6.4. Theorem. There exist recursive functions $\mathbf{T}_0 : \mathcal{U}_A \longrightarrow \mathcal{U}_B$, $\mathbf{T}_1 : \mathcal{U}_B \longrightarrow \mathcal{U}_L$, $\mathbf{T}_2 : \mathcal{U}_L \longrightarrow \mathcal{U}_{AC}$, $\mathbf{T}_3 : \mathcal{U}_{AC} \longrightarrow \mathcal{U}_{RC}$ and $\mathbf{T}_4 : \mathcal{U}_{RC} \longrightarrow \mathcal{U}_A$ such that 6.4a,b,c,d,e are satisfied for all nontrivial rings R with unit.

6.4a. $\Gamma \in \mathcal{U}_A(R)$ iff $\mathbf{T}_0(\Gamma) \in \mathcal{U}_B(R)$ for all Γ in $\mathcal{U}_A$.

6.4b. $\Gamma \in \mathcal{U}_B(R)$ iff $\mathbf{T}_1(\Gamma) \in \mathcal{U}_L(R)$ for all Γ in $\mathcal{U}_B$.

6.4c. $\Gamma \in \mathfrak{U}_L(R)$ iff $\mathbf{T}_2(\Gamma) \in \mathfrak{U}_{AC}(R)$ for all Γ in $\mathfrak{U}_L$.

6.4d. $\Gamma \in \mathfrak{U}_{AC}(R)$ iff $\mathbf{T}_3(\Gamma) \in \mathfrak{U}_{RC}(R)$ for all Γ in $\mathfrak{U}_{AC}$.

6.4e. $\Gamma \in \mathfrak{U}_{RC}(R)$ iff $\mathbf{T}_4(\Gamma) \in \mathfrak{U}_A(R)$ for all Γ in $\mathfrak{U}_{RC}$.

We have already discussed the translation functions $\mathbf{T}_0$ (inclusion; see 4.10) and $\mathbf{T}_1$ (see 4.8). The translation $\mathbf{T}_2$ is based on standard methods for representing subobjects by monomorphisms and lattice operations by exactness conditions in an abelian category (see [GH,§3]). For $\mathbf{T}_3$, we use the identification of R-**Mod** with **P**(R-**Rel**) as in 5.12c. Finally, $\mathbf{T}_4$ is obtained by adapting 3.16 and the construction of $\tilde{\mathcal{C}}$ in 5.8 and 5.9. We describe $\mathbf{T}_2$, $\mathbf{T}_3$ and $\mathbf{T}_4$ in the following; additional analysis of the translation functions is given in Appendix E.

To construct $\mathbf{T}_2:\mathfrak{U}_L \longrightarrow \mathfrak{U}_{AC}$, consider short exact sequences in R-**Mod** as below:

$$A_2 \xrightarrow{f_{2,h}} A_h \xrightarrow{f_{h,1}} A_1 \xrightarrow{f_{1,h+1}} A_{h+1} \xrightarrow{f_{h+1,3}} A_3,$$

where h is an even number, $h \geq 4$, and A_2 and A_3 represent zero objects. If A_1 represents an R-module M, then $f_{h,1}$ represents a monomorphism whose image is an element of Su(M), and $f_{1,h+1}$ represents an epimorphism whose kernel is the same element of Su(M). In order to define $\mathbf{T}_2(\Gamma)$ for Γ in $\mathfrak{U}_L$, we must make lattice polynomials occurring in Γ correspond to short exact sequences as above. These interconnected short exact sequences form a commutative diagram in R-**Mod**, in the strong sense that there is a most one diagram morphism $f_{i,j}:A_i \longrightarrow A_j$ for each pair $\langle A_i, A_j\rangle$ of diagram objects, with $f_{i,i}$ always the unit for A_i. Suppose Γ is $(\forall x_1, x_2, \ldots, x_n)(W)$, W an open formula. For some $m \geq 0$, W equals

$$(p_1 = q_1 \wedge p_2 = q_2 \wedge \cdots \wedge p_m = q_m) \Rightarrow p_0 = q_0,$$

for lattice polynomials $p_i = p_i(x_1, x_2, \ldots, x_n)$ and $q_i = q_i(x_1, x_2, \ldots, x_n)$, $0 \leq i \leq m$. (If $m = 0$, then W is $p_0 = q_0$.) We begin the definition of $\mathbf{T}_2(\Gamma)$ by recursively constructing from Γ a list $r_1, r_2, \ldots, r_s$ $(s \geq n)$ of lattice polynomials, $r_i = r_i(x_1, x_2, \ldots, x_n)$, such that

$$\{p_0, p_1, \ldots, p_m\} \cup \{q_0, q_1, \ldots, q_m\} \subseteq \{r_1, r_2, \ldots, r_s\},$$

r_i is the variable x_i for $i \le n$, and for each i, $n < i \le s$, there exist j and k, $1 \le j,k < i$, such that r_i is $r_j \wedge r_k$ or is $r_j \vee r_k$. We intend to make each lattice polynomial r_i correspond to a short exact sequence with $h = 2i+2$ in the format above (monomorphism $f_{2i+2,1}$ and epimorphism $f_{1,2i+3}$). Suppose symbols A_i and $f_{j,k}$ $(i,j,k \ge 1)$ represent systematically chosen distinct variables in $X = \{x_1, x_2, \ldots\}$. The open formula W_2 for $T_2(\Gamma)$ has the form:

$$U_1 \wedge U_2 \wedge \ldots \wedge U_m \wedge V_1 \wedge V_2 \wedge \ldots \wedge V_t \Rightarrow U_0,$$

with terms $U_1, U_2, \ldots, U_m$ omitted if $m = 0$. For R-modules in our context, the equation $r_j = r_k$, $1 \le j,k \le s$, is equivalent to the exactness of

$$A_{2j+2} \xrightarrow{f_{2j+2,1}} A_1 \xrightarrow{f_{1,2k+3}} A_{2k+3}.$$

For $0 \le i \le m$, therefore, let U_i denote the atomic formula:

$$\mathrm{Exact}_{AC}(f_{2j+2,1},\ f_{1,2k+3},\ A_{2j+2},\ A_1,\ A_{2k+3}),$$

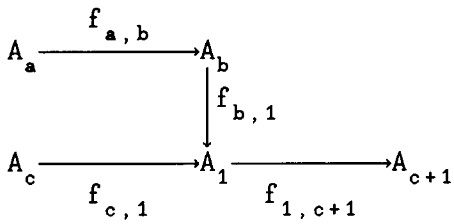
where j is the smallest integer such that $r_j = p_i$ and k is the smallest integer such that $r_k = q_i$. (Note that U_i corresponds to $p_i = q_i$ by the discussion above.) The hypotheses $V_1, V_2, \ldots, V_t$ describe the commutative diagram of interconnected short exact sequences. First, let V_1, V_2, V_3 and V_4 be

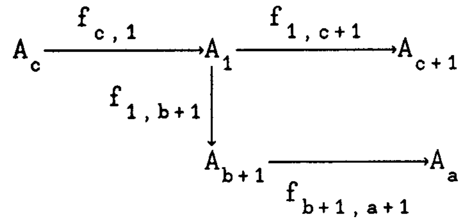
$$\mathrm{Unit}_{AC}(f_{2,2}, A_2),\ \mathrm{Zero}_{AC}(f_{2,2}, A_2, A_2),\ \mathrm{Unit}_{AC}(f_{3,3}, A_3) \text{ and } \mathrm{Zero}_{AC}(f_{3,3}, A_3, A_3),$$

respectively. These four conditions force A_2 and A_3 to be zero objects. Next, for $i = 1,2,\ldots,s$, we require the 3s short exact sequence conditions:

$$\mathrm{Exact}_{AC}(f_{2,2i+2},\ f_{2i+2,1},\ A_2,\ A_{2i+2},\ A_1),$$
$$\mathrm{Exact}_{AC}(f_{2i+2,1},\ f_{1,2i+3},\ A_{2i+2},\ A_1,\ A_{2i+3}),$$
$$\mathrm{Exact}_{AC}(f_{1,2i+3},\ f_{2i+3,3},\ A_1,\ A_{2i+3},\ A_3).$$

Finally, we must add morphism variables and conditions to the diagram corresponding to the sequence of lattice polynomials $r_1, r_2, \ldots, r_s$. For $1 \le i \le n$, r_i is just a variable x_i, and nothing further is needed. Suppose $n < i \le s$, so r_i is $r_j \wedge r_k$ or $r_j \vee r_k$ for some j and k, $1 \le j,k < i$. Then we use the diagrams below, with $a = 2i+2$, $b = 2j+2$ and $c = 2k+2$.





On the left, assume that r_i, r_j and r_k correspond to the images of $f_{a,1}$, $f_{b,1}$ and $f_{c,1}$, respectively, and introduce $f_{a,b}$ which is required to be a kernel of $f_{b,1}f_{1,c+1}$ and to satisfy $f_{a,1} = f_{a,b}f_{b,1}$. For R-modules, we see that r_i will then correspond to $r_j \wedge r_k$. This can be expressed in the logical language $\mathbf{L}(\tau_{AC})$ by the three conditions below:

$\text{Composition}_{AC}(f_{b,c+1},\ f_{b,1},\ f_{1,c+1},\ A_b,\ A_1,\ A_{c+1})$,
$\text{Composition}_{AC}(f_{a,1},\ f_{a,b},\ f_{b,1},\ A_a,\ A_b,\ A_1)$,
$\text{Exact}_{AC}(f_{a,b},\ f_{b,c+1},\ A_a,\ A_b,\ A_{c+1})$.

Dually on the right, $r_i = r_j \vee r_k$ if r_i, r_j and r_k correspond to the kernels of $f_{1,a+1}$, $f_{1,b+1}$ and $f_{1,c+1}$ respectively, $f_{b+1,a+1}$ is a cokernel of $f_{c,1}f_{1,b+1}$, and $f_{1,a+1} = f_{1,b+1}f_{b+1,a+1}$. This yields join conditions:

$\text{Composition}_{AC}(f_{c,b+1},\ f_{c,1},\ f_{1,b+1},\ A_c,\ A_1,\ A_{b+1})$,
$\text{Composition}_{AC}(f_{1,a+1},\ f_{1,b+1},\ f_{b+1,a+1},\ A_1,\ A_{b+1},\ A_{a+1})$,
$\text{Exact}_{AC}(f_{c,b+1},\ f_{b+1,a+1},\ A_c,\ A_{b+1},\ A_{a+1})$.

The complete list $V_1, V_2, \ldots, V_t$ consists of the $4 + 6s - 3n$ atomic formulas given above. So, W_2 has been defined, and $\mathbf{T}_2(\Gamma)$ is $(\forall x_1, x_2, \ldots, x_u)(W_2)$, with u chosen minimal so that $\mathbf{T}_2(\Gamma)$ is closed. This completes our description of $\mathbf{T}_2$.

The translation $\mathbf{T}_3 : \mathfrak{U}_{AC} \longrightarrow \mathfrak{U}_{RC}$ from exact additive category formulas to additive relation category formulas is straightforward. Suppose Λ is $(\forall x_1, x_2, \ldots x_n)(W)$ in $\mathfrak{U}_{AC}$ with open formula W equal to

$$P_1 \wedge P_2 \wedge \ldots \wedge P_m \Rightarrow P_0,$$

for atomic formulas $P_0, P_1, \ldots, P_m$ of $\mathbf{L}(\tau_{AC})$. (If m = 0, W is P_0.) The open formula W_3 for $\mathbf{T}_3(\Lambda)$ will have the form

$$U_1 \wedge U_2 \wedge \ldots \wedge U_m \wedge V_1 \wedge V_2 \wedge \ldots \wedge V_{2t} \Rightarrow U_0.$$

(If $m = t = 0$, then W_3 is U_0.) Recall that each atomic predicate for $L(\tau_{AC})$ has a corresponding atomic predicate for $L(\tau_{RC})$. Define U_i to be the atomic formula corresponding to P_i, with the same argument list, for $i = 0,1,\ldots,m$. (For example, U_0 is $\mathrm{Sum}_{RC}(x_1,x_5,x_2,x_8,x_9)$ if P_0 is $\mathrm{Sum}_{AC}(x_1,x_5,x_2,x_8,x_9)$.) The terms $V_1,V_2,\ldots,V_{2t}$ correspond to the list $f_1,f_2,\ldots,f_t$ of morphism arguments appearing in the list $P_0,P_1,\ldots,P_m$ of atomic formulas. Suppose that $\langle f_k,A_k,B_k\rangle$ is the morphism triple corresponding to f_k, $k \le t$. Then V_{2k-1} is $\mathrm{Leftproper}_{RC}(f_k,A_k,B_k)$ and V_{2k} is $\mathrm{Rightproper}_{RC}(f_k,A_k,B_k)$, for $k = 1,2,\ldots,t$. For example, if P_0 is $\mathrm{Exact}_{AC}(x_7,x_8,x_3,x_4,x_5)$, there are four atomic formulas from the two triples $\langle x_7,x_3,x_4\rangle$ and $\langle x_8,x_4,x_5\rangle$. Treating P_1 to P_m similarly, we obtain $V_1,V_2,\ldots,V_{2t}$ $(t \ge 0)$. This defines the open formula W_3, and $\mathbf{T}_3(\Lambda)$ is then $(\forall x_1,x_2,\ldots,x_n)(W_3)$. Note that $\mathbf{T}_3(\Lambda)$ is a closed sentence, hence is in $\mathfrak{U}_{RC}$, since no new variables were introduced.

This completes our description of $\mathbf{T}_3$. As previously noted, its properties are based on the isomorphism between R-**Mod** and **P**(R-**Rel**).

To define $\mathbf{T}_4:\mathfrak{U}_{RC}\longrightarrow\mathfrak{U}_A$, we first identify certain sentences Δ of $\mathfrak{U}_{RC}$ which are called *superficial* sentences. Each superficial sentence is either true for all R-**Rel** or is false for all R-**Rel**, R nontrivial, and we can determine recursively which case holds. For Δ superficial, define $\mathbf{T}_4(\Delta)$ in $\mathfrak{U}_A$ by:

$$\mathbf{T}_4(\Delta) = \begin{cases} (\forall x_1)(x_1 = x_1) & \text{if } \Delta \text{ is true for all R-}\mathbf{Rel}. \\ (\forall x_1,x_2)(x_1 = x_2) & \text{if } \Delta \text{ is false for all R-}\mathbf{Rel}. \end{cases}$$

Superficial sentences are of six types. Any Δ of form $(\forall x_1,x_2,\ldots,x_n)(P_0)$ for P_0 an atomic formula is called *superficial of type 1*. Note that such a Δ is false for all R-**Rel** except when P_0 is of the form $x_j = x_j$. Hereafter, suppose Δ is of form $(\forall x_1,x_2,\ldots,x_n)(Q \Rightarrow P_0)$, where Q equals $P_1 \wedge P_2 \wedge \ldots \wedge P_m$, for atomic formulas $P_0,P_1,\ldots,P_m$. Then Δ is *superficial of type 2* if there is a variable x_j such that $d(x_j)$ must be both an object and a morphism under any interpretation d satisfying $Q \wedge P_0$. Such a sentence is essentially meaningless for us, because of the requirement that no element of R-**Rel** be both an object and a morphism. Calculation shows that Δ is then false for all R-**Rel** or vacuously true for all R-**Rel**. For example, the following vacuously true sentence is superficial of type 2 because of x_3 and x_8:

$$(\forall x_1, x_2, \ldots, x_9)((\text{Negative}_{RC}(x_1, x_3, x_8, x_9) \wedge x_8 = x_3) \Rightarrow \text{Unit}_{RC}(x_4, x_6)).$$

Excluding types 1 and 2, say that Δ is *superficial of type 3* if P_0 has the form $\text{Object}_{RC}(x_i)$, and is *superficial of type 4* if P_0 has the form $\text{Morphism}_{RC}(x_i, x_j, x_k)$. Some calculation with object arguments and morphism triples of $P_1, P_2, \ldots, P_m$, together with consideration of terms P_j which are equations, shows that $\mathbf{T}_4(\Delta)$ can be recursively computed for superficial sentences of these two types. Excluding types 1 to 4, say that Δ is *superficial of type 5* if there is an object variable x_j for P_0 such that

$$(\forall x_1, x_2, \ldots, x_n)(Q \Rightarrow \text{Object}_{RC}(x_j)),$$

which is superficial of type 3, is false for all R-**Rel**. Finally, excluding types 1 to 5, Δ is *superficial of type 6* if there is a morphism triple $\langle x_i, x_j, x_k \rangle$ for P_0 such that

$$(\forall x_1, x_2, \ldots, x_n)(Q \Rightarrow \text{Morphism}_{RC}(x_i, x_j, x_k)),$$

which is superficial of type 4, is false for all R-**Rel**. Obviously, each superficial sentence of type 5 or 6 is false in all R-**Rel**.

We are finally prepared to define $\mathbf{T}_4(\Delta)$ in the general case, when Δ is not superficial. Based on 3.16 and 5.8ff, we will define below an equation of τ_A-polynomials corresponding to each atomic formula P for $\mathbf{L}(\tau_{RC})$, called the *primary* equation for P. Also, P may have additional corresponding equations of τ_A-polynomials that are called *structural* equations. There is one structural equation $A = AA^{\#}$ corresponding to each object argument A for P. (Category objects of $\tilde{Y}$ correspond to symmetric idempotents of the additive relation algebra Y.) There are two structural equations, $f = Af$ and $f = fB$, corresponding to each morphism triple $\langle f, A, B \rangle$ for P. (The Hom set in $\tilde{Y}$ corresponding to symmetric idempotents A and B in Y consists of triples $\langle A, f, B \rangle$ with $Af = f = fB$.) The primary τ_A-polynomial equations corresponding to each P are shown below, together with the number of associated structural equations. (Recall that $\mathbf{p}(x)$ denotes $(x+(-x))^{\#}(x+(-x))$, $\mathbf{q}(x)$ denotes $(x+(-x))(x+(-x))^{\#}$, and both $x \le y$ and $y \ge x$ denote $x = x \wedge y$.)

$x = y$	$x = y$, no structural equations
$\mathrm{Object}_{RC}(A)$	$A = A$, 1 structural equation
$\mathrm{Morphism}_{RC}(f,A,B)$	$f = f$, 2 structural equations
$\mathrm{Unit}_{RC}(f,A)$	$f = A$, 3 structural equations
$\mathrm{Composition}_{RC}(f,g,h,A,B,C)$	$f = gh$, 9 structural equations
$\mathrm{Converse}_{RC}(f,g,A,B)$	$f = g^{\#}$, 6 structural equations
$\mathrm{Inclusion}_{RC}(f,g,A,B)$	$f \le g$, 6 structural equations
$\mathrm{Meet}_{RC}(f,g,h,A,B)$	$f = g \wedge h$, 8 structural equations
$\mathrm{Join}_{RC}(f,g,h,A,B)$	$f = g \vee h$, 8 structural equations
$\mathrm{Smallest}_{RC}(f,A,B)$	$f = \mathbf{p}(A)\mathbf{p}(B)$, 4 structural equations
$\mathrm{Largest}_{RC}(f,A,B)$	$f = \mathbf{q}(A)\mathbf{q}(B)$, 4 structural equations
$\mathrm{Sum}_{RC}(f,g,h,A,B)$	$f = g + h$, 8 structural equations
$\mathrm{Negative}_{RC}(f,g,A,B)$	$f = -g$, 6 structural equations
$\mathrm{Zero}_{RC}(f,A,B)$	$f = \mathbf{q}(A)\mathbf{p}(B)$, 4 structural equations
$\mathrm{Exact}_{RC}(f,g,A,B,C)$	$f^{\#}\mathbf{q}(A)f = g\mathbf{p}(B)g^{\#}$, 7 structural equations
$\mathrm{Leftproper}_{RC}(f,A,B)$	$ff^{\#} \ge A$, 4 structural equations
$\mathrm{Rightproper}_{RC}(f,A,B)$	$f^{\#}f \le B$, 4 structural equations

For example, $\mathrm{Zero}_{RC}(x_3,x_5,x_6)$ has the primary equation $x_3 = \mathbf{q}(x_5)\mathbf{p}(x_6)$ and the structural equations $x_5 = x_5x_5{}^{\#}$, $x_6 = x_6x_6{}^{\#}$, $x_3 = x_5x_3$ and $x_3 = x_3x_6$. Note that the primary equations correspond to the morphism graph formulas in $\tilde{Y}$ for Y an additive relation algebra.

For Δ not superficial, the open formula W_4 for $\mathbf{T}_4(\Delta)$ has the form

$$U_1 \wedge U_2 \wedge \ldots \wedge U_m \wedge V_1 \wedge V_2 \wedge \ldots \wedge V_t \Rightarrow U_0.$$

Here, U_i is the primary equation corresponding to P_i, for $i = 0,1,\ldots,m$. The list $V_1,V_2,\ldots,V_t$ $(t \ge 0)$ is obtained by aggregating all the structural formulas corresponding to the atomic formulas $P_1, P_2, \ldots, P_m$. Finally, $T_4(\Delta)$ is $(\forall x_1, x_2, \ldots, x_n)(W_4)$, which is a closed sentence, hence is in in $\mathfrak{U}_A$, because no new variables were introduced.

The atomic formula P_0 of Δ for $\tilde{Y}$ corresponds to a conjunction of its primary equation and its structural equations. If Δ is not superficial, the structural equations for P_0 are consequences of Q, because types 5 and 6 are excluded. In that case, P_0 is true iff its primary equation is true. To see why this approach is necessary, suppose Δ is

$$(\forall x_1, x_2, x_3)(\text{Leftproper}_{RC}(x_1, x_2, x_3) \Rightarrow \text{Leftproper}_{RC}(x_1, x_2, x_2)).$$

This sentence is superficial of type 6, hence false for all R-**Rel**. If we computed by the procedure for non-superficial sentences, $\mathbf{T}_4(\Delta)$ would be a trivially true sentence, with both U_0 and U_1 equal to $x_1 x_1^{\#} \geq x_2$.

This completes the description of $\mathbf{T}_4$, closing the loop of recursive translation functions.

§7. Equivalence and Inclusion of Rings for Module Theories.

In the introduction, we asserted that our five external theories for modules determined a unified theory for universal sentences. We have showed that basic universal Horn sentences can be translated between these theories, with truth preserved relative to any ring. We can compare two rings R and S according to the unified theory: we say that R is *smaller* than S if it has *fewer* models, say $\mathfrak{B}(R) \subseteq \mathfrak{B}(S)$, or *more* true universal basic Horn sentences, say $\mathfrak{U}_B(R) \supseteq \mathfrak{U}_B(S)$. (It seems natural to make trivial rings minimal, and the ring **Z** of integers maximal; **Z** corresponds to the most general module theory, that of abelian groups.) In the following, we will show that these ring comparisons can be characterized by the existence of exact embedding functors. This makes many abelian category techniques available for analysis of such comparisons.

7.1. Definitions and Properties. For rings R and S with unit, say that R is *diagram-smaller* than S, written $R \precsim S$, if there exists an exact embedding functor R-**Mod**$\longrightarrow$S-**Mod**. If $R \precsim S$ and $S \precsim R$, say that R is *diagram-equivalent* to S, and write $R \sim S$. The relation $\precsim$ is called *diagram inclusion*, and the relation $\sim$ is called *diagram equivalence*.

7.1a. Diagram inclusion is a symmetric and transitive relation (quasiorder) on the class of all rings. Diagram equivalence is an equivalence relation on the class of all rings.

It is convenient to state the unification results at this point, although part of the proof must be deferred.

7.2. Theorem. Suppose R and S are rings with unit. Then the following are all equivalent:

7.2a. $R \precsim S$, that is, there exists an exact embedding functor R-**Mod**$\longrightarrow$S-**Mod**.

7.2b. There exists an embedding relation functor R-**Rel**$\longrightarrow$S-**Rel**.

7.2c. $\mathcal{L}(R) \subseteq \mathcal{L}(S)$. (Every lattice which is representable by an R-module is representable by an S-module.)

7.2d. $\mathfrak{A}(R) \subseteq \mathfrak{A}(S)$. (Every additive relation algebra which is representable by an R-module is representable by an S-module.)

7.2e. $\mathcal{B}(R) \subseteq \mathcal{B}(S)$. (Every additive relation algebra with unit which is representable by an R-module is representable by an S-module.)

7.2f. $\mathcal{U}_{AC}(R) \supseteq \mathcal{U}_{AC}(S)$. (Every basic universal Horn sentence of type τ_{AC} which is satisfied for S-**Mod** is satisfied for R-**Mod**.)

7.2g. $\mathcal{U}_{RC}(R) \supseteq \mathcal{U}_{RC}(S)$. (Every basic universal Horn sentence of type τ_{RC} which is satisfied for S-**Rel** is satisfied for R-**Rel**.)

7.2h. $\mathcal{U}_{L}(R) \supseteq \mathcal{U}_{L}(S)$. (Every basic universal Horn sentence of type τ_{L} which is satisfied in every Su(M), M an R-module, is satisfied in every Su(N), N an S-module.)

7.2i. $\mathcal{U}_{A}(R) \supseteq \mathcal{U}_{A}(S)$. (Every basic universal Horn sentence of type τ_{A} which is satisfied in every Su(M$\oplus$M), M an R-module, is satisfied in every Su(N$\oplus$N), N an S-module.)

7.2j. $\mathcal{U}_{B}(R) \supseteq \mathcal{U}_{B}(S)$. (Every basic universal Horn sentence of type τ_{B} which is satisfied in every Su(M$\oplus$M), M an R-module, is satisfied in every Su(N$\oplus$N), N an S-module.)

We have already proved many parts of this theorem. We know that 7.2a,b are equivalent by 5.18, and that 7.2c,d,e are equivalent by 4.11. From the loop of translation functions (Theorem 6.4), it follows that 7.2f,g,h,i,j are all equivalent. Clearly 7.2c,h are equivalent by 2.9. Also, 7.2b ⇒ 7.2e by 5.6b. We will show 7.2e ⇒ 7.2a later in this section, to complete the proof.

We give several immediate consequences of Theorem 7.2 next.

7.3. Corollary. If R and S are rings with unit, then $R \sim S$ iff there exist embedding relation functors R-**Rel**⟶S-**Rel** and S-**Rel**⟶R-**Rel** iff $\mathcal{L}(R) = \mathcal{L}(S)$ iff $\mathcal{A}(R) = \mathcal{A}(S)$ iff $\mathcal{B}(R) = \mathcal{B}(S)$ iff $\mathcal{U}_{AC}(R) = \mathcal{U}_{AC}(S)$ iff $\mathcal{U}_{RC}(R) = \mathcal{U}_{RC}(S)$ iff $\mathcal{U}_{L}(R) = \mathcal{U}_{L}(S)$ iff $\mathcal{U}_{A}(R) = \mathcal{U}_{A}(S)$ iff $\mathcal{U}_{B}(R) = \mathcal{U}_{B}(S)$.

7.4. Corollary. Each ring is diagram-equivalent to some countable ring (2.10).

7.5. Corollary. Suppose R and S are rings with unit, and Γ is a (closed, prenex) universal sentence in one of the languages $L(\tau_{AC})$, $L(\tau_{RC})$, $L(\tau_{L})$,

$\mathbf{L}(\tau_A)$ or $\mathbf{L}(\tau_B)$. If Γ is satisfied for R-modules (that is, throughout R-**Mod**, R-**Rel**, $\mathcal{L}(R)$, $\mathcal{A}(R)$ or $\mathcal{B}(R)$, respectively), then there exist basic universal Horn sentences $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ satisfied for R-modules, in the same language as Γ, such that Γ is a *logical* consequence of $\Gamma_1 \wedge \Gamma_2 \wedge \ldots \wedge \Gamma_n$. If $R \precsim S$ and Γ is satisfied for S-modules, then it is satisfied for R-modules.

Since $\mathcal{L}(R)$, $\mathcal{A}(R)$ and $\mathcal{B}(R)$ are quasivarieties by 2.9 and 4.9, 7.5 follows immediately for Γ in $\mathcal{L}(\tau_L)$, $\mathcal{L}(\tau_A)$ or $\mathcal{L}(\tau_B)$. See Appendix E for Γ in $\mathcal{L}(\tau_{AC})$ or $\mathcal{L}(\tau_{RC})$.

We will prove Theorem 7.2 using the methods of Fuller and Hutchinson [xx], which exploit the special properties of endomorphism rings $\mathrm{End}({}_R R^{(\beta)})$, where β is a sufficiently large infinite cardinal number. An abbreviated treatment is given below; consult [xx] for the motivating ideas from the theory of left coherent rings.

7.6. Definitions and Properties. Let R be a ring, let β be a cardinal number with $\beta \geq \aleph_0 + |R|$, and let T denote the endomorphism ring $\mathrm{End}({}_R R^{(\beta)})$. Let B be a set of free generators for ${}_R R^{(\beta)}$, with $|B| = \beta$.

7.6a. If M is a submodule of ${}_R R^{(\beta)}$, there is an endomorphism t in T such that Im t = M (since $|{}_R R^{(\beta)}| = \beta$ because $\beta \geq \aleph_0 + |R|$, and ${}_R R^{(\beta)}$ is free on β generators).

7.6b. For t and u in T, u is in Tt iff Im u $\leq$ Im t.

7.6c. The principal left ideals $\{Tt\colon t \in T\}$ of T form a sublattice of the lattice $\mathrm{Su}({}_T T)$ of left ideals of T, and this sublattice is isomorphic to $\mathrm{Su}({}_R R^{(\beta)})$ via the map $Tt \longmapsto \mathrm{Im}\, t$ (7.6a,b). Similarly, $\mathrm{Su}({}_T T)$ is isomorphic to the lattice of lattice ideals of $\mathrm{Su}({}_R R^{(\beta)})$.

7.6d. If $P = {}_R R^{(\beta)}$, then for each $n \geq 1$, $P \equiv P^{(n)}$ in R-**Mod**. (Partition B into subsets $B_1, B_2, \ldots, B_n$, each of cardinality β, and let P_i denote the submodule of P generated by B_i. Then $P_1 \oplus P_2 \oplus \ldots \oplus P_n$ is an internal direct sum for P such that $P \approx P_i$ for each $i \leq n$.)

7.7. Proposition. If R is a ring with unit and $\beta \geq 1$, then $R \sim \mathrm{End}({}_R R^{(\beta)})$.

Proof: Suppose B is a free generating set for ${}_R R^{(\beta)}$ and $T = \mathrm{End}({}_R R^{(\beta)})$. Note that ${}_R R_T^{(\beta)}$ is a bimodule and ${}_R R^{(\beta)}$ is a projective generator. So, $G = \mathrm{Hom}_R({}_R R_T^{(\beta)}, -)$ is an exact embedding functor R-**Mod**⟶T-**Mod**, and $R \precsim T$. Also, $\varphi: R \longrightarrow T$ given by $\varphi(r)(b) = rb$ for r in R and b in B is a ring homomorphism preserving 1. So, φ induces an exact embedding functor H_φ:T-**Mod**⟶R-**Mod** by *change of rings* ($rv = \varphi(r)v$ for v in ${}_T M$ and r in R). This proves $R \sim T$. ■

Some special definitions are given next.

7.8. Definitions and Properties. For M in R-**Mod** and N in S-**Mod**, a ring homomorphism $\psi: \mathrm{End}({}_R M) \longrightarrow \mathrm{End}({}_S N)$ is said to *preserve exactness* if for f and g in $\mathrm{End}({}_R M)$ such that $\langle f, g\rangle$ is exact in R-**Mod**, $\langle \psi(f), \psi(g)\rangle$ is exact in S-**Mod**. We say that ψ *reflects epimorphisms* if whenever $\psi(f)$ is an epimorphism (onto), then f must be an epimorphism.

7.9. Definitions and Properties. A right T-module N_T is called *1-flat* if for any $t_1, t_2, \ldots, t_n$ in T ($n \geq 1$), there exist $s_1, s_2, \ldots, s_n$ in T such that $\bigcup_{i=1}^{n} s_i t_i = 0$, and for any $v_1, v_2, \ldots, v_n$ in N such that $\bigcup_{i=1}^{n} v_i t_i = 0$, there exists u in N such that $us_i = v_i$ for $i \leq n$. Note that $s_1, s_2, \ldots, s_n$ depend only upon $t_1, t_2, \ldots, t_n$, and not on any particular choice of $v_1, v_2, \ldots, v_n$.
7.9a. If N_T is 1-flat, then it is flat.

We are now prepared to complete the proof of Theorem 7.2, and also obtain two more technical characterizations of diagram inclusion.

7.10. Theorem. Suppose R and S are rings with unit, and $T = \mathrm{End}({}_R R^{(\beta)})$ for some cardinal $\beta \geq \aleph_0 + |R|$. Then $R \precsim S$ iff $\mathcal{B}(R) \subseteq \mathcal{B}(S)$ iff 7.10a iff 7.10b.
7.10a. For some S-module N, there exists a (unit-preserving) ring monomorphism $\psi: T \longrightarrow \mathrm{End}({}_S N)$ which preserves exactness and reflects epimorphisms.
7.10b. There exists a bimodule ${}_S N_T$ such that N_T is 1-flat and $Nt \neq N$ for each t in T which is not an epimorphism.

Proof: As previously noted, $R \precsim S$ implies $\mathcal{B}(R) \subseteq \mathcal{B}(S)$ by 5.18 and 5.6b.

Assume $\mathcal{B}(R) \subseteq \mathcal{B}(S)$, so there exists a τ_B-embedding $\xi: \mathrm{Rel}({}_R R^{(\beta)}) \longrightarrow \mathrm{Rel}({}_S N)$ for some S-module N. By 3.16i, the ring of endomorphisms of any module M is isomorphic to hom(1,1), the ring of proper maps in Rel(M). Restricting ξ to proper morphisms produces an ψ satisfying 7.10a. This proves that $\mathcal{B}(R) \subseteq \mathcal{B}(S)$ implies 7.10a.

Assume that $\psi:T \longrightarrow \mathrm{End}({}_SN)$ satisfies 7.10a. It is convenient to write certain function evaluations using a binary infix symbol and reverse order: $x * f$ denotes $f(x)$. A bimodule structure ${}_SN_T$ can be defined using $vt = v * \psi(t)$ for $v \in N$ and $t \in T$. Suppose $t_1, t_2, \ldots, t_n$ are in T, $n \geq 1$. Since ${}_RP \cong {}_RP^{(n)}$ by 7.6d if ${}_RP$ denotes ${}_RR^{(\beta)}$, we can select insertions κ_i and projections π_i in T for $i \leq n$ such that $\kappa_i\pi_i = 1$ for $i \leq n$, $\kappa_i\pi_j = 0$ for $i \neq j$, $1 \leq i,j \leq n$, and $\bigcup_{j=1}^{n} \pi_j\kappa_j = 1$. Let $t = \bigcup_{j=1}^{n} \pi_j t_j$, so that $\kappa_i t = t_i$ for $i \leq n$. By 7.6a, there exists s in T such that Im s = Ker t. Define $s_i = s\pi_i$ for $i \leq n$. So,

$$\bigcup_{i=1}^{n} s_i t_i = \bigcup_{i=1}^{n} s\pi_i\kappa_i t = s(\bigcup_{i=1}^{n} \pi_i\kappa_i)t = s1t = 0.$$

Now suppose $\bigcup_{i=1}^{n} v_i t_i = 0$ for $v_1, v_2, \ldots, v_n$ in N. Let $v = \bigcup_{i=1}^{n} v_i\kappa_i$, so

$$v * \psi(t) = (\bigcup_{i=1}^{n} v_i * \psi(\kappa_i)) * \psi(t) = \bigcup_{i=1}^{n} v_i * \psi(\kappa_i t) = \bigcup_{i=1}^{n} v_i t_i = 0.$$

Since ψ preserves exactness, Im $\psi(s)$ = Ker $\psi(t)$, and so there exists u in N such that $u * \psi(s) = v$. But then

$$us_i = us\pi_i = (u * \psi(s)) * \psi(\pi_i) = v * \psi(\pi_i) = (\bigcup_{j=1}^{n} v_j * \psi(\kappa_j)) * \psi(\pi_i) = v_i,$$

for $i = 1,2,\ldots,n$. Therefore, N_T is 1-flat. If t in T is not onto, then $\psi(t)$ is not onto by hypothesis, so Nt = Im $\psi(t) \neq N$. Therefore, 7.10a implies 7.10b.

Assume 7.10b, and let F denote the composite functor

$$R\text{-}\mathbf{Mod} \xrightarrow{G} T\text{-}\mathbf{Mod} \xrightarrow{H} S\text{-}\mathbf{Mod},$$

where G is $\mathrm{Hom}({}_RR_T^{(\beta)}, -)$ and H is ${}_SN \otimes_T -$. Since G is exact by 7.7 and H is exact because N_T is flat (7.9a), F is exact also.

Let K be a proper left ideal of R. Suppose B is a free generating set for ${}_RP$, and b_0 is in B. Let $K_0 = Kb_0 \vee P_0$ in $\mathrm{Su}({}_RP)$, where P_0 is generated by the set $B - \{b_0\}$. By 7.6a, there exists t in T such that Im t = K_0. Since $w \in Tt$ iff Im w $\leq$ Im t by 7.6b, there is a T-linear monomorphism $\lambda: T/Tt \longrightarrow G(R/K)$, given by $\lambda(1 + Tt) = \lambda_0$ in $\mathrm{Hom}({}_RP, {}_RR/K)$, where $\lambda_0(b_0) = 1 + K$ and $\lambda_0(b) = 0$ for all b in $B - \{b_0\}$. Since t is not onto, NTt = Nt $\neq$ N by hypothesis, and so

${}_S N \otimes_T (T/Tt) \approx N/NTt \neq 0$. But then

$$F(R/K) = H(G(R/K)) = {}_S N \otimes_T G(R/K) \neq 0,$$

via the monomorphism $N \otimes_T \lambda$. Then F is an exact embedding functor, and so 7.10b implies $R \precsim S$. ∎