§10. Free Word Problems for Additive Relation Algebras of Modules.

We are now ready to develop the algorithm for free word problems with respect to $\mathcal{L}(R)$, $\mathcal{A}(R)$ and $\mathcal{B}(R)$. From any τ_B-polynomials d and e, we will recursively compute integers $m \geq 0$ and $n \geq 1$ such that $d \leq e$ is satisfied in every additive relation algebra of $\mathcal{B}(R)$ if and only if $m\cdot 1$ divides $n\cdot 1$ in R. Using the analysis of **Z**-image divisibility conditions in §9, it then follows that $\mathcal{B}(R)$ has solvable free word problems iff the predicate $dgr_R(p) \leq k$ is recursively decidable for all primes p and $k \geq 0$. This is always true if R has nonzero characteristic. Essentially the same analysis holds for $\mathcal{L}(R)$ and $\mathcal{A}(R)$. The varieties of additive relation algebras (or lattices) generated by the quasivarieties above are uniquely determined by the **Z**-image divisibility pattern $\langle zchar_R, dgr_R\rangle$ of R in $\mathcal{J}$, as in 9.10. This free word problem analysis was originally given for lattices in [TISL], and for additive relation algebras in [FWPARAM].

10.1. Definitions and Properties. The variables $X = \{x_1, x_2, x_3, \ldots\}$ will represent additive relation algebra elements. As in 3.1, $P(X,\tau_B)$ denotes the τ_B-algebra of all τ_B-polynomials on X. Suppose $d = d(x_1, x_2, \ldots, x_n)$ is in $P(X,\tau_B)$. Let $\|d\|$ denote the number of symbols in d (count x_i as one symbol, supply any omitted product symbols $(\cdot)$, and exclude parentheses).

Sets $B = \{b_1, b_2, b_3, \ldots\}$ and $C = \{c_1, c_2, c_3, \ldots\}$ contain variables that will represent module elements. Let $\mathbf{Z}_B$ denote the free **Z**-module of **Z**-linear combinations of elements of B, and similarly for $\mathbf{Z}_C$ and C.

A recursive decomposition of d is given next, using finite sequences of elements of $W_B = \mathbf{Z}_B \times \mathbf{Z}_B \times P(X,\tau_B)$. The m-tuple $\mathbf{w}_{B,k}(d) = \langle u_1, u_2, \ldots, u_m\rangle$ of triples u_i in W_B is defined by induction on $k \geq 0$, where m is a function of k and d. For each d, $\mathbf{w}_{B,0}(d) = \langle u_1\rangle$ for $u_1 = \langle b_1, b_2, d\rangle$. Assume $\mathbf{w}_{B,k}(d) = \langle u_1, u_2, \ldots, u_m\rangle$ has been defined. If $m \leq k$, then $\mathbf{w}_{B,j}(d) = \mathbf{w}_{B,k}(d)$ for all $j \geq k$ and we also write $\mathbf{w}_B(d) = \mathbf{w}_{B,k}(d)$. Otherwise, we suppose that $u_{k+1} = \langle p_{k+1}, q_{k+1}, d_{k+1}\rangle$ and define $\mathbf{w}_{B,k+1}(d)$ by adding zero, one or two triples to the end of $\mathbf{w}_{B,k}(d)$, as described below. We also need to know the first *unused* variable b_j, that is, the smallest value of j, $j \geq 3$, such that b_j has zero coefficient in the first and second coordinates of all of the triples $u_1, u_2, \ldots, u_m$. Define $\mathbf{w}_{B,k+1}(d)$ by cases as follows:

If d_{k+1} is 0, 1, **0**, **I** or a variable x_i in X, then $\mathbf{w}_{B,k+1}(d) = \mathbf{w}_{B,k}(d)$.

If d_{k+1} is $f+g$ for τ_B-polynomials f and g, then add the two triples:

$u_{m+1} = \langle p_{k+1},\ b_j,\ f\rangle$ and $u_{m+2} = \langle p_{k+1},\ q_{k+1} - b_j,\ g\rangle$.

If d_{k+1} is $-f$, then add the triple:

$u_{m+1} = \langle p_{k+1},\ -q_{k+1},\ f\rangle$.

If d_{k+1} is fg, then add the two triples:

$u_{m+1} = \langle p_{k+1},\ b_j,\ f\rangle$ and $u_{m+2} = \langle b_j,\ q_{k+1},\ g\rangle$.

If d_{k+1} is $f^{\#}$, then add the triple:

$u_{m+1} = \langle q_{k+1},\ p_{k+1},\ f\rangle$.

If d_{k+1} is $f \wedge g$, add the two triples:

$u_{m+1} = \langle p_{k+1},\ q_{k+1},\ f\rangle$ and $u_{m+2} = \langle p_{k+1},\ q_{k+1},\ g\rangle$.

If d_{k+1} is $f \vee g$, then add the two triples:

$u_{m+1} = \langle b_j, b_{j+1}, f\rangle$ and $u_{m+2} = \langle p_{k+1} - b_j,\ q_{k+1} - b_{j+1},\ g\rangle$.

This completes the inductive definition of $\mathbf{w}_{B,k}(d)$ for $k \geq 0$.

Given e in $P(X,\tau_B)$, we define $\mathbf{w}_{C,k}(e)$ and $\mathbf{w}_C(e)$ similarly, except that elements of $\mathbf{Z}_C$ are used rather than elements of $\mathbf{Z}_B$. That is, the first triple is $\langle c_1, c_2, e\rangle$, and we use elements c_j rather than b_j in the cases above, so that all triples of $\mathbf{w}_{C,k}(d)$ belong to $W_C = Z_C \times Z_C \times P(X,\tau_B)$.

Let $\ell(d)$ equal 2 plus the number of occurrences of relational sum (+) and product ($\cdot$) symbols in d plus twice the number of join ($\vee$) symbols in d, and similarly for $\ell(e)$ and e.

10.1a. For all d and e in $P(X,\tau_B)$, $\mathbf{w}_B(d)$ is defined and is a $\|d\|$-tuple of elements of W_B, and similarly $\mathbf{w}_C(e)$ is defined and is an $\|e\|$-tuple on W_C. Also, $\mathbf{w}_B(d)$ and $\ell(d)$ are recursively computable from d, and similarly $\mathbf{w}_C(e)$ and $\ell(e)$ are recursively computable from e. A variable b_j occurs (with nonzero coefficient) in the first or second coordinate of a triple in $\mathbf{w}_B(d)$ iff $j \le \ell(d)$, and similarly for c_j, $\mathbf{w}_C(e)$ and $\ell(e)$.

Given additive relation algebra polynomials d and e in $P(X,\tau_B)$, we will consider free word problems of form $d \le e$. (Obviously, we can solve for all equations $d = e$ iff we can solve all such inclusions.) Our word problem procedure begins with the computation of $\mathbf{w}_B(d)$ and $\mathbf{w}_C(e)$. To assist the reader, a sample computation is shown below.

10.2. Example. Let $d = (-x_1)0 \wedge x_2\mathbf{I}$ in $P(X,\tau_B)$. Then $\mathbf{w}_B(d) = \langle u_1, u_2, \ldots, u_8\rangle$ as shown below:

$$u_1 = \langle b_1,\ b_2,\ (-x_1)0 \wedge x_2\mathbf{I}\rangle \qquad u_5 = \langle b_3,\ b_2,\ 0\rangle$$
$$u_2 = \langle b_1,\ b_2,\ (-x_1)0\rangle \qquad u_6 = \langle b_1,\ b_4,\ x_2\rangle$$
$$u_3 = \langle b_1,\ b_2,\ x_2\mathbf{I}\rangle \qquad u_7 = \langle b_4,\ b_2,\ \mathbf{I}\rangle$$
$$u_4 = \langle b_1,\ b_3,\ -x_1\rangle \qquad u_8 = \langle b_1,\ -b_3,\ x_1\rangle$$

Secondly, let $e = (x_1 + x_1)\mathbf{0} \vee (1 + x_2{}^{\#})$ in $P(X,\tau_B)$, so $\mathbf{w}_C(e) = \langle v_1, v_2, \ldots, v_{10}\rangle$ as shown below:

$$v_1 = \langle c_1,\ c_2,\ (x_1 + x_1)\mathbf{0} \vee (1 + x_2{}^{\#})\rangle \qquad v_6 = \langle c_1 - c_3,\ c_6,\ 1\rangle$$
$$v_2 = \langle c_3,\ c_4,\ (x_1 + x_1)\mathbf{0}\rangle \qquad v_7 = \langle c_1 - c_3,\ c_2 - c_4 - c_6,\ x_2{}^{\#}\rangle$$
$$v_3 = \langle c_1 - c_3,\ c_2 - c_4,\ 1 + x_2{}^{\#}\rangle \qquad v_8 = \langle c_3,\ c_7,\ x_1\rangle$$
$$v_4 = \langle c_3,\ c_5,\ x_1 + x_1\rangle \qquad v_9 = \langle c_3,\ c_5 - c_7,\ x_1\rangle$$
$$v_5 = \langle c_5,\ c_4,\ \mathbf{0}\rangle \qquad v_{10} = \langle c_2 - c_4 - c_6,\ c_1 - c_3,\ x_2\rangle$$

These two decompositions show applications of all the rules in 10.1. Note that $\ell(d) = 4$ and $\ell(e) = 7$.

Obviously, the decompositions of 10.1 are related to the structure of unary and binary τ_B-operations for Rel(M), M an R-module. Next, we recover information from $\mathbf{w}_B(d)$ relating to the occurrences of constants and variables in d. Then $\mathbf{w}_C(e)$ and e are treated similarly.

10.3. Definitions and Properties. Suppose $d = d(x_1, x_2, \ldots, x_n)$ in $P(X,\tau_B)$ and $\mathbf{w}_B(d) = \langle u_1, u_2, \ldots, u_k\rangle$ with $u_j = \langle p_j, q_j, d_j\rangle$ in W_B for $j \le k$.

Define $V_B(d) = V_1 \cup V_2 \cup V_3 \subseteq \mathbf{Z}_B$ by:

$V_1 = \{p_j: j \le k \text{ and } d_j = \mathbf{0}\} \cup \{q_j: j \le k \text{ and } d_j = \mathbf{0}\}$,

$V_2 = \{q_j: j \le k \text{ and } d_j = 0\}$ and

$V_3 = \{p_j - q_j: j \le k \text{ and } d_j = 1\}$.

Since I contains all pairs of Rel(M), no restriction conditions in $V_B(d)$ are needed for the cases $d_j = I$.

For i = 1,2,...,n, define subsets $U_{B,i}(d)$ of $\mathbf{Z}_B \times \mathbf{Z}_B$ by:

$U_{B,i}(d) = \{\langle p_j, q_j\rangle: j \le k \text{ and } d_j = x_i\}$.

Assuming that $e = e(x_1,x_2,\ldots,x_n)$ is in $P(X,\tau_B)$ also, similarly define subsets $V_C(e)$ of $\mathbf{Z}_C$ and $U_{C,i}(e)$ of $\mathbf{Z}_C \times \mathbf{Z}_C$ for i = 1,2,...,n, using $\mathbf{w}_C(e)$.

10.3a. For d as above, $V_B(d)$ and $U_{B,i}(d)$ for i = 1,2,...,n are finite sets which are recursively computable from d. Similarly, $V_C(e)$ and $U_{C,i}(e)$ for i = 1,2,...,n are finite sets which are recursively computable from e.

10.4. Example. Defining d and e as in 10.2, we obtain:

$V_B(d) = \{b_2\}$

$U_{B,1}(d) = \{\langle b_1, -b_3\rangle\}$

$U_{B,2}(d) = \{\langle b_1, b_4\rangle\}$

$V_C(e) = \{c_4, c_5, c_1 - c_3 - c_6\}$

$U_{C,1}(e) = \{\langle c_3, c_7\rangle, \langle c_3, c_5 - c_7\rangle\}$

$U_{C,2}(e) = \{\langle c_2 - c_4 - c_6, c_1 - c_3\rangle\}$

Since only x_1 and x_2 occur in d and e, we take n = 2.

Our decomposition procedure was constructed to satisfy the next result.

10.5. Proposition. Suppose $d = d(x_1,x_2,\ldots,x_n)$ is in $P(X,\tau_B)$. Let M be an R-module, with a_1 and a_2 in M and $z_1,z_2,\ldots,z_n$ in Rel(M). Then $\langle a_1,a_2\rangle$ is in $d(z_1,z_2,\ldots,z_n)$ iff there exists a Z-linear map $\varphi:\mathbf{Z}_B \longrightarrow M$ satisfying conditions 10.5a,b,c,d below.

10.5a. For all $j > \ell(d)$, $\varphi(b_j) = 0$.

10.5b. For i = 1 and 2, $a_i = \varphi(b_i)$.

10.5c. For all x in $V_B(d)$, $\varphi(x) = 0$.

10.5d. For each $i \le n$, $\langle x,y\rangle$ in $U_{B,i}(d)$ implies $\langle\varphi(x), \varphi(y)\rangle$ is in z_i.

Proof: Assume the hypotheses, suppose $\mathbf{w}_B(d) = \langle u_1,u_2,\ldots,u_k\rangle$ with $u_j = \langle p_j,q_j,d_j\rangle$ for $j \le k = \|d\|$, and let $s = \ell(d)$. If $\varphi:\mathbf{Z}_B \longrightarrow M$ is Z-linear and satisfies 10.5b,c,d, then $\langle\varphi(p_j), \varphi(q_j)\rangle$ is in $d_j(z_1,\ldots,z_n)$ by induction backwards through $\mathbf{w}_B(d)$, from u_k to u_1. (Use 10.5c if d_j is a constant 0, 1, $\mathbf{0}$ or I, and 10.5d if d_j is a variable x_i.) But then $\langle a_1,a_2\rangle$ is in $d(z_1,\ldots,z_n)$ via u_1 and 10.5b.

Conversely, if $\langle a_1,a_2\rangle$ is in $d(z_1,\ldots,z_n)$, then we use induction forward through $\mathbf{w}_B(d)$, from u_1 to u_k, to define $\varphi(b_1),\varphi(b_2),\ldots,\varphi(b_s)$ in M which uniquely determine a Z-linear map $\varphi:\mathbf{Z}_B\longrightarrow M$ satisfying 10.5a,b,c,d and such that $\langle\varphi(p_j),\ \varphi(q_j)\rangle$ is in $d_j(z_1,\ldots,z_n)$ for $j \le k$. ∎

Proposition 10.5 remains true if we relabel so that d is replaced by e, B by C, and b_i by c_i throughout.

We are now ready to make the first direct connection between our analysis and free additive relation algebra word problems. Our procedure here was originally motivated by the method of R. Wille for constructing Malcev conditions characterizing certain universal algebra congruence Horn formulas ([KG, Satz 6.16, p. 76]; also see [TISL, Thm. 1, p. 276]).

10.6. Definitions. Suppose R is a ring, and recall the Z-image map $\zeta_R:\mathbf{Z}\longrightarrow R$ from 9.1. Let $d = d(x_1,x_2,\ldots,x_n)$ be in $P(X,\tau_B)$, and $s = \ell(d)$. Then define:

$\xi:\mathbf{Z}_B\longrightarrow R^s$ to be the Z-linear map such that $\xi(b_j) = 0$ if $j > s$ and

$\xi(m_1b_1 + m_2b_2 + \ldots + m_sb_s) = \langle\zeta_R(m_1),\ \zeta_R(m_2),\ldots,\zeta_R(m_s)\rangle$,

N to be the R-submodule of R^s generated by $\{\xi(v)\colon v \in V_B(d)\}$

with $N = 0$ if $V_B(d) = \emptyset$,

$\eta:R^s\longrightarrow R^s/N$ to be the canonical R-linear epimorphism, and

$\kappa:\mathbf{Z}_B\longrightarrow R^s/N$ to be the (Z-linear) composite function, $\kappa = \xi\eta$.

For each $i \le n$, let y_i denote the element of $\mathrm{Rel}(R^s/N)$ which is generated as a submodule of $R^s/N \oplus R^s/N$ by the set of pairs:

$\{\langle\kappa(u),\kappa(v)\rangle\colon \langle u,v\rangle \in U_{B,i}(d)\}$,

with $y_i = \mathbf{0}$ if $U_{B,i}(d) = \emptyset$.

10.7. Proposition. Let R be a ring with unit, and suppose $d = d(x_1,\ldots,x_n)$ and $e = e(x_1,\ldots,x_n)$ are τ_B-polynomials. Define $s = \ell(d)$, ξ, N, η, κ and y_i

for $i \leq n$ from d as in 10.6. Then the following conditions are equivalent:

10.7a. $d \leq e$ is satisfied in all additive relation algebras in $\mathcal{B}(R)$.

10.7b. $\langle\kappa(b_1), \kappa(b_2)\rangle$ is in $e(y_1, y_2, \ldots, y_n)$.

10.7c. There exists a Z-linear map $\psi:\mathbf{Z}_C \longrightarrow R^s/N$ such that $\psi(c_j) = 0$ for all $j > \ell(e)$, $\psi(c_1) = \kappa(b_1)$, $\psi(c_2) = \kappa(b_2)$, $\psi(x) = 0$ for all x in $V_C(e)$, and for each $i \leq n$, $\langle x,y\rangle$ in $U_{C,i}(e)$ implies $\langle\psi(x), \psi(y)\rangle$ is in y_i.

Proof: Assume the hypotheses, and check that κ and $y_1, y_2, \ldots, y_n$ satisfy the conditions 10.5a,c,d for d, and so $\langle\kappa(b_1), \kappa(b_2)\rangle$ is in $d(y_1, \ldots, y_n)$ by 10.5. But then 10.7a implies 10.7b. Obviously, 10.7b and 10.7c are equivalent by the relabelled version of 10.5.

Assuming 10.7b, 10.7a can be proved by showing that for any R-module M,

$$d(z_1, z_2, \ldots, z_n) \leq e(z_1, z_2, \ldots, z_n)$$

for any $z_1, z_2, \ldots, z_n$ in Rel(M). (The quasivariety $\mathcal{B}(R)$ is generated by $\{\mathrm{Rel}(M)\colon M \text{ in } R\text{-}\mathbf{Mod}\}$.) Let $\langle a_1, a_2\rangle \in d(z_1, \ldots, z_n)$. By 10.5, there exists a Z-linear map $\varphi:\mathbf{Z}_B \longrightarrow M$ satisfying 10.5a,b,c,d. Define $\mu:R^s \longrightarrow M$ by

$$\mu(\langle r_1, r_2, \ldots, r_s\rangle) = \sum_{i=1}^{s} r_i\varphi(b_i).$$

Clearly μ is R-linear, and $\varphi = \xi\mu$ by 10.5a. Then $\mu(\xi(x)) = \varphi(x) = 0$ for x in $V_B(d)$ by 10.5c. So, $N \subseteq \mathrm{Ker}\,\mu$, and there exists an R-linear map $\nu:R^s/N \longrightarrow M$ such that $\mu = \eta\nu$. We have the homomorphism diagram

$$\mathbf{Z}_B \xrightarrow{\ \xi\ } R^s \xrightarrow{\ \eta\ } R^s/N \xrightarrow{\ \nu\ } M,$$

with $\kappa = \xi\eta$, $\mu = \eta\nu$ and $\varphi = \xi\eta\nu$. By induction on τ_B-polynomial length $\|f\|$ using 10.5d, we can show that $\langle x,y\rangle$ in $f(y_1, \ldots, y_n)$ implies $\langle\nu(x), \nu(y)\rangle$ is in $f(z_1, \ldots, z_n)$ for any $f = f(x_1, \ldots, x_n)$ in $P(X, \tau_B)$. By 10.5b and 10.7b,

$$\langle a_1, a_2\rangle = \langle\varphi(b_1), \varphi(b_2)\rangle = \langle\nu(\kappa(b_1)), \nu(\kappa(b_2))\rangle \in e(z_1, z_2, \ldots, z_n).$$

So, $d(z_1, \ldots, z_n) \leq e(z_1, \ldots, z_n)$, proving that 10.7b implies 10.7a. ∎

Our next objective is to recursively construct a nonhomogeneous system of linear equations with Z-image coefficients which is satisfied in R iff the criterion 10.7c is true. We can then apply 9.14 and 10.7 to connect the free word problem for $\mathcal{B}(R)$ with the **Z**-image divisibility analysis of §9. First, we examine our special case.

10.8. Example. For d and e as in 10.2 and 10.4, $\mathbf{V}_B(d) = \{b_2\}$, so

$$N = R\langle 0,1,0,0\rangle \subseteq R^4$$

Since $U_{B,1}(d) = \{\langle b_1, -b_3\rangle\}$, y_1 is generated by one element:

$$\langle\langle 1,0,0,0\rangle + N,\ \langle 0,0,-1,0\rangle + N\rangle.$$

Using $U_{B,2}(d) = \{\langle b_1, b_4\rangle\}$ similarly, y_2 is generated by one element:

$$\langle\langle 1,0,0,0\rangle + N,\ \langle 0,0,0,1\rangle + N\rangle.$$

Furthermore, 10.7c is satisfied in this case if and only if there exist elements $\psi(c_1),\psi(c_2),\ldots,\psi(c_7)$ of R^4/N such that:

$\psi(c_1) = \langle 1,0,0,0\rangle + N$, $\psi(c_2) = \langle 0,1,0,0\rangle + N$ and

$\psi(c_4) = \psi(c_5) = \psi(c_1) - \psi(c_3) - \psi(c_6) = 0$ in R^4/N,

$\langle\psi(c_3),\ \psi(c_7)\rangle$ and $\langle\psi(c_3),\ \psi(c_5) - \psi(c_7)\rangle$ are in y_1, and

$\langle\psi(c_2) - \psi(c_4) - \psi(c_6),\ \psi(c_1) - \psi(c_3)\rangle$ is in y_2.

Of course, 0, 1 and -1 represent **Z**-images in R above.

Assume 10.7c for the example d and e, and let a_{ij} for $i \le 7$ and $j \le 4$ be variables representing elements of R such that:

$$\psi(c_i) = \langle a_{i1},\ a_{i2},\ a_{i3},\ a_{i4}\rangle + N \text{ in } R^4/N, \text{ for } i \le 7.$$

We can take $\langle a_{i1},a_{i2},a_{i3},a_{i4}\rangle$ to be $\langle 1,0,0,0\rangle$ for i = 1 and $\langle 0,1,0,0\rangle$ for i = 2. So, the first part of the system equivalent to 10.7c consists of linear equations $a_{ij} = \delta_{ij}$ (Kronecker delta) for i = 1,2 and $j \le 4$.

Now N is generated by $\langle 0,1,0,0\rangle$, so $\langle w_1,w_2,w_3,w_4\rangle$ is in N iff there exists r_0 in R such that

$$\langle w_1,w_2,w_3,w_4\rangle = r_0\langle 0,1,0,0\rangle \text{ in } R^4.$$

Then for the source equations $\psi(c_4) = \psi(c_5) = 0$, we have linear equations $a_{4j} = a_{5j} = 0$ for j = 1,3,4, plus $a_{42} = e_0$ and $a_{52} = e_1$ for auxiliary variables e_0 and e_1. Similarly, $\psi(c_1) - \psi(c_3) - \psi(c_6) = 0$ in R^4/N is equivalent to equations of form $a_{1j} - a_{3j} - a_{6j} = 0$ for j = 1,3,4, plus one equation $a_{12} - a_{32} - a_{62} = e_2$. These equations are linear with **Z**-image coefficients, and form the second part of the system equivalent to 10.7c.

The process is similar for conditions of the form $\langle\psi(x),\ \psi(y)\rangle \in y_i$. This condition is equivalent to two source equations, one for each coordinate of the pair. Since y_1 is generated by

$$\langle\xi(b_1) + N,\ -\xi(b_3) + N\rangle = \langle\langle 1,0,0,0\rangle + N,\ \langle 0,0,-1,0\rangle + N\rangle,$$

membership of $\langle\psi(c_3),\ \psi(c_5) - \psi(c_7)\rangle$ in y_1 is equivalent to existence of f_1 in R such that $\psi(c_3) = f_1\xi(b_1) + N$ and $\psi(c_5) - \psi(c_7) = f_1(-\xi(b_3)) + N$. Again we introduce auxiliary variables, say g_1 and h_1, for the generating element of N, and we also consider f_1 to be another kind of auxiliary variable. Then the first source equation is equivalent to the linear equations $a_{31} = f_1$, $a_{32} = g_1$ and $a_{3j} = 0$ for $j = 3,4$. The second source equation is equivalent to $a_{52} - a_{72} = h_1$, $a_{53} - a_{73} = -f_1$ and $a_{5j} - a_{7j} = 0$ for $j = 1,4$. Conditions $\langle\psi(c_3),\ \psi(c_5)\rangle$ in y_1 and $\langle\psi(c_2) - \psi(c_4) - \psi(c_6),\ \psi(c_1) - \psi(c_3)\rangle$ in y_2 can be treated similarly using further auxiliary variables. This will complete the system of linear equations equivalent to 10.7c.

In general, the left side of each source equation is a **Z**-linear combination of $\ell(d)$-tuples representing elements $\psi(c_j)$. The right side of each equation is an R-linear combination of $\ell(d)$-tuples with **Z**-image coordinates. All source equation right sides have $\ell(d)$-tuples representing generators of N, and some also have $\ell(d)$-tuples representing first or second coordinates of pairs generating one of the additive relations y_i. Since **Z**-images are central elements of R by 9.1a, each source equation is equivalent to a system of $\ell(d)$ linear equations with **Z**-image coefficients.

The general case may be better understood by formulating matrix equations equivalent to the criterion 10.7c.

10.9. Definitions and Properties. Let R be a ring with unit. For $k \geq 2$, let $\mathbf{Y}_k = [y_{ij}]$ denote the $2 \times k$ matrix on R with $y_{ij} = \delta_{ij}$ (Kronecker delta).

Let $s = \ell(d)$ and $t = \ell(e)$. Define $\xi_B = \xi : \mathbf{Z}_B \longrightarrow R^s$ as in 10.6, and define

$\xi_C:\mathbf{Z}_C \longrightarrow R^t$ similarly, that is, ξ_C is $\mathbf{Z}$-linear, $\xi_C(c_j) = 0$ for $j > \ell(e)$, and

$$\xi_C(i_1c_1 + i_2c_2 + \ldots + i_tc_t) = \langle \zeta_R(i_1), \zeta_R(i_2), \ldots, \zeta_R(i_t) \rangle.$$

Let p denote $|V_B(d)|^+$, where $p = 1$ if $V_B(d) = \emptyset$ and p is equal to the cardinality of $V_B(d)$ otherwise. Similarly, let q denote $|V_C(e)|^+$. For each $i \le n$, let $m_i = |U_{B,i}(d)|^+$ and $n_i = |U_{C,i}(e)|^+$.

Number the elements of $V_B(d)$ systematically, and let $\mathbf{V}_B$ denote the $p \times s$ matrix on R such that if x is the j-th element of $V_B(d)$, then the j-th row of $\mathbf{V}_B$ is the s-vector $\xi_B(x)$. If $V_B(d)$ is empty, let $\mathbf{V}_B$ be a $1 \times s$ matrix of zeros. Similarly, number $V_C(e)$ and let $\mathbf{V}_C$ be the $q \times t$ matrix which has j-th row $\xi_C(y)$ if y is the j-th element of $V_C(e)$. Again, $\mathbf{V}_C$ is a $1 \times t$ zero matrix if $V_C(e) = \emptyset$.

Now number $U_{B,i}(d)$ and $U_{C,i}(e)$ systematically for each $i \le n$. For each such i, define $m_i \times s$ matrices $\mathbf{U}_{BL,i}$ and $\mathbf{U}_{BR,i}$ on R by:

$U_{BL,i}$ has j-th row $\xi_B(u_0)$ if $\langle u_0,v_0 \rangle$ is the j-th pair of $U_{B,i}(d)$,

$U_{BR,i}$ has j-th row $\xi_B(v_0)$ if $\langle u_0,v_0 \rangle$ is the j-th pair of $U_{B,i}(d)$,

and $n_i \times t$ matrices $\mathbf{U}_{CL,i}$ and $\mathbf{U}_{CR,i}$ on R by:

$U_{CL,i}$ has j-th row $\xi_C(u_1)$ if $\langle u_1,v_1 \rangle$ is the j-th pair of $U_{C,i}(d)$,

$U_{CR,i}$ has j-th row $\xi_C(v_1)$ if $\langle u_1,v_1 \rangle$ is the j-th pair of $U_{C,i}(d)$,

Let $\mathbf{U}_{BL,i}$ and $\mathbf{U}_{BR,i}$ be $1 \times s$ zero matrices if $U_{B,i}(d) = \emptyset$. Similarly, $\mathbf{U}_{BL,i}$ and $\mathbf{U}_{CR,i}$ are $1 \times t$ zero matrices if $U_{C,i}(e) = \emptyset$.

10.9a. The matrices $\mathbf{Y}_s$, $\mathbf{Y}_t$, $\mathbf{V}_B$, $\mathbf{V}_C$ and $\mathbf{U}_{BL,i}$, $\mathbf{U}_{BR,i}$, $\mathbf{U}_{CL,i}$ and $\mathbf{U}_{CR,i}$ for $i \le n$ are recursively computable from d and e, as are their dimension parameters s, t, p, q and m_i and n_i for $i \le n$.

10.10. Proposition. Let R be a ring with unit, and suppose $d = d(x_1,\ldots,x_n)$ and $e = e(x_1,\ldots,x_n)$ are τ_B-polynomials. Define s, t, $\mathbf{Y}_s$, $\mathbf{Y}_t$, p, q, $\mathbf{V}_B$, $\mathbf{V}_C$, and for each $i \le n$, m_i, n_i, $\mathbf{U}_{BL,i}$, $\mathbf{U}_{BR,i}$, $\mathbf{U}_{CL,i}$ and $\mathbf{U}_{CR,i}$ as in 10.9. Then $d \le e$ is satisfied in every additive relation algebra of $\mathcal{B}(R)$ iff there exist matrices on R, consisting of a $t \times s$ matrix $\mathbf{A}$, a $p \times q$ matrix $\mathbf{E}$, and an $m_i \times n_i$ matrix $\mathbf{F}_i$ and $m_i \times q$ matrices $\mathbf{G}_i$ and $\mathbf{H}_i$ for each $i \le n$, satisfying matrix equations 10.10a,b,c,d below.

10.10a. $\mathbf{Y}_t\mathbf{A} = \mathbf{Y}_s$, dimensions $(2\times t)(t\times s) = 2\times s$.

10.10b. $\mathbf{V}_C\mathbf{A} = \mathbf{E}\mathbf{V}_B$, dimensions $(q\times t)(t\times s) = (q\times p)(p\times s)$.

10.10c. $\mathbf{U}_{CL,i}\mathbf{A} = \mathbf{F}_i\mathbf{U}_{BL,i} + \mathbf{G}_i\mathbf{V}_B$ for $i \le n$, dimensions:

$(n_i\times t)(t\times s) = (n_i\times m_i)(m_i\times s) + (n_i\times p)(p\times s)$.

10.10d. $\mathbf{U}_{CR,i}\mathbf{A} = \mathbf{F}_i\mathbf{U}_{BR,i} + \mathbf{H}_i\mathbf{V}_B$ for $i \le n$, same dimensions as 10.10c.

We omit the routine calculations verifying 10.10, which follow the methods previously described. The matrix $\mathbf{A} = [a_{ij}]$ has been discussed, and the matrix entries of $\mathbf{E}$, $\mathbf{G}_i$ and $\mathbf{H}_i$ correspond to the auxiliary variables for generators of N above. The entries of $\mathbf{F}_i$ correspond to the auxiliary variables of the second kind, like f_1 above. Condition 10.10a corresponds to the conditions $\psi(c_1) = \kappa(b_1)$ and $\psi(c_2) = \kappa(b_2)$, and 10.10b corresponds to the conditions $\psi(x) = 0$ for x in $V_C(e)$. Conditions $\langle\psi(x), \psi(y)\rangle$ in y_i for $\langle x,y\rangle$ in $U_{C,i}(e)$ correspond to 10.10c and 10.10d, treating first and second coordinates of the pairs in separate matrix equations.

10.11. Example. Again, consider d and e as in 10.2, 10.4 and 10.8. Note that $s = 4$, $t = 7$, $p = 1$, $q = 3$, $m_1 = 1$, $n_1 = 2$ and $m_2 = n_2 = 1$. Matrix equation 10.10a is shown below:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Give names to the auxiliary variables as follows:

$$\mathbf{E} = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \end{bmatrix}, \quad \mathbf{F}_1 = \begin{bmatrix} f_0 \\ f_1 \end{bmatrix}, \quad \mathbf{G}_1 = \begin{bmatrix} g_0 \\ g_1 \end{bmatrix} \quad \text{and} \quad \mathbf{H}_1 = \begin{bmatrix} h_0 \\ h_1 \end{bmatrix},$$

together with $\mathbf{F}_2 = [f_2]$, $\mathbf{G}_2 = [g_2]$ and $\mathbf{H}_2 = [h_2]$. Now, observe that:

$$\mathbf{V}_C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{V}_B = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix},$$

so that 10.10b is the matrix equation

$$\begin{bmatrix} a_{41} & a_{42} & a_{43} & a_{44} \\ a_{51} & a_{52} & a_{53} & a_{54} \\ a_{11}-a_{31}-a_{61} & a_{12}-a_{32}-a_{62} & a_{13}-a_{33}-a_{63} & a_{14}-a_{34}-a_{64} \end{bmatrix} = \begin{bmatrix} 0 & e_0 & 0 & 0 \\ 0 & e_1 & 0 & 0 \\ 0 & e_2 & 0 & 0 \end{bmatrix}$$

From the definitions, we have

$$\mathbf{U}_{CL,1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{U}_{BL,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{U}_{CL,2} = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & -1 & 0 \end{bmatrix} \text{ and } \mathbf{U}_{BL,2} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the two matrix equations of 10.10c are:

$$\begin{bmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} f_0 & g_0 & 0 & 0 \\ f_1 & g_1 & 0 & 0 \end{bmatrix} \text{ and}$$

$$\begin{bmatrix} a_{21}-a_{41}-a_{61} & a_{22}-a_{42}-a_{62} & a_{23}-a_{43}-a_{63} & a_{24}-a_{44}-a_{64} \end{bmatrix} = \begin{bmatrix} f_2 & g_2 & 0 & 0 \end{bmatrix}.$$

Finally, we have

$$\mathbf{U}_{CR,1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \text{ and } \mathbf{U}_{BR,1} = \begin{bmatrix} 0 & 0 & -1 & 0 \end{bmatrix},$$

$$\mathbf{U}_{CR,2} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{U}_{BR,2} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

Then 10.10d may be given by the two matrix equations:

$$\begin{bmatrix} a_{71} & a_{72} & a_{73} & a_{74} \\ a_{51}-a_{71} & a_{52}-a_{72} & a_{53}-a_{73} & a_{54}-a_{74} \end{bmatrix} = \begin{bmatrix} 0 & h_0 & -f_0 & 0 \\ 0 & h_1 & -f_1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} a_{11}-a_{31} & a_{12}-a_{32} & a_{13}-a_{33} & a_{14}-a_{34} \end{bmatrix} = \begin{bmatrix} 0 & h_2 & 0 & f_2 \end{bmatrix}$$

The above system of equations is satisfied in R iff $\mathrm{Div}_R(0,2)$ iff R has characteristic 1 or 2. To prove this, suppose we have a solution. Then $a_{14} = 0$ by 10.10a and $a_{34} = 0$ by 10.10c, so $f_2 = a_{14} - a_{34} = 0$ by 10.10d. Therefore, $a_{21} - a_{41} - a_{61} = f_2 = 0$ by 10.10c. But $a_{21} = 0$ by 10.10a and $a_{41} = 0$ by 10.10b, so $a_{61} = 0$. Now $a_{11} - a_{31} - a_{61} = 0$ by 10.10b, so $a_{31} = a_{11} = 1$ by 10.10a. But then $f_0 = f_1 = a_{31} = 1$ using 10.10c. Finally, $a_{73} = -f_0$ and $a_{53} - a_{73} = -f_1$ by 10.10d, so $1 + 1 = f_0 + f_1 = -a_{53} = 0$ by the above and 10.10b.

Conversely, suppose R has characteristic 1 or 2. Then there is a solution of the above system of linear equations with the following variables set to 1:

a_{11}, a_{22}, a_{31}, a_{32}, a_{73}, e_2, f_0, f_1, g_0, g_1, g_2 and h_2.

All other variables are set to 0, including:

e_0, e_1, f_2, h_0, h_1 and the variables a_{ij} not listed above.

This completes the analysis of the example.

As previously described, we now obtain the general reduction of free word problems for $\mathcal{B}(R)$ to **Z**-image divisibility conditions.

10.12. Theorem. Let R be a ring with unit and suppose $d = d(x_1,x_2,\ldots,x_n)$ and $e = e(x_1,x_2,\ldots,x_n)$ are τ_B-polynomials. Then there exist integers $m \geq 0$ and $n \geq 1$, recursively computable from d and e, such that $d \leq e$ is satisfied in every additive relation algebra representable by an R-module if and only if $m\cdot 1$ divides $n\cdot 1$ in R.

Proof: We have $\mathcal{B}(R) \models d \leq e$ iff the matrix equations 10.10a,b,c,d have a solution in R. Systematically arrange all the matrix entries of **A**, **E**, and $\mathbf{F}_i$, $\mathbf{G}_i$ and $\mathbf{H}_i$ for each $i \leq n$, into a single vector $Z = \langle z_1,z_2,\ldots,z_k\rangle$ of variables. Then recursively compute a matrix M, in $\mathfrak{M}_{m,k}(\mathbf{Z})$ say, and a vector V in $\mathfrak{M}_{m,1}(\mathbf{Z})$ such that the equations 10.10a,b,c,d are solvable for R iff there exists Z in $\mathfrak{M}_{k,1}(R)$ such that $M_R Z = V_R$. Applying 9.14, recursively compute integers $m \geq 0$ and $n \geq 1$ so that $M_R Z = V_R$ for some such Z iff $\mathrm{Div}_R(m,n)$ is true. Then $\mathcal{B}(R) \models d \leq e$ iff $\mathrm{Div}_R(m,n)$. ■

Computer programs for the free word problem algorithm are in Appendix F. There are also many theoretical consequences of Theorem 10.12. Using 4.10 with 10.12 extends the result to the quasivarieties $\mathcal{A}(R)$ and $\mathcal{L}(R)$.

10.13. Corollary. Let R be a ring with unit. For any τ_A-polynomials d and e, there exist $m \geq 0$ and $n \geq 1$, recursively computable from d and e, such that $\mathcal{A}(R) \models d \leq e$ iff $\mathrm{Div}_R(m,n)$. Similar results hold for $\mathcal{L}(R)$ if d and e are lattice polynomials.

Recall that for any quasivariety of algebras $\mathcal{V}$, the class $H\mathcal{V}$ of homomorphic images of algebras in $\mathcal{V}$ is the smallest variety containing $\mathcal{V}$. Since a variety is determined by algebraic equations, it follows from 9.7b, 10.12 and 10.13 that $\mathbf{H}\mathcal{B}(R)$, $\mathbf{H}\mathcal{A}(R)$ and $\mathbf{H}\mathcal{L}(R)$ are completely determined by the **Z**-image divisibility pattern for R, that is by $\langle \mathrm{zchar}_R, \mathrm{dgr}_R \rangle$ in $\mathcal{J}$, for each ring R. We show next that the class of such varieties $\mathbf{H}\mathcal{B}(R)$ forms a lattice isormorphic to $\mathcal{J}$, and similarly for the varieties $\mathbf{H}\mathcal{A}(R)$ and $\mathbf{H}\mathcal{L}(R)$. In particular, $\mathbf{H}\mathcal{L}(R) \neq \mathbf{H}\mathcal{L}(S)$ for rings R and S such that $\langle \mathrm{zchar}_R, \mathrm{dgr}_R \rangle$ and $\langle \mathrm{zchar}_S, \mathrm{dgr}_S \rangle$ are different. (By 4.10, $\mathbf{H}\mathcal{L}(R) \neq \mathbf{H}\mathcal{L}(S)$ implies $\mathbf{H}\mathcal{A}(R) \neq \mathbf{H}\mathcal{A}(S)$ and $\mathbf{H}\mathcal{B}(R) \neq \mathbf{H}\mathcal{B}(S)$.) Given any $m \geq 0$ and $n \geq 1$, we contruct a lattice equation satisfied throughout $\mathbf{H}\mathcal{L}(R)$ iff $\mathrm{Div}_R(m,n)$ is true. C. Herrmann and A. Huhn [ ] gave such lattice identities discriminating ring characteristic and invertible primes (as in 9.4d). This method was extended in two ways in [TISL] to obtain all the needed equations; we simplify one of these ways to obtain the version below.

10.14. Definitions. Specify lattice polynomials on x_1, x_2, x_3, x_4 as follows:

$$d = (x_1 \vee x_2) \wedge (x_3 \vee x_4),$$

and by recursion on k, f_k for $k \geq 0$ given by

$$f_0 = x_2,$$

$$f_{k+1} = (((f_k \vee d) \wedge (x_1 \vee x_3)) \vee x_4) \wedge (x_2 \vee x_3).$$

For $m \geq 0$ and $n \geq 1$, define the lattice equation

$$\Delta_L(m,n): \quad d \leq x_1 \vee ((f_m \vee d) \wedge (x_1 \vee x_3)) \vee f_n.$$

(The inclusion $d \leq e$ represents the equation $d = d \wedge e$ here.)

In some cases, free word problems $d \leq e$ are most efficiently solved by direct computation of 10.7b. We use this method below.

10.15. Proposition. Suppose R is a ring with unit, $m \geq 0$ and $n \geq 1$. Then $\Delta_L(m,n)$ is satisfied throughout $\mathbf{H}\mathcal{L}(R)$ iff $\mathrm{Div}_R(m,n)$ is true.

Proof: Assuming the hypotheses, compute $\mathbf{w}_B(d)$ for $d = (x_1 \vee x_2) \wedge (x_3 \vee x_4)$. Let $a_i = \kappa(b_i)$ for $i \leq \ell(d) = 6$. Since $N = 0$, R^6/N is a free R-module on

$\{a_1, a_2, a_3, a_4, a_5, a_6\}$. Furthermore, we have:

$$y_1 = R\langle a_3, a_4\rangle, \qquad y_3 = R\langle a_5, a_6\rangle,$$

$$y_2 = R\langle a_1 - a_3,\ a_2 - a_4\rangle, \qquad y_4 = R\langle a_1 - a_5,\ a_2 - a_6\rangle.$$

For each element of $d(y_1, y_2, y_3, y_4)$, there exist r_1, r_2, r_3, r_4 in R such that

$$r_1a_3 + r_2(a_1 - a_3) = r_3a_5 + r_4(a_1 - a_5),$$

hence $r_2 = r_4$, $r_1 - r_2 = 0$ and $r_3 - r_4 = 0$, since a_1, a_3 and a_5 are free generators of R^6/N. Therefore,

$$d(y_1, y_2, y_3, y_4) = R\langle a_1, a_2\rangle.$$

By induction on $k \geq 0$, similar arguments show that:

$$f_k(y_1, y_2, y_3, y_4) = R\langle a_1 - a_3 - ka_5,\ a_2 - a_4 - ka_6\rangle \text{ and}$$

$$(f_k(y_1, y_2, y_3, y_4) \vee d(y_1, y_2, y_3, y_4)) \wedge (y_1 \vee y_3) = R\langle a_3 + ka_5,\ a_4 + ka_6\rangle.$$

By 4.10 and 10.7, $\mathbf{H}\mathcal{L}(R) \mid d \leq e$ iff $\langle a_1, a_2\rangle \in e = y_1 \vee ((f_m \vee d) \wedge (y_1 \vee y_3)) \vee f_n$. But $\langle a_1, a_2> \in e(y_1, y_2, y_3, y_4)$ implies that

$$a_1 \in Ra_3 + R(a_3 + ma_5) + R(a_1 - a_3 - na_5).$$

Hence $a_1 = t_1a_3 + t_2(a_3 + ma_5) + t_3(a_1 - a_3 - na_5)$ for some t_1, t_2, t_3 in R, from which it follows that $t_3 = 1$, $t_1 + t_2 - t_3 = 0$ and $mt_2 - nt_3 = 0$. But such t_1, t_2 and t_3 exist in R iff $\mathrm{Div}_R(m,n)$ is true, and then $\langle a_1, a_2\rangle$ equals

$$t_1\langle a_3, a_4\rangle + t_2\langle a_3 + ma_5,\ a_4 + ma_6\rangle + t_3\langle a_1 - a_3 - na_5,\ a_2 - a_4 - na_6\rangle,$$

and so is in $e(y_1, y_2, y_3, y_4)$. Therefore, $\Delta_L(m,n)$ is satisfied in all members of $\mathbf{H}\mathcal{L}(R)$ iff $\mathrm{Div}_R(m,n)$ is true. ∎

10.15. Corollary. Suppose R and S are rings with unit. Then $\mathbf{H}\mathcal{L}(R) \subseteq \mathbf{H}\mathcal{L}(S)$ iff $\mathbf{H}\mathfrak{A}(R) \subseteq \mathbf{H}\mathfrak{A}(S)$ iff $\mathbf{H}\mathfrak{B}(R) \subseteq \mathbf{H}\mathfrak{B}(S)$ iff $\langle \mathrm{zchar}_R,\ \mathrm{dgr}_R\rangle \leq \langle \mathrm{zchar}_S,\ \mathrm{dgr}_S\rangle$. In particular, $\mathbf{H}\mathcal{L}(R) = \mathbf{H}\mathcal{L}(S)$ iff $\mathbf{H}\mathfrak{A}(R) = \mathbf{H}\mathfrak{A}(S)$ iff $\mathbf{H}\mathfrak{B}(R) = \mathbf{H}\mathfrak{B}(S)$ iff $\langle \mathrm{zchar}_R,\ \mathrm{dgr}_R\rangle = \langle \mathrm{zchar}_S,\ \mathrm{dgr}_S\rangle$.

Identities simpler than $\Delta_L(m,n)$ are available to discriminate Z-image divisibility conditions for varieties of additive relation algebras.

10.16. Definitions and Properties. For f in any additive relation algebra (with or without unit), recursively define $n\cdot f$ for integers n by $0\cdot f = f+(-f)$, $(n+1)\cdot f = n\cdot f+f$ if $n > 0$ and $n\cdot f = -|n|\cdot f$ if $n < 0$.

Suppose $m \geq 0$ and $n \geq 1$. Define equations of τ_B-polynomials as shown below. (Again, $d \leq e$ represents $d = d\wedge e$.)

$\Delta_B(m,n)$: $n\cdot 1 \leq \mathbf{I}(m\cdot 1)$ (contains only constants).

Now let $e = x_1x_1^{\#}$ and $z = (e+(-e))(e+(-e))^{\#}$, and define equations of τ_A-polynomials as shown below.

$\Delta_A(m,n)$: $n\cdot e \leq z(m\cdot e)$ (contains only x_1).

10.16a. For integers j and k, $(j+k)\cdot f = j\cdot f+k\cdot f$.

10.16b. $\Delta_B(m,n)$ is satisfied throughout $\mathbf{H}\mathcal{B}(R)$ iff $\mathrm{Div}_R(m,n)$ is true. (Check that $\mathrm{Div}_R(m,n)$ implies that $\Delta_B(m,n)$ is satisfied in Rel(M) for M an R-module, and that $\Delta_B(m,n)$ satisfied in $\mathrm{Rel}({}_RR)$ implies $\mathrm{Div}_R(m,n)$.)

10.16c. $\Delta_A(m,n)$ is satisfied throughout $\mathcal{A}(R)$ iff $\mathrm{Div}_R(m,n)$. (Take $x_1 = 1$ in Rel(M), then apply 3.11 and 10.16b.)

To conclude the analysis, we compare the lattice operations in $\mathcal{J}$ with the corresponding lattice operations in the lattices of all varieties of algebraic types τ_L, τ_A and τ_B.

10.17. Definitions. Roughly speaking, $\mathbf{L}(\tau)$ denotes the lattice of all varieties of algebras of a fixed algebraic type τ, ordered by inclusion. (In order to avoid foundational difficulties, $\mathbf{L}(\tau)$ is formally defined to be the order dual of the lattice of all fully invariant congruences on $P(X,\tau)$; see [ ].)

Let $\mathcal{J}_L \subseteq \mathbf{L}(\tau_L)$, $\mathcal{J}_A \subseteq \mathbf{L}(\tau_A)$ and $\mathcal{J}_B \subseteq \mathbf{L}(\tau_B)$ be given by:

$\mathcal{J}_L = \{\mathbf{H}\mathcal{L}(R)\text{: R a ring with unit}\}$,

$\mathcal{J}_A = \{\mathbf{H}\mathcal{A}(R)\text{: R a ring with unit}\}$,

$\mathcal{J}_B = \{\mathbf{H}\mathcal{B}(R)\text{: R a ring with unit}\}$.

Note that $\mathcal{J}_L$, $\mathcal{J}_A$ and $\mathcal{J}_B$ are partially ordered by inclusion.

10.18. Proposition. There is a lattice isomorphism between $\mathcal{J}_L$ and $\mathcal{J}$ given by $H\mathcal{L}(R) \longmapsto \langle zchar_R, dgr_R \rangle$, and $\mathcal{J}_A$ and $\mathcal{J}_B$ are complete distributive lattices isomorphic to $\mathcal{J}$ similarly. In $L(\tau_L)$, $\mathcal{J}_L$ is a subsemilattice admitting finite and infinite joins, and similarly for $\mathcal{J}_A$ in $L(\tau_A)$ and $\mathcal{J}_B$ in $L(\tau_B)$. However, $\mathcal{J}_L$ is not a meet subsemilattice of $L(\tau_L)$, and similarly for $\mathcal{J}_A$ and $L(\tau_A)$.

Proof: The first part follows from 10.15. Given a family $\{R_j\}_{j \varepsilon J}$ of rings with unit, let $R = \prod_{j \varepsilon J} R_j$. From 10.12, we can easily verify that $H\mathcal{B}(R)$ is the join of all the varieties $H\mathcal{B}(R_j)$, $j \in J$, since the identities satisfied throughout $H\mathcal{B}(R)$ are just those identities satisfied in $H\mathcal{B}(R_j)$ for all $j \in J$. Therefore, $\mathcal{J}_B$ is a complete join subsemilattice of $L(\tau_B)$. By 10.13, $\mathcal{J}_A$ is a complete join subsemilattice of $L(\tau_A)$, and similarly for $\mathcal{J}_L$ and $L(\tau_L)$.

Clearly $\langle \mathbf{0}, expt_1 \rangle = \langle \mathbf{0}, expt_2 \rangle \wedge \langle \mathbf{0}, expt_3 \rangle$ in $\mathcal{J}$. However, $S(\mathbf{0}, expt_1)$ is a trivial ring, so $H\mathcal{L}(S(\mathbf{0}, expt_1))$ is the variety of trivial lattices. But $H\mathcal{L}(S(\mathbf{0}, expt_2)) \cap H\mathcal{L}(S(\mathbf{0}, expt_3))$ contains nontrivial lattices, for example, all distributive lattices. This proves that $\mathcal{J}_L$ is not a meet subsemilattice of $L(\tau_L)$. Using D^2 as in 3.14 and 3.15 for D any nontrivial distributive lattice, we can show similarly that the meet of $H\mathcal{A}(S(\mathbf{0}, expt_2))$ and $H\mathcal{A}(S(\mathbf{0}, expt_3))$ in $L(\tau_A)$ is not in $\mathcal{J}_A$. ■

It is unknown whether or not $\mathcal{J}_B$ is a meet subsemilattice of $L(\tau_B)$.