§11. Extensions of Solvability by Coretracts of Free Algebras.

By 8.3, there is no recursive solution for all finitely presented word problems, for any nontrivial quasivariety $\mathcal{L}(R)$, $\mathcal{A}(R)$ or $\mathcal{B}(R)$. However, there might still be recursive solutions for useful subsets of the general problem, extending beyond the methods already given. In this section, we describe a known standard method using coretracts of free algebras (sometimes called *projective* algebras) to extend word problem solvability. If the divisibility of $\mathbf{Z}$-images is computable for R, these extensions yield classes of solvable finitely presented word problems for $\mathcal{L}(R)$, $\mathcal{A}(R)$ and $\mathcal{B}(R)$, based on the solvability of free word problems. Using members of $\mathcal{A}(R)$ called *tower* algebras, free algebra coretracts also provide solutions of certain diagram-chasing problems for the additive relation category R-**Rel**.

11.1. Example. Let R be a nontrivial ring with $\mathrm{Div}_R(m,n)$ a computable predicate, and let Ψ be the open conjunctive formula

$$(x_1 \wedge x_2 = x_1 \wedge x_3) \land (x_1 \wedge x_2 = x_2 \wedge x_3) \land (x_1 \vee x_2 = x_1 \vee x_3) \land$$
$$(x_1 \vee x_2 = x_2 \vee x_3) \land (x_1 \wedge x_2 \le x_4) \land (x_4 \le x_1 \vee x_2).$$

Note that Ψ is equivalent to asserting that x_1, x_2 and x_3 generate an $\mathbf{M}_3$ sublattice spanning $[a,b]$ for $a = x_1 \wedge x_2 \wedge x_3$ and $b = x_1 \vee x_2 \vee x_3$ (except in the trival case $a = b$), and that x_4 is in $[a,b]$. Suppose that we wanted to study the consequences of Ψ for the quasivariety of additive relation algebras $\mathcal{A}(R)$. For τ_A-polynomials p and q on the set of variables $X_4 = \{x_1,x_2,x_3,x_4\}$, the universal Horn sentences $\Lambda(p,q)$ with the fixed set of hypotheses Ψ has the form

$$(\forall x_1,x_2,x_3,x_4)(\Psi \Rightarrow (p(x_1,x_2,x_3,x_4) = q(x_1,x_2,x_3,x_4))).$$

Now let A be the free $\mathcal{A}(R)$-algebra on the generating set X_4, and define a unique τ_A-homomorphism $\lambda: A \longrightarrow A$ by

$$a_0 = (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3),$$
$$b_0 = (x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (x_2 \vee x_3) \text{ and}$$
$$\lambda(x_i) = (a_0 \vee x_i) \wedge b_0 \text{ for } i = 1,2,3,4.$$

Using only modular lattice properties, we can verify that Ψ is equivalent to the conjunction of the formulas $x_i = \lambda(x_i)$ for $i \le 4$, so that $\Lambda(p,q)$ is equivalent for $\mathfrak{A}(R)$ to the universal closure $\Phi(p,q)$ of the open formula

$$(x_1 = \lambda(x_1)) \ \&\ (x_2 = \lambda(x_2)) \ \&\ (x_3 = \lambda(x_3)) \ \&\ (x_4 = \lambda(x_4)) \Rightarrow (p(x_1,x_2,x_3,x_4) = q(x_1,x_2,x_3,x_4)).$$

Now λ has a convenient special property: $\lambda = \lambda\lambda$ can be shown by calculations proving $\lambda(\lambda(x_i)) = \lambda(x_i)$ for $i \le 4$. Using $\lambda = \lambda\lambda$, we see that $\Phi(p,q)$ is true in A iff the universal closure of the identity

$$p(\lambda(x_1),\lambda(x_2),\lambda(x_3),\lambda(x_4)) = q(\lambda(x_1),\lambda(x_2),\lambda(x_3),\lambda(x_4))$$

is true in A. But an identity in A corresponds to a free word problem for $\mathfrak{A}(R)$, which we know how to solve by §10. Therefore, we can compute whether or not $\Lambda(p,q)$ is satisfied throughout $\mathfrak{A}(R)$ for any τ_A-polynomials p and q on the variables of X_4.

The argument above can be put into a general framework using coretracts of free algebras.

11.2. Definition and Properties. Suppose τ is an algebraic type, and recall that $P(X_n,\tau)$ denotes the τ-algebra of all τ-polynomials on the generating set $X_n = \{x_1,x_2,\dots,x_n\}$, $n \ge 1$. Let $\mathbf{W}$ be a quasivariety of τ-algebras, W a free $\mathbf{W}$-algebra on the generating set X_n, and $\eta: P(X_n,\tau) \longrightarrow W$ the τ-homomorphism such that $\eta(x_i) = x_i$ for $i \le n$. An n-tuple $\langle u_1,u_2,\dots,u_n\rangle$ of elements of $P(X_n,\tau)$ is called a *free coretract sequence* for $\mathbf{W}$ if the unique τ-homorphism $\lambda: W \longrightarrow W$ such that $\lambda(x_i) = \eta(u_i)$ for $i \le n$ is idempotent: $\lambda = \lambda\lambda$. We say that $\langle u_1,u_2,\dots,u_n\rangle$ *determines* λ.

If Ψ is an open conjunctive formula such that each term of Ψ is an equation p = q for p and q in $P(X_n,\tau)$, say that Ψ is *compatible* with λ (or with $\langle u_1,u_2,\dots,u_n\rangle$ determining λ as above) if for any assignment of the variables in X_n to a member of $\mathbf{W}$, Ψ is satisfied if and only if $x_i = \lambda(x_i)$ for $i = 1,2,\dots,n$.

Recall that X is a *coretract* of W if there exist τ-homomorphisms $\iota: X \longrightarrow W$

and $\kappa: W \longrightarrow X$ such that $\iota\kappa = 1_X$. Note that κ must be onto and ι must be one-one.

11.2a. If X is any coretract of $\mathbb{W}$ via $\kappa: W \longrightarrow X$ and $\iota: X \longrightarrow W$, then $\lambda = \kappa\iota$ is an idempotent τ-homomorphism $W \longrightarrow W$, and a τ-isomorphism of X to $\lambda[W]$ is induced by ι. If $\lambda: W \longrightarrow W$ is any idempotent, then $\lambda[W]$ is a coretract of W via $\kappa: W \longrightarrow \lambda[W]$ induced by λ and the inclusion map $\iota: \lambda[W] \longrightarrow W$. For such a λ, any n-tuple $\langle u_1, u_2, \ldots, u_n \rangle$ on $P(X_n, \tau)$ such that $\nu(u_i) = \lambda(x_i)$ for $i \le n$ is a free coretract sequence. If $\langle u_1, u_2, \ldots, u_n \rangle$ is any free coretract sequence for $\mathbb{W}$, then the set $\{\eta(u_i): i \le n\}$ generates a coretract of the free $\mathbb{W}$-algebra W.

11.2b. An n-tuple $\langle u_1, u_2, \ldots, u_n \rangle$ on $P(X_n, \tau)$ is a free coretract sequence for $\mathbb{W}$ iff

$$\eta(u_i(x_1, x_2, \ldots, x_n)) = \eta(u_i(u_1(x_1, x_2, \ldots, x_n), \ldots, u_n(x_1, x_2, \ldots, x_n)))$$

for $i = 1, 2, \ldots, n$.

11.2c. A coretract of a coretract of W is a coretract of W. If $\lambda: W \longrightarrow W$ and $\mu: \lambda[W] \longrightarrow \lambda[W]$ are idempotent τ-homomorphisms and an n-tuple $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ on $P(X_n, \tau)$ satisfies $\eta(u_i) = \mu(\lambda(x_i))$ for $i \le n$, then $\mathbf{u}$ is a free coretract sequence for $\mathbb{W}$, with associated coretract $\mu[\lambda[W]]$ of W.

11.2d. Suppose $\mathbb{W}$ and $\mathbb{W}'$ are quasivarieties of τ-algebras, with $\mathbb{W}' \subseteq \mathbb{W}$. If $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ is a free coretract sequence for $\mathbb{W}$, then it is a free coretract sequence for $\mathbb{W}'$. If Ψ is an open conjunction of τ-equations which is compatible with $\mathbf{u}$ for $\mathbb{W}$, then Ψ is compatible with $\mathbf{u}$ for $\mathbb{W}'$.

11.2e. Suppose $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ is a free coretract sequence for a quasivariety of τ-algebras $\mathbb{W}$, and Ψ is an open conjunctive formula compatible with $\mathbf{u}$. For any p and q in $P(X_n, \tau)$, the universal Horn sentence

$$(\forall x_1, x_2, \ldots, x_n)(\Psi \Rightarrow p(x_1, x_2, \ldots, x_n) = q(x_1, x_2, \ldots, x_n))$$

is satisfied in all members of $\mathbb{W}$ iff the identity

$$p(u_1(x_1, \ldots, x_n), \ldots, u_n(x_1, \ldots, x_n)) = q(u_1(x_1, \ldots, x_n), \ldots, u_n(x_1, \ldots, x_n))$$

is satisfied in all members of $\mathbb{W}$.

Given Ψ and $\mathbf{W}$ as in 11.2e such that all free word problems for $\mathbf{W}$ are solvable, we can seek a free coretract sequence **u** for $\mathbf{W}$ such that Ψ is compatible with **u**. If we find such a **u**, then we can solve all word problems with hypothesis Ψ and conclusion an arbitrary equation $p = q$ on $P(X_n, \tau)$. The results 11.2a,b,c,d may assist in verifying the hypotheses of 11.2e.

11.3. (Techniques and examples for free coretract method.)

In order to extend our methods to diagram-chasing problems for R-**Rel**, we make use of the flexibility of τ_A-subalgebras. Suppose M_0 and M_1 are distinct R-modules, and $M = M_0 \oplus M_1$. Identify M_0 with $M_0 \oplus 0$ and M_1 with $0 \oplus M_1$ in Su(M), as usual. Obviously, $\mathrm{Rel}(M_0)$ is τ_B-isomorphic to the interval $I_{00} = [\mathbf{0}, M_0 \oplus M_0]$ of Rel(M), as is $\mathrm{Rel}(M_1)$ to $[\mathbf{0}, M_1 \oplus M_1]$. However, we can also take M_1 *above* M_0 in Rel(M) by using the τ_B-isomorphism of $\mathrm{Rel}(M_1)$ to $I_{11} = [M_0 \oplus M_0, M \oplus M]$, since $M/M_0 \approx M_1$. Then elements of I_{00} correspond to morphisms $M_0 \longrightarrow M_0$ in R-**Rel**, and similarly elements of I_{11} correspond to morphisms $M_1 \longrightarrow M_1$. Furthermore, morphisms $M_0 \longrightarrow M_1$ in R-**Rel** can be regarded as elements of $I_{01} = [0 \oplus M_0, M_0 \oplus M]$, and similarly for morphisms $M_1 \longrightarrow M_0$ and $I_{10} = [M_0 \oplus 0, M \oplus M_0]$. Therefore, the full subcategory $\mathcal{C}_{0,1}$ of R-**Rel** for the objects $\{M_0, M_1\}$ corresponds to the set union $U = I_{00} \cup I_{11} \cup I_{01} \cup I_{10}$. The key point: U is a τ_A-subalgebra of Rel(M), and the operations of U extend those of $\mathcal{C}_{0,1}$.

It is sometimes convenient to force the intervals I_{ij} to be disjoint. This can be done using a nonzero module K for spacing: Redefine $M = M_0 \oplus M_1 \oplus K$ and let $N = M_0 \oplus 0 \oplus K$, so $M/N \approx M_1$. Then redefine U to be the set union of $[\mathbf{0}, M_0 \oplus M_0]$, $[N \oplus N, M \oplus M]$, $[0 \oplus N, M_0 \oplus M]$ and $[N \oplus 0, M \oplus M_0]$, which are pairwise disjoint intervals. Again U is a τ_A-subalgebra of Rel(M), with operations extending those of U.

In general, the full subcategory of R-**Rel** determined by an indexed set $\{M_j\}_{j \varepsilon J}$ of R-modules can be realized as an appropriate τ_A-subalgebra A of Rel(M), where M is a direct sum of the modules M_j, $j \in J$, plus spacing modules as needed. We call A a 'tower' algebra, because a total ordering of

$\{M_j\}_{j\varepsilon J}$ must be provided in order to define it. The forced choice of such an ordering can lead to certain irrelevant complications. On the other hand, this wholly algebraic approach eliminates the need for model theory predicates to treat the partial operations of an additive relation category, like the type τ_{RC} of §6. In particular, certain diagram-chasing properties of R-**Rel** can be computed by the free algebra coretract method for $\mathfrak{a}(R)$.

11.4. Definitions and Properties. Let A be an additive relation algebra. Symmetric idempotents d and e of A are said to be *separated* if $\mathbf{q}(d) \le \mathbf{p}(e)$ or $\mathbf{q}(e) \le \mathbf{p}(d)$ (3.9), and to be *strictly separated* if $\mathbf{q}(d) < \mathbf{p}(e)$ or $\mathbf{q}(e) < \mathbf{p}(d)$. A *tower* of A is a set T of symmetric idempotents of A such that any two distinct elements of T are separated; T is a *strict tower* if any two elements are strictly separated. An element f of A is called T-*admissible* if there exist elements d and e in T such that f = df = fe. (Recall that rel(d,e) in 3.16 denotes the set of f in A such that f = df = fe, so f is T-admissible iff f is in rel(d,e) for some d and e in T.) If T is a tower of A and every element of A is T-admissible, then A is called a *tower algebra*, with *spanning tower* T. If A has a strict spanning tower, then it is called a *strict tower algebra*.

11.4a. If T is a tower of A, then T is a chain (totally ordered subset) of A. (By 3.9f, $\mathbf{p}(e) \le e \le \mathbf{q}(e)$ for e a symmetric idempotent.)

11.4b. If f is T-admissible for a strict tower T of A, then there exist *unique* elements d and e of T such that f is in rel(d,e). (Suppose cf = f = df for c and d in T. If c ≠ d, we can suppose $\mathbf{q}(c) < \mathbf{p}(d)$. Using 3.9a,f:

$$ff^{\#} = cff^{\#}c \le \mathbf{q}(c)ff^{\#}\mathbf{q}(c) = \mathbf{q}(c) < \mathbf{p}(d) = \mathbf{p}(d)ff^{\#}\mathbf{p}(d) \le dff^{\#}d = ff^{\#},$$

which is a contradiction. The uniqueness of e is dual.)

11.4c. A strict tower algebra has a unique strict spanning tower. (If T and T′ are strict spanning towers for A and c is in T, then there exists d in T′ such that c = cd, and e in T such that d = ed. It follows from 3.10i that c and e are not strictly separated, hence c = d = e.)

11.4d. A tower algebra has a unit iff it has a singleton strict spanning tower. (Clearly, u is a unit iff $\{u\}$ is a strict spanning tower.)

11.5. Proposition. Suppose T is a tower of an additive relation algebra A. Then the set U of T-admissible elements of A is a τ_A-subalgebra of A containing T, and U is a tower algebra with spanning tower T.

Proof: Assume the hypotheses, and suppose f and g are in U. Obviously, $f^{\#}$ is in U by 3.1d, $-f$ is in U by 3.1e, and fg is in U by associativity. Suppose $f = bf$ and $g = cg$ for b and c in T, and $b \le c$. Choose symmetric null y and z such that b, c, f and g are in [y,z] by 3.10g. Then $by \le bf = f \le bz$ and $by \le cg = g$, so $by \le f \wedge g \le bz$. Then $b(f \wedge g) = f \wedge g$ by 3.10e, and we can find d in T with $(f \wedge g)d = f \wedge g$ similarly. Therefore, $f \wedge g$ is T-admissible, and dual arguments show that $f \vee g$ is T-admissible. Using 3.4d, 3.7 and 3.8,

$$by = by + by \le bf + cg = f + g \le (f+g)z = fz \wedge gz \le fz = bfz \le bz.$$

But then $b(f+g) = f+g$, and dual arguments show that $(f+g)e = f+g$ for a suitable e in T. Then $f+g$ is in U, and U is a τ_A-subalgebra of A. Clearly $T \subseteq U$, so U is a tower algebra with spanning tower T. ∎

We now construct an almost strongly exact relation category from any τ_A-subalgebra A, with symmetric idempotents as objects and morphism sets isomorphic to rel(d,e). (See 3.16, and compare $\tilde{\mathcal{C}}$ in 5.8 and 5.9.)

11.6. Definitions and Properties. Suppose A is an additive relation algebra and V is a nonempty set of symmetric idempotents of A. Let $\mathbf{C}_V(A)$ denote the system with additive relation category structures such that V is the set of objects of $\mathbf{C}_V(A)$, morphisms $d \longrightarrow e$ are triples $\langle d,f,e \rangle$ such that f is in rel(d,e), and composition, converse, sum, negative, meet and join are defined as in 5.8, and $1_d = \langle d,d,d \rangle$, $0_{de} = \langle d, \mathbf{q}(d)\mathbf{p}(e), e \rangle$, $\mathbf{O}_{de} = \langle d, \mathbf{p}(d)\mathbf{p}(e), e \rangle$ and $\mathbf{I}_{de} = \langle d, \mathbf{q}(d)\mathbf{q}(e), e \rangle$.

11.6a. $\mathbf{C}_V(A)$ is an almost strongly exact relation category. If $h: A \longrightarrow B$ is a τ_A-homomorphism and $W = \{h(d)\colon d \in V\}$, then W is a set of symmetric idempotents of B and h induces a τ_{RC}-functor $H: \mathbf{C}_V(A) \longrightarrow \mathbf{C}_W(B)$ given by

$H(b) = h(b)$ and $H\langle b,f,c\rangle = \langle h(b),h(f),h(c)\rangle$.

11.6b. If e is in V, then e is a zero object of $\mathbf{C}_V(A)$ iff e is null in A.

11.6c. If A has a unit u and V is the set of all symmetric idempotents of A, then $\mathbf{C}_V(A)$ equals $\tilde{A}$.

11.7. Proposition. Suppose T is a totally ordered set of R-modules, for R a nontrivial ring. Then there exists a strict tower algebra A in $\mathcal{Q}(R)$, with strict spanning tower V that is order isomorphic to T, such that $\mathbf{C}_V(A)$ is τ_{RC}-isomorphic to the small full subcategory $\mathcal{C}_T$ of R-**Rel** determined by the R-modules in T.

Proof: Assume the hypotheses, and let M be the direct sum of R-modules $N \oplus R_N$ ($R_N \approx {}_R R$ is a spacing module) for all N in T. For each N in T, define N_0 in Su(M) to be the join of all $K \oplus R_K$ for $K < N$ in T, interpreting $K \oplus R_K$ as a submodule of M as usual. Similarly, define $N_1 = N_0 \vee N$ in Su(M), so $N_1/N_0 \approx N$, and let e_N be the symmetric idempotent of Rel(M) containing pairs $\langle x,y\rangle$ such that x and y are in N_1 and $x - y$ is in N_0 (3.4h). Verify that $V = \{e_N\colon N \in T\}$ is a strict tower of Rel(M) order isomorphic to T by $N \longmapsto e_N$, and $\mathbf{C}_V(A)$ is isomorphic to $\mathcal{C}_T$ for A the set of V-admissible elements of Rel(M). ∎

If two R-modules of T always have a nonzero module of T strictly between them, then the spacing modules may be omitted above.

We may be unable to determine the structure of A from $\mathbf{C}_V(A)$, even if V contains all the symmetric idempotents of A.

11.8. Example. Suppose A contains only null elements, so $A \approx L^2$ where L is the lattice of symmetric (null) idempotents of A, by 3.15. Then $\mathbf{C}_L(A)$ is an almost exact additive relation category with $|L|$ zero objects by 11.6b. But L may be any lattice of $\mathcal{L}(R)$ (or even any modular lattice) by 3.14a, so that the lattice structure of L is not recoverable from the τ_{RC}-structure of $\mathbf{C}_L(A)$.

If A has a unit u, then A is recoverable from $\mathbf{C}_V(A)$ if u is in V, since $A = \mathrm{rel}(u,u)$. More generally, suppose A is a tower algebra with spanning tower T. We show next that the structure of A can be recovered from $\mathbf{C}_T(A)$ plus the total order on the chain T, using the recovery formulas below.

11.9. Definitions. Suppose A is a tower algebra with spanning tower T, with f in rel(b,c) and g in rel(d,e) for b, c, d and e in T. Let $\mathbf{f}$ equal $\langle b,f,c\rangle$ and $\mathbf{g} = \langle d,g,e\rangle$ be the morphisms of $\mathbf{C}_T(A)$ corresponding to f and g. In the tables below, each table entry is a morphism of $\mathbf{C}_T(A)$, which can be used to determine a τ_A-operation in A when the elements of T are ordered as shown above the table entry. The pair below the table entry shows the domain and codomain of the table entry in $\mathbf{C}_T(A)$. For example, the first entry in the sum table represents the equation $\langle b,h,e\rangle = \mathbf{f}0_{ce} + 0_{bd}\mathbf{g}$ in $C_T(A)$ if $h = f+g$ in A and $b < d$ and $c < e$ in T. Each τ_A-operation of A is called T-*standard* if it agrees with the given table.

11.9a. Sum is T-standard in A if f + g is specified by the table:

$b<d$, $c<e$	$b<d$, $c=e$	$b<d$, $c>e$	$b=d$, $c<e$	$b=d$, $c=e$	$b=d$, $c>e$	$b>d$, $c<e$	$b>d$, $c=e$	$b>d$, $c>e$
$\mathbf{f}0_{ce}+0_{bd}\mathbf{g}$	$\mathbf{f}+0_{bd}\mathbf{g}$	$\mathbf{f}$	$\mathbf{f}0_{ce}+\mathbf{g}$	$\mathbf{f}+\mathbf{g}$	$\mathbf{f}+\mathbf{g}0_{ec}$	$\mathbf{g}$	$0_{db}\mathbf{f}+\mathbf{g}$	$0_{db}\mathbf{f}+\mathbf{g}0_{ec}$
$\langle b,e\rangle$	$\langle b,c\rangle$	$\langle b,c\rangle$	$\langle b,e\rangle$	$\langle b,c\rangle$	$\langle b,c\rangle$	$\langle d,e\rangle$	$\langle d,c\rangle$	$\langle d,c\rangle$

11.9b. Negative is T-standard in A if -f is specified by the table:

all cases
$-\mathbf{f}$
$\langle b,c\rangle$

11.9c. Composition is T-standard in A if fg is specified by the table:

$c<d$	$c=d$	$c>d$
$\mathbf{f}0_{cd}\mathbf{g}$	$\mathbf{f}\mathbf{g}$	$\mathbf{f}0^{\#}_{dc}\mathbf{g}$
$\langle b,e\rangle$	$\langle b,e\rangle$	$\langle b,e\rangle$

11.9d. Converse is T-standard in A if $f^{\#}$ is specified by the table:

all cases
$\mathbf{f}^{\#}$
$\langle c,b\rangle$

11.9e. Meet is T-standard in A if $f \wedge g$ is specified by the table:

$b < d$, $c < e$	$b < d$, $c = e$	$b < d$, $c > e$	$b = d$, $c < e$	$b = d$, $c = e$	$b = d$, $c > e$	$b > d$, $c < e$	$b > d$, $c = e$	$b > d$, $c > e$
$\mathbf{f}$	$\mathbf{f} \wedge 0_{bd}\mathbf{g}$	$\mathbf{f}0^{\#}_{ec} \wedge 0_{bd}\mathbf{g}$	$\mathbf{f} \wedge \mathbf{g}0^{\#}_{ce}$	$\mathbf{f} \wedge \mathbf{g}$	$\mathbf{f}0^{\#}_{ce} \wedge \mathbf{g}$	$0_{db}\mathbf{f} \wedge \mathbf{g}0^{\#}_{ce}$	$0_{db}\mathbf{f} \wedge \mathbf{g}$	$\mathbf{g}$
$\langle b,c\rangle$	$\langle b,c\rangle$	$\langle b,e\rangle$	$\langle b,c\rangle$	$\langle b,c\rangle$	$\langle b,e\rangle$	$\langle d,c\rangle$	$\langle d,c\rangle$	$\langle d,e\rangle$

11.9f. Join is T-standard in A if $f \vee g$ is specified by the table:

$b < d$, $c < e$	$b < d$, $c = e$	$b < d$, $c > e$	$b = d$, $c < e$	$b = d$, $c = e$	$b = d$, $c > e$	$b > d$, $c < e$	$b > d$, $c = e$	$b > d$, $c > e$
$\mathbf{g}$	$0^{\#}_{bd}\mathbf{f} \vee \mathbf{g}$	$0^{\#}_{bd}\mathbf{f} \vee \mathbf{g}0_{ec}$	$\mathbf{f}0_{ce} \vee \mathbf{g}$	$\mathbf{f} \vee \mathbf{g}$	$\mathbf{f} \vee \mathbf{g}0_{ec}$	$\mathbf{f}0_{ce} \vee 0^{\#}_{db}\mathbf{g}$	$\mathbf{f} \vee 0^{\#}_{db}\mathbf{g}$	$\mathbf{f}$
$\langle d,e\rangle$	$\langle d,c\rangle$	$\langle d,c\rangle$	$\langle b,e\rangle$	$\langle b,c\rangle$	$\langle b,c\rangle$	$\langle b,e\rangle$	$\langle b,c\rangle$	$\langle b,c\rangle$

This completes the T-standard operation tables.

11.10. Proposition. Suppose A is a tower algebra with spanning tower T. Then all the operations of A are T-standard. Therefore, the τ_A-structure of A is uniquely determined from $\mathbf{C}_T(A)$ and the ordering of T.

Proof: Assume the hypotheses, and suppose f is in rel(b,c) and g is in rel(d,e) for b, c, d and e in T. Let $\mathbf{f} = \langle b,f,d\rangle$ and $\mathbf{g} = \langle d,g,e\rangle$ in $\mathbf{C}_T(A)$. Clearly, $\langle b,-f,c\rangle = -\mathbf{f}$ and $\langle c,f^{\#},b\rangle = \mathbf{f}^{\#}$ in all cases, proving that negative and converse are T-standard.

Suppose $c < d$, so $\mathbf{q}(c) \le \mathbf{p}(d)$ because c and d are separated. Using 3.9a and 3.16a,

$$fg = fgg^{\#}g \ge f\mathbf{p}(d)g \ge f\mathbf{q}(c)\mathbf{p}(d)g \ge f\mathbf{q}(c)g \ge ff^{\#}fg = fg.$$

Since $0_{cd} = \langle c,\ \mathbf{q}(c)\mathbf{p}(d),\ d\rangle$, $\langle b,fg,e\rangle = \mathbf{f}0_{cd}\mathbf{g}$ in $\mathbf{C}_T(A)$. If $c > d$, then $g^{\#}f^{\#} = g^{\#}\mathbf{q}(d)\mathbf{p}(c)f^{\#}$ by the same argument, so $\langle b,fg,e\rangle = \mathbf{f}0_{dc}{}^{\#}\mathbf{g}$ by taking converses. Since $\langle b,fg,e\rangle = \mathbf{fg}$ if $c = d$, composition is T-standard.

For sum, meet and join, observe that the cases with $b = d$ and $c = e$ follow from the definition of $\mathbf{C}_T(A)$. Furthermore, the cases with $b > d$ or with $b = d$ and $c > e$ can be obtained from other cases by relabelling, exchanging f with g, b with d and c with e. We give some sample cases for the twelve remaining equations. By 3.10g, we choose symmetric null y and z such that f, g, b, c, d and e are in [y,z].

Suppose b < d and c > e, so $\mathbf{q}(b) \le \mathbf{p}(d)$ and $\mathbf{q}(e) \le \mathbf{p}(c)$. Now $fzy \le \mathbf{q}(b)\mathbf{q}(c)zy = \mathbf{q}(b)y \le \mathbf{p}(d)\mathbf{p}(e)$ by 3.16a and 3.4c, and $\mathbf{q}(d)\mathbf{q}(e) \le zyf$ similarly, so by 3.10d and 3.16a we have

$$f = f + fzy \le f + \mathbf{p}(d)\mathbf{p}(e) \le f + g \le f + \mathbf{q}(d)\mathbf{q}(e) \le f + zyf = f.$$

This proves the third equation $\langle b, f+g, c\rangle = \langle b,f,c\rangle$ of 11.9a.

Suppose b < d and c = e, so $\mathbf{q}(b) \le \mathbf{p}(d)$. Then $f = ff^{\#}f \le \mathbf{q}(b)z$, so

$$f \wedge g = f \wedge \mathbf{q}(b)z \wedge g \le f \wedge \mathbf{q}(b)g \le f \wedge \mathbf{q}(b)\mathbf{p}(d)g \le f \wedge \mathbf{p}(d)g \le f \wedge g,$$

using 3.9a, 3.10e and 3.16a. Since $0_{bd} = \langle b,\ \mathbf{q}(b)\mathbf{p}(d),\ d\rangle$, this verifies the second equation $\langle b, f \wedge g, c\rangle = \langle b,\ f \wedge \mathbf{q}(b)\mathbf{p}(d)g,\ c\rangle = \mathbf{f} \wedge 0_{bd}\mathbf{g}$ for 11.9e.

Again suppose b < d and c > e. Then $f \ge y\mathbf{p}(c)$ and $g \ge \mathbf{p}(d)y$, so

$$f \vee g = f \vee y\mathbf{p}(c) \vee g \vee \mathbf{p}(d)y \ge \mathbf{p}(d)f \vee g\mathbf{p}(c) \ge f \vee g,$$

using arguments similar to the meet case above. But $\mathbf{p}(d)f \ge \mathbf{p}(d)\mathbf{q}(b)f \ge \mathbf{p}(d)ff^{\#}f = \mathbf{p}(d)f$ and similarly $g\mathbf{p}(c) = g\mathbf{q}(e)\mathbf{p}(c)$. Then the third equation $\langle d, f \vee g, c\rangle = 0^{\#}_{bd}\mathbf{f} \vee \mathbf{g}0_{ec} = \langle d,\ \mathbf{p}(d)\mathbf{q}(b)f \vee g\mathbf{q}(e)\mathbf{p}(c),\ c\rangle$ of 11.9f follows.

Routine calculations proving the remaining nine cases will be omitted; they complete the verification that sum, meet and join are T-standard. Clearly, every τ_A-operation of A is uniquely determined by the total order on T and the τ_{RC}-structure of $\mathbf{C}_T(A)$. ■

We showed above that small full subcategories of R-**Rel** can be represented by strict tower algebras in $\mathcal{Q}(R)$. More generally, there is a connection between small full subcategories $\mathcal{C}$ of strongly exact relation categories and strict tower algebras. Suppose we have such a $\mathcal{C}$, which may be any small almost strongly exact relation category by 5.2 and 5.9, and we choose a total ordering T of the set of objects of $\mathcal{C}$. In Chapter IV, we will show that the set of all morphisms of $\mathcal{C}$ can be made into a strict tower algebra A with spanning tower T, identifying each object B of C with its unit 1_B and using the T-standard formulas of 11.9 to define the operations of A. This construction is essentially reciprocal to the construction of T and $\mathbf{C}_T(A)$ from A.

(Use of tower algebras to do diagram-chasing in R-**Rel**.)