A Note On the Equational Theory of Modular Ortholattices

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today

ABSTRACT. We prove that every atomic modular ortholattice is in the variety generated by its finite dimensional members.

1 Introduction

An ortholattice, abbreviated OL, is an algebra \((L;+,\cdot,0,1)\) where \((L;+,\cdot,0,1)\) is a bounded lattice and \(\complement : L \to L\) is an orthocomplementation, ie. \(x + x' = 1\), \(x \cdot x' = 0\), and \(x \leq y\) implies \(y' \leq x'\), for all \(x,y \in L\). Since the last condition, in the presence of the other two, is equivalent to DeMorgan's laws \(((x + y) = x'\cdot y'\) and \((xy)' = x' + y')\), the class of ortholattices forms a variety. An OL, \(L\), is an orthomodular lattice, abbreviated OML, iff it satisfies the identity \(y(xy + y') = xy\).

This is a weak, or 'orthogonal', version of the modular law. An OML is a modular ortholattice, abbreviated MOL, iff it is modular. For background on these classes of algebras the reader is referred to [4], and for background on modular lattices to [2], for example.

The height of a modular lattice is the length of any maximal chain in the lattice. For our purposes this height is a non negative integer or \(\infty\), with \(n < \infty\) for all non negative integers \(n\). In, [1], Bruns made the following conjecture (stated slightly differently),

Conjecture 1 (Bruns' Conjecture). Every variety of MOLs which contains a subdirectly irreducible algebra of height greater than two, contains a subdirectly irreducible algebra of height 3.

A partial confirmation of Bruns' Conjecture is given in [5]. In this note we prove,

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1We will follow the common convention of usually writing the meet operation as juxtaposition, ie. \('x \cdot y'\) as \('xy'\).
Proposition 2 Every variety of MOLs which is generated by its atomistic members is already generated by its finite dimensional members.

We present this result as motivation for the following strengthening of Bruns' Conjecture,

Conjecture 3 Every variety of MOLs is generated by its finite dimensional members.

2 A Lemma

We begin with a lemma, the proof of which is essentially in the proof of Frink's Embedding Theorem, [3].

Lemma 4 Let $L$ be an atomic complemented modular lattice. Then, for any lattice polynomial $p = p(y_1, ..., y_n)$ and $b_i \in L$, $i = 1, ..., n$, if $0 < p(b_1, ..., b_n)$, then there exist $c_i \in L$, $i = 1, ..., n$, of finite height, so that $c_i \leq b_i$, $i = 1, ..., n$, and $0 < p(c_1, ..., c_n)$.

Proof. We actually prove the following stronger statement by induction on the length of the polynomial $p$.

If $a$ is an atom of $L$ and $a \leq p(b_1, ..., b_n)$, then there exist $c_i \leq b_i$ of finite height so that $a \leq p(c_1, ..., c_n)$.

If $p$ is one of the constants then the claim is vacuously true. If $p$ is a single variable then setting $c_i = a$ does the trick.

If $p = p_1p_2$ then $a \leq p(b_1, ..., b_n)^2$ implies $a \leq p_k(b_1, ..., b_n)$, $k = 1, 2$. By inductive hypothesis, there exist $c_{ki} \leq b_i$, $k = 1, 2$, of finite height, with $a \leq p_k(c_{k1}, ..., c_{kn})$, $k = 1, 2$. Setting $c_i = c_{i1} + c_{i2}$, $i = 1, ..., n$, gives $a \leq p_1(c_{11}, ..., c_{1n}) \cdot p_2(c_{21}, ..., c_{2n}) \leq p_1(c_1, ..., c_n) \cdot p_2(c_1, ..., c_n) = p(c_1, ..., c_n)$.

If $p = p_1 + p_2$ then, for convenience, we set $d_k = p_k(b_1, ..., b_n)$, $k = 1, 2$. Choose $e_1$ as a relative complement of $d_1d_2$ in $[0, d_1]$. Set $e_2 = d_2$ and, for $k = 1, 2$, $a_k = e_k(a + e_1)$, where $\{k, l\} = \{1, 2\}$. One easily computes using modularity that $\{0, a, a_1, a_2, a_1 + a_2\}$ form an $M_3$ in $[0, a_1 + a_2]$ and, consequently, the $a_k$, $k = 1, 2$, are atoms of $L$. Now, for $k = 1, 2$, $a_k \leq e_k \leq d_k = p_k(b_1, ..., b_n)$, so by inductive hypothesis, there exist $c_{ki} \leq b_i$, $i = 1, ..., n$, of finite height, with $a_k \leq p_k(c_{k1}, ..., c_{kn})$. Again, set $c_i = c_{i1} + c_{i2} \leq b_i$, $i = 1, ..., n$. This gives $a \leq a_1 + a_2 \leq p_1(c_{11}, ..., c_{1n}) + p_2(c_{21}, ..., c_{2n}) \leq p_1(c_1, ..., c_n) + p_2(c_1, ..., c_n) = p(c_1, ..., c_n)$.

\[\text{Formally every polynomial is a polynomial on the whole countable set of variables, with all but finitely many set to 0. Our notation is a matter of convenience then, and not part of the induction.}\]
3 Orthoimplications

Let \( L \) be an ortholattice. Elements \( x, y \in L \) are orthogonal, written \( x \perp y \), iff \( x \leq y' \). More generally, two sequences \( (x_1, \ldots, x_n), (y_1, \ldots, y_n) \) of elements of \( L \) are orthogonal, written \( (x_1, \ldots, x_n) \perp (y_1, \ldots, y_n) \), iff \( x_i \perp y_i \), \( i = 1, \ldots, n \). An orthoimplication is a sentence formed by the universal quantification of a formula of the form,

\[
(x_1, \ldots, x_n) \perp (y_1, \ldots, y_n) \implies r(x_1, y_1, \ldots, x_n, y_n) = 0,
\]

where \( r \) is a bounded lattice term.

**Lemma 5** For any two ortholattice terms \( p(x_1, \ldots, x_n) \) and \( q(x_1, \ldots, x_n) \), there is a bounded lattice term \( r(x_1, y_1, \ldots, x_n, y_n) \) such that for all orthomodular lattices the equation

\[
p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n)
\]

holds in \( L \) iff the orthoimplication

\[
(x_1, \ldots, x_n) \perp (y_1, \ldots, y_n) \implies r(x_1, y_1, \ldots, x_n, y_n) = 0
\]

holds in \( L \).

*Proof:* By orthomodulararity the ortholattice identity \( p = q \) holds in an OML \( L \) iff the identity \( p(p' + q') + q(p' + q') = 0 \) holds in \( L \). Repeated application of De Morgan's laws (which hold in any OL) allow one to bring all occurrences of ' inside all brackets, so that any ortholattice term \( t(x_1, \ldots, x_n) \) is equivalent to a bounded lattice term \( r(x_1, x'_1, \ldots, x_n, x'_n) \). These two observations are easily combined to prove the lemma.

4 Proof of the Proposition

If \( c_1, \ldots, c_n \) are elements of finite height in an MOL \( L \), then \( u = \sum_{i=1}^{n} c_i \) is of finite height, \([0, u] \times [u', 1]\) is a subalgebra of \( L \), containing \( c_1, \ldots, c_n \), and \([0, u]\) is a homomorphic image of this subalgebra. These elementary facts will be used in our proof of Proposition 2 which we are now in a position to give.

*Proof of Proposition 2.*

Let \( u \in L \) of finite height. From the above comments, \([0, u]\) is in the variety generated by \( L \). Let \( p = q \) be an ortholattice identity which does not hold in \( L \) and let \( (x_1, \ldots, x_n) \perp (y_1, \ldots, y_n) \) implies \( r(x_1, y_1, \ldots, x_n, y_n) = 0 \) be its associated orthoimplication. By Lemma 5, there exist \( (x_1, \ldots, x_n) \perp (y_1, \ldots, y_n) \) in \( L \) so that \( r(x_1, y_1, \ldots, x_n, y_n) > 0 \). By Lemma 4, there exist \( c_i, d_i \in L \), \( i = 1, \ldots, n \), of finite height, so that \( c_i \leq x_i, d_i \leq y_i \) for each \( i \), and \( r(c_1, d_1, \ldots, c_n, d_n) \geq 0 \). Let \( u = \sum_{i=1}^{n} (c_i + d_i) \) and note that \( c_i \perp d_i \) in \([0, u]\), so the orthoimplication \((x_1, \ldots, x_n) \perp (y_1, \ldots, y_n) \) implies \( r(x_1, y_1, \ldots, x_n, y_n) = 0 \) fails in \([0, u]\). By Lemma 5, the identity \( p = q \), does not hold in \([0, u]\).
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