ON THE EQUATIONAL THEORY OF PROJECTION LATTICES OF
FINITE VON-NEUMANN FACTORS

CHRISTIAN HERRMANN

Abstract. For a finite von-Neumann algebra factor $M$, the projections form a modular ortholattice $L(M)$. We show that the equational theory of $L(M)$ coincides with that of some resp. all $L(C^{n \times n})$ and is decidable. In contrast, the uniform word problem for the variety generated by all $L(C^{n \times n})$ is shown to be undecidable.

§1. Introduction. Projection lattices $L(M)$ of finite von-Neumann algebra factors $M$ are continuous orthocomplemented modular lattices and have been considered as logics resp. geometries of quantum mechanics cf. [25]. In the finite dimensional case, the correspondence between irreducible lattices and algebras, to wit the matrix rings $C^{n \times n}$, has been completely clarified by Birkhoff and von Neumann [5]. Combining this with Tarski’s [27] decidability result for the reals and elementary geometry, decidability of the first order theory of $L(M)$ for a finite dimensional factor $M$ has been observed by Dunn, Hagge, Moss, and Wang [7].

The infinite dimensional case has been studied by von Neumann and Murray in the landmark series of papers on ‘Rings of Operators’ [23], von Neumann’s lectures on ‘Continuous Geometry’ [28], and in the treatment of traces resp. transition probabilities in a ring resp. lattice-theoretic framework [20, 29].

The key to an algebraic treatment is the coordinatization of $L(M)$ by a $*$-regular ring $U(M)$ derived from $M$ and having the same projections: $L(M)$ is isomorphic to the lattice of principal right ideals of $U(M)$ (cf. [8] for a thorough discussion of coordinatization theory). For finite factors this has been achieved in [23], more generally for finite $AW^*$-algebras and certain Baer-$*$-rings by Berberian in [2, 3].

In the present note we show that the equational theory of $L(M)$ coincides with that of $L(C^{n \times n})$ if $L(M)$ is $n + 1$- but not $n$-distributive for some $n$; and with that of all $L(C^{n \times n})$, $n < \infty$, otherwise - which applies to the type $II_1$ factors. In the latter case, the equational theory is decidable, but the theory of quasi-identities is not.

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§2. Modular ortholattices: Equations and representations. An algebraic structure \( (L, \cdot, \cup, 0, 1) \) is an ortholattice (shortly OL) if there is a partial order \( \leq \) on \( L \) such that, for all \( a, b \in L \), \( 0 \leq a \leq 1 \), \( a \cdot b = ab = \inf \{a, b\} \), \( a + b = \sup \{a, b\} \), \( a'' = a \), and \( a \leq b \) iff \( b' \leq a' \). It is a modular ortholattice (shortly: MOL) if, in addition, \( a \geq b \) implies \( a(b + c) = b + ac \). One can define this class by a finite set of equations, easily ([4, 5]).

If \( V \) is a unitary space then the subspaces of finite dimensions together with their orthogonal complements form an MOL \( L(V) \) - a sublattice of the lattice of all subspaces. For \( V \) of finite dimension \( n \), we have \( L(V) \cong L(\mathbb{C}^n) \) for \( \mathbb{C}^n \) endowed with the canonical scalar product. A lattice is \( n \)-distributive if and only if it satisfies

\[
\sum_{i=0}^{n} x_i = \sum_{i=0}^{n} x \sum_{j \neq i} y_j.
\]

**Lemma 2.1.** \( L(\mathbb{C}^k) \) is \( n \)-distributive if and only if \( k \leq n \).

**Proof.** Huhn [18, p. 304] cf. [13].

For a class \( \mathcal{C} \) of algebraic structures, e.g. ortholattices, let \( \mathcal{VC} \) denote the smallest equationally definable class (variety) containing \( \mathcal{C} \) cf. [6]. By Tarski’s version of Birkhoff’s Theorem, \( \mathcal{VC} = \mathcal{HSPC} \) where \( \mathcal{HC}, \mathcal{SC} \), and \( \mathcal{FC} \) denote the classes of all homomorphemic images, subalgebras, and direct products, resp., of members of \( \mathcal{C} \). Define

\[
\mathcal{N} = \mathcal{V}\{L(\mathbb{C}^k) \mid k < \infty\}.
\]

Clearly, \( L(\mathbb{C}^k) \in \mathcal{SHL}(\mathbb{C}^n) \) for \( k \leq n \). Within the variety of MOLS, each ortholattice identity is equivalent to an identity \( t = 0 \) (namely, \( a = b \) if and only if \( a(ab)'' + b(ab)'' = 0 \)). If \( L \) is an MOL and \( u \in L \) then the section \([0, u]\) is naturally an MOL with orthocomplement \( x \mapsto x^n = x' u \).

**Lemma 2.2.** An ortholattice identity \( t = 0 \) with \( m \) occurrences of variables holds in a given atomic MOL \( L \) if and only if it holds in all sections \([0, u]\) of \( L \) with \( \dim u \leq m \).

**Proof.** As usual, we write \( \overline{x} \) for sequences \((x_1, \ldots, x_n)\) with \( n \) varying according to the context. We show by induction on complexity: if \( f(\overline{x}) \) is a lattice term with each variable occurring exactly once and if \( p \) is an atom of \( L \) and \( a_i \) in \( L \) with \( p \leq f(\overline{x}) \) in \( L \) then there are \( p_i \leq a_i \) in \( L \) which are atoms or 0 such that \( p \leq f(\overline{x}) \). Indeed, if \( f = x_1 \) let \( p_1 = p \). Now, let \( \overline{x} = \overline{x'} \) and \( \overline{\overline{x}} = \overline{x'} \), accordingly. If \( f(\overline{x}) = f_1(\overline{x}) \cdot f_2(\overline{x}) \) then \( p \leq f_1(\overline{\overline{\overline{x}}}) \) and \( p \leq f_2(\overline{\overline{\overline{x}}}) \) and we may choose the \( q_i \leq b_i \) and and \( r_j \leq c_j \) by inductive hypothesis and put \( \overline{p} = \overline{f(\overline{x})} \). On the other hand, consider \( f(\overline{x}) = f_1(\overline{x}) + f_2(\overline{x}) \). If \( f_2(\overline{x}) = 0 \) then \( p \leq f_1(\overline{x}) \) and we may choose \( q_i \leq b_i \) by induction and \( r_j = 0 \). Similarly, if \( f_1(\overline{x}) = 0 \). Otherwise, there are atoms \( q_i \) such that \( p^1 \leq f_1(\overline{x}), p^2 \leq f_2(\overline{x}) \) and \( p \leq p^1 + p^2 \) (cf. [1]). Applying the inductive hypothesis, we may choose \( q_i \leq b_i \) and \( r_j \leq c_j \), atoms or 0, such that \( p^1 \leq f_1(\overline{x}) \) and \( p^2 \leq f_2(\overline{x}) \) whence \( p \leq f(\overline{x}) \) where \( \overline{p} = \overline{f(\overline{x})} \).
Now, consider an identity \( t(\overline{\alpha}) = 0 \). By de Morgan’s laws, we may assume that 
\( t \) is in so called negation normal form, i.e. there is a lattice term \( f(\overline{\alpha}) \) with each variable occurring exactly once from which \( t(\overline{\alpha}) \) arises substituting the variable \( x_{\alpha i} \) for \( y_i \), the negated variable \( x'_{\beta j} \) for \( z_j \) (with suitable functions \( \alpha \) and \( \beta \)).

Assume \( t(\overline{\alpha}) > 0 \) in \( L \). Since \( L \) is atomic, there is an atom \( p \) with \( p \leq t(\overline{\alpha}) \).

With \( b_i = a_{\alpha i} \) and \( c_j = a'_{\beta j} \) one obtains \( t(\overline{\alpha}) = f(\overline{\beta}) \). As shown above, there are \( q_i \leq b_i \) and \( r_j \leq c_j \) such that \( p \leq f(\overline{\beta}) \). Put

\[
 u_k = \sum_{\alpha i = k} q_i, \quad v_k = \sum_{\beta j = k} r_j, \quad w = \sum_{k=1}^{n} u_k + v_k.
\]

Then \( u_k \leq a_k \leq w \) and \( v_k \leq a'_k \leq w \). Thus, \( a'_k \leq u'_k \) and \( v_k \leq u''_k \). For the MOL \( [0, w] \) it follows by monotonicity that \( 0 < p \leq f(\overline{\beta}) \leq t(\overline{\alpha}) \).

A unitary representation of an MOL \( L \) is a 0-lattice embedding \( \varepsilon : L \rightarrow L(V) \) into the lattice of all subspaces of a unitary space such that

\[
 \varepsilon(a') = \varepsilon(a)^\perp \quad \text{for all} \quad a \in L.
\]

This means that \( \varepsilon \) is an embedding of the ortholattice \( L \) into the orthostable lattice associated with the unitary space \( V \) in the sense of Herbert Gross [10].

**Corollary 2.3.** \( L \in \mathcal{N} \) for any MOL admitting a unitary representation.

**Proof.** By [14, Thm.2.1] \( L \) embeds into an atomic MOL \( \hat{L} \) such that the sections \( [0, u] \), \( \dim u < \infty \) are subspace ortholattices of finite dimensional unitary spaces (namely, if \( L \) is represented in \( V \) then \( \hat{L} \) consists of all closed subspaces \( X \) such that \( \dim[X \cap e a, X + e a] < \infty \) for some \( a \in L \)). By Lemma 2.2, \( \hat{L} \) whence also \( L \) belong to the variety \( \mathcal{N} \) generated by these.

**Corollary 2.4.** \( \mathcal{N} = VL(V) \) for any unitary space of infinite dimension.

§3. **Regular rings with positive involution.** An associative ring (with or without unit) \( R \) is (von Neumann) regular if for any \( a \in R \) there is a quasi-inverse \( x \in R \) such that \( axa = a \) cf. [28, 22, 9]. A \( * \)-ring is a ring with an involution \( * \) as additional operation:

\[
 (x + y)^* = x^* + y^*, \quad (xy)^* = y^* x^*, \quad x^{**} = x.
\]

A \( * \)-ring is \( * \)-regular if it is regular and, moreover, positive: \( xx^* = 0 \) only for \( x = 0 \). Equivalently, for any \( a \in R \) there is a (unique) projection \( e \) (i.e. \( e = e^* = e^2 \)) such that \( aR = eR \). Particular examples are the rings \( \mathbb{C}^{n \times n} \) of all complex \( n \times n \)-matrices with \( r^* \) the adjoint matrix, i.e. the transpose of the conjugate.

The projections of a \( * \)-regular ring with unit form a modular ortholattice \( L(R) \) where \( e \leq f \iff e = ef = e^2 \) such that \( eR = eR \). Particular examples are the rings \( \mathbb{C}^{n \times n} \) of all complex \( n \times n \)-matrices with \( r^* \) the adjoint matrix, i.e. the transpose of the conjugate.

The projections of a \( * \)-regular ring with unit form a modular ortholattice \( L(R) \) where \( e \leq f \iff e = ef \) and \( e^2 = 1 - e \). Now, \( e \mapsto e^2 \) is an isomorphism of \( L(R) \) onto the ortholattice of principal right ideals of \( R \) and we may use the same notation for both. Observe that \( L(\mathbb{C}^n) \cong L(\mathbb{C}^{n \times n}) \), canonically, where a subspace \( X \) corresponds to the set of all matrices with columns in \( X \) cf. the following Proposition.

**Proposition 3.1.** (Giudici). Let \( M \) be a right module over a ring \( S \) and let \( R \) be a regular subring of the endomorphism ring \( \text{End}(M_S) \). Then \( L(R) \) embeds into the lattice of submodules of \( M_S \) via \( \varepsilon(\phi R) = \text{Im} \phi \).
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Proof. This is (1) in the proof of [8, Thm.4.2.1] in the thesis of Luca Giudici, cf. [15, Prop.9.1]. ⊥

Corollary 3.2. If R and S are *-regular rings, R a *-subring of S, then L(R) is a sub-OL of L(S).

Proof. R embeds into End S via r → ⃗r where ⃗r(x) = rx for x ∈ S. By Prop.3.1 this yields an embedding of L(R) into L(S) with eR → ⃗me = eS for e ∈ L(R). Since e′ = 1 − e in both OLs, we have L(R) a sub-OL of L(S). ⊥

Corollary 3.3. For any *-regular ring S,

VL(S) = {VL(R) | R at most countable, regular *-subring of S}

Proof. ‘⊇’ follows from Cor.3.2. Conversely, L(S) belongs to the variety generated by its finitely generated sub-OLs L. Endow S with a unary operation q such that aq(a) = a for all a in S. Now, for any such L there is an at most countable *-subring R of S containing L and also closed under the operation q. Observe that for e, f ∈ L(R) one has e ≤ f if and only if ef = e, i.e. e ≤ f in L(S). Thus L is also a sublattice of L(R): assume we have join e ∨ f = g in L and h ∈ L(R) with h ≥ e, f in L(R). Then h ≥ g in L(S) whence h ≥ g which means e ∨ f = g also in L(R). Similarly for meets. Also, since L is closed under the orthocomplement e → 1 − e in L(S), the same is true in L(R). It follows, that L is a sub-OL of L(R).

Let V be a unitary space. Denote by φ∗ the adjoint of φ - if it exists. A unitary representation of a *-ring R is a ring embedding τ : R → End(V) such that τ(r∗) = τ(r)∗ for any r ∈ R.

Corollary 3.4. If τ : R → End(V) is a unitary representation of the *-regular ring R, then

ε(eR) = im(τ(e))

is a unitary representation of the MOL L(R) in V.

Proof. The lattice embedding follows from Prop.3.1. Now, observe that

ε(eR)† = im(id − τ(e)) = ε((1 − e)R) = ε(eR)∗

since e and τ(e) are selfadjoint idempotents. ⊥

§4. Finite von-Neumann algebras. A von-Neumann algebra (cf. [17]) M is an unital involutive C-subalgebra of the algebra B(H) of all bounded operators of a separable Hilbert space H with M = M′′ where A′ = {φ ∈ B(H) | φψ = ψφ ∀φ ∈ A} is the commutant of A. M is finite if rr∗ = 1 implies r∗r = 1. For such, the projections e of M, i.e. the e = e2 = e∗, form a (continuous) modular ortholattice L(M). Here, the order is given by e ≤ f ⇔ e = ef and one has e′ = 1 − e. A finite von-Neumann algebra is a factor if its center is C·1. Particular examples of a finite factors are the algebras C××n of all complex n-by-n-matrices.

Theorem 4.1. (Murray-von-Neumann.) Any finite von-Neumann algebra factor is either isomorphic to C××n for some n < 1 (type I1) or contains for any n < 1 a subalgebra isomorphic to C××n (type II1).
For any operator \( \phi \) defined on some linear subspace of \( H \), write \( \phi \eta \mathcal{M} \) if \( \psi \phi \psi^{-1} = \phi \) for all unitary \( \psi \in \mathcal{M}' \) (cf \[23, Def.4.2.1\]). Let \( U(\mathcal{M}) \) consist of all closed linear operators \( \phi \) with \( \phi \eta \mathcal{M} \) and having dense linear domain. Consider the following operations with domain \( U(\mathcal{M}) \)

\[
(\phi, \psi) \mapsto [\phi + \psi], \quad (\phi, \psi) \mapsto [\phi \circ \psi], \quad \phi \mapsto [\phi^*]
\]

where \([\chi]\) denotes the closure of the linear operator \( \chi \).

**Theorem 4.2.** *(Murray-von-Neumann.)* For every finite factor \( \mathcal{M} \), \( U(\mathcal{M}) \) is a \( * \)-regular ring having \( \mathcal{M} \) as \( * \)-subring and such that \( \phi^* \) is adjoint to \( \phi \). Moreover, \( \mathcal{M} \) and \( U(\mathcal{M}) \) have the same projections.

**Proof.** This is trivial for type \( I \). For II factors this is \[23, Thm. XV\] together with \[28, Part II, Ch.II, App 2.(VI)\] and \[29, p.191 \] for \( * \)-regularity. Now, consider \( \pi : D \to H \) in \( U(\mathcal{M}) \) such that \( \pi = \pi^* = \pi^2 \). Then \( U = \text{Im} \pi \subseteq D \) so \( \pi \) is a projection of \( D \), i.e. \( D = U \oplus \perp V \). By density of \( D \) it follows \( U \perp \perp \oplus \perp V \perp \perp = H \) and \( \pi \) extends to a projection \( \hat{\pi} \) of \( H \) onto \( U \perp \perp \). From \( \pi \eta \mathcal{M} \) it follows \( \hat{\pi} \eta \mathcal{M} \), whence \( \hat{\pi} \in U(\mathcal{M}) \) and \( \pi = \hat{\pi} \in \mathcal{M} \) by \[23\] Lemmas 16.4.2 and 4.2.1.

An important concept in the Murray-von-Neumann construction is that of an essentially dense linear subspace \( X \) of \( H \) (w.r.t. \( \mathcal{M} \)). Here, we need only the following properties:

1. Essentially dense \( X \) is dense in \( H \) \[23, Lemma 16.2.1\].
2. The domains of members of \( U(\mathcal{M}) \) are essentially dense \[23, Lemma 16.4.3\].
3. For any \( \phi \in U(\mathcal{M}) \) and essentially dense \( X \), the preimage \( \phi^{-1}(X) \) is essentially dense \[23, Lemma 16.2.3\].
4. Any finite or countable intersection of essentially dense \( X_n \) is essentially dense \[23, Lemma 16.2.2\].

**Theorem 4.3.** *(Luca Guidici.)* Any countable \( * \)-subring of \( U(\mathcal{M}) \) is representable.

**Proof.** Consider any countable \( * \)-subring \( R \) of \( U(\mathcal{M}) \). A representation of \( R \) is constructed from the given Hilbert space \( H \). Let \( H_0 \) be the intersection of all domains of operators \( \phi \in R \). By (2), \( H_0 \) is essentially dense. Define, recursively, \( H_{n+1} \) as the intersection of \( H_n \) and all preimages \( \phi^{-1}(H_n) \) where \( \phi \in R \). By (3) and (4), \( H_{n+1} \) is essentially dense. By (4), the intersection \( H_\omega = \bigcap_{n<\omega} H_n \) is essentially dense and, by (1), dense in \( H \). By construction, \( H_\omega \) is invariant under \( R \).

Now, for \( \phi \in R \) define \( \epsilon(\phi) = \phi|H_\omega \). Then \( \epsilon : R \to \text{End}_H(H_\omega) \) is a \( * \)-ring homomorphism. Indeed, e.g. \( |\phi + \psi| \) is an extension of \( \phi|H_\omega + \psi|H_\omega \) and equality holds since both are maps with the same domain. Also \( \epsilon(\phi^*) \) is the restriction of the adjoint \( \phi^* \) in \( H \), whence the adjoint in \( H_\omega \). If \( \epsilon(\phi) = 0 \), then \( H_\omega \) is contained in the closed subspace \( \ker \phi \) and it follows \( \phi = 0 \) by density. Thus, \( \epsilon \) is a representation.

§5. Equational theory of projection lattices.
Theorem 5.1. For any class $\mathcal{M}$ of finite von-Neumann algebra factors, and $V = V(L(M) \mid M \in \mathcal{M})$ one has $V = VL(\mathbb{C}^n)$ if and only if $V$ satisfies the $n + 1$-distributive law but not the $n$-distributive law. Moreover, $V = N$ if and only if $V$ satisfies no $n$-distributive law. In any case, the equational theory of $V$ is decidable.

Proof. Let $M$ be a finite von-Neumann algebra factor. In view of Thm.4.2 and Cor.3.3, we have to consider countable regular *-subrings of $U(M)$. By Thm.4.3, each such $R$ is representable. By Cor.3.4 and Cor.2.3 we have $L(R) \in N$ and it follows $L(M) \in N$.

By Lemma 2.1, Cor.3.2, and Thm.4.1, $\mathcal{M}$ contains factors of arbitrarily large finite dimensions or a type II$_1$ factor if and only if $V$ is $n$-distributive for no $n$. In this case, $V = N$. Otherwise, there is a maximal $n$ such that $V$ is $n$-distributive, in particular all members of $\mathcal{M}$ are of the form $\mathbb{C}^k \times k$ with $k \leq n$ and $k = n$ occurs, so $V = VL(\mathbb{C}^{n \times n})$.

Recall that according to Tarski [27] the (ordered) field $\mathbb{R}$ has a decidable first order theory. This extends to the field $\mathbb{C}$ endowed with the unary operation of conjugation and then (uniformly) to the involutive $\mathbb{C}$-algebras $\mathbb{C}^{n \times n}$. Encoding the geometry in Tarski style into $\mathbb{C}$ or von-Neumann style into $\mathbb{C}^{n \times n}$, it follows, that there is a uniform decision procedure for the first order theories of the $L(\mathbb{C}^n) \cong L(\mathbb{C}^{n \times n})$. This settles the case of $V = VL(\mathbb{C}^{n \times n})$. To decide whether an identity $t = 0$ holds in $N$, by Lemma 2.2 it suffices to decide validity in $L(\mathbb{C}^{m \times n})$, $m$ the number of occurences of variables in $t$.

§6. Von-Neumann frames. Let $n \geq 3$ fixed. An $n$-frame, in the sense of von-Neumann [28], in a lattice $L$ is a list $\overline{\pi} : a_1, a_{ij}, 1 \leq i, j \leq n, i \neq j$ of elements of $L$ such that for any 3 distinct $j, k, l$

$$a_j \sum_{i \neq j} a_i = a_j a_{jk}, \quad \sum_i a_i = 1$$

$$a_j + a_{jk} = a_j + a_k, \quad a_{ji} = a_{ij} = (a_j + a_l)(a_{jk} + a_{kl}).$$

If $L$ is modular and $n \geq 4$ then

$$R(L, \overline{\pi}) = \{ r \in L \mid ra_2 = 0, \ r + a_2 = a_1 + a_2 \}$$

can be turned into a ring, the coordinate ring. For the present purpose it suffices to know that $R(L, \overline{\pi})$ is a semigroup under the multiplication

$$s \otimes r = [(r + a_{21})(a_1 + a_3) + (s + a_{13})(a_2 + a_3)](a_1 + a_2)$$

cf. [21] where $R(L, \overline{\pi})$ is denoted by $L_{12}$ and $r = r_{12}$ replaced by the array of the $r_{ij}$ obtained via the perspectivities provided by the $a_{kl}$. Thus, for each multiplicative term $t(\overline{\pi}) = x_n \cdot (\ldots x_2) \cdot x_1$ there is a lattice polynomial

$$\tilde{t}(\overline{\pi}, \overline{\tau}) = x_n \otimes (\ldots \otimes x_2) \otimes x_1$$

such that $\tilde{t}(\overline{\pi}, \overline{\tau}) = t(\overline{\tau})$ for all substitutions $\overline{\tau}$ in $R(L, \overline{\pi})$.

In the sequel, orthocomplementation is no longer an issue and we write $L(V)$ for the lattice of all subspaces of $V$, $L(R)$ for the lattice of all right ideals of $R$. If $R^{n \times n}$ is the $n \times n$-matrix ring of some ring $R$ with unit and $L = L(R^{n \times n})$ with the canonical $n$-frame $\overline{\pi}$ then $R(L, \overline{\pi})$ is isomorphic to $R$ - here $\overline{\pi}$ consists...
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of the $E_{ij}R^{n\times n}$ and $(E_{ij} - E_{ij})R^{n\times n}$ where the $E_{ij}$ form the canonical basis of the $R$-module $R^{n\times n}$. Indeed, one has a 1-1-correspondence between $R$, $R(L, \pi)$, and certain right submodules of $R^n$ given by

$$ r \leftrightarrow (E_{11} - rE_{21})R^{n\times n} \leftrightarrow (e_1 - re_2)R $$

where $e_1, \ldots, e_n$ is the canonical basis of $R^n$. Using the notations $(rx, sx, tx) = (e_1r + e_2s + e_3t)R$ and $\tilde{r} = (e_1 - re_2)R$ we compute

$$ (\tilde{r} + a_{23})(a_1 + a_3) = (x, y - rx, -y) \cap (u, 0, v) = (x, 0, -rx) $$

$$ (\tilde{s} + a_{13})(a_2 + a_3) = (x - y, -sx, y) \cap (0, u, v) = (0, -sy, y) $$

$$ \tilde{s} \otimes \tilde{r} = (x, -sy, y - rx) \cap (u, v, 0) = (x, -srx, 0). $$

This translates back into $L(R^{n\times n})$ and shows that $r \leftrightarrow \tilde{r}$ is an isomorphism between the semigroups $R$ and $R(L, \pi)$.

§7. Quasivarieties and word problems. A quasi-identity is a sentence

$$ \forall \mathfrak{T}. \bigwedge_{j=1}^{k} s_j(\mathfrak{T}) = t_j(\mathfrak{T}) \Rightarrow s(\mathfrak{T}) = t(\mathfrak{T}) $$

where the $s_j(\mathfrak{T})$ and so on are terms. A quasivariety is a class of algebraic structures defined by quasi-identities, equivalently an axiomatic class closed under substructures and direct products.

A solution of the uniform word problem for a class $C$ consists in a decision procedure for quasi-identities (i.e. a solution for all finite presentations). The restricted word problem is unsolvable for $C$ if for some fixed premise the associated set of quasi-identities is undecidable within $C$. In other words, within the quasivariety $QC$ generated by $C$ there is a finitely presented member having unsolvable word problem.

Unsolvability of the restricted word problem has been established by Hutchinson [19] and Lipshitz [21] for any class $C$ of modular lattices with $L(V) \in QC$ for some infinite dimensional vector space $V$. Also, based on analogous results of Gurevich [11] for semigroups, Lipshitz has shown unsolvability for classes $\{L(F^n) \mid F \in \mathcal{F}, n < \infty\}$, $\mathcal{F}$ any class of fields, and for $C$ the class of finite (complemented) modular lattices. These results extend to classes having the appropriate lattice reducts.

For sufficiently large classes of modular ortholattices (e.g. containing all 14-distributives) unsolvability in 3 generators has been shown by M.S. Roddy [26] and this has been used in [16] to prove undecidability of the equational theory for the class of all n-distributives for fixed $n \geq 14$.

Let $S$ ($S_{fin}$) denote the class of all (finite) semigroups, and $S_p$ the set of semigroups $F_p^{n\times n}$ ($n \geq 1$) where $F_p$ is the prime field of characteristic $p$, prime or 0. Let $\mathcal{M}$ denote the class of all modular lattices, $\mathcal{M}_p$ the set of lattices $L(F_p^n) \cong L(F_p^{n\times n})$ ($n \geq 1$). For a class $C$ denote by $R_3C$ and $R_LC$ the class of all semigroup resp. lattice reducts of structures in $C$ and by $Th_pC$ the set of all quasi-identities valid in $C$.

**Theorem 7.1.** A quasivariety $Q$ has unsolvable uniform word problem if $S_p \subseteq SR_3Q \subseteq S$ or $\mathcal{M}_p \subseteq SR_LQ \subseteq \mathcal{M}$ for some $p$. 

Proof. Given a finite semigroup $S$, one may consider the semigroup ring $F_p[S]$ as an $F_p$-vector space $V$ and thus embed $S$ into $\text{End}_{F_p}(V) \cong F_p^{n \times n}$ where $n = |S|$. It follows $\text{Th}_qS_p \subseteq \text{Th}_qS_{fin}$ for all $p$ and equality for $p > 0$. Since $Q^{n \times n} \in \mathcal{SP}_u\{F_p^{n \times n} \mid p \text{ prime}\}$, one has

$$\text{Th}_qS_p = \text{Th}_qS_{fin} \text{ for all } p.$$ 

This is contained in Lipshitz [21, Lemma 3.5]. The claim in the semigroup case follows from the result of Gurevich and Lewis [12] that there is no recursive $\Gamma$ such that $\text{Th}_qS \subseteq \Gamma \subseteq \text{Th}_qS_{fin}$.

According to the preceding section and again following Lipshitz [21], one may associate with each quasi-identity $\phi$ as above in the semigroup language a quasi-identity $\bar{\phi}$ in the lattice language

$$\forall \bar{\pi} \forall \bar{\tau} \exists \bar{a} \in \tilde{L} \left( \bar{a} \neq \bar{0} \land \bar{a} \neq \bar{1} \right) \land \forall j \forall i \left( \bar{a}_j \bar{a}_i = \bar{a} \right)$$

where $\alpha(\bar{a})$ states that $\bar{a}$ is a 4-frame. Since $R(\tilde{L}, \bar{a})$ is a semigroup for any modular lattice $L$, it follows that $\bar{\phi} \in \text{Th}_qM$ for all $\phi \in \text{Th}_qS$. On the other hand, if $\phi$ holds in $L(R^{p \times p})$, substituting the canonical 4-frame for $\bar{a}$, then $\phi$ holds in $R$. In particular, for the ring $R = F_p^{n \times n}$ we encode equality of products of $n \times n$-matrices over $F_p$ into equality of particular lattice elements. Thus, considering all $R = F_p^{n \times n}$, $n \geq 1$, it follows $\phi \in \text{Th}_qS_p$ for $\phi \in \text{Th}_qM_p$. This proves that $\phi \in \text{Th}_qS_p$ if and only if $\bar{\phi} \in \text{Th}_qM_p$.

Now, given $\text{Th}_qM \subseteq \Delta \subseteq \text{Th}_qM_p$, define $\Gamma$ as the set of those quasi-identities $\phi$ in semigroup language with $\bar{\phi} \in \Delta$. Then

$$\text{Th}_qS \subseteq \Gamma \subseteq \text{Th}_qS_p$$

and if $\Delta$ is recursive then so is $\Gamma$.

Corollary 7.2. $\mathcal{N}$ as well as the class of projection lattices of finite factors have an undecidable uniform word problem. The quasivariety $\mathcal{Q}$ generated by all ortholattices $L(C^{n \times n})$ $(n < \omega)$ has an undecidable restricted word problem and is not a variety.

Proof. The undecidability claim is immediate by Thm.7.1 resp. the quoted result of Lipshitz [21, Thm.3.6]. By decidability of the $L(C^{n \times n})$, the complement of $\text{Th}_q\mathcal{Q}$ within the set of quasi-identities is recursively enumerable. If $\mathcal{Q}$ were a variety, then by Thm.5.1 it would coincide with $\mathcal{N}$ and be recursively axiomatizable. Thus $\text{Th}_q\mathcal{Q}$ would be recursively enumerable, too, and this would imply solvability of the uniform word problem.

Problem 7.3. Is the restricted word problem solvable for (a) $\mathcal{N}$ resp. (b) the class of projection lattices of finite factors?

References


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TECHNISCHE UNIVERSITÄT DARMSTADT
FB MATHEMATIK
SCHLOSSGARTENSTR. 7, D 64289 DARMSTADT, GERMANY
E-mail: herrmann@mathematik.tu-darmstadt.de