Algebra Universalis, **34** (1995) 35–60 0002-5240/95/010035-26$01.50 + 0.20/0

Alan Day's work on modular and Arguesian lattices

C. HERRMANN

Dedicated to the memory of Alan Day

The present state of art in the theory of modular lattices is to a great extent due to Alan Day's contributions. The purpose of the present paper is to outline the most important ones and their impact on further developments. This is accompanied by problems of various degress of relevance and difficulty. For the better ones, full credit should be given to Alan. For a more detailed account of some of the subjects the reader may consult Alan's excellent surveys [7, 11, 16].

1. Modular lattices with four generators

The first thing I learned from Alan about modular lattices was the following diagram drawn on the ground in the fall of 1971 – see Figure 1. The lattice is called A_∞.

THEOREM 1. *A_∞ is the modular lattice freely generated by a, b, c, d subject to the relations*

$$ab = ac = ad = bc = bd = cd = 0$$

$$a + c = a + d = b + c = b + d = 1.$$

As it turned out, this diagram is basic for the understanding of 4-generated modular lattices. In the joint paper [17] with R. Wille the subdirectly irreducible factor R_∞ of A_∞ was characterized as the free modular lattice generated by a, b, c, d satisfying the additional relation $a + b = 1$. Also, the interval sublattices $S(n, 4) = [e_n, 1]$ were

Presented by J. Sichler.
Received December 7, 1992; accepted in final form October 10, 1994.

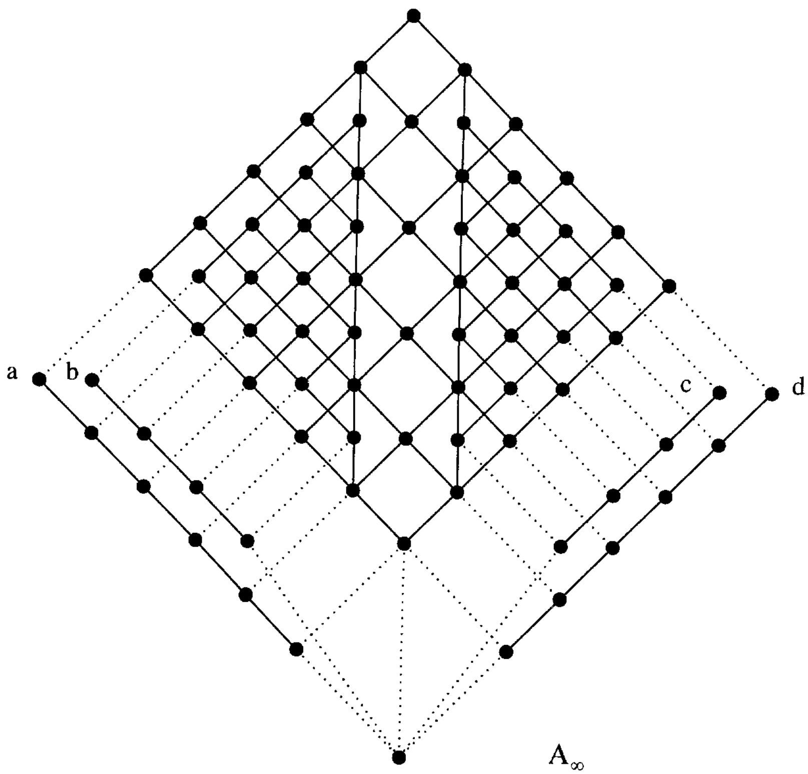

Figure 1

discovered as further subdirectly irreducibles with four generators (the doubly irreducible elements) – see Figure 2.

Freese [30] continued this investigation showing that R_∞ and its dual, the $S(n, 4)$, and the 6-element length two lattice M_4 are the only subdirectly irreducible breadth two modular lattices with four generators. In [43] and [44] the same list appeared for subdirectly irreducible modular lattices generated by two pairs of complements and for subdirectly irreducible modular 2-distributive lattices with four generators. Here, following A. Huhn [53], a modular lattice L is 2-distributive if it satisfies

$$w(x + y + z) = w(x + y) + w(x + z) + w(y + z).$$

Equivalent conditions are that L contains no proper 3-frame (cf. section 4) resp. no projective plane in its variety – so this covers that part of modular lattice theory where drawing of order diagrams makes sense.

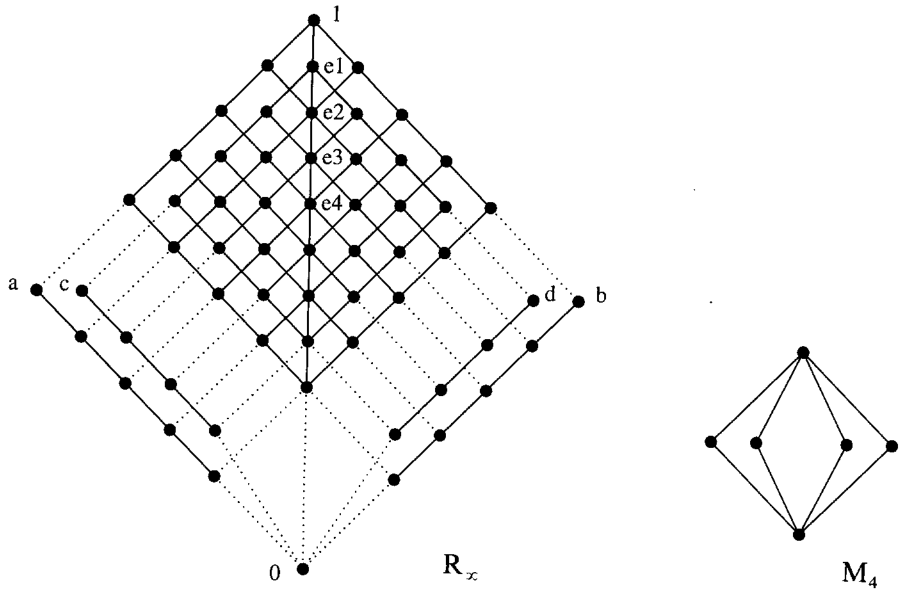

Figure 2

Knowledge about 4-generated modular lattices has proven helpful with various algebraic problems. For example, a lattice theoretic version of the Baer Refinement Theorem for direct decompositions and so an alternative proof of the Krull–Remak–Schmidt Theorem [40] in congruence modular varieties has been based on the list of lattices generated by two pairs of complements – and started also the investigation of permutability properties of the commutator.

Another example is the (linear) representation theory of posets resp. the free modular lattice they generate. Here a *representation* of a lattice L is a homomorphism into a vector space lattice and *indecomposable* if it does not arise as the direct product of two such. For the free modular lattice $FM(4)$ on four generators a complete list of finite dimensional indecomposable representations has been provided by Gelfand and Ponomarev [36]. The image $FM(4)$ under such a representation turned out to be D_2, M_3, M_4, one of the $S(n, 4)$, or the full lattice of subspaces of a vector space over a prime field having dimension $3 \leq d < \infty$. And so one had a complete list of 4-generated subdirectly irreducibles in the variety generated by complemented Arguesian lattices (cf. [45]). The lattice A_∞ occurred when considering ultraproducts of finite dimensional representations of $FM(4)$ and this has been used for determining the *perfect elements* (i.e. elements providing direct decompositions) cf Gelfand and Ponomarev [37], Dlab and Ringel [29], and [45]. This structural analysis of $FM(4)$ can even be carried over to lattices generated by posets of *tame representation type* (for which, roughly speaking, classifying the finite dimensional representations does not involve classifying matrix pairs) see Cylke [2].

A_{∞} is also behind the construction of a finitely generated modular ortholattice with an infinite set of perspective orthogonal elements by G. Bruns and M. Roddy [1].

Checking through the list of subdirectly irreducibles was the first step in the classification of pairs consisting of a finite dimensional Hermitean space E of characteristic 2 and a subspace F, see H. Gross, R. Moresi et al. [39]. Here, the lattice generators are $F, F^{\perp}, E^*, E^{*\perp}$ where E^* is the subspace of trace valued vectors. Lattice relations on these generators allowed to single out the possible indecomposable quadruples of such subspaces and lead finally to the 13 subdirectly irreducible polarity lattices serving as the primary isometry invariants for orthogonally indecomposable pairs (E, F).

Combining examples from the list (generalized from vector spaces to abelian groups) and twisting their gluing structure resulted in the proof of the unsolvability of the word problem for $FM(4)$.

Problems. Is every 4-generated Arguesian lattice embeddable into the subgroup lattice of an abelian group?

Is the word problem for free 4-generated Arguesian lattices solvable?

Is the 4-generator word problem solvable within the congruence variety of abelian groups of exponent p^2?

2. Splitting modular lattices

Following McKenzie [61] a finite subdirectly irreducible lattice L is *splitting* (within a given variety) if there is a greatest variety not containing L. For a subdirectly irreducible lattice L a sufficient condition for being splitting is to be a *bounded homomorphic image*: for every finitely generated (free) lattice M and homomorphism π of M onto L each preimage class $\pi^{-1}(x)$ has a least and greatest element. In the variety of all lattices, the splitting lattices are exactly the subdirectly irreducible bounded images [61]. An intrinsic characterization has been given by B. Jónsson and J. B. Nation [57].

Alan's results on nonmodular splitting lattices are well known. Yet, also most of what is known about splitting modular lattices can be traced back to his paper [4], cf. E. T. Schmidt [68, 69, 70].

THEOREM 2. *Let $\mathcal{V}$ be any modular lattice variety containing all 2-distributives and L a finite planar modular lattice. Then the following are equivalent:*

(1) *L is a subdirect product of splitting lattices in $\mathcal{V}$*

(2) *L is acyclic*
(3) *L is finitely projected in* $\mathcal{V}$.
Any acyclic lattice in $\mathcal{V}$ *is a bounded homomorphic image whence a subdirect product of splitting lattices in* $\mathcal{V}$.

Here, a finite 2-distributive modular lattice is called *acyclic* if it contains neither a sublattice M_4 nor a cycle of M_3's. The lattices in Figures 3a and 4 are acyclic, that in Figure 3b is not. Alan showed that acyclic planar lattices are *finitely projected*: for each π as above there is a finite sublattice L' of M with $\pi(L') = L$ or,

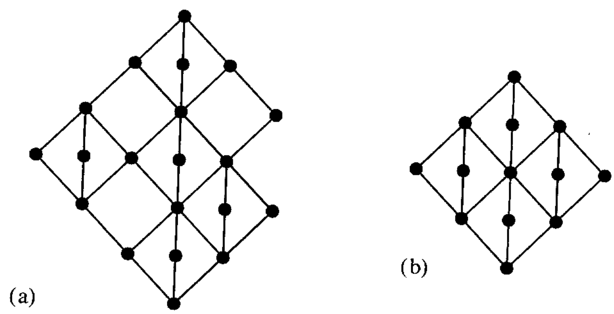

Figure 3

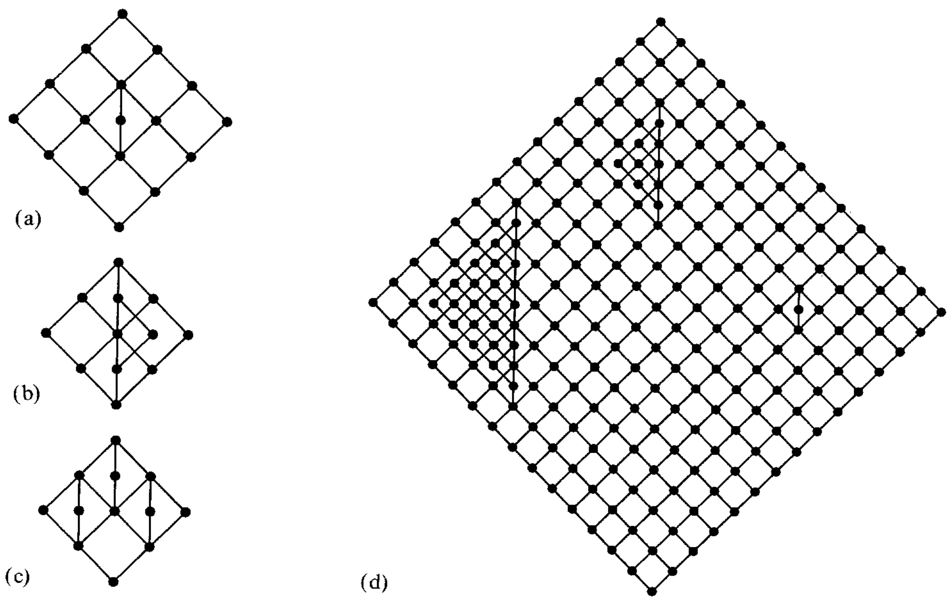

Figure 4

equivalently, L has a finite projective cover (the subdirectly irreducibles are then splitting, too). In his proof he considered a planar modular lattice L as a planar distributive lattice D with some new irreducibles inserted in "boxes". D is finitely projected, obviously, with a planar distributive preimage D'. For a single M_3 in L one gets a preimage using the fact that Figure 4a presents the free modular lattice with generators a, b, c such that $ab \leq c \leq a + b$. The preimages already obtained can be adjusted since putting a chain on an edge of an M_3 yields a subdirect power as in Figure 4b. Having no cycle, this adjustment comes to an end (Figure 4c,d) – otherwise, one would spiral for ever. Finally, M_4 is not a splitting modular lattice since it is a homomorphic image of R_∞ which is in the variety generated by the S_n and similarly, for every planar lattice containing an interval M_n with $n \geq 4$.

Acyclic breadth 2 lattices have been studied by A. Mitschke and R. Wille [62]. In [52] acyclic lattices have been described in terms of the "geometry" on the set of join irreducibles: let a *line* be a maximal, at least 3-element set of join irreducibles such that any pair out of this set yields the same join, the *linetop*. A finite modular lattice is acyclic if and only if each of its lines is 3-element and it admits a choice of one line per linetop such that the resulting set of lines does not contain a cycle (the irreducibles can be presented by points in the plane and the lines by segments such that the resulting figure is a union of trees). With that description, Alan's approach to the planar case could be mimicked to prove that acyclic lattices are bounded homomorphic images within the variety of modular lattices. Acyclic lattices proved important for the representation theory of modular lattices and the application to quadratic forms in infinite dimensions by M. Wild [72]. In [52] it has been shown that a finite 2-distributive modular lattice is of *finite representation type* (i.e. there are, up to isomorphisms induced by linear maps, only finitely many indecomposable representations over a fixed field) if and only if it is acyclic. And, for acyclic lattices the representations (over a given field) are in 1-1 correspondence to the subdirectly irreducible factors.

The "spiral" phenomenon has been elaborated by E. T. Schmidt [67] and R. Freese [31], later on, to produce a finitely generated simple modular lattice with no prime quotients and to show that the variety of modular lattices of primitive breadth 2 is not generated by its finite dimensional members.

Problems. Are acyclic breadth 2 modular lattices finitely projected?

How are the following properties of finite lattices related to each other within the variety of all modular (Arguesian) lattices: "Subdirect product of splitting lattices", "Bounded homomorphic image", "Lower bounded homomorphic image", "Finite representation type"? Are they equivalent for modular 2-distributives?

Is every finite uniquely representable lattice splitting?

Find a version of acyclicity beyond 2-distributives!

3. Desargues' law

Beyond 2-distributivity, modular lattice theory relies heavily on some kind of (at least local) geometric point of view, for positive and negative results as well. In order to proceed to a (local) coordinatization, as in Projective Geometry one needs sufficiently high dimension or some kind of Desargues' law. Such has been provided by B. Jónsson [54, 56]: given "triangles" $\mathbf{a} = (a_0, a_1, a_2)$ and $\mathbf{b} = (b_0, b_1, b_2)$ let $c_i = (a_j + a_k)(b_j + b_k)$ for $\{i, j, k\} = \{0, 1, 2\}$ and $c = c_2(c_0 + c_1)$. A lattice is *Arguesian* if it satisfies the identity

$$(a_0 + b_0)(a_1 + b_1)(a_2 + b_2) \leq a_0(a_1 + c) + b_0(b_1 + c).$$

This law holds in any lattice consisting of permuting equivalence relations [54] or belonging to a modular congruence variety [34]. In a more geometric version one says that $\mathbf{a}, \mathbf{b}$ are *centrally perspective* if "the center of perspectivity" $p = (a_0 + b_0)(a_1 + b_1) \leq a_2 + b_2$ and *axially perspective* if $c_2 \leq c_0 + c_1$. Then, an equivalent definition is that any centrally perspective pair of triangles is also axially perspective.

The hypothesis can be made more specific: $\mathbf{a}, \mathbf{b}$ form a *perspectivity configuration*, shortly PC, if $a_i + p = b_i + p$ for all i and if the quadruples $\mathbf{a}, p$ and $\mathbf{b}, p$ are in general position – cf. Figure 5. Here, $(x_0, \ldots, x_3)$ is in general position if $x_i(x_j + x_k)$ is the same u for any choice of 3 distinct indices. Equivalently, $x_0 \cdots x_3$

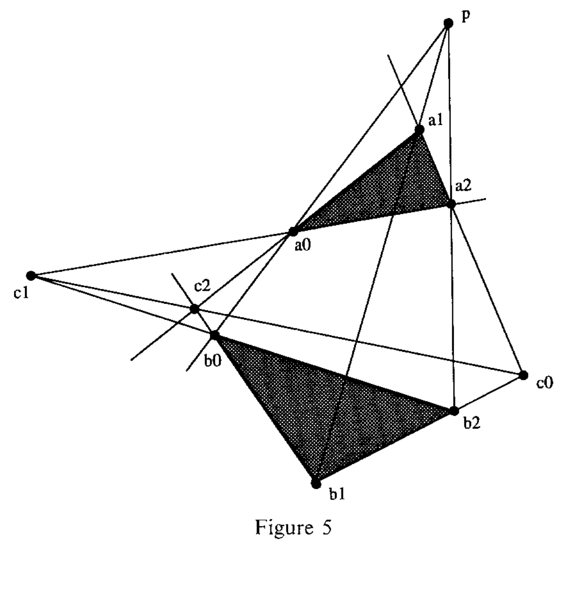

Figure 5

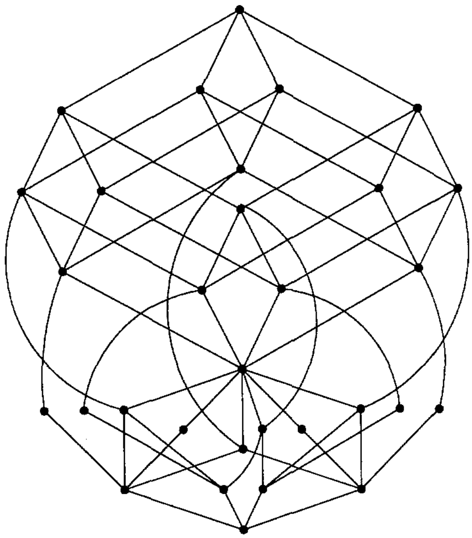

Figure 6

generate a sublattice which is a subdirect product of a 3-frame generated lattice and factors D_2 where $x_i = 0$ for all but one index i – cf. Figure 6.

In a PC one has $a_i + b_i = a_i + p, p = (a_i + b_i)(a_j + b_j)$, and (a_i, b_i, c_j, c_k) in general position for $\{i, j, k\} = \{0, 1, 2\}$. Moreover, PC's are projective configurations within the variety of modular lattices [21]. In [9] Alan showed

THEOREM 3. *A modular lattice is Arguesian if and only if every PC is axially perspective. 2-distributive modular lattices are Arguesian.*

So one obtains the following hierarchy of modular lattice varieties, all selfdual:

distributive $\Rightarrow$ 2-distributive modular $\Rightarrow$ Arguesian $\Rightarrow$ modular.

Experience has proven that, in an axiomatic approach, Arguesian lattices or any equivalent version, cf. Alan's [25], are the proper concept: it is strong enough to allow results, and handy enough if one is going to derive them. This is more so since the important semantically defined lattice classes cannot be finitely axiomatized, see M. Haiman [42]. Behind this there is a series of higher Arguesian laws of increasing number of variables and strength (which can be separated by modified lattices of vector subspaces over any given field), each satisfied in lattices permuting equivalences and modular congruence varieties (this is in [41] and R. Freese [33]).

Motivated by the same question, namely whether every Arguesian lattice has a permuting equivalence representation, Alan and St. Tschantz [28] studied an identity due to Schuetzenberger and showed that it defined a lattice variety between permutable equivalence lattices and Arguesian lattices.

In another direction, Alan found that in the orthomodular lattice of closed subspaces of a Hilbert space the Arguesian identity is valid for triangles consisting of orthogonal points. From this he derived the orthoarguesian identity which is satisfied by Hilbert space lattices but not in general cf. [58]. Also, he realized that orthogonal frames in Arguesian ortholattices play a central role for the investigation of varieties of modular ortholattices – which then was undertaken by M. Roddy [66].

Problems. Is there a list of finite configurations, projective within the variety of modular lattices, the exclusion of which (as relative substructures) defines the class of Arguesian lattices? Can this list be chosen consisting of enrichments of PC's?

Is Schuetzenberger's identity equivalent to the Arguesian law? Is it valid in all 2-distributives?

Is there a decision procedure for the equational theory of Arguesian lattices?

Is there a finitely based modular lattice variety which is generated by its finite members and contains the subspace lattices of vector spaces over a fixed finite field?

Which representability classes of lattices form varieties? Consider e.g. vector space, (abelian) group, and permuting equivalence representations – cf. [3, 49]?

Is every orthoarguesian lattice representable by means of an orthomodular space?

4. Frames and coordinate rings

A lattice theoretic concept of coordinate system has been introduced by v.Neumann [64] and in an equivalent, but sometimes more convenient form, by G. Bergmann and A. Huhn: A *n-frame* or n-diamond (originally called a n-1-diamond) in a modular lattice L is a sequence $\mathbf{d} = (d_1, \dots d_{n+1}) = (x_1, \dots x_{n-1}, z, t)$ of elements of L such that

$$\Sigma_{j \neq i} d_j = v \qquad \text{for all } i$$

$$d_i \Sigma_{k \neq i,j} d_k = u \qquad \text{for all } i \neq j.$$

Figure 7a and b show the corresponding matroidal (consider only joins and those meets which satisfy the rank formula) and geometric diagrams (four points in general position and two coordinate axes with unit points and points at infinity). If

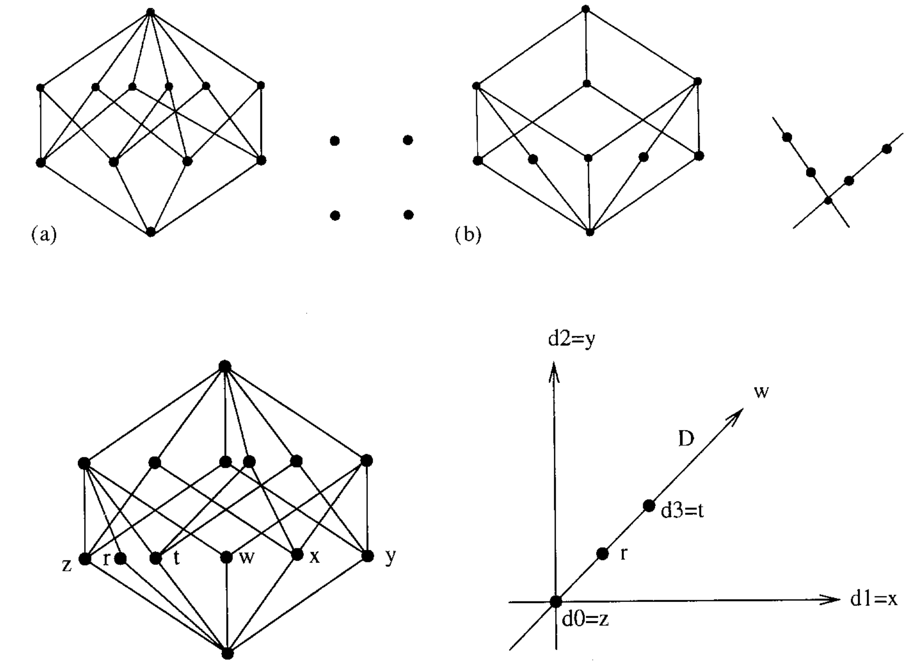

Figure 8

$u = 0_L$ and $v = 1_L$ one speaks of a *spanning* frame. In the standard example, L is the lattice $\mathscr{L}({}_R R^n)$ of all submodules of a free module with basis $e_0, \ldots e_{n-1}$, $x_i = Re_i$ for $1 \leq i \leq n$, $z = Re_0$, and $t = R(\Sigma_{1 \leq i < n} e_i)$. Projectively, these are the directions of the coordinate axes, the zero, and the unit point on the diagonal.

It is the diagonal, D, consisting of all complements of the point at infinity, $w = (z + t)\Sigma x_i$, in the interval $[u, z + t]$ which serves as *coordinate domain* – see Figure 8. In the example, $D = \{R(e_0 + a\ \Sigma_{0 < i}\, e_i \mid a \in R\}$. With $b_0 = (y + z)(x + b)$ and $b_1 = (y + t)(x + b)$, addition and multiplication are defined by

$$a \oplus b = (z + t)(x + (y + a)(w + b_0))$$

$$a \otimes b = (z + t)(x + (y + a)(z + b_1))$$

– see Figures 9 and 10. The following central result is due to v.Neumann [64] and R. Freese [32] for the modular part, to Alan and D. Pickering [26, 8] for the Arguesian part.

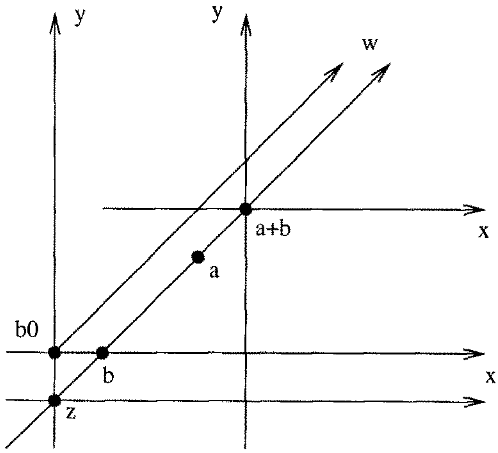

Figure 9

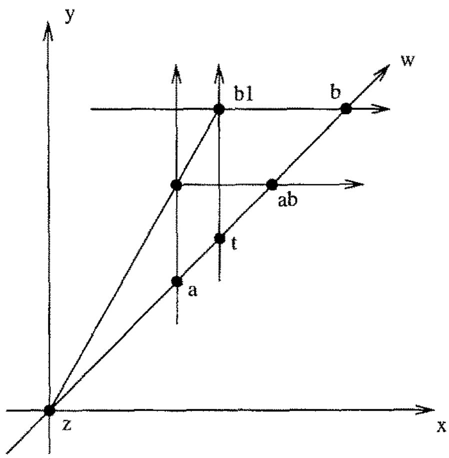

Figure 10

THEOREM 4. *For an n-frame in a modular lattice L, if $n \geq 4$ or if $n \geq 3$ and L is Arguesian, then the coordinate domain D is a ring with unit under the above definitions.*

If the coordinate ring has prime characteristic p, then the sublattice generated by the frame is an n-1-dimensional projective geometry over the p-element field.

This and Alan's help were crucial for the determination of the sublattices generated by the frame, in general, [47] and applications to the equational theory of modular lattices – cf. [11]. That the coordinate ring is to a high degree independent of the particular frame chosen was shown by Alan in [12].

Problems. Determine the Arguesian lattices (or just lattices of submodules) generated by a frame and a subring of its coordinate ring – at least for some decent rings and assuming that the subring is the full coordinate ring!

For a recursive ring, does the Arguesian lattice freely generated by a frame and a ring admit a solution of the word problem?

Analyze the relationships between the various rings associated with a skew frame and its subquotient frames in an Arguesian lattice – cf. [46]!

Is there a solution to the word problem for the modular lattice freely generated by a 3-frame?

5. Coordinatization

In their joint paper [26] Alan and D. Pickering proceeded to coordinatize hyperplanes. This required some additional hypotheses, of course. They used *upper complementability* which is satisfied in the standard example: for any join g over elements from the n-frame and p with $p+g=1$ there is a complement $s \leq p$ of g – see Figure 11. A *hyperplane* is a join of n-2 elements from the frame.

THEOREM 5. *For any Arguesian lattice L with upper complementable spanning n-frame, $n \geq 3$, and for every hyperplane h of such there is a lattice homomorphism of* $[0, h]$ *into* $\mathscr{L}({}_D D^{n-1})$ *having all finitely generated submodules in its image. Here, D is the coordinate ring of the frame.*

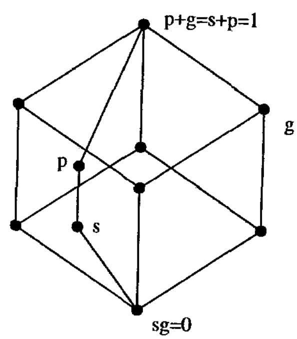

Figure 11

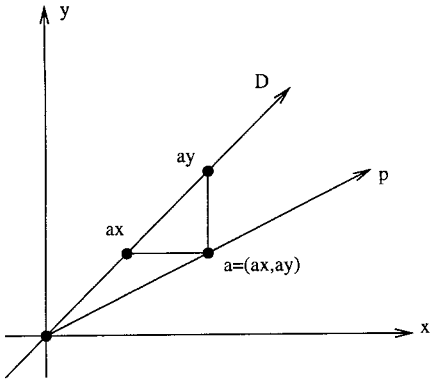

Figure 12

Their coordinatization is the natural one in the case of planes – see Figure 12. From this they derived that L is isomorphic to the lattice $\mathscr{L}_f({}_D D^n)$ of finitely generated submodules of $({}_D D^n)$ with regular resp. completely primary uniserial D in case L is complemented resp. primary Arguesian (in a primary lattice join and meet irreducibles form trees under order and reverse order, respectively, and any interval is either a chain or has at least 3 atoms). These are, basically, the classical coordinatization results of v.Neumann [64] and Jónsson-Monk [56]. An alternative (short) proof of v.Neumann's result also came out of their analysis: consider L embedded into the subgroup lattice of an abelian group (via Frink's embedding) and show that the canonical map from $\mathscr{L}_f({}_D D^n)$ is an isomorphism [48].

In another direction, the coordinatization of hyperplanes was extended by M. Greferath [38] to Arguesian lattices with an additional geometric structure tailored for submodule lattices, see [71].

Problems. Is there a common generalization of the coordinatization of complemented and primary lattices which also has the potential of computing sublattices?

For a modular lattice with spanning n-frame, $n \geq 4$, and hyperplane h, is the interval $[0, h]$ Arguesian or even embeddable into $L(D^{n-1})$?

Has every Arguesian lattice with spanning n-frame, $n \geq 3$, a permuting equivalence representation?

6. Quasiplanes

From an equational point of view, projective planes may be seen as modular lattices of primitive length ≤ 3 or as 3-distributive lattices. In the first approach, the

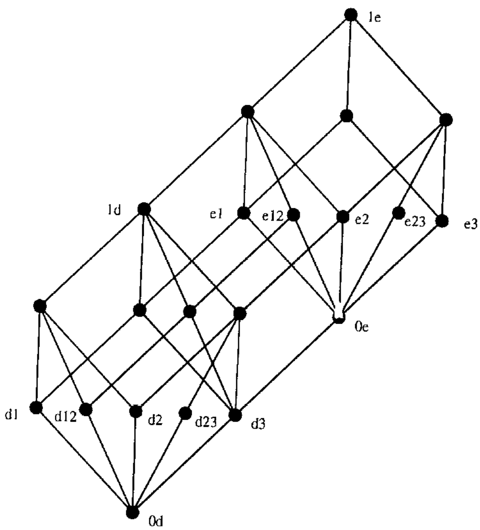

Figure 13

model class is close to the intended one but working with the equations becomes rather unhandy. The second approach includes too many models much more complicated than projective planes. Therefore, Alan [6, 7, 13] suggested to understand them as *quasiplanes* – modular lattices not containing any 2-dimensional gluing of nondegenerate 3-frames of Figure 13.

Here, two n-frames **d** and **e** in a modular lattice from a n-1-dimensional gluing, if $\Sigma_{\mathbf{d}}/d_{n+1}$ transposes up to $\Sigma_{i \leq n} d_i/\Pi\mathbf{e}$ matching the canonically induced n-1-frames (see [13, 11] for the generalization to k-dimensional gluings of an n and an m-frame). Gluings of non-degenerate frames can be understood as configurations projective within modular lattices, so their exclusion defines a variety. This variety can be also defined via a handy distributivity condition relatively to nondegenerate frames.

$$L \text{ modular 2-distributive} \;\Rightarrow\; L \text{ quasiplane} \;\Rightarrow\; L \text{ modular 3-distributive.}$$

Relatively complemented modular 3-distributive lattices are quasiplanes. Alan's central result [8, 6] shows how to characterize projective planes.

THEOREM 6. *For a modular lattice L the following are equivalent*:
(1) *L is an irreducible projective plane*

(2) *L is a subdirectly irreducible quasiplane generated by a 3-frame and its coordinate domain*
(3) *L has a spanning 3-frame with respect to which all nonzeros are invertible.*

Here, the coordinate domain of a 3-frame in a modular lattice is endowed with a ternary operator from which an additive loop and a multiplication with unit can be derived [26]. In the case of projective planes this yields a ternary ring, indeed.

Problems. Is a subdirectly irreducible quasiplane generated by a 3-frame and a subset of its coordinate domain necessarily a projective plane?

Is the word problem for (free) quasiplanes (or modular 2-distributives) in more than 4 generators decidable?

Characterize quasiplanes in the geometric language for spatial modular lattices [51]!

Do all Arguesian quasiplanes admit a permutable equivalence representation?

Is every quasiplane (modular 2-distributive) variety of finite height generated by a finite lattice?

Let L a finitely generated modular lattice of finite length such that every of its complemented intervals is locally finite. Is L finite (at least if it is a quasiplane)? The answer is 'yes' for 2-distributives.

7. Properties of rings

The example of embedding the lattice of a 3-dimensional space over the quaternions in that of a 6-dimensional space over the complex numbers shows that commutativity of the coordinatizing ring cannot be equationally expressed without a dimension restriction. In [6] Alan studied some possible approaches, each based on a projective configuration. A *strong line pair* consists of a_0, a_1, b_0, b_1 subject to relations which, in a modular lattice, grant that the sublattice generated is a subdirect product of a factor where these form a 3-frame and a factor where $a_0 = a_1$ and $b_0 = b_1$ – cf. Figure 14.

A modular lattice is (SLP-) *Pappian* if for each strong line pair and $a_2 \leq a_0 + a_1, b_2 \leq b_0 + b_1$ one has

$$(a_0 + b_1)(a_1 + b_0)(a_0 + a_1 + b_2)(b_0 + b_1 + a_2)$$
$$\leq (a_0 + b_2)(a_2 + b_0) + (a_1 + b_2)(a_2 + b_1).$$

See Figure 15 where λ denotes the left hand side. This, of course, reflects Pappus' Law from Projective Geometry. The dimension restriction implied by this identity

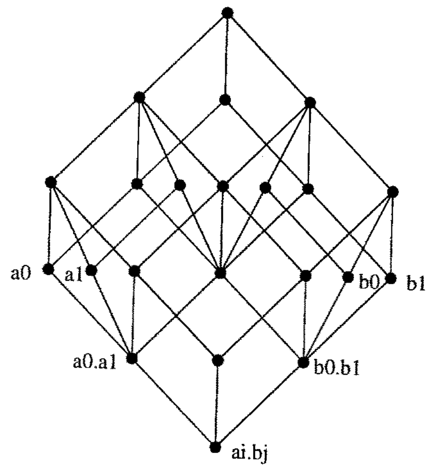

Figure 14

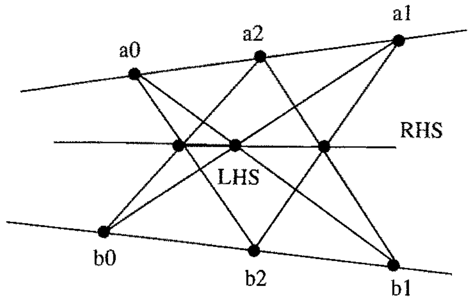

Figure 15

is 3-distributivity. Requiring a frame, even, one gets still 5-distributivity (which cannot be improved in view of the above example).

THEOREM 7. *R is strongly regular if and only if $\mathscr{L}({}_RR^3)$ is a quasiplane.*
If $\mathscr{L}({}_RR^3)$ is Pappian then R is commutative. For regular R the converse is also true.

Here, a ring is strongly regular if for every a there is x with $a^2x = a$ or, equivalently, it is regular and $a^2 = 0$ only for $a = 0$. These rings are of interest since they allow a decent sheaf representation.

Problems. Are there, besides the above and divisibility conditions for integers, any other properties of rings which can be expressed via identities for the lattice $\mathscr{L}({}_R R^3)$? How about commutativity and regularity?

Is every Pappian lattice Arguesian?

Is every finite Arguesian quasiplane Pappian?

8. Critical configurations for the Arguesian identity

In a series of joint papers with B. Jónsson [20, 21, 22, 23] Alan took up a theme from his earlier note [9]: How to understand the failure of the Arguesian law in terms of well described forbidden configurations, if not sublattices? The sublattice approach turned out inviable since D. Pickering [65] found later on an infinite list of finite minimal non-Arguesian modular lattices and even a modular non-Arguesian variety whose finite members all are Arguesian.

So, what they did, is to study non-Arguesian (i.e. $c_2 \not\leq c_0 + c_1$) PC's and to gain as much information as possible about them. Here, the p, c_i, $a_i b_j$ are also included into the concept of a PC. They undertook an intricate analysis of how to construct new non-Arguesian PC's from given ones aiming at *primes* where $a_1 b_1 (c_2 + c_2) \prec a_1 b_1$. More conceptually, such a PC has a specific lower cover in the pointwise ordered set of PC's.

THEOREM 8. *Every modular non-Arguesian lattice contains a prime PC at least in its ideal lattice.*

For a prime PC in a modular lattice L there are order preserving maps σ, π (defined in terms of the PC) of $\mathbf{2}^5$ into L such that $I_x = [\sigma x, \pi x]$ is a projective geometry of dimension at most 3 for $x = 0, 1$, an irreducible projective plane, otherwise. The union of these intervals contains the PC and the p, c_i, $a_i b_j$.

If the PC is stable and Boolean then σ and π factor through some $\mathbf{2}^r$, $r \leq 3$, and $\bigcup_x I_x$ is a sublattice.

Here, the PC is *Boolean* if σ and π are lattice homomorphisms. Of course, some of the intervals I_x may coincide. If that happens for all transposed I_x, I_y with $0 \prec x \prec y$ then the *PC* is called *stable*.

From each prime non-stable PC one obtains a stable and Boolean one by passing down in the order of PC's. Examples for any $r \leq 3$ can be found in the next section.

On the other hand, for a stable non-Boolean prime PC all of the twenty planes I_x, x of rank 2 or 3, are distinct. An example where this occurs has been constructed in [19], see Figure 16.

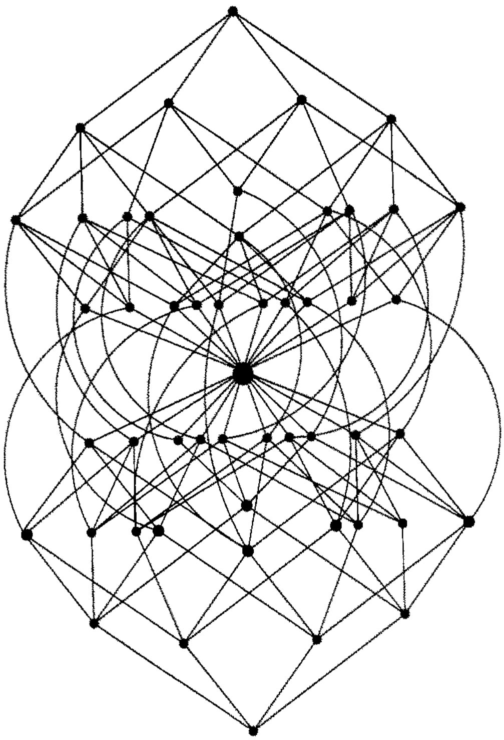

Figure 16

Problems. Are there stable prime PC's with 23 or more intervals, none contained in any other?

Can the Arguesian law be decently expressed in the geometric language for spatial modular lattices [51]?

Is there a configuration of join irreducibles (points) and meet irreducibles (hyperplanes), or finitely many such, the exclusion of which characterizes Arguesian lattices among finite length modular lattices?

9. Arguesian lattices and gluing

The above suggests to consider how the gluing structure and the Arguesian law interact. Alan, of course, had his own, categorical, view of gluing which finally in [18] resulted in a more general and more accessible presentation. Analytically, the most convenient way is to think an external "skeleton" lattice S and join resp. meet preserving 1-1 maps σ, π of S into L with $\sigma x \leq \pi x$ such that L is covered by the blocks $L_x = [\sigma x, \pi x]$. If necessary, one resorts to "imaginary" bounds in a lattice containing both, all ideals and filters of L. The tolerance γ on S indicates whether, for $x \leq y$, the blocks overlap: $x\gamma y$ iff $\sigma y \leq \pi x$. One speaks of a *gluing* if the equivalence relation generated by γ is total.

Synthetically, given S, γ, the blocks L_x, and a pair of adjunctions between each L_x, L_y, $x \leq y$ one can form the disjoint union with the order defined by that on the blocks and the adjunctions (a variant of Plonka's sum). Then, provided appropriate axioms are satisfied, one obtains the glued sum (with overlaps as encoded in γ) as homomorphic image. The gluing of modular lattices is always modular.

For Arguesian lattices, the situation is much more complicated. Still, Alan's papers [18, 23] provide some sufficient conditions (for the generalization one just has to notice that each complemented interval is contained in some block). Call a non-Arguesian PC *minimal* if it is so in the pointwise order of PC's.

THEOREM 9. *Let L be a gluing of Arguesian lattices. If S is a chain, or S is of length or width* 2, *or S is of width* ≤ 9 *then every minimal non-Arguesian PC is Boolean with at most* 2, 4 *and* 8 *intervals, respectively.*

On the other hand, L is Arguesian provided that $S = \mathbf{2}$ *and* $L_0 \cap L_1$ *is distributive or, else, S is modular of finite length and primitive breadth* ≤ 2 *and* $\bigcup_{x \in T} L_x$ *is Arguesian for each sublattice* $T \cong \mathbf{2}^n$, $n < 6$, *of S.*

Problems. Is there a non-Boolean minimal non-Arguesian PC in a finite length modular lattice L with $S(L)$ modular and/or 3-distributive?

In the second half of the Theorem: are the cases $n = 4, 5$ to be considered; is it sufficient to have S modular and/or 2-distributive?

Is there a non-2-distributive Arguesian lattice variety closed under gluings with $S = \mathbf{2}$? According to D. Pickering [65] frames in such a variety have characteristic 2, here, whence the variety cannot be closed under gluings with S the projective plane of order 2.

10. Arguesian lattices of finite length and representability

If L is modular of finite length, then one has a particular gluing the blocks of which are the maximal complemented intervals. S is then also addressed to as the *prime skeleton* $S(L)$.

Using the gluing of Arguesian blocks, Alan and his coauthors produced in [19, 23] typical, and partly also minimal, non-Arguesian lattices illustrating the theory of the preceding sections. These examples also give some insight into conditions granting representability.

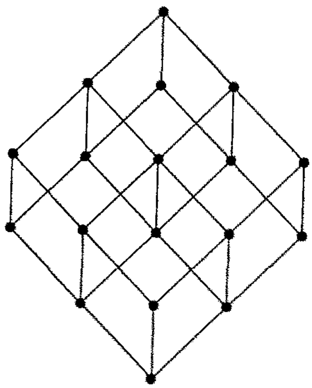

Figure 17

The basic observation dates back to B. Jónsson [55], in essence: Frames glued 2-dimensionally in an Arguesian lattice have isomorphic coordinate rings. If the blocks are coordinatized as in the standard example then L is Arguesian (and embeddable into a submodule lattice over the coordinate ring) if and only if the gluing map is induced by a semilinear map.

For L of length ≤ 5, by the preceding theorem, every minimal non-Arguesian PC has at most 4 intervals. Accordingly, assume that $S(L)$ is an M_n, $n \geq 2$, and that for any pair of neighbours we have Arguesian 2-dimensional gluings of irreducible projective planes – cf. Figure 17.

Then the gluing maps are induced by division ring isomorphisms. Under these hypotheses, L is Arguesian (and representable over a division ring) if and only if the composition of the isomorphisms along a gluing path depends on start and goal, only – cf. G. Kurinnoi [60]. As a consequence of this and B. Jónsson [54, 55], Alan and his coauthors derived (cf. [63]).

THEOREM 10. *A finite length modular lattice L of length ≤ 5 or with $S(L)$ a chain is Arguesian if and only if it has a permuting equivalence representation.*

In contrast, an Arguesian lattice of length 8 or with $S(L)$ distributive is not necessarily representable – this is witnessed by examples due to M. Haiman [42].

For L of length 6, if L is not Arguesian, then either there is a prime Boolean PC or $S(L)$ is an irreducible plane and blocks are pairwise isomorphic planes. The first kind of example, with $S(L) = \mathbf{2}^3$, appeared in D. Pickering [65] (cf. Fig. 18) and was generalized there to produce his infinite list of minimal non-Arguesian lattices. The second can be also chosen as a minimal non-Arguesian lattice which, moreover, is primary and so a projective Hjelmslev plane, cf. [27] for more on such.

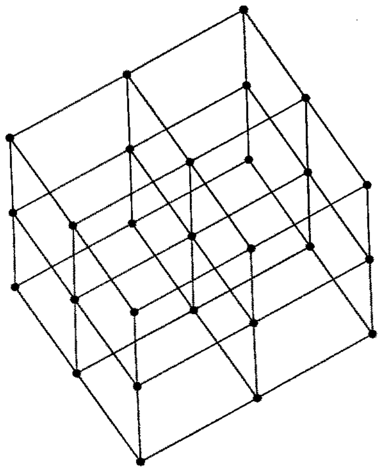

Figure 18

Problems. Has every Arguesian lattice of length 6 (7) a permuting equivalence representation?

Can representability be understood in terms of gluing of coordinatizable parts? On this there are some unpublished results of M. Haiman and J. B. Nation.

For finite lattices, is representability a decidable property?

For fixed n, is the class of representable lattices of length $\leq n$ finitely axiomatizable? Again, consider your favorite representability.

11. Databases

Alan was among the first to realize the connection of lattice theory to dependencies in relational databases. In [14] he gave a detailed analysis of the relationship between functional dependencies and semilattice congruences. As he pointed out, modular lattice theory necessarily comes in when studying embedded multivalued dependencies. From a mathematical point of view, a database is a family $(\theta_A)_{A \in U}$ of partitions on a set, each partition named with an "attribute". The underlying set can be seen as the set of rows and two rows having the same value of a given attribute will be in the same class of the partition. A functional dependency $X \to Y$ now amounts to $\theta_X \subseteq \theta_Y$ where $\theta_X = \bigcap_{A \in X} \theta_A$. Following Alan [15], an embedded multivalued or symmetric dependency $[X, Y]$ is satisfied if $\theta_X \circ \theta_Y = \theta_{X \cap Y}$. A multivalued dependency is one where $X \cup Y = U$, the total set of attributes. An *implication* $\Sigma \Rightarrow \sigma$ between a finite set of dependencies and a dependency is (finitely) valid if σ is satisfied in all (finite) databases satisfying Σ. An important question for each kind of dependencies is to find a sound and complete aziomatization of this semantic consequence operator or an algorithm deciding whether an implication is (finitely) valid: the (finite) *implication problem*, see the survey of P. Kanellakis [59]. Positive and negative results have been obtained in various cases.

On the basis of what happens in terms of meets and products of equivalence relations, Alan suggested in [15] a *natural lattice interpretation* of dependencies via sets of relations: A map $\varphi : U \to L$ is a model for $X \to Y$ if $\Pi\varphi X \leq \Pi\varphi Y$ and a model for $[X, Y]$ if for all $Z \subseteq X \cup Y$ one has $\Pi\varphi(Z \cup (X \cap Y)) = \Pi\varphi(Z \cup X) + \Pi\varphi(Z \cup Y)$. So, with an implication of dependencies there corresponds a conjunction of universal Horn lattice formulas. A lattice variety $\mathscr{V}$ provides a *sound and complete* natural interpretation if this conjunction is valid in $\mathscr{V}$ if and only if the implication is valid for databases. Alan showed in [15]:

THEOREM 11. *For the class of functional and multivalued dependencies, every nontrivial lattice variety provides a sound and complete natural interpretation – and this yields a new solution of the (finite) implication problem.*

For embedded multivalued dependencies such interpretation has to be in terms of a non-distributive modular variety.

In particular, the rather effective methods of solving word problems in the class of all lattices can be applied. But the following remains open:

Problems. Is there a non-distributive modular lattice variety providing a sound and complete (natural) interpretation of embedded multivalued dependencies? Does such have to be Arguesian?

Though, in the same mood one sees that the (finite) implication problem is unsolvable for embedded multivalued dependencies [50]: given an n-frame, $n \geq 3$, in a partition lattice and finitely many elements of the coordinate domain plus some auxiliary ones, if one requires sufficiently, but finitely many permutability and meet relations between these partitions, then they generate a sublattice of the partition lattice which is a sublattice of the congruence lattice of a suitably chosen module [48]. Since meet and permutability relations can be captured by embedded multivalued dependencies (it needs a little trick of Vardi's to eliminate functional dependencies) one can interpret an unsolvable word problem for semigroups into implications.

12. Frames in congruence modular varieties

The ring showing up in the above had been identified by Alan in a fine paper with E. Kiss [24], earlier. Even, they clarified the relationships between the various rings arising in Co-ordinatization and Commutator Theory – establishing one of the essential links between Lattice Theory and Universal Algebra.

THEOREM 12. *For a n-frame, $n \geq 3$, with top τ and bottom β, of congruences of an algebra* $\mathbf{A}$ *in a congruence modular variety* $\mathscr{V}$ *the following rings are canonically isomorphic*:

(1) *The v.Staudt–v.Neumann projective coordinate ring.*
(2) *The Day–Pickering coordinate ring.*
(3) *The subdirect product of the Euclid–Hilbert affine coordinate rings of the classes of* τ/β.
(4) *The Jónsson ring of equivalence relations on* A/β *derived from any of the associated 2-frames.*
(5) *The Day–Kiss ring of equivalence classes of endomorphisms of the pair* $(\mathbf{A}/\beta, \tau/\beta)$.
(6) *The Freese–McKenzie ring* $R(\mathscr{V})$ *of the variety, provided the frame is free in the variety.*

If the variety $\mathscr{V}$ *is residually small and either solvable or locally finite then its congruence variety is that of* $R(\mathscr{V})$*-modules or distributive.*

Problems. Has the congruence variety of groups a decidable equational theory? Is it generated by its finite members, even?

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Technische Hochschule Darmstadt
Fachbereich Mathematik
Schlossgartenstrasse 7
D-64289 Darmstadt
Germany