Mathematical Analysis of the Euler and Navier-Stokes Equations, and other Geophysical Models

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Abstract

In this series of lectures I will be covering two main topics. The first part will be concerning the Navier-Stokes and Euler equations. The second part will discuss the question of global regularity of certain geophysical flows.

The basic problem faced in geophysical fluid dynamics is that a mathematical description based only on fundamental physical principles, which are called the "Primitive Equations", is often prohibitively expensive computationally, and hard to study analytically. In these lectures I will survey the main obstacles in proving the global regularity for the three dimensional Navier–Stokes equations and their geophysical counterparts. However, taking advantage of certain geophysical balances and situations, such as geostrophic balance and the shallowness of the ocean and atmosphere, geophysicists derive more simplified and manageable models which are easier to study analytically. In particular, I will present the global well-posedness for the three dimensional Benard convection problem in porous media, and the global regularity for a three-dimensional viscous planetary geostrophic models. Furthermore, these systems will be shown to have finite dimensional global attractors. Based on the tools developed for attacking these problems I will also prove the global regularity for the three-dimensional Primitive equations of large scale oceanic and atmospheric dynamics.

In the inviscid case I will survey the-state-of-the-art theory concerning the three-dimensional Euler equations, and the role the rotation plays in extending the life span of the solutions. Moreover, I will also present a basic example of shear flow that was introduced by DiPerna and Majda to

study the weak limit of oscillatory solutions of the Euler equations of incompressible ideal fluids. In particular, they proved by means of this example that weak limit of solutions of Euler equations may, in some cases, fail to be a solution of Euler equations. I will use this shear flow example to provide non-generic, yet nontrivial, examples concerning the loss of smoothness of solutions of the three-dimensional Euler equations, for initial data that do not belong to $C^{1,\alpha}$. Moreover, I will show by means of this shear flow example the existence of weak solutions for the three-dimensional Euler equations with vorticity that is having a nontrivial density concentrated on non-smooth surface. This is very different from what has been proven for the two-dimensional Kelvin-Helmholtz problem where a minimal regularity implies the real analyticity of the interface. Eventually, we use this shear flow to provide explicit examples of non-regular solutions of the three-dimensional Euler equations that conserve the energy, an issue which is related to the Onsager conjecture.