WEAK NEUMANN IMPLIES STOKES

MATTHIAS GEISSERT, HORST HECK, MATTHIAS HIEBER, AND OKIHIRO SAWADA

ABSTRACT. Consider a domain $\Omega \subset \mathbb{R}^n$ with possibly non compact but uniform C^3-boundary and assume that the Helmholtz projection P exists on $L^p(\Omega)$ for *some* $1 < p < \infty$. It is shown that the Stokes operator in $L^p(\Omega)$ generates an analytic semigroup on $L^p_\sigma(\Omega)$ admitting maximal L^q-L^p-regularity. Moreover, for $u_0 \in L^p_\sigma(\Omega)$ there exists a unique local mild solution to the Navier-Stokes equations on domains of this form provided $p > n$.

1. INTRODUCTION

Given an open set $\Omega \subset \mathbb{R}^n$, it is well known that the Stokes operator is a selfadjoint and semibounded operator in $L^2_\sigma(\Omega)$. Hence, it is the generator of an analytic semigroup $(e^{tA})_{t\geq 0}$ on $L^2_\sigma(\Omega)$. Here $L^2_\sigma(\Omega)$ is defined by the Helmholtz decomposition of $L^2(\Omega)$ into $L^2_\sigma(\Omega) \oplus G_2(\Omega)$, which is valid in $L^2(\Omega)$ for all open sets $\Omega \subset \mathbb{R}^n$. The question whether $(e^{tA})_{t\geq 0}$ extends to an analytic semigroup on an L^p-space for some $1 < p < \infty$ and whether there are maximal L^q-L^p-estimates for the solution of the associated Stokes equation is more difficult to answer. In particular, the question whether the Stokes operator generates an analytic semigroup on $L^p(\Omega)$ for domains with noncompact boundaries recently gained quite some attention; see e.g. [FKS05].

An affirmative answer to the above question for bounded or exterior domains with smooth boundaries was first given by Solonnikov ([Sol77]). His proof makes use of potential theoretic arguments. Later on, further proofs were obtained e.g. by combining Giga's result on bounded imaginary powers of the Stokes operator ([Gig85]) with the Dore-Venni theorem, by Giga and Sohr [GS91a], by Grubb and Solonnikov [GS91b] using pseudo-differential techniques, by Dan, Kobayashi and Shibata [DKS98], [DS99] by local energy decay estimates and by Fröhlich [Fro01] making use of the concept of weighted estimates with respect to Muckenhoupt weights. For related results see also [Gig81], [FS94] and [SS08]. The half-space case was studied e.g. in [Uka87] and [DHP01]. For results concerning infinite layers we refer to the work of Abe and Shibata [AS03], Abels [Abe05] and Abels and Wiegner [AW05]. In [Fra00], [His04] the case of an aperture domain is discussed and in [FR08] it was shown that the Stokes operator has maximal L^q-L^p-regularity estimates on tube-like domains. For applications of these results to the equations of Navier-Stokes, see e.g.[Kat84], [Ama00] and [Soh01].

Considering unbounded domains with noncompact boundaries, no a priori estimates for the Stokes problem or no generation result for analytic semigroups on the classical function space $L^p(\Omega)$ seem to be known in general, unless $p = 2$. A key problem in the investigation of the Stokes problem in such general unbounded domains is that the Helmholtz decomposition of $L^p(\Omega)$ into $L^p_\sigma(\Omega) \oplus G_p(\Omega)$ is not possible, in general. Indeed, Bogovskiĭ gave in [Bog86] examples of unbounded domains Ω with smooth boundaries for which the Helmholtz decomposition of $L^p(\Omega)$ exists only for certain values of p. For details, see also [Gal94].

For results on weak or strong solutions to the Stokes and Navier-Stokes equations on *special* domains with noncompact boundaries, e.g. domains with strip-like or cylindrical outlets at infinity or parabolically growing layers, we refer to the works of Heywood [Hey76], Solonnikov [Sol81], Pileckas [Pil05], [Pil07], [Pil08]. In [AT09] Abels and Terasawa considered the reduced Stokes operator in unbounded

2000 *Mathematics Subject Classification.* 35Q30, 76D03.

Key words and phrases. Stokes equations, domains with noncompact boundaries, maximal $L^q - L^p$-estimates, Navier-Stokes equations.

The work of Matthias Hieber is supported by the Center of Excellence *Smart Interfaces* at TU Darmstadt.

domains with the additional assumption on Ω that the associated space for the pressure can be decomposed suitably. In [Abe10] the Stokes operator was studied under similar restrictons on the domain Ω.

One way to overcome the difficulties described above was shown in [FKS05] by Farwig, Kozono and Sohr by replacing the usual $L^p(\Omega)$-space by

$$\tilde{L}^p(\Omega) := \begin{cases} L^2(\Omega) \cap L^p(\Omega), & 2 \leq p < \infty, \\ L^2(\Omega) + L^p(\Omega), & 1 < p < 2 \end{cases}$$

for domains $\Omega \subset \mathbb{R}^n$ with uniform C^2-boundaries. They proved that the Helmholtz projection exists in the space $\tilde{L}^p(\Omega)$ and possesses the properties which are known for $L^p(\Omega)$. Moreover, it was shown by them that the Stokes operator $P\Delta$ is well defined in $\tilde{L}^p(\Omega)$ and generates an analytic semigroup on $\tilde{L}^p(\Omega)$. Furthermore, they showed the maximal L^q-L^p-regularity estimate

$$\|u_t\|_{L^q(J;\tilde{L}^p(\Omega))} + \|u\|_{L^q(J;\tilde{L}^p(\Omega))} + \|\nabla^2 u\|_{L^q(J;\tilde{L}^p(\Omega))} + \|\nabla \pi\|_{L^q(J;\tilde{L}^p(\Omega))} \leq C\|f\|_{L^q(J;\tilde{L}^p(\Omega))},$$

where $J = (0,T)$ for some $T > 0$, for the solution of the Stokes equation in domains Ω, i.e. for

$$\begin{aligned} u_t - \Delta u + \nabla \pi &= f && \text{in } \Omega \times (0,T), \\ \operatorname{div} u &= 0 && \text{in } \Omega \times (0,T), \\ u &= 0 && \text{on } \partial\Omega \times (0,T), \\ u(0) &= u_0 && \text{in } \Omega \end{aligned} \tag{1.1}$$

with $u_0 = 0$.

In this paper we will follow a *different* approach and will consider the above Stokes equation in the setting of *usual* L^p-spaces. We will consider domains $\Omega \subset \mathbb{R}^n$ with a uniformly C^3-boundary and assume that the Helmholtz projection P exists for $L^p(\Omega)$. We then show that the Stokes operator A_p, defined as in (2.1) below, generates an analytic semigroup on $L^p_\sigma(\Omega)$ and that the solution of the Stokes equation (1.1) satisfies the maximal L^q-L^p-regularity estimate.

Applying the well known iteration scheme (see [Kat84], [Gig86]) to our situation, we obtain as our second main result the existence of a unique, local mild solution to the Navier-Stokes equations defined on domains of the above form provided $p > n$.

This paper is organized as follows. In Sections 2 and 3 we state our main results concerning the Stokes and the Navier-Stokes equations, respectively. In the following Section 4 we discuss the strategy of our approach before we present in Section 5 certain tools for the proof of the main result which will be needed later on. Section 6 deals with representation formulas and estimates for the Stokes equations in the half space. The gain of regularity of weak solutions to the Neumann problem is shown in Section 7, whereas Section 8 is devoted to the proof of the linear result.

The authors would like to thank Prof. Yoshihiro Shibata for valuable discussions during the preparation of this paper.

2. Main Results for the Stokes equation

We start with the definition of a domain $\Omega \subset \mathbb{R}^n$ having a uniform smooth boundary. Given $k \in \mathbb{N}$, a domain $\Omega \subset \mathbb{R}^n$ is called a *uniform C^k-domain*, if there exist constants $K, \alpha, \beta > 0$ such that for each $x_0 \in \partial\Omega$ there exists a Cartesian coordinate system with origin at x_0, coordinates $y = (y', y_n)$ and $h \in C^k((-\alpha,\alpha)^{n-1})$ with $\|h\|_{C^k} \leq K$ such that the neighborhood

$$U(x_0) := \{(y', y_n) \in \mathbb{R}^n : h(y') - \beta < y_n < h(y') + \beta, |y'| < \alpha\}$$

of x_0 satisfies

$$U^-(x_0) := \{(y', y_n) \in \mathbb{R}^n : h(y') - \beta < y_n < h(y'), |y'| < \alpha\} = U(x_0) \cap \Omega$$

and $\partial\Omega \cap U(x_0) = \{(y', h(y')) : |y'| < \alpha\}$.

Secondly, given an open set $\Omega \subset \mathbb{R}^n$ and $1 < p < \infty$, we set

$$G_p(\Omega) := \{u \in L^p(\Omega) : u = \nabla\pi \text{ for some } \pi \in W^{1,p}_{\mathrm{loc}}(\Omega)\}$$
$$L^p_\sigma(\Omega) := \overline{\{u \in C_c^\infty(\Omega) : \operatorname{div} u = 0 \text{ in } \Omega\}}^{\|\cdot\|_p}.$$

We say that the *Helmholtz projection* exists for $L^p(\Omega)$ whenever $L^p(\Omega)$ can be decomposed into

$$L^p(\Omega) = L^p_\sigma(\Omega) \oplus G_p(\Omega),$$

where $\oplus$ denotes the direct sum operation. In this case, there exists a unique projection operator $P_p : L^p(\Omega) \to L^p_\sigma(\Omega)$ having $G_p(\Omega)$ as its null space. Setting $p' = \frac{p}{p-1}$ it is well known (see e.g. [Gal94]) that the Helmholtz projection exists for $L^p(\Omega)$ if and only if for every $f \in L^p(\Omega)$, there exists a unique function $u \in \widehat{W}^{1,p}(\Omega) := \{v \in L^1_{\mathrm{loc}}(\Omega) : \nabla v \in L^p(\Omega)\}$ satisfying

$$\langle \nabla u, \nabla\varphi\rangle = \langle f, \nabla\varphi\rangle, \quad \varphi \in \widehat{W}^{1,p'}(\Omega).$$

Thus the Helmholtz projection exists for $L^p(\Omega)$ if and only if for every $f \in L^p(\Omega)$ the above weak Neumann problem is uniquely solvable within the class $\widehat{W}^{1,p}(\Omega)$.

The following theorem is one of the main results of this paper.

2.1. Theorem. *Let $n \geq 2$, $p,q \in (1,\infty)$ and $J = (0,T)$ for some $T > 0$. Assume that $\Omega \subset \mathbb{R}^n$ is a domain with uniform C^3-boundary and that the Helmholtz projection P exists for $L^p(\Omega)$. Let $f \in L^q(J; L^p_\sigma(\Omega))$. Then equation (1.1) with $u_0 = 0$ admits a unique solution $(u,\pi) \in W^{1,q}(J; L^p(\Omega)) \cap L^q(J; W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap L^p_\sigma(\Omega)) \times L^q(J; \widehat{W}^{1,p}(\Omega))$ and there exists a constant $C > 0$ such that*

$$\|u_t\|_{L^q(J;L^p(\Omega))} + \|u\|_{L^q(J;L^p(\Omega))} + \|\nabla^2 u\|_{L^q(J;L^p(\Omega))} + \|\nabla\pi\|_{L^q(J;L^p(\Omega))} \leq C\|f\|_{L^q(J;L^p(\Omega))}.$$

Assuming as in the above theorem that the Helmholtz projection P exists for $L^p(\Omega)$, we may define the Stokes operator A_p in $L^p_\sigma(\Omega)$ as

$$\begin{aligned} D(A_p) &:= W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap L^p_\sigma(\Omega), \\ A_p u &:= P\Delta u \quad \text{for } u \in D(A_p). \end{aligned} \tag{2.1}$$

Then the following corollary concerning the Cauchy problem in $L^p_\sigma(\Omega)$

$$\begin{aligned} u'(t) - A_p u(t) &= f(t), \quad t > 0, \\ u(0) &= u_0 \end{aligned} \tag{2.2}$$

holds true.

2.2. Corollary. *Let $n \geq 2$, $p,q \in (1,\infty)$ and $J = (0,T)$ for some $T > 0$. Assume that $\Omega \subset \mathbb{R}^n$ is a domain with uniform C^3-boundary and that the Helmholtz projection P exists for $L^p(\Omega)$. Then the Stokes operator defined as in (2.1) generates an analytic C_0-semigroup T_p on $L^p_\sigma(\Omega)$ with generator A_p. Moreover, the solution u to the problem (2.2) satisfies*

$$\|u'\|_{L^q(J;L^p(\Omega))} + \|A_p u\|_{L^q(J;L^p(\Omega))} \leq C(\|f\|_{L^q(J;L^p(\Omega))} + \|u_0\|_{X_0})$$

for some constant $C > 0$ independent of $f \in L^q(J; L^p_\sigma(\Omega))$ and $u_0 \in X_0 = (L^p_\sigma(\Omega), D(A_p))_{1-1/q,q}$.

Setting $\nabla\pi = (Id - P)\Delta R(\lambda, A_p)f$, we also obtain the following result for the Stokes resolvent problem

$$\begin{aligned} \lambda u - \Delta u + \nabla\pi &= f \quad \text{in } \Omega, \\ \operatorname{div} u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.3}$$

for $\lambda \in \Sigma_\theta := \{\lambda \in \mathbb{C}; \lambda \neq 0, |\arg\lambda| < \theta\}$ for some $\theta \in (0,\pi)$.

2.3. Corollary. *Let $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ as above and $\theta \in (0,\pi)$. Then there exists $\lambda_0 \in \mathbb{R}$ such that for all $\lambda \in \lambda_0 + \sum_\theta$ and $f \in L^p_\sigma(\Omega)$ there exists a unique $(u,\pi) \in (W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap L^p_\sigma(\Omega)) \times \widehat{W}^{1,p}(\Omega)$ satisfying (2.3). Moreover, there exists $C > 0$ such that*

$$|\lambda|\|u\|_{L^p(\Omega)} + \|\nabla^2 u\|_{L^p(\Omega)} + \|\nabla\pi\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}, \quad \lambda \in \lambda_0 + \Sigma_\theta, f \in L^p_\sigma(\Omega).$$

3. Main Result for the Navier-Stokes equations

The semigroup $e^{tA_p} := T_p$ on $L^p_\sigma(\Omega)$ described in Corollary 2.2 admits the following L^p-L^q smoothing properties, which are well known for the situation of bounded or exterior domains.

3.1. Proposition. *Let $p, r, s \in (1,\infty)$ such that $s \le p \le r$ and let $T > 0$. Then there exists a constant $C > 0$ such that for $f \in L^s(\Omega)$*

$$\begin{aligned}
\|e^{tA_p}Pf\|_{L^r(\Omega)} &\le Ct^{-\frac{n}{2}(\frac{1}{s}-\frac{1}{r})}\|f\|_{L^s(\Omega)}, && \frac{1}{p}-\frac{2}{n} \le \frac{1}{r}, \quad \frac{1}{s} \le \frac{1}{p}+\frac{2}{n}, \quad 0<t<T.\\
\|\nabla e^{tA_p}Pf\|_{L^r(\Omega)} &\le Ct^{-\frac{n}{2}(\frac{1}{s}-\frac{1}{r})-\frac{1}{2}}\|f\|_{L^s(\Omega)}, && \frac{1}{p}-\frac{1}{n} \le \frac{1}{r}, \quad \frac{1}{s} \le \frac{1}{p}+\frac{1}{n}, \quad 0<t<T.\\
\|e^{tA_p}P \operatorname{div} f\|_{L^r(\Omega)} &\le Ct^{-\frac{n}{2}(\frac{1}{s}-\frac{1}{r})-\frac{1}{2}}\|f\|_{L^s(\Omega)}, && \frac{1}{p}-\frac{1}{n} \le \frac{1}{r}, \quad \frac{1}{s} \le \frac{1}{p}+\frac{1}{n}, \quad 0<t<T.
\end{aligned}$$

Proof. Note that by Theorem 5.1 of [Ste70] and our assumption on Ω, there exists a continuous extension operator $E : L^r(\Omega) \to L^r(\mathbb{R}^n)$ which is also continuous with respect to the $H^{2,p}$-norm. Hence, setting $\alpha = n(\frac{1}{p} - \frac{1}{r})$ it follows from Sobolev's embeddings and the continuity of the above extension operator that

$$\begin{aligned}
\|e^{tA_p}Pf\|_{L^r(\Omega)} &\le C\|Ee^{tA_p}Pf\|_{L^r(\mathbb{R}^n)} \le C\|Ee^{tA_p}Pf\|_{H^{\alpha,p}(\mathbb{R}^n)}\\
&\le C\|Ee^{tA_p}Pf\|^{1-\alpha/2}_{L^p(\mathbb{R}^n)}\,\|Ee^{tA_p}Pf\|^{\alpha/2}_{H^{2,p}(\mathbb{R}^n)}\\
&\le C\|e^{tA_p}Pf\|^{1-\alpha/2}_{L^p(\Omega)}\,\|e^{tA_p}Pf\|^{\alpha/2}_{H^{2,p}(\Omega)}\\
&\le Ct^{-\alpha/2}\|f\|_{L^p(\Omega)}, \quad f \in L^p(\Omega),\, 0<t<T,
\end{aligned}$$

for some constant $C > 0$. Note that we used the boundedness of the Helmholtz projection in $L^p(\Omega)$ only. In order to prove the estimate for $s \le p$ let $e^{tA'_p}$ be the dual semigroup of e^{tA_p} defined on $L^{p'}(\Omega)$, $f, \varphi \in C^\infty_c(\Omega)$ and P' the Helmholtz projection on $L^{p'}(\Omega)$; see Lemma 5.1 below. Then

$$\langle e^{tA_p}Pf, \varphi\rangle = \langle f, P'e^{tA'_p}P'\varphi\rangle = \langle f, e^{tA'_p}P'\varphi\rangle$$

and thus

$$\begin{aligned}
\|e^{tA_p}Pf\|_{L^p(\Omega)} &\le \|f\|_{L^s(\Omega)} \sup_{\|\varphi\|_{p'}=1} \|e^{tA'_p}P'\varphi\|_{L^{s'}(\Omega)}\\
&\le Ct^{-\frac{n}{2}(\frac{1}{p'}-\frac{1}{s'})}\|f\|_{L^s(\Omega)} \sup_{\|\varphi\|_{p'}=1} \|\varphi\|_{L^{p'}(\Omega)}\\
&\le Ct^{-\frac{n}{2}(\frac{1}{p'}-\frac{1}{s'})}\|f\|_{L^s(\Omega)}.
\end{aligned}$$

Since $\frac{1}{p'} - \frac{1}{s'} = \frac{1}{s} - \frac{1}{p}$, the proof of the first assertion is complete. The other assertions follow in a similar way. □

We finally consider the equations of Navier-Stokes

$$\begin{aligned}
u_t - \Delta u + (u\cdot\nabla)u + \nabla\pi &= 0 \quad \text{in } \Omega \times (0,T),\\
\operatorname{div} u &= 0 \quad \text{in } \Omega \times (0,T),\\
u &= 0 \quad \text{on } \partial\Omega \times (0,T),\\
u(0) &= u_0 \quad \text{in } \Omega,
\end{aligned} \tag{3.1}$$

and prove the following local wellposedness result for equation (3.1). To this end, assume that $\Omega \subset \mathbb{R}^n$ is a domain such that the Helmholtz projection P exists for $L^p(\Omega)$. Then, by a *mild solution* of (3.1) we understand a function $u \in C([0,T); L^p_\sigma(\Omega))$ for some $T > 0$ satisfying the integral equation

$$u(t) = e^{tA_p}u_0 - \int_0^t e^{(t-s)A_p} P \operatorname{div}(u(s) \otimes u(s)) ds, \qquad 0 \le t < T.$$

3.2. Theorem. *Let $n \ge 2$. Assume that $\Omega \subset \mathbb{R}^n$ is a domain with uniform C^3-boundary and that the Helmholtz projection P exists for $L^p(\Omega)$ for some $p > n$. Let $u_0 \in L^p_\sigma(\Omega)$. Then there exist $T_0 > 0$ and a unique mild solution u of* (3.1).

The proof follows the lines of the well known iteration procedure described in [Kat84] and [Gig86]. We will not give a detailed proof here and note only that the two main linear estimates for e^{tA_p} needed for the proof, namely the L^s-L^r-smoothing property and the gradient estimate for e^{tA_p} are provided by Proposition 3.1.

4. Comments on Localizations and The Divergence Equation

Before starting with the proof of our main theorem some comments about our localization procedure and the divergence equation are in order.

Starting from the corresponding result for the halfspace $\mathbb{R}^n_+$, the main problem is that the usual localization procedure known from elliptic problems does not transfer to the situation of the Stokes equation. Indeed, the usual localization procedure does not respect the divergence free condition. In [GHHSS08] a new localization procedure for the Stokes resolvent problem (2.3), respecting the divergence free condition, was introduced.

Before explaining the main idea, let us note that our assumption implies that one may choose for some $r \in (0, \alpha)$, depending only on α, β, K, balls $B_j := B_r(x_j)$ with centers $x_j \in \overline{\Omega}$ and C^3-functions h_j, $j = 1, 2, \dots, N$ if Ω is bounded and $j \in \mathbb{N}$ if Ω is unbounded, such that

$$\overline{\Omega} \subset \cup_{j=1}^\infty B_j, \quad \overline{B_j} \subset U(x_j) \text{ if } x_j \in \partial\Omega, \quad \overline{B_j} \subset \Omega \text{ if } x_j \in \Omega.$$

Moreover, we may construct this covering in such a way that not more than a finite fixed number $N_0 \in \mathbb{N}$ of these balls can have a nonempty intersection. Thus, choosing $N_0 + 1$ different balls $B_1, B_2, \dots$, their common intersection is empty. If Ω is bounded, we may choose $N_0 = N$.

Given the covering (B_j), there exists a partion of unity $\varphi_j \in C_c^\infty(\mathbb{R}^n)$ satisfying $\operatorname{supp} \varphi_j \subset B_j$ and $0 \le \varphi_j \le 1$.

In order to explain our main idea, let us consider

$$\tilde{u} := \sum_{j=1}^N \varphi_j u_j, \quad \tilde{\pi} = \sum_{j=1}^N \varphi_j \pi_j,$$

where φ_j are cut-off functions and (u_j, π_j) is the push-forward of the solution $(\hat{u}_j, \hat{\pi}_j)$ to

$$\begin{aligned} \lambda \hat{u}_j - \Delta \hat{u} + \nabla \hat{\pi}_j &= \hat{f}_j \quad \text{in } \mathbb{R}^n_+, \\ \operatorname{div} \hat{u}_j &= 0 \quad \text{in } \mathbb{R}^n_+, \\ \hat{u} &= 0 \quad \text{on } \partial\mathbb{R}^n_+, \end{aligned}$$

with a suitable right hand side $\hat{f}_j$. Since we assume that Ω has boundary of class C^3, we may construct the pull-back and push-forward mappings in such a way that they preserve the condition on the divergence. Hence, u_j is solenoidal by construction. But $\tilde{u}$ is not solenoidal in general, since

$$\operatorname{div} \tilde{u} = \sum_{j=1}^N (\nabla \varphi_j) u_j \neq 0.$$

Therefore, we use the modified ansatz

$$u := \sum_{j=1}^{N} \left(\varphi_j u_j + B_D(\operatorname{div}(\varphi_j u_j)) \right), \tag{4.1}$$

where B_D denotes Bogovskiĭ's operator on an open set $D \subset \Omega$ such that $\bigcup_{j=0}^{N} \operatorname{supp}(\nabla\varphi_j) \subset D$.

Inserting (u, π) in (2.3), we obtain

$$\begin{aligned} \lambda u - \Delta u + \nabla\pi &= f + T_\lambda f && \text{in } \Omega, \\ \operatorname{div} u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where T_λ denotes the correction terms. In order to show that T_λ is small for λ large, it is crucial to estimate the correction terms involving the pressure π and Bogovskiĭ's operator.

Note that, for domains with compact boundary it is enough to consider the divergence problem on suitable bounded domains, only. In particular, D as defined above is bounded. If the domain does not have a compact boundary it seems to be necessary to correct the divergence term on an unbounded domain. It would be tempting to extend this approach to countable many cut-off functions. However, in this case one would need estimates for the Bogovskiĭ operator in suitable higher order Sobolev spaces for the unbounded set D.

Recently, Diening, Růžička and Schumacher developed in [DRS08] a technique to decompose L^p functions on very rough domains Ω. These domains Ω are allowed to be unbounded, e.g. some fractal domains satisfy a condition which is related to John's condition. Then, they constructed a solution $u \in W^{1,p}(\Omega)$ of the divergence problem for suitable $f \in L^p(\Omega)$ by using a decomposition technique. Their approach allows to give a solution to the divergence problem for certain unbounded and rough domains. However, it seems to be unclear whether estimates of the form

$$\|B_D g\|_{W_0^{s+1,p}(D)^n} \leq C\|g\|_{W_0^{s,p}(D)}, \quad g \in W_0^{s,p}(D),$$

for higher order as well as negative Sobolev spaces, which would be needed, hold true in our situation.

In order to circumvent these difficulties, we present an approach to the Stokes problem on domains which noncompact boundaries which relies on the above localization procedure where, however, the Bogovskiĭ correction term is replaced by the weak solution of the *Neumann problem*:

$$\begin{aligned} \Delta v &= \operatorname{div} f && \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} &= f \cdot \nu && \text{on } \partial\Omega. \end{aligned} \tag{4.2}$$

To be more precise, we use the ansatz

$$u := \sum_{j=1}^{\infty} \left(\varphi_j u_j + \nabla v_j \right),$$

where v_j is a weak solution to the Neumann problem (4.2) with $f = \varphi_j u_j$. Note that the existence and uniqueness of v_j is guaranteed since the Helmholtz projection exists by assumption. By construction we then obtain

$$\operatorname{div} u = \sum_{j=1}^{\infty} \operatorname{div}(\varphi_j u_j) + \Delta v_j = 0.$$

However, the tangential component of u does not vanish at the boundary anymore. This leads to additional correction terms. In our main linear result we show that (2.3) has a unique solution for any $f \in L^q_\sigma(\Omega)$ satisfying the usual resolvent estimates.

Replacing norm bounds by $\mathcal{R}$-bounds (see e.g. [DHP03] or [KW04]) in the arguments above, we even obtain the maximal L^q-L^p-estimate in view of the vector-valued version of Mikhlin's theorem due to Weis [Wei01].

5. Tools for the Proof

This section is devoted to the presentation of several tools needed later on in the proof of our main result.

First, we consider the Helmholtz decomposition in $L^p(\Omega)$. More precisely, we prove that this decomposition exists in $L^p(\Omega)$ if and only if it exists for the dual space $L^{p'}(\Omega)$ where $1/p + 1/p' = 1$.

5.1. Lemma. *Let $\Omega \subset \mathbb{R}^n$ be open and assume that the Helmholtz projection exists on $L^p(\Omega)$ for some $p \in (1,\infty)$. Then the Helmholtz projection exists on $L^{p'}(\Omega)$, where p' denotes the dual exponent of p.*

Proof. It is well-known that the existence of the Helmholtz-projection is equivalent to the unique solvability of the following weak Neumann problem ($\mathrm{WN_p}$): Given $f \in L^p(\Omega)$, there exists a unique solution $u \in \widehat{W}^{1,p}(\Omega)$ satisfying

$$\langle \nabla u, \nabla \varphi \rangle = \langle f, \nabla \varphi \rangle, \quad \varphi \in \widehat{W}^{1,p'}(\Omega).$$

Since $\widehat{W}^{1,p}(\Omega)$ can be identified with a closed subspace of $L^p(\Omega)$, for given $g \in \widehat{W}^{-1,p}(\Omega) := \left(\widehat{W}^{1,p}(\Omega)\right)'$ there exists $f \in L^p(\Omega)$ such that

$$\langle f, \nabla \varphi \rangle = \langle g, \varphi \rangle, \quad \varphi \in \widehat{W}^{1,p'}(\Omega). \tag{5.1}$$

It thus follows from (5.1) that the existence of the Helmholtz-projection is equivalent to the solvability of the following weak Neumann problem ($\mathrm{WN2_p}$), which is different from ($\mathrm{WN_p}$): For given $g \in W^{-1,p}(\Omega)$ there exists a unique $u \in \widehat{W}^{1,p}(\Omega)$ satisfying

$$\langle \nabla u, \nabla \varphi \rangle = \langle g, \varphi \rangle, \quad \varphi \in \widehat{W}^{1,p'}(\Omega).$$

By duality, ($\mathrm{WN2_p}$) is uniquely solvable if and only if ($\mathrm{WN2_{p'}}$) is uniquely solvable. Hence, the Helmholtz projection exists on $L^{p'}(\Omega)$. □

In the following we make use of the concept of $\mathcal{R}$-bounded families of bounded operators. Here we only state the definition and refer to [DHP03] or [KW04] for further properties. Given Banach spaces X and Y, we call a family $\mathcal{T} \subset \mathcal{L}(X;Y)$ *$\mathcal{R}$-bounded*, if there exists a positive constant C such that for all $L \in \mathbb{N}$, $\mathcal{T}_\ell \in \mathcal{T}$, $x_\ell \in X$ for $\ell \in \{1,\dots,L\}$ and for all independent, symmetric, $\{-1,1\}$-valued random variables ε_ℓ on a probability space M the following inequality holds:

$$\|\sum_{\ell=1}^{L} \varepsilon_\ell \mathcal{T}_\ell x_\ell\|_{L^2(M;Y)} \le C \|\sum_{\ell=1}^{L} \varepsilon_\ell x_\ell\|_{L^2(M;X)}. \tag{5.2}$$

The smallest constant C such that (5.2) holds is called $\mathcal{R}$-bound of $\mathcal{T}$ and will be denoted by $\mathcal{R}_{X\to Y}(\mathcal{T})$. We simply write $\mathcal{R}_X(\mathcal{T}) = \mathcal{R}_{X\to X}(\mathcal{T})$. A sectorial operator B is called $\mathcal{R}$-sectorial, if $\mathcal{R}\{\lambda(\lambda-\mathcal{B})^{-1} : \lambda \in \Sigma_\theta\} < \infty$.

Next, we will state well known properties of the solution of the Stokes resolvent problem on $\mathbb{R}^n$. More precisely, consider

$$\begin{aligned} \lambda u - \Delta u + \nabla q &= f \qquad \text{in } \mathbb{R}^n, \\ \nabla \cdot u &= 0 \qquad \text{in } \mathbb{R}^n. \end{aligned} \tag{5.3}$$

Then the following result holds true.

5.2. Lemma. *For $p \in (1,\infty)$, $f \in L^p_\sigma(\mathbb{R}^n)$, $\theta \in (0,\pi)$ and $\lambda \in \Sigma_\theta$ there exists a unique solution (u,q) of (5.3) in the class $(W^{2,p}(\mathbb{R}^n) \cap L^p_\sigma(\mathbb{R}^n)) \times \widehat{W}^{1,p}(\mathbb{R}^n)$. Moreover, there exists a constant $C > 0$ such that*

$$\mathcal{R}_{L^p(\mathbb{R}^n)}\{\lambda(\lambda-\Delta)^{-1} \ : \ \lambda \in \Sigma_\theta\} \le C. \tag{5.4}$$

In particular, there exists a constant $C > 0$ such that

$$|\lambda| \|u\|_p + |\lambda|^{1/2} \|\nabla u\|_p + \|\nabla^2 u\|_p \le C \|f\|_p, \quad f \in L^p_\sigma(\mathbb{R}^n),\ \lambda \in \Sigma_\theta. \tag{5.5}$$

Next, we consider the Stokes resolvent equations in the half space with homogeneous boundary data

$$\begin{aligned} \lambda u - \Delta u + \nabla q &= f && \text{in } \mathbb{R}^n_+, \\ \nabla \cdot u &= 0 && \text{in } \mathbb{R}^n_+ \\ u &= 0 && \text{on } \partial\mathbb{R}^n_+. \end{aligned} \tag{5.6}$$

Then the following result is also known.

5.3. **Proposition.** *Let $p \in (1,\infty)$, $\theta \in (0,\pi)$, $\lambda \in \Sigma_\theta$ and $f \in L^p_\sigma(\mathbb{R}^n_+)$. Then there exists a unique solution $(u,q) \in (W^{2,p}(\mathbb{R}^n_+) \cap W^{1,p}_0(\mathbb{R}^n_+) \cap L^p_\sigma(\mathbb{R}^n_+)) \times \widehat{W}^{1,p}(\mathbb{R}^n_+)$ of equation (5.6).*

For a proof of these facts we refer e.g. to [Sol77], [Uka87] or [DHP01].

The following lemma contains further estimates on the solution of the Stokes resovent equations. In order to formulate the assertion precisely, let $\Omega_0 \subset \mathbb{R}^n_+$ be a bounded Lipschitz domain. We set

$$\begin{aligned} \widehat{U}^1_\lambda &: L^p_\sigma(\mathbb{R}^n_+) \to L^p_\sigma(\mathbb{R}^n_+), \quad \widehat{U}^1_\lambda f := u \\ \widehat{\Pi}^1_\lambda &: L^p_\sigma(\mathbb{R}^n_+) \to L^p(\Omega_0), \quad \widehat{\Pi}^1_\lambda f := q, \end{aligned}$$

where (u,q) is a solution (5.6) with $f \in L^p_\sigma(\mathbb{R}^n_+)$ satifying $\int_{\Omega_0} q = 0$.

5.4. **Lemma.** *Let $p \in (1,\infty)$, $s \in [0,2]$, $\alpha \in (0, \frac{1}{2p'})$ and $\theta \in (0,\pi)$. Then there exists a constant $C > 0$ such that*

$$\mathcal{R}_{L^p_\sigma(\mathbb{R}^n_+) \to W^{s,p}(\mathbb{R}^n_+)}\{(1+\lambda)^{1-s/2}\widehat{U}^1_\lambda \ : \ \lambda \in \Sigma_\theta\} \le C, \tag{5.7}$$

$$\mathcal{R}_{L^p_\sigma(\mathbb{R}^n_+) \to \widehat{W}^{1,p}(\mathbb{R}^n_+)}\{\widehat{\Pi}^1_\lambda \ : \ \lambda \in \Sigma_\theta\} \le C, \tag{5.8}$$

$$\mathcal{R}_{L^p_\sigma(\mathbb{R}^n_+) \to L^p(\Omega_0)}\{\lambda^\alpha \widehat{\Pi}^1_\lambda \ : \ \lambda \in \Sigma_\theta\} \le C. \tag{5.9}$$

For a proof of this lemma, see [GHHSS08].

Next, we establish $\mathcal{R}$-bounds for the operators which appear in the representation formula of the solution of the Stokes resolvent problem with inhomogeneous boundary data. Here Δ' denotes the Laplacian with respect to the coordinates $x' = (x_1, \cdots, x_{n-1})$.

5.5. **Lemma.** *Let $p \in (1,\infty), \alpha \in (0, \frac{1}{2p})$, $\lambda_0 > 0$ and $\theta \in (0,\pi)$. Then there exists a constant $C > 0$ such that*

$$\mathcal{R}_{L^p(\mathbb{R}^{n-1}) \to L^p(\mathbb{R}^n_+)}\{\lambda^\alpha e^{-\sqrt{\lambda-\Delta'}\cdot} \ : \ \lambda \in \lambda_0 + \Sigma_\theta\} \le C, \tag{5.10}$$

$$\mathcal{R}_{W^{2-1/p,p}(\mathbb{R}^{n-1}) \to L^p(\mathbb{R}^n_+)}\{\Delta' e^{-\sqrt{\lambda-\Delta'}\cdot} \ : \ \lambda \in \lambda_0 + \Sigma_\theta\} \le C. \tag{5.11}$$

Proof. We recall from [DHP01] that Δ' admits an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus in $L^p(\mathbb{R}^{n-1})$. Note that

$$\mathcal{R}_{L^p(\mathbb{R}^{n-1})}\{\lambda^\alpha e^{-\sqrt{\lambda-\Delta'}x_n} \ : \ \lambda \in \lambda_0 + \Sigma_\theta\} \le \sup_{\lambda \in \lambda_0 + \Sigma_\theta, z \in \Sigma_\phi} |\lambda^\alpha e^{-\sqrt{\lambda+z}x_n}| \le C x_n^{-2\alpha} e^{-c\sqrt{\lambda_0}x_n}, \quad x_n > 0$$

for any $\phi \in (0, \pi-\theta)$ and some constant $c > 0$. Let $L \in \mathbb{N}$, $\lambda_\ell \in \lambda_0 + \Sigma_\theta$ and ε_ℓ symmetric, independent $\{-1,1\}$-valued random variables on a probability space $\mathcal{P}$ and $a_\ell \in L^p(\mathbb{R}^{n-1})$ for $\ell \in \{1, \cdots, L\}$. Then

$$\begin{aligned} \|\sum_{\ell=1}^L \varepsilon_\ell \lambda_\ell^\alpha e^{-\sqrt{\lambda_\ell-\Delta'}\cdot} a_\ell\|^p_{L^p(\mathcal{P};L^p(\mathbb{R}^n_+))} &= \int_{\mathcal{P}} \int_0^\infty \int_{\mathbb{R}^{n-1}} \left| \sum_{\ell=1}^L \varepsilon_\ell \lambda_\ell^\alpha e^{-\sqrt{\lambda_\ell-\Delta'}x_n} a_\ell(x') \right|^p \mathrm{d}x' \, \mathrm{d}x_n \, \mathrm{d}\omega \\ &\le C \int_{\mathcal{P}} \int_0^\infty x_n^{-2p\alpha} e^{-c\sqrt{\lambda_0}x_n} \, \mathrm{d}x_n \|\sum_{\ell=1}^L \varepsilon_\ell a_\ell\|^p_{L^p(\mathbb{R}^{n-1})} \, \mathrm{d}\omega \\ &\le C \|\sum_{\ell=1}^L \varepsilon_\ell a_\ell\|^p_{L^p(\mathcal{P};L^p(\mathbb{R}^{n-1}))}. \end{aligned}$$

This proves (5.10).

In order to prove (5.11) we use similar arguments as above. The boundedness of the term $(\sqrt{\lambda}+\sqrt{z})/(\sqrt{\lambda+z})$ for $\lambda \in \lambda_0 + \Sigma_\theta$ and $z \in \Sigma_\phi$ implies

$$e^{-\sqrt{\lambda+z}x_n} \le Ce^{-(\sqrt{\lambda}+\sqrt{z})x_n}.$$

Hence, using the $\mathcal{R}$-bounded H^∞-calculus for Δ' we obtain

$$\begin{aligned}
\|\sum_{\ell=1}^{L} \varepsilon_\ell \Delta' e^{-\sqrt{\lambda_\ell-\Delta'}\cdot} a_\ell\|^p_{L^p(\mathcal{P},L^p(\mathbb{R}^n_+))} &\le \int_{\mathcal{P}}\int_{\mathbb{R}^n_+} \Big|\sum_{\ell=1}^{L} \varepsilon_\ell \sqrt{-\Delta'} e^{-\sqrt{\lambda_\ell-\Delta'}x_n} \sqrt{-\Delta'} a_\ell(x')\Big|^p \,\mathrm{d}x\,\mathrm{d}\omega \\
&\le C\int_{\mathcal{P}}\int_{\mathbb{R}^n_+} \Big|\sum_{\ell=1}^{L} \varepsilon_\ell \sqrt{-\Delta'} e^{-(\sqrt{\lambda_\ell}+\sqrt{-\Delta'})x_n} \sqrt{-\Delta'} a_\ell(x')\Big|^p \,\mathrm{d}x\,\mathrm{d}\omega \\
&\le \int_{\mathcal{P}}\int_{\mathbb{R}^n_+} e^{-p\sqrt{\lambda_0}x_n} \Big|\sum_{\ell=1}^{L} \varepsilon_\ell \sqrt{-\Delta'} e^{-\sqrt{-\Delta'}x_n} \sqrt{-\Delta'} a_\ell(x')\Big|^p \,\mathrm{d}x\,\mathrm{d}\omega \\
&\le \|\sum_{\ell=1}^{L} \varepsilon_\ell \sqrt{-\Delta'} e^{-\sqrt{-\Delta'}\cdot} \sqrt{-\Delta'} a_\ell\|^p_{L^p(\mathcal{P},L^p(\mathbb{R}^n_+))} \\
&\le C\|\sum_{\ell=1}^{L} \varepsilon_\ell \sqrt{-\Delta'} a_\ell\|^p_{L^p(\mathcal{P},W^{1-1/p,p}(\mathbb{R}^{n-1}))} \\
&\le C\|\sum_{\ell=1}^{L} \varepsilon_\ell a_\ell\|^p_{L^p(\mathcal{P},W^{2-1/p,p}(\mathbb{R}^{n-1}))}.
\end{aligned}$$

Note that in the second last inequality we used the maximal regularity property of $\sqrt{-\Delta'}$. □

The final lemma of this section gives a tool to calculate $\mathcal{R}$-bounds of expressions appearing during the localization procedure desribed later on. Given Banach spaces X_j and Y_j for $j \in \mathbb{N}$, we set

$$\ell^p(X_j) = \{f = (f_j)_{j\in\mathbb{N}} : f_j \in X_j,\ \|f\| := (\sum_{j=1}^{\infty} \|f_j\|^p_{X_j})^{1/p} < \infty\}.$$

5.6. **Lemma.** *Let $(X_j),(Y_j)$, $j \in \mathbb{N}$, be sequences of Banach spaces and I be an arbitrary index set. Let $L_t \in \mathcal{L}(\ell^p(X_j),\ell^p(Y_j))$ for any $t \in I$ be a diagonal operator, i.e. $L_t x = (L_t^{(j)}x_j)_{j\in\mathbb{N}}$, where $L_t^{(j)} \in \mathcal{L}(X_j,Y_j)$. If for any $j \in \mathbb{N}$ the set $\{L_t^{(j)} : t \in I\}$ is $\mathcal{R}$-bounded and* $\max_{j\in\mathbb{N}}\{\mathcal{R}\{L_t^{(j)} : t \in I\}\} = R < \infty$, *then $\{L_t : t \in I\}$ is $\mathcal{R}$-bounded with $\mathcal{R}\{L_t : t \in I\} \le R$.*

Proof. Note first that by Kahane's inequality it is possible to replace the exponent 2 in the definition of the $\mathcal{R}$-bound by p. Let M be a probability space and let (ε_k) denote a sequence of independent symmetric $\{-1,1\}$-valued random variables. We write $X = \ell^p(X_j)$. Further let $K \in \mathbb{N}$ and $t_k \in I$, $f_k = (f_k^{(j)})_{j\in\mathbb{N}} \in X$, for $k = 1,\cdots,K$. Setting $L_k := L_{t_k}$ we calculate

$$\begin{aligned}
\|\sum_{k=1}^{K} \varepsilon_k L_k f_k\|_{L^p(M;Y)} &= \Big(\int_M \Big(\sum_{j=1}^{\infty} \Big\|\sum_{k=1}^{K} \varepsilon_k(\omega) L_k^{(j)} f_k^{(j)}\Big\|^p_{Y_j}\Big)\,\mathrm{d}\omega\Big)^{1/p} \\
&= \Big(\sum_{j=1}^{\infty}\int_M \Big\|\sum_{k=1}^{K} \varepsilon_k(\omega) L_k^{(j)} f_k^{(j)}\Big\|^p_{Y_j}\,\mathrm{d}\omega\Big)^{1/p} \\
&\le \Big(\sum_{j=1}^{\infty} C_j^p \int_M \Big\|\sum_{k=1}^{K} \varepsilon_k(\omega) f_k^{(j)}\Big\|^p_{X_j}\,\mathrm{d}\omega\Big)^{1/p} \\
&\le R\Big(\int_M \Big\|\sum_{k=1}^{K} \varepsilon_k(\omega) f_k\Big\|^p_X\,\mathrm{d}\omega\Big)^{1/p}.
\end{aligned}$$

Therefore, $\{L_t : t \in I\}$ is $\mathcal{R}$-bounded. □

6. The Half space with Inhomogeneous Boundary Data

In this section we consider the resolvent problem for the Stokes equation in the half space with inhomogeneous boundary data. In particular, we prove $\mathcal{R}$-boundedness of our solution operator with respect to certain Sobolev norms and show moreover decay estimates with respect to λ. To this end, we first introduce the scaling matrices

$$K_\lambda := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^{1-\frac{1}{3p}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad K'_\lambda := \begin{pmatrix} \lambda^{1-\frac{1}{3p}} & 0 \\ 0 & 1 \end{pmatrix}$$

and set

$$\widehat{X}_{a,b} := \{a \in W^{1-1/p,p}(\partial\mathbb{R}^n_+)^n : a \cdot \nu = 0\} \times \{b \in W^{2-1/p,p}(\partial\mathbb{R}^n_+)^n : b \cdot \nu = 0\}.$$

Since $a \cdot \nu = a_n = 0$ and $b \cdot \nu = b_n = 0$ for $(a,b) \in \widehat{X}_{a,b}$, we denote for simplicity the first $n-1$ components of a also by a, i.e. $a = a'$, if no confusion seems to be likely.

We now define $\widehat{U}^2_\lambda : \widehat{X}_{a,b} \to W^{2,p}(\mathbb{R}^n_+)^n$ and $\widehat{\Pi}^2_\lambda : \widehat{X}_{a,b} \to \widehat{W}^{1,p}(\mathbb{R}^n_+)$ by

(6.1)

$$\left(\widehat{\Pi}^2_\lambda(a,b)\right)(\cdot, x_n) := -\frac{\sqrt{\lambda - \Delta'} - \sqrt{-\Delta'}}{(\lambda - \Delta')} \mathrm{e}^{-\sqrt{-\Delta'}x_n} \left(\lambda \frac{\nabla' \cdot a}{\sqrt{-\Delta'}} - \Delta' \frac{\nabla' \cdot b}{\sqrt{-\Delta'}}\right),$$

$$\left(\widehat{U}^2_\lambda(a,b)\right)(\cdot, x_n) := \begin{pmatrix} (\lambda - \Delta_D)^{-1}(-\nabla'\widehat{\Pi}^2_\lambda(a,b)) \\ (\lambda - \Delta_N)^{-1}(-\partial_n\widehat{\Pi}^2_\lambda(a,b)) \end{pmatrix} + \mathrm{e}^{-\sqrt{\lambda-\Delta'}x_n} \begin{pmatrix} \frac{\lambda}{\lambda-\Delta'}a - \frac{\Delta'}{\lambda-\Delta'}b \\ \frac{\nabla'\cdot}{\sqrt{\lambda-\Delta'}}\left(\frac{\lambda}{\lambda-\Delta'}a - \frac{\Delta'}{\lambda-\Delta'}b\right) \end{pmatrix}.$$

Later on, we will set $a = b$. The reason to use different variables here is due to the fact that we will estimate the boundary data with respect to different norms. For a we will use the $W^{1-1/p,p}$-norm and exploit the decay in λ of this norm. For b we will use the $W^{2-1/p,p}$-norm with no decay in λ. Since we have to estimate $\mathcal{R}$-bounds of sets parameterized by λ we cannot use λ dependend norms like $|\lambda|\|u\|_p + \sqrt{|\lambda|}\|\nabla u\|_p + \|\Delta u\|_p$ (cf. Lemma 6.1.(c)).

Mapping as well as decay properties of $\widehat{U}^2_\lambda$ and $\widehat{\Pi}^2_\lambda$, respectively, are being described in the following lemma.

6.1. Lemma. (a) *Let $(a,b) \in \widehat{X}_{a,b}$. Then*

$$\begin{aligned} \nabla \cdot \widehat{U}^2_\lambda(a,b) &= 0 \quad \textit{in } \mathbb{R}^n_+, \\ \nu \cdot \widehat{U}^2_\lambda(a,b) &= 0 \quad \textit{on } \partial\mathbb{R}^n_+. \end{aligned}$$

In particular, $U^2_\lambda : \widehat{X}_{a,b} \to W^{2,p}(\mathbb{R}^n_+) \cap L^p_\sigma(\mathbb{R}^n_+)$.

(b) *Let $a \in W^{2-1/p,p}(\mathbb{R}^{n-1})^n$ with $\nu \cdot a = 0$. Then, $(u,q) := (\widehat{U}^2_\lambda(a,a), \widehat{\Pi}^2_\lambda(a,a))$ is the unique solution of*

$$\begin{aligned} \lambda u - \Delta u + \nabla q &= 0 \quad \textit{in } \mathbb{R}^n_+, \\ \nabla \cdot u &= 0 \quad \textit{in } \mathbb{R}^n_+, \\ u &= a \quad \textit{on } \partial\mathbb{R}^n_+. \end{aligned}$$

(c) *For $X_n > 0$ there exists $C > 0$ such that*

$$\begin{aligned} \mathcal{R}_{\widehat{X}_{a,b} \to L^p(\mathbb{R}^{n-1}\times(0,X_n))} \left\{\lambda^{\frac{1}{2p'}} \widehat{\Pi}^2_\lambda {K'}^{-1}_\lambda : \lambda \in 1 + \Sigma_\theta\right\} &\le C, \\ \mathcal{R}_{\widehat{X}_{a,b} \to L^p(\mathbb{R}^n_+)} \left\{\nabla\widehat{\Pi}^2_\lambda {K'}^{-1}_\lambda : \lambda \in 1 + \Sigma_\theta\right\} &\le C, \\ \mathcal{R}_{\widehat{X}_{a,b} \to W^{k,p}(\mathbb{R}^n_+)} \left\{\lambda^{\frac{2-k}{2}} \widehat{U}^2_\lambda {K'}^{-1}_\lambda : \lambda \in 1 + \Sigma_\theta\right\} &\le C, \quad k = 0,1,2. \end{aligned}$$

Proof. a) We first prove that $\nu\cdot\widehat{U}^2_\lambda(a,b)_{|\partial\mathbb{R}^n_+}=0$. We set $v:=(\lambda-\Delta_N)^{-1}(-\partial_n\Pi^2_\lambda(a,b))$. In order to see this, we rewrite v in view of the second term in the formula for $\widehat{U}^2_\lambda$ in (6.1). Let $(a,b)\in\widehat{X}_{a,b}\cap C_c^\infty(\partial\mathbb{R}^n_+)^2$. We set $\partial_n\widehat{\Pi}^2_\lambda(a,b)=:\mathrm{e}^{-\sqrt{-\Delta'}x_n}c=:w_1$ where

$$c:=\frac{(\sqrt{\lambda-\Delta'}-\sqrt{-\Delta'})}{\lambda-\Delta'}(\lambda\nabla'a-\Delta'\nabla'b).$$

Then obviously $\Delta w_1=0$. Setting $w_2=-v-\lambda^{-1}w_1$ it follows that

$$\begin{aligned}(\lambda-\Delta)w_2&=(\lambda-\Delta)v-\frac{1}{\lambda}(\lambda w_1-\Delta w_1)\\&=w_1-w_1+\frac{1}{\lambda}\Delta w_1=0 &&\text{in } \mathbb{R}^n_+\\ \partial_n w_2&=\partial_n v-\frac{1}{\lambda}\partial_n w_1=\frac{1}{\lambda}\sqrt{-\Delta'}c &&\text{on } \partial\mathbb{R}^n_+.\end{aligned}$$

Hence, $w_2=-\frac{1}{\lambda}\frac{\sqrt{-\Delta'}}{\sqrt{\lambda-\Delta'}}\mathrm{e}^{-\sqrt{\lambda-\Delta'}x_n}c$ and therefore

$$v=-w_2-\frac{1}{\lambda}w_1=-\frac{1}{\lambda}\frac{\sqrt{-\Delta'}}{\sqrt{\lambda-\Delta'}}\mathrm{e}^{-\sqrt{\lambda-\Delta'}x_n}c-\frac{1}{\lambda}\mathrm{e}^{-\sqrt{-\Delta'}x_n}c.$$

In particular, we have

$$\begin{aligned}v(\cdot,0)&=-\frac{1}{\lambda}\left(Id+\frac{\sqrt{-\Delta'}}{\sqrt{\lambda-\Delta'}}\right)\left(\frac{\sqrt{\lambda-\Delta'}-\sqrt{-\Delta'}}{\lambda-\Delta'}\right)(\lambda\nabla'\cdot a-\Delta'\nabla'\cdot b)\\&=-\frac{1}{\sqrt{\lambda-\Delta'}^3}(\lambda\nabla'\cdot a-\Delta'\nabla'\cdot b).\end{aligned}$$

Therefore $\nu\cdot\widehat{U}^2_\lambda(a,b)=0$ on $\partial\mathbb{R}^n_+$. A density argument proves the first assertion for all $(a,b)\in\widehat{X}_{a,b}$. Note that $\partial_n(\lambda-\Delta_N)^{-1}=(\lambda-\Delta_D)^{-1}\partial_n$. This gives $\nabla\cdot\widehat{U}^2_\lambda(a,b)=0$ in $\mathbb{R}^n_+$.
b) The representation

$$\begin{aligned}\widehat{\Pi}^2_\lambda(a,a)(\cdot,x_n)&=-\left(\sqrt{\lambda-\Delta'}-\sqrt{-\Delta'}\right)\mathrm{e}^{-\sqrt{-\Delta'}x_n}\frac{\nabla'\cdot a}{\sqrt{-\Delta'}},\\ \widehat{U}^2_\lambda(a,a)(\cdot,x_n)&=\begin{pmatrix}(\lambda-\Delta_D)^{-1}(-\nabla'\widehat{\Pi}^2_\lambda(a,a))\\(\lambda-\Delta_N)^{-1}(-\partial_n\widehat{\Pi}^2_\lambda(a,a))\end{pmatrix}+\mathrm{e}^{-\sqrt{\lambda-\Delta'}x_n}\begin{pmatrix}a\\\frac{\nabla\cdot a}{\sqrt{\lambda-\Delta'}}\end{pmatrix}\end{aligned}$$

yields $\lambda u-\Delta u+\nabla p=0$ and $\widehat{U}^2_\lambda(a,a)'=a'$ on $\partial\mathbb{R}^n_+$. Together with (a) this proves (b).
c) For $X_n>0$ we define

$$\begin{aligned}A_\lambda&:=\begin{cases}L^p(\mathbb{R}^n_+)&\to L^p(\mathbb{R}^n_+)\\ f&\mapsto\frac{\sqrt{\lambda-\Delta'}-\sqrt{-\Delta'}}{\lambda-\Delta'}f\end{cases}\\ B_1&:=\begin{cases}W^{1-1/p,p}(\partial\mathbb{R}^n_+)&\to L^p(\mathbb{R}^{n-1}\times(0,X_n))\cap\widehat{W}^{1,p}(\mathbb{R}^n_+)\\ a&\mapsto(\cdot,x_n)\mapsto\mathrm{e}^{-\sqrt{-\Delta'}x_n}\sqrt{-\Delta'}^{-1}\nabla'\cdot a\end{cases}\\ B_2&:=\begin{cases}W^{2-1/p}(\partial\mathbb{R}^n_+)&\to L^p(\mathbb{R}^n)\\ b&\mapsto(\cdot,x_n)\mapsto\mathrm{e}^{-\sqrt{-\Delta'}x_n}\sqrt{-\Delta'}^{-1}\Delta'\nabla'\cdot b.\end{cases}\end{aligned}$$

We thus may write

$$\begin{aligned}\widehat{\Pi}^2_\lambda(a,b)&=-A_\lambda\left(\lambda B_1(a)-B_2(b)\right)\\ \nabla'\widehat{\Pi}^2_\lambda(a,b)&=-A_\lambda\nabla'\left(\lambda B_1(a)-B_2(b)\right).\end{aligned}$$

Since $C_A^1 := \mathcal{R}_{L^p(\mathbb{R}^n_+)}\{\lambda^{\frac{1}{2}}A_\lambda : \lambda \in 1+\Sigma_\theta\} < \infty$, $C_A^2 := \mathcal{R}_{L^p(\mathbb{R}^n_+)}\{\nabla' A_\lambda : \lambda \in 1+\Sigma_\theta\} < \infty$ and B_1 and B_2 are bounded, we obtain

$$\mathcal{R}_{\widehat{X}_{a,b}\to L^p(\mathbb{R}^{n-1}\times(0,X_n))}\left\{\lambda^{\frac{1}{2p'}}\widehat{\Pi}^2_\lambda {K'}_\lambda^{-1} : \lambda \in 1+\Sigma_\theta\right\} \le C_A^1\left(\|B_1\| + \|B_2\|\right) < \infty,$$
$$\mathcal{R}_{\widehat{X}_{a,b}\to L^p(\mathbb{R}^{n-1}\times(0,X_n))}\left\{\nabla\widehat{\Pi}^2_\lambda {K'}_\lambda^{-1} : \lambda \in 1+\Sigma_\theta\right\} \le C_A^1\|B_1\| + C_A^2\|B_2\| < \infty.$$

Recall that $\partial_n\widehat{\Pi}^2_\lambda = -\sqrt{-\Delta'}\widehat{\Pi}^2_\lambda$. Finally, the estimates for $\widehat{U}^2_\lambda$ follow easily from Lemma 5.5 and well-known resolvent estimates for the Dirichlet- and Neumann-Laplacian. □

Next, let $\widehat{X}_f = L^p_\sigma(\mathbb{R}^n_+)$. We set

$$\widehat{X} := \widehat{X}_f \times \widehat{X}_{a,b},$$
$$\widehat{\Pi}_\lambda := \begin{cases} \widehat{X} & \to L^p(\Omega_0) \cap \widehat{W}^{1,p}(\mathbb{R}^n_+) \\ (f,a,b) & \mapsto \widehat{\Pi}^1_\lambda(f) + \Pi^2_\lambda(a,b), \end{cases}$$
$$\widehat{U}_\lambda := \begin{cases} \widehat{X} & \to W^{2,p}(\mathbb{R}^n_+) \cap L^p_\sigma(\mathbb{R}^n_+) \\ (f,a,b) & \mapsto \widehat{U}^1_\lambda(f) + \widehat{U}^2_\lambda(a,b). \end{cases}$$

Then the subsequent lemma follows by combining Lemma 5.4 with Lemma 6.1.

6.2. Lemma. *For $\alpha \in (0, \frac{1}{2p'})$ there exists $C > 0$ such that*

$$\mathcal{R}_{\widehat{X}\to L^p(\Omega_0)}\left\{\lambda^\alpha\widehat{\Pi}_\lambda K_\lambda^{-1} : \lambda \in 1+\Sigma_\theta\right\} \le C,$$
$$\mathcal{R}_{\widehat{X}\to L^p(\mathbb{R}^n_+)}\left\{\nabla\widehat{\Pi}_\lambda K_\lambda^{-1} : \lambda \in 1+\Sigma_\theta\right\} \le C,$$
$$\mathcal{R}_{\widehat{X}\to W^{k,p}(\mathbb{R}^n_+)}\left\{\lambda^{\frac{k-2}{2}}\widehat{U}_\lambda K_\lambda^{-1} : \lambda \in 1+\Sigma_\theta\right\} \le C, \quad k = 0,1,2.$$

7. Regularity of the Neumann problem

We consider the following Neumann problem in Ω:

$$\text{(N)} \qquad \begin{cases} \Delta v &= \nabla\cdot g \quad \text{in } \Omega, \\ \nu\cdot\nabla v &= \nu\cdot g \quad \text{on } \partial\Omega. \end{cases}$$

Here v is a scalar-valued function and g a vector-valued function. It is known (see e.g. [FS94], [SS96]) that the Helmholtz decomposition for $L^p(\Omega)$ exists if and only if for $g \in L^p(\Omega)$ there exists a unique weak solution $v \in \widehat{W}^{1,p}(\Omega)$ to (N), i.e. $\langle\nabla v, \nabla\varphi\rangle = \langle g, \nabla\varphi\rangle$, $\varphi \in \widehat{W}^{1,p}(\Omega)$. In this case there exists $C > 0$ such that

$$\|\nabla v\|_p \le C\|g\|_p. \tag{7.1}$$

The next proposition shows that higher order estimates hold as well provided the boundary of Ω and the right hand side g are smooth enough. We start with a Poincaré type inequality.

7.1. Lemma. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with diameter δ and let $1 < p < \infty$. Assume that there exists a ball $B := B_\varrho(y_0) \subset \Omega$ for some $y_0 \in \Omega$ and some $\varrho > 0$ such that Ω is starshaped with respect to any $y \in B$. Then*

$$\|u - \frac{1}{|B|}\int_B u(y)dy\|_{L^p(\Omega)} \le \frac{\delta^{n+1}}{n}\frac{|S^{n-1}|}{|B|}\|\nabla u\|_{L^p(\Omega)}, \quad u \in W^{1,p}(\Omega),$$

where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$.

Proof. Let $x \in \Omega, y \in B$ with $x \ne y$. By assumption, $x + t(y-x)/|y-x| \in \Omega$ for all $t \in [0, |x-y|]$. Assuming first that $u \in C^1(\Omega)$, we have

$$u(x) - u(y) = \int_0^{|x-y|}\frac{x-y}{|y-x|}\nabla u(x + t\frac{y-x}{|y-x|})dt.$$

Integrating with respect to y and extending ∇u by 0 to all of $\mathbb{R}^n$ we obtain

$$\begin{aligned}|B||u(x) - \frac{1}{|B|}\int_B u(y)\,\mathrm{d}y| &= |\int_B \int_0^{|x-y|} \frac{x-y}{|y-x|}\nabla u(x+t\frac{y-x}{|y-x|})\,\mathrm{d}t\,\mathrm{d}y| \\ &\le \int_{\mathbb{R}^n}\int_0^{|x-y|} 1_B(y)|\nabla u(x+t\frac{y-x}{|y-x|})|\,\mathrm{d}t\,\mathrm{d}y \\ &\le \int_{\overline{B}_\delta(0)}\int_0^\delta |\nabla u(x+t\frac{z}{|z|})|\,\mathrm{d}t\,\mathrm{d}z \\ &= \frac{\delta^n}{n}\int_\Omega |x-z|^{1-n}|\nabla u(z)|\,\mathrm{d}z.\end{aligned}$$

Hence, by Young's inequality

$$\|u - \frac{1}{|B|}\int_B u(y)dy\|_{L^p(\Omega)} \le \frac{\delta^n}{n|B|}\|g\|_{L^1(\mathbb{R}^n)}\|\nabla u\|_{L^p(\Omega)},$$

where $g(x) = 1_K|x|^{1-n}$ with $K = \overline{\Omega - \Omega}$ and thus the assertion for $u \in C^1(\Omega)$. Approximating $u \in W^{1,p}(\Omega)$ by a sequence $(u_n) \in C^1(\Omega) \cap W^{1,p}(\Omega)$ which converges by the Meyers-Serrin theorem towards u in $W^{1,p}(\Omega)$, the proof is complete. □

7.2. **Proposition.** *Let $p \in (1,\infty)$ and assume that $\Omega \subset \mathbb{R}^n$ is a domain with uniform C^3-boundary and that the Helmholtz decomposition exists. Then for $k_0 = 2,3$ there exists $C > 0$ such that for $g \in W^{k_0-1,p}(\Omega)$ with $\nu \cdot g = 0$ on $\partial\Omega$ the weak solution $v \in \widehat{W}^{1,p}(\Omega)$ of (N) satisfies the estimate*

$$\sum_{k=1}^{k_0} \|\nabla^k v\|_p \le C\left(\|g\|_p + \|\nabla \cdot g\|_{W^{k_0-2,p}}\right). \tag{7.2}$$

Proof. Let $p \in (1,\infty)$ and let $v \in \widehat{W}^{1,p}(\Omega)$ be a weak solution to (N). Let $\{B_j\}_{j=1}^\infty$ denote the open covering of Ω described in the beginning of Section 4, where $r > 0$ is chosen later to be small enough. By definition of a uniform C^3-boundary, we may assume that after a suitable rotation and translation the boundary of $B_j \cap \partial\Omega$ can be described by a height functions h_j satisfying

$$\|h_j\|_{W^{1,\infty}} \le \varepsilon_r, \quad \|h_j\|_{W^{3,\infty}} \le C, \quad j \in \mathbb{N}. \tag{7.3}$$

Here and in the following $C > 0$ denotes a constant independent of r and j and $\varepsilon_r > 0$ denotes a constant independent of j but depending on r and satisfying $\varepsilon_r \to 0$ as $r \to 0$. For $j \in \mathbb{N}$, we set $U_j := B_j \cap \Omega$ and choose cut-off functions $\theta_j \in C_c^\infty(\overline{\Omega})$, supp $\theta_j \subset B_j$ with $\bigcup_{j\in\mathbb{N}}\{\theta_j \equiv 1\} \supset \Omega$, $\|D^\alpha\theta_j\|_{L^\infty(B_j)} \le C_r$, $|\alpha| = 1,2,3$, $\|\theta_j\|_{L^\infty(B_j)} \le 1$, $j \in \mathbb{N}$ and $\nu \cdot \nabla\theta_j = 0$ on $\partial\Omega$. Here, C_r denotes a constant independent of j but depending on r which may grow as $r \to 0$. Note that $\{\theta_j\}_{j\in\mathbb{N}}$ is *not* a partition of unity.

By [Gal94, Lemma III.3.4] it follows that U_j is starshaped with respect to a ball $\tilde{B}_j$ provided the radius r of the balls B_j is small enough. Let us now consider $v_j := v - \frac{1}{|\tilde{B}_j|}\int_{\tilde{B}_j} v$. Clearly, v_j still solves (N).

Integrating by parts yields

$$\begin{aligned}\langle\nabla(\theta_j v_j), \nabla\varphi\rangle_{U_j} &= \langle(\nabla\theta_j)v_j, \nabla\varphi\rangle_{U_j} + \langle\theta_j(\nabla v_j), \nabla\varphi\rangle_{U_j} \\ &= \langle(\nabla\theta_j)v_j, \nabla\varphi\rangle_{U_j} + \langle\nabla v_j, \nabla(\theta_j\varphi)\rangle_\Omega - \langle\nabla v_j, (\nabla\theta_j)\varphi\rangle_{U_j} \\ &= -\langle\nabla\cdot((\nabla\theta_j)v_j), \varphi\rangle_{U_j} - \langle\theta_j\nabla\cdot g, \varphi\rangle_\Omega - \langle(\nabla\theta_j)\cdot\nabla v_j, \varphi\rangle_{U_j} \\ &= -\langle(\Delta\theta_j)v_j, \varphi\rangle_{U_j} - \langle\theta_j\nabla\cdot g, \varphi\rangle_{U_j} - 2\langle(\nabla\theta_j)\cdot\nabla v_j, \varphi\rangle_{U_j} \\ &= \langle\tilde{g}_j, \varphi\rangle_{U_j}, \quad \varphi \in W^{1,p'}(U_j),\end{aligned}$$

with $\tilde{g}_j = -(\Delta\theta_j)v_j - \theta_j\nabla\cdot g - 2(\nabla\theta_j)\cdot\nabla v_j$. Therefore, $\theta_j v_j$ is the weak solution of the Neumann-Laplace problem on U_j with right hand side $\tilde{g}_j$. Changing U_j to a set $\tilde{U}_j$ in $(\operatorname{supp}\theta_j)^c$ such that $\tilde{U}_j$ has a smooth boundary, it follows form standard elliptic regularity theory that $\theta_j v_j \in W^{2,p}(\tilde{U}_j)$ and

$$\|\theta_j v_j\|_{W^{2,p}(\tilde{U}_j)} \leq C_j \left(\|\tilde{g}\|_{L^p(\tilde{U}_j)} + \|\theta_j v_j\|_{L^p(\tilde{U}_j)} \right).$$

In order to show that $\{C_j\}_{j\in\mathbb{N}}$ is uniformly bounded, we transfer the Neumann-Laplace problem on $\tilde{U}_j$ to a Neumann-Laplace problem on a fixed domain $S \subset \mathbb{R}^n$ with smooth boundary satisfying

$$\{x := (x_1, \cdots, x_n) \in \mathbb{R}^n : |x| < 1/2, x_n > 0\} \subset S \subset \{x := (x_1, \cdots, x_n) \in \mathbb{R}^n : |x| < 1, x_n > 0\}.$$

We define $w_j(x) := (\theta_j v_j)(H_j(x)) - u_*$ for $x \in H_j^{-1}(U_j)$ and $w_j(x) = 0$ for $x \in S \setminus H_j^{-1}(U_j)$, where

$$H_j(x', x_n) := \begin{pmatrix} x' \\ x_n \end{pmatrix} + \tilde{H}_j(x', x_n), \quad \tilde{H}_j(x', x_n) := \begin{pmatrix} 0 \\ -h_j(x') \end{pmatrix}, \quad u_* := \sum_{k=1}^{n-1} E((\partial_k h_j)(\partial_k \theta_j v_j) \circ H_j).$$

and $E : W^{1-1/p,p}(\partial S) \to W^{2,p}(S)$ denotes an extension operator satisfying $\partial_\nu E f = f$ on ∂S.

Note that in the change of coordinates we have neglected translation and rotation and that the function $\partial_j h(\partial_j \theta_j v_j) \circ H_j$ is extended by 0 to ∂S. The Jacobian J_{H_j} of H_j satisfies

$$J_{H_j(x)} = Id + J_{\tilde{H}_j}(x), \quad \|J_{\tilde{H}_j}(x)\| \leq \varepsilon_r, \quad \|\nabla^2 H_j(x)\| \leq C, \quad j \in \mathbb{N}, \tag{7.4}$$

and, moreover, w_j solves

$$\begin{aligned} \Delta w_j &= \tilde{g}_j \circ H_j + \langle \nabla w_j, \Delta H \rangle + Tr \left((J_{\tilde{H}_j}^T + J_{\tilde{H}_j} + J_{\tilde{H}_j} J_{\tilde{H}_j}^T) \nabla^2 w_j \right) - \Delta u_*, && \text{in } S, \\ \nu \cdot \nabla w_j &= 0, && \text{on } \partial S. \end{aligned} \tag{7.5}$$

We estimate

$$\begin{aligned} \|u_*\|_{W^{2,p}(S)} &\leq \sum_{k=1}^{n-1} C \|\partial_k h_j (\partial_k \theta_j v_j) \circ H_j\|_{W^{1-1/p,p}(\partial S)} \leq C \sum_{k=1}^{n-1} \|\partial_k h_j (\partial_k \theta_j v_j) \circ H_j\|_{W^{1,p}(S)} \\ &\leq C \sum_{k=1}^{n-1} \left(\|\nabla h_j\|_\infty \|(\partial_k \nabla \theta_j v_j) \circ H_j \cdot J_{H_j}\|_{L^p(S)} + C_r \|\nabla h_j\|_{W^{1,\infty}} \|v_j\|_{W^{1,p}(U_j)} \right) \\ &\leq C \left(\varepsilon_r \|\nabla^2 \theta_j v_j\|_{L^p(U_j)} + C_r \|v_j\|_{W^{1,p}(U_j)} \right), \quad j \in \mathbb{N}. \end{aligned}$$

Since w_j solves (7.5), we again use elliptic regularity theory and (7.4) to get

$$\begin{aligned} \|w_j\|_{W^{2,p}(S)} &\leq C \left(\|\tilde{g}_j \circ H_j\|_{L^p(S)} + \|w_j\|_{W^{1,p}(S)} + \varepsilon_r \|w_j\|_{W^{2,p}(S)} + \|u_*\|_{W^{2,p}(S)} \right) \\ &\leq C \left(\|\tilde{g}_j\|_{L^p(U_j)} + C_r \|v_j\|_{W^{1,p}(U_j)} + \varepsilon_r \|w_j\|_{W^{2,p}(S)} + \|u_*\|_{W^{2,p}(S)} \right) \\ &\leq C \left(\|\nabla \cdot g\|_{L^p(U_j)} + C_r \|v_j\|_{W^{1,p}(U_j)} + \varepsilon_r \|w_j\|_{W^{2,p}(S)} + \varepsilon_r \|\nabla^2 (\theta_j v)\|_{L^p(U_j)} \right), \quad j \in \mathbb{N}. \end{aligned}$$

Hence, by choosing $r := r_0$ small enough, we obtain

$$\begin{aligned} \|\nabla^2 (\theta_j v_j)\|_{L^p(U_j)} &\leq C_{r_0} \|v_j\|_{W^{1,p}(U_j)} + C \|\nabla \cdot g\|_{L^p(U_j)} \\ &\leq C_{r_0} \|\nabla v_j\|_{L^p(U_j)} + C \|\nabla \cdot g\|_{L^p(U_j)}, \quad j \in \mathbb{N}, \end{aligned}$$

where we used Lemma 7.1 in the last step. We finally obtain

$$\begin{aligned} \|\nabla^2 v\|_{L^p(\Omega)} &\leq C \sum_{j=1}^\infty \|\nabla^2 (\theta_j v_j)\|_{L^p(U_j)} \leq C \sum_{j=1}^\infty \left(\|\nabla v_j\|_{L^p(\tilde{U}_j)} + \|\nabla \cdot g\|_{L^p(\tilde{U}_j)} \right) \leq C \left(\|\nabla v\|_p + \|\nabla \cdot g\|_p \right) \\ &\leq C \left(\|g\|_p + \|\nabla \cdot g\|_p \right) \end{aligned}$$

This proves (7.2) with $k_0 = 2$. The case $k_0 = 3$ now follows similarly. □

8. Proof of the Linear Estimates

The proof of Theorem 2.1 is, roughly speaking, based on the localization method described in Section 4. Note that due to the given situation of domains with possible noncompact boundaries, we need to consider a countable covering of Ω.

For each $x_0 \in \partial\Omega$ local coordinates corresponding to x_0 are defined as coordinates obtained from the original ones by a rotation and a shift which moves x_0 into the origin and after which the positive x_n-axis has the direction of the interior normal to $\partial\Omega$ at x_0.

Let now $x_0 \in \partial\Omega$ and choose local coordinates corresponding to x_0. By definition of a uniform C^3-boundary, there exists an open neighbourhood $U = U_1 \times U_2 \subset \mathbb{R}^n$ containing $x_0 = 0$ with $U_1 \subset \mathbb{R}^{n-1}$ and $U_2 \subset \mathbb{R}$ open and a *height* function $h \in C^3(\overline{U_1})$ satisfying $\partial\Omega \cap U = \{x = (x', x_n) \in U : x_n = h(x')\}$ and $\Omega \cap U = \{x \in U : x_n > h(x')\}$. Note that choosing U_1 small , we may assume that $\|h\|_\infty + \|\nabla h\|_\infty$ is as small as we like. Next we define

$$g(x) := \begin{pmatrix} x' \\ x_n - h(x') \end{pmatrix}, \quad x \in U. \tag{8.1}$$

Since $\partial\Omega$ is a uniform C^3-boundary, all derivatives of g and of g^{-1} (defined on $\hat{U} := g(U)$) up to order 3 may be assumed to be bounded by a constant independent of x_0.

For a function $u\colon U \cap \Omega \to \mathbb{R}$ we define the *push-forward* $v = \mathcal{G}u$ on $\hat{U} \cap \mathbb{R}^n_+$ by $v(y) := u(g^{-1}(y))$. Due to the regularity of the boundary, this transformation is an isomorphism $W^{s,p}(U \cap \Omega) \to W^{s,p}(\hat{U} \cap \mathbb{R}^n_+)$ for all $p \in (1, \infty)$ and $s \in [-2, 2]$.

Similarly, for a function $u\colon U \cap \Omega \to \mathbb{R}^n$ we define the *push-forward* $v_\sigma = \mathcal{G}_\sigma u$ for the solenoidal spaces by $v_\sigma(y) := J_g(u(g^{-1}(y)))$, where J_g denotes the Jacobian of g. In fact, the linear transformation $\mathcal{G}_\sigma$ is an isomorphisms from $L^p_\sigma(U \cap \Omega)$ to $L^p_\sigma(\hat{U} \cap \mathbb{R}^n_+)$. Furthermore, it is an isomorphism from $W^{s,p}(U \cap \Omega)$ to $W^{s,p}(\hat{U} \cap \mathbb{R}^n_+)$ for all $p \in (1, \infty)$ and $s \in [-2, 2]$. The corresponding *pull-back* mappings $\mathcal{G}_\sigma^{-1}$ and $\mathcal{G}^{-1}$ are defined in a similar way. Note, that we may choose $h = 0$ if $U \cap \partial\Omega = \emptyset$.

For any $\varepsilon \in (0, 1)$ let $\{\Omega_j^\varepsilon : j \in \mathbb{N}\}$ of Ω denote a family of locally finite covers, cf. [GHHSS08], such that

$$\|\nabla h_j^\varepsilon\|_\infty < \varepsilon, \tag{8.2}$$

$$\sum_{j \in \mathbb{N}} \chi_{\Omega_j^\varepsilon}(x) \le C, \quad x \in \overline{\Omega} \tag{8.3}$$

where h_j^ε is the height function corresponding to Ω_j^ε and $C > 0$ is independent of ε. For each such covering $\{\Omega_j^\varepsilon\}_{j \in \mathbb{N}}$ we choose a partition of unity $\{\varphi_j^\varepsilon : j \in \mathbb{N}\}$ subordinate to this covering. Furthermore, denote by $\mathcal{G}_j^\varepsilon$, $\mathcal{G}_{\sigma,j}^\varepsilon$, $\mathcal{G}_j^{-1,\varepsilon}$, $\mathcal{G}_{\sigma,j}^{-1,\varepsilon}$ the corresponding push-forward mappings and pull-back mappings.

The commutator $[\Delta, \mathcal{G}_{j,\sigma}^{-1,\varepsilon}]\widehat{u}$, $\widehat{u} \in W^{2,p}(\mathbb{R}^n_+)$, of Δ and $\mathcal{G}_{j,\sigma}^{-1,\varepsilon}$ can be split into two parts: $[\Delta, \mathcal{G}_{j,\sigma}^{-1,\varepsilon}]_h\widehat{u}$ contains second order terms of $\widehat{u}$ only and $[\Delta, \mathcal{G}_{j,\sigma}^{-1,\varepsilon}]_l\widehat{u}$ contains all lower order terms. In particular, by (8.2) there exists a constant $C > 0$ such that

$$\|[\Delta, \mathcal{G}_{j,\sigma}^{-1,\varepsilon}]_h \hat{u}\|_{L^p(\Omega_j)^n} \le C\varepsilon \|\hat{u}\|_{\widehat{W}^{2,p}(\hat{\Omega}_j^\varepsilon)^n}, \quad \varepsilon \in (0,1),\ j \in \mathbb{N},\ \hat{u} \in W^{2,p}(\hat{\Omega}_j)^n, \tag{8.4}$$

$$\|[\Delta, \mathcal{G}_{j,\sigma}^{-1,\varepsilon}]_l \hat{u}\|_{L^p(\Omega_j)^n} \le C \|\hat{u}\|_{W^{1,p}(\hat{\Omega}_j^\varepsilon)^n}, \quad \varepsilon \in (0,1),\ j \in \mathbb{N},\ \hat{u} \in W^{2,p}(\hat{\Omega}_j)^n. \tag{8.5}$$

Here and in the following $\hat{\Omega}_j^\varepsilon$ denotes the transformation by the j-th push forward map of Ω_j^ε. In the same way $\hat{u}_j^\varepsilon$ denotes the function living on the half space $\mathbb{R}^n_+$ which is connected with u_j^ε through the j-th push forward map. Similar to (8.4), there exists a constant $C > 0$ such that

$$\|(\nabla \mathcal{G}_j^{-1,\varepsilon} - \mathcal{G}_{j,\sigma}^{-1,\varepsilon}\nabla)\hat{q}\|_{L^p(\Omega_j)^n} \le C\varepsilon \|\hat{q}\|_{\widehat{W}^{1,p}(\mathbb{R}^n_+)}, \quad \varepsilon \in (0,1),\ j \in \mathbb{N},\ \hat{q} \in \widehat{W}^{1,p}(\hat{\Omega}_j^\varepsilon). \tag{8.6}$$

As in [GHHSS08] we use Bogovskiĭ's operator to construct localized data for our localization procedure. For a bounded Lipschitz domain $\Omega' \subset \Omega$ and $g \in L^p(\Omega')$ with $\int_{\Omega'} g = 0$ Bogovskiĭ's operator $B_{\Omega'}$

is a solution operator to the problem

$$\left\{\begin{array}{rcll} \operatorname{div} u &=& g & \text{in } \Omega', \\ u &=& 0 & \text{on } \partial\Omega', \end{array}\right. \tag{8.7}$$

see [Bog86], [Gal94] or [GHHSS08]. By [GHHSS08], there exists $C>0$, independent of $j \in \mathbb{N}$, such that

$$\|B_{\Omega_j^\varepsilon} f\|_{L^p(\Omega_j^\varepsilon)^n} \le C\|f\|_{L^p(\Omega_j^\varepsilon)}, \quad \varepsilon \in (0,1),\ j \in \mathbb{N},\ f \in L^p(\Omega_j^\varepsilon). \tag{8.8}$$

We finally choose cut-off functions $\psi_j^\varepsilon \in C_c^\infty(\Omega_j^\varepsilon)$ such that $\psi_j^\varepsilon \equiv 1$ on $\operatorname{supp}\varphi_j^\varepsilon$; see [GHHSS08]. For $f \in X_f := L^p_\sigma(\Omega)$ we define the local data by

$$f_j^\varepsilon := \psi_j^\varepsilon f - B_{\Omega_j^\varepsilon}\left((\nabla\psi_j^\varepsilon) f\right)$$

and let $\widehat{f}_j^\varepsilon$ denote the extension to $\mathbb{R}^n_+$ by 0 of the push-forward $\mathcal{G}^\varepsilon_{\sigma,j} f_j^\varepsilon$. By (8.8), we obtain $\widehat{f}_j^\varepsilon \in L^p_\sigma(\mathbb{R}^n_+)$ and

$$\|\widehat{f}_j^\varepsilon\|_{L^p(\mathbb{R}^n_+)^n} \le C\|f\|_{L^p(\Omega_j^\varepsilon)^n}, \tag{8.9}$$

where $C>0$ is independent of ε, j and f. Hence, (8.3) yields that

$$((S_j^{1,\varepsilon})_{j\in\mathbb{N}})_{\varepsilon\in(0,1)} \subset \mathcal{L}(X_f, \ell^p(\widehat{X}_f)) \tag{8.10}$$

is uniformly bounded, where $S_j^{1,\varepsilon} f := \widehat{f}_j^\varepsilon$. Similarly, for $(a,b) \in X_{a,b} := \{a \in W^{1-1/p,p}(\partial\Omega)^n : a\cdot\nu = 0\} \times \{b \in W^{2-1/p,p}(\partial\Omega)^n : b\cdot\nu = 0\}$, we define the local data $a_j^\varepsilon = \Psi_j^\varepsilon a$, $b_j^\varepsilon = \Psi_j^\varepsilon b$ and $\widehat{a}_j^\varepsilon = \mathcal{G}^{\partial\Omega,\varepsilon}_{j,\sigma}\Psi_j^\varepsilon a$, $\widehat{b}_j^\varepsilon = \mathcal{G}^{\partial\Omega,\varepsilon}_{j,\sigma}\Psi_j^\varepsilon b$. Here, $\mathcal{G}^{\partial\Omega,\varepsilon}_{j,\sigma}$ is the restriction of $\mathcal{G}^\varepsilon_{j,\sigma}$ to the boundary of Ω. Again, we have that

$$((S_j^{2,\varepsilon}(a,b))_{j\in\mathbb{N}})_{\varepsilon\in(0,1)} \subset \mathcal{L}(X_{a,b}, \ell^p(\widehat{X}_{a,b})) \tag{8.11}$$

is uniformly bounded, where $S_j^{2,\varepsilon}(a,b) := (\widehat{a}_j^\varepsilon, \widehat{b}_j^\varepsilon)$. We set

$$U_\lambda^\varepsilon(f,a,b) := \sum_{j\in\mathbb{N}} \varphi_j^\varepsilon \mathcal{G}^{-1,\varepsilon}_{j,\sigma} \widehat{U}_\lambda S_j^\varepsilon(f,a,b) - \nabla N\Big(\sum_{j\in\mathbb{N}} \varphi_j^\varepsilon \mathcal{G}^{-1,\varepsilon}_{j,\sigma} \widehat{U}_\lambda S_j^\varepsilon(f,a,b)\Big),$$

where N is the solution operator of the weak Neumann problem (N) and $S_j^\varepsilon(f,a,b) := (S_j^{1,\varepsilon} f, S_j^{2,\varepsilon}(a,b))$. Here, similarly as in [GHHSS08], we add a correction term in order to have a solenoidal ansatz U_λ^ε. However, in contrast to [GHHSS08] the correction term is based on the solution operator of the weak Neumann problem instead of Bogovskiĭ's operator. Inserting $u := U_\lambda^\varepsilon(f,a,a)$, we calculate

$$\begin{array}{rll} \lambda u - P\Delta u = f + \mathcal{T}_\lambda^{1,\varepsilon}(f,a,a) & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = a + \mathcal{T}_\lambda^{2,\varepsilon}(f,a,a) & \text{in } \partial\Omega, \end{array} \tag{8.12}$$

where

$$(\mathcal{T}_\lambda^{1,\varepsilon}(f,a,b), \mathcal{T}_\lambda^{2,\varepsilon}(f,a,b), \mathcal{T}_\lambda^{3,\varepsilon}(f,a,b)) = \mathcal{T}_\lambda^\varepsilon(f,a,b) := T^\varepsilon_{1,\lambda}(f,a,b) + \cdots + T^\varepsilon_{6,\lambda}(f,a,b)$$

with

$$
\begin{aligned}
T^{\varepsilon}_{1,\lambda}(f,a,b) &:= (P_{\Omega}\sum_{j=1}^{\infty}\varphi^{\varepsilon}_j\left(\nabla\mathcal{G}^{-1,\varepsilon}_j-\mathcal{G}^{-1,\varepsilon}_{j,\sigma}\nabla\right)\widehat{\Pi}_{\lambda}S^{\varepsilon}_j(f,a,b),0,0),\\
T^{\varepsilon}_{2,\lambda}(f,a,b) &:= (P_{\Omega}\sum_{j=1}^{\infty}(\nabla\varphi^{\varepsilon}_j)\mathcal{G}^{-1,\varepsilon}_j\widehat{\Pi}_{\lambda}S^{\varepsilon}_j(f,a,b),0,0),\\
T^{\varepsilon}_{3,\lambda}(f,a,b) &:= -(P_{\Omega}\sum_{j=1}^{\infty}[\Delta,\varphi^{\varepsilon}_j]\mathcal{G}^{-1,\varepsilon}_{j,\sigma}\widehat{U}_{\lambda}S^{\varepsilon}_j(f,a,b),0,0),\\
T^{\varepsilon}_{4,\lambda}(f,a,b) &:= -(P_{\Omega}\sum_{j=1}^{\infty}\varphi^{\varepsilon}_j[\Delta,\mathcal{G}^{-1.\varepsilon}_j]_h\widehat{U}_{\lambda}S^{\varepsilon}_j(f,a,b),0,0),\\
T^{\varepsilon}_{5,\lambda}(f,a,b) &:= -(P_{\Omega}\sum_{j=1}^{\infty}\varphi^{\varepsilon}_j[\Delta,\mathcal{G}^{-1,\varepsilon}_j]_l\widehat{U}_{\lambda}S^{\varepsilon}_j(f,a,b),0,0),\\
T^{\varepsilon}_{6,\lambda}(f,a,b) &:= -(0,\nabla N(\sum_{j\in\mathbb{N}}\varphi^{\varepsilon}_j\mathcal{G}^{-1,\varepsilon}_{j,\sigma}\widehat{U}_{\lambda}S^{\varepsilon}_j(f,a,b))|_{\partial\Omega},\nabla N(\sum_{j\in\mathbb{N}}\varphi^{\varepsilon}_j\mathcal{G}^{-1,\varepsilon}_{j,\sigma}\widehat{U}_{\lambda}S^{\varepsilon}_j(f,a,b))|_{\partial\Omega}).
\end{aligned}
$$

This means that we obtain a solution of the Stokes resolvent problem which is given by

$$
R^{\varepsilon}(\lambda)f := U^{\varepsilon}_{\lambda}(1+\mathcal{T}^{\varepsilon}{}_{\lambda})^{-1}(f,0,0) = U^{\varepsilon}_{\lambda}\sum_{n\in\mathbb{N}_0}(-\mathcal{T}^{\varepsilon}_{\lambda})^n(f,0,0), \tag{8.13}
$$

provided the above sum is convergent.

In the following we show that the Neumann series $\sum_{n\in\mathbb{N}_0}(\mathcal{T}^{\varepsilon}_{\lambda})^n(f,0,0)$ exists for some $\varepsilon\in(0,1)$, which hence yields the existence of a solution to (8.12). The uniqueness of the solution follows from a standard duality argument. Hence, we finally obtain

$$
R^{\varepsilon}(\lambda) = (\lambda - A_q)^{-1}.
$$

In order to estimate the above Neumann series, we set $X := X_f \times X_{a,b}$. Then, the representation formula (8.13) can be written as

$$
\begin{aligned}
R^{\varepsilon}(\lambda)f &= U^{\varepsilon}_{\lambda}\sum_{n\in\mathbb{N}_0}(\mathcal{T}^{\varepsilon}_{\lambda})^n(f,0,0) = U^{\varepsilon}_{\lambda}K_{\lambda}^{-1}\sum_{n\in\mathbb{N}_0}(K_{\lambda}\mathcal{T}^{\varepsilon}_{\lambda}K_{\lambda}^{-1})^nK_{\lambda}(f,0,0)\\
&= U^{\varepsilon}_{\lambda}K_{\lambda}^{-1}\sum_{n\in\mathbb{N}_0}(K_{\lambda}\mathcal{T}^{\varepsilon}_{\lambda}K_{\lambda}^{-1})^n(f,0,0)
\end{aligned}
$$

provided the above series converges. In the following lemma we show that

$$
\mathcal{R}_X\{K_{\lambda}\mathcal{T}^{\varepsilon}_{\lambda}K_{\lambda}^{-1} : \lambda\in\lambda_0+\Sigma_{\theta}\} < 1
$$

for some $\lambda_0 > 0$. Hence, $R^{\varepsilon}(\lambda)$ is well defined for some $\varepsilon\in(0,1)$ and all $\lambda\in\lambda_0+\Sigma_{\theta}$ with λ_0 large enough.

8.1. **Lemma.** *For $\alpha\in(0,1/2p')$ there exist $\varepsilon_0\in(0,1)$ and $C>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$:*

(a) $\mathcal{R}_X\{K_{\lambda}T^{\varepsilon}_{1,\lambda}K_{\lambda}^{-1} : \lambda\in 1+\Sigma_{\theta}\} \le \frac{1}{4}$,
(b) $\mathcal{R}_X\{\lambda^{\alpha}K_{\lambda}T^{\varepsilon}_{2,\lambda}K_{\lambda}^{-1} : \lambda\in 1+\Sigma_{\theta}\} \le C$,
(c) $\mathcal{R}_X\{\lambda^{\frac{1}{2}}K_{\lambda}T^{\varepsilon}_{3,\lambda}K_{\lambda}^{-1} : \lambda\in 1+\Sigma_{\theta}\} \le C$,
(d) $\mathcal{R}_X\{K_{\lambda}T^{\varepsilon}_{4,\lambda}K_{\lambda}^{-1} : \lambda\in 1+\Sigma_{\theta}\} \le \frac{1}{4}$,
(e) $\mathcal{R}_X\{\lambda^{\frac{1}{2}}K_{\lambda}T^{\varepsilon}_{5,\lambda}K_{\lambda}^{-1} : \lambda\in 1+\Sigma_{\theta}\} \le C$,
(f) $\mathcal{R}_X\{\lambda^{\frac{1}{2p}}K_{\lambda}T^{\varepsilon}_{6,\lambda}K_{\lambda}^{-1} : \lambda\in 1+\Sigma_{\theta}\} \le C$.

Proof. By (8.10) and (8.11), $((S_j^\varepsilon)_{j\in\mathbb{N}})_{\varepsilon\in(0,1)} \subset \mathcal{L}(X, l^p(\widehat{X}))$ is uniformly bounded. Moreover, since $\|\varphi_j^\varepsilon\|_\infty \le 1$, $j \in \mathbb{N}$, $\varepsilon \in (0,1)$, it follows from (8.3) and (8.6) that

$$\|P_\Omega \sum_{j=1}^\infty \varphi_j^\varepsilon [\nabla, \mathcal{G}_j^{-1,\varepsilon}]\widehat{g}_j\|_{L^p(\Omega)}^p \le C \sum_{j=1}^\infty \|[\nabla, \mathcal{G}_j^{-1,\varepsilon}]\widehat{g}_j\|_{L^p(\Omega_j^\varepsilon)}^p \le C \sum_{j=1}^\infty \Big(\varepsilon_j \|\widehat{g}_j\|_{\widehat{W}^{1,p}(\mathbb{R}_+^n)}\Big)^p$$
$$\le C\varepsilon^p \|(\widehat{g}_j)\|_{\ell^p(\widehat{W}^{1,p}(\mathbb{R}_+^n))}^p, \quad \varepsilon \in (0,1).$$

Hence, by Lemma 5.6, Lemma 6.2 and (8.2), we obtain

$$\mathcal{R}_X\{K_\lambda T_{1,\lambda}^\varepsilon K_\lambda^{-1} : \lambda \in \Sigma_\theta\} \le C\varepsilon \mathcal{R}_{\widehat{X}\to\widehat{W}^{1,p}(\mathbb{R}_+^n)}\{\Pi_\lambda K_\lambda : \lambda \in 1 + \Sigma_\theta\} \le \frac{1}{4}$$

for $\varepsilon \in (0, \varepsilon_1)$ and ε_1 small enough. This shows (a).

By similar arguments as above it follows from (8.4) that

$$\|P_\Omega \sum_{j=1}^\infty \varphi_j^\varepsilon [\Delta, \mathcal{G}_{j,\sigma}^{-1,\varepsilon}]_h \widehat{g}_j\|_{L^p(\Omega)} \le C\varepsilon \|(\widehat{g}_j)\|_{\ell^p(\widehat{W}^{2,p}(\mathbb{R}_+^n))}$$

and, therefore, there exists $\varepsilon_0 \in (0, \varepsilon_1)$ such that (d) holds.

Now, choose $X_n > 0$ such that $\widehat{\Omega}_j^{\varepsilon_0} \subset \mathbb{R}^{n-1} \times (0, X_n)$ for $\varepsilon \in (0, \varepsilon_1)$ and $j \in \mathbb{N}$. Then, by (8.5) there exists $C > 0$ such that

$$\|P_\Omega \sum_{j=1}^\infty (\nabla \varphi_j^{\varepsilon_0}) \mathcal{G}_j^{-1,\varepsilon_0} \widehat{g}_j\|_{L^p(\Omega)} \le C \|(g_j)\|_{\ell^p(L^p(\mathbb{R}^{n-1}\times(0,X_n)))},$$
$$\|(P_\Omega \sum_{j=1}^\infty [\varphi_j^{\varepsilon_0}, \Delta] \mathcal{G}_{j,\sigma}^{-1,\varepsilon_0} \widehat{g}_j\|_{L^p(\Omega)} \le C \|(\widehat{g}_j)\|_{\ell^p(L^p(\mathbb{R}_+^n))}$$
$$\|P_\Omega \sum_{j=1}^\infty \varphi_j^{\varepsilon_0} [\Delta, \mathcal{G}_{j,\sigma}^{-1,\varepsilon_0}]_l \widehat{g}_j\|_{L^p(\Omega)} \le C \|(\widehat{g}_j)\|_{\ell^p(W^{1,p}(\Omega))},$$
$$\|\nabla N \sum_{j=1}^\infty \varphi_j^{\varepsilon_0} \mathcal{G}_{j,\sigma}^{-1,\varepsilon_0} \widehat{g}_j|_{\partial\Omega}\|_{W^{k+1-\frac{1}{p},p}(\partial\Omega)} \le C \|(\widehat{g}_j)\|_{\ell^p(W^{k,p}(\mathbb{R}_+^n)\cap L_\sigma^p(\mathbb{R}_+^n))}, \quad k = 0, 1,$$

and (b), (c), (e), (f) are proved as above. □

Summing up, Lemma 8.1 and Lemma 6.2 imply

$$(\lambda - P\Delta) R^{\varepsilon_0}(\lambda) f = f, \quad \lambda \in \lambda_0 + \Sigma_\theta, \ f \in L_\sigma^p(\Omega), \tag{8.14}$$

and $\mathcal{R}_{L_\sigma^p(\Omega)}\{\lambda R^{\varepsilon_0}(\lambda) : \lambda \in \lambda_0 + \Sigma_\theta\} < C$ for $\lambda_0 > 0$ large enough. Finally, thanks to Lemma 5.1, uniqueness of the solution of (RS) follows from standard duality arguments. The proof of Theorem 2.1 is complete.

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Fachbereich Mathematik, Angewandte Analysis, Technische Universität Darmstadt, Schlossgartenstr. 7, D-64289 Darmstadt, Germany

E-mail address: `geissert@mathematik.tu-darmstadt.de`

Fachbereich Mathematik, Angewandte Analysis, Technische Universität Darmstadt, Schlossgartenstr. 7, D-64289 Darmstadt, Germany

E-mail address: `heck@mathematik.tu-darmstadt.de`

Fachbereich Mathematik, Angewandte Analysis, Technische Universität Darmstadt, Schlossgartenstr. 7, D-64289 Darmstadt, Germany, and
Center of Smart Interfaces, Petersenstr. 32, D-64289 Darmstadt

E-mail address: `hieber@mathematik.tu-darmstadt.de`

Fachbereich Mathematik, Angewandte Analysis, Technische Universität Darmstadt, Schlossgartenstr. 7, D-64289 Darmstadt, Germany

E-mail address: `sawada@mathematik.tu-darmstadt.de`