$\mathcal{H}^\infty$-CALCULUS FOR CYLINDRICAL BOUNDARY VALUE PROBLEMS

TOBIAS NAU AND JÜRGEN SAAL

ABSTRACT. In this note an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus for linear operators associated to *cylindrical* boundary value problems is proved. The obtained results are based on an abstract result on operator-valued functional calculus by N. Kalton and L. Weis, cf. [26]. Cylindrical in this context means that both domain and differential operator possess a certain cylindrical structure. Compared to standard methods (e.g. localization procedures), our approach looks rather elegant and provides short proofs. Besides, we are even able to deal with some classes of equations on rough domains. For instance, we can extend the well-known (and in general sharp) range for p such that the Dirichlet Laplacian admits an $\mathcal{H}^\infty$-calculus on $L^p(\Omega)$, from $(3+\varepsilon)' < p < 3+\varepsilon$ to $(4+\varepsilon)' < p < 4+\varepsilon$ for three dimensional bounded or unbounded Lipschitz cylinders Ω. Our approach even admits mixed Dirichlet Neumann boundary conditions in this situation.

1. INTRODUCTION

Consider the boundary value problem

$$\begin{aligned} \lambda u + A(x,D)u &= f \text{ in } \Omega, \\ B_j(x,D)u &= 0 \text{ on } \partial\Omega \quad (j=1,...,m), \end{aligned} \tag{1.1}$$

of order $2m$. Many situations in mathematics and in applied sciences naturally lead to problems of type (1.1) in cylindrical domains. In the simplest case Ω might be a rectangle, a cube, or a cylinder. Hence Ω is of the form $\Omega = V_1 \times V_2$ with V_1 an interval in $\mathbb{R}$ and V_2 an interval, a rectangle, or a circle in $\mathbb{R}^2$. We refer, e.g., to the textbook [8] and [9] for a demonstration of the significance of problems on such Ω. More generally, in this note V_i will always assumed to be a standard domain, i.e. $\mathbb{R}^n$, $\mathbb{R}^n_+$, or a domain with compact boundary in $\mathbb{R}^n$ for $n \geq 1$. Furthermore, our approach is not restricted to products of just two domains. In fact, we will consider domains of the form

$$\Omega = \prod_{i=1}^{k} V_i$$

for $k \in \mathbb{N}$ and standard domains $V_i \subset \mathbb{R}^{n_i}$, $n_i \in \mathbb{N}$, $i = 1, \dots, k$. To take full advantage of the cylindrical shape of Ω, also the corresponding differential operators are required to possess cylindrical structure. Roughly speaking, we assume that $A(x,D)$ (and in fact also $B_j(x,D)$) resolves into k parts

$$A = A_1 + A_2 + \cdots + A_k$$

such that A_i merely acts on V_i, $i = 1, \dots, k$. Note that many standard systems, such as the heat equation with Dirichlet or Neumann boundary conditions, are of this form.

Of course, several cases, e.g. infinite cylinders $\Omega = V \times \mathbb{R}^n$, could be handled via standard localization procedures employing an infinite partition of the unity. However, such procedures are generally extensive and take quite some pages of exhaustive calculations and estimations. This is not the strategy we pursue here. In fact, we essentially take advantage of the cylindrical structure of domain and operator and employ operator-valued functional calculus. Roughly speaking, by this method the treatment of (1.1) in Ω is reduced to corresponding results on the cross-sections V_i, for which they are well-known (see e.g. [11], [10]). This approach reveals a much shorter and more elegant way to prove properties such as an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus for linear operators associated to cylindrical boundary value problems. Note that in the regarded situation an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus implies the important maximal regularity, a rather useful tool for the treatment of related semi- and quasilinear problems, cf. [11]. We refer to Section 4 for precise definitions and the obtained results on general cylindrical boundary value problems (on sufficiently smooth domains). Our main result in this direction will be Theorem 4.6.

The idea to apply operator-valued theory for the treatment of boundary value problems in infinite cylinders to our knowledge first appeared in [18] and [19], see also [28]. The results in these papers are based on operator-valued multiplier theorems in Besov spaces obtained in [1] and are therefore restricted to infinite cylinders of the form $\Omega = V \times \mathbb{R}^n$ and to operators with constant coefficients on $\mathbb{R}^n$. It is interesting to note that, although it is more abstract, the approach based on functional calculus presented here yields not only results of the same quality, but we can also deal with finite cylinders and general operators in the cross-sections. Hence, this method admits a much larger class of boundary value problems. In [18] and [19] also a couple of further applications are discussed. In particular, a generalized form of fundamental solutions for operator-valued problems is derived. The results in [19] also demonstrate that the approach to cylindrical problems based on operator-valued theory is by no means restricted to elliptic or parabolic equations.

Note that X-valued parabolic boundary value problems in standard domains were extensively studied in [11]. There a bounded H^∞-calculus and hence maximal regularity for the operator of the associated Cauchy problem is proved in the situation when X is of class $\mathcal{HT}$. The results obtained in the paper at hand also extend the maximal regularity results proved in [11] to a class of domains with non-compact boundary. For classical papers on scalar-valued boundary value problems we refer to [12], [3], [4], and [32] in the elliptic case and to [6] and [2] in the parameter-elliptic case. (For a more comprehensive list see also [11].) For an approach to a class of elliptic differential operators with Dirichlet-boundary conditions in uniform C^2-domains we refer to [25]. We want to remark that all cited results above are based on standard localization procedures for the domain, contrary to the approach presented in this paper.

The development of the approach just described, specifically working for cylindrical boundary value problems, might already be justified by the significance and the frequent appearance of cylindrical problems. On the other hand, this approach reveals a couple of further advantages. In fact, we will be able to handle also some classes of boundary value problems on rough domains or with degenerate coefficients. More precisely, in Section 5 the cross-sections V_i are allowed to be bounded (graph) Lipschitz domains. This relies on the fact that the abstract results we apply 'only' require an $\mathcal{H}^\infty$-calculus for the operators on the cross-sections, no matter how domain or operator look in detail. An $\mathcal{H}^\infty$-calculus can be derived e.g. for the Dirichlet- or the Neumann-Laplacian on bounded Lipschitz domains. Our main results on domains with Lipschitz cross-sections, giving an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus for the Laplacian, are Theorem 5.1 and Theorem 5.3. With regard on existing literature, it is interesting to note that here large classes of unbounded Lipschitz domains and (simultaneously) of mixed Dirichlet Neumann type boundary conditions are included. We refer to [22], [21], and to the literature cited therein for more information on problems with mixed boundary conditions in Lipschitz domains.

As a further interesting outcome (see Theorem 5.3, also Corollary 5.5), we can easily extend the range for p such that the Dirichlet Laplacian admits this property on $L^p(\Omega)$, from $(3+\varepsilon)' < p < 3+\varepsilon$ to $(4+\varepsilon)' < p < 4+\varepsilon$ for three (or higher) dimensional Lipschitz cylinders Ω. Recall that for general three (or higher) dimensional Lipschitz domains the smaller range is sharp, in the sense that for every $\varepsilon > 0$ there exists a Lipschitz domain Ω such that the Dirichlet problem in $L^p(\Omega)$ is ill-posed, cf. [24]. In $\mathbb{R}^2$, however, it is known that the 3 in the range condition can be replaced by 4, see [24]. Hence, our technique preserves the larger range, provided every cross-section admits this range for p. Another interesting outcome is that we obtain highest possible regularity in the directions of smooth cross-sections (see Remark 5.2). This is also known to fail for general bounded Lipschitz domains. For literature on general bounded Lipschitz domains we refer to [24], [36] for the Dirichlet problem and to [17], [36] for the Neumann problem and to the references therein.

The proofs of the results in Sections 4 and 5 are based on an abstract result on operator-valued functional calculus obtained by N. Kalton and L. Weis in [26] and its generalization given in [27]. An $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus is a very strong tool in the theory of linear an nonlinear PDE's. Particularly, for the treatment of free boundary value problems the result of Kalton and Weis turned out to be of crucial importance, see e.g. [16], [30]. The principle symbols arising in the treatment of such problems are in general of intricate non-homogeneous structure and cannot be handled by classical methods. We refer to [13] for more information on that. In Section 2 we recall basic notions related to an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus and give a precise statement of the Kalton-Weis theorem. From this general result we derive specific versions on sums and products of operators suitable for our purposes.

An essential assumption for the results in Sections 4 and 5 is that the operators on the cross-sections are resolvent commuting. In Section 6, however, we will demonstrate that our approach is not restricted to this situation. There we allow for disturbances of the cylindrical structure of the form

$$A(x,D) = A_1(x^1, D_{x_1}) + r(x^1)A_2(x^2, D_{x_2}).$$

Our main results in this context are represented by Theorem 6.1 and Theorem 6.2. Even some sorts of degenerate coefficients r are admitted. Their proofs are based on a non-commutative version of the Kalton-Weis theorem, see [33] and [31].

2. Sectorial operators and $\mathcal{R}$-boundedness

We employ standard notation throughout this article. The symbols X, Y, E, and F stand for Banach spaces. Given a closed operator A, we denote by $D(A), \ker(A)$, and $R(A)$ domain of definition, kernel, and range, respectively. Furthermore, by $\rho(A)$ and $\sigma(A)$ we denote resolvent set and spectrum of A. The symbol $\mathcal{L}(X,Y)$ stands for the Banach space of all bounded linear operators from X to Y equipped with operator norm $\|\cdot\|_{\mathcal{L}(X,Y)}$. As an abbreviation we set $\mathcal{L}(X) := \mathcal{L}(X,X)$.

For $p \in [1,\infty)$ and a domain $G \subset \mathbb{R}^n$, $L^p(G,F)$ denotes the F-valued Lebesgue space of all p-Bochner-integrable functions with norm

$$\|f\|_{L^p(G,F)} := \left(\int_G \|f(x)\|_F^p dx \right)^{\frac{1}{p}}.$$

We also write $L^\infty(G,F)$ for the space consisting of all functions f satisfying $\|f\|_\infty := \operatorname{ess\,sup}_{x\in G} \|f(x)\|_F < \infty$. The F-valued Sobolev space of order $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ is denoted by $W^{m,p}(G,F)$ and its norm is given by

$$\|f\|_{W^{m,p}(G,F)} := \left(\sum_{|\alpha|\le m} \|D^\alpha f\|^p_{L^p(G,F)} \right)^{\frac{1}{p}},$$

where $\alpha \in \mathbb{N}_0^n$ is a multiindex. We write $\|\cdot\|_p := \|\cdot\|_{L^p(G,F)}$ and $\|\cdot\|_{p,m} := \|\cdot\|_{W^{m,p}(G,F)}$ for short. Finally, for $m \in \mathbb{N}_0 \cup \{\infty\}$, $C^m(G,F)$ denotes the space of all m times continously differentiable functions.

In order to avoid confusion with different definitions for the notion of sectoriality occurring in the literature, next we precisely clarify what we mean by a sectorial operator.

Definition 2.1. A closed linear operator A in a Banach space X is called *sectorial*, if

(1) $\overline{D(A)} = X$, $\ker(A) = \{0\}$, $\overline{R(A)} = X$,
(2) $(-\infty, 0) \subset \rho(A)$ and there is some $C > 0$ such that $\|t(t+A)^{-1}\|_{\mathcal{L}(X)} \le C$ for all $t > 0$.

In this case it is well-known, see e.g. [11], that there exists a $\phi \in [0,\pi)$ such that the uniform estimate in (2) extends to all

$$\lambda \in \Sigma_{\pi-\phi} := \{z \in \mathbb{C}\backslash\{0\};\ |\arg(z)| < \pi - \phi\}.$$

The number

$$\phi_A := \inf\{\phi : \rho(-A) \supset \Sigma_{\pi-\phi}, \sup_{\lambda\in\Sigma_{\pi-\phi}} \|\lambda(\lambda + A)^{-1}\|_{\mathcal{L}(X)} < \infty\}$$

is called *spectral angle* of A. The class of sectorial operators is denoted by $\mathcal{S}(X)$.

Observe that $\sigma(A)\backslash\{0\} \subset \Sigma_{\phi_A}$. As is well-known, in case that $\phi_A < \frac{\pi}{2}$, $-A$ generates a bounded analytic C_0-semigroup on X. For a suitable treatment of quasilinear problems, however, this property might not be enough. Then *maximal regularity* is required, which implies that the solution of the related Cauchy problem

$$\begin{cases} u' + Au &= f \quad \text{in } \mathbb{R}_+, \\ u(0) &= 0, \end{cases}$$

satisfies the estimate

$$\|u'\|_{L^p(\mathbb{R}_+,X)} + \|Au\|_{L^p(\mathbb{R}_+,X)} \leq C\|f\|_{L^p(\mathbb{R}_+,X)} \tag{2.1}$$

with a $C > 0$ independent of $f \in L^p(\mathbb{R}_+, X)$. We denote this class of operators by $\mathcal{MR}(X)$.

Boundedness and analyticity of the generated semigroup in general are not enough to guarantee (2.1) (except if X is a Hilbert space). Therefore other sufficient criteria implying maximal regularity have been successfully established in recent years. For instance, if the Banach space X is of class $\mathcal{HT}$ in [34, Theorem 4.2] it is shown that the property of maximal regularity is equivalent to the $\mathcal{R}$-sectoriality of an operator A. This notion is based on the concept of $\mathcal{R}$-bounded operator families introduced below. Also recall that a Banach space X is of class $\mathcal{HT}$ or, equivalently, a UMD space, if there exists a $q \in (1, \infty)$ such that the Hilbert transform $Hf := \frac{1}{\pi}PV\left(\frac{1}{t}\right) * f$ acts as a bounded operator on $L^q(\mathbb{R}, X)$. Typical examples of spaces of class $\mathcal{HT}$ are given by Hilbert spaces and reflexive $L^p(G)$-spaces. Also $L^p(G, X)$ is of class $\mathcal{HT}$ provided X is of class $\mathcal{HT}$ and $1 < p < \infty$. We refer to [11] and [27] for a comprehensive introduction to the notion of $\mathcal{R}$-bounded operator families and its relation to maximal regularity and to other functional analytic properties.

Definition 2.2. A family $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-*bounded*, if there exist a $C > 0$ and a $p \in [1, \infty)$ such that for all $N \in \mathbb{N}, T_j \in \mathcal{T}, x_j \in X$, and all independent symmetric $\{-1, 1\}$-valued random variables ε_j on a probability space $(G, \mathcal{M}, P)$ for $j = 1, ..., N$ we have that

$$\|\sum_{j=1}^N \varepsilon_j T_j x_j\|_{L^p(G,Y)} \leq C\|\sum_{j=1}^N \varepsilon_j x_j\|_{L^p(G,X)}. \tag{2.2}$$

The smallest $C > 0$ such that (2.2) is satisfied is called $\mathcal{R}$-bound of $\mathcal{T}$ and denoted by $\mathcal{R}(\mathcal{T})$.

Definition 2.3. A closed operator A in X satisfying condition (1) of Definition 2.1 is called $\mathcal{R}$-*sectorial*, if there exist an angle $\phi \in [0, \pi)$ and a constant $C_\phi > 0$ such that

$$\mathcal{R}(\{\lambda(\lambda + A)^{-1} : \lambda \in \Sigma_{\pi-\phi}\}) \leq C_\phi. \tag{2.3}$$

The class of $\mathcal{R}$-sectorial operators is denoted by $\mathcal{RS}(X)$ and we call $\phi_A^{\mathcal{RS}}$ given as the infimum over all angles ϕ such that (2.3) holds the *$\mathcal{R}$-angle* of A.

Note that $\mathcal{R}$-boundedness implies uniform boundedness. This yields $\mathcal{RS}(X) \subset \mathcal{S}(X)$ and $\phi_A \leq \phi_A^{\mathcal{RS}}$. However, the converse in general is false, except for Hilbert spaces.

Since we will use it frequently in the sequel, we recall the following two properties of $\mathcal{R}$-bounded families. The first one shows that $\mathcal{R}$-bounds behave like uniform bounds and will be used without any further notice. It is an easy consequence of the definition. The second one is known as the contraction principle of Kahane. A proof can be found in [27] or [11].

Lemma 2.4. *a) Let X,Y, and Z be Banach spaces and let $\mathcal{T}, \mathcal{S} \subset \mathcal{L}(X,Y)$ as well as $\mathcal{U} \subset \mathcal{L}(Y,Z)$ be $\mathcal{R}$-bounded. Then $\mathcal{T}+\mathcal{S} \subset \mathcal{L}(X,Y)$ and $\mathcal{UT} \subset \mathcal{L}(X,Z)$ are $\mathcal{R}$-bounded as well and we have*

$$\mathcal{R}(\mathcal{T}+\mathcal{S}) \leq \mathcal{R}(\mathcal{S}) + \mathcal{R}(\mathcal{T}), \quad \mathcal{R}(\mathcal{UT}) \leq \mathcal{R}(\mathcal{U})\mathcal{R}(\mathcal{T}).$$

b) Let $p \in [1,\infty)$. Then for all $N \in \mathbb{N}, x_j \in X, \varepsilon_j$ as above, and for all $a_j, b_j \in \mathbb{C}$ with $|a_j| \leq |b_j|$ for $j = 1,\dots,N$, we have

$$\|\sum_{j=1}^N a_j\varepsilon_j x_j\|_{L^p(G,X)} \leq 2\|\sum_{j=1}^N b_j\varepsilon_j x_j\|_{L^p(G,X)}. \tag{2.4}$$

3. Functional calculus and sums of closed operators

Next we introduce an operator valued holomorphic functional calculus for sectorial operators. This will lead to the notion of an $\mathcal{H}^\infty$-calculus and of an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus, which represent two further important subclasses of $\mathcal{RS}(X)$. For a comprehensive introduction to this concept we refer to [7], [26], [11], [27], and [15].

Let $\mathcal{A} \subset \mathcal{L}(X)$ denote the subalgebra of bounded operators on X which commute with the resolvent $(\mu - A)^{-1}$. For $\sigma \in (0,\pi]$ we denote by $\mathcal{H}^\infty(\Sigma_\sigma, \mathcal{A})$ the commutative algebra of bounded, $\mathcal{A}$-valued, holomorphic functions on Σ_σ, that is,

$$\mathcal{H}^\infty(\Sigma_\sigma, \mathcal{A}) := \{f : \Sigma_\sigma \to \mathcal{A};\ f \text{ is holomorphic}, |f|_\infty^\sigma < \infty\}$$

where

$$|f|_\infty^\sigma := \sup\{\|f(z)\|_{\mathcal{L}(X)};\ z \in \Sigma_\sigma\}.$$

Using $\rho(z) := \frac{z}{(1+z)^2}$ we define the subalgebra

$$\begin{aligned}\mathcal{H}_0(\Sigma_\sigma, \mathcal{A}) := \{f \in \mathcal{H}^\infty(\Sigma_\sigma, \mathcal{A}) :\ &\text{there are } C, \varepsilon > 0 \text{ such that} \\ &\|f(z)\|_{\mathcal{L}(X)} \leq C|\rho(z)|^\varepsilon \text{ for all } z \in \Sigma_\sigma\}.\end{aligned}$$

Let A be a sectorial operator in X with spectral angle ϕ_A. Pick $\sigma \in (\phi_A, \pi]$ and $\psi \in (\phi_A, \sigma)$. The path $\Gamma := (\infty, 0]e^{i\psi} \cup [0,\infty)e^{-i\psi}$ oriented counterclockwise, i.e. the positive real axis $\mathbb{R}_+$ lies to the left, stays with the only possible exception at zero in the resolvent set of A. Hence, by Cauchy's integral formula and the sectoriality of A, the Bochner integral

$$f(A) := \frac{1}{2\pi i}\int_\Gamma f(\mu)(\mu - A)^{-1}d\mu$$

represents a well-defined element in $\mathcal{L}(X)$ for every $f \in \mathcal{H}_0(\Sigma_\sigma, \mathcal{A})$. As f is supposed to take values in $\mathcal{A}$ the above formula defines an algebra homomorphism

$$\Phi_A : \mathcal{H}_0(\Sigma_\sigma, \mathcal{A}) \to \mathcal{L}(X), \quad f \mapsto f(A), \tag{3.1}$$

known as Dunford calculus. For arbitrary $f \in \mathcal{H}^\infty(\Sigma_\sigma, \mathcal{A})$ we set

$$f(A) := \rho(A)^{-1}(\rho f)(A).$$

This definition gives rise to a closed, densely defined operator in X. Moreover, by Cauchy's theorem it is consistent with the former one for $f \in \mathcal{H}_0(\Sigma_\sigma, \mathcal{A})$. Note that in the scalar valued case we have $\mathcal{A} = \mathbb{C}$. Then we write $\mathcal{H}^\infty(\Sigma_\sigma)$ and $\mathcal{H}_0(\Sigma_\sigma)$. For the purposes in this note, it will be sufficient to introduce an $\mathcal{H}^\infty$-calculus for this situation.

Definition 3.1. a) Let $\mathcal{A} = \mathbb{C}$. The operator $A \in \mathcal{S}(X)$ is said to admit a bounded $\mathcal{H}^\infty$-calculus on X, if there exists $\sigma > \phi_A$ such that Φ_A given in (3.1) is bounded (w.r.t. the topologies on $\mathcal{H}^\infty(\Sigma_\sigma)$ and $\mathcal{L}(X)$). We denote the class of operators admitting a bounded $\mathcal{H}^\infty$-calculus in X by $\mathcal{H}^\infty(X)$. The bound for Φ_A in general depends on σ. The infimum over all $\sigma > \phi_A$ such that this bound remains finite is called $\mathcal{H}^\infty$-angle of A and denoted by ϕ_A^∞.
b) Accordingly, $A \in \mathcal{S}(X)$ is said to admit an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus on X, if there exist a $\sigma > \phi_A$ and a constant $C_\sigma > 0$ such that

$$\mathcal{R}(\{f(A);\ f \in \mathcal{H}^\infty(\Sigma_\sigma),\ |f|_\infty^\sigma \le 1\}) \le C_\sigma. \tag{3.2}$$

This class is denoted by $\mathcal{RH}^\infty(X)$ and the corresponding $\mathcal{RH}^\infty$-angle by $\phi_A^{\mathcal{R},\infty}$.

The following result is known as convergence lemma (see e.g. [7, Lemma 2.1], [20], [15, Theorem 4.7]).

Lemma 3.2. *Let $f \in \mathcal{H}^\infty(\Sigma_\sigma, \mathcal{A})$ and may $\rho_n \in \mathcal{H}_0(\Sigma_\sigma)$ be defined by $\rho_n(z) := n^2z/(1+nz)(n+z)$. Then, $f(A) \in \mathcal{L}(X)$ if and only if $\sup_{n\in\mathbb{N}} \|(\rho_n f)(A)\| < \infty$. In this case $f(A)x = \lim_{n\to\infty}(\rho_n f)(A)x$ for all $x \in X$.*

Thanks to the convergence lemma it is easy to see that, in case that $A \in \mathcal{H}^\infty(X)$, Φ_A extends boundedly from $\mathcal{H}_0(\Sigma_\sigma)$ to $\mathcal{H}^\infty(\Sigma_\sigma)$.

Given an operator admitting a bounded or an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus, naturally the question arises, whether Φ_A can be extended to some class of $\mathcal{A}$-valued functions. By combining the notion of $\mathcal{R}$-boundedness with operator-valued functional calculus an affirmative answer to this question is given by N. Kalton and L. Weis in [26]. For the precise formulation of this result we need the following property of Banach space geometry.

Definition 3.3. A Banach space X is said to have property (α), if there exists a $C > 0$ such that for all $n \in \mathbb{N}$, all $\alpha_{ij} \in \mathbb{C}$ with $|\alpha_{ij}| \leq 1$, all $x_{ij} \in X$, and all independent symmetric $\{-1,1\}$-valued random variables ε_i^1 on a probability space $(G_1, \mathcal{M}_1, P_1)$ and ε_j^2 on a probability space $(G_2, \mathcal{M}_2, P_2)$ for $i, j = 1, ..., N$, we have that

$$\int_{G_1} \int_{G_2} \| \sum_{i,j=1}^{N} \varepsilon_i^1(u) \varepsilon_j^2(v) \alpha_{ij} x_{ij} \|_X du dv \leq C \int_{G_1} \int_{G_2} \| \sum_{i,j=1}^{N} \varepsilon_i^1(u) \varepsilon_j^2(v) x_{ij} \|_X du dv.$$

Again standard spaces such as Hilbert spaces and reflexive $L^p(G)$-spaces enjoy property (α). Moreover, the spaces $L^p(G, X)$ enjoy property (α) for $1 \leq p < \infty$, if X does so (cf. [27]). Note that for Banach spaces X of class $\mathcal{HT}$ we have the relations

$$\mathcal{S}(X) \supset \mathcal{RS}(X) \supset \mathcal{H}^\infty(X) \supset \mathcal{RH}^\infty(X)$$

between the single classes. For sectorial A on X we also have

$$\phi_A \leq \phi_A^{\mathcal{RS}} \leq \phi_A^\infty \leq \phi_A^{\mathcal{R},\infty}$$

for the corresponding angles, cf. [11]. In [26, Theorem 5.3] it is proved that $\mathcal{RH}^\infty(X) = \mathcal{H}^\infty(X)$ and $\phi_A^\infty = \phi_A^{\mathcal{R},\infty}$, provided that X has additionally property (α).

The mentioned result of Kalton and Weis reads as follows (see [26, Corollary 5.4]).

Proposition 3.4. *Let X be a Banach space having property (α) and $A \in S(X)$. Given an $\mathcal{R}$-bounded subset $\tau \subset \mathcal{L}(X)$, we put*

$$\mathcal{H}^\infty(\Sigma_\sigma, \tau) := \{f \in \mathcal{H}^\infty(\Sigma_\sigma, \mathcal{A});\ f(z) \in \tau\ (z \in \Sigma_\sigma)\}.$$

If A admits a bounded $\mathcal{H}^\infty$-calculus, then for $\sigma > \phi_A^\infty$ we have

$$\mathcal{R}(\{f(A);\ f \in \mathcal{H}^\infty(\Sigma_\sigma, \tau)\}) < \infty.$$

For a long time it remained an open question, under what circumstances in a Banach space X the sum of two closed operators is closed again. A first suitable answer was given by the celebrated result of Dore and Venni in [14] stating that if both A and B have bounded imaginary powers with sum of power angles less than π and if X is of class $\mathcal{HT}$, then $A + B$ is closed and invertible. Another answer to this question is also contained in Proposition 3.4 as a special case. In fact this result yields the same assertion if one of the operators is $\mathcal{R}$-sectorial and if the other one admits a bounded $\mathcal{H}^\infty$-calculus, see [26, Theorem 6.3]. This is in particular important for the situation of Cauchy problems (i.e. $B = \partial_t$), since then the assumption for A of having bounded imaginary powers is reduced to the weaker property of $\mathcal{R}$-sectoriality. In view of our applications, in this note we want sums $A + B$ or products AB not only to be closed again, but even to admit an $\mathcal{H}^\infty$-calculus. This leads to the following result, which is also obtained as a consequence of Proposition 3.4.

Proposition 3.5. *Let X be a Banach space of class $\mathcal{HT}$ with property (α). Let $A, B \in \mathcal{H}^\infty(X)$ with $\phi_A^\infty + \phi_B^\infty < \pi$ be two resolvent commuting operators.*

(a) *Then $A+B$ admits an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus and for the $\mathcal{RH}^\infty$-angle we have $\phi_{A+B}^{\mathcal{R},\infty} \le \max\{\phi_A^\infty, \phi_B^\infty\}$.*
(b) *Let further $0 \in \rho(A)$. Then AB admits an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus with $\phi_{AB}^{\mathcal{R},\infty} \le \phi_A^\infty + \phi_B^\infty$.*

Remark 3.6. By iteration it readily follows that the assertions remain true for finite sums (resp. finite products) as long as in each step the condition for the $\mathcal{H}^\infty$-angles and commutativity of the resolvents is satisfied.

Proof. We give a detailed proof of part (b) and omit the very similar proof of part (a). Let $\phi \in (\phi_B^\infty, \pi)$, $\sigma \in (\phi_A^\infty, \pi-\phi)$, and fix $z \in \Sigma_\sigma$. we set $g_z(\lambda) := z\lambda$ for $\lambda \in \Sigma_\phi$. Let $\theta \in (\sigma+\phi, \pi)$ and $f \in \mathcal{H}_0(\Sigma_\theta)$ with $|f|_\infty^\theta \le 1$. Then $f \circ g_z \in \mathcal{H}^\infty(\Sigma_\phi)$. By the fact that X has property (α) we also have $B \in \mathcal{RH}^\infty(X)$ and $\phi_B^{\mathcal{R},\infty} = \phi_B^\infty$. Hence

$$\mathcal{R}(\{f \circ g_z(B);\ z \in \Sigma_\sigma,\ |f|_\infty^\theta \le 1\}) < \infty \tag{3.3}$$

follows. By permanence properties for sectorial operators we also have that $g_z(B) = zB$ is sectorial with $\phi_{zB} \le \arg z + \phi_B \le \theta$. Thus $f(g_z(B))$ is well-defined and a straight forward calculation shows that

$$f \circ g_z(B) = f(g_z(B)) = f(zB) \quad (z \in \Sigma_\sigma)$$

(see also [27, Proposition 15.11]). Relation (3.3) therefore implies that zB admits an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus uniformly in $z \in \Sigma_\sigma$, i.e.

$$\mathcal{R}(\{f(zB);\ z \in \Sigma_\sigma,\ |f|_\infty^\theta \le 1\}) < \infty.$$

In particular, we have $\phi_{zB}^{\mathcal{R},\infty} \le \sigma + \phi$.

Now let $\psi \in (\sigma+\phi, \theta)$ and $\Gamma := (-\infty, 0]e^{i\psi} \cup [0,\infty)e^{-i\psi}$. We set $R_\lambda(z) := (\lambda - zB)^{-1}$ for $\lambda \in R(\Gamma)$ and $z \in \Sigma_\sigma$ and define

$$H_f(z) := f(zB) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)R_\lambda(z)d\lambda.$$

Obviously R_λ is holomorphic on Σ_σ. Lebesgue's dominated convergence theorem therefore yields continuity of H_f. Furthermore, for any simple closed curve $\gamma \subset \Sigma_\sigma$ Fubini's theorem and Cauchy's theorem give us

$$\int_\gamma H_f(z)dz = \frac{1}{2\pi i}\int_\Gamma\int_\gamma f(\lambda)R_\lambda(z)dzd\lambda = 0.$$

Thanks to Morera's theorem holomorphy of H_f on Σ_σ follows. By assumption $R_\lambda(z)$ commutes with the resolvent of A for all $\lambda \in R(\Gamma)$ and $z \in \Sigma_\sigma$. Continuity of the resolvent therefore implies the same to be true for $H_f(z)$ $(z \in \Sigma_\sigma)$. From Proposition 3.4 we infer that the family $\{H_f(A);\ |f|_\infty^\sigma \le 1\}$ is $\mathcal{R}$-bounded.

It remains to show that

$$H_f(A) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - AB)^{-1}d\lambda. \tag{3.4}$$

We only show $R_\lambda(A) = (\lambda - AB)^{-1}$ by an approximation argument. If this is proved, (3.4) follows easily by the same arguments. Employing the Dunford approximation sequence ρ_n from Lemma 3.2 and the identity

$$(\lambda - \mu B)^{-1}(\mu - A)^{-1}(\lambda - AB) = \left((\mu - A)^{-1} + B(\lambda - \mu B)^{-1}\right),$$

we deduce for $x \in D(AB)$ that

$$(\rho_n R_\lambda)(A)(\lambda - AB)x = \frac{1}{2\pi i}\int_\Gamma \rho_n(\mu)(\lambda - \mu B)^{-1}(\mu - A)^{-1}(\lambda - AB)x d\mu$$
$$= \frac{1}{2\pi i}\int_\Gamma \rho_n(\mu)(\mu - A)^{-1}x d\mu + \frac{1}{2\pi i}\int_\Gamma \rho_n(\mu)B(\lambda - \mu B)^{-1}x d\mu.$$

As the second integral vanishes by Cauchy's theorem we arrive at

$$(\rho_n R_\lambda)(A)(\lambda - AB)x = \rho_n(A)x \to x.$$

This proves $R_\lambda(A)$ to be a left inverse to $(\lambda - AB)$. Since $(\lambda - AB)$ is closed, the same approximation argument shows that $R_\lambda(A)$ is a right inverse as well. The proof is now complete. □

In Section 6 we will demonstrate that our approach is not restricted to the resolvent commuting situation. For this purpose, we employ a corresponding result to Proposition 3.5(a) for the non-commuting case, cf. [31]. Indeed, the same assertion holds, if the following so-called *Labbas-Terreni condition* is satisfied.

$$\left.\begin{array}{l} \text{Let } 0 \in \rho(A) \text{ and let there exist constants } c > 0,\ 0 \le \alpha < \beta < 1, \\ \psi_a > \phi_A, \psi_B > \phi_B, \psi_A + \psi_B < \pi, \\ \text{ such that for all } \lambda \in \Sigma_{\pi - \psi_A},\ \mu \in \Sigma_{\pi - \psi_B} \text{it holds that} \\ \|A(\lambda + A)^{-1}[A^{-1}, (\mu + B)^{-1}]\| \le c/(1 + |\lambda|)^{1-\alpha}|\mu|^{1+\beta}. \end{array}\right\} \tag{3.5}$$

Here $[S, T] = ST - TS$. The result given in [31] then reads as follows.

Proposition 3.7. *Let X be a Banach space of class $\mathcal{HT}$ with property (α), let $A, B \in \mathcal{H}^\infty(X)$ and suppose that* (3.5) *holds for some angles $\psi_A > \phi_A^\infty$, $\psi_B > \phi_B^\infty$ with $\psi_A + \psi_B < \pi$. Then there exists $\delta \ge 0$ such that $A + B + \delta$ is invertible and such that $A + B + \delta \in \mathcal{RH}^\infty(X)$ with $\phi_{A+B+\delta}^\infty \le \max\{\psi_A, \psi_B\}$. In case that the resolvents commute or if c in (3.5) is small enough, we can take $\delta = 0$.*

Remark 3.8. Again iteration is possible, provided the angle and commutator conditions are satisfied in every step.

Remark 3.9. Both Proposition 3.5 and Proposition 3.7 exist in slightly different versions if X is an arbitrary Banach space, cf. [26], [31]. In our applications in the subsequent sections, however, the assumptions of class $\mathcal{HT}$ and of property (α) are always satisfied.

4. Cylindrical parameter-elliptic boundary value problems

First let us recall the notion of parameter-ellipticity from [11]. Let F be a Banach space, $G \subset \mathbb{R}^n$ be a domain, and

$$A(x,D) := \sum_{|\alpha|\le 2m} a_\alpha(x) D^\alpha,$$

where $m \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^n$, and $a_\alpha : G \to \mathcal{L}(F)$. For $\lambda \in \mathbb{C}$ and boundary operators

$$B_j(x,D)u := \sum_{|\beta|\le m_j} b_{j,\beta}(x)(D^\beta u)|_{\partial G},$$

where $m_j < 2m$, $\beta \in \mathbb{N}_0^n$, and $b_{j,\beta} : \partial G \to \mathcal{L}(F)$ for $j = 1, ..., m$, we consider the boundary value problem

$$\begin{aligned} \lambda u + A(x,D)u &= f \text{ in } G, \\ B_j(x,D)u &= 0 \text{ on } \partial G \quad (j = 1, ..., m). \end{aligned} \tag{4.1}$$

Definition 4.1. In the setting introduced above the $\mathcal{L}(F)$-valued homogeneous polynomial

$$a(\xi) := \sum_{|\alpha|=2m} a_\alpha \xi^\alpha \quad (\xi \in \mathbb{R}^n)$$

is called *$\mathcal{L}(F)$-valued parameter-elliptic*, if there exists an angle $\phi \in [0, \pi)$ such that the spectrum $\sigma(a(\xi))$ of $a(\xi)$ in $\mathcal{L}(F)$ satisfies

$$\sigma(a(\xi)) \subset \Sigma_\phi \quad (\xi \in \mathbb{R}^n,\ |\xi| = 1). \tag{4.2}$$

Then

$$\varphi := \inf\{\phi : \ (4.2) \text{ holds}\}$$

is called *angle of ellipticity* of a. A differential operator $A(x,D) := \sum_{|\alpha|\le 2m} a_\alpha(x) D^\alpha$ with coefficients $a_\alpha : G \to \mathcal{L}(F)$ is called $\mathcal{L}(F)$-valued parameter-elliptic in G with angle of ellipticity φ, if the principle part of its symbol

$$a^\#(x,\xi) := \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha$$

for every $x \in \overline{G}$ is $\mathcal{L}(F)$-valued parameter-elliptic with this angle of ellipticity.

Definition 4.2. Let F be a Banach space and let $G \subset \mathbb{R}^n$ be a C^1-domain. Let $a_\alpha : G \to \mathcal{L}(F)$ and $b_{j,\beta} : \partial G \to \mathcal{L}(F)$. Set $B_j^\#(x,D)u := \sum_{|\beta|=m_j} b_{\beta,j}(x)(D^\beta u)|_{\partial G}$ and let $A^\#(x,D) := \sum_{|\alpha|=2m} a_\alpha(x) D^\alpha$ be $\mathcal{L}(F)$-valued parameter-elliptic in G of angle of ellipticity $\varphi \in [0,\pi)$. For each $x_0 \in \partial G$ we write the boundary value problem in local coordinates corresponding to x_0. The boundary value problem (4.1) is said to satisfy the *Lopatinskii-Shapiro condition*, if for every $\phi > \varphi$ the ODE

$$\begin{aligned} (\lambda + A^\#(x_0,\xi',D_{x_n}))v(x_n) &= 0, \quad x_n > 0, \\ B_j^\#(x_0,\xi',D_{x_n})v(0) &= h_j, \quad j = 1, ..., m, \\ v(x_n) &\to 0, \quad x_n \to \infty, \end{aligned}$$

has a unique solution $v \in C((0,\infty),F)$ for each $(h_1,...,h_m)^T \in F^m$ and for each $\lambda \in \overline{\Sigma}_{\pi-\phi}$ and $\xi' \in \mathbb{R}^{n-1}$ with $|\xi'|+|\lambda| \neq 0$.

We refer to [35] for an introduction to the Lopatinskii-Shapiro condition for scalar valued boundary value problems and to [11] for an extensive treatment of the F-valued case. Parameter-ellipticity of a boundary value problem now reads as follows.

Definition 4.3. The boundary value problem (A,B) given through (4.1) is called $\mathcal{L}(F)$-valued parameter-elliptic in G of angle $\varphi \in [0,\pi)$, if $A(\cdot,D)$ is $\mathcal{L}(F)$-valued parameter-elliptic in G of angle $\varphi \in [0,\pi)$ and if the Lopatinskii-Shapiro condition holds. To indicate that φ is the angle of ellipticity of the boundary value problem (A,B) we use the subscript notation $\varphi_{(A,B)}$.

Next, consider a domain $\Omega \subset \mathbb{R}^n$ given as product of finitely many domains $V_i \subset \mathbb{R}^{n_i}$ with $n_i \in \mathbb{N}$ and $\sum_{i=1}^k n_i = n$, that is $\Omega = \prod_{i=1}^k V_i$. For $x \in \Omega$ we write $x = (x^1,...,x^k)$ with $x^i \in V_i$ for $x \in \Omega$ and $i = 1,...,k$, whenever we want to refer to the cylindrical geometry of Ω. Accordingly, we write $\alpha = (\alpha^1,...,\alpha^k) \in \prod_{i=1}^k \mathbb{N}_0^{n_i}$ for a multiindex $\alpha \in \mathbb{N}^n$. Finally we set

$$\partial\mathcal{V}_i := V_1 \times \ldots \times V_{i-1} \times \partial V_i \times V_{i+1} \times \ldots \times V_k \tag{4.3}$$

and $\partial\Omega := \bigcup\limits_{i=1}^{k} \partial\mathcal{V}_i$. As $\partial\mathcal{V}_i \cap \partial\mathcal{V}_j = \emptyset$ for $i \neq j$, all points belonging to the Lebesgue zero set of edges of Ω are neglected in this definition of $\partial\Omega$.

In the sequel we consider the $\mathcal{L}(F)$-valued boundary value problem

$$\begin{aligned} \lambda u + A(x,D)u &= f \text{ in } \Omega, \\ B_j(x,D)u &= 0 \text{ on } \partial\Omega \quad (j=1,...,m), \end{aligned} \tag{4.4}$$

with $A(x,D) = \sum_{|\alpha|\leq 2m} a_\alpha(x)D^\alpha$, $m \in \mathbb{N}$, an $\mathcal{L}(F)$-valued differential operator in the interior and $\mathcal{L}(F)$-valued operators $B_j(x,D)u = \sum_{|\beta|<2m} b_\beta(x)(D^\beta u)|_{\partial\Omega}$ on the boundary. In this section we particularly deal with the following class of operators.

Definition 4.4. Let $m_i \in \mathbb{N}$ for $i = 1,...,k$ and set $m := \max\{m_i : i = 1,...,k\}$ and $B_{i,j}(\cdot,D) = 0$ for $j > m_i$. The boundary value problem (4.4) is called *cylindrical* if the operator $A(\cdot,D)$ is represented as

$$A(x,D) = \sum_{i=1}^k A_i(x^i,D) = \sum_{i=1}^k \sum_{|\alpha^i|\leq 2m_i} a^i_{\alpha^i}(x^i) D^{(0,\ldots,0,\alpha^i,0,\ldots 0)}$$

and the boundary operator on $\partial\Omega$ is given as

$$\begin{aligned} B_j(x,D)u &= \sum_{i=1}^k \chi_{\partial\mathcal{V}_i}(x) B_{i,j}(x^i,D)u \\ &= \sum_{i=1}^k \chi_{\partial\mathcal{V}_i}(x) \sum_{|\beta^i|\leq m_{i,j}} b^i_{j,\beta^i}(x^i)(D^{(0,..,0,\beta^i,0,\ldots,0)}u)|_{\partial\Omega} \end{aligned}$$

for $m_{i,j} < m_i$ and $j = 1, ..., m$, where $\chi_{\partial\mathcal{V}_i}$ denotes the characteristic function of the set $\partial\mathcal{V}_i$. In other words, the differential operators $A(x, D)$ and $B_j(x, D)$ resolve completely into k parts $A_i(x^i, D)$ and $B_{i,j}(x^i, D)$ of which each one acts merely on V_i.

As the $L^p(\Omega, F)$-realization of the boundary value problem

$$(A, B) := (A(\cdot, D), B_1(\cdot, D), ..., B_m(\cdot, D))$$

given by (4.4) we define for $1 < p < \infty$,

$$\begin{aligned} D(A) :=& \Big\{ u \in L^p(\Omega); D^\alpha u \in L^p(\Omega) \text{ for } \sum_{i=1}^k \frac{|\alpha^i|}{2m_i} \leq 1 \\ & \qquad \text{and } B_j(\cdot, D)u = 0 \quad (j = 1, ..., m) \Big\} \\ Au :=& A(\cdot, D)u, \quad u \in D(A). \end{aligned}$$

In case that $m_i = m$ for all $i = 1, ..., k$, we obviously have

$$D(A) := \{u \in W^{2m,p}(\Omega, F);\ B_j(\cdot, D)u = 0 \quad (j = 1, ..., m)\}.$$

For $i = 1, ..., k$ we consider the boundary value problems

$$(A_i, B_i) := (A_i(\cdot, D), B_{i,1}(\cdot, D), ..., B_{i,m_i}(\cdot, D))$$

given by

$$\begin{aligned} \lambda u + A_i(x, D)u &= f \text{ in } V_i, \\ B_{i,j}(x, D)u &= 0 \text{ on } \partial V_i \quad (j = 1, ..., m_i), \end{aligned} \tag{4.5}$$

which occur naturally by cylindrical decomposition of (A, B). For the cross-sections V_i, we will admit the following types of domains.

Definition 4.5. Let $m, n \in \mathbb{N}$. The domain $G \subset \mathbb{R}^n$ is called a standard domain in $\mathbb{R}^n$, if it is given as the whole space $\mathbb{R}^n$, the half space $\mathbb{R}^n_+$ or as a domain in $\mathbb{R}^{n_i}$ with compact boundary, that is, a bounded or an exterior domain. If a standard domain G is of class C^m, it is called a C^m standard domain.

From now on we assume every $V_i \subset \mathbb{R}^{n_i}$ to be given as a C^{2m_i} standard domain. Furthermore, for some $\gamma_i \in (0, 1)$, $i = 1, ..., k$, the following smoothness assumptions on the coefficients of (A_i, B_i) may hold:

$$\left.\begin{aligned} & a^i_{\alpha^i} \in BUC^{\gamma_i}(\overline{V_i}, \mathcal{L}(F)) \text{ for } |\alpha^i| = 2m_i \quad a^i_{\alpha^i}(\infty) := \lim_{|x^i|\to\infty,\ x^i\in V_i} a^i_{\alpha^i}(x^i) \\ & \qquad \text{exists} \quad \text{and } \|a^i_{\alpha^i}(x^i) - a^i_{\alpha^i}(\infty)\| \leq C|x^i|^{-\gamma_i} \quad (x^i \in V_i, |x^i| \geq 1)), \\ & a^i_{\alpha^i} \in [L^\infty + L^{r_\nu}](V_i, \mathcal{L}(F)) \text{ for } |\alpha^i| = \nu < 2m_i, \\ & \qquad \text{where } r_\nu \geq p,\ \frac{2m_i - \nu}{n_i} > \frac{1}{r_\nu}, \\ & b^i_{j,\beta^i} \in C^{2m_i - m_{i,j}}(\partial V_i, \mathcal{L}(F)) \quad (j = 1, ..., m_i;\ |\beta^i| \leq m_{i,j}). \end{aligned}\right\} \tag{4.6}$$

Our main theorem of this section reads as follows.

Theorem 4.6. *Let $1 < p < \infty$ and let F be a Banach space of class $\mathcal{HT}$ enjoying property (α). Let $\Omega := \prod_{i=1}^k V_i$, $\sum_{i=1}^k n_i = n$, and let every V_i be a C^{2m_i} standard domain in $\mathbb{R}^{n_i}$, $n_i \in \mathbb{N}$, $i = 1, \dots, k$. Furthermore, we assume that*

(i) *the boundary value problem (A, B) is cylindrical,*
(ii) *the coefficients of (A_i, B_i) satisfy (4.6),*
(iii) *(A_i, B_i) is $\mathcal{L}(F)$-valued parameter-elliptic of angle $\varphi_i := \varphi_{(A_i,B_i)} \in [0, \pi)$ in V_i,*
(iv) *(A_i, B_i) is $\mathcal{L}(F)$-valued parameter-elliptic of angle φ_i in $V_i \cup \{\infty\}$ in the sense of (4.6), line 1 in case of V_i being unbounded,*
(v) *$\varphi_i + \varphi_j < \pi$ for $i, j = 1, ..., k$, $i \neq j$,*
(vi) *$a^i_{\alpha^i}(x^i) a^j_{\alpha^j}(x^j) = a^j_{\alpha^j}(x^j) a^i_{\alpha^i}(x^i)$ in $\mathcal{L}(F)$ for $i, j = 1, ..., k$, $i \neq j$ and a.e. $x \in \Omega$.*

Then for every $\phi > \max_{i=1,\dots,k}\{\varphi_i\}$, there exists $\delta = \delta_\phi > 0$ such that $A + \delta \in \mathcal{RH}^\infty(L^p(\Omega, F))$ and $\phi^{\mathcal{R},\infty}_{A+\delta} \leq \phi$. Moreover we have

$$\mathcal{R}(\{\lambda^{1-\sum_{i=1}^k \frac{|\alpha^i|}{2m_i}} D^\alpha(\lambda + A + \delta)^{-1};\ \lambda \in \Sigma_{\pi-\phi},\ 0 \leq \sum_{i=1}^k \frac{|\alpha^i|}{2m_i} \leq 1\}) < \infty. \tag{4.7}$$

Remark 4.7. a) Note that no continuity of the boundary conditions at the edges of Ω has to be assumed.
b) It is worthwhile to mention that another advantage of our approach lies in the fact that it easily generalizes to the case of different p-integrability in the single cross-sections V_i. In fact, if $p = (p_1, \dots, p_k) \in (1, \infty)^k$ we set

$$L^p(\Omega, F) := L^{p_1}(V_1, L^{p_2}(V_2, \dots L^{p_k}(V_k, F) \dots)).$$

In the third line of the smoothness assumptions (4.6) then we have to replace p by p_i. The remaining definitions, such as the domain of A, remain exactly the same. Also the statement of Theorem 4.6 holds without any change. Observe that step IIa) of the proof implies that A_1 and A_2 are resolvent commuting on $C_c^\infty(\Omega)$. Since this space is dense in $L^p(\Omega)$ for $p = (p_1, \dots, p_k)$, A_1 and A_2 are resolvent commuting in the case of different p_i as well. The remaining parts of the proof then copy verbatim.

We remark that according extensions to different p in each cross-section are valid also for the results obtained in the subsequent sections.

Proof. Step I: cylindrical decomposition.
We define $L^p(V_i, F)$-realizations of the boundary value problems (A_i, B_i) by

$$\begin{aligned} D(A_i) &:= \{u \in W^{2m_i,p}(V_i, F);\ B_{i,j}(\cdot, D)u = 0 \quad (j = 1, ..., m_i)\} \\ A_i u &:= A_i(\cdot, D)u, \quad u \in D(A_i). \end{aligned}$$

As F is Banach space of class $\mathcal{HT}$ assumptions $(ii)-(iv)$ and [10, Theorem 2.3] show that for every $\phi > \varphi_i$ there exists $\delta_i = \delta_i(\phi) \geq 0$ such that $A_i + \delta_i \in \mathcal{H}^\infty(L^p(V_i, F))$ and $\phi^\infty_{A_i+\delta_i} \leq \phi$. Moreover, it is proved in [11, Theorem 8.2] that

$$\mathcal{R}(\{\lambda^{1-\frac{|\alpha|}{2m_i}} D^\alpha(\lambda + A_i + \delta_i)^{-1};\ \lambda \in \Sigma_{\pi-\phi},\ 0 \leq |\alpha| \leq 2m_i\}) < \infty. \tag{4.8}$$

These statements remain true for the canonical extension of A_i to $L^p(\Omega, F)$ which, for simplicity, we will denote by A_i again. Note that the domain of A_i in $L^p(\Omega, F)$ reads as

$$D(A_i) := \Big\{ u \in L^p\big(V_1 \times \cdots \times V_{i-1}, W^{2m_i,p}(V_i, L^p(V_{i+1} \times \cdots \times V_k, F))\big); \\ B_{i,j}(\cdot, D)u = 0 \quad (j = 1, ..., m_i) \Big\} \tag{4.9}$$

Step II: case $k = 2$.

a) We first show that resolvents of the extensions A_1 and A_2 commute. To this end, we will frequently make use of the following observation. If $T \in \mathcal{L}(E_1, E_2)$ and $u \in W^{k,p}(G, E_1)$ for Banach spaces E_1, E_2, and $G \subset \mathbb{R}$ open, then

$$D^\alpha T u = T D^\alpha u \quad (|\alpha| \le k) \tag{4.10}$$

in $L^p(G, E_2)$. This follows easily by the fact that a derivative $\partial_{x_j} u$ represents the limit of a convergent sequence in $L^p(G, E_1)$ and by the continuity of T. For the following argumentation it will be convenient to introduce the notation

$$D(A_1, X) := \{u \in W^{2m_1,p}(V_1, X);\ B_{1,j}(\cdot, D)u = 0,\ j = 1, \ldots, m_1\}$$

for the domain of A_1 in the X-valued space $L^p(V_1, X)$. Here we are particularly interested in the case $X = D(A_2, F)$. According to step I,

$$\lambda + A_1 : D(A_1, D(A_2, F)) \to L^p(V_1, D(A_2, F))$$

is an isomorphism for $\lambda \in \rho(-A_1)$. Fubini's theorem yields

$$D(A_1, D(A_2, F)) \hookrightarrow W^{2m_1,p}(V_1, W^{2m_2,p}(V_2, F)) \cong W^{2m_2,p}(V_2, W^{2m_1,p}(V_1, F)).$$

First we set $E_1 := W^{2m_1,p}(V_1, F)$, $E_2 := W^{1-1/p,p}(\partial V_1, F)$, and $T = B_1$. Then relation (4.10) implies

$$D^{\alpha_2} B_1 u = B_1 D^{\alpha_2} u \quad (u \in W^{2m_2,p}(V_2, E_1),\ |\alpha_2| \le 2m_2).$$

This shows that

$$D(A_1, D(A_2, F)) \hookrightarrow W^{2m_2,p}(V_2, D(A_1, F)).$$

Since $B_2 u = 0$ for $u \in D(A_1, D(A_2, F))$, we even have

$$D(A_1, D(A_2, F)) \hookrightarrow D(A_2, D(A_1, F)).$$

Interchanging the roles of A_1 and A_2 we obtain the converse embedding. Hence we have

$$D(A_1, D(A_2, F)) \cong D(A_2, D(A_1, F))$$

with equivalent norms. The above arguments also include that

$$L^p(V_1, D(A_2, F)) \cong D(A_2, L^p(V_1, F)).$$

From this we conclude that

$$\lambda + A_1 : D(A_2, D(A_1, F)) \to D(A_2, L^p(V_1, F)) \tag{4.11}$$

is isomorphic. Setting $E_1 = D(A_1, F)$, $E_2 = L^p(V_1, F)$, and $T = \lambda + A_1$, relation (4.10) gives us

$$D^{\alpha_2}(\lambda + A_1)u = (\lambda + A_1)D^{\alpha_2}u \quad (u \in D(A_2, D(A_1, F))).$$

Setting $E_1 = E_2 = F$ and $T = a^1_{\alpha^1}$, in view of (4.10) we also see that D^{α_2} and the coefficients $a^1_{\alpha^1}$ commute. By our assumption (vi) on the coefficients this yields

$$(\mu + A_2)(\lambda + A_1)u = (\mu + A_1)(\lambda + A_2)u \quad (u \in D(A_2, D(A_1, F))). \tag{4.12}$$

Now pick $f \in L^p(V_2, L^p(V_1, F))$ and $\mu \in \rho(-A_2)$. Then we have $(\mu + A_2)^{-1}f \in D(A_2, L^p(V_1, F))$. Since (4.11) is isomorphic, we obtain

$$(\lambda + A_1)^{-1}(\mu + A_2)^{-1}f \in D(A_2, D(A_1, F)).$$

Hence the application of $(\lambda + A_1)(\mu + A_2)$ on this expression makes sense and we obtain by virtue of (4.12) that

$$(\lambda + A_1)^{-1}(\mu + A_2)^{-1}f = (\lambda + A_2)^{-1}(\mu + A_1)^{-1}f.$$

b) Let $\phi > \max\{\varphi_1, \varphi_2\}$. Thanks to assumption (v) and part I) of the proof, for $i = 1, 2$ there exist $\phi > \phi_i > \varphi_i$ and $\delta_i = \delta_i(\phi_i) \geq 0$ such that $A_i + \delta_i \in \mathcal{H}^\infty(L^p(\Omega, F))$, $\phi^\infty_{A_i+\delta_i} < \phi_i$, and $\phi^\infty_{A_1+\delta_1} + \phi^\infty_{A_2+\delta_2} < \pi$. Setting $\delta := \delta_1 + \delta_2$, $D(\tilde{A}) := D(A_1) \cap D(A_2)$ and $\tilde{A} := A_1 + A_2 + \delta$, Proposition 3.5 yields $\tilde{A} \in \mathcal{RH}^\infty(L^p(\Omega, F))$ with $\phi^{\mathcal{R},\infty}_{\tilde{A}} < \max\{\phi_1, \phi_2\} < \phi$.
c) It remains to show $D(\tilde{A}) \subset D(A)$ and the $\mathcal{R}$-boundedness statement (4.7). As F has property (α), $A_2 + \delta_2 \in \mathcal{RH}^\infty(L^p(V_2, F))$ by [26, Theorem 5.3]. For $\frac{|\alpha^1|}{2m_1} + \frac{|\alpha^2|}{2m_2} \leq 1$ we consider the family of operators

$$\{\lambda^{1-(\frac{|\alpha^1|}{2m_1}+\frac{|\alpha^2|}{2m_2})}D^{(\alpha^1,\alpha^2)}(\lambda + \tilde{A})^{-1};\ \lambda \in \Sigma_{\pi-\phi}\}.$$

By the fact that $A_1 + \delta_1 \in \mathcal{H}^\infty(L^p(\Omega, F))$ has bounded imaginary powers, we obtain $D(A_1^\nu) = [L^p(\Omega, F), D(A_1)]_\nu$ for $\nu \in [0, 1]$ (see [11, Theorem 2.5]), where $[L^p(\Omega, F), D(A_1)]_\nu$ denotes the complex interpolation space between $L^p(\Omega, F)$ and $D(A_1)$ of order ν. From this we deduce

$$D(A_1^\nu) = [L^p(\Omega, F), D(A_1)]_\nu \hookrightarrow W^{2m_1\nu,p}(V_1, L^p(V_2, F)).$$

Choosing δ_1 suitably large this shows that $D^{(\alpha^1,0)}(A_1 + \delta_1)^{-\frac{|\alpha^1|}{2m_1}}$ is bounded for $|\alpha^1| \leq 2m_1$. Thus, thanks to Lemma 2.4a), it suffices to show that the family

$$\{\lambda^{1-(\frac{|\alpha^1|}{2m_1}+\frac{|\alpha^2|}{2m_2})}(A_1 + \delta_1)^{\frac{|\alpha^1|}{2m_1}}D^{(0,\alpha^2)}(\lambda + \tilde{A})^{-1};\ \lambda \in \Sigma_{\pi-\phi},\ \frac{|\alpha^1|}{2m_1} + \frac{|\alpha^2|}{2m_2} \leq 1\}$$

is $\mathcal{R}$-bounded. To this end, pick $\sigma \in (\phi_1, \phi)$. For any $\lambda \in \Sigma_{\pi-\phi}$ we define the holomorphic function

$$G_\lambda(z) := \lambda^{1-(\frac{|\alpha^1|}{2m_1}+\frac{|\alpha^2|}{2m_2})}z^{\frac{|\alpha^1|}{2m_1}}D^{(0,\alpha^2)}(\lambda + z + A_2 + \delta_2)^{-1} \quad (z \in \Sigma_\sigma).$$

A homogeneity argument yields the existence of $C = C(\phi, \sigma) > 0$ such that

$$|\lambda^{1-(\frac{|\alpha^1|}{2m_1}+\frac{|\alpha^2|}{2m_2})}z^{\frac{|\alpha^1|}{2m_1}}| \leq C|\lambda + z|^{1-\frac{|\alpha^2|}{2m_2}}.$$

By virtue of Lemma 2.4b) and relation (4.8) we conclude

$$\mathcal{R}(\{G_\lambda(z);\ z \in \Sigma_\sigma,\ \lambda \in \Sigma_{\pi-\varphi}\}) < \infty.$$

From step II b) we also know that

$$\begin{aligned} &D^{(0,\alpha^2)}(\lambda + z + A_2 + \delta_2)^{-1}(\mu - A_1)^{-1} \\ &= (\mu - A_1)^{-1} D^{(0,\alpha^2)}(\lambda + z + A_2 + \delta_2)^{-1}. \end{aligned}$$

Hence we may apply Theorem 3.4 to the result

$$\mathcal{R}(\{G_\lambda(A_1);\ \lambda \in \Sigma_{\pi-\varphi}\}) < \infty.$$

By an approximation argument very similar to the final part of the proof of Proposition 3.5 we therefore see that

$$G_\lambda(A_1) = \lambda^{1-(\frac{|\alpha^1|}{2m_1}+\frac{|\alpha^2|}{2m_2})}(A_1+\delta_1)^{\frac{|\alpha^1|}{2m_1}} D^{(0,\alpha^2)}(\lambda + \tilde{A})^{-1}.$$

Consequently, relation (4.7) follows. This, in turn, yields $D(\tilde{A}) \subset D(A)$, hence $A = \tilde{A}$.

Step III: case $k > 2$.

Given $f \in L^p(\Omega, F)$, Lemma 3.2 and part IIa) of the proof imply

$$\begin{aligned} &(\zeta - A_l)^{-1}(\lambda - (A_i + A_j))^{-1} f \\ &\quad = \lim_{n\to\infty} \frac{1}{2\pi i} \int_\Gamma \rho_n(\mu)(\zeta - A_l)^{-1}(\lambda - \mu - A_i)^{-1}(\mu - A_j)^{-1} d\mu\ f \\ &\quad = \lim_{n\to\infty} \frac{1}{2\pi i} \int_\Gamma \rho_n(\mu)(\lambda - \mu - A_i)^{-1}(\mu - A_j)^{-1}(\zeta - A_l)^{-1} d\mu\ f \\ &\quad = (\lambda - (A_i + A_j))^{-1}(\zeta - A_l)^{-1} f. \end{aligned}$$

Hence the resolvent of an extension commutes with the resolvent of finite sums of extensions. As the bounded $\mathcal{H}^\infty$-calculus as well as relation (4.7) are preserved in each iteration step, the claim for arbitrary k follows by induction. □

5. The Laplacian on cylindrical Lipschitz domains with mixed boundary conditions

In this section we still consider $\Omega := \prod_{i=1}^k V_i$, however, with the difference that $V_i \subset \mathbb{R}^{n_i}$ now each may be a bounded Lipschitz domain. Recall from [24] that a bounded domain $G \subset \mathbb{R}^n$ is called a bounded (graph) Lipschitz domain if its boundary locally (and eventually after a suitable coordinate transformation) is described as the graph of a Lipschitz function. On such domains we consider the resolvent problem for the Laplacian with mixed Dirichlet and Neumann boundary conditions

$$\begin{aligned} \lambda u - \Delta u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \Gamma_0, \\ \partial_\nu u &= 0 \text{ on } \Gamma_1, \end{aligned} \tag{5.1}$$

where $\partial_\nu u$ denotes the outer normal derivative of u, $\Gamma_0 := \prod_{i\in N_0} \mathcal{V}_i$ and $\Gamma_1 := \prod_{i\in N_1} \mathcal{V}_i$ with $N_0 \cup N_1 = \{1,\dots,k\}$, $N_0 \cap N_1 = \emptyset$, and $\mathcal{V}_i$ as in (4.3). The boundary value problem (5.1) decomposes into

$$\begin{aligned} \lambda u - \Delta u &= f \text{ in } V_i, \\ B_i u &= 0 \text{ on } \partial V_i, \end{aligned} \tag{5.2}$$

where $B_i u := u$ for $i \in N_0$ and $B_i u := \partial_\nu u$ for $i \in N_1$.

Given a Banach space E, as L^p-realizations we define

$$\begin{aligned} D(\Delta_{p,i}, E) &:= \{u \in W_0^{1,p}(V_i, E);\ \Delta u \in L^p(V_i, E)\} \\ \Delta_{p,i} u &:= \Delta u \end{aligned} \tag{5.3}$$

for the E-valued Dirichlet-Laplacian on V_i, i.e. in case $i \in N_0$, and

$$\begin{aligned} D(\Delta_{p,i}, E) := \{u \in W^{1,p}(V_i, E);\ \exists v \in L^p(V_i, E)\ \forall \varphi \in W^{1,p'}(V_i) : \\ -\int_{V_i} \nabla u \nabla \varphi = \int_{V_i} v\varphi\} \\ \Delta_{p,i} u := v \end{aligned}$$

for the E-valued Neumann-Laplacian on V_i, i.e. in case $i \in N_1$. If $E = \mathbb{C}$, we just write $D(\Delta_{p,i})$. To emphasize that we mean the Dirichlet-Laplacian or the Neumann-Laplacian we also use occasionally the notation $\Delta^D_{p,i}$ or $\Delta^N_{p,i}$, respectively. Recall that for smooth u in view of Green's formula $\partial_\nu u = 0$ on ∂V if and only if

$$\int_V \Delta u \varphi = -\int_V \nabla u \nabla \varphi \quad (\varphi \in W^{1,p'}(V)).$$

As before, we use the same symbol for the canonical extensions of $\Delta_{p,i}$ to $L^p(\Omega)$. We define the (scalar-valued) L^p-realization of the Laplacian with mixed boundary conditions on Ω by

$$\begin{aligned} D(\Delta_p) &:= \bigcap_{i=1}^k D(\Delta_{p,i}) \\ \Delta_p u &:= \sum_{i=1}^k \Delta_{p,i} u. \quad (u \in D(\Delta_p)). \end{aligned} \tag{5.4}$$

Theorem 5.1. *For $i = 1,\dots,k$ let V_i be a C^2 standard domain in $\mathbb{R}^{n_i}$, $n_i \in \mathbb{N}$, (see Definition 4.5), or a bounded Lipschitz domain in $\mathbb{R}^{n_i}$, $n_i \geq 2$. On two-dimensional Lipschitz cross-sections V_i we assume $\Delta_{p,i}$ to be the Dirichlet-Laplacian. Set $\Omega := \prod_{i=1}^k V_i$. Then there exists $\varepsilon > 0$ depending only on the Lipschitz character of the different V_i such that for all $(3+\varepsilon)' < p < 3+\varepsilon$ there exists $\delta \geq 0$ with $-\Delta_p + \delta \in \mathcal{RH}^\infty(L^p(\Omega))$ and $\phi^{\mathcal{R},\infty}_{-\Delta_p+\delta} < \frac{\pi}{2}$. In the case of pure Dirichlet boundary conditions, the assertion remains true for $\delta = 0$ and we even have $\phi^{\mathcal{R},\infty}_{-\Delta_p} = 0$.*

Remark 5.2. a) Compared to existing literature, it is worthwhile to highlight two facts concerning the outcome of Theorem 5.1: the result includes (simultaneously) classes of unbounded Lipschitz domains and of mixed boundary conditions.

b) Note that $W_0^{1,p}(V) = \{u \in W^{1,p}(V);\ \gamma_{\partial V} u = 0\}$ for any bounded Lipschitz-domain V, cf. [29]. This implies

$$D(\Delta_p^D) \subset \{u \in W_0^{1,p}(\Omega);\ \Delta u \in L^p(\Omega)\} \tag{5.5}$$

in case we have pure Dirichlet boundary conditions. Fubini's theorem further implies that

$$\begin{aligned} &D(\Delta_p^N) \\ &\subset \{u \in W^{1,p}(\Omega);\ \exists v \in L^p(\Omega)\ \forall \varphi \in W^{1,p'}(\Omega):\ -\int_\Omega \nabla u \nabla \varphi = \int_\Omega v\varphi\} \end{aligned} \tag{5.6}$$

in case of pure Neumann conditions. Hence, Δ_p^D and Δ_p^N coincide with the usual (weak) Dirichlet- and Neumann-Laplacian respectively, defined on Lipschitz domains, cf. [24], [36].

c) It is interesting to remark that Theorem 5.1 yields more regularity for $u \in D(\Delta_p)$ than indicated by the right hand sides of (5.5) and (5.6), if at least one cross-section V_i is smooth. In fact, in each smooth cross-section in view of (4.8) u belongs to $W^{2,p}$. Thus, for the special situation of cylindrical domains this improves previous results in the literature, since for general Lipschitz domains mixed second order derivatives are not expected to belong to $L^p(\Omega)$, cf. [24].

Proof. Step I: cylindrical decomposition.

If V_i is a C^2 standard domain, it is well known that the boundary value problems (5.2) are parameter-elliptic with $\varphi = 0$. As noted in the first step of the proof of Theorem 4.6, there exist $\delta \geq 0$ such that $-\Delta_{p,i} + \delta_i \in \mathcal{H}^\infty(L^p(V_i))$ with $\phi^\infty_{-\Delta_{p,i}+\delta_i} = 0$.

In case that V_i is a bounded Lipschitz domain in $\mathbb{R}^{n_i}$, it is shown in [36] that there exists $\varepsilon > 0$ depending only on the Lipschitz character of V_i such that for all $(3+\varepsilon)' < p < 3+\varepsilon$ we have that $\Delta_{p,i}$ generates a positive C_0-semigroup of contractions. According to a result of Duong, cf. [23, Corollary 1], for $\delta > 0$ this implies $-\Delta_{p,i} + \delta \in \mathcal{H}^\infty(L^p(V_i))$ with $\phi^\infty_{-\Delta_{p,i}+\delta} < \frac{\pi}{2}$. Note that the shift δ is inserted to assure injectivity in case of Neumann boundary conditions. By the fact that $L^p(V_i)$ has property (α) this yields $-\Delta_{p,i} + \delta \in \mathcal{RH}^\infty(L^p(V_i))$ and $\phi^{\mathcal{R},\infty}_{-\Delta_{p,i}+\delta} < \frac{\pi}{2}$. Again this remains true for the canonical extensions of the operators to $L^p(\Omega)$.

Step II: case $k = 2$.

Unlike in the proof of Theorem 4.6 we have $-\Delta_{p,i} + \delta_i \in \mathcal{RH}^\infty(L^p(V_i, E))$ a priori only for $E := \mathbb{C}$ instead of general UMD spaces E. Moreover, $D(\Delta_{p,i}, E)$ in general is no longer a subset of $W^{2,p}(V_i, E)$. By these facts we first have to show that

$$\lambda - \Delta_{p,1} : D(\Delta_{p,1}, D(\Delta_{p,2})) \to L^p(V_1, D(\Delta_{p,2})) \tag{5.7}$$

is isomorphic, which before was guaranteed by known results (see step 2 of the proof of Theorem 4.6).

Let $\Delta_{p,1}$ be arbitrary, that is, $\Delta_{p,1}$ is either the Dirichlet- or the Neumann-Laplacian in $L^p(V_1)$ and let $\lambda \in \rho(\Delta_{p,1})$. By Fubini's theorem, we see that

$$-\Delta_{p,1} + \delta_1 \in \mathcal{RH}^\infty(L^p(V_1, W^{k,p}(V_2))), \quad \phi^{\mathcal{R},\infty}_{-\Delta_{p,1}+\delta_1} < \frac{\pi}{2}, \tag{5.8}$$

for $k = 0, 1$. In particular, $\lambda - \Delta_{p,1} + \delta_1 : D(\Delta_{p,1}, W^{1,p}(V_2)) \to L^p(V_1, W^{1,p}(V_2))$ is isomorphic. For the sake of readability, in what follows we assume the shift $\delta_1 = 0$ to be included in λ. In order to show that (5.7) is isomorphic as well, it remains to prove surjectivity. To this end, pick $f \in L^p(V_1, D(\Delta_{p,2}))$. In view of (5.8) there exists $u \in D(\Delta_{p,1}, W^{1,p}(V_2))$ such that $(\lambda - \Delta_{p,1})u = f$.

First assume $\Delta_{p,2}$ to be given as a Neumann-Laplacian. Then there exists $v \in L^p(V_1, L^p(V_2))$ such that for all $\varphi \in W^{1,p'}(V_2)$ it holds that

$$-\int_{V_2} \nabla_2(\lambda - \Delta_{p,1})u\nabla_2\varphi = -\int_{V_2} \nabla_2 f \nabla_2\varphi = \int_{V_2} v\varphi.$$

Set $w := (\lambda - \Delta_{p,1})^{-1}v \in D(\Delta_{p,1}, L^p(V_2))$. Since $(\lambda - \Delta_{p,1})^{-1}$ is bounded on $L^p(V_1, L^p(V_2))$, we deduce

$$\begin{aligned} -\int\limits_{V_2} \nabla_2 u\nabla_2\varphi &= -(\lambda - \Delta_{p,1})^{-1} \int\limits_{V_2} \nabla_2(\lambda - \Delta_{p,1})u\nabla_2\varphi \\ &= (\lambda - \Delta_{p,1})^{-1} \int_{V_2} v\varphi = \int_{V_2} (\lambda - \Delta_{p,1})^{-1} v\varphi = \int_{V_2} w\varphi. \end{aligned}$$

Observe that here $(\lambda - \Delta_{p,1})^{-1}$ can be replaced by the terms $\nabla_1(\lambda - \Delta_{p,1})^{-1}$ or $\Delta_{p,1}(\lambda - \Delta_{p,1})^{-1}$ and u by $\nabla_1 u$ or $\Delta_{p,1}u$, respectively. Hence, we obtain $u \in D(\Delta_{p,1}, (\Delta_{p,2}))$. In a very similar way surjectivity of (5.7) can be proved, if both $\Delta_{p,1}$ and $\Delta_{p,1}$ are given as Dirichlet-Laplacians.

Now we continue as in step IIa) of the proof of Theorem 4.6. Indeed, Fubini's theorem yields

$$D(\Delta_{p,1}, D(\Delta_{p,2})) \hookrightarrow W^{1,p}(V_1, W^{1,p}(V_2)) \cong W^{1,p}(V_2, W^{1,p}(V_1))$$

and by very similar calculations as above we obtain

$$u, \nabla_2 u, \Delta_{p,2}u \in L^p(V_2, D(\Delta_{p,1}))$$

for $u \in D(\Delta_{p,1}, D(\Delta_{p,2}))$. Thus, we arrive at

$$D(\Delta_{p,1}, D(\Delta_{p,2})) \cong D(\Delta_{p,2}, D(\Delta_{p,1})).$$

Now we are in the same situation as in step IIa) of the proof of Theorem 4.6. The same arguments therefore show that $\Delta_{p,1}$ and $\Delta_{p,2}$ are resolvent commuting. Exactly as in step IIb), Proposition 3.5(a) now proves the claim for $k = 2$.

Step III: case $k > 2$.

W.l.o.g. we can rearrange the different sets V_i such that $\Gamma_0 = \prod_{i=1}^{l} \mathcal{V}_i$ and $\Gamma_1 = \prod_{i=l+1}^{k} \mathcal{V}_i$ with some $0 < l < k$. Set $\Omega_0 := \prod_{i=1}^{l} V_i$ and $\Omega_1 := \prod_{i=l+1}^{k} V_i$ and let $\Delta^D_{p,0}$ and $\Delta^N_{p,1}$ denote the extended $L^p(\Omega)$-realizations of the Dirichlet problem on Ω_0 and of the Neumann problem on Ω_1, respectively. Then according to the first part of the proof, by iteration we see that there exist $\delta_D, \delta_N \geq 0$ such that $-\Delta_{p,0}^D + \delta_D$ and $-\Delta_{p,1}^N + \delta_N$ admit bounded $\mathcal{RH}^\infty$-caluli with angles less than $\frac{\pi}{2}$. Moreover, the argumentation on commutativity of resolvents performed in step II) applies to $\Delta_{p,0}^D$ and $\Delta_{p,1}^N$. As $\Delta_p = \Delta_{p,0}^D + \Delta_{p,1}^N$, Proposition 3.5(a) gives the result.

The additional assertion on pure Dirichlet boundary conditions can be seen as follows. For every appearing domain $V_i \subset \mathbb{R}^{n_i}$ it is well known, that $-\Delta_{p,i}^D$ as defined in (5.3) is injective, i.e. no shift is required. Hence, in this case the result remains true for $\delta = 0$. Furthermore, For bounded Lipschitz domains V_i the semigroup generated by $\Delta_{p,i}^D$ satisfies an appropriate Gaussian estimate, cf. [5, Theorem 5.7]. Thanks to results derived in [27, Chapter 11], we therefore even have $\phi^{\mathcal{R},\infty}_{-\Delta_{p,i}^D} = 0$. Combining this with the first part of step I, we therefore can achieve that $\phi^{\mathcal{R},\infty}_{-\Delta_p} = 0$ in case of pure Dirichlet conditions. The proof is now complete. □

Given a bounded Lipschitz domain $V \subset \mathbb{R}^2$, the range for p such that $-\Delta_p^D \in \mathcal{RH}^\infty(L^p(V))$ extends to $(4+\varepsilon)' < p < 4+\varepsilon$. This activates the natural question, whether this range is preserved for a higher dimensional domain provided the roughness of the boundary is of two dimensional character. By our technique we can give a positive respond to this question with a relatively short proof for the case of higher dimensional Lipschitz cylinders.

Theorem 5.3. *Let V_i be a C^2 standard domain in $\mathbb{R}^{n_i}$, $n_i \in \mathbb{N}$, or a bounded Lipschitz domain in $\mathbb{R}^2$. Set $\Omega := \prod_{i=1}^k V_i$ and assume that $\Delta_{p,i}$ is the Dirichlet-Laplacian on Lipschitz cross-sections V_i. Then there exists $\varepsilon > 0$ depending only on the Lipschitz character of the different V_i such that for all $(4+\varepsilon)' < p < 4+\varepsilon$ there exists $\delta \geq 0$ with $-\Delta_p + \delta \in \mathcal{RH}^\infty(L^p(\Omega))$ and $\phi^{\mathcal{R},\infty}_{-\Delta_p+\delta} < \pi/2$, and where Δ_p is defined as in (5.4). In the case of pure Dirichlet boundary conditions, the assertion remains true for $\delta = 0$ and we even have $\phi^{\mathcal{R},\infty}_{-\Delta_p} = 0$.*

Remark 5.4. Observe that for the regarded cylindrical Lipschitz domains we can not only extend the range for p, but all observations given in Remark 5.2 apply also here, in particular the increase of regularity in smooth cross-sections.

Proof. If V_i is a bounded Lipschitz domain in $\mathbb{R}^2$, it is shown in [36] that there exists $\varepsilon > 0$ depending only on the Lipschitz character of V_i such that for all $(4+\varepsilon)' < p < 4+\varepsilon$ we have that $\Delta_{p,i}$ generates a positive C_0-semigroup of contractions. Now we can go on as in proof of Theorem 5.1. □

We close this section by giving a simple example in a Lipschitz cylinder in $\mathbb{R}^3$.

Corollary 5.5. *Let V be a bounded Lipschitz domain in $\mathbb{R}^2$ and $-\infty \leq a < b \leq \infty$. Then the negative Dirichlet-Laplacian $-\Delta_p^D$ admits an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus on $L^p(V \times (a,b))$ with $\phi^{\mathcal{R},\infty}_{-\Delta_p^D} = 0$, provided $(4+\varepsilon)' < p < 4+\varepsilon$ for a certain $\varepsilon > 0$ depending only on the Lipschitz character of V.*

Remark 5.6. The assertion of Corollary 5.5 remains true, if we assume Dirichlet conditions on the barrel and Neumann conditions on top and bottom of the cylinder.

6. A non-commuting example

In this section we consider a situation where operators on cross-sections do not necessarily commute. We emphasize that we do not aim for the greatest generality. The purpose of this section is just to demonstrate that our approach is not restricted to the commuting situation. Improvements and generalizations in one or the other direction are certainly possible. In particular, we restrict ourselves to the case of two domains and of two operators. So, let $\Omega = V_1 \times V_2$ and differential operators $A_1(x^1, D)$ and $A_2(x^2, D)$ such as in Theorem 4.6 be given. Then, for $\phi_i > \varphi_i$ there exist $\delta_i \geq 0$ such that the canonical extensions fullfill $A_i + \delta_i \in \mathcal{RH}^\infty(L^p(\Omega, F))$ and $\phi^{\mathcal{R},\infty}_{A_i+\delta_i} \leq \phi_i$. For the sake of simplicity we will assume $\delta_i = 0$ and the operators A_i to be subject to Dirichlet boundary conditions for $i = 1, 2$.

We assume the cylindrical structure to be disturbed in the following way: given a function r on V_1, we consider the differential operator

$$A_1(x^1, D) + r(x^1)A_2(x^2, D)$$

in Ω subject to Dirichlet boundary conditions. Associated to r we define an operator of pointwise multiplication in $L^p(\Omega, F)$ as

$$\begin{aligned} D(M_r) &:= \{u \in L^p(\Omega, F);\ ru \in L^p(\Omega, F)\} \\ M_r u &:= ru \quad (u \in D(M_r)). \end{aligned}$$

As before we will investigate the operator

$$\begin{aligned} D(A_r) &:= D(A_1) \cap D(M_r A_2) \\ A_r &:= A_1 + M_r A_2 \quad (u \in D(A_r)). \end{aligned}$$

The main difference to previous boundary value problems is that the operators A_1 and $M_r A_2$ are no longer resolvent commuting on $L^p(\Omega, F)$. Therefore we have to impose conditions on r that allow for an application of Proposition 3.7.

Theorem 6.1. *Let $1 < p < \infty$ and let F, V_1, V_2, as well as A_1 and A_2 fullfill the assumptions of Theorem 4.6 subject to Dirichlet boundary conditions. Let $\vartheta > 0$ with $\varphi_1 + \varphi_2 + \vartheta < \pi$ and assume that*

(i) *$r \in [W^{2m_1,p} + W^{2m_1,\infty}](V_1)$ if $2m_1 p > n_1$ and $r \in W^{2m_1,\infty}(V_1)$ else,*
(ii) *$r(x^1) \in \Sigma_\vartheta$ for all $x^1 \in V_1$, and*
(iii) *$r^{-1}D^\eta r \in L^\infty(V_1)$ for all $|\eta| \leq 2m_1$.*

Then for every $\phi > \max\{\varphi_1, \varphi_2 + \vartheta\}$, there exists $\delta = \delta_\phi > 0$ such that $A_r + \delta \in \mathcal{RH}^\infty(L^p(\Omega, F))$ and $\phi^{\mathcal{R},\infty}_{A_r+\delta} \leq \phi$.

Proof. Step I. For r subject to assumption (i) we have $M_r \in \mathcal{L}(L^p(\Omega, F))$. This implies

$$D(M_r A_2) := \{u \in D(A_2) :\ A_2 u \in D(M_r)\} = D(A_2),$$

hence $D(A_r) = D(A_1) \cap D(A_2)$. In addition, assumption (ii) yields $0 \in \rho(M_r)$ and $M_r \in \mathcal{H}^\infty(L^p(\Omega, F))$ with $\phi^\infty_{M_r} \leq \vartheta$. Thus we can apply Proposition 3.5(b) to the result that $M_r A_2 \in \mathcal{H}^\infty(L^p(\Omega, F))$ and $\phi^\infty_{M_r A_2} \leq \varphi_2 + \phi^\infty_{M_r}$.

Step II. We show that A_1 and $M_r A_2$ satisfy the Labbas-Terreni condition. To this end, we may assume $0 \in \rho(M_r A_2)$, since this can always be derived by a shift which we can compensate at the end by choosing $\delta \geq 0$ eventually a bit larger. By the fact that we assume Dirichlet boundary conditions (at this point general boundary conditions would cause more trouble) and in view of assumption (i), we obtain $M_r\big(D(A_1)\big) \subset D(A_1)$. This implies

$$D(M_r A_2 A_1) = D(A_2 A_1) = D(A_1 A_2) \subset D(A_1 M_r A_2). \tag{6.1}$$

For $u \in D(A_1 A_2)$ therefore the equality

$$M_r A_2(\mu + A_1)u = (\mu + A_1 - R) M_r A_2 u \tag{6.2}$$

with $R := [A_1, M_r] M_{r^{-1}}$ makes sense in $L^p(\Omega, F)$. Thanks to (6.1) we may also identify R as

$$Ru = [A_1(x^1, D), r(x^1)] r(x^1)^{-1} u =: R(x^1, D)u$$

for all $u \in M_r A_2 D(A_1 A_2) \subset D(A_1)$. Due to assumption (ii) the differential operator $R(x^1, D)$, and hence also R, is well defined on all of $D(A_1)$ and represented as a linear combination of differential operators of the form

$$R_\gamma(x^1, D) = a_{\alpha^1}(x^1) \prod_{\eta \in \mathcal{M}_\gamma} \big(r^{-1} D^\eta r\big)^{l_\eta}(x^1) D^\gamma,$$

with $\gamma < \alpha^1$, some $\mathcal{M}_\gamma \subset \{\eta \in \mathbb{N}_0^{n_1};\ \eta \neq 0,\ 0 \leq \eta_i \leq \alpha^1_i \text{ for } i = 1, \ldots, n_1\}$ and integers $l_\eta \in \mathbb{N}$ such that $\sum_{\eta \in \mathcal{M}_\gamma} l_\eta \eta = \alpha^1 - \gamma$. This shows that $R(x^1, D)$ is of lower order w.r.t. A_1. In view of assumption (iii) we also see that the coefficients of $R(x^1, D)$ satisfy condition (4.6). Hence there is a $\delta_1 \geq 0$ such that $A_1 - R + \delta_1 \in \mathcal{S}(L^p(\Omega, F))$ with $\phi_{A_1 - R + \delta_1} = \phi_{A_1}$.

Next, let $\phi_1 > \phi_{A_1}$, $\mu \in \Sigma_{\pi - \phi_1}$, and $v \in D(A_2)$. Inserting $u = (\mu + A_1)^{-1} v \in D(A_1 A_2)$ into (6.2) and applying $(\mu + A_1 - R + \delta_1)^{-1}$ to the resulting equation gives us

$$M_r A_2 (\mu + A_1)^{-1} v = (\mu + A_1 - R + \delta_1)^{-1} M_r A_2 v.$$

From this for $v \in D(A_2)$ and $\mu \in \Sigma_{\pi - \phi_1}$ we infer that

$$[M_r A_2, (\mu + A_1)^{-1}] v = (\mu + A_1)^{-1} (R + \delta_1)(\mu + A_1 - R + \delta_1)^{-1} M_r A_2 v.$$

Let $\phi_2 > \phi_{M_r A_2}$, and $\lambda \in \Sigma_{\pi - \phi_2}$. With the above relation, the expression appearing in the Labbas-Terreni commutator condition turns into

$$\begin{aligned} &M_r A_2 (\lambda + M_r A_2)^{-1} [(M_r A_2)^{-1}, (\mu + A_1)^{-1}] \\ &= -(\lambda + M_r A_2)^{-1} (\mu + A_1)^{-1} (R + \delta_1)(\mu + A_1 - R + \delta_1)^{-1}. \end{aligned}$$

This formula can easily be estimated to the result

$$\| M_r A_2 (\lambda + M_r A_2)^{-1} [(M_r A_2)^{-1}, (\mu + A_1)^{-1}] \|$$

$$\leq \frac{C}{(1+|\lambda|)|\mu|^{1+\frac{1}{2m_2}}} \quad (\mu \in \Sigma_{\pi-\phi_1},\ \lambda \in \Sigma_{\pi-\phi_2}),$$

where we employed $0 \in \rho(M_r A_2)$ and the fact that R is relatively bounded by A_1. The assertion now follows from Proposition 3.7. □

A counterpart of the above result for cylindrical Lipschitz domains reads as follows.

Theorem 6.2. *Let $1 < p < \infty$ and V_i, $i = 1, 2$, be given as in Theorem 5.1 (resp. Theorem 5.3). Assume that*

(i) $r \in [W^{2,p} + W^{2,\infty}](V_1)$ *if* $2p > n_1$ *and* $r \in W^{2,\infty}(V_1)$ *else,*
(ii) $r(x^1) \in \Sigma_\vartheta$ *for all* $x^1 \in V_1$ *and some* $0 \leq \vartheta < \pi$,
(iii) $\frac{\nabla r}{r}$, $\frac{\Delta r}{r} - 2\frac{|\nabla r|^2}{r^2} \in L^\infty(V_1)$,

and let $\phi > \vartheta$. Then, there exists $\varepsilon > 0$ depending only on the Lipschitz character of V_i such that for all $(3+\varepsilon)' < p < 3+\varepsilon$ (resp. $(4+\varepsilon)' < p < 4+\varepsilon$) there is a $\delta \geq 0$ such that for $-\Delta_{r,p}+\delta := -\Delta_{p,1} - M_r\Delta_{p,2} + \delta$ defined on $D(\Delta_{r,p}) := D(\Delta_{p,1}) \cap D(M_r\Delta_{p,2})$ we have that $-\Delta_{r,p} + \delta \in \mathcal{RH}^\infty(L^p(\Omega))$ with $\phi^{\mathcal{R},\infty}_{-\Delta_{r,p}+\delta} \leq \phi$.

Proof. We try to mimic the proof of Theorem 6.1. By the fact that

$$\Delta r u = u\Delta r + \nabla r \cdot \nabla u + r\Delta u$$

we see that also here we have $M_r(D(\Delta_{p,1})) \subset D(\Delta_{p,1})$. Completely analogous as before we therefore arrive at (6.2) with $A_1 = -\Delta_{p,1}$, $A_2 = -\Delta_{p,2}$, and

$$Ru = \frac{\nabla r}{r} \cdot \nabla u + \left(\frac{\Delta r}{r} - 2\frac{|\nabla r|^2}{r^2}\right) u.$$

We have to show that R is relatively bounded. Since we do not have $D(\Delta_{p,1}) \subset W^{2,p}$ here, this is not so obvious as above. Recall that by the results obtained in [24] we know that $D((-\Delta_{p,1})^{1/2}) = W^{1,p}_0(V_1)$. Since $\Delta_{p,1} \in \mathcal{H}^\infty(L^p(V_1))$ this yields

$$W^{1,p}_0(V_1) = D((-\Delta_{p,1})^{1/2}) = [L^p(V_1),\, D(\Delta_{p,1})]_{1/2}.$$

Hence the interpolation inequality

$$\|u\|_{W^{1,p}} \leq C\|u\|^{1/2}_{D(\Delta_{p,1})}\|u\|^{1/2}_p \leq C(\varepsilon\|u\|_{D(\Delta_{p,1})} + C(\varepsilon)\|u\|_p)$$

holds for all $u \in D(\Delta_{p,1})$ and $\varepsilon > 0$. This implies

$$\|Ru\|_p \leq C\|u\|_{W^{1,p}} \leq C(\varepsilon\|\Delta_{p,1}u\|_p + C(\varepsilon)\|u\|_p) \quad (u \in D(\Delta_{p,1}),\ \varepsilon > 0).$$

Thus R is a Kato perturbation of $-\Delta_{p,1}$ from which we deduce that $\mu - \Delta_{p,1} - R$ is sectorial for some $\mu > 0$. The remaining proof then copies verbatim from Theorem 6.1.

□

Example 6.3. Theorem 6.2 in particular covers the case of heat conduction in a Lipschitz cylinder with longitudinal or in cross-sections non constant heat conductivity coefficient.

In Theorems 6.1 and 6.2 there appears no explicit non-degeneracy condition on r. Therefore a brief discussion when r may degenerate is in order.

Remark 6.4. (Examples for r) Consider $V_1 \subset \mathbb{R}$. Let r satisfy the assumptions of Theorem 6.2 and assume that $\frac{r'}{r} = g$ with $g \in C_b(V_1)$. In case that $V_1 = I := (a, b)$ with $-\infty < a < b < \infty$ we have

$$r(x) = c_1 \exp(\int_a^x g(\tau) d\tau + c_2), \tag{6.3}$$

and in view of condition (iii) every possible r has this structure. Hence no degeneration is possible. In case $I := (a, \infty)$ equation (6.3) holds as before and no degeneration at a is possible. On the other hand, $c_2 = 0$ and $\lim_{x\to\infty} \int_a^x g(\tau)d\tau = -\infty$ implies $r(x) \to 0$ for $x \to \infty$. Thanks to

$$\frac{r''}{r} - 2\frac{r'r'}{r^2} = g' - g^2,$$

$g \in C_b^1(V_1)$ suffices for equation (6.3) to define r subject to assumption (iii).

In more explicit situations we may even allow r to grow at infinity as the next result shows.

Theorem 6.5. *Let the assumptions of Theorem 6.2 be satisfied. In case that $V_1 = \mathbb{R}^{n_1}$ the assertions of Theorem 6.2 remain true, if condition (i) is replaced by $r \in C^\infty(V_1)$.*

Proof. The proof follows by the fact that in this situation relation (6.2) holds in the sense of distributions. By standard arguments we also can show that $M_r A_2 D(A_1) \subset D(A_1)$. But then we can argue in the same way as in the proof of Theorem 6.1 (see also [31, Chapter 5]). □

Remark 6.6. Theorem 6.5 includes the special coefficient $r(x) = \exp(cx)$ with $c \in \mathbb{R}$ and $V_1 = \mathbb{R}$. The significance of this degenerate example lies in the fact that it appears, for instance, in the treatment of contact angle problems (see [31, Chapter 5]).

Acknowledgements. We would like to thank Robert Haller-Dintelmann and Joachim Rehberg for helpful hints concerning the content of Section 5. Also we wish to thank Robert Denk for many fruitful discussions.

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University of Konstanz
Department of Mathematics
Box D 187, 78457 Konstanz
Germany

E-mail address: `tobias.nau@uni-konstanz.de`

Center of Smart Interfaces
TU Darmstadt
Petersenstrasse 32, 64287 Darmstadt
Germany

E-mail address: `saal@csi.tu-darmstadt.de`