On L^q-estimates for the stationary Oseen equations in a rotating frame

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Abstract

Consider a body moving in a three-dimensional Navier-Stokes liquid with a prescribed constant non-zero velocity $\xi \in \mathbb{R}^3 \setminus \{0\}$ and non-zero angular velocity $\omega \in \mathbb{R}^3 \setminus \{0\}$. Linearizing the associated equations of motion written in a frame attached to the body, we obtain the three-dimensional Oseen equations in a rotating frame of reference. We will consider the corresponding stationary problem in the whole space. Our main result concerns elliptic L^q-estimates of the solutions. Such estimates have been established by R. Farwig in Tohoku Math. J., Vol. 58, 2006. We introduce a new method resulting in a much simplified proof of these estimates. Moreover, our method yields more detailed information on the dependency of the involved constants on ξ and ω.

1 Introduction

Consider a body moving in a Navier-Stokes liquid with a prescribed constant non-zero velocity $\xi \in \mathbb{R}^3 \setminus \{0\}$ and non-zero angular velocity $\omega \in \mathbb{R}^3 \setminus \{0\}$. We assume that ξ and ω are parallel and directed along the x_3-axis. Due to a simple transformation (see [6, Section 2]), this assumption can be made without loss of generality whenever $\xi \cdot \omega \neq 0$. After a suitable non-dimensionalization, the

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corresponding equations of motion in a frame attached to the body $\mathcal{B} \subset \mathbb{R}^3$ are

$$\begin{cases} \partial_t v - \Delta v + \nabla p + \mathcal{R}\big(v \cdot \nabla v - \partial_3 v\big) & \\ \qquad\qquad + \mathcal{T}\big(\,\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big) = f & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{B}} \times (0,\infty), \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{B}} \times (0,\infty), \\ v = v_* & \text{on } \partial\mathcal{B} \times (0,\infty), \\ v(x,0) = v_0, & \end{cases}$$

where $\mathcal{R}, \mathcal{T} > 0$ are non-dimensional constants. Here, v and p denotes the velocity field and the pressure of the liquid, respectively.

We consider in this paper the corresponding stationary linearized whole space problem, that is,

$$\text{(1.1)} \qquad \begin{cases} -\Delta v + \nabla p - \mathcal{R}\partial_3 v + \mathcal{T}\big(\,\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big) = f & \text{in } \mathbb{R}^3, \\ \operatorname{div} v = 0 & \text{in } \mathbb{R}^3. \end{cases}$$

The above system is the classical stationary Oseen problem with the extra term $\mathcal{T}\big(\,\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big)$, which stems from the rotating frame of reference. Note that due to the unbounded coefficient, this term can not be treated at a perturbation of the Oseen operator.

We will prove elliptic L^q-estimates of the solutions (v,p) to (1.1) in terms of the data f. Our main result reads:

Theorem 1.1. *[Main Theorem] Let $1 < q < \infty$, let $\mathcal{R}_0 > 0$, and consider $0 < \mathcal{R} < \mathcal{R}_0$ and $\mathcal{T} > 0$. For any $f \in L^q(\mathbb{R}^3)$ there exists a solution $(v,p) \in D^{2,q}(\mathbb{R}^3)^3 \times D^{1,q}(\mathbb{R}^3)$ to (1.1) that satisfies*

$$\text{(1.2)} \qquad \|\nabla^2 v\|_q + \|\nabla p\|_q \leq C_1 \|f\|_q,$$

with C_1 independent on $\mathcal{R}_0$, $\mathcal{R}$, and $\mathcal{T}$. Moreover,

$$\text{(1.3)} \qquad \|\mathcal{R}\partial_3 v\|_q + \|\mathcal{T}\big(\,\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big)\|_q \leq C_2 \bigg(1 + \frac{1}{\mathcal{T}^2}\bigg) \|f\|_q,$$

with $C_2 = C_2(\mathcal{R}_0)$. If $1 < q < 4$, then

$$\text{(1.4)} \qquad \|\nabla v\|_{\frac{4q}{4-q}} \leq C_3 \bigg(\mathcal{R}^{-\frac{1}{4}} + \mathcal{T}^{-\frac{1}{2}}\bigg) \|f\|_q,$$

with $C_3 = C_3(\mathcal{R}_0)$. If $1 < q < 2$, then

$$\text{(1.5)} \qquad \|v\|_{\frac{2q}{2-q}} \leq C_4 \bigg(\mathcal{R}^{-\frac{1}{2}} + \mathcal{T}^{-1}\bigg) \|f\|_q,$$

with $C_4 = C_4(\mathcal{R}_0)$. Moreover, if $(\tilde{v},\tilde{p}) \in D^{2,r}(\mathbb{R}^3)^3 \times D^{1,r}(\mathbb{R}^3)$ is another solution to (1.1), then

$$\text{(1.6)} \qquad \tilde{v} = v + \alpha\,\mathrm{e}_3 + \beta\,\mathrm{e}_3 \wedge x \quad \textit{and} \quad \tilde{p} = p + \gamma$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$.

The estimates and uniqueness statement in Theorem 1.1 have already been established in [2]. Due to the term $\mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla v - \mathrm{e}_3 \wedge v\big)$, the estimates do not follow, as opposed to the classical Oseen system, from a standard application of well-known Fourier multiplier theorems. Therefore, in [2] the estimates are established by a very technical application of the Littlewood-Payley decomposition. The purpose of this paper is to give a simpler proof of the estimates using a different method. More specifically, we utilize an idea going back to [4] of transforming (1.1) into a time-dependent Oseen problem. After the transformation, we are able to use standard Fourier multiplier theory to obtain the estimates. In addition to being simpler, our method also yields more detailed information than in [2] on the dependency of the involved constants on $\mathcal{R}$ and $\mathcal{T}$.

Before we in section 2 give a proof of the main theorem, we first introduce some basic notation. By $L^q(\mathbb{R}^3)$ we denote the usual Lebesgue space with norm $\|\cdot\|_q$. By $W^{m,q}(\mathbb{R}^3)$ we standard Sobolev spaces, and by $D^{m,q}(\mathbb{R}^3)$ the homogeneous Sobolev space with semi-norm $|\cdot|_{m,q}$, that is,

$$|v|_{m,q} := \Big(\sum_{|\alpha|=m} \int_{\mathbb{R}^3} |\partial^\alpha v(x)|^q \,\mathrm{d}x \Big)^{\frac{1}{q}}, \; D^{m,q} := \{v \in L^1_{loc}(\mathbb{R}^3) \mid |v|_{m,q} < \infty\}.$$

For functions $u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$, $\operatorname{div} u(x,t) := \operatorname{div}_x u(x,t)$, $\Delta u(x,t) := \Delta_x u(x,t)$ etc., that is, unless otherwise indicated, differential operators act in the spatial variable x only. We use $\mathcal{F} f = \widehat{f}$ to denote the Fourier transformation, and $\mathscr{S}(\mathbb{R}^n)$ to denote the class of Schwartz functions. Finally note that constants in capital letters in the proofs and theorems are global, while constants in small letters are local to the proof in which they appear.

2 Proof of Main Theorem

We will make use a simple transformation that transforms solutions to (1.1) into time-periodic solutions to the classical time-dependent Oseen problem. For this purpose, we introduce the rotation-matrix corresponding to the angular velocity $\mathcal{T}\,\mathrm{e}_3$. More specifically, let $\mathrm{E}_3 \in \operatorname{skew}_{3\times 3}(\mathbb{R})$ denote the skew-symmetric adjoint of e_3 and put

$$Q(t) := \exp(\mathcal{T}\mathrm{E}_3 t) = \begin{pmatrix} \cos(\mathcal{T}t) & -\sin(\mathcal{T}t) & 0 \\ \sin(\mathcal{T}t) & \cos(\mathcal{T}t) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For a sufficiently smooth solution (v,p) to (1.1), the transformation $u(x,t) := Q(t)v(Q(t)^T x)$, $\mathfrak{p}(x,t) := p(Q(t)^T x)$ yields a $\frac{2\pi}{\mathcal{T}}$-periodic (in time t) solution to the time-dependent Oseen problem in the whole space. In order to prove (1.2), we split this solution into a solution to a Cauchy problem with zero initial value, and a Cauchy problem with zero forcing term, respectively. We then prove (1.2) by a simple analysis of these two systems. The main idea behind our proof of (1.3)–(1.5) is to exploit the time-periodicity and expand $(u, \mathfrak{p})$ in a Fourier-series. We will then analyze the L^q-norm of v in terms the resulting Fourier coefficients. As we shall see below, these coefficients each solves (in space) a resolvent Oseen-equation. This information enables us to estimate their L^q-norms using standard multiplier theorems.

We split the proof into several lemmas. We start by establishing existence and higher order estimates in the case $q = 2$. This can be shown by an argument based on the Galerkin method (see for example [10]), but we choose here an approach using the ideas described above.

Lemma 2.1. *Let $f \in C_0^\infty(\mathbb{R}^3)^3$. There exists a solution*

$$\begin{aligned} &v \in D^{2,2}(\mathbb{R}^3)^3 \cap D^{1,2}(\mathbb{R}^3)^3 \cap L^6(\mathbb{R}^3)^3 \cap C^\infty(\mathbb{R}^3)^3, \\ &p \in D^{1,2}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3). \end{aligned} \tag{2.1}$$

to (1.1) *that satisfies*

$$\|\nabla^2 v\|_2 + \|\nabla p\|_2 \le C_5 \|f\|_2, \tag{2.2}$$

with C_5 independent on $\mathcal{R}$ and $\mathcal{T}$.

Proof. Since $f \in C_0^\infty(\mathbb{R}^3)^3$ there is a $h \in L^2(\mathbb{R}^3)^{3\times 3}$ with $\operatorname{div} h = f$ (see for example [11, Lemma 1.6.2]). For $k \in \mathbb{Z}$ put

$$\begin{aligned} F_k(x) &:= \frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} Q(t) f\big(Q(t)^T x\big)\, \mathrm{e}^{-i\mathcal{T}kt}\, \mathrm{d}t, \\ H_k(x) &:= \frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} Q(t) h\big(Q(t)^T x\big)\, \mathrm{e}^{-i\mathcal{T}kt}\, \mathrm{d}t. \end{aligned} \tag{2.3}$$

Note that $F_k \in C_0^\infty(\mathbb{R}^3)^3$ and $\operatorname{div} H_k = F_k$. Now define

$$\begin{aligned} u_k &:= \mathcal{F}^{-1}\Big[\frac{1}{i(\mathcal{T}k - \mathcal{R}\xi_3) + |\xi|^2}\Big(I - \frac{\xi \otimes \xi}{|\xi|^2}\Big)\widehat{F_k}\Big], \\ \mathfrak{p}_k &:= \mathcal{F}^{-1}\Big[\frac{\xi}{|\xi|^2} \cdot \widehat{F_k}\Big]. \end{aligned} \tag{2.4}$$

Both of these expressions are well-defined as the inverse Fourier transformation of a tempered distribution multiplied with the Schwartz function $\widehat{F_k}$. Clearly, by Plancherel's identity, $(u_k, \mathfrak{p}_k) \in D^{2,2}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$. In fact, we see that $u_k \in W^{2,2}(\mathbb{R}^3)$ for $k \neq 0$. Moreover, inserting $\widehat{H_k} \cdot \xi$ for $\widehat{F_k}$ in (2.4), we see, again by Plancherel's identity, that $\mathfrak{p}_k \in L^2(\mathbb{R}^3)$, and, by the mapping properties of the Riesz potential ([8, Theorem 6.1.3]) that $u_0 \in L^6(\mathbb{R}^3)$. By construction of $(u_k, \mathfrak{p}_k)$ we also have

$$\begin{cases} i\mathcal{T}k u_k - \Delta u_k + \nabla \mathfrak{p}_k - \mathcal{R}\partial_3 u_k = F_k & \text{in } \mathbb{R}^3, \\ \operatorname{div} u_k = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.5}$$

Now put for $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$:

$$
\begin{aligned}
&u(x,t) := \sum_{k\in\mathbb{Z}} u_k(x)\,\mathrm{e}^{i\mathcal{T}kt}, \quad \mathfrak{p}(x,t) := \sum_{k\in\mathbb{Z}} \mathfrak{p}_k(x)\,\mathrm{e}^{i\mathcal{T}kt},\\
&F(x,t) := \sum_{k\in\mathbb{Z}} F_k(x)\,\mathrm{e}^{i\mathcal{T}kt}.
\end{aligned} \tag{2.6}
$$

Note that since

$$
F_k(x) = \frac{1}{k^2}\frac{1}{\mathcal{T}2\pi} \int_0^{2\pi/\mathcal{T}} \partial_t^2\big[Q(t)f\big(Q(t)^T x\big)\big]\,\mathrm{e}^{-i\mathcal{T}kt}\,\mathrm{d}t,
$$

the series above converge even point-wise. Observe that

$$
\begin{cases}
\partial_t u - \Delta u + \nabla\mathfrak{p} - \mathcal{R}\partial_3 u = F & \text{in } \mathbb{R}^3\times\mathbb{R},\\
\operatorname{div} u = 0 & \text{in } \mathbb{R}^3\times\mathbb{R}.
\end{cases} \tag{2.7}
$$

Finally, we define $v(x,t) := Q(t)^T u\big(Q(t)x,t\big)$ and $p(x,t) := \mathfrak{p}\big(Q(t)x,t\big)$. As one easily verifies, v and p are time independent and solve (1.1). Moreover, repeatedly using Plancherel's identity, it follows that

$$
\begin{aligned}
\|\Delta v\|_2^2 + \|\nabla p\|_2^2 &= \frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}}\int_{\mathbb{R}^3} |\Delta v(x)|^2 + |\nabla p(x)|^2\,\mathrm{d}x\mathrm{d}t\\
&= \int_{\mathbb{R}^3}\frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} |\Delta u(x,t)|^2 + |\nabla\mathfrak{p}(x,t)|^2\,\mathrm{d}t\mathrm{d}x\\
&= \int_{\mathbb{R}^3}\sum_{k\in\mathbb{Z}} |\Delta u_k(x)|^2 + |\nabla\mathfrak{p}_k(x)|^2\,\mathrm{d}x\\
&\leq c_1\sum_{k\in\mathbb{Z}}\|F_k\|_2^2 = c_1\int_{\mathbb{R}^3}\frac{\mathcal{T}}{2\pi}\int_0^{2\pi/\mathcal{T}} \big|Q(t)f\big(Q(t)^T x\big)\big|^2\,\mathrm{d}t\mathrm{d}x\\
&= c_1\|f\|_2^2,
\end{aligned}
$$

where c_1 is independent on $\mathcal{R}$ and $\mathcal{T}$. Thus, (2.2) follows. Similarly, we find that $v \in D^{1,2}(\mathbb{R}^3)\cap L^6(\mathbb{R}^3)$ and $p \in D^{1,2}(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)$. By standard elliptic regularity theory, we also find, since (v,p) solves (1.1), that both v and p lie in $C^\infty(\mathbb{R}^3)$. This concludes the lemma. □

In the next lemma, we establish higher order L^q-estimates for the solution found above.

Lemma 2.2. *Let $1 < q < \infty$, and let $f \in C_0^\infty(\mathbb{R}^3)^3$. The solution (v,p) from Lemma 2.1 satisfies (1.2).*

Proof. Assume first that $q > 2$. Let $T > 0$. For $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ put

$$u(x,t) := Q(t)v\big(Q(t)^T x - \mathcal{R}t\,\mathrm{e}_3\big), \quad \mathfrak{p}(x,t) := p\big(Q(t)^T x - \mathcal{R}t\,\mathrm{e}_3\big),$$
$$F(x,t) := Q(t)f\big(Q(t)^T x - \mathcal{R}t\,\mathrm{e}_3\big).$$

Then

$$\begin{cases} \partial_t u - \Delta u + \nabla \mathfrak{p} = F & \text{in } \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ u(x,0) = v(x) & \text{in } \mathbb{R}^3. \end{cases} \tag{2.8}$$

We denote by $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ and $(\xi_0, \xi) \in \mathbb{R}^4$, and consider the operator

$$\Phi : \mathscr{S}(\mathbb{R}^4)^3 \to \mathscr{S}'(\mathbb{R}^4)^3, \quad \Phi(\psi) := \mathcal{F}^{-1}\left[\frac{1}{i\xi_0 + |\xi|^2 + \frac{1}{T}}\left(I - \frac{\xi \otimes \xi}{|\xi|^2}\right)\widehat{\psi}\right].$$

By the Hörmander-Mihlin multiplier theorem ([7, Theorem 5.2.7]), we deduce that Φ extends to a bounded operator $\Phi : L^r(\mathbb{R}^4) \to L^r(\mathbb{R}^4)$ for all $1 < r < \infty$. We then put

$$u_1(x,t) := \mathrm{e}^{t/T}\,\Phi\big(F(x,t)\chi_{[0,T]}(t)\,\mathrm{e}^{-t/T}\big),$$

where $\chi_{[0,T]}$ denotes the indicator function of the interval $[0,T]$. Note that $F(x,t)\chi_{[0,T]}(t)\,\mathrm{e}^{-t/T} \in L^r(\mathbb{R}^4)$ and thus $u_1 \in L^r(\mathbb{R}^3 \times (0,T))$ for all $1 < r < \infty$. As one may verify,

$$\begin{cases} \partial_t u_1 - \Delta u_1 + \nabla \mathfrak{p} = F & \text{in } \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u_1 = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ \lim\limits_{t \to 0^+} \|u_1(\cdot,t)\|_r = 0. \end{cases} \tag{2.9}$$

for all $1 < r < \infty$. In particular, $(2.9)_3$ follows by a standard argument (see for example [9, Sec. 5, Theorem 6]). Furthermore, again by the Hörmander-Mihlin multiplier theorem, we obtain

$$\|\nabla^2 u_1\|_{L^r(\mathbb{R}^3 \times (0,T))} \le c_1 \|F\|_{L^r(\mathbb{R}^3 \times (0,T))}, \tag{2.10}$$

with c_1 independent on T. Next, put

$$u_2(x,t) := (4\pi t)^{-3/2} \int_{\mathbb{R}^3} \mathrm{e}^{-|x-y|^2/4t}\, v(y)\,\mathrm{d}y. \tag{2.11}$$

An elementary calculation shows that $u_2 \in L^6(\mathbb{R}^3 \times (0,T))$, $\partial_t u_2, \nabla u_2, \nabla^2 u_2 \in L^6_{loc}(\mathbb{R}^3 \times (0,T))$, and that u_2 solves

$$\begin{cases} \partial_t u_2 - \Delta u_2 = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ \operatorname{div} u_2 = 0 & \text{in } \mathbb{R}^3 \times (0,T), \\ \lim\limits_{t \to 0^+} \|u_2(\cdot,t) - v(\cdot)\|_6 = 0. \end{cases} \tag{2.12}$$

Taking second order derivatives on both sides in (2.11) and applying Young's inequality, we obtain

$$\|\nabla^2 u_2(\cdot,t)\|_{L^q(\mathbb{R}^3)} \leq c_2\, t^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{q})}\, \|\nabla^2 v\|_2, \tag{2.13}$$

with c_2 independent on T. Next, we claim that $u = u_1 + u_2$ in $\mathbb{R}^3 \times (0,T)$. This follows from the fact that $u_1 + u_2$ satisfies (2.8) combined with a uniqueness argument (see [5, Lemma 3.6]). We can now calculate

$$\begin{aligned} T\|\Delta v\|_q^q &= \int_0^T \int_{\mathbb{R}^3} |\Delta u(x,t)|^q \,\mathrm{d}x\mathrm{d}t \\ &\leq c_3\Big(\|\Delta u_1\|^q_{L^q(\mathbb{R}^3\times(0,T))} + \int_0^T \|\Delta u_2(\cdot,t)\|_q^q \,\mathrm{d}t\Big) \\ &\leq c_4\Big(\|F\|^q_{L^q(\mathbb{R}^3\times(0,T))} + \int_0^T t^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{q})} \|\nabla^2 v\|_2 \,\mathrm{d}t\Big) \\ &= c_4\big(T\|f\|_q^q + T^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{q})+1} \|\nabla^2 v\|_2\big), \end{aligned}$$

with c_4 independent on T, and of course also on $\mathcal{R}$ and $\mathcal{T}$. Dividing both sides with T and subsequently letting $T \to \infty$ (note that $q > 2$ by assumption), we conclude that $\|\Delta v\|_q \leq c_4 \|f\|_q$. Finally, we deduce directly from (1.1), by taking div on both sides in $(1.1)_1$, that $-\Delta p = \operatorname{div} f$. From this it follows that also $\|\nabla p\|_q \leq c_5 \|f\|_q$, with c_5 independent on $\mathcal{R}$ and $\mathcal{T}$. Hence (1.2) follows in the case $q > 2$.

The case $q = 2$ was shown in Lemma 2.1. Consider now $1 < q < 2$. In this case will establish (1.2) by a duality argument. Consider for this purpose $\varphi \in C_0^\infty(\mathbb{R}^3)$. Just as in Lemma 2.1, one can show the existence of a solution (ψ, η) in the class (2.1) to the adjoint problem

$$\begin{cases} -\Delta\psi - \nabla\eta + \mathcal{R}\partial_3\psi + \mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla\psi - \mathrm{e}_3 \wedge \psi\big) = \varphi & \text{in } \mathbb{R}^3, \\ \operatorname{div}\psi = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{2.14}$$

By arguments as above, we can also show for $2 < r < \infty$ that

$$\|\nabla^2 \psi\|_r + \|\nabla \eta\|_r \leq c_6 \|\varphi\|_r, \tag{2.15}$$

with c_6 independent on $\mathcal{R}$ and $\mathcal{T}$. We now exploit that

$$\int_{\mathbb{R}^3} \Delta v \cdot \varphi \,\mathrm{d}x = \int_{\mathbb{R}^3} \Delta v \cdot \big[-\Delta\psi - \nabla\eta + \mathcal{R}\partial_3\psi + \mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla\psi - \mathrm{e}_3 \wedge \psi\big)\big] \,\mathrm{d}x.$$

As one may verify, the summability properties of (v,p) and (ψ,η), ensured by the fact that both pairs lie in the class (2.1), suffice for us to integrate partially in the integral on the right-hand side above. Consequently,

$$\int_{\mathbb{R}^3} \Delta v \cdot \varphi \,\mathrm{d}x = \int_{\mathbb{R}^3} \big[-\Delta\Delta v - \mathcal{R}\partial_3 \Delta v + \mathcal{T}\big(\mathrm{e}_3 \wedge x \cdot \nabla \Delta v - \mathrm{e}_3 \wedge \Delta v\big)\big] \cdot \psi \,\mathrm{d}x.$$

Note that $\Delta\big[\mathrm{e}_3 \wedge x \cdot \nabla v\big]_i = \big[\mathrm{e}_3 \wedge x \cdot \nabla \Delta v\big]_i + 2\nabla^2 v_i : \mathrm{E}_3 = \big[\mathrm{e}_3 \wedge x \cdot \nabla \Delta v\big]_i$. We thus deduce from the above that

$$\int_{\mathbb{R}^3} \Delta v \cdot \varphi \,\mathrm{d}x = \int_{\mathbb{R}^3} \Delta f \cdot \psi \,\mathrm{d}x = \int_{\mathbb{R}^3} f \cdot \Delta \psi \,\mathrm{d}x.$$

Using (2.15), we then obtain

$$\Big|\int_{\mathbb{R}^3} \Delta v \cdot \varphi \,\mathrm{d}x\Big| \leq \|f\|_q \|\Delta\psi\|_{q'} \leq \|f\|_q \|\varphi\|_{q'},$$

where q' denotes the Hölder conjugate of q. It follows that $\|\Delta v\|_q \leq c_6 \|f\|_q$, and thus $\|\nabla^2 v\|_q \leq c_7 \|f\|_q$, with c_7 independent on $\mathcal{R}$ and $\mathcal{T}$. Again, the estimate $\|\nabla p\|_q \leq c_8 \|f\|_q$ follows simply from the fact that $-\Delta p = \operatorname{div} f$. This concludes the lemma. □

Having dealt with the higher order terms, we now establish estimates for the other terms on the left-hand side of (1.1).

Lemma 2.3. *Let* $1 < q < \infty$*, and let* $f \in C_0^\infty(\mathbb{R}^3)^3$*. The solution* (v, p) *from Lemma 2.1 satisfies* (1.3)*.*

Proof. Consider first $1 < q \leq 2$. We let $(u, \mathfrak{p}, F)$ and $(u_k, \mathfrak{p}_k, F_k)$ be as in the proof of Lemma 2.1, that is, as in (2.3), (2.4), and (2.6). Since $(u_k, \mathfrak{p}_k)$ satisfies (2.5), we have

$$u_k = \mathcal{F}^{-1}\Big[\frac{1}{i\mathcal{T}k + |\xi|^2}\Big(I - \frac{\xi \otimes \xi}{|\xi|^2}\Big)\mathcal{F}\big[F_k + \mathcal{R}\partial_3 u_k\big]\Big]$$

The multiplier $m_k(\xi) := \frac{1}{i\mathcal{T}k + |\xi|^2}$ satisfies $|\xi|^{|\alpha|} |\partial^\alpha m_k(\xi)| \leq \frac{c_1}{|\mathcal{T}k|}$ for all multi-indices $\alpha \in \mathbb{N}_0^3$. Thus, by the Hörmander-Mihlin multiplier theorem ([7, Theorem 5.2.7]), we obtain for all $k \neq 0$:

$$\|u_k\|_q \leq \frac{c_2}{|\mathcal{T}k|}\big(\|F_k\|_q + \|\mathcal{R}\partial_3 u_k\|_q\big),$$

with c_2 independent on $\mathcal{R}$ and $\mathcal{T}$. By interpolation, it follows that ($k \neq 0$)

$$\|u_k\|_q \leq \frac{c_3}{|\mathcal{T}k|}\big(\|F_k\|_q + \mathcal{R}\varepsilon\|u_k\|_q + \frac{\mathcal{R}}{\varepsilon}\|\nabla^2 u_k\|_q\big) \tag{2.16}$$

for all $\varepsilon > 0$. Recall that $u(x,t) = Q(t)v(Q(t)^T x)$ and, by definition of u as the Fourier series with respect to Fourier coefficients u_k,

$$u_k(x) = \frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} u(x,t)\, \mathrm{e}^{-i\mathcal{T}kt}\, \mathrm{d}t.$$

Consequently, using Lemma 2.2, it follows that $\|\nabla^2 u_k\|_q \leq c_4 \|f\|_q$, with c_4 independent on $\mathcal{R}$ and $\mathcal{T}$. Clearly, $\|F_k\|_q \leq \|f\|_q$. Thus, choosing $\varepsilon = \frac{|\mathcal{T}k|}{2\mathcal{R}c_3}$ in (2.16), we conclude that $(k \neq 0)$

$$\|u_k\|_q \leq \frac{c_5}{|\mathcal{T}k|}\left(1 + \frac{\mathcal{R}^2}{|\mathcal{T}k|}\right)\|f\|_q,$$

with c_5 independent on $\mathcal{R}$ and $\mathcal{T}$. We can now estimate

$$\begin{aligned}\left(\sum_{k\in\mathbb{Z}\setminus\{0\}} \|u_k\|_q^q\right)^{1/q} &\leq c_6 \left(\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{|\mathcal{T}k|^q}\left(1 + \frac{\mathcal{R}^2}{|\mathcal{T}k|}\right)^q \|f\|_q^q\right)^{1/q} \\ &\leq \frac{c_7}{|\mathcal{T}|}\left(1 + \frac{\mathcal{R}^2}{|\mathcal{T}|}\right)\|f\|_q \end{aligned} \tag{2.17}$$

where c_7 is independent on $\mathcal{R}$ and $\mathcal{T}$. We now put $U(x,t) := u(x,t) - u_0(x)$. Recall that $1 < q \leq 2$. Let $q' = \frac{q}{q-1}$ denote the corresponding Hölder conjugate. Using the Hausdorff-Young inequality for Fourier-series (see for example [1, Proposition 4.2.7]), we obtain the estimate

$$\left(\frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} |U(x,t)|^{q'}\, \mathrm{d}t\right)^{\frac{1}{q'}} \leq \left(\sum_{k\in\mathbb{Z}\setminus\{0\}} |u_k(x)|^q\right)^{\frac{1}{q}},$$

which we write as

$$\left(\frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} \left(|U(x,t)|^q\right)^{\frac{1}{q-1}} \mathrm{d}t\right)^{q-1} \leq \sum_{k\in\mathbb{Z}\setminus\{0\}} |u_k(x)|^q.$$

Integrating now both sides above over $\mathbb{R}^3$ and subsequently using the Minkowski inequality (recall that $1 < q \leq 2$), we deduce that

$$\left(\frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} \left(\int_{\mathbb{R}^3} |U(x,t)|^q\, dx\right)^{\frac{1}{q-1}} \mathrm{d}t\right)^{q-1} \leq \sum_{k\in\mathbb{Z}\setminus\{0\}} \|u_k\|_q^q. \tag{2.18}$$

Define

$$v_1(x) := v(x) - u_0. \tag{2.19}$$

Recall the definition of u_0 and observe that $Q(t)^T u_0(Q(t)x,t) = u_0(x)$ for all $t \in \mathbb{R}$. It follows that $v_1(x) = Q(t)^T U(Q(t)x,t)$. We now see that the inner integral on the right-hand side in (2.18) evaluates to $\|v_1\|_q^q$. Combined with (2.17), we thus have

$$\|v_1\|_q \leq \frac{c_7}{|\mathcal{T}|}\left(1+\frac{\mathcal{R}^2}{|\mathcal{T}|}\right)\|f\|_q. \tag{2.20}$$

Furthermore, since also $Q(t)^T F_0(Q(t)x,t) = F_0(x)$, it follows that v_1 satisfies (1.1) with $f - F_0$ as the right-hand side. Consequently, Lemma (2.2) yields

$$\|\nabla^2 v_1\|_q \leq C_1\|f-F_0\|_q \leq c_8\|f\|_q, \tag{2.21}$$

where c_8 is independent on $\mathcal{R}$ and $\mathcal{T}$. Next, we observe that u_0 is a solution to the classical whole space Oseen problem. From standard theory (see for example [3, Theorem VII.4.1]), we have

$$\|\nabla^2 u_0\|_q + \mathcal{R}\|\partial_3 u_0\|_q \leq c_9\|F_0\|_q \leq c_9\|f\|_q, \tag{2.22}$$

where c_9 is independent on $\mathcal{R}$ and $\mathcal{T}$. Combining (2.20), (2.21), and (2.22), we can finally estimate

$$\begin{aligned}
\|\mathcal{R}\partial_3 v\|_q &\leq \mathcal{R}\|\partial_3 u_0\|_q + \mathcal{R}\|\partial_3 v_1\|_q \\
&\leq c_{10}\|f\|_q + \mathcal{R}(\|v_1\|_q + \|\nabla^2 v_1\|_q) \\
&\leq c_{11}\left(1+\frac{\mathcal{R}}{\mathcal{T}}+\frac{\mathcal{R}^3}{\mathcal{T}^2}+\mathcal{R}\right)\|f\|_q \\
&\leq c_{12}\left(1+\frac{1}{\mathcal{T}^2}\right)\|f\|_q,
\end{aligned}$$

where $c_{12} = c_{12}(\mathcal{R}_0)$, but is otherwise independent on $\mathcal{R}$ and $\mathcal{T}$. This concludes the proof in the case $1 < q \leq 2$. The case $2 < q < \infty$ follows by a duality argument similar to that in the proof of Lemma 2.2. □

Using a simple interpolation argument, we will now show estimates for the lower order terms of the solution.

Lemma 2.4. *Let $1 < q < \infty$, and let $f \in C_0^\infty(\mathbb{R}^3)^3$. The solution (v,p) from Lemma 2.1 satisfies* (1.4)*-*(1.5)*.*

Proof. The proof will follow from the decomposition (2.19) of v into a part u_0, which satisfies the classical Oseen problem and thus enjoys corresponding L^q-estimates, and a part v_1, which satisfies (2.20) and (2.21). Note that (2.20) and (2.21) were established in Lemma 2.3 under the assumption that $1 < q \leq 2$. It is, however, immediately clear from the argument in Lemma 2.3 that (2.21) holds for all $1 < q < \infty$. Moreover, by a duality argument similar to that in the proof of Lemma 2.2, one readily shows that (2.20) also holds for all $1 < q < \infty$.

Consider now $1 < q < 4$. By well known theory (see for example [3, Theorem VII.4.1]),

$$\|\nabla u_0\|_{\frac{4q}{4-q}} \leq c_1 \mathcal{R}^{1/4} \|F_0\|_q \leq c_1 \mathcal{R}^{1/4} \|f\|_q. \tag{2.23}$$

By Sobolev embedding (see for example [3, Lemma II.2.2]), (2.20), and (2.21) it follows that

$$\begin{aligned} \|\nabla v_1\|_{\frac{4q}{4-q}} &\leq c_2 \|\nabla v_1\|_q^{\frac{1}{4}} \|\nabla^2 v_1\|_q^{\frac{3}{4}} \\ &\leq c_3 \left[1 + \frac{1}{|\mathcal{T}|}\left(1 + \frac{\mathcal{R}^2}{|\mathcal{T}|}\right)\right]^{\frac{1}{4}} \|f\|_q \leq c_4 \left(1 + \mathcal{T}^{-\frac{1}{2}}\right) \|f\|_q, \end{aligned} \tag{2.24}$$

with $c_4 = c_4(\mathcal{R}_0)$. Combining (2.19), (2.23), and (2.24) gives us (1.4).

Consider next $1 < q < 2$. It is well known that ([3, Theorem VII.4.1])

$$\|u_0\|_{\frac{2q}{2-q}} \leq c_5 \mathcal{R}^{1/2} \|F_0\|_q \leq c_5 \mathcal{R}^{1/2} \|f\|_q. \tag{2.25}$$

Again by Sobolev embedding we find that

$$\begin{aligned} \|v_1\|_{\frac{2q}{2-q}} &\leq c_6 \|v_1\|_{\frac{3q}{3-q}}^{\frac{1}{2}} \|\nabla v_1\|_{\frac{3q}{3-q}}^{\frac{1}{2}} \leq c_7 (\|v_1\|_q + \|\nabla v_1\|_q)^{\frac{1}{2}} \|\nabla^2 v_1\|_q^{\frac{1}{2}} \\ &\leq c_8 \left[1 + \frac{1}{|\mathcal{T}|}\left(1 + \frac{\mathcal{R}^2}{|\mathcal{T}|}\right)\right]^{\frac{1}{2}} \|f\|_q \leq c_9 \left(1 + \mathcal{T}^{-1}\right) \|f\|_q, \end{aligned} \tag{2.26}$$

with $c_9 = c_9(\mathcal{R}_0)$. Combining (2.19), (2.25), and (2.26) yields (1.5) □

We can now finalize the proof of Theorem 1.1.

Proof of Theorem 1.1. Lemma 2.1–2.4 establish the theorem, except for the uniqueness statement, in the case $f \in C_0^\infty(\mathbb{R}^3)$. We shall now extend the statements to the general case $f \in L^q(\mathbb{R}^3)$. Therefore, let $f \in L^q(\mathbb{R}^3)$ and choose $\{f_n\}_{n=1}^\infty \subset C_0^\infty(\mathbb{R}^3)$ with $\lim_{n\to\infty} f_n = f$ in $L^q(\mathbb{R}^3)$. Let (v_n, p_n) be the solution from Lemma 2.1 to (1.1) with f_n as the right-hand side. Then choose $\alpha_n, \beta_n, \kappa_n, \iota_n \in \mathbb{R}^3$ such that

$$0 = \int_{\mathrm{B}_1} \partial_1 v_n - \alpha_n \,\mathrm{d}x = \int_{\mathrm{B}_1} \partial_2 v_n - \beta_n \,\mathrm{d}x, \tag{2.27}$$

$$0 = \int_{\mathrm{B}_1} v_n - (\kappa_n + \alpha_n x_1 + \beta_n x_2) \,\mathrm{d}x, \tag{2.28}$$

and $0 = \int_{\mathrm{B}_1} p_n - \iota_n \,\mathrm{d}x$. Put $r_n := \kappa_n + \alpha_n x_1 + \beta_n x_2$. From Lemma 2.2–2.3 we see, using Poincaré's inequality, that $\{(v_n - r_n, p_n - \iota_n)\}_{n=1}^\infty$ is a Cauchy sequence in the Banach-space

$$\begin{aligned} &X_m := \{(v,p) \in L^1_{loc}(\mathbb{R}^3)^3 \times L^1_{loc}(\mathbb{R}^3) \mid \|(v,p)\|_{X_m} < \infty\}, \\ &\|(v,p)\|_{X_m} := \|\nabla^2 v\|_q + \|\nabla p\|_q + \mathcal{R}\|\partial_3 v\|_q + \|v\|_{L^q(\mathrm{B}_m)} + \|p\|_{L^q(\mathrm{B}_m)} \end{aligned}$$

all $m \in \mathbb{N}$. Here, $\mathrm{B}_m := \{x \in \mathbb{R}^3 \mid |x| < m\}$. Consequently, there is an element $(v,p) \in \cap_{m\in\mathbb{N}} X_m$ with the property that $\lim_{n\to\infty}(v_n - r_n, p_n - \iota_n) = (v,p)$ in X_m for all $m \in \mathbb{N}$. Put

$$L(V,P) := \big(- \Delta V + \nabla P - \mathcal{R}\partial_3 V + \mathcal{T}(\mathrm{e}_3 \wedge x \cdot \nabla V - \mathrm{e}_3 \wedge V), \operatorname{div} V\big).$$

It follows that $\lim_{n\to\infty} L(v_n - r_n, p_n - \iota_n) = L(v,p)$ in $\mathcal{D}'(\mathbb{R}^3)$. By construction, we have $\lim_{n\to\infty} L(v_n, p_n) = \lim_{n\to\infty}(f_n, 0) = (f,0)$ in $L^q(\mathbb{R}^3)$. We thus deduce that $\lim_{n\to\infty} L(r_n, \iota_n) = (f,0) - L(v,p)$ in $\mathcal{D}'(\mathbb{R}^3)$. Consequently, $(f,0) - L(v,p)$ must be equal to $L(r,\iota)$ for some first order polynomial r and constant ι. It follows that $(v+r, p+\iota) \in D^{2,q}(\mathbb{R}^3) \times D^{1,q}(\mathbb{R}^3)$ solves (1.1). Moreover, since (v_n, p_n) satisfies (1.2)–(1.3), so does $(v+r, p+r)$. This proves the first part of the theorem.

If $1 < q < 4$, we repeat the argument above with the modification that we ignore (2.27) (put $\alpha_n = \beta_n = 0$) and add the term $\|\nabla v\|_{\frac{4q}{4-q}}$ to the X_m-norm. We then obtain a solution to (1.1) that also also satisfies (1.4). If $1 < q < 2$, we ignore both (2.27) and (2.28) (put $\alpha_n = \beta_n = \kappa_n = 0$), and add the term $\|v\|_{\frac{2q}{2-q}}$ to the X_m-norm. We then obtain a solution to (1.1) satisfying (1.5).

Finally, we prove uniqueness. Assume that $(\tilde{v}, \tilde{p}) \in D^{2,r}(\mathbb{R}^3)^3 \times D^{1,r}(\mathbb{R}^3)$ is another solution to (1.1). Put $w := v - \tilde{v}$ and $\mathfrak{q} := p - \tilde{p}$. It immediately follows that $\Delta \mathfrak{q} = 0$, which, since $\mathfrak{q} \in D^{1,q}(\mathbb{R}^3) + D^{1,r}(\mathbb{R}^3)$, implies that $\mathfrak{q}$ is a constant. Now put $U(x,t) := Q(t) w(Q(t)^T x)$ for $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$. Since U is smooth and $2\pi/\mathcal{T}$-periodic in t, we can write U in terms of its Fourier-series

$$U(x,t) = \sum_{k\in\mathbb{Z}} U_k(x)\, \mathrm{e}^{i\mathcal{T}kt}, \; U_k(x) := \frac{\mathcal{T}}{2\pi} \int_0^{2\pi/\mathcal{T}} U(x,t)\, \mathrm{e}^{-i\mathcal{T}kt} \, \mathrm{d}t.$$

As one may easily verify, U_k satisfies $i\mathcal{T}kU_k - \Delta U_k - \mathcal{R}\partial_3 U_k = 0$. Thus, taking the Fourier transform yields $\big(i(\mathcal{T}k - \mathcal{R}\xi_3) + |\xi|^2\big)\widehat{U_k} = 0$. It follows that $U_k = 0$ for all $k \neq 0$. Moreover, since $\big(- i\mathcal{R}\xi_3 + |\xi|^2\big)\widehat{U_0} = 0$, it follows that $\operatorname{supp}(\widehat{U_0}) \subset \{0\}$. Consequently, since $U_0 \in D^{2,q}(\mathbb{R}^3) + D^{2,r}(\mathbb{R}^3)$, $U_0 = Ax + b$ for some $A \in \mathbb{R}^{3\times 3}$ and $b \in \mathbb{R}^3$. Note that $U(x,t) = U_0(x) = Q(t) w(Q(t)^T x)$ for all $t \in \mathbb{R}$. Thus, $Q(t)^T \big(AQ(t)x + b\big)$ is t-independent. Combining this property with the fact that $\operatorname{div}(Ax) = 0$ and $\partial_3(Ax) = 0$, we find that $A_{3i} = A_{i3} = 0$ $(i = 1,2,3)$, $A_{12} = -A_{21}$, $A_{ii} = 0$ $(i = 1,2,3)$, and $b \wedge \mathrm{e}_3 = 0$. We thus see that $w = U_0 = \beta\, \mathrm{e}_3 \wedge x + \alpha\, \mathrm{e}_3$ and $\mathfrak{q} = \gamma$ for some $\alpha, \beta, \gamma \in \mathbb{R}$. □

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