ON THE STATIONARY NAVIER-STOKES FLOWS AROUND A ROTATING BODY

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ABSTRACT. Consider the stationary motion of an incompressible Navier-Stokes fluid around a rotating body $\mathcal{K} = \mathbb{R}^3 \backslash \Omega$ which is also moving in the direction of the axis of rotation. We assume that the translational and angular velocities U, ω are constant and the external force is given by $f = \operatorname{div} F$. Then the motion is described by a variant of the stationary Navier-Stokes equations on the exterior domain Ω for the unknown velocity u and pressure p, with U, ω, F being the data. We first prove the existence of at least one solution (u, p) satisfying $\nabla u, p \in L_{3/2,\infty}(\Omega)$ and $u \in L_{3,\infty}(\Omega)$ under the smallness condition on $|U| + |\omega| + \|F\|_{L_{3/2,\infty}(\Omega)}$. Then the uniqueness is shown for solutions (u, p) satisfying $\nabla u, p \in L_{3/2,\infty}(\Omega) \cap L_{q,r}(\Omega)$ and $u \in L_{3,\infty}(\Omega) \cap L_{q^*,r}(\Omega)$ provided that $3/2 < q < 3$ and $F \in L_{3/2,\infty}(\Omega) \cap L_{q,r}(\Omega)$. Here $L_{q,r}(\Omega)$ denotes the well-known Lorentz space and $q^* = 3q/(3 - q)$ is the Sobolev exponent to q.

INTRODUCTION

Let Ω be an exterior domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Consider the motion of an incompressible Navier-Stokes fluid around the rigid body $\mathcal{K} = \mathbb{R}^3 \setminus \Omega$ which is rotating about an axis with constant angular velocity $\omega = ce_3 = (0, 0, c)^T$. We also assume that the body $\mathcal{K}$ is moving in the direction of the axis of rotation with constant velocity $U = ke_3$. Then with respect to a coordinate system attached to the body, the velocity $u = (u_1, u_2, u_3)^T$ and pressure p of the fluid is governed by the following initial boundary value problem for a variant of the Navier-Stokes equations in Ω (see [18, 12, 7] for a detailed derivation):

$$
(0.1) \qquad \begin{cases} u_t + \operatorname{div}(u \otimes u) + Lu + \nabla p = \operatorname{div} F & \text{in } \Omega \times (0, \infty), \\ \operatorname{div} u = 0 & \text{in } \Omega \times (0, \infty), \\ u = \omega \wedge x - U & \text{on } \partial\Omega \times (0, \infty), \\ u(x, t) \to 0 & \text{as } |x| \to \infty, \ t > 0, \\ u(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases}
$$

where L is the linear differential operator defined by

$$
Lu = -\Delta u + (U - \omega \wedge x) \cdot \nabla u + \omega \wedge u.
$$

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Here u_0 and $f = \operatorname{div} F$ denote the given initial velocity and the external force, respectively.

The nonstationary problem (0.1) has been studied from the mathematical point of view by Hishida [18], Galdi [12], Galdi and Silvestre [14], Geissert, Heck and Hieber [17] and Hishida and Shibata [20], since the global existence of weak solutions was established by Borchers [3] in 1992. Of particular interest are the global existence and stability results in [14] and [20] for the problem (0.1) with $U = 0$, which corresponds to the fluid motion around a purely rotating rigid body. In particular, Hishida and Shibata [20] showed that if u_0 is sufficiently close in $L_{3,\infty}(\Omega)$ to a small stationary solution u_S of (0.1) with $U = 0$, then there exists a unique global solution u which tends to u_S as $t \to \infty$. This is a highly nontrivial extension of the previous stability result by Kozono and Yamazaki [25] for the classical Navier-Stokes problem, i.e, the problem (0.1) with $U = \omega = 0$. It is also very important and challenging to extend these stability results to the general case of possibly nonzero U and/or ω, which requires a detailed study of stationary solutions of (0.1).

In this paper, we shall study the steady motion[1] of the fluid around $\mathcal{K}$, which is described by stationary solutions of the problem (0.1); thus assuming that F is time-independent, we consider the stationary problem in the exterior domain Ω:

$$\begin{cases} Lu + \operatorname{div}(u \otimes u) + \nabla p = \operatorname{div} F & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = \omega \wedge x - U & \text{on } \partial\Omega, \\ u(x) \to 0 \quad \text{as } |x| \to \infty. \end{cases} \tag{0.2}$$

The existence of a weak solution u of (0.2) satisfying finite Dirichlet integral (i.e., $\nabla u \in L_2(\Omega)$) and the energy inequality can be shown for arbitrarily large data by applying the classical Galerkin method as in [3, 12, 30]. The solution u has a complete local regularity property: it becomes smooth up to the boundary $\partial\Omega$ if F is smooth enough. However there have been very few results on the asymptotic behavior of $u(x)$ as $|x| \to \infty$ even for the classical Navier-Stokes equations. This makes it extremely difficult to establish the uniqueness and stability of weak solutions of (0.2) even though the data are assumed to be suitably small. Two approaches have been developed to investigate such basic mathematical questions for the classical Navier-Stokes equations in exterior domains.

The first approach due to Galdi and his collaborators relies on pointwise estimates obtained by a detailed analysis of the volume potentials associated with the Stokes and Oseen equations. In fact, Galdi and Simader [16] (see also [10, 11]) showed that if $\omega = 0$ and $\|(1 + |x|^2)F\|_{L_\infty(\Omega)}$ is sufficiently small, then there exists a unique weak solution u of (0.2) with finite Dirichlet integral and moreover the solution u satisfies the decay estimate $|u(x)| = O(|x|^{-1})$ at infinity, which is the same as those of the fundamental solutions. An extension was obtained by Galdi [13] to the

[1] It should be noted that the steady motion with respect to a coordinate system attached to the rigid body corresponds to a time-periodic motion with respect to the original coordinate system. See [7, 8] for more details.

rotating problem (0.1) with $U = 0$. Finally, Galdi and Silvestre [15] showed that if $|U| + |\omega| + \|\operatorname{div} F\|_{L_2(\Omega)} + \|(1+|x|^2)\, F\|_{L_\infty(\Omega)}$ is sufficiently small, then there exists a unique strong solution (u,p) of (0.2) satisfying $\nabla^2 u, \nabla p \in L_2(\Omega)$, $(1+|x|)\, u \in L_\infty(\Omega)$ and $p \in L_s(\Omega)$ $(s > 3/2)$.

The second approach is of functional analytic nature and utilizes the theory of weak L_q-spaces or more generally Lorentz spaces. The Lorentz spaces $L_{q,r}(\Omega)$ have been introduced by Borchers and Miyakawa [4] and by Kozono and Yamazaki [24, 26, 25] in order to establish the existence, uniqueness and stability of stationary solutions of the classical Navier-Stokes equations. In particular, Kozono and Yamazaki showed in [24] that if $U = \omega = 0$ and $\|F\|_{L_{3/2,\infty}(\Omega)}$ is small, then the problem (0.2) has at least one solution (u,p) satisfying $\nabla u, p \in L_{3/2,\infty}(\Omega)$ and $u \in L_{3,\infty}(\Omega)$. This result was extended by Farwig and Hishida [8] to the case of non-zero angular velocity ω. However, it remains still open to prove the uniqueness of solutions (u,p) of (0.2) satisfying $\nabla u, p \in L_{3/2,\infty}(\Omega)$ and $u \in L_{3,\infty}(\Omega)$; see e.g. [22] for a relevant discussion. Instead, as observed recently by Kim and Kozono [22], a bootstrap argument enables us to deduce that if $U = \omega = 0$, $3/2 < q < 3$, $F \in L_{3/2,\infty}(\Omega) \cap L_{q,r}(\Omega)$ and $\|F\|_{L_{3/2,\infty}(\Omega)}$ is small, then the problem (0.2) has a unique solution (u,p) satisfying $\nabla u, p \in L_{3/2,\infty}(\Omega) \cap L_{q,r}(\Omega)$ and $u \in L_{3,\infty}(\Omega) \cap L_{q^*,r}(\Omega)$, where $q^* = 3q/(3-q)$ is the Sobolev exponent to q. Moreover, if $q = r = 2$, then the solution (u,p) satisfies the energy equality and coincides with any weak solution satisfying the energy inequality whose existence was established long ago in Leray's celebrated paper [28].

The purpose of this paper is to establish the existence and uniqueness of solutions of the problem (0.2) in the framework of Lorentz spaces, which extends the results in [24, 8, 22] to the more general fluid model of possibly nonzero U and/or ω. In fact, we shall show (see Theorems 1.1 and 1.2 in the next section) that

- (*Existence*) if $|U| + |\omega| + \|F\|_{L_{3/2,\infty}(\Omega)}$ is small, then the problem (0.2) has at least one solution (u,p) satisfying $\nabla u, p \in L_{3/2,\infty}(\Omega)$ and $u \in L_{3,\infty}(\Omega)$; and

- (*Unique solvability*) if $F \in L_{3/2,\infty}(\Omega) \cap L_{q,r}(\Omega)$ for some $3/2 < q < 3$ in addition, then the problem (0.2) has a unique solution (u,p) satisfying $\nabla u, p \in L_{3/2,\infty}(\Omega) \cap L_{q,r}(\Omega)$ and $u \in L_{3,\infty}(\Omega) \cap L_{q^*,r}(\Omega)$.

Our theorems are extensions of the previous existence and uniqueness results given in [24, 8, 22] to the more general problem (0.2). Moreover, the unique solvability of (0.2) is shown for more general external forces $f = \operatorname{div} F$ than in [15], but with lack of such a pointwise estimate as $|u(x)| = O\left(|x|^{-1}\right)$ at infinity. On the other hand, it was shown in [22] that weak solutions of (0.2) satisfying the energy inequality are unique if $U = \omega = 0$, $F \in L_{3/2,\infty}(\Omega) \cap L_2(\Omega)$ and $\|F\|_{L_{3/2,\infty}(\Omega)}$ is small. The proof is based on a uniqueness criterion due to Kozono and Yamazaki [26] which we have a serious difficulty in extending to the rotating case $\omega \neq 0$. This difficulty, which is caused by the presence of the unbounded term $(\omega \wedge x) \cdot \nabla u$ in Lu, has not been resolved yet. Hence the uniqueness of weak solutions satisfying the energy inequality remains still open for the problem (0.2) with nonzero angular velocity ω.

Our results on the nonlinear problem (0.2) are deduced from the corresponding results on its linearized problem by means of a standard fixed point theorem together with a bootstrap argument. The linearized problem in Lorentz spaces has been studied in great detail by Kozono and Yamazaki [24] and Farwig and Hishida [8] for the special cases of $U = \omega = 0$ and $U = 0$, respectively. We will obtain a complete $L_{q,r}$-result (see Theorem 1.3 below) for the linearized problem of (0.2), by following the argument of Farwig and Hishida [8] with the help of the L_q-estimate due to Kračmar, Nečasová and Penel [27].

The outline of this paper is as follows. In Section 1, we shall state all of our main results with some basic definitions introduced. Section 2 is devoted to proving the unique solvability result in $L_{q,r}(\Omega)$ for the linearized problems in the whole space and bounded domains. In Section 3, we then obtain a complete $L_{q,r}$-result for the linearized problem in exterior domains. Finally we complete the proofs of the main results for the nonlinear problem (0.2) in Section 4.

1. Results

To begin with, we rewrite the problem (0.2) as an equivalent problem with homogeneous boundary conditions by using a simple change of variables. Let $\eta \in C_0^\infty(\mathbb{R}^3)$ be a fixed cut-off function with $\eta = 1$ near $\partial\Omega$, and let us define

$$b(x) = \frac{1}{2}\,\mathrm{rot}\left[\eta(x)\left(U \wedge x - |x|^2\omega\right)\right].$$

It is easy to show that

$$\left\{\begin{array}{c} b \in C_0^\infty(\mathbb{R}^3), \quad \mathrm{div}\, b = 0 \quad \text{in } \mathbb{R}^3, \quad b|_{\partial\Omega} = \omega \wedge x - U, \\ \|b\|_{L^\infty(\mathbb{R}^3)} + \|\nabla b\|_{L^\infty(\mathbb{R}^3)} \le C\left(|U| + |\omega|\right) \end{array}\right. \tag{1.3}$$

for some constant $C = C(\Omega)$. Hence any solution (u,p) of (0.2) is determined uniquely by $(u,p) = (v + b, \pi)$ for a solution (v,π) of the following homogeneous problem in the exterior domain Ω:

$$\text{(NS)} \qquad \left\{\begin{array}{ll} Lv + \nabla\pi = \mathrm{div}\,(F - Q_b(v)) & \text{in } \Omega, \\ \mathrm{div}\, v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ v(x) \to 0 & \text{as } |x| \to \infty, \end{array}\right.$$

where Q_b is the nonlinear operator defined by

$$Q_b(v) = (v + b) \otimes (v + b) - \nabla b + (U - \omega \wedge x) \otimes b + b \otimes (\omega \wedge x). \tag{1.4}$$

From now on, we shall study the solvability of the exterior nonlinear problem (NS) in the framework of Lorentz spaces.

Let D be any bounded or unbounded domain in $\mathbb{R}^3$. For $1 < q < \infty$ and $1 \le r \le \infty$, $L_q(D)$ and $L_{q,r}(D)$ denote the usual Lebesgue and Lorentz spaces over D with norms $\|\cdot\|_{q;D}$ and $\|\cdot\|_{q,r;D}$, respectively. The completion of $C_0^\infty(D)$ with respect to the norm $\|\nabla\cdot\|_{q;D}$ is denoted by $\dot{H}_q^1(D)$. By real interpolation, we define

$\dot{H}^1_{q,r}(D)$ by $\dot{H}^1_{q,r}(D) = (\dot{H}^1_{q_0}(D), \dot{H}^1_{q_1}(D))_{\theta,r}$, where $1 < q_0 < q < q_1 < \infty$ and $0 < \theta < 1$ satisfy $1/q = (1-\theta)/q_0 + \theta/q_1$. It is well-known (see [1] for instance) that $C_0^\infty(D)$ is dense in both $L_{q,r}(D)$ and $\dot{H}^1_{q,r}(D)$ if $1 \le r < \infty$. Denote by $\hat{L}_{q,r}(D)$ and $\hat{H}^1_{q,r}(D)$ the closures of $C_0^\infty(D)$ in $L_{q,r}(D)$ and $\dot{H}^1_{q,r}(D)$, respectively; of course, $\hat{L}_{q,r}(D) = L_{q,r}(D)$ and $\hat{H}^1_{q,r}(D) = \dot{H}^1_{q,r}(D)$ for $1 \le r < \infty$. Note that $L_{q,r}(D) = \hat{L}_{q',r'}(D)^*$, where q' and r' denote the Hölder exponents to q and r, respectively. We also define $\dot{H}^{-1}_{q,r}(D) = \hat{H}^1_{q',r'}(D)^*$ and $\dot{H}^{-1}_q(D) = \dot{H}^{-1}_{q,q}(D) = \dot{H}^1_{q'}(D)^*$, so that $\dot{H}^{-1}_{q,r}(D) = (\dot{H}^{-1}_{q_0}(D), \dot{H}^{-1}_{q_1}(D))_{\theta,r}$. We denote by $\langle \cdot, \cdot \rangle$ simultaneously the duality pairings between $L_{q,r}(D)$ and $L_{q',r'}(D)$ as well as between $\hat{H}^1_{q,r}(D)$ and $\dot{H}^{-1}_{q',r'}(D)$. Finally, the norm of $\dot{H}^{\pm 1}_{q,r}(D)$ is denoted by $\|\cdot\|_{\pm 1,q,r;D}$ or simply by $\|\cdot\|_{\pm 1,q,r}$ if D is the exterior domain Ω under consideration.

It was shown by Kozono and Yamazaki [24] (see Lemma 3.1 below) that if $w \in \dot{H}^1_{q,r}(\Omega)$ for some (q,r) satisfying either $1 < q < 3$ or $(q,r) = (3,1)$, then $w(x) \to 0$ as $|x| \to \infty$ and $w = 0$ on $\partial\Omega$ in some weak sense. Hence we define weak solutions of (NS) as follows:

Definition 1.1. Let (q,r) satisfy either $1 < q < 3, 1 \le r \le \infty$ or $(q,r) = (3,1)$. Suppose that $F \in L_{q,r}(\Omega)$. Then a pair (v,π) in $\dot{H}^1_{q,r}(\Omega) \times L_{q,r}(\Omega)$ is called a *weak solution* or simply a *solution* of (NS) if

(1) $\operatorname{div} v = 0 \quad \text{in } \Omega$,
(2) $(U - \omega \wedge x) \cdot \nabla v + \omega \wedge v \in \dot{H}^{-1}_{q,r}(\Omega)$,
(3) $\langle \nabla v, \nabla w \rangle + \langle (U - \omega \wedge x) \cdot \nabla v + \omega \wedge v, w \rangle = \langle \pi, \operatorname{div} w \rangle - \langle F - Q_b(v), \nabla w \rangle$ for all $w \in C_0^\infty(\Omega)$.

The main purpose of this paper is to establish the existence and uniqueness of solutions of (NS) under the smallness condition on $|U| + |\omega| + \|F\|_{3/2,\infty}$. We first prove the existence of at least one solution in $\dot{H}^1_{3/2,\infty}(\Omega) \times L_{3/2,\infty}(\Omega)$ of (NS).

Theorem 1.1. *There are small positive constants $\delta_0 = \delta_0(\Omega)$ and $\varepsilon_0 = \varepsilon_0(\Omega)$ such that if $F \in L_{3/2,\infty}(\Omega)$ and $|U| + |\omega| + \|F\|_{3/2,\infty} \le \delta_0$, then there exists a unique weak solution $(v,\pi) \in \dot{H}^1_{3/2,\infty}(\Omega) \times L_{3/2,\infty}(\Omega)$ of* (NS) *satisfying the estimate*

$$\|v\|_{3,\infty} \le \varepsilon_0. \tag{1.5}$$

Moreover we have

$$\|v\|_{3,\infty} + \|\nabla v\|_{3/2,\infty} + \|\pi\|_{3/2,\infty} \le C_0 \left(|U| + |\omega| + \|F\|_{3/2,\infty} \right) \tag{1.6}$$

for some constant $C_0 = C_0(\Omega) > 0$.

Remark 1.1. If (v,π) is a solution of (NS) obtained by Theorem 1.1, then $(u,p) = (v + b, \pi)$ is a weak solution of the original stationary problem (0.2).

Remark 1.2. Theorem 1.1 was first proved by Kozono and Yamazaki [24] for the special case $U = \omega = 0$ and then by Farwig and Hishida [8] for the more general case $U = 0$. Our result is an extension of the previous results in [24, 8] to the more general problem (NS) with possibly nonzero U and/or ω.

As an application of Theorem 1.1, we deduce the continuous dependence of the solution of (NS) on the data with respect to the weak-* topology.

Corollary 1.1. *Let $F \in L_{3/2,\infty}(\Omega)$ and $(U,\omega) = (ke_3, ce_3) \in \mathbb{R}^3 \times \mathbb{R}^3$. Let δ_0 be the same constant as in Theorem 1.1, and suppose that $F_n \to F$ weakly-* in $L_{3/2,\infty}(\Omega)$, $(k_n, c_n) \to (k, c)$ in $\mathbb{R}^2$ and $\|F_n\|_{3/2,\infty} + |k_n| + |c_n| \le \delta_0$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we denote by (v_n, π_n) the weak solution in $\dot{H}^1_{3/2,\infty}(\Omega) \times L_{3/2,\infty}(\Omega)$ of* (NS) *satisfying* (1.5)*, with (F, U, ω) replaced by $(F_n, k_n e_3, c_n e_3)$. Further let (v, π) be the weak solution in $\dot{H}^1_{3/2,\infty}(\Omega) \times L_{3/2,\infty}(\Omega)$ of* (NS) *satisfying* (1.5)*. Then the sequence $\{(v_n, \pi_n)\}$ converges to (v, π) weakly-* in $\dot{H}^1_{3/2,\infty}(\Omega) \times L_{3/2,\infty}(\Omega)$.*

Remark 1.3. Corollary 1.1 extends a stability result due to Shibata and Yamazaki [29] to the more general, possibly rotating, Navier-Stokes flows.

It remains still open to prove the uniqueness of solutions in $\dot{H}^1_{3/2,\infty}(\Omega) \times L_{3/2,\infty}(\Omega)$ of (NS) for small $\|F\|_{3/2,\infty}$ without the smallness condition (1.5) on the solutions themselves, even in case of the classical Navier-Stokes problem, i.e., (NS) with $U = \omega = 0$; see [22] for more details. We shall however establish the uniqueness of solutions of (NS) in some supercritical solution spaces. To state our result precisely, let us introduce the following function spaces: for $3/2 \le q < 3$ and $1 \le r \le \infty$, we define

$$V_{q,r} = \dot{H}^1_{3/2,\infty}(\Omega) \cap \dot{H}^1_{q,r}(\Omega) \quad \text{and} \quad \Pi_{q,r} = L_{3/2,\infty}(\Omega) \cap L_{q,r}(\Omega).$$

Both $V_{q,r}$ and $\Pi_{q,r}$ are Banach spaces equipped with the natural norms

$$\|v\|_{V_{q,r}} = \|\nabla v\|_{3/2,\infty} + \|\nabla v\|_{q,r} \quad \text{and} \quad \|\pi\|_{\Pi_{q,r}} = \|\pi\|_{3/2,\infty} + \|\pi\|_{q,r},$$

respectively.

Theorem 1.2. *Suppose that $3/2 < q < 3$ and $1 \le r \le \infty$. Then there is a small positive constant $\delta = \delta(\Omega, q, r)$ such that if $F \in \Pi_{q,r}$ and $|U| + |\omega| + \|F\|_{3/2,\infty} \le \delta$, then there exists a unique weak solution $(v, \pi) \in V_{q,r} \times \Pi_{q,r}$ of* (NS)*. Moreover, we have*

$$\|v\|_{3,\infty} + \|\nabla v\|_{3/2,\infty} + \|\pi\|_{3/2,\infty} \le C \left(|U| + |\omega| + \|F\|_{3/2,\infty} \right)$$

and

$$\|v\|_{q^*,r} + \|\nabla v\|_{q,r} + \|\pi\|_{q,r} \le C' \left(|U| + |\omega| + \|F\|_{q,r} \right)$$

for some constants $C = C(\Omega)$ and $C' = C'(\Omega, q, r)$, where $q^* = 3q/(3-q)$ is the Sobolev exponent to q.

Remark 1.4. (1) The existence of a unique solution (v, π) of (NS) was proved first by Galdi and Silvestre [15] under a stronger hypothesis that $|U| + |\omega| + \|\operatorname{div} F\|_2 + \|(1 + |x|^2) F\|_\infty$ is sufficiently small. They also obtained the pointwise estimate $|v(x)| = O(|x|^{-1})$ at infinity.

(2) In Theorem 1.2, we prove the unique solvability of (NS) for more general F than in [15], but with lack of a pointwise estimate at infinity.

(3) Theorem 1.2 also extends recent existence and uniqueness results by Kim and Kozono [22] for the case $U = \omega = 0$.

Remark 1.5. $V_{3/2,\infty} \times \Pi_{3/2,\infty} = \dot{H}^1_{3/2,\infty}(\Omega) \times L_{3/2,\infty}(\Omega)$ is a critical solution space for the classical Navier-Stokes equations from the viewpoint of scaling invariance, while $V_{q,r} \times \Pi_{q,r}$ is supercritical if $q > 3/2$. On the other hand, the smallness on F needs to be assumed only in the scaling invariant space $L_{3/2,\infty}(\Omega)$.

As a consequence of Theorem 1.2, we also obtain the following continuous dependence result.

Corollary 1.2. *Let $F \in \Pi_{q,r}$ and $(U, \omega) = (ke_3, ce_3) \in \mathbb{R}^3 \times \mathbb{R}^3$, where $3/2 < q < 3$ and $1 \le r \le \infty$ are fixed. Suppose that $F_n \to F$ weakly-$*$ in $\Pi_{q,r}$, $(k_n, c_n) \to (k, c)$ in $\mathbb{R}^2$ and $\|F_n\|_{3/2,\infty} + |k_n| + |c_n| \le \delta$ for each $n \in \mathbb{N}$, where δ is the same constant as in Theorem 1.2. For each $n \in \mathbb{N}$, we denote by (v_n, π_n) the weak solution in $V_{q,r} \times \Pi_{q,r}$ of (NS) with (F, U, ω) replaced by $(F_n, k_n e_3, c_n e_3)$. Further let (v, π) be the weak solution in $V_{q,r} \times \Pi_{q,r}$ of (NS). Then the sequence $\{(v_n, \pi_n)\}$ converges to (v, π) weakly-* in $V_{q,r} \times \Pi_{q,r}$.*

In order to obtain our results on the nonlinear problem (NS), we need to study the corresponding linearized problem. Consider the following linear problem in the exterior domain Ω:

$$
\text{(S)} \qquad \begin{cases} Lv + \nabla \pi = f & \text{in } \Omega, \\ \operatorname{div} v = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ v(x) \to 0 & \text{as } |x| \to \infty. \end{cases}
$$

Following the argument of Farwig and Hishida [8] with the help of the L_q-estimate due to Kračmar, Nečasová and Penel [27], we shall establish the complete $L_{q,r}$-result for the linear problem (S) in Section 3.

Theorem 1.3. *Let (q, r) satisfy one of the three conditions*

$$
(1.7) \qquad \begin{array}{ll} (i) & q = \tfrac{3}{2},\ r = \infty; \\ (ii) & \tfrac{3}{2} < q < 3,\ 1 \le r \le \infty; \\ (iii) & q = 3,\ r = 1. \end{array}
$$

Then for every $f \in \dot{H}^{-1}_{q,r}(\Omega)$ and $g \in L_{q,r}(\Omega)$ with $(U - \omega \wedge x)g \in \dot{H}^{-1}_{q,r}(\Omega)$, there exists a unique weak solution $(v, \pi) \in \dot{H}^1_{q,r}(\Omega) \times L_{q,r}(\Omega)$ of (S). Moreover, for $|U| + |\omega| \le M < \infty$, there is a positive constant $C = C(\Omega, q, r, M)$ such that

$$
(1.8) \qquad \|\nabla v\|_{q,r} + \|\pi\|_{q,r} \le C \left(\|f\|_{-1,q,r} + \|g\|_{q,r} + \|(U - \omega \wedge x)g\|_{-1,q,r} \right).
$$

Remark 1.6. Theorem 1.3 extends the previous result by Kozono and Yamazaki [24] as well as the result by Farwig and Hishida [8] to the more general case of possibly nonzero U and/or ω.

2. Linear problems in the whole space and bounded domains

In this section, we study the unique solvability in the Lorentz spaces $L_{q,r}$ of the following linear problems in the whole space $\mathbb{R}^3$ and on a smooth bounded domain $D \subset \mathbb{R}^3$:

$$(\mathrm{S}_{\mathbb{R}^3}) \qquad \begin{cases} Lv + \nabla\pi = f & \text{in } \mathbb{R}^3, \\ \operatorname{div} v = g & \text{in } \mathbb{R}^3 \end{cases}$$

and

$$(\mathrm{S}_{\mathrm{D}}) \qquad \begin{cases} Lv + \nabla\pi = f & \text{in } D, \\ \operatorname{div} v = g & \text{in } D, \\ v = 0 & \text{on } \partial D. \end{cases}$$

2.1. **The whole space problem** $(\mathrm{S}_{\mathbb{R}^3})$**.** Denote by $\mathcal{S}'(\mathbb{R}^3)$ the space of all tempered distributions on $\mathbb{R}^3$. Since $L_{q,r}(\mathbb{R}^3) \subset L_1(\mathbb{R}^3) + L_\infty(\mathbb{R}^3)$, it is obvious that $L_{q,r}(\mathbb{R}^3) \subset \mathcal{S}'(\mathbb{R}^3)$. It is quite well-known (see [6, Proposition 1.2.1] e.g.) that if v is a distribution with $\nabla v \in \mathcal{S}'(\mathbb{R}^3)$, then $v \in \mathcal{S}'(\mathbb{R}^3)$. Hence it follows that $\dot{H}^1_{q,r}(\mathbb{R}^3) \subset \mathcal{S}'(\mathbb{R}^3)$.

By a direct calculation, we derive

$$\begin{aligned} &\operatorname{div}\left((U - \omega\wedge x)\cdot\nabla v - \omega\wedge v\right) \\ &\qquad = (U-\omega\wedge x)\cdot\nabla\left(\operatorname{div} v\right) = \operatorname{div}\left((U-\omega\wedge x)\operatorname{div} v\right) \end{aligned} \tag{2.9}$$

for all $v \in \mathcal{S}'(\mathbb{R}^3)$. Hence if $(v,\pi) \in \mathcal{S}'(\mathbb{R}^3)$ is a distributional solution of $(\mathrm{S}_{\mathbb{R}^3})$, then

$$\Delta\pi = \operatorname{div} h \quad \text{in } \mathbb{R}^3 \tag{2.10}$$

and

$$Lv = f_0 \quad \text{in } \mathbb{R}^3, \tag{2.11}$$

where

$$h = f + \nabla g - (U - \omega\wedge x)g \quad \text{and} \quad f_0 = f - \nabla\pi.$$

The unique solvability in $L_{q,r}$ of (2.10) was already shown by Kozono and Yamazaki [24, Lemmas 2.4 and 2.5].

Lemma 2.1. *Let $1 < q < \infty$ and $1 \le r \le \infty$. Then for every $h \in \dot{H}^{-1}_{q,r}(\mathbb{R}^3)$, there exists a unique (very weak) solution $\pi \in L_{q,r}(\mathbb{R}^3)$ of* (2.10)*. Moreover, we have*

$$\|\pi\|_{q,r;\mathbb{R}^3} \le C\|h\|_{-1,q,r;\mathbb{R}^3}$$

for some constant $C = C(q,r)$.

Sketch of the proof. The uniqueness follows from the classical Liouville theorem for harmonic functions in $\mathbb{R}^3$ or Lemma 2.2 (b) below. To prove the existence, let $h \in \dot{H}^{-1}_{q,r}(\mathbb{R}^3)$ be given. By Lemma 2.2 in [24], there is a matrix-valued function $H = \{H^j_k\}_{j,k=1,2,3}$ in $L_{q,r}(\mathbb{R}^3)$ such that $h = \operatorname{div} H$ and $\|H\|_{q,r;\mathbb{R}^3} \le C(q,r)\|h\|_{-1,q,r;\mathbb{R}^3}$. Then the solution in $L_{q,r}(\mathbb{R}^3)$ of (2.10) is given by $\pi = \sum_{j,k=1}^3 R_j R_k H^j_k$, where $R =$

$\mathcal{F}^{-1}((i\xi/|\xi|)\mathcal{F}\cdot)$ denotes the Riesz transform. It is well-known that R is bounded on $L_{q,r}(\mathbb{R}^3)$. This completes the proof of Lemma 2.1. □

The unique solvability in L_q of (2.11) has been studied by Farwig [7] and by Kračmar, Nečasová and Penel [27] for strong solutions and weak solutions, respectively. The uniqueness of solutions can be deduced from the following Liouville-type results, the proofs of which are given in [7, p. 142](see also [9, 19]).

Lemma 2.2. *(a) If $v \in \mathcal{S}'(\mathbb{R}^3)$ and $Lv = 0$ in $\mathbb{R}^3$, then v is a polynomial vector function.*

(b) If $w \in \mathcal{S}'(\mathbb{R}^3)$ and $-\Delta w + (U - \omega \wedge x) \cdot \nabla w = 0$ in $\mathbb{R}^3$, then w is a polynomial.

For the existence of weak solutions in L_q of (2.11), we recall the following result due to Kračmar, Nečasová and Penel [27].

Lemma 2.3. *Let $1 < q < \infty$. Then for every $f_0 \in \dot{H}_q^{-1}(\mathbb{R}^3)$, there exists a unique weak solution $v \in \dot{H}_q^1(\mathbb{R}^3)$ of (2.11). Moreover, we have*

$$\|\nabla v\|_{q;\mathbb{R}^3} \le C\|f_0\|_{-1,q;\mathbb{R}^3}$$

for some constant $C = C(q)$.

By real interpolation, we can prove the unique solvability in $L_{q,r}(\mathbb{R}^3)$ of (2.11).

Lemma 2.4. *Let $1 < q < \infty$ and $1 \le r \le \infty$. Then for every $f_0 \in \dot{H}_{q,r}^{-1}(\mathbb{R}^3)$, there exists a unique weak solution $v \in \dot{H}_{q,r}^1(\mathbb{R}^3)$ of (2.11). Moreover, we have*

$$\|\nabla v\|_{q,r;\mathbb{R}^3} \le C\|f_0\|_{-1,q,r;\mathbb{R}^3}$$

for some constant $C = C(q, r)$.

Proof. It follows from Lemma 2.3 that for each $1 < q < \infty$, there exists a bounded linear operator $T_q : \dot{H}_q^{-1}(\mathbb{R}^3) \to \dot{H}_q^1(\mathbb{R}^3)$ such that $L(T_q f_0) = f_0$ for all $f_0 \in \dot{H}_q^{-1}(\mathbb{R}^3)$.

Let $1 < q < \infty$ and $1 \le r \le \infty$ be fixed, and let us choose q_0, q_1, θ, with $1 < q_0 < q < q_1 < \infty$, $0 < \theta < 1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$, so that $\dot{H}_{q,r}^1(\mathbb{R}^3) = (\dot{H}_{q_0}^1(\mathbb{R}^3), \dot{H}_{q_1}^1(\mathbb{R}^3))_{\theta,r}$ and $\dot{H}_{q,r}^{-1}(\mathbb{R}^3) = (\dot{H}_{q_0}^{-1}(\mathbb{R}^3), \dot{H}_{q_1}^{-1}(\mathbb{R}^3))_{\theta,r}$. Suppose that $f_0 \in \dot{H}_{q_0}^{-1}(\mathbb{R}^3) \cap \dot{H}_{q_1}^{-1}(\mathbb{R}^3)$. Then $v = T_{q_1} f_0 - T_{q_0} f_0$ satisfies

$$v \in \dot{H}_{q_0}^1(\mathbb{R}^3) + \dot{H}_{q_1}^1(\mathbb{R}^3) \quad \text{and} \quad Lv = 0 \quad \text{in } \mathbb{R}^3.$$

It follows from Lemma 2.2 (a) that v is a polynomial. Since $\nabla v \in L_{q_0}(\mathbb{R}^3) + L_{q_1}(\mathbb{R}^3)$, ∇v must be identically zero. This shows that $T_{q_0} = T_{q_1}$ on $\dot{H}_{q_0}^{-1}(\mathbb{R}^3) \cap \dot{H}_{q_1}^{-1}(\mathbb{R}^3)$. Hence there exists a unique linear operator

$$T : \dot{H}_{q_0}^{-1}(\mathbb{R}^3) + \dot{H}_{q_1}^{-1}(\mathbb{R}^3) \to \dot{H}_{q_0}^1(\mathbb{R}^3) + \dot{H}_{q_1}^1(\mathbb{R}^3)$$

such that $T = T_{q_i}$ on $\dot{H}_{q_i}^{-1}(\mathbb{R}^3)$ for each $i = 0, 1$. Recall that $T|_{\dot{H}_{q_i}^{-1}(\mathbb{R}^3)} = T_{q_i}$ is bounded from $\dot{H}_{q_i}^{-1}(\mathbb{R}^3)$ into $\dot{H}_{q_i}^1(\mathbb{R}^3)$ for each $i = 0, 1$. Therefore by real interpolation theory, we deduce that T is bounded from $\dot{H}_{q,r}^{-1}(\mathbb{R}^3)$ into $\dot{H}_{q,r}^1(\mathbb{R}^3)$. On the other hand, it is obvious that for each $f_0 \in \dot{H}^{-1}_{q_0}(\mathbb{R}^3)+\dot{H}^{-1}_{q_1}(\mathbb{R}^3)$, $v = Tf_0$ is a distributional solution of $Lv = f_0$ in $\mathbb{R}^3$. This proves the existence and a priori estimate of a weak solution $v \in \dot{H}^1_{q,r}(\mathbb{R}^3)$ of (2.11) for every $f_0 \in \dot{H}^{-1}_{q,r}(\mathbb{R}^3)$. The uniqueness of weak solutions in $\dot{H}^1_{q,r}(\mathbb{R}^3)$ follows immediately from Lemma 2.2 (a) since $\dot{H}^1_{q,r}(\mathbb{R}^3) \subset \mathcal{S}'(\mathbb{R}^3)$. This completes the proof of Lemma 2.4. □

We are now ready to prove the unique solvability result in $L_{q,r}$ for the whole space problem $(\mathrm{S}_{\mathbb{R}^3})$.

Proposition 2.1. *Let $1 < q < \infty$ and $1 \le r \le \infty$. Then for every $f \in \dot{H}^{-1}_{q,r}(\mathbb{R}^3)$ and $g \in L_{q,r}(\mathbb{R}^3)$ with $(U - \omega \wedge x)g \in \dot{H}^{-1}_{q,r}(\mathbb{R}^3)$, there exists a unique weak solution $(v, \pi) \in \dot{H}^1_{q,r}(\mathbb{R}^3) \times L_{q,r}(\mathbb{R}^3)$ of $(\mathrm{S}_{\mathbb{R}^3})$. Moreover, we have*

$$\|\nabla v\|_{q,r;\mathbb{R}^3} + \|\pi\|_{q,r;\mathbb{R}^3} \le C\left(\|f\|_{-1,q,r;\mathbb{R}^3} + \|g\|_{q,r;\mathbb{R}^3} + \|(U - \omega \wedge x)g\|_{-1,q,r;\mathbb{R}^3}\right)$$

for some constant $C = C(q, r)$.

Proof. Let (v, π) be a weak solution in $\dot{H}^1_{q,r}(\mathbb{R}^3) \times L_{q,r}(\mathbb{R}^3)$ of $(\mathrm{S}_{\mathbb{R}^3})$ with $(f, g) = (0, 0)$. Then it follows from (2.10) and (2.11) that $\Delta\pi = 0$ and $Lv = 0$ in $\mathbb{R}^3$. Using Lemma 2.2 again, we easily deduce that $\pi = 0$ and $\nabla v = 0$ in $\mathbb{R}^3$. This proves the uniqueness assertion.

To prove the existence and a priori estimate, let us suppose that $f \in \dot{H}^{-1}_{q,r}(\mathbb{R}^3)$, $g \in L_{q,r}(\mathbb{R}^3)$ and $(U - \omega \wedge x)g \in \dot{H}^{-1}_{q,r}(\mathbb{R}^3)$. Then

$$h \equiv f + \nabla g - (U - \omega \wedge x)g \in \dot{H}^{-1}_{q,r}(\mathbb{R}^3) \tag{2.12}$$

and

$$\|h\|_{\dot{H}^{-1}_{q,r}(\mathbb{R}^3)} \le \|f\|_{-1,q,r;\mathbb{R}^3} + \|g\|_{q,r;\mathbb{R}^3} + \|(U - \omega \wedge x)g\|_{-1,q,r;\mathbb{R}^3}.$$

By Lemma 2.1, there exists a unique $\pi \in L_{q,r}(\mathbb{R}^3)$ such that

$$\Delta\pi = \operatorname{div} h \quad \text{and} \quad \|\pi\|_{q,r;\mathbb{R}^3} \le C\|h\|_{-1,q,r;\mathbb{R}^3}. \tag{2.13}$$

Note that $f_0 \equiv f - \nabla\pi \in \dot{H}^{-1}_{q,r}(\mathbb{R}^3)$ and $\|f_0\|_{\dot{H}^{-1}_{q,r}(\mathbb{R}^3)} \le \|f\|_{-1,q,r;\mathbb{R}^3} + \|\pi\|_{q,r;\mathbb{R}^3}$. Hence by Lemma 2.4, there exists a unique $v \in \dot{H}^1_{q,r}(\mathbb{R}^3)$ such that

$$Lv = f - \nabla\pi \quad \text{and} \quad \|\nabla v\|_{q,r;\mathbb{R}^3} \le C\left(\|f\|_{-1,q,r;\mathbb{R}^3} + \|\pi\|_{q,r;\mathbb{R}^3}\right). \tag{2.14}$$

To complete the proof, it now remains to show that $\operatorname{div} v = g$. Set $w = \operatorname{div} v - g$. Then by virtue of (2.13) and (2.14), we easily have

$$w \in L_{q,r}(\mathbb{R}^3) \quad \text{and} \quad -\Delta w + (U - \omega \wedge x) \cdot \nabla w = 0.$$

Hence it follows from Lemma 2.2 (b) that $w = 0$, i.e., $\operatorname{div} v = g$. We have completed the proof of Proposition 2.1. □

We finish this section with Galdi and Silvestre's existence result [15, Theorem 2], which will be used to prove the energy equality and uniqueness of weak solutions of the linear problem in exterior domains.

Proposition 2.2. *Assume that* $g \equiv 0$. *If* $f = \operatorname{div} F \in L_2(\mathbb{R}^3)$ *and* $(1+|x|^2)\,F \in L_\infty(\mathbb{R}^3)$, *then there exists a unique strong solution* (v,π) *of* $(\mathrm{S}_{\mathbb{R}^3})$ *satisfying*

$$\begin{gathered} v \in \dot{H}^1_2(\mathbb{R}^3), \quad (1+|x|)\,v \in L_\infty(\mathbb{R}^3), \\ \nabla^2 v, \nabla\pi \in L_2(\mathbb{R}^3), \quad \pi \in L_s(\mathbb{R}^3)\ \left(s > \tfrac{3}{2}\right). \end{gathered}$$

2.2. **The boundary value problem** $(\mathrm{S_D})$**.** We next study the linear problem $(\mathrm{S_D})$, where D is a bounded domain in $\mathbb{R}^3$ with smooth boundary. Since D is bounded, $L = -\Delta + (U - \omega\wedge x)\cdot\nabla + \omega\wedge$ is a compact perturbation of the Laplace operator $-\Delta$. Hence the unique solvability in L_q of $(\mathrm{S_D})$ can be easily deduced from classical Cattabriga's L_q-theory in [5] for the usual Stokes problem. Then by real interpolation, we can establish the unique solvability in $L_{q,r}$ for $(\mathrm{S_D})$. To begin with, let us introduce

$$\dot{H}^1_{q,\sigma}(D) = \overline{C^\infty_{0,\sigma}(D)}^{\|\nabla\cdot\|_{p;D}} = \left\{ v \in \dot{H}^1_q(D) : \operatorname{div} v = 0 \text{ in } D \right\}.$$

Lemma 2.5. *Let* $1 < q < \infty$. *Then for every* $f \in \dot{H}^{-1}_q(D)$, *there exists a unique* $v \in \dot{H}^1_{q,\sigma}(D)$ *such that*

$$\int_D \nabla v\cdot\nabla\varphi\,dx + \int_D ((U-\omega\wedge x)\cdot\nabla v + \omega\wedge v)\cdot\varphi\,dx = \langle f,\varphi\rangle \tag{2.15}$$

for all $\varphi \in C^\infty_{0,\sigma}(D)$. *Moreover, for* $|U| + |\omega| \le M < \infty$, *there is a positive constant* $C = C(D,q,M)$ *such that*

$$\|\nabla v\|_{q;D} \le C\|f\|_{-1,q;D}. \tag{2.16}$$

Proof. Note that

$$\int_D ((U-\omega\wedge x)\cdot\nabla v + \omega\wedge v)\cdot v\,dx = 0 \quad \text{for all } v \in \dot{H}^1_{2,\sigma}(D).$$

Hence by the Lax-Milgram theorem, we deduce that for each $f \in \dot{H}^{-1}_2(D)$, there exists a unique $v \in \dot{H}^1_{2,\sigma}(D)$ satisfying (2.15) and (2.16) with $C = 1$.

Suppose now that $2 \le q < \infty$, and let $f \in \dot{H}^{-1}_q(D)$ be given. Then there exists a unique $v \in \dot{H}^1_{2,\sigma}(D)$ satisfying (2.15) and $\|\nabla v\|_{2;D} \le \|f\|_{-1,2;D}$. Let L_0 be the operator defined by $L_0 v = (U-\omega\wedge x)\otimes v + v\otimes(\omega\wedge x)$. Then since $\operatorname{div} v = 0$ and $v \in \dot{H}^1_2(D) \hookrightarrow L_6(D)$, it follows that $(U-\omega\wedge x)\cdot\nabla v + \omega\wedge v = \operatorname{div} L_0 v$ and $L_0 v \in L_6(D)$. Hence it follows from the classical L_q-result due to Cattabriga [5] that $v \in \dot{H}^1_{q_0}(D)$, where $q_0 = \min(6,q)$. If $2 \le q \le 6$, then $v \in \dot{H}^1_{q,\sigma}(D)$ and

$$\begin{aligned} \|\nabla v\|_{q;D} &\le C\left(\|L_0 v\|_{q;D} + \|f\|_{-1,q;D}\right) \\ &\le C(|U|+|\omega|)\|v\|_{q;D} + C\|f\|_{-1,q;D} \\ &\le C(|U|+|\omega|)\|\nabla v\|_{2;D} + C\|f\|_{-1,q;D} \\ &\le C(|U|+|\omega|+1)\|f\|_{-1,q;D} \end{aligned}$$

for some $C = C(D,q)$. If $6 < q < \infty$, then $v \in \dot{H}^1_{q,\sigma}(D) \hookrightarrow L_\infty(D)$ and

$$\|v\|_{q;D} \le \|v\|_{6;D}^{6/q} \|v\|_{\infty;D}^{1-6/q} \le C\|f\|_{-1,q;D}^{6/q} \|\nabla v\|_{q;D}^{1-6/q},$$

so that

$$\begin{aligned}\|\nabla v\|_{q;D} &\le C\left(\|L_0 v\|_{q;D} + \|f\|_{-1,q;D}\right)\\ &\le C\|f\|_{-1,q;D}^{6/q}\|\nabla v\|_{q;D}^{1-6/q} + C\|f\|_{-1,q;D}\\ &\le C\|f\|_{-1,q;D} + \frac{1}{2}\|\nabla v\|_{q;D},\end{aligned}$$

which also implies the estimate (2.16) too. This proves the lemma for the case that $2 \le q < \infty$. The remaining case can be easily proved by a duality argument because the formal adjoint L^* of L is given by $L^* = -\Delta - (U - \omega \wedge x)\cdot\nabla - \omega\wedge$ which is essentially the same as L. The proof of Lemma 2.5 is complete. □

Fix a cut-off function $\eta_D \in C_0^\infty(D)$ with $\int_D \eta_D\, dx = 1$. Then by real interpolation, we easily deduce the following basic lemma from the classical result due to Bogovoskiĭ [2] (see also [8, 10, 31]).

Lemma 2.6. *For $1 < q < \infty$ and $1 \le r \le \infty$, there exists a bounded linear operator*

$$\mathcal{B} = \mathcal{B}_D : L_{q,r}(D) \to \dot{H}^1_{q,r}(D),$$

called the Bogovoskiĭ operator on D, such that

$$\mathcal{B}\left(C_0^\infty(D)\right) \subset C_0^\infty(D) \tag{2.17}$$

and

$$\operatorname{div}\mathcal{B}g = g - \left(\int_D g\, dx\right)\eta_D \quad \textit{for all } g \in L_{q,r}(D).$$

Remark 2.1. It also follows from (2.17) that $\mathcal{B}$ is a bounded operator from $\hat{L}_{q,r}(D)$ to $\hat{H}^1_{q,r}(D)$.

Using Lemma 2.6, we obtain

Lemma 2.7. *Let $1 < q < \infty$ and $1 \le r \le \infty$. If $f \in \dot{H}^{-1}_{q,r}(D)$ is a vector satisfying*

$$\langle f, w\rangle = 0 \quad \textit{for all } w \in C_{0,\sigma}^\infty(D), \tag{2.18}$$

then there exists a scalar $\pi \in L_{q,r}(D)$ such that

$$f = \nabla\pi, \quad \textit{i.e.,} \quad \langle f, w\rangle = -\int_D \pi \operatorname{div} w\, dx \quad \textit{for all } w \in C_0^\infty(D)$$

and

$$\|\pi\|_{q,r;D} \le C\|f\|_{-1,q,r;D}$$

for some constant $C = C(D,q,r)$.

Proof. Following the proof of [21, Lemma 7], we define $\hat{\pi} : \hat{L}_{q',r'}(D) \to \mathbb{R}$ by

$$\langle \hat{\pi}, g \rangle = -\langle f, \mathcal{B}g \rangle \quad \text{for all } g \in \hat{L}_{q',r'}(D). \tag{2.19}$$

By Lemma 2.6 and Remark 2.1, $\hat{\pi}$ is a well-defined bounded linear functional on $\hat{L}_{q',r'}(D)$. Since $L_{q,r}(D) = \hat{L}_{q',r'}(D)^*$, there exists $\pi \in L_{q,r}(D)$ such that $\langle \hat{\pi}, g \rangle = \int_D \pi g \, \mathrm{d}x$ for all $g \in \hat{L}_{q',r'}(D)$. Moreover, since $|-\langle f, \mathcal{B}g \rangle| \le \|f\|_{\dot{H}^{-1}_{q,r}(D)} \|\mathcal{B}g\|_{\hat{H}^1_{q',r'}(D)} \le C\|f\|_{\dot{H}^{-1}_{q,r}(D)} \|g\|_{\hat{L}_{q',r'}(D)}$ for all $g \in \hat{L}_{q,r}(D)$, it follows that $\|\pi\|_{L_{q,r}(D)} \le C\|f\|_{\dot{H}^{-1}_{q,r}(D)}$. Hence to complete the proof, it remains to prove that $f = \nabla \pi$. Given $w \in C_0^\infty(D)$, we set $g = \operatorname{div} w$. It is obvious that $g \in C_0^\infty(D)$ and $\int_D g \, dx = 0$. Hence $\mathcal{B}g \in C_0^\infty(D)$ and $\operatorname{div} \mathcal{B}g = g = \operatorname{div} w$. This implies that $\mathcal{B}g - w \in C_{0,\sigma}^\infty(D)$ and $< f, \mathcal{B}g - w >= 0$. Therefore, using (2.19) and (2.18), we have

$$\int_D \pi \operatorname{div} w \, dx = \langle \hat{\pi}, g \rangle = -\langle f, \mathcal{B}g \rangle = -\langle f, w \rangle.$$

This completes the proof of Lemma 2.7. □

We are now ready to prove the unique solvability result in $L_{q,r}$ for the boundary value problem ($\mathrm{S_D}$).

Proposition 2.3. *Let $1 < q < \infty$ and $1 \le r \le \infty$. Then for every $f \in \dot{H}^{-1}_{q,r}(D)$ and $g \in L_{q,r}(D)$ with $\int_D g \, dx = 0$, there exists a unique weak solution $(v, \pi) \in \dot{H}^1_{q,r}(D) \times L_{q,r}(D)$ of ($\mathrm{S_D}$) with $\int_D \pi \, dx = 0$. Moreover, for $|U| + |\omega| \le M < \infty$, there is a positive constant $C = C(D, q, r, M)$ such that*

$$\|\nabla v\|_{q,r;D} + \|\pi\|_{q,r;D} \le C \left(\|f\|_{-1,q,r;D} + \|g\|_{q,r;D} \right).$$

Proof. Suppose that (v, π) is a weak solution in $\dot{H}^1_{q,r}(D) \times L_{q,r}(D)$ of ($\mathrm{S_D}$) with $(f, g) = (0, 0)$. Then since $v \in \dot{H}^1_{q_0,\sigma}(D)$ for any $1 < q_0 < q$, it follows from Lemma 2.5 that $v = 0$ and so $\nabla \pi = 0$. This proves the uniqueness assertion of the proposition. To prove the existence and a priori estimate, let $S_q : \dot{H}^{-1}_q(D) \to \dot{H}^1_{q,\sigma}(D), 1 < q < \infty$, be the operator such that for each $f \in \dot{H}^{-1}_q(D)$, $v = S_q f$ satisfies (2.15) and (2.16). Then S_q is linear and bounded on $\dot{H}^{-1}_q(D)$, and $S_{q_0} = S_{q_1}$ on $\dot{H}^{-1}_{q_0}(D) \cap \dot{H}^{-1}_{q_1}(D)$ for any $1 < q_0 < q_1 < \infty$. Hence by real interpolation, there exists a bounded linear operator $S_{q,r} : \dot{H}^{-1}_{q,r}(D) \to \dot{H}^1_{q,r}(D)$ such that for each $f \in \dot{H}^{-1}_{q,r}(D)$, $v = S_{q,r} f$ satisfies (2.15), $\operatorname{div} v = 0$ and $\|\nabla v\|_{q,r;D} \le C\|f\|_{-1,q,r;D}$. By Lemma 2.7, there also exists $\pi \in L_{q,r}(D)$ such that $Lv - f = -\nabla \pi$ and $\|\pi\|_{q,r;D} \le C\|Lv - f\|_{-1,q,r;D}$. This completes the proof of Proposition 2.3 in case when $g = 0$. The proof of the general case based on Lemma 2.6 is quite easy and omitted. □

3. The linear problem in exterior domains

The purpose of this section is to establish both existence and uniqueness assertions of Theorem 1.3. To do so, we need to combine the solvability results for the whole space problem ($\mathrm{S}_{\mathbb{R}^3}$) and the boundary value problem ($\mathrm{S_D}$), by using suitable cut-off functions. Here we shall closely follow the cut-off procedure developed by Farwig-Hishida [8].

3.1. **Preliminaries.** Let $R > 5$ be a fixed number so large that $\mathbb{R}^3 \setminus \Omega \subset B_{R-5}$, where $B_r = \{x \in \mathbb{R}^3 : |x| < r\}$ for $r > 0$. We also set $\Omega_r = \Omega \cap B_r$ for any $r \geq R-5$.

The following result was obtained by Kozono and Yamazaki [24, Lemma 2.1] using the real interpolation theory.

Lemma 3.1. *Let D be the whole space $\mathbb{R}^3$, a smooth bounded domain in $\mathbb{R}^3$ or a smooth exterior domain in $\mathbb{R}^3$.*

(a) Let $1 < q < 3$ and $1 \leq r \leq \infty$. If $w \in \dot{H}^1_{q,r}(D)$, then $w|_{\partial D} = 0$, $w \in L_{q^*,r}(D)$ and $\|w\|_{q^*,r;D} \leq C\|\nabla w\|_{q,r;D}$ for some constant $C = C(q,r) > 0$, where $q^* = 3q/(3-q)$ is the Sobolev exponent to q.

(b) If $w \in \dot{H}^1_{3,1}(D)$, then $w|_{\partial D} = 0$, $w \in L^\infty(D) \cap C(\overline{D})$, $\|w\|_{\infty;D} \leq \frac{1}{3}\|\nabla w\|_{3,1;D}$ and $w(x) \to 0$ uniformly as $|x| \to \infty$.

Using Lemma 3.1, we first obtain the following result for our cut-off techniques.

Lemma 3.2. *Let $\psi \in C_0^\infty(B_R)$ be a cut-off function with $\psi = 1$ in B_{R-5}. Suppose that $1 < q < \infty$ and $1 \leq r \leq \infty$.*

(a) For every $v \in \dot{H}^1_{q,r}(\Omega)$, we have

$$\psi v \in \dot{H}^1_{q,r}(\Omega_R), \quad (1-\psi)v \in \dot{H}^1_{q,r}(\mathbb{R}^3)$$

and

$$\|\nabla(\psi v)\|_{q,r;\Omega_R} + \|\nabla\left((1-\psi)v\right)\|_{q,r;\mathbb{R}^3} \leq C\|\nabla v\|_{q,r}$$

for some constant $C = C(\Omega, \psi, q, r) > 0$.

(b) For every $v_1 \in \dot{H}^1_{q,r}(\Omega_R)$ and $v_2 \in \dot{H}^1_{q,r}(\mathbb{R}^3)$, we have

$$\psi v_1 \in \dot{H}^1_{q,r}(\Omega), \quad \|\nabla(\psi v_1)\|_{q,r} \leq C\|\nabla v_1\|_{q,r;\Omega_R},$$

$$(1-\psi)v_2 \in \dot{H}^1_{q,r}(\Omega) \quad \textit{and} \quad \|\nabla((1-\psi)v_2)\|_{q,r} \leq C\|\nabla v_2\|_{q,r;\mathbb{R}^3}$$

for some constant $C = C(\Omega, \psi, q, r) > 0$.

(c) For every $f \in \dot{H}^{-1}_{q,r}(\Omega)$, we have

$$\psi f \in \dot{H}^{-1}_{q,r}(\Omega_R), \quad (1-\psi)f \in \dot{H}^{-1}_{q,r}(\mathbb{R}^3)$$

and

$$\|\psi f\|_{-1,q,r;\Omega_R} + \|(1-\psi)f\|_{-1,q,r;\mathbb{R}^3} \leq C\|f\|_{-1,q,r}$$

for some constant $C = C(\Omega, \psi, q, r) > 0$.

Proof. By real interpolation, it suffices to prove the lemma for the special case when $1 < q = r < \infty$. Assume thus that $1 < q = r < \infty$.

(a) Let $v \in C_0^\infty(\Omega)$ be fixed. Then since $v = 0$ on $\partial\Omega$, there holds the Poincaré inequality

$$\|v\|_{q;\Omega_R} \leq C\|\nabla v\|_{q;\Omega_R}$$

with C depending only on Ω and q. Using this, we have

$$\|\nabla(\psi v)\|_{q;\Omega_R} + \|\nabla\left((1-\psi)v\right)\|_{q;\mathbb{R}^3} \le C\left(\|v\|_{q;\Omega_R} + \|\nabla v\|_{q;\Omega}\right) \le C\|\nabla v\|_q.$$

Recall from the definition that $C_0^\infty(D)$ is dense in $\dot{H}^1_q(D)$, where $D = \Omega_R$, Ω or $\mathbb{R}^3$. Hence by a density argument, we easily deduce that if $v \in \dot{H}^1_q(\Omega)$, then $\psi v \in \dot{H}^1_q(\Omega_R)$, $(1-\psi)v \in \dot{H}^1_q(\mathbb{R}^3)$ and $\|\nabla(\psi v)\|_{q;\Omega_R} + \|\nabla\left((1-\psi)v\right)\|_{q;\mathbb{R}^3} \le C\|\nabla v\|_q$. This proves (a) for the case $q = r$.

(b) It is quite easy to prove the assertions for v_1 because $\dot{H}^1_q(\Omega_R) \hookrightarrow L_q(\Omega_R)$. Hence we provide only the proof of the assertions for $v_2 \in \dot{H}^1_q(\mathbb{R}^3)$.

Suppose first that $1 < q < 3$. Then by Sobolev's inequality (see Lemma 3.1), we have

$$\|\nabla\left((1-\psi)v_2\right)\|_{q;\Omega} \le C\left(\|v_2\|_{q;\Omega_R} + \|\nabla v_2\|_{q;\Omega}\right) \le C\|\nabla v_2\|_{q;\mathbb{R}^3}$$

for all $v_2 \in C_0^\infty(\mathbb{R}^3)$. This prove (b) for the case $1 < q = r < 3$, by a standard density argument.

Suppose next that $3 \le q < \infty$. Denote by $\mathcal{D}_0$ the set of all $\varphi \in C_0^\infty(\mathbb{R}^3)$ with $\int_{\Omega_R} \varphi\, dx = 0$. Then by the Poincaré inequality, we have

$$\|\nabla((1-\psi)v_2)\|_{q;\Omega} \le C\left(\|v_2\|_{q;\Omega_R} + \|\nabla v_2\|_{q;\Omega}\right) \le C\|\nabla v_2\|_{q;\mathbb{R}^3}$$

for all $v_2 \in \mathcal{D}_0$. But since $3 \le q < \infty$, it follows from the proof of [23, Lemma 2.5] (see also [24, Lemma 2.3]) that $\mathcal{D}_0$ is dense in $\dot{H}^1_q(\mathbb{R}^3)$. Hence by a density argument, we easily prove (b) for the case $3 \le q = r < \infty$.

(c) This follows immediately from (b) by a duality argument. Indeed, for all $v_2 \in \dot{H}^1_{q'}(\mathbb{R}^3)$, we have

$$\begin{aligned} < (1-\psi)f, v_2 > &= < f, (1-\psi)v_2 > \\ &\le \|f\|_{-1,q}\|\nabla\left((1-\psi)v_2\right)\|_{q'} \\ &\le C\|f\|_{-1,q}\|\nabla v_2\|_{q';\mathbb{R}^3}, \end{aligned}$$

which implies that $(1-\psi)f \in \dot{H}^{-1}_q(\mathbb{R}^3)$ and $\|(1-\psi)f\|_{-1,q;\mathbb{R}^3} \le C\|f\|_{-1,q}$. The assertions for ψf can be proved similarly. This proves (c) for the case $1 < q = r < \infty$.

We have proved the lemma for the special case when $1 < q = r < \infty$. The general case then follows by real interpolation. This completes the proof of Lemma 3.2. □

3.2. **Proofs of the uniqueness and a priori estimate.**

We first prove the uniqueness assertion and a priori estimate of Theorem 1.3. In fact, the uniqueness assertion of Theorem 1.3 is an immediate consequence of the following result, which is inspired by [15, Theorem 3] and [8, Propositions 5.1 and 5.2].

Lemma 3.3. *For each $i = 1, 2$, let (q_i, r_i) satisfy either $1 < q_i < 3, 1 \le r_i \le \infty$ or $(q_i, r_i) = (3, 1)$. Suppose that*

$$\left\{ \begin{array}{c} f = \operatorname{div} F \in L_2(\Omega), \quad (1+|x|^2)\, F \in L_\infty(\Omega); \\ v \in \dot{H}^1_{q_1,r_1}(\Omega) + \dot{H}^1_{q_2,r_2}(\Omega), \quad \pi \in L_{q_1,r_1}(\Omega) + L_{q_2,r_2}(\Omega); \\ Lv + \nabla\pi = f, \quad \operatorname{div} v = 0 \quad \text{in } \Omega. \end{array} \right. \tag{3.20}$$

Then we have

$$v \in \dot H^1_2(\Omega), \quad (1+|x|)\, v \in L_\infty(\Omega), \\ \nabla^2 v, \nabla\pi \in L_2(\Omega), \quad \pi \in L_s(\Omega)\ \big(s > \tfrac{3}{2}\big) \tag{3.21}$$

and

$$\int_\Omega |\nabla v|^2\, dx = -\int_\Omega F \cdot \nabla v\, dx. \tag{3.22}$$

It thus follows that if $F \equiv 0$, then $(v,\pi) = (0,0)$ in Ω.

Proof. It follows from (3.20) that $\nabla v \in L_{q,loc}(\overline{\Omega})$ for any $q < \min(q_1, q_2)$. Hence applying a local regularity theory for the Stokes equations (see [10] for instance), we easily deduce that

$$\nabla v, \pi \in H^1_{2,loc}(\overline{\Omega});\ \text{that is,}\ \nabla v,\ \nabla^2 v,\ \pi,\ \nabla\pi \in L_{2,loc}(\overline{\Omega}). \tag{3.23}$$

Choosing a fixed cut-off function $\psi \in C_0^\infty(\mathbb{R}^3)$ such that

$$\psi(x) = \begin{cases} 1, & |x| \le R-3, \\ 0, & |x| \ge R-2, \end{cases} \tag{3.24}$$

we define

$$\overline{v} = (1-\psi)\, v + \mathcal{B}(v\cdot\nabla\psi) \quad \text{and} \quad \overline{\pi} = (1-\psi)\,\pi,$$

where $\mathcal{B}$ is the Bogovoskiĭ operator on the annulus $B_{R-1} \setminus \overline{B}_{R-4}$; see Lemma 2.6. Then it follows from (3.20), (3.23), Lemma 3.2 (a) and Lemma 2.6 that

$$\begin{cases} \overline{v} \in \Big[\dot H^1_{q_1,r_1}(\mathbb{R}^3) + \dot H^1_{q_2,r_2}(\mathbb{R}^3)\Big] \cap H^2_{2,loc}(\mathbb{R}^3), \\ \overline{\pi} \in [L_{q_1,r_1}(\mathbb{R}^3) + L_{q_2,r_2}(\mathbb{R}^3)] \cap H^1_{2,loc}(\mathbb{R}^3); \\ L\overline{v} + \nabla\overline{\pi} = \overline{f}, \quad \operatorname{div}\overline{v} = 0 \quad \text{in } \mathbb{R}^3, \end{cases} \tag{3.25}$$

where

$$\overline{f} = (1-\psi) f + 2\nabla\psi\cdot\nabla v + [\Delta\psi - (U - \omega\wedge x)\cdot\nabla\psi]\, v - \pi\nabla\psi + L\mathcal{B}(v\cdot\nabla\psi).$$

Note here that $\overline{f} = \overline{f}_1 + \overline{f}_2$, $\overline{f}_1 = \operatorname{div}((1-\psi)F) \in L_2(\mathbb{R}^3)$, $(1+|x|^2)((1-\psi)F) \in L_\infty(\mathbb{R}^3)$, $\overline{f}_2 \in L_6(\mathbb{R}^3)$ and $\operatorname{supp}\overline{f}_2 \subset \overline{B}_{R-2}$. Setting $\overline{F}_2 = \nabla\big((1/4\pi|x|) * \overline{f}_2\big)$, we easily deduce that $\overline{f}_2 = \operatorname{div}\overline{F}_2$ and $(1+|x|^2)\overline{F}_2 \in L_\infty(\mathbb{R}^3)$; see [15, p. 396] e.g. Hence by virtue of Proposition 2.2, there exists a unique pair $(\tilde v,\ \tilde\pi)$ such that

$$\begin{cases} \tilde v \in \dot H^1_2(\mathbb{R}^3), \quad (1+|x|)\,\tilde v \in L_\infty(\mathbb{R}^3), \\ \nabla^2\tilde v,\ \nabla\tilde\pi \in L_2(\mathbb{R}^3), \quad \tilde\pi \in L_s(\mathbb{R}^3)\ \big(s > \tfrac{3}{2}\big); \\ L\tilde v + \nabla\tilde\pi = \overline{f}, \quad \operatorname{div}\tilde v = 0 \quad \text{in } \mathbb{R}^3. \end{cases} \tag{3.26}$$

Let us now define $(v_0, \pi_0) = (\overline{v} - \tilde v, \overline{\pi} - \tilde\pi)$. Then by virtue of (3.25) and (3.26), we have

$$\begin{cases} v_0 \in \dot H^1_{q_1,r_1}(\mathbb{R}^3) + \dot H^1_{q_2,r_2}(\mathbb{R}^3) + \dot H^1_2(\mathbb{R}^3); \\ \pi_0 \in L_{q_1,r_1}(\mathbb{R}^3) + L_{q_2,r_2}(\mathbb{R}^3) + L_2(\mathbb{R}^3); \\ Lv_0 + \nabla\pi_0 = 0, \quad \operatorname{div} v_0 = 0 \quad \text{in } \mathbb{R}^3. \end{cases}$$

Since $\operatorname{div} v_0 = 0$, it follows that $\Delta \pi_0 = 0$ in $\mathbb{R}^3$. Hence using Lemma 2.2, we easily deduce that $\pi_0 = 0$ and $v_0 = 0$. This proves that $(\overline{v}, \overline{\pi}) = (\tilde{v}, \tilde{\pi})$ in $\mathbb{R}^3$. Moreover, since $(v, \pi) = (\overline{v}, \overline{\pi})$ for $|x| \geq R - 2$, we conclude from (3.23) and (3.26) that (v, π) satisfies the regularity (3.21).

Next, to prove the energy equality (3.22), we apply the method of cut-off functions with an anisotropic decay. Following Galdi-Silvestre [15, Lemma 3], we choose a number $\alpha > 1$ and a non-increasing function $\tilde{\psi} \in C^\infty([0, \infty))$ with $\tilde{\psi}(t) = 1$ for $0 \leq t \leq 1$ and $\tilde{\psi}(t) = 0$ for $t \geq 2$. For any $\varrho \geq R$, we then define

$$\psi_\varrho(x) = \tilde{\psi}\left(\sqrt{\frac{x_1^2 + x_2^2}{\varrho^2} + \frac{x_3^2}{\varrho^{2\alpha}}}\right) \quad \text{for } x \in \mathbb{R}^3.$$

By direct calculations, we easily obtain

$$(\omega \wedge x) \cdot \nabla \psi_\varrho(x) = -c x_2 \frac{\partial \psi_\varrho}{\partial x_1}(x) + c x_1 \frac{\partial \psi_\varrho}{\partial x_2}(x) = 0,$$

$$\left|\frac{\partial \psi_\varrho}{\partial x_1}(x)\right| + \left|\frac{\partial \psi_\varrho}{\partial x_2}(x)\right| \leq \frac{C}{\varrho} \quad \text{and} \quad \left|\frac{\partial \psi_\varrho}{\partial x_3}(x)\right| \leq \frac{C}{\varrho^\alpha} \tag{3.27}$$

for all $x \in \mathbb{R}^3$, where the constant C depends only on α and ψ. Hence multiplying the system in (3.20) by $\psi_\varrho v$ and integrating by parts, we have

$$\begin{aligned} &\int_\Omega |\nabla v|^2 \psi_\varrho \, dx + \int_{\Sigma_\varrho} (\nabla v \cdot \nabla \psi_\varrho) \cdot v - \frac{1}{2}|v|^2 U \cdot \nabla \psi_\varrho \, dx \\ &\quad = \int_{\Sigma_\varrho} \pi (v \cdot \nabla \psi_\varrho) - (F \cdot \nabla \psi_\varrho) \cdot v \, dx - \int_\Omega (F \cdot \nabla v) \psi_\varrho \, dx, \end{aligned}$$

where Σ_ϱ denotes the support of $\nabla \psi_\varrho$. It follows immediately from (3.21) that

$$\int_\Omega |\nabla v|^2 \psi_\varrho \, dx \to \int_\Omega |\nabla v|^2 \, dx \quad \text{and} \quad \int_\Omega (F \cdot \nabla v) \psi_\varrho \, dx \to \int_\Omega (F \cdot \nabla v) \, dx$$

as $\varrho \to \infty$. To treat the remaining terms, we observe that

$$\begin{aligned} \Sigma_\varrho &\subset \left\{ x = (x', x_3) \in \mathbb{R}^3 : \varrho^2 \leq |x'|^2 + \varrho^{2-2\alpha} x_3^2 \leq 4\varrho^2 \right\} \\ &\subset \left\{ x : |x_3| \leq 2\varrho^\alpha, \quad \varrho^2 - \varrho^{2-2\alpha} x_3^2 \leq |x'|^2 \leq 4\varrho^2 - \varrho^{2-2\alpha} x_3^2 \right\}. \end{aligned}$$

Hence for all large ϱ with $\rho^{2-2\alpha} < 1/2$

$$\begin{aligned}\int_{\Sigma_\varrho} \frac{1}{|x|^2}\,dx &\le 2\int_0^{2\varrho^\alpha}\int_{\varrho^2-\varrho^{2-2\alpha}x_3^2}^{4\varrho^2-\varrho^{2-2\alpha}x_3^2} \frac{2\pi r}{r^2+x_3^2}\,drdx_3\\ &= 2\pi\int_0^{2\varrho^\alpha} \ln\left(\frac{4\varrho^2+(1-\varrho^{2-2\alpha})x_3^2}{\varrho^2+(1-\varrho^{2-2\alpha})x_3^2}\right)dx_3\\ &\le 2\pi\int_0^{2\varrho^\alpha}\ln\left(\frac{4\varrho^2+x_3^2/2}{\varrho^2+x_3^2/2}\right)dx_3\\ &= 2\pi\int_0^{2\varrho^{\alpha-1}}\ln\left(\frac{4+t^2/2}{1+t^2/2}\right)\varrho\,dt\\ &\le 2\pi\rho\int_0^\infty \ln\left(\frac{4+t^2/2}{1+t^2/2}\right)dt = C\varrho,\end{aligned}$$

where C is an absolute constant. Using this estimate together with (3.21) and (3.27), we thus obtain

$$\begin{aligned}&\int_{\Sigma_\varrho} |(\nabla v\cdot\nabla\psi_\varrho)\cdot v| + |\pi(v\cdot\nabla\psi_\varrho)| + |(F\cdot\nabla\psi_\varrho)\cdot v|\,dx\\ &\qquad\le C\left(\int_{\Sigma_\varrho}|\nabla\psi_\varrho|^2|v|^2\,dx\right)^{1/2} \le \frac{C}{\varrho}\left(\int_{\Sigma_\varrho}\frac{1}{|x|^2}\,dx\right)^{1/2} \le \frac{C}{\varrho^{1/2}}\to 0\end{aligned}$$

and

$$\int_{\Sigma_\varrho}|v|^2\,|U\cdot\nabla\psi_\varrho|\,dx \le \frac{C}{\varrho^\alpha}\int_{\Sigma_\varrho}\frac{1}{|x|^2}\,dx \le \frac{C}{\varrho^{\alpha-1}}\to 0$$

as $\varrho\to\infty$. This proves the energy equality (3.22). We have completed the proof of Lemma 3.3. □

We next prove the a priori estimate of Theorem 1.3.

Lemma 3.4. *Let (q,r) satisfy either $1<q<3, 1\le r\le\infty$ or $(q,r)=(3,1)$. Then for $M>0$, there is a positive constant $C=C(\Omega,q,r,M)$ such that if $|\omega|+|U|\le M$, $f\in\dot H^{-1}_{q,r}(\Omega)$, $g\in L_{q,r}(\Omega)$ and $(U-\omega\wedge x)g\in\dot H^{-1}_{q,r}(\Omega)$, and if (v,π) is a weak solution of (S) in $\dot H^1_{q,r}(\Omega)\times L_{q,r}(\Omega)$, then*

$$\|\nabla v\|_{q,r}+\|\pi\|_{q,r}\le C\left(\|f\|_{-1,q,r}+\|g\|_{q,r}+\|(U-\omega\wedge x)g\|_{-1,q,r}\right). \tag{3.28}$$

Proof. Let us define

$$\begin{cases} v_1=\psi v,\\ \pi_1=\psi\pi\end{cases}\quad\text{and}\quad\begin{cases} v_2=(1-\psi)v,\\ \pi_2=(1-\psi)\pi,\end{cases}$$

where ψ is the same cut-off function as in the proof of Lemma 3.3. Then it follows from Lemma 3.2 (a) that for each $i=1,2$, $(v_i,\pi_i)\in\dot H^1_{q,r}(\Omega_i)\times L_{q,r}(\Omega_i)$, where

$\Omega_1 = \Omega_R$ and $\Omega_2 = \mathbb{R}^3$. Moreover, the pair (v_i, π_i) satisfies

$$(S)_{\Omega_i} \qquad \begin{cases} Lv_i + \nabla\pi_i = f_i & \text{in } \Omega_i, \\ \operatorname{div} v_i = g_i & \text{in } \Omega_i, \end{cases}$$

where

$$\begin{gathered} f_1 = \psi\, f + f_0, \quad f_2 = (1-\psi)\, f - f_0, \\ f_0 = -2\nabla\psi \cdot \nabla v + [-\Delta\psi + (U - \omega \wedge x)\cdot\nabla\psi]\, v + \pi\nabla\psi, \\ g_1 = \psi\, g + g_0, \quad g_2 = (1-\psi)g - g_0 \quad \text{and} \quad g_0 = \nabla\psi\cdot v. \end{gathered}$$

By virtue of Propositions 2.1 and 2.3, we obtain

$$\begin{aligned} &\|\nabla v_1\|_{q,r;\Omega_R} + \|\pi_1\|_{q,r;\Omega_R} \\ &\quad\le\ C\left(\|f_1\|_{-1,q,r;\Omega_R} + \|g_1\|_{q,r;\Omega_R} + \left|\int_{\Omega_R} \pi_1\, dx\right|\right) \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} &\|\nabla v_2\|_{q,r;\mathbb{R}^3} + \|\pi_2\|_{q,r;\mathbb{R}^3} \\ &\quad\le\ C\left(\|f_2\|_{-1,q,r;\mathbb{R}^3} + \|g_2\|_{q,r;\mathbb{R}^3} + \|(U-\omega\wedge x)g_2\|_{-1,q,r;\mathbb{R}^3}\right). \end{aligned} \tag{3.30}$$

Since $\nabla\psi$ is supported in $A_R = B_R \setminus \overline{B}_{R-5}$, it follows that

$$\operatorname{supp} f_0 \cup \operatorname{supp} g_0 \subset A_R.$$

Moreover, adapting the proof of Lemma 3.2, we deduce that

$$\|f_0\|_{-1,q,r;\mathbb{R}^3} \le C\left(\|v\|_{q,r;\Omega_R} + \|\pi\|_{-1,q,r;\Omega_R}\right)$$

and

$$\|g_0\|_{-1,q,r;\mathbb{R}^3} + \|(U-\omega\wedge x)g_0\|_{-1,q,r;\mathbb{R}^3} \le C\|v\|_{q,r;\Omega_R}.$$

Using these together with Lemma 3.2 (c), we thus have

$$\begin{aligned} &\|f_1\|_{-1,q,r;\Omega_R} + \|f_2\|_{-1,q,r;\mathbb{R}^3} + \left|\int_{\Omega_R}\pi_1\, dx\right| \\ &\qquad\le C\left(\|f\|_{-1,q,r} + \|v\|_{q,r;\Omega_R} + \|\pi\|_{-1,q,r;\Omega_R}\right) \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} &\|g_1\|_{-1,q,r;\Omega_R} + \|g_2\|_{-1,q,r;\mathbb{R}^3} + \|(U-\omega\wedge x)g_2\|_{-1,q,r;\mathbb{R}^3} \\ &\qquad\le C\left(\|g\|_{q,r} + \|(U-\omega\wedge x)g\|_{-1,q,r} + \|v\|_{q,r;\Omega_R}\right). \end{aligned} \tag{3.32}$$

Substituting (3.31) and (3.32) into (3.29) and (3.30), we have derived

$$\begin{aligned} \|\nabla v\|_{q,r} + \|\pi\|_{q,r} \le\ & C\left(\|f\|_{-1,q,r} + \|g\|_{q,r} + \|(U-\omega\wedge x)g\|_{-1,q,r}\right. \\ &\left. + \|v\|_{q,r;\Omega_R} + \|\pi\|_{-1,q,r;\Omega_R}\right). \end{aligned} \tag{3.33}$$

We can now deduce the desired estimate (3.28) from (3.33) by using the uniqueness result, Lemma 3.3. To do so, we argue by contradiction. Suppose thus that there are sequences $\{U_n\}, \{\omega_n\} \subset \mathbb{R}^3$, $\{f_n\} \subset \dot{H}^{-1}_{q,r}(\Omega)$, $\{g_n\} \subset L_{q,r}(\Omega)$ $\{v_n\} \subset \dot{H}^1_{q,r}(\Omega)$ and $\{\pi_n\} \subset L_{q,r}(\Omega)$ such that

$$|U_n| + |\omega_n| \le M; \quad L_n v_n + \nabla \pi_n = f_n, \quad \operatorname{div} v_n = g_n \quad \text{in } \Omega,$$

where $L_n v = -\Delta v + (U_n - \omega_n \wedge x) \cdot \nabla v + \omega_n \wedge v$, and

$$1 = \|\nabla v_n\|_{q,r} + \|\pi_n\|_{q,r} \ge n \left(\|f_n\|_{-1,q,r} + \|g_n\|_{q,r} + \|(U_n - \omega_n \wedge x) g_n\|_{-1,q,r}\right).$$

Then from the a priori estimate (3.33), we deduce that

$$1 \le C \left(\frac{1}{n} + \|v_n\|_{q,r;\Omega_R} + \|\pi_n\|_{-1,q,r;\Omega_R} \right). \tag{3.34}$$

Moreover, by standard compactness results, we may assume that $\{(U_n, \omega_n)\} \to \{(U, \omega)\}$ in $\mathbb{R}^3 \times \mathbb{R}^3$ and $\{(v_n, \pi_n)\} \to \{(v, \pi)\}$ in the weak-* topology of $\dot{H}^1_{q,r}(\Omega) \times L_{q,r}(\Omega)$. It is then easy to check that (v, π) is a weak solution in $\dot{H}^1_{q,r}(\Omega) \times L_{q,r}(\Omega)$ of (S) with the trivial data $(f, g) = (0, 0)$. Hence by Lemma 3.3, we must have $(v, \pi) = (0, 0)$. However, since the embeddings $\dot{H}^1_{q,r}(\Omega) \hookrightarrow L_{q,r}(\Omega_R) \hookrightarrow \dot{H}^{-1}_{q,r}(\Omega_R)$ are compact, it follows that $\{(v_n, \pi_n)\} \to \{(v, \pi)\}$ strongly in $L_{q,r}(\Omega_R) \times \dot{H}^{-1}_{q,r}(\Omega_R)$. Hence letting $n \to \infty$ in (3.34), we obtain

$$1 \le C \left(\|v\|_{q,r;\Omega_R} + \|\pi\|_{-1,q,r;\Omega_R}\right),$$

which is a contradiction. The proof of Lemma 3.4 is complete. □

3.3. **Proof of the existence.** We finally prove the existence assertion of Theorem 1.3 by using suitable cut-off functions; see also [8], [17] and [19].

Let ψ be the cut-off function satisfying (3.24). We also choose cut-off functions $\phi_1, \phi_2 \in C^\infty(\mathbb{R}^3)$ such that

$$\phi_1(x) = \begin{cases} 1, & |x| \le R-2, \\ 0, & |x| \ge R-1 \end{cases} \quad \text{and} \quad \phi_2(x) = \begin{cases} 0, & |x| \le R-4, \\ 1, & |x| \ge R-3. \end{cases} \tag{3.35}$$

Then it is obvious that $\psi\phi_1 + (1-\psi)\phi_2 = 1$ on $\mathbb{R}^3$ and $\nabla\psi, \nabla\phi_1, \nabla\phi_2$ are all supported in the annulus $A_R = B_R \setminus \overline{B}_{R-5}$.

Let (q, r) be a fixed pair satisfying one of the three conditions in (1.7). Denote by X the space of all pairs $(f, g) \in \dot{H}^{-1}_{q,r}(\Omega) \times L_{q,r}(\Omega)$ with $(U - \omega \wedge x) g \in \dot{H}^{-1}_{q,r}(\Omega)$. It is easy to show that X is a Banach space equipped with the norm

$$\|(f, g)\|_X = \|f\|_{-1,q,r} + \|g\|_{q,r} + \|(U - \omega \wedge x) g\|_{-1,q,r}.$$

Let $(f, g) \in X$ be given. For each $i = 1, 2$, we denote by $(S_i(f, g), P_i(f, g))$ the solution (v_i, π_i) in $\dot{H}^1_{q,r}(\Omega_i) \times L_{q,r}(\Omega_i)$ of $(S)_{\Omega_i}$ with data $(f_i, g_i) = (\phi_i f, \phi_i g)$, where $\Omega_1 = \Omega_R$ and $\Omega_2 = \mathbb{R}^3$ are as in the proof of Lemma 3.4. Then in view of Propositions 2.1, 2.3 and Lemma 3.2, we have

$$\sum_{i=1}^{2} \left(\|\nabla S_i(f, g)\|_{q,r;\Omega_i} + \|P_i(f, g)\|_{q,r;\Omega_i}\right) \le C \|(f, g)\|_X. \tag{3.36}$$

We define a *parametrix*, an approximate solution of (S), as

$$S(f,g) := \psi S_1(f,g) + (1-\psi)S_2(f,g),$$
$$P(f,g) := \psi P_1(f,g) + (1-\psi)P_2(f,g).$$

It then follows from Lemma 3.2 and (3.36) that

$$(S(f,g), P(f,g)) \in \dot{H}^1_{q,r}(\Omega) \times L_{q,r}(\Omega)$$

and

$$\|\nabla S(f,g)\|_{q,r} + \|P(f,g)\|_{q,r} \le C\|(f,g)\|_X \tag{3.37}$$

By a direct calculation, we also obtain

$$\begin{cases} LS(f,g) + \nabla P(f,g) = f + E_1(f,g) & \text{in } \Omega, \\ \operatorname{div} S(f,g) = g + E_2(f,g) & \text{in } \Omega, \end{cases}$$

where

$$\begin{aligned} E_1(f,g) = & -2\nabla\psi \cdot \nabla(S_1(f,g) - S_2(f,g)) + (P_1(f,g) - P_2(f,g))\nabla\psi \\ & + ((U - \omega \wedge x)\cdot\nabla\psi - \Delta\psi)\,(S_1(f,g) - S_2(f,g)) \end{aligned}$$

and

$$E_2(f,g) = \nabla\psi \cdot (S_1(f,g) - S_2(f,g)).$$

We now show that $E = (E_1, E_2)$ is a compact linear operator on X. To begin with, we observe that

$$\operatorname{supp} E_1(f,g) \cup \operatorname{supp} E_2(f,g) \subset \operatorname{supp} \nabla\psi \subset A_R,$$

which implies in particular that

$$\|E_2(f,g)\|_{q,r} = \|E_2(f,g)\|_{q,r;A_R} \le C\|\nabla E_2(f,g)\|_{q,r;A_R}. \tag{3.38}$$

On the other hand, it follows from Lemma 3.1 that if $1 < s < 3$ or $(s,t) = (3,1)$, then

$$\|\varphi\|_{s,t;A_R} \le \begin{cases} C\|\nabla\varphi\|_{s,t} & \text{for all } \varphi \in \dot{H}^1_{s,t}(\Omega), \\ C\|\nabla\varphi\|_{s,t;\Omega_i} & \text{for all } \varphi \in \dot{H}^1_{s,t}(\Omega_i);\ i = 1,2. \end{cases} \tag{3.39}$$

Using (3.39) and (3.36), we have

$$\begin{aligned} &\|E_1(f,g)\|_{q,r;A_R} + \|\nabla E_2(f,g)\|_{q,r;A_R} \\ &\quad \le C\|\nabla(S_1(f,g) - S_2(f,g))\|_{q,r;A_R} + C\|P_1(f,g) - P_2(f,g)\|_{q,r;A_R} \\ &\qquad + C\|S_1(f,g) - S_2(f,g)\|_{q,r;A_R} \\ &\quad \le \sum_{i=1}^{2} (\|\nabla S_i(f,g)\|_{q,r;\Omega_i} + \|P_i(f,g)\|_{q,r;\Omega_i}) \le C\|(f,g)\|_X. \end{aligned} \tag{3.40}$$

Let us then choose any $\xi \in C_0^\infty(A_R)$ with $\xi = 1$ on $\operatorname{supp} \nabla\psi$. Recall also from (1.7) that (q', r') indeed satisfies either $1 < q' < 3$ or $(q', r') = (3, 1)$. Hence using (3.39) again, we have

$$\begin{aligned}\langle E_1(f,g), \varphi\rangle &= \langle E_1(f,g), \xi\varphi\rangle \\ &\le \|E_1(f,g)\|_{-1,q,r;A_R} \|\nabla(\xi\varphi)\|_{q',r';A_R} \\ &\le C\|E_1(f,g)\|_{-1,q,r;A_R} \left(\|\varphi\|_{q',r';A_R} + \|\nabla\varphi\|_{q',r'}\right) \\ &\le C\|E_1(f,g)\|_{-1,q,r;A_R} \|\nabla\varphi\|_{q',r'}\end{aligned}$$

and similarly

$$\langle (U - \omega \wedge x)E_2(f,g), \varphi\rangle \le C\|E_2(f,g)\|_{-1,q,r;A_R} \|\nabla\varphi\|_{q',r'}$$

for all $\varphi \in C_0^\infty(\Omega)$, which implies that

$$\begin{aligned}&\|E_1(f,g)\|_{-1,q,r} + \|(U - \omega \wedge x)E_2(f,g)\|_{-1,q,r} \\ &\qquad \le C\left(\|E_1(f,g)\|_{-1,q,r;A_R} + \|E_2(f,g)\|_{-1,q,r;A_R}\right).\end{aligned} \tag{3.41}$$

Finally, since the embeddings $\dot{H}^1_{q,r}(A_R) \hookrightarrow L_{q,r}(A_R) \hookrightarrow \dot{H}^{-1}_{q,r}(A_R)$ are compact, it follows from (3.38), (3.40) and (3.41) that E maps X into X compactly and of course linearly.

We now show that $Id + E$ is injective on X. Suppose that $(f,g) \in X$ and $(Id + E)(f,g) = 0$. Then since $(S(f,g), P(f,g))$ is a solution in $\dot{H}^1_{q,r}(\Omega) \times L_{q,r}(\Omega)$ of (S) with trivial data, it follows from the uniqueness result in Lemma 3.3 that $(S(f,g), P(f,g)) = (0,0)$ in Ω. From the definitions of $S(f,g)$ and $P(f,g)$, it follows that $(S_1(f,g), P_1(f,g)) = (0,0)$ in Ω_{R-3} and $(S_2(f,g), P_2(f,g)) = (0,0)$ in $\mathbb{R}^3 \setminus \overline{B}_{R-2}$. Arguing as in the proof of Lemma 3.2, we easily deduce that $(S_i(f,g), P_i(f,g)) \in \dot{H}^1_{q,r}(B_R) \times L_{q,r}(B_R)$ for each $i = 1, 2$, where $(S_1(f,g), P_1(f,g))$ is extended to B_R by defining zero outside Ω_R. Moreover, by the definitions of $S_i(f,g)$ and $P_i(f,g)$, we deduce that $(f_1, g_1) = (0,0)$ in Ω_{R-3} and $(f_2, g_2) = (0,0)$ in $\mathbb{R}^3 \setminus \overline{B}_{R-2}$. It then follows from (3.35) that $(f,g) = (0,0)$ in $\Omega_{R-3} \cup (\Omega \setminus \overline{B}_{R-2})$ and $(f_1, g_1) = (f_2, g_2) = (f,g)$ in Ω_R. Hence both $(S_1(f,g), P_1(f,g))$ and $(S_2(f,g), P_2(f,g))$ are solutions in $\dot{H}^1_{q,r}(B_R) \times L_{q,r}(B_R)$ of the problem

$$Lv + \nabla\pi = f, \quad \operatorname{div} v = g \quad \text{in } B_R; \quad v = 0 \quad \text{on } \partial B_R.$$

It thus follows from Proposition 2.3 that $S_1(f,g) = S_2(f,g)$ and $\nabla P_1(f,g) = \nabla P_2(f,g)$ in B_R. Finally, using the definitions of $S(f,g)$ and $P(f,g)$ again, we deduce that $S_i(f,g) = S(f,g) = (0,0)$ and $P_i(f,g) = P(f,g) = 0$ in Ω_R for $i = 1, 2$. This allows us to conclude that $(f,g) = (0,0)$ in Ω_R and so in the whole domain Ω. We have shown the injectivity of $Id + E$ on X. Therefore, it follows from the Fredholm theory that $Id + E$ has a bounded inverse.

Given $(f,g) \in X$, we define $(\overline{f}, \overline{g}) = (Id + E)^{-1}(f,g)$. Then a solution in $\dot{H}^1_{q,r}(\Omega) \times L_{q,r}(\Omega)$ of (S) is given by $(v, \pi) = (S(\overline{f}, \overline{g}), P(\overline{f}, \overline{g}))$. This proves the existence assertion of Theorem 1.3. We have completed the proof of Theorem 1.3. $\square$

4. Proofs of the main theorems

To begin with, we deduce two useful results from Theorem 1.3 and Lemma 3.3.

Theorem 4.1. *Assume that $g \equiv 0$, and let (q, r) satisfy one of the three conditions in* (1.7). *Then for every $f = \operatorname{div} F$ with $F \in \Pi_{q,r}$, there exists a unique weak solution $(v, \pi) \in V_{q,r} \times \Pi_{q,r}$ of* (S).

Proof. The uniqueness assertion follows immediately from Theorem 1.3(or Lemma 3.3). To prove the existence, we use Theorem 1.3 again to deduce the existence of two weak solutions $(v_1, \pi_1) \in \dot{H}^1_{3/2,\infty}(\Omega) \times L_{3/2,\infty}(\Omega)$ and $(v_2, \pi_2) \in \dot{H}^1_{q,r}(\Omega) \times L_{q,r}(\Omega)$ of (S). Then $(v, \pi) = (v_1 - v_2, \pi_1 - \pi_2)$ satisfies

$$\begin{cases} v \in \dot{H}^1_{3/2,\infty}(\Omega) + \dot{H}^1_{q,r}(\Omega), \quad \pi \in L_{3/2,\infty}(\Omega) + L_{q,r}(\Omega); \\ \qquad Lv + \nabla\pi = 0, \quad \operatorname{div} v = 0 \quad \text{in } \Omega. \end{cases}$$

Hence it follows from Lemma 3.3 that $(v, \pi) = (0, 0)$ and so $(v_1, \pi_1) = (v_2, \pi_2) \in V_{q,r} \times \Pi_{q,r}$. This completes the proof of Theorem 4.1. □

Lemma 4.1. *Assume that $g \equiv 0$, $f = \operatorname{div} F$ and $F \in L_2(\Omega)$. If $(v, \pi) \in \dot{H}^1_2(\Omega) \times L_2(\Omega)$ is the weak solution of* (S) *obtained by Theorem 1.3, then v satisfies the energy equality:*

$$\int_\Omega |\nabla v|^2\,dx = -\int_\Omega F \cdot \nabla v\,dx.$$

Proof. Choose a sequence $\{F_k\}$ in $C_0^\infty(\Omega)$ with $F_k \to F$ in $L_2(\Omega)$. Then by virtue of Theorem 1.3, there exists a unique weak solution $(v_k, \pi_k) \in \dot{H}^1_2(\Omega) \times L_2(\Omega)$ of (S) with F replaced by F_k. Moreover, there is a constant $C = C(\Omega, |U| + |\omega|)$ such that $\|\nabla(v_k - v)\|_2 \le C\|F_k - F\|_2$. On the other hand, since $F_k \in C_0^\infty(\Omega)$, it follows from Lemma 3.3 that $\int_\Omega |\nabla v_k|^2\,dx = -\int_\Omega F_k \cdot \nabla v_k\,dx$. Letting $k \to \infty$, we obtain the energy equality. □

To prove the main results, we also need Kozono and Yamazaki's result [26, Proposition 2.1] for Hölder inequalities in Lorentz spaces.

Lemma 4.2. *Let $1 < q, q_1, q_2 < \infty$, $1 \le r_1, r_2 \le \infty$ and $1/q = 1/q_1 + 1/q_2$. If $f \in L_{q_1,r_1}(\Omega)$ and $g \in L_{q_2,r_2}(\Omega)$, then $fg \in L_{q,r}(\Omega)$, where $r = \min(r_1, r_2)$, and*

$$\|fg\|_{q,r} \le C\|f\|_{q_1,r_1}\|g\|_{q_2,r_2}$$

for some constant $C = C(q_1, r_1, q_2, r_2) > 0$.

An immediate consequence of Lemmas 4.1 and 4.2 is the following bilinear estimate whose easy proof is omitted.

Lemma 4.3. *Suppose that $1 < q < 3$, $1 \le r \le \infty$ and $v \in L_{3,\infty}(\Omega)$. Then for every $w \in \dot{H}^1_{q,r}(\Omega)$, we have*

$$v \otimes w \in L_{q,r}(\Omega) \quad \text{and} \quad \|v \otimes w\|_{q,r} \le C\|v\|_{3,\infty}\|\nabla w\|_{q,r}$$

for some constant $C = C(q,r) > 0$. *Moreover, if* $\operatorname{div} v = 0$ *in* Ω *and* $w \in \dot{H}^1_2(\Omega)$, *then*

$$\int_\Omega v \otimes w : \nabla w \, dx = 0.$$

We can now prove an existence result which will play a crucial role in our proofs of both Theorems 1.1 and 1.2.

Proposition 4.1. *Suppose that either* $(q,r) = (3/2, \infty)$ *or* $3/2 < q < 3, 1 \le r \le \infty$. *Then there are positive constants* $\delta_0' = \delta_0'(\Omega, q, r)$, $C_0 = C_0'(\Omega)$ *and* $C_0'' = C_0''(\Omega, q, r)$ *such that if* $F \in \Pi_{q,r}$ *and* $|U| + |\omega| + \|F\|_{3/2,\infty} \le \delta_0'$, *then there exists at least one weak solution* $(v, \pi) \in V_{q,r} \times \Pi_{q,r}$ *of* (NS) *satisfying the estimates*

$$\|v\|_{3,\infty} + \|\nabla v\|_{3/2,\infty} + \|\pi\|_{3/2,\infty} \le C_0' \left(|U| + |\omega| + \|F\|_{3/2,\infty}\right) \tag{4.42}$$

and

$$\|v\|_{q^*,r} + \|\nabla v\|_{q,r} + \|\pi\|_{q,r} \le C_0'' \left(|U| + |\omega| + \|F\|_{q,r}\right). \tag{4.43}$$

Proof. We may assume that $|U| + |\omega| + \|F\|_{3/2,\infty} \le 1$. Let $v \in V_{q,r}$ be fixed. Then since $v \in \dot{H}^1_{3/2,\infty}(\Omega) \hookrightarrow L_{3,\infty}(\Omega)$, it follows from (1.3), (1.4), Lemmas 4.3 and 3.1 that

$$\|Q_b(v)\|_{3/2,\infty} \le C \left(|U| + |\omega| + \|\nabla v\|^2_{3/2,\infty}\right) \tag{4.44}$$

and

$$\|Q_b(v)\|_{q,r} \le C_{q,r} \left(|U| + |\omega|\right) + C_{q,r} \left(|U| + |\omega| + \|\nabla v\|_{3/2,\infty}\right) \|\nabla v\|_{q,r} \tag{4.45}$$

for some $C = C(\Omega)$ and $C = C(\Omega, q, r)$. Hence by Theorem 4.1, there exists a unique $\overline{v} = \mathcal{T}(v) \in V_{q,r}$ such that for some unique $\overline{\pi} \in \Pi_{q,r}(\Omega)$, the pair $(\overline{v}, \overline{\pi})$ is a weak solution of the following linear problem:

$$\begin{cases} L\overline{v} + \nabla \overline{\pi} = \operatorname{div}\left(F - Q_b(v)\right) & \text{in } \Omega, \\ \operatorname{div} \overline{v} = 0 & \text{in } \Omega, \\ \overline{v} = 0 & \text{on } \partial\Omega, \\ \overline{v}(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{4.46}$$

Moreover, it follows from Theorem 1.3, (4.44) and (4.45) that for all $v, v_1, v_2 \in V_{q,r}$, we have

$$\begin{aligned} \|\nabla \mathcal{T}(v)\|_{3/2,\infty} &\le C \|F - Q_b(v)\|_{3/2,\infty} \\ &\le C^* \left(|U| + |\omega| + \|F\|_{3/2,\infty}\right) + C^* \|\nabla v\|^2_{3/2,\infty}, \end{aligned}$$

$$\begin{aligned} \|\nabla \mathcal{T}(v)\|_{q,r} &\le C \|F - Q_b(v)\|_{q,r} \\ &\le C^*_{q,r} \left(|U| + |\omega| + \|F\|_{q,r}\right) \\ &\quad + C^*_{q,r} \left(|U| + |\omega| + \|\nabla v\|_{3/2,\infty}\right) \|\nabla v\|_{q,r} \end{aligned}$$

and

$$\begin{aligned}
&\|\mathcal{T}(v_1) - \mathcal{T}(v_2)\|_{V_{q,r}} \\
&\qquad \le C\|Q_b(v_1) - Q_b(v_2)\|_{\Pi_{q,r}} \\
&\qquad \le C^*_{q,r}\left(|U| + |\omega| + \|\nabla v_1\|_{3/2,\infty} + \|\nabla v_2\|_{3/2,\infty}\right)\|v_1 - v_2\|_{V_{q,r}}
\end{aligned}$$

for some constants $C^* = C^*(\Omega)$ and $C^*_{q,r} = C^*_{q,r}(\Omega, q, r)$ with $C^*_{q,r} \ge C^* > 1$.

Let us now suppose that

$$|U| + |\omega| + \|F\|_{3/2,\infty} < \delta_0' \equiv \frac{1}{8}\left(C^*_{q,r}\right)^{-2}, \tag{4.47}$$

and let B be the closed set of all $v \in V_{q,r}$ such that

$$\|\nabla v\|_{3/2,\infty} \le 2C^*\left(|U| + |\omega| + \|F\|_{3/2,\infty}\right)$$

and

$$|U| + |\omega| + \|\nabla v\|_{q,r} \le 2C^*_{q,r}\left(|U| + |\omega| + \|F\|_{q,r}\right).$$

Then for all $v, v_1, v_2 \in B$, we easily obtain

$$\|\nabla \mathcal{T}(v)\|_{3/2,\infty} \le 2C^*\left(|U| + |\omega| + \|F\|_{3/2,\infty}\right),$$

$$\|\nabla \mathcal{T}(v)\|_{q,r} \le 2C^*_{q,r}\left(|U| + |\omega| + \|F\|_{q,r}\right)$$

and

$$\|\mathcal{T}(v_1) - \mathcal{T}(v_2)\|_{V_{q,r}} \le \frac{1}{2}\|v_1 - v_2\|_{V_{q,r}}.$$

Therefore, $\mathcal{T}$ is a contraction on the complete metric space B and thus has a fixed point v in B by the Banach fixed point theorem.

To complete the proof, it remains to derive the estimates (4.42) and (4.43) for the solution (v, π) of (NS), where π is the pressure associated with v. To do so, we can argue as before using Theorem 1.3, (4.44), (4.45) and (4.47) together with the fact that $v = \mathcal{T}(v) \in B$. Indeed, we have

$$\begin{aligned}
\|v\|_{3,\infty} + \|\nabla v\|_{3/2,\infty} + \|\pi\|_{3/2,\infty} &\le C\|F - Q_b(v)\|_{3/2,\infty} \\
&\le C\left(|U| + |\omega| + \|F\|_{3/2,\infty}\right) + C\|\nabla v\|^2_{3/2,\infty} \\
&\le C_0'\left(|U| + |\omega| + \|F\|_{3/2,\infty}\right)
\end{aligned}$$

for some $C_0' = C_0'(\Omega) > 0$, which proves (4.42). The proof of (4.43) is similar and omitted. This completes the proof of Proposition 4.1. □

We are now ready to prove Theorems 1.1, 1.2 and their corollaries.

Proof of Theorem 1.1. Let δ_0 be any positive number less than or equal to the small constant $\delta_0' = \delta_0'(\Omega, 3/2, \infty)$ in Proposition 4.1. Suppose that $F \in L_{3/2,\infty}(\Omega) = \Pi_{3/2,\infty}$ and $|U| + |\omega| + \|F\|_{3/2,\infty} \le \delta_0$. Then by Proposition 4.1, there exists at least one solution $(v_1, \pi_1) \in \dot{H}^1_{3/2,\infty}(\Omega) \times L_{3/2,\infty}(\Omega)$ of (NS) satisfying the estimate (4.42).

Let $(v_2,\pi_2)\in \dot H^1_{3/2,\infty}(\Omega)\times L_{3/2,\infty}(\Omega)$ be a solution of (NS) which is possibly different from (v_1,π_1). Then arguing as in the proof of Proposition 4.1, we have

$$\|\nabla(v_1-v_2)\|_{3/2,\infty}+\|\pi_1-\pi_2\|_{3/2,\infty}\le C\|Q_b(v_1)-Q_b(v_2)\|_{3/2,\infty}$$

for some $C=C(\Omega)$. By Lemma 4.3, we also have

$$\begin{aligned}&C\|Q_b(v_1)-Q_b(v_2)\|_{3/2,\infty}\\&\qquad\le C'\left(|U|+|\omega|+\|v_1\|_{3,\infty}+\|v_2\|_{3,\infty}\right)\|\nabla(v_1-v_2)\|_{3/2,\infty}\end{aligned}$$

for some $C'=C'(\Omega)>1$. Therefore, taking

$$\delta_0=\min\left(\delta_0',\frac{1}{4C'}\right)\quad\text{and}\quad\varepsilon_0=\frac{1}{4C'}$$

and assuming that

$$\|v_1\|_{3,\infty},\|v_2\|_{3,\infty}\le\varepsilon_0,$$

we conclude that $(v_1,\pi_1)=(v_2,\pi_2)$. This completes the proof of Theorem 1.1. □

Proof of Corollary 1.1. Since the constant C_0 in the estimate (1.6) is independent of n, it follows from Alaoglu's compactness theorem that there exist a subsequence of $\{(v_n,\pi_n)\}$, which we denote by $\{(v_n,\pi_n)\}$ again, and a pair $(\tilde v,\tilde\pi)$ in $\dot H^1_{3/2,\infty}(\Omega)\times L_{3/2,\infty}(\Omega)$ such that $v_n\to\tilde v$ weakly-* in $L_{3,\infty}(\Omega)$ and $(\nabla v_n,\pi_n)\to(\nabla\tilde v,\tilde\pi)$ weakly-* in $L_{3/2,\infty}(\Omega)$. It is easy to show that $\tilde v$ also satisfies the estimate (1.5). On the other hand, it is well-known that $\dot H^1_{3/2,\infty}(\Omega)\hookrightarrow H^1_q(\Omega')=\{u\in L_q(\Omega'):\nabla u\in L_q(\Omega')\}$ for any $q<3/2$ and bounded $\Omega'\subset\Omega$. Hence it follows from the Rellich compactness theorem that $v_n\to\tilde v$ in $L_q(\Omega')$ for any $q<3$ and bounded $\Omega'\subset\Omega$. These convergence properties enable us to deduce, by a standard argument, that $(\tilde v,\tilde\pi)$ is a weak solution in $\dot H^1_{3/2,\infty}(\Omega)\times L_{3/2,\infty}(\Omega)$ of (NS) with data (F,U,ω). But since both (v,π) and $(\tilde v,\tilde\pi)$ satisfy the smallness condition (1.5), it follows from the uniqueness assertion of Theorem 1.1 that $(v,\pi)=(\tilde v,\tilde\pi)$.

In fact, the above argument yields that every subsequence of $\{(v_n,\pi_n)\}$ has a subsequence that converges weakly-* in $\dot H^1_{3/2,\infty}(\Omega)\times L_{3/2,\infty}(\Omega)$ to the same limit (v,π). Therefore by a standard contradiction argument using Alaoglu's compactness theorem we easily deduce the convergence of the full sequence $\{(v_n,\pi_n)\}$ to (v,π). □

Proof of Theorem 1.2. Let $3/2<q<3$ and $1\le r\le\infty$, and suppose that $F\in\Pi_{q,r}$ and $|U|+|\omega|+\|F\|_{3/2,\infty}<\delta_0'=\delta_0'(\Omega,q,r)$. Then by Proposition 4.1, there exists at least one solution $(v_1,\pi_1)\in V_{q,r}\times\Pi_{q,r}$ of (NS) satisfying the estimates (4.42) and (4.43). This proves the existence assertion of the theorem, in particular. To prove the uniqueness, let us suppose that $(v_2,\pi_2)\in V_{q,r}\times\Pi_{q,r}$ is a solution of (NS) which is possibly different from (v_1,π_1). Let us define $(v,\pi)\in V_{q,r}\times\Pi_{q,r}$ by

$$(v,\pi)=(v_1-v_2,\pi_1-\pi_2).$$

Then (v,π) is a solution in $V_{q,r}\times\Pi_{q,r}$ of the following linear problem:

$$\begin{cases} Lv+\nabla\pi=\operatorname{div}G & \text{in } \Omega,\\ \operatorname{div}v=0 & \text{in } \Omega,\\ v=0 & \text{on } \partial\Omega,\\ v(x)\to 0 & \text{as } |x|\to\infty, \end{cases} \tag{4.48}$$

where

$$G=Q_b(v_2)-Q_b(v_1)=v\otimes(v_1+b)+(v_2+b)\otimes v.$$

Assume for the moment that (v,π) has the following additional regularity

$$(v,\pi)\in \dot H^1_2(\Omega)\times L_2(\Omega). \tag{4.49}$$

Then since $v_1,v_2\in L_{3,\infty}(\Omega)$ and $b\in C^\infty_{0,\sigma}(\Omega)$, it follows from Lemmas 4.3 and 4.1 that

$$\begin{aligned}\|\nabla v\|_2^2=-\int_\Omega G:\nabla v\,dx&=-\int_\Omega v\otimes(v_1+b):\nabla v\,dx\\ &\le C\left(|U|+|\omega|+\|v_1\|_{3,\infty}\right)\|\nabla v\|_2^2\end{aligned}$$

for some C. Since (v_1,π_1) satisfies the estimate (4.42), we thus obtain

$$\|\nabla v\|_2^2\le C_0\left(|U|+|\omega|+\|F\|_{3/2,\infty}\right)\|\nabla v\|_2^2$$

for some $C_0=C_0(\Omega)$. Therefore, assuming that

$$|U|+|\omega|+\|F\|_{3/2,\infty}<\delta\equiv\min(\delta_0,1/2C_0),$$

we conclude that $\|\nabla v\|_2^2=0$ and so $(v_1,\pi_1)=(v_2,\pi_2)$ in Ω.

Therefore, to complete the uniqueness proof, it remains to prove (4.49). This can be shown by a bootstrap argument based on Theorems 1.3 and 4.1. First of all, noting that

$$v,v_1,v_2\in V_{q,r}\hookrightarrow V_{q,\infty}\hookrightarrow L_{3,\infty}(\Omega)\cap L_{q^*,\infty}(\Omega),$$

we deduce from Lemma 4.2 that $G\in L_{3/2,\infty}(\Omega)\cap L_{s_1,\infty}(\Omega)$, where $s_1=q^*/2>3/2$. Suppose that $s_1<3$. Then since $3/2<s_1<3$ and $G\in\Pi_{s_1,\infty}$, it follows from Theorems 1.3, 4.1, Lemmas 3.1 and 4.2 that

$$v\in V_{s_1,\infty}\hookrightarrow L_{3,\infty}(\Omega)\cap L_{s_1^*,\infty}(\Omega)\quad\text{and}\quad G\in L_{3/2,\infty}(\Omega)\cap L_{s_2,\infty}(\Omega),$$

where

$$\frac{1}{s_2}=\frac{1}{q^*}+\frac{1}{s_1^*}=\frac{1}{s_1}+\left(\frac{1}{q^*}-\frac{1}{3}\right)<\frac{1}{s_1}.$$

Similarly, if $s_2<3$, then we have

$$v\in V_{s_2,\infty}\quad\text{and}\quad G\in L_{3/2,\infty}(\Omega)\cap L_{s_3,\infty}(\Omega),$$

where

$$\frac{1}{s_3}=\frac{1}{q^*}+\frac{1}{s_2^*}=\frac{1}{s_2}+\left(\frac{1}{q^*}-\frac{1}{3}\right)<\frac{1}{s_2}.$$

Hence by a simple induction, we conclude that

$$v\in V_{s_j,\infty}\quad\text{and}\quad G\in L_{3/2,\infty}(\Omega)\cap L_{s_{j+1},\infty}(\Omega)$$

for all j with $s_j < 3$, where $\{s_j\}$ is a sequence defined recursively by

$$s_1 = \frac{q^*}{2} \quad \text{and} \quad \frac{1}{s_{j+1}} = \frac{1}{q^*} + \frac{1}{s_j^*} = \frac{1}{s_j} + \left(\frac{1}{q^*} - \frac{1}{3}\right) \quad (j \geq 1).$$

Since $3 < q^* < \infty$, it follows that $0 < 1/s_1 < 2/3$ and $1/s_j > 1/s_{j+1} > 1/s_j - 1/3$ for all $j \geq 1$. Hence there exists the smallest $j = j_0 \geq 1$ such that $0 < 1/s_j \leq 1/3$ or equivalently $s_j \geq 3$. By definition of j_0, we deduce that $G \in L_{3/2,\infty}(\Omega) \cap L_{s_{j_0},\infty}(\Omega)$ and $s_{j_0} \geq 3$. It follows from the reiteration theorem in real interpolation theory that $G \in L_s(\Omega)$ for all $3/2 < s < 3$. Hence by Theorems 1.3 and 4.1, we have

$$(v, \pi) \in \dot{H}_s^1(\Omega) \times L_s(\Omega) \quad \text{for all} \quad 3/2 < s < 3,$$

which proves (4.49). This completes the proof of Theorem 1.2. □

Proof of Corollary 1.2. The proof is exactly the same as that of Corollary 1.1 and so omitted. □

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