REGULARITY OF WEAK SOLUTIONS TO THE NAVIER-STOKES EQUATIONS IN EXTERIOR DOMAINS

REINHARD FARWIG, CHRISTIAN KOMO

ABSTRACT. Let u be a weak solution of the Navier-Stokes equations in an exterior domain $\Omega \subset \mathbb{R}^3$ and a time interval $[0,T[\,,\ 0 < T \le \infty$, with initial value u_0, external force $f = \mathrm{div} F$, and satisfying the strong energy inequality. It is well known that global regularity for u is an unsolved problem unless we state additional conditions on the data u_0 and f or on the solution u itself such as Serrin's condition $\|u\|_{L^s(0,T;L^q(\Omega))} < \infty$ with $2 < s < \infty\,,\ \frac{2}{s} + \frac{3}{q} = 1$. In this paper, we generalize results on local in time regularity for bounded domains, see [2], [5], [6], to exterior domains. If e.g. u fulfills Serrin's condition in a left-side neighborhood of t or if the norm $\|u\|_{L^{s'}(t-\delta,t;L^q(\Omega))}$ converges to 0 sufficiently fast as $\delta \to 0+$, where $\frac{2}{s'} + \frac{3}{q} > 1$, then u is regular at t. The same conclusion holds when the kinetic energy $\frac{1}{2}\|u(t)\|_2^2$ is locally Hölder continuous with exponent $\alpha > \frac{1}{2}$.

1. INTRODUCTION AND MAIN RESULTS

In this paper, $\Omega \subset \mathbb{R}^3$ is an exterior domain, i.e. an open, connected subset having a nonempty, compact complement in $\mathbb{R}^3$, with smooth boundary $\partial\Omega \in C^{2,1}$, and $[0,T[\,, 0 < T \le \infty$, denotes the time interval. In $[0,T[\times\Omega$ we consider the instationary Navier-Stokes equations

$$\begin{aligned} u_t - \nu\Delta u + u\cdot\nabla u + \nabla p = f \quad & \text{in }]0,T[\times\Omega \\ \mathrm{div}\ u = 0 \quad & \text{in }]0,T[\times\Omega \\ u = 0 \quad & \text{on }]0,T[\times\partial\Omega \\ u = u_0 \quad & \text{at } t = 0 \end{aligned} \tag{1.1}$$

with constant viscosity $\nu > 0$ (fixed throughout this paper), external force $f = \mathrm{div} F = (\sum_{i=1}^3 \partial_i F_{i,j})_{j=1}^3$ and initial value u_0. First we recall the definition of weak and strong solutions. The space of test functions is defined to be

$$C_0^\infty([0,T[; C_{0,\sigma}^\infty(\Omega)) := \{\, u|_{[0,T[\times\Omega}\ ; u \in C_0^\infty(]-1,T[\times\Omega)\ ; \mathrm{div}\, u = 0\}.$$

Definition 1.1. Let $\Omega \subset \mathbb{R}^3$ be an exterior domain and let $u_0 \in L^2_\sigma(\Omega)$, $f = \mathrm{div} F$ with $F \in L^1_{\mathrm{loc}}([0,T[; L^2(\Omega))$ where $0 < T \le \infty$. Then a vector field $u \in LH_T$, where LH_T denotes the *Leray-Hopf class*

$$LH_T := L^\infty_{\mathrm{loc}}([0,T[; L^2_\sigma(\Omega)) \cap L^2_{\mathrm{loc}}([0,T[; W^{1,2}_{0,\sigma}(\Omega))\,, \tag{1.2}$$

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is called *weak solution* (in the sense of *Leray-Hopf*) of the instationary Navier-Stokes system (1.1) with data $f\,, u_0$, if the following identity is satisfied for all test functions $w \in C_0^\infty([0,T[; C_{0,\sigma}^\infty(\Omega))$:

$$\begin{aligned} \int_0^T & \big(- \langle u, w_t\rangle_\Omega + \nu\langle \nabla u, \nabla w\rangle_\Omega + \langle u\cdot\nabla u, w\rangle_\Omega\big)\, dt \\ & = \langle u_0, w(0)\rangle_\Omega - \int_0^T \langle F, \nabla w\rangle_\Omega\, dt. \end{aligned} \tag{1.3}$$

As a consequence of (1.2), (1.3), $u : [0,T[\to L^2_\sigma(\Omega)$ is - after a possible redefinition on a set of Lebesgue measure 0 - weakly continuous and the initial value u_0 is attained in the sense

$$\langle u(t), \phi\rangle \to \langle u_0, \phi\rangle\,, \quad t\to 0+ \quad \forall \phi \in L^2_\sigma(\Omega).$$

Moreover, there exists a distribution p, called an associated pressure, such that the equality

$$u_t - \nu\Delta u + u\cdot\nabla u + \nabla p = f$$

holds in the sense of distributions on $]0,T[\times\Omega$, see [14, V.1.7].

A weak solution of (1.1) is called a *strong solution* if there exist exponents s, q with $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$ such that additionally *Serrin's condition*

$$u \in L^s(0,T; L^q(\Omega)) \tag{1.4}$$

is satisfied. By Hölder's inequality, such a strong solution u satisfies $u\otimes u \in L^2_{\mathrm{loc}}([0,T[; L^2(\Omega))$. Moreover, by Serrin's Uniqueness Theorem [14, V. Theorems 1.5.1, 1.4.1], a weak solution with (1.4) is unique within the class of weak solutions satisfying the energy inequality, i.e., fulfilling (1.5) below with $s = 0$. Finally, $u : [0,T[\to L^2_\sigma(\Omega)$ is strongly continuous and satisfies the energy identity (1.15) below.

For sufficiently smooth $\Omega\,, f\,, u_0$ a strong solution u has the regularity property

$$u \in C^\infty(]0,T[\times\bar\Omega)\,, \quad p \in C^\infty(]0,T[\times\bar\Omega)\,,$$

see [14, Theorem V.1.8.2], and therefore a strong solution is also called a *regular solution*. We call a weak solution u of (1.1) *regular at* t, if there exists a $\delta = \delta(t) > 0$ with $u \in L^s(t-\delta, t+\delta; L^q(\Omega))$ where $s\,, q$ satisfy $\frac{2}{s} + \frac{3}{q} = 1$.

Now let $\Omega \subset \mathbb{R}^3$ be an exterior domain with smooth boundary. We know, see [13], that there exists at least one weak solution u of (1.1) satisfying the *strong energy inequality*

$$\frac{1}{2}\|u(t)\|_2^2 + \nu\int_s^t \|\nabla u\|_2^2\, d\tau \le \frac{1}{2}\|u(s)\|_2^2 - \int_s^t \langle F, \nabla u\rangle_\Omega\, d\tau \tag{1.5}$$

for almost all $s\in[0,T[$ and all $t \in [s,T[$.

Our first main theorem states that if u fulfills the Serrin condition in a left-side neighborhood of t then u is regular at t. Furthermore, it gives conditions depending on $\|u\|_{L^{s'}(0,T;L^q(\Omega))}$ with $\frac{2}{s'} + \frac{3}{q} > 1$ to imply regularity of u at t; note that in this case, the integrability of u is weaker than in Serrin's condition.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$, let $2 < s < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$, $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$ and let $1 \le s' < s$. Assume that $f = \mathrm{div} F$ with $F \in L^s(0,T; L^r(\Omega)) \cap L^4(0,T; L^2(\Omega))$, $u_0 \in L^2_\sigma(\Omega)$, $0 < T < \infty$, and let $u \in LH_T$ be a weak solution of the Navier-Stokes equations (1.1) satisfying the strong energy inequality (1.5). Then we have:*

(1) Left-side $L^s(L^q)$-condition. *If for $t \in]0,T[$*

$$u \in L^s(t-\delta, t; L^q_\sigma(\Omega)) \quad \text{for some } 0 < \delta = \delta(t) < t\,, \tag{1.6}$$

then u is regular at t.

(2) Left-side $L^{s'}(L^q)$-condition. *The condition*

$$\liminf_{\delta \to 0+} \frac{1}{\delta^{1-\frac{s'}{s}}} \int_{t-\delta}^{t} \|u(\tau)\|_q^{s'}\, d\tau = 0 \tag{1.7}$$

is necessary and sufficient for regularity of u at t.

(3) Global $L^{s'}(L^q)$-condition. *There exists a constant $\epsilon_* = \epsilon_*(q, s', \Omega) > 0$, independent of f, u_0, T with the following property: If $u_0 \in L^2_\sigma(\Omega) \cap L^q_\sigma(\Omega)$, $u \in L^{s'}(0,T; L^q_\sigma(\Omega))$ and the conditions*

$$\int_0^T \|F(\tau)\|_r^s\, d\tau \le \epsilon_* \nu^{2s-1} \quad \text{and} \quad \int_0^T \|u(\tau)\|_q^{s'}\, d\tau \le \epsilon_* \frac{\nu^{s-1}}{\|u_0\|_q^{s-s'}} \tag{1.8}$$

are satisfied, then $u \in L^s(0,T; L^q(\Omega))$.

The following theorem states that Hölder continuity of the kinetic energy with exponent $\alpha \in]\frac{1}{2}, 1[$ implies regularity of u at t. In the case $\alpha = \frac{1}{2}$ we need a smallness condition for the corresponding Hölder term under which we can prove regularity of u at t.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with boundary $\partial\Omega \in C^{2,1}$, let $0 < T < \infty$ and let u be a weak solution of the Navier-Stokes equations (1.1) satisfying the strong energy inequality (1.5) with initial value $u_0 \in L^2_\sigma(\Omega)$ and external force $f = \mathrm{div} F$ which will be specified below. Furthermore, we assume that the kinetic energy $E(t) := \frac{1}{2}\|u(t)\|_2^2$ is a continuous function of $t \in [0,T[$. Then we have:*

(1) *Let $\alpha \in]\frac{1}{2}, 1[\,, 2 < s' < 4\alpha\,, 3 < q < 6\,, \frac{2}{s'} + \frac{3}{q} = \frac{3}{2}\,, \frac{2}{s} + \frac{3}{q} = 1$, $f \in L^{\frac{s}{s'}}(0,T; L^2(\Omega))$ and $F \in L^4(0,T; L^2(\Omega)) \cap L^s(0,T; L^r(\Omega))$, where $\frac{1}{3} + \frac{1}{q} = \frac{1}{r}$, and let u satisfy at $t \in]0,T[$ the left-side condition*

$$\sup_{t-\mu < t' < t} \frac{|E(t) - E(t')|}{|t - t'|^\alpha} < \infty \tag{1.9}$$

with a $\mu > 0$. Then u is regular at t.

(2) (The case $\alpha = \frac{1}{2}$) *Let $f \in L^2(0,T; L^2(\Omega))$, $F \in L^4(0,T; L^2(\Omega))$. Then there exists a constant $\gamma_* = \gamma_*(\Omega)$ such that the left-side condition*

$$\sup_{t-\mu < t' < t} \frac{|E(t) - E(t')|}{|t - t'|^{\frac{1}{2}}} \le \gamma_* \nu^{\frac{5}{2}} \tag{1.10}$$

with a $\mu > 0$ implies regularity of u at t.

Remark. (1) The proof of Theorem 1.3, in particular see (4.8), will yield the following regularity criteria using the dissipation energy: If

$$\alpha \in]\frac{1}{2}, 1[\quad \text{and} \quad \liminf_{\delta \to 0+} \frac{1}{\delta^\alpha} \int_{t-\delta}^t \|\nabla u(\tau)\|_2^2 \, d\tau < \infty \,, \tag{1.11}$$

or

$$\liminf_{\delta \to 0+} \frac{1}{\delta^{\frac{1}{2}}} \int_{t-\delta}^t \|\nabla u(\tau)\|_2^2 \, d\tau \le \gamma_* \nu^{\frac{3}{2}} \tag{1.12}$$

then u is regular at t.

(2) In the case $\alpha = \frac{1}{2}$ a smallness condition as in (1.10) and (1.12) is necessary. Indeed, if $f = 0$ and $]0, t[$ is a maximal regularity interval of u, then

$$\|\nabla u(\tau)\|_2 \ge \frac{c_0}{(t-\tau)^{\frac{1}{4}}} \,, \quad 0 < \tau < t,$$

where $c_0 = c_0(\Omega) > 0$, see [8]. Hence

$$\liminf_{\delta \to 0+} \frac{1}{\delta^{\frac{1}{2}}} \int_{t-\delta}^t \|\nabla u(\tau)\|_2^2 \, d\tau \ge 2c_0^2 > 0 \,,$$

and due to the strong energy inequality (1.5) it holds for all $\mu > 0$

$$\sup_{t-\mu < t' < t} \frac{|E(t) - E(t')|}{|t-t'|^{\frac{1}{2}}} \ge 2\nu c_0^2 > 0.$$

The proofs of the regularity criteria formulated in this paper are based on a local or global identification of a weak solution with a very weak solution, a concept described in Definition 2.3 below. The following key result, Theorem 1.4, gives conditions under which a given very weak solution is also a weak solution in the sense of Leray-Hopf and, therefore, yields under certain smallness conditions on the data f and u_0 the existence of a unique strong solution of (1.1) on $[0, T[\times\Omega$.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$, let $2 < s < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$ and let $\frac{1}{3} + \frac{1}{q} = \frac{1}{q*}$. Then there exists a constant $\epsilon_* = \epsilon_*(q, \Omega) > 0$ with the following property: Given $0 < T < \infty$ and data $u_0 \in L^2_\sigma(\Omega) \cap L^q_\sigma(\Omega)$ and $f = \operatorname{div} F$ with $F \in L^s(0, T; L^{q*}(\Omega)) \cap L^4(0, T; L^2(\Omega))$ satisfying the following two conditions:*

$$\int_0^T \|F(\tau)\|_{q*}^s \, d\tau \le \epsilon_* \nu^{2s-1} \,, \tag{1.13}$$

$$\int_0^T \|e^{-\nu\tau A_q} u_0\|_q^s \, d\tau \le \epsilon_* \nu^{s-1}. \tag{1.14}$$

In this case, there exists a unique weak solution $u \in LH_T$ of (1.1) satisfying the Serrin condition $u \in L^s(0, T; L^q(\Omega))$. After a possible redefinition on a set of Lebesgue measure 0, we get that $u : [0, T[\to L^2_\sigma(\Omega)$ is strongly continuous and it holds the energy identity

$$\frac{1}{2}\|u(t)\|_2^2 + \nu \int_0^t \|\nabla u\|_2^2 \, d\tau = \frac{1}{2}\|u_0\|_2^2 - \int_0^t \langle F, \nabla u \rangle_\Omega \, d\tau \tag{1.15}$$

for all $t \in [0, T[$.

The proof of this crucial result is the content of Section 3 and differs from the case of bounded domains, see [4], [6], where the trivial inclusion $L^q(\Omega) \subset L^r(\Omega)$, $q > r$, yielding also better embedding properties of fractional powers of the Stokes operator, was used several times. The main idea of the proof is to construct a very weak solution $v \in L^s(0,T; L^q_\sigma(\Omega))$ for the given data u_0, f and to identify u and v by Serrin's Uniqueness Theorem; for this reason, we have to show that v lies in the Leray-Hopf class LH_T.

After some preliminaries to be discussed in Section 2 we prove Theorem 1.4 in Section 3. Finally, Section 4 deals with the proofs of the main results Theorem 1.2 and 1.3.

2. Preliminaries

Given $1 \le q \le \infty\,, k \in \mathbb{N}$ we need the usual Lebesgue and Sobolev spaces, $L^q(\Omega)\,, W^{k,q}(\Omega)$ with norm $\|\cdot\|_{L^q(\Omega)} = \|\cdot\|_q$ and $\|\cdot\|_{W^{k,q}(\Omega)} = \|\cdot\|_{k,q}$, respectively. For two measurable functions $f\,,g$ with the property $f \cdot g \in L^1(\Omega)$, where $f \cdot g$ means the usual scalar product of vector or matrix fields, we set

$$\langle f, g\rangle_\Omega := \int_\Omega f(x) \cdot g(x)\, dx.$$

Note that the same symbol $L^q(\Omega)$ etc. will be used for spaces of scalar-, vector or matrix-valued functions. Let $C^m(\Omega)\,, m = 0, 1, \dots, \infty$, denote the usual space of functions for which all partial derivatives of order $|\alpha| \le m$ exist and are continuous. As usual, $C^m_0(\Omega)$ is the set of all functions from $C^m(\Omega)$ with compact support in Ω. Further we need the space of smooth solenoidal vector fields

$$C^\infty_{0,\sigma}(\Omega) := \{\, v \in C^\infty_0(\Omega)^3;\ \operatorname{div} v = 0 \,\}$$

and define the spaces

$$L^q_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}, \quad W^{1,2}_{0,\sigma}(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{W^{1,2}}}.$$

For $1 \le q \le \infty$ let $q' \in [1, \infty]$ denote its dual exponent. It is well known that $L^q_\sigma(\Omega)' = L^{q'}_\sigma(\Omega)$ using the standard pairing $\langle \cdot, \cdot\rangle_\Omega$. Moreover, let us write $[d, v]_\Omega$ for the application of a distribution $d \in C^\infty_0(\Omega)'$ on a test function $v \in C^\infty_0(\Omega)$.

Given a Banach space X and an interval $[0, T]$, $0 < T \le \infty$, we denote by $L^p(0, T; X)$, $1 \le p \le \infty$, the space of all equivalence classes of strongly measurable functions $f : [0, T) \to X$ such that

$$\|f\|_p := \left(\int_0^T \|f(t)\|^p_X\, dt \right)^{\frac{1}{p}} < \infty$$

if $p < \infty$, and $\|f\|_\infty := \operatorname{ess\,sup}_{[0,T[} \|f(\cdot)\|_X$, if $p = \infty$. Moreover, we define the set of *locally integrable* L^p-functions on $[0, T[$ as

$$\begin{aligned} L^p_{\mathrm{loc}}([0,T[; X) := \{\, u : [0,T[\to X \text{ strongly measurable},& \\ u \in L^p(0, T'; X) \text{ for all } 0 < T' < T \}.& \end{aligned}$$

When $X = L^q(\Omega)$, $1 \le q \le \infty$, we denote the norm in $L^p(0, T; L^q(\Omega))$ by $\|\cdot\|_{q,p,\Omega;T}$. For $1 < p\,, q < \infty$ it holds

$$L^p(0, T; L^q(\Omega))' = L^{p'}(0, T; L^{q'}(\Omega))$$

and we define

$$\langle f, g\rangle_{\Omega,T} := \int_0^T \int_\Omega f(t,x)\cdot g(t,x)\,dx\,dt$$

for $f \in L^p(0,T;L^q(\Omega))\,, g \in L^{p'}(0,T;L^{q'}(\Omega))$.

Given an exterior domain $\Omega \subset \mathbb{R}^3$ with $\partial\Omega \in C^{2,1}$ and $1 < q < \infty$, there exists a bounded, linear projection $P_q : L^q(\Omega)) \to L^q_\sigma(\Omega)$ with range $\mathcal{R}(P_q) = L^q_\sigma(\Omega)$ and nullspace $N(P_q) = \{\, \nabla p \in L^q(\Omega)\,; p \in L^q_{\mathrm{loc}}(\overline{\Omega})\,\}$. The operator P_q is called *Helmholtz projection* and is *consistent* in the sense that

$$P_q f = P_r f \qquad \forall f \in L^q(\Omega)\cap L^r(\Omega). \tag{2.1}$$

Furthermore, we get $P_q' = P_{q'}$ for the dual operator, i.e.,

$$\langle P_q f, g\rangle_\Omega = \langle f, P_{q'} g\rangle_\Omega \quad \forall f \in L^q(\Omega) \quad \forall g \in L^{q'}(\Omega). \tag{2.2}$$

For $1 < q < \infty$ we define the *Stokes operator* A_q on $L^q_\sigma(\Omega)$ by

$$\mathcal{D}(A_q) = L^q_\sigma(\Omega)\cap W^{1,q}_0(\Omega)\cap W^{2,q}(\Omega), \tag{2.3}$$

$$A_q u := -P_q\Delta u\,, \quad u \in \mathcal{D}(A_q). \tag{2.4}$$

The Stokes operator is *consistent* in the sense that for $1 < q, r < \infty$ it holds

$$A_q u = A_r u \quad \forall u \in \mathcal{D}(A_q)\cap\mathcal{D}(A_r). \tag{2.5}$$

In general, $\mathcal{D}(A_q)$ will be equipped with the graph norm $\|u\|_{\mathcal{D}(A_q)} := \|u\|_q + \|A_q\|_q$ for $u \in D(A_q)$ which makes $\mathcal{D}(A_q)$ to a Banach space since the Stokes operator is closed. For simplicity, we use the notation $A = A_2$.

For $\alpha \in [-1,1]$ the fractional power $A^\alpha_q : \mathcal{D}(A^\alpha_q) \to L^q_\sigma(\Omega)$ with dense domain $\mathcal{D}(A^\alpha_q) \subseteq L^q_\sigma(\Omega))$ is a well defined, injective, closed operator such that

$$(A^\alpha_q)^{-1} = A^{-\alpha}_q\,, \quad \mathcal{R}(A^\alpha_q) = \mathcal{D}(A^{-\alpha}_q)\,\text{and} \quad (A^\alpha_q)' = A^\alpha_{q'}.$$

It holds $\mathcal{D}(A^{1/2}_q) = W^{1,q}_0(\Omega)\cap L^q_\sigma(\Omega)$ for $1 < q < 3$, and the estimate

$$\|\nabla u\|_{q,\Omega} \le c\|A^{1/2}_q u\|_{q,\Omega} \quad \text{for } 1 < q < 3,\ \ u \in \mathcal{D}(A^{1/2}_q), \tag{2.6}$$

with a constant $c = c(\Omega, q) > 0$. Moreover,

$$\|u\|_{\gamma,\Omega} \le c\|A^\alpha_q u\|_{q,\Omega} \quad \text{where } 0 \le \alpha \le \frac{1}{2}\,, 1 < q < 3\,, 2\alpha + \frac{3}{\gamma} = \frac{3}{q}\,, \tag{2.7}$$

for all $u \in \mathcal{D}(A^\alpha_q)$ with a constant $c = c(\Omega, q, \gamma) > 0$. It is well known that $-A_q$ generates a uniformly bounded analytic semigroup $\{\, e^{-tA_q} : t \ge 0\,\}$ on $L^q_\sigma(\Omega)$ satisfying the decay estimate

$$\|A^\alpha_q e^{-tA_q}\|_q \le ct^{-\alpha} \quad \forall t > 0\,, \tag{2.8}$$

where $\alpha \ge 0\,, 1 < q < \infty$ and $c = c(\Omega, q, \alpha) > 0$.

Lemma 2.1. *Let $d \in C^\infty_0(\Omega)'$ be a distribution, well defined for all $v \in \mathcal{D}(A^\alpha_{q'})$ where $1 < q < \infty\,, 0 < \alpha \le 1$. We assume that there exists a constant $c \ge 0$, independent of $v \in \mathcal{D}(A^\alpha_{q'})$, such that*

$$|[d, v]_\Omega| \le c\|A^\alpha_{q'} v\|_{q',\Omega} \quad \forall v \in \mathcal{D}(A^\alpha_{q'}). \tag{2.9}$$

Then there exists a unique element $\tilde{d} \in L^q_\sigma(\Omega)$, to be denoted by $A^{-\alpha}_q P_q d$, with the properties

$$\langle A^{-\alpha}_q P_q d, A^\alpha_{q'} v\rangle_\Omega = [d, v]_\Omega \quad \forall v \in \mathcal{D}(A^\alpha_{q'}) \quad \textit{and } \|A^{-\alpha}_q P_q d\|_q \le c \tag{2.10}$$

with the constant c from (2.9). In particular, if $F \in L^q(\Omega)$, and $\frac{3}{2} < q < \infty$, then $A_q^{-\frac{1}{2}} P_q \mathrm{div} F \in L^q_\sigma(\Omega)$ and

$$\|A_q^{-\frac{1}{2}} P_q \mathrm{div} F\|_q \le c\|F\|_q \,. \tag{2.11}$$

Proof. We define for $w \in \mathcal{R}(A^\alpha_{q'})$

$$[\widetilde{d}, w]_\Omega := [d, v]_\Omega \,, \quad \text{where } w = A^\alpha_{q'} v \,, v \in \mathcal{D}(A^\alpha_{q'}).$$

By the density of $\mathcal{R}(A^\alpha_{q'})$ in $L^{q'}_\sigma(\Omega)$, we extend $\tilde{d}$ to a functional defined on $L^{q'}_\sigma(\Omega)$. We use $L^{q'}_\sigma(\Omega)' = L^q_\sigma(\Omega)$ to obtain a unique element $A_q^{-\alpha} P_q d \in L^q_\sigma(\Omega)$ satisfying the identity in (2.10). For the proof of (2.11) we exploit (2.6) with q replaced by $q' \in]1, 3[$. □

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$, let $1 < q\,, s < \infty$ and $0 < T < \infty$. Furthermore, let $f \in L^s(0, T; L^q_\sigma(\Omega))$ and $u_0 \in L^q_\sigma(\Omega)$ such that $\int_0^\infty \|A_q e^{-tA_q} u_0\|^s_{q,\Omega}\, dt < \infty$. Then the instationary Stokes system*

$$\begin{aligned} u_t + \nu A_q u &= f \quad \text{in } (0, T) \\ u(0) &= u_0 \end{aligned} \tag{2.12}$$

has a unique strong solution $u \in L^s(0, T; D(A_q))$ with $u_t \in L^s(0, T; L^q_\sigma(\Omega))$ and $u \in C([0, T[; L^q_\sigma(\Omega))$. Moreover, u satisfies the maximal regularity estimate

$$\|u_t\|_{q,s,\Omega;T} + \|\nu A_q u\|_{q,s,\Omega;T} \le c \left(\left(\int_0^T \|\nu A_q e^{-\nu t A_q} u_0\|^s_{q,\Omega}\, dt \right)^{\frac{1}{s}} + \|f\|_{q,s,\Omega;T} \right) \tag{2.13}$$

with a constant $c = c(\Omega, q, s)$ independent of T und ν. It holds the representation

$$u(t) = e^{-\nu t A_q} u_0 + \int_0^t e^{-\nu(t-\tau)A_q} f(\tau)\, d\tau \tag{2.14}$$

for all $t \in [0, T[$. In the case $T = \infty$ we get a unique strong solution $u \in L^s_{\mathrm{loc}}(0, \infty; D(A_q))$ of (2.12) satisfying $u_t \in L^s(0, \infty; L^q_\sigma(\Omega))$ and $u \in C([0, \infty[; L^q_\sigma(\Omega))$ and it holds the estimate (2.13) and the representation (2.14) for all $t \in [0, \infty[$.

Proof. See [10, Theorem 4.2]. □

A major tool for the proof of Theorem 1.4 is the theory of very weak solutions. In this context we refer to [3] for exterior domains and to [4] for bounded domains. In the following definition let

$$C^1_0([0, T[; C^2_{0,\sigma}(\bar{\Omega})) := \{\, w\,|_{[0,T[\times\bar{\Omega}} \text{ with } w \in C^{1,2}_0(-]1, T[\times\mathbb{R}^3); \tag{2.15}$$

$$\mathrm{div}\ w = 0, w\,|_{\partial\Omega} = 0 \ \text{ for all } t \in [0, T[\,\} \tag{2.16}$$

denote the space of test functions and let

$$\mathcal{J}^{q,s}(\Omega) := \{\, u_0 \in C^\infty_0(\Omega)'; \tag{2.17}$$

$$A_q^{-1} P_q u_0 \in L^q_\sigma(\Omega), \int_0^\infty \|A_q e^{-tA_q}(A_q^{-1} P_q u_0)\|^s_{q,\Omega}\, dt < \infty\,\} \tag{2.18}$$

denote the space of initial values.

Definition 2.3. Let $\Omega \subset \mathbb{R}^3$ be an exterior domain, let $F \in L^s(0,T;L^r(\Omega))$ and $u_0 \in \mathcal{J}^{q,s}(\Omega)$ where $2 < s < \infty$, $\frac{2}{s} + \frac{3}{q} = 1\,, \frac{1}{3} + \frac{1}{q} = \frac{1}{r}$. Then $u \in L^s(0,T;L^q_\sigma(\Omega))$ is called *very weak solution* of the instationary Navier-Stokes equations (1.1) if

$$\int_0^T \langle -u, w_t\rangle_\Omega - \nu\langle u, \Delta w\rangle_\Omega - \langle u\otimes u, \nabla w\rangle_\Omega \, dt = [u_0, w(0)]_\Omega - \int_0^T \langle F, \nabla w\rangle_\Omega \, dt \tag{2.19}$$

holds for all test functions $w \in C^1_0([0,T[;C^2_{0,\sigma}(\bar{\Omega}))$.

In the corresponding definition of very weak solutions to the linear, instationary Stokes system where the nonlinear term $u\cdot\nabla u$ is absent, we may omit in Definition 2.3 the restriction $\frac{2}{s}+\frac{3}{q} = 1$, and in (2.19) the term $\langle u\otimes u, \nabla w\rangle_{\Omega,T}$ is absent. A proof of the following Theorem can be found in [3], [12].

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$ and let $2 < s < \infty\,, \frac{2}{s}+\frac{3}{q} = 1,\ \frac{1}{3}+\frac{1}{q} = \frac{1}{r}$. Then there exists a constant $c = c(q,\Omega) > 0$ with the following property: For data $f = \mathrm{div}F$ with $F \in L^s(0,T;L^r(\Omega))$ and $u_0 \in \mathcal{J}^{q,s}(\Omega)$, satisfying the condition*

$$\Big(\int_0^T \|\nu A_q e^{-\nu t A_q}(A_q^{-1}P_q u_0)\|^s_{q,\Omega}\, dt\Big)^{\frac{1}{s}} + \|F\|_{r,s,\Omega;T} \le c\nu^{1+\alpha} \tag{2.20}$$

with $\alpha := \frac{3}{2q}+\frac{1}{2} = 1-\frac{1}{s}$, there exists a unique very weak solution $u \in L^s(0,T;L^q_\sigma(\Omega))$ of the instationary Navier-Stokes system (1.1). Moreover, u has the representation $u = E+\tilde{u}$, where $E \in L^s(0,T;L^q_\sigma(\Omega))$ is the unique very weak solution of the linear Stokes system with data $f\,, u_0$ and $\tilde{u}$ is the unique solution in $L^s(0,T;L^q_\sigma(\Omega))$ of the nonlinear fixed point equation

$$\tilde{u}(t) = -\int_0^t A_q^\alpha e^{-\nu(t-\tau)A_q} A_q^{-\alpha} P_q \mathrm{div}\big((\tilde{u}(\tau)+E(\tau))\otimes(\tilde{u}(\tau)+E(\tau))\big)\, d\tau \tag{2.21}$$

for almost all $t \in [0,T[$.

Finally we recall the Hardy-Littlewood inequality [14, II Lemma 3.3.2]. Let $0 < \alpha < 1, 1 < r < q < \infty$ with $\alpha + \frac{1}{q} = \frac{1}{r}$ and let $f \in L^r(\mathbb{R})$. Then the integral

$$u(t) := \int_{\mathbb{R}} |t-\tau|^{\alpha-1} f(\tau)\, d\tau$$

converges absolutely for almost all $t \in \mathbb{R}$ and it holds

$$\|u\|_{L^q(\mathbb{R})} \le c\|f\|_{L^r(\mathbb{R})} \tag{2.22}$$

with a constant $c = c(\alpha,q) > 0$.

3. Proof of Theorem 1.4

Before proving Theorem 1.4 we discuss the nonlinear term arising in the nonlinear fixed point problem (2.21). We denote by $\mathrm{div}(u\otimes u)$ the functional defined for suitable vector fields w by

$$[\mathrm{div}(u\otimes u), w]_\Omega := -\langle u\otimes u, \nabla w\rangle_\Omega.$$

The following lemma is technical but essential for Lemma 3.2 below.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with boundary $\partial\Omega \in C^{2,1}$, let $3 < q < \infty\,, r \in [\frac{q}{2}, q]$ and $\beta := \frac{3}{q} - \frac{3}{2r} + \frac{1}{2}$.*

(1) *There exists a constant $c = c(\Omega, q, r) > 0$ such that for all $u \in L^q_\sigma(\Omega)$*

$$\|A_r^{-\beta} P_r \operatorname{div}(u \otimes u)\|_{r,\Omega} \leq c\|u\|^2_{q,\Omega}. \tag{3.1}$$

(2) *For $2 < s < \infty\,, 3 < q < \infty\,, 0 < T \leq \infty$ there exists a constant $c = c(\Omega, q, r) > 0$ such that for all $u \in L^s(0, T; L^q_\sigma(\Omega))$*

$$\|A_r^{-\beta} P_r \operatorname{div}(u \otimes u)\|_{r,\frac{s}{2},\Omega;T} \leq c\|u\|^2_{q,s,\Omega;T}. \tag{3.2}$$

Proof. The assumptions of the lemma imply

$$2(\beta - \frac{1}{2}) + \frac{3}{\left(\frac{q}{2}\right)'} = \frac{3}{r'} \quad \text{with } 1 < r' < 3\,, \frac{1}{2} \leq \beta < 1. \tag{3.3}$$

Then we get for arbitrary $w \in \mathcal{D}(A^\beta_{r'})$ by (2.6) using $1 < \left(\frac{q}{2}\right)' < 3$, (2.7) and (2.5) (applied to $A^{1/2}$ instead of A)

$$\begin{aligned}
|[\operatorname{div}(u \otimes u), w]| &= |-\langle u \otimes u, \nabla w\rangle| \\
&\leq \|u \otimes u\|_{\frac{q}{2}}\, \|\nabla w\|_{\left(\frac{q}{2}\right)'} \\
&\leq c\|u\|^2_q\, \|A^{1/2}_{(q/2)'} w\|_{\left(\frac{q}{2}\right)'} \\
&\leq c\|u\|^2_q\, \|A_{r'}^{(\beta-\frac{1}{2})}(A^{1/2}_{(q/2)'} w)\|_{r'} \\
&\leq c\|u\|^2_q\, \|A^\beta_{r'} w\|_{r'}.
\end{aligned}$$

It is possible to choose the constant $c > 0$ in the above estimate depending only on $\Omega\,, q$ and r. For the second assertion we use (3.1), which holds for almost all $t \in [0, T[$, and integrate over $[0, T]$. □

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$, let $0 < T \leq \infty$, $2 < s < \infty\,, \frac{2}{s} + \frac{3}{q} = 1$ and let $u \in L^s(0, T; L^q(\Omega))$. We define for $r \in [\frac{q}{2}, q]$ and $\beta := \frac{3}{q} - \frac{3}{2r} + \frac{1}{2}$ the term $\Lambda^r(u)$ by*

$$\Lambda^r u(t) := -\int_0^t A^\beta_r e^{-\nu(t-\tau)A_r} A_r^{-\beta} P_r \operatorname{div}(u(\tau) \otimes u(\tau))\, d\tau\,. \tag{3.4}$$

Then the following statements are satisfied.

(1) *For almost all $t \in [0, T[$ we get*

$$\int_0^t \|A^\beta_r e^{-\nu(t-\tau)A_r} A_r^{-\beta} P_r \operatorname{div}(u(\tau) \otimes u(\tau))\|_r\, d\tau < \infty \tag{3.5}$$

and

$$\begin{aligned}
&- A^\beta_r \int_0^t e^{-\nu(t-\tau)A_r} A_r^{-\beta} P_r \operatorname{div}(u(\tau) \otimes u(\tau))\, d\tau \\
&= -\int_0^t A^\beta_r e^{-\nu(t-\tau)A_r} A_r^{-\beta} P_r \operatorname{div}(u(\tau) \otimes u(\tau))\, d\tau.
\end{aligned} \tag{3.6}$$

(2) *For all* $r_1, r_2 \in [\frac{q}{2}, q]$ *with* $\beta_1 := \frac{3}{q} - \frac{3}{2r_1} + \frac{1}{2}$, $\beta_2 := \frac{3}{q} - \frac{3}{2r_2} + \frac{1}{2}$ *it holds*

$$\Lambda^{r_1} u(t) = \Lambda^{r_2} u(t) \qquad \text{for almost all } \ t \in [0, T[. \tag{3.7}$$

Therefore, we can denote the expression in (3.4), independently of $r \in [\frac{q}{2}, q]$, *by* $\Lambda(u)$.

(3) *For all* $q_1 \in [\frac{q}{2}, q]$ *with* $3 < q_1 < \infty$ *and* $s_1 > 2$ *defined by* $\frac{2}{s_1} + \frac{3}{q_1} = 1$ *we have*

$$\Lambda u \in L^{s_1}(0, T; L^{q_1}(\Omega)) . \tag{3.8}$$

(4) *If* $q \in]3, 6[$ *then*

$$\Lambda u \in L^{\frac{s}{2}}(0, T; L^{q_2}(\Omega)) \tag{3.9}$$

where $q_2 > 3$ *satisfies* $\frac{1}{3} + \frac{1}{q_2} = \frac{1}{(\frac{q}{2})}$ *and consequently* $\frac{2}{(\frac{s}{2})} + \frac{3}{q_2} = 1$.

Proof. (1) By (2.8) and (3.1) we know that for all $t \in [0, T[$

$$\begin{aligned} &\int_0^t \|A_r^\beta e^{-\nu(t-\tau)A_r} A_r^{-\beta} P_r \text{ div } (u(\tau) \otimes u(\tau))\|_r \, d\tau \\ &\qquad \le c(\Omega, q, r)\nu^{-\beta} \int_0^T |t-\tau|^{-\beta} \|u(\tau)\|_q^2 \, d\tau. \end{aligned} \tag{3.10}$$

Moreover, as for almost all $t \in [0, T[$ the integral in (3.10) is finite (see the application of the Hardy-Littlewood inequality (2.22) in the proof of part (3) below) and

$$\int_0^t \|e^{-\nu(t-\tau)A_r} A_r^{-\beta} P_r \text{div}(u \otimes u)\|_r \, d\tau \le c \int_0^t \|A_r^{-\beta} P_r \text{div}(u \otimes u)\|_r \, d\tau < \infty ,$$

the closedness of the operator A_r^β yields the identity (3.6).

(2) To prove (3.7) for $t \in (0, T[$ as in (1) let

$$f_t^r(\tau) := A_r^\beta e^{-\nu(t-\tau)A_r} A_r^{-\beta} P_r \text{div}(u(\tau) \otimes u(\tau)) \quad \text{for almost all } \ \tau \in]0, t[,$$

where $\beta = \beta(r) = \frac{3}{q} - \frac{3}{2r} + \frac{1}{2}$. Since for all $\phi \in C_{0,\sigma}^\infty(\Omega)$

$$\int_0^t \langle f_t^{r_1}(\tau), \phi\rangle_\Omega \, d\tau = -\int_0^t \langle u(\tau) \otimes u(\tau), \nabla e^{-\nu(t-\tau)A_{r'}} \phi\rangle_\Omega \, d\tau ,$$

we see that

$$\int_0^t \langle f_t^{r_1}(\tau), \phi\rangle_\Omega \, d\tau = \int_0^t \langle f_t^{r_2}(\tau), \phi\rangle_\Omega \, d\tau ;$$

for details of the proof we refer to [12]. A density argument finishes the proof of (3.7).

(3) We consider (3.10) and use the Hardy-Littlewood inequality (2.22) with $(1-\beta) + \frac{1}{s_1} = \frac{1}{(\frac{s}{2})}$ to conclude with $\Lambda^{q_1} u = \Lambda u$ and (3.2) that

$$\begin{aligned} &\|\Lambda u\|_{q_1, s_1, \Omega; T} \\ &\quad \le \left(\int_0^T \left(c\nu^{-\beta} \int_0^T |t-\tau|^{-\beta} \|A_{q_1}^{-\beta} P_{q_1} \text{div}(u(\tau) \otimes u(\tau))\|_{q_1} \, d\tau \right)^{s_1} dt \right)^{\frac{1}{s_1}} \\ &\quad \le c\nu^{-\beta} \|A_{q_1}^{-\beta} P_{q_1} \text{div}(u(\tau) \otimes u(\tau))\|_{q_1, \frac{s}{2}, \Omega; T} \\ &\quad \le c(q, q_1, \Omega)\nu^{-\beta} \|u\|_{q,s,\Omega;T}^2 < \infty. \end{aligned}$$

(4) From $2\frac{1}{2}+\frac{3}{q_2}=\frac{3}{(\frac{q}{2})}$ and (2.7) it follows with (3.6) and $\beta=\frac{1}{2}$, $r=\frac{q}{2}$, for almost all $t\in[0,T[$

$$\begin{aligned}\|\Lambda^{q_2}u(t)\|_{q_2} &\le \|A_{\frac{q}{2}}^{1/2}\Lambda u(t)\|_{\frac{q}{2}}\\ &= \|A_{\frac{q}{2}}\int_0^t e^{-\nu(t-\tau)A_{\frac{q}{2}}}A_{\frac{q}{2}}^{-1/2}P_{\frac{q}{2}}\mathrm{div}(u(\tau)\otimes u(\tau))\,d\tau\|_{\frac{q}{2}}.\end{aligned} \tag{3.11}$$

Since by (3.2)

$$A_{\frac{q}{2}}^{-1/2}P_{\frac{q}{2}}\mathrm{div}(u\otimes u)\in L^{\frac{s}{2}}(0,T;L^{\frac{q}{2}}(\Omega)), \tag{3.12}$$

the maximal regularity estimate (2.13) yields the last statement of the lemma. □

Proof of Theorem 1.4. Given the smallness conditions (1.13) and (1.14), Theorem 2.4 implies the existence of a unique very weak solution $u\in L^s(0,T;L^q_\sigma(\Omega))$ of (1.1). Moreover, we know the representation $u=E+\tilde u$, where the linear part E satisfies

$$E(t)=e^{-\nu tA_q}u_0+A_q\int_0^t e^{-\nu(t-\tau)A_q}(A_q^{-1}P_q\,\mathrm{div}\;F(\tau))\,d\tau \tag{3.13}$$

in $[0,T[$ and the nonlinear part $\tilde u\in L^s(0,T;L^q_\sigma(\Omega))$ solves the fixed point equation

$$\tilde u(t)=-\int_0^t A_q^\alpha e^{-\nu(t-\tau)A_q}A_q^{-\alpha}P_q\mathrm{div}\big((\tilde u(\tau)+E(\tau))\otimes(\tilde u(\tau)+E(\tau))\big)\,d\tau \tag{3.14}$$

with $\alpha:=\frac{3}{2q}+\frac{1}{2}$ for almost all $t\in[0,T[$. Since $F\in L^2(0,T;L^2(\Omega))$ and $u_0\in L^2_\sigma(\Omega)$ it follows with (2.5) that

$$E(t)=E_1(t)+E_2(t):=e^{-\nu tA}u_0+A^{1/2}\int_0^t e^{-\nu(t-\tau)A}A^{-1/2}P\,\mathrm{div}F(\tau)\,d\tau \tag{3.15}$$

almost everywhere. We use [14, IV Theorems 2.3.1, 2.4.1] to obtain that E lies in the Leray-Hopf class (1.2) and is a weak solution of the linear stationary Stokes system with data f, u_0. To finish the proof, we want to show that

$$u\in L^8(0,T;L^4(\Omega)). \tag{3.16}$$

The validity of the above property implies

$$u\otimes u\in L^2(0,T;L^2(\Omega)). \tag{3.17}$$

As a consequence of (3.14) and (3.17) we conclude that $\tilde u$ lies in the Leray-Hopf class (1.2) and $\tilde u$ is the unique weak solution of the linear, stationary Stokes system with the external force $\mathrm{div}(u\otimes u)$ and vanishing initial value, see [14, IV Theorems 2.3.1, 2.4.1]. Furthermore, from these two Theorems and $\langle u\otimes u,\nabla u\rangle(\tau)=0$ almost everywhere, it follows that u is, after a possible redefinition on a set of Lebesgue measure 0, strongly continuous and satisfies the energy equality (1.15).

Since in the case $q=4$ (and $s=8$) there is nothing left to be proved, we may assume in the proof of (3.16) that $q\neq 4$.

Assertion 1: $E = E_1 + E_2 \in L^8(0,T;L^4(\Omega))$.
Proof. In the case $4 < q < \infty$ it is easily seen since $L^2_\sigma(\Omega) \cap L^q_\sigma(\Omega) \subset L^4_\sigma(\Omega)$ that $E_1(t) = e^{-\nu t A}u_0 = e^{-\nu t A_q}u_0 \in L^8(0,T;L^4(\Omega))$. If $3 < q < 4$ we use [11, Theorem 1.2 (ii)] to find a constant $c > 0$, independent of t, such that

$$\|e^{-\nu t A_4}u_0\|_4 \le c\, t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{4})}\|u_0\|_q$$

for all $t > 0$. The estimate

$$\int_0^T \|e^{-\nu t A_4}u_0\|_4^8\, dt \le c\|u_0\|_q^8 \int_0^T t^{-12(\frac{1}{q}-\frac{1}{4})}\, dt < \infty$$

implies $E_1 \in L^8(0,T;L^4(\Omega))$. To get the property $E_2 \in L^8(0,T;L^4(\Omega))$ we estimate for almost all $t \in [0,T[$, using (2.7), (2.8) and (2.11), that

$$\begin{aligned}\|E_2(t)\|_4 &\le c\|A^{3/8}E_2(t)\|_2\\ &= c\left\|\int_0^t A^{7/8}e^{-\nu(t-\tau)A}A^{-1/2}P\mathrm{div}\, F(\tau)\, d\tau\right\|_2\\ &\le c\nu^{-7/8}\int_0^T |t-\tau|^{-7/8}\|F(\tau)\|_2\, d\tau\,.\end{aligned} \tag{3.18}$$

Then an application of the Hardy-Littlewood inequality (2.22) yields

$$\|E_2\|_{4,8,\Omega;T} \le c\nu^{-\frac{7}{8}}\|F\|_{2,4,\Omega;T} < \infty.$$

Assertion 2: Let $3 < q < 4$. Then $\tilde u \in L^8(0,T;L^4(\Omega))$.
Proof. We use an iterative argument to improve the regularity in space step by step. Assume that for almost all $t \in [0,T[$ with certain parameters $s_k\,, r_k\,, \beta_k$

$$\tilde u(t) = -\int_0^t A_{r_k}^{\beta_k}e^{-\nu(t-\tau)A_{r_k}}A_{r_k}^{-\beta_k}P_{r_k}\mathrm{div}((\tilde u + E)\otimes(\tilde u + E))\, d\tau\,, \tag{3.19}$$

$$\tilde u\,, E \in L^{s_k}(0,T;L^{r_k}(\Omega)) \text{ with } 3 < r_k < 4\,, \frac{2}{s_k} + \frac{3}{r_k} = 1\,, \beta_k \in [\frac{1}{2}, 1]. \tag{3.20}$$

For $k = 1$ the iteration starts with $s_1 := s$, $r_1 := q$ and $\beta_1 := \frac{3}{2q} + \frac{1}{2} = \alpha$, see (3.14). We denote by $r_{k+1} > r_k$ the unique element satisfying $\frac{1}{3} + \frac{1}{r_{k+1}} = \frac{1}{r_k/2}$ and set $s_{k+1} := \frac{s_k}{2}$. Then (3.9) implies that

$$\tilde u \in L^{s_{k+1}}(0,T;L^{r_{k+1}}(\Omega)). \tag{3.21}$$

We define $\beta_{k+1} := \frac{3}{r_{k+1}} - \frac{3}{2r_{k+1}} + \frac{1}{2} = \frac{3}{2r_{k+1}} + \frac{1}{2}$ and get with (3.7)

$$\tilde u(t) = -\int_0^t A_{r_{k+1}}^{\beta_{k+1}}e^{-\nu(t-\tau)A_{r_{k+1}}}A_{r_{k+1}}^{-\beta_{k+1}}P_{r_{k+1}}\mathrm{div}((\tilde u{+}E)\otimes(\tilde u{+}E))\, d\tau. \tag{3.22}$$

From the first step of the proof we know that $E \in L^8(0,T;L^4(\Omega))$. There can occur two different possibilities. If $4 \le r_{k+1} < \infty$ we get by an interpolation argument $\tilde u\,, E \in L^8(0,T;L^4(\Omega))$. Otherwise, if $3 < r_{k+1} < 4$, an interpolation argument yields $E \in L^{s_{k+1}}(0,T;L^{r_{k+1}}(\Omega))$. Looking at (3.21), (3.22), we see that (3.19) and (3.20) are satisfied with the parameters $s_{k+1}\,, r_{k+1}\,, \beta_{k+1}$. Therefore, we can start a new step of this iterative argument. Repeating this step finitely many times, we get $\tilde u \in L^8(0,T;L^4(\Omega))$ which finishes the proof of Assertion 2.

Assertion 3: Let $4 < q < \infty$. Then $\tilde{u} \in L^8(0,T;L^4(\Omega))$.
Proof. Assume that we have for almost all $t \in [0,T[$ with certain parameters $s_k\,,r_k\,,\beta_k$

$$\tilde{u}(t) = -\int_0^t A_{r_k}^{\beta_k} e^{-\nu(t-\tau)A_{r_k}} A_{r_k}^{-\beta_k} P_{r_k} \mathrm{div}((\tilde{u}+E)\otimes(\tilde{u}+E))\,d\tau\,, \tag{3.23}$$

$$\tilde{u}\,,E \in L^{s_k}(0,T;L^{r_k}(\Omega)) \text{ with } 4 < r_k < \infty\,, \frac{2}{s_k} + \frac{3}{r_k} = 1\,, \beta_k \in [\frac{1}{2},1]. \tag{3.24}$$

Again, for $k=1$, the iteration starts with $s_1 := s$, $r_1 := q$ and $\beta_1 := \frac{3}{2q} + \frac{1}{2} = \alpha$, see (3.14). We set $r_{k+1} := \frac{3}{4}r_k$ and $\beta_{k+1} := \frac{3}{r_k} - \frac{3}{2r_{k+1}} + \frac{1}{2} = \frac{1}{r_k} + \frac{1}{2}$. Let $s_{k+1} > 2$ be the unique element which satisfies the relation $\frac{2}{s_{k+1}} + \frac{3}{r_{k+1}} = 1$. Then (3.7) implies that $\tilde{u}$ has the representation (3.22) with the new parameters $s_{k+1}\,,r_{k+1}\,,\beta_{k+1}$. From (3.22) we conclude with (3.8) that

$$\tilde{u} \in L^{s_{k+1}}(0,T;L^{r_{k+1}}(\Omega)). \tag{3.25}$$

From the first step of the proof we know that $E \in L^8(0,T;L^4(\Omega))$. There can occur two different possibilities. If $3 < r_{k+1} \le 4$ we get by an interpolation argument $\tilde{u}\,,E \in L^8(0,T;L^4(\Omega))$. Otherwise, if $4 < r_{k+1} < \infty$, we use an interpolation argument to get $E \in L^{s_{k+1}}(0,T;L^{r_{k+1}}(\Omega))$. If we look at (3.22), (3.25) we see that the equations (3.23) and (3.24) are satisfied with the parameters $s_{k+1}\,,r_{k+1}\,,\beta_{k+1}$. Therefore, we can start a new step of this iterative argument. Repeating this step finitely many times, we get $\tilde{u} \in L^8(0,T;L^4(\Omega))$ which finishes the proof of Assertion 3.

Now the claim (3.16) for $u = \tilde{u} + E$ follows, and the proof of this theorem is complete. ☐

4. Proof of Regularity Results

Before proving Theorems 1.2 and 1.3 we need a useful, but technical lemma. In this lemma we assume that u satisfies the strong energy inequality (1.5) to consider the term $u(t)$ for almost all $t \in [0,T]$ as initial value of a local strong solution which can be identified locally with u. Therefore, the proof will be based on Theorem 1.4. We will use the notation

$$⨍_a^b f(x)\,dx := \frac{1}{b-a}\int_a^b f(x)\,dx$$

for the mean value of an integral.

Lemma 4.1. *Let $\Omega\,,q\,,s\,,f\,,u_0\,,T$ satisfy the assumptions of Theorem 1.4, let $1 \le s' \le s$, and let u be a weak solution of (1.1) satisfying the strong energy inequaltiy (1.5). Then there exists a constant $\epsilon_* = \epsilon_*(q,s',\Omega) > 0$ with the following property: If $0 < t_0 < t \le t_1 \le T$, and if*

$$\int_{t_0}^{t_1} \|F(\tau)\|_{q*}^s\,d\tau \le \epsilon_* \nu^{2s-1}\,, \tag{4.1}$$

$$⨍_{t_0}^{t} (t_1-\tau)^{\frac{s'}{s}} \|u(\tau)\|_q^{s'}\,d\tau \le \epsilon_* \nu^{s'-\frac{s'}{s}}\,, \tag{4.2}$$

then there exists a $\delta = \delta(t) > 0$ such that $u \in L^s(t-\delta,t_1;L^q(\Omega))$. In particular, if $t_1 > t$, then t is a regular point of u.

Proof. We may assume that $u(\tau) \in L^2(\Omega)$ for all $\tau \in [0, T[$. From (4.2) and the fact that u satisfies the strong energy inequality we find a null set $N \subset]t_0, t[$ such that for $\tau_0 \in]t_0, t[\setminus N$ it holds $u(\tau_0) \in L^q_\sigma(\Omega)$ and

$$\frac{1}{2}\|u(\tau_1)\|_2^2 + \nu \int_{\tau_0}^{\tau_1} \|\nabla u\|_2^2 \, d\tau \le \frac{1}{2}\|u(\tau_0)\|_2^2 - \int_{\tau_0}^{\tau_1} \langle F, \nabla u\rangle_\Omega \, d\tau \tag{4.3}$$

for all τ_1 mit $\tau_0 \le \tau_1 < T$. Moreover, the condition (4.2) yields the existence of $\tau_0 \in]t_0, t[\setminus N$ which fulfills the inequality

$$(t_1 - \tau_0)^{\frac{s'}{s}} \|u(\tau_0)\|_q^{s'} \le \fint_{t_0}^{t} (t_1 - \tau)^{\frac{s'}{s}} \|u(\tau)\|_q^{s'} \, d\tau \le \epsilon_* \nu^{s' - \frac{s'}{s}}.$$

It follows with a constant $c = c(\Omega, q) > 0$ that

$$\begin{aligned}\int_0^{t_1 - \tau_0} \|e^{-\nu\tau A_q} u(\tau_0)\|_q^s \, d\tau &\le \int_0^{t_1-\tau_0} c\|u(\tau_0)\|_q^s \, d\tau \\ &= c(t_1 - \tau_0)\|u(\tau_0)\|_q^s \le c\,\epsilon_*^{\frac{s}{s'}} \nu^{s-1}.\end{aligned}$$

Hence with a new constant $\widetilde{\epsilon_*} := (\frac{\epsilon_*}{c})^{\frac{s'}{s}}$, where ϵ_* is the constant from Theorem 1.4, the conditions of Theorem 1.4 are satisfied. We get the existence of a unique weak solution $v \in L^s([\tau_0, t_1[; L^q_\sigma(\Omega))$ to the Navier-Stokes system (1.1) with initial value $v(\tau_0) = u(\tau_0)$. Considering u as a weak solution to the Navier-Stokes system with initial value $u(\tau_0)$ on $[0, t_1 - \tau_0]$, we use Serrin's Uniqueness Theorem to get that $u = v \in L^s(\tau_0, t_1; L^q_\sigma(\Omega))$. The proof is complete. □

Proof of Theorem 1.2. (1) Let $s := s'$, $t_0 := t - \delta$, $t_1 := t + \delta$ where $\delta > 0$ is chosen so small that, see (1.6),

$$\begin{gathered}\fint_{t-\delta}^{t} (t_1 - \tau)\|u(\tau)\|_q^s \, d\tau \le 2\int_{t-\delta}^{t} \|u(\tau)\|_q^s \, d\tau \le \epsilon_* \nu^{s-1}, \\ \int_{t-\delta}^{t+\delta} \|F(\tau)\|_r^s \, d\tau \le \epsilon_* \nu^{2s-1}.\end{gathered}$$

The assertion follows with Lemma 4.1.

(2) Because of (1.7) it is possible to choose a $\delta > 0$ such that with $t_0 := t - \delta$, $t_1 := t + \delta$ the estimate

$$\begin{aligned}\fint_{t-\delta}^{t} (t_1 - \tau)^{\frac{s'}{s}} \|u(\tau)\|_q^{s'} \, d\tau &\le \frac{1}{\delta}\int_{t-\delta}^{t} (2\delta)^{\frac{s'}{s}} \|u(\tau)\|_q^{s'} \, d\tau \\ &= \frac{2^{\frac{s'}{s}}}{\delta^{1-\frac{s'}{s}}} \int_{t-\delta}^{t} \|u(\tau)\|_q^{s'} \, d\tau \le \epsilon_* \nu^{s' - \frac{s'}{s}}\end{aligned}$$

holds. This shows (4.2). Furthermore, condition (4.1) on F can be fulfilled as well. Then Lemma 4.1 proves the sufficiency of (1.7) to imply regularity of u at t. Since by Hölder's inequaltiy

$$\frac{1}{\delta^{1-\frac{s'}{s}}} \int_{t-\delta}^{t} \|u(\tau)\|_q^{s'} \, d\tau \le \left(\int_{t-\delta}^{t} \|u(\tau)\|_q^s \, d\tau\right)^{\frac{s'}{s}}$$

we get that the condition (1.7) is also necessary for regularity of u at t.

(3) The constant $\epsilon_* = \epsilon_*(q, \Omega) > 0$ will be determined in the proof; therefore, we begin with considering ϵ_* as an arbitrary, fixed positive number. Let $\varepsilon_1 = \varepsilon_1(q, \Omega) > 0$ denote the constant from Theorem 1.4 which in (1.13), (1.14) is called ϵ_*, and let $\epsilon_2 = \epsilon_2(s', \Omega)$ be the constant in Lemma 1.5 called ϵ_* in (4.1), (4.2). We assume $\epsilon_* \le \varepsilon_1$ and $u_0 \neq 0$. It holds

$$\int_0^{\delta_1} \|e^{-\nu\tau A_q} u_0\|_q^s \, d\tau \le c\delta_1 \|u_0\|_q^s, \quad c = c(\Omega, q) > 0.$$

We define

$$\delta_1 := \min\Big(\frac{\varepsilon_1 \nu^{s-1}}{c\|u_0\|_q^s}, T\Big). \tag{4.4}$$

If $\delta_1 = T$, we already know that $u \in L^s(0, T; L^q(\Omega))$. So let us assume that $\delta_1 = \frac{\varepsilon_1 \nu^{s-1}}{c\|u_0\|_q^s}$. With this choice of δ_1, Theorem 1.4 yields the existence of a unique weak solution $v \in L^s(0, \delta_1; L^q(\Omega))$ of (1.1), which coincides by Serrin's Uniqueness with u on $[0, \delta_1[$. For an arbitrary $t \in \big[\frac{\delta_1}{2}, T - \frac{\delta_1}{2}\big]$, we get with $t_0 := t - \frac{\delta_1}{2}$, $t_1 := t + \frac{\delta_1}{2}$

$$\begin{aligned}
\int_{t_0}^{t} (t_1 - \tau)^{\frac{s'}{s}} \|u(\tau)\|_q^{s'} \, d\tau &\le \frac{2}{\delta_1^{1-\frac{s'}{s}}} \int_0^T \|u(\tau)\|_q^{s'} \, d\tau \\
&\le 2\left(\frac{\varepsilon_1 \nu^{s-1}}{c\|u_0\|_q^s}\right)^{\frac{s'}{s}-1} \epsilon_* \frac{\nu^{s-1}}{\|u_0\|_q^{s-s'}} \\
&= 2\Big(\frac{\varepsilon_1}{c}\Big)^{\frac{s'}{s}-1} \epsilon_* \nu^{s'-\frac{s'}{s}}.
\end{aligned} \tag{4.5}$$

From this estimate it follows that we may define

$$\epsilon_* := \min\Big(\frac{\varepsilon_2}{2}\Big(\frac{\varepsilon_1}{c}\Big)^{1-\frac{s'}{s}}, \varepsilon_1, \varepsilon_2\Big). \tag{4.6}$$

We see that ϵ_* depends only on Ω, q, s'. Using Lemma 4.1 we find a $\delta = \delta(t) > 0$ such that

$$u \in L^s(t - \delta(t), t + \frac{\delta_1}{2}; L^q(\Omega)). \tag{4.7}$$

With (4.7) and $u \in L^s(0, \delta_1; L^q(\Omega))$ we obtain due to the compactness of the interval $[0, T]$ that $u \in L^s(0, T; L^q(\Omega))$.

Now the theorem is completely proved. □

Proof of Theorem 1.3. By interpolation, in both cases the weak solution u satisfies $u \in L^{s'}(0, T; L^q(\Omega))$. The idea of the proof is to use Lemma 4.1. To control the term in (4.2) we use the interpolation inequality, see [1, Theorem 4.3.1],

$$\|v\|_q \le c\|v\|_2^{1-\frac{2}{s'}} \|\nabla v\|_2^{\frac{2}{s'}}, \quad v \in H_0^1(\Omega),$$

where $c = c(\Omega, q) > 0$. For $\delta \in]0, \delta_0[$ with a small $\delta_0 > 0$ we get with $t_0 := t - \delta$, $t_1 := t + \delta$ the estimate

$$\begin{aligned} I(\delta) &:= \fint_{t-\delta}^{t} (t_1 - \tau)^{\frac{s'}{s}} \|u(\tau)\|_q^{s'} \, d\tau \\ &\le c\delta^{\frac{s'}{s}-1} \int_{t-\delta}^{t} \left(\|u(\tau)\|_2^{1-\frac{2}{s'}} \|\nabla u(\tau)\|_2^{\frac{2}{s'}} \right)^{s'} d\tau \\ &\le c\delta^{\frac{s'}{s}-1} \|u\|_{2,\infty;T}^{s'-2} \int_{t-\delta}^{t} \|\nabla u(\tau)\|_2^2 \, d\tau \end{aligned} \tag{4.8}$$

with a constant $c = c(\Omega, q) > 0$. Since u is supposed to satisfy the strong energy inequality (1.5), we may proceed for almost all $\delta \in]0, \delta_0[$ as follows:

$$I(\delta) \le \frac{c}{\nu} \, \delta^{\frac{s'}{s}-1} \left(|E(t-\delta) - E(t)| + \left| \int_{t-\delta}^{t} \langle f, u \rangle \, d\tau \right| \right) \tag{4.9}$$

where the constant c depends on $\|u\|_{2,\infty;T}$ in the case $\alpha > \frac{1}{2}$ and $c = c(\Omega)$ if $\alpha = \frac{1}{2}$. By Hölder's inequality we get that

$$\left| \frac{1}{\delta^{\frac{s'}{4}}} \int_{t-\delta}^{t} \langle f(\tau), u(\tau) \rangle \, d\tau \right| \;\le\; \|u\|_{2,\infty;T} \left(\int_{t-\delta}^{t} \|f\|_2^{\frac{4}{4-s'}} \, d\tau \right)^{\frac{4-s'}{4}} . \tag{4.10}$$

As $\frac{s}{s'} = \frac{4}{4-s'}$ and consequently $f \in L^{\frac{4}{4-s'}}(0, T; L^2(\Omega))$, the left-hand side in the previous inequality converges to 0 as $\delta \to 0+$.

First consider the case $\alpha > \frac{1}{2}$ and choose $\epsilon > 0$ with $s' = 4\alpha - \epsilon$. Due to the assumption (1.9) we get with $1 - \frac{s'}{s} = \frac{s'}{4} = \alpha - \frac{\epsilon}{4}$

$$\lim_{\delta \to 0+} \frac{c}{\nu} \, \delta^{-\frac{s'}{4}} \, |E(t-\delta) - E(t)| = \lim_{\delta \to 0+} \frac{c}{\nu} \, \delta^{\frac{\epsilon}{4}} \, \frac{|E(t-\delta) - E(t)|}{\delta^{\alpha}} = 0. \tag{4.11}$$

Consequently the right hand side of (4.9) converges to 0 as $\delta \to 0+$. Hence we can fulfill (4.2) and, due to the assumption $F \in L^s(0, T; L^r(\Omega))$, it is also possible to satisfy (4.1). Altogether, Lemma 4.1 yields regularity of u at t.

Secondly, consider the case $\alpha = \frac{1}{2}$ in which $s' = 2$, $s = 4$. We will choose the constant $\gamma_* = \gamma_*(\Omega) > 0$ below. Let $\epsilon_* = \epsilon_*(q) > 0$ denote the constant from Lemma 4.1. The assumption (1.10) implies that for all $0 < \delta < \mu$

$$\frac{1}{\nu} \frac{|E(t-\delta) - E(t)|}{\delta^{\frac{1}{2}}} \le \gamma_* \nu^{\frac{3}{2}} . \tag{4.12}$$

Then by (4.9), (4.10) and (4.12) we get with a constant $c = c(\Omega) > 0$ for almost all $\delta \in]0, \delta_0[$ that

$$I(\delta) \le c\gamma_* \nu^{\frac{3}{2}} + \frac{c}{\nu} \|u\|_{2,\infty;T} \left(\int_{t-\delta}^{t} \|f\|_2^2 \, d\tau \right)^{\frac{1}{2}} .$$

Now with $\gamma_* := \frac{\epsilon_*}{2c}$ we find $0 < \delta < \mu$ such that $I(\delta) \le \epsilon_* \nu^{\frac{3}{2}}$, cf. (4.2), and that (4.1) is satisfied. Hence Lemma 4.1 implies regularity of u at t. □

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Reinhard Farwig, Darmstadt University of Technology, Department of Mathematics, 64283 Darmstadt, Germany
E-mail address: `farwig@mathematik.tu-darmstadt.de`

Christian Komo, Darmstadt University of Technology, Department of Mathematics, 64283 Darmstadt, Germany
E-mail address: `komo@mathematik.tu-darmstadt.de`