Leray’s inequality in general multi-connected domains in $\mathbb{R}^n$

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Dedicated to Professor Izumi Takagi on the occasion of his 60th birthday.

Abstract
Consider the stationary Navier-Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^n$ whose boundary $\partial \Omega$ consists of $L + 1$ smooth $n - 1$ dimensional closed hypersurfaces $\Gamma_0, \Gamma_1, \cdots, \Gamma_L$, where $\Gamma_1, \cdots, \Gamma_L$ lie inside of $\Gamma_0$ and outside of one another. The Leray inequality of the given boundary data $\beta$ on $\partial \Omega$ plays an important role for the existence of solutions. It is known that if the flux $\gamma_i \equiv \int_{\Gamma_i} \beta \cdot \nu dS = 0$ on $\Gamma_i (\nu$: the unit outer normal to $\Gamma_i)$ is zero for each $i = 0, 1, \cdots, L$, then the Leray inequality holds. We prove that if there exists a sphere $S$ in $\Omega$ separating $\partial \Omega$ in such a way that $\Gamma_1, \cdots, \Gamma_k$ (1 $\leq$ $k$ $\leq$ $L$) are contained inside of $S$ and that the others $\Gamma_{k+1}, \cdots, \Gamma_L$ are outside of $S$, then the Leray inequality necessarily implies that $\gamma_1 + \cdots + \gamma_k = 0$. In particular, suppose that there are $L$ spheres $S_1, \cdots, S_L$ in $\Omega$ such that $\Gamma_i$ lies inside of $S_i$ for all $i = 1, \cdots, L$. Then the Leray inequality holds if and only if $\gamma_0 = \gamma_1 = \cdots = \gamma_L = 0$.

1 Introduction.
We consider Leray’s problem on the stationary Navier-Stokes equations with the inhomogeneous boundary data under the general flux condition. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$ with smooth boundary $\partial \Omega$. Throughout this paper, we impose the following assumption on $\Omega$.

**Assumption.** The boundary $\partial \Omega$ has $L + 1$ connected components $\Gamma_0, \Gamma_1, \cdots, \Gamma_L$ of $n - 1$ dimensional $C^\infty$-closed hypersurfaces such that $\Gamma_1, \cdots, \Gamma_L$ lie inside of $\Gamma_0$ and outside of one other;

$$\partial \Omega = \bigcup_{j=0}^{L} \Gamma_j.$$
In the most interesting case when $n = 3$, it is often called that $\Omega$ has the second Betti number $L$. In $\Omega$ we consider the boundary value problem for the stationary Navier-Stokes equations:

\begin{equation}
\begin{aligned}
&-\mu \Delta v + v \cdot \nabla v + \nabla p = 0 \quad \text{in } \Omega, \\
&\text{div } v = 0 \quad \text{in } \Omega, \\
&v = \beta \quad \text{on } \partial \Omega,
\end{aligned}
\end{equation}

where $v = v(x) = (v_1(x), \ldots, v_n(x))$ and $p = p(x)$ denote the unknown velocity vector and the unknown pressure at the point $x = (x_1, \ldots, x_n) \in \Omega$, while $\mu > 0$ is the given viscosity constant, and $\beta = \beta(x) = (\beta_1(x), \ldots, \beta_n(x))$ is the given boundary data on $\partial \Omega$. We use the standard notation as $\Delta v = \sum_{j=1}^{n} \frac{\partial^2 v}{\partial x_j^2}$, $\nabla p = \left( \frac{\partial p}{\partial x_1}, \ldots, \frac{\partial p}{\partial x_n} \right)$, $\text{div } v = \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}$, and $v \cdot \nabla v = \sum_{j=1}^{n} v_j \frac{\partial v}{\partial x_j}$.

Since the solution $v$ satisfies $\text{div } v = 0$ in $\Omega$, the given boundary data $\beta$ on $\partial \Omega$ is required to fulfill the following compatibility condition which we call the general flux condition:

\begin{equation}
\sum_{j=0}^{L} \int_{\Gamma_j} \beta \cdot \nu dS = 0,
\end{equation}

where $\nu$ denotes the unit outer normal to $\partial \Omega$. Leray [11] proposed to solve the following problem.

**Leray’s problem.** Let $n = 2, 3$. Suppose that $\beta \in H^{1/2}(\partial \Omega)$ satisfies the general flux condition (G.F.). Does there exist at least one weak solution $v \in H^1(\Omega)$ of (N-S) ?

Up to now, we are not yet successful to give a complete answer to this question. However, some partial answer has been proved by Leray [11], Fujita[3] and Ladyzhenskaya [10] under the restricted flux condition (R.F.) on $\beta$:

\begin{equation}
\gamma_i \equiv \int_{\Gamma_j} \beta \cdot \nu dS = 0 \quad \text{for all } j = 0, 1, \ldots, L.
\end{equation}

Indeed, under the restricted flux condition (R.F.) on $\beta$, they showed that there exists at least one weak solution $v$ of (N-S). Although more refined existence results were given by Galdi [6, Chapter VIII, Theorem 4.1] and the second and the third authors [9], it is still an open problem to prove an existence theorem under the general flux condition (G.F.).

If the given boundary data $\beta$ satisfies the general flux condition (G.F.), then there exists an extension $b$ into $\Omega$ with $b|_{\partial \Omega} = \beta$ such that $\text{div } b = 0$. See e.g., Borchers-Sohr [2]. We call such $b$ a solenoidal extension into $\Omega$ of $\beta$. Introducing a new unknown variable $u \equiv v - b$, we can reduce the original equations (N-S) to the following ones with the homogeneous boundary condition:

\begin{equation}
\begin{aligned}
&-\mu \Delta u + b \cdot \nabla u + u \cdot \nabla b + u \cdot \nabla u + \nabla p = \mu \Delta b - b \cdot \nabla b \quad \text{in } \Omega, \\
&\text{div } u = 0 \quad \text{in } \Omega, \\
&u = 0 \quad \text{on } \partial \Omega.
\end{aligned}
\end{equation}

To solve (N-S’) we need to handle the linear convection term $b \cdot \nabla u + u \cdot \nabla b$ as a perturbation of $-\mu \Delta u$. More precisely, to prove the existence of the solution $u$ of (N-S’), we rely on the following Leray inequality.
**Definition 1** Let $\Omega$ be as in the Assumption, and let $\beta \in H^{1/2}(\partial \Omega) \cap W^{1-\frac{2}{n}, \frac{2}{n}}(\Omega)$. Suppose that $\beta$ fulfills (G.F.) . We say that $\beta$ satisfies the Leray inequality in $\Omega$ if for every $\varepsilon > 0$ there exists $b_\varepsilon \in H^{1}(\Omega) \cap W^{1, \frac{2}{2}}(\Omega)$ with $\text{div} \ b_\varepsilon = 0$ in $\Omega$ and $b_\varepsilon = \beta$ on $\partial \Omega$ such that

$$(L.I.) \quad |(u \cdot \nabla b_\varepsilon, u)| \leq \varepsilon \|\nabla u\|^2_{L^2(\Omega)} \text{ for all } u \in H^1_{0,\sigma}(\Omega),$$

where $H^1_{0,\sigma}(\Omega) = \{ u \in H^1(\Omega); \text{div} \ u = 0 \}$ and $(\cdot, \cdot)$ denotes the usual inner product in $L^2(\Omega)$.

For $u \in H^1_{0,\sigma}(\Omega)$ and $b_\varepsilon \in H^1(\Omega) \cap W^{1, \frac{2}{2}}(\Omega)$, the left hand side of (L.I.) is well-defined since we have by the Sobolev imbedding $H^1_{0}(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega)$ that

$$|(u \cdot \nabla b_\varepsilon, u)| \leq \|u\|^2_{L^{\frac{2n}{n-2}}(\Omega)} \|\nabla b_\varepsilon\|_{L^{\frac{2}{2}}(\Omega)} \leq C\|\nabla u\|^2_{L^2(\Omega)}\|\nabla b_\varepsilon\|_{L^{\frac{2}{2}}(\Omega)}.$$

Notice that it holds a continuous imbedding $H^{1/2}(\partial \Omega) \subset W^{1-\frac{2}{n}, \frac{2}{n}}(\partial \Omega)$ provided $n = 2, 3, 4$. So, the space $W^{1-\frac{2}{n}, \frac{2}{n}}(\partial \Omega)$ for $\beta$ is meaningful only for $n \geq 5$.

For a moment, let us assume that $n = 2$ or $n = 3$. Once (L.I.) is established, by the well-known identities $(b \cdot \nabla u, u) = (u \cdot \nabla u, u) = 0$ and $(\nabla p, u) = 0$ for $u \in H^1_{0,\sigma}(\Omega)$, we obtain from (L.I.) with $\varepsilon = \mu/2$ such an apriori estimate that

$$\|\nabla u\|_{L^2(\Omega)} \leq 2\mu^{-1}\|\mu \Delta b - b \cdot \nabla b\|_{H^{-1}(\Omega)},$$

which yields the solution $u$ of (N-S') with the aid of the Leray-Schauder fixed point theorem.

If the given boundary data $\beta$ satisfies the restricted flux condition (R.F.), then we see that $\beta$ fulfills (L.I.). Indeed, under the hypothesis of the restricted flux condition (R.F.) on $\beta$, the solenoidal extension $b$ into $\Omega$ of $\beta$ has a vector potential $w \in H^2(\Omega)$, which means that $b$ can be expressed as $b = \text{rot} \ w$. Taking a family $\{\theta_\varepsilon\}_{\varepsilon > 0}$ of cut-off functions with $\theta_\varepsilon(x) \equiv 1$ for $x$ near the boundary $\partial \Omega$ so that the support of $\theta_\varepsilon$ is confined in an arbitrarily narrow closed strip to $\partial \Omega$ as $\varepsilon \to +0$, and then redefining $b_\varepsilon$ as $b_\varepsilon(x) \equiv \text{rot} \ (\theta_\varepsilon(x)w(x))$, we see that $\beta$ satisfies (L.I.). For instance, see Temam [14, Chapter II, Lemma 1.8] and Galdi [6, Chapter VIII, Lemma 4.2]. It should be noted that the boundary value of $b_\varepsilon$ is invariant under the multiplication of $w$ by $\theta_\varepsilon$. In the previous work [8, Remark 1 (4)](see also [9]), we showed that the solenoidal extension $b$ into $\Omega$ of $\beta$ has a vector potential if and only if $\beta$ satisfies the restricted flux condition (R.F.).

Now the natural question arises whether the general flux condition (G.F.) implies (L.I.). Unfortunately, Takeshita[13] gave a negative answer to this question. Indeed, he treated the annular domain $\Omega = \{ x \in \mathbb{R}^n; R_1 < |x| < R_0 \}$ with $\Gamma_0 = \{ x \in \mathbb{R}^n; |x| = R_0 \}$, $\Gamma_1 = \{ x \in \mathbb{R}^n; |x| = R_1 \}$, and proved that $\beta$ satisfies (L.I.) in such an annulus $\Omega$ if and only if

$$\int_{\Gamma_0} \beta \cdot \nu dS = \int_{\Gamma_1} \beta \cdot \nu dS = 0.$$

Another refined proof in the 2D annular region was given by Galdi [6, page 23]. In the last part of Takeshita’s paper [13, Theorem 2], he treated the domain $\Omega$ as in the Assumption and stated without any detail that if for each $i = 0, 1, \cdots, L$, $\Gamma_i$ is diffeomorphically deformed to the sphere in $\bar{\Omega}$, then the Leray inequality (L.I.) is equivalent to (R.F.).
In this paper, we generalize Galdi-Takeshita’s result with a simple proof. Although our result is not altogether new, we do not need to impose any topological restriction on the boundary, while Takeshita [13] requires that each $\Gamma_i$, $i = 0, 1, \cdots, L$, is diffeomorphic to the sphere. The main theorem now reads:

**Theorem 1** Let $n \geq 3$ and let $\Omega$ be as in the Assumption. Suppose that $\beta \in H^{1/2}(\partial \Omega) \cap W^{1-\frac{2}{n}, \frac{n}{2}}(\partial \Omega)$ and that $\beta$ satisfies (G.F.). Assume that there is a sphere $S$ in $\Omega$ such that $\Gamma_1, \cdots, \Gamma_k$ lie inside of $S$ and such that the others $\Gamma_{k+1}, \cdots, \Gamma_L$ and $\Gamma_0$ lie outside of $S$. If $\beta$ satisfies (L.I.) in $\Omega$, then we have

$$
\gamma_1 + \cdots + \gamma_k = 0, \quad \gamma_{k+1} + \cdots + \gamma_L + \gamma_0 = 0.
$$

As an immediate consequence of this theorem, we obtain the following necessary and sufficient condition on the Leray inequality.

**Corollary 1** Let $n \geq 3$ and let $\Omega$ be as in the Assumption. Suppose that $\beta \in H^{1/2}(\partial \Omega) \cap W^{1-\frac{2}{n}, \frac{n}{2}}(\partial \Omega)$ and that $\beta$ satisfies (G.F.). Assume that there exist $L$ spheres $S_1, \cdots, S_L$ in $\Omega$ such that $S_i$ contains only $\Gamma_i$ in its inside and the rests $\partial \Omega \setminus \Gamma_i$ lie in the outside of $S_i$ for all $i = 1, \cdots, L$. Then $\beta$ satisfies (L.I.) in $\Omega$ if and only if (R.F.) holds.

**Remarks.**

1. Corollary 1 may be regarded as a generalization of Takeshita [13, Theorem 2] since it is only assumed that each component $\Gamma_i$, $i = 1, \cdots, L$ is a smooth $n-1$ dimensional closed hypersurface in $\mathbb{R}^n$ with $n \geq 3$.

2. The assumption on regularity of the boundary $\partial \Omega$ can be relaxed so that the Stokes integral formula holds for vector fields on $\Omega$. For instance, Theorem 1 holds for bounded locally Lipschitz domains $\Omega$. More generally, we may treat the case when $\Omega$ is a bounded domain in $\mathbb{R}^n$ with locally finite perimeter as in Ziemer [15, Theorem 5.8.2].

3. A similar argument to make use of the sphere covering each component of the boundary was established by Kobayashi [7] in the two-dimensional multi-connected domains. Indeed, he proved the corresponding result to Corollary 1 in the plane. However, it seems difficult to apply his method directly to our higher-dimensional case.

4. Under some hypothesis on symmetry of the multi-connected domain $\Omega$ in $\mathbb{R}^2$, Amick [1] and Fujita [4] proved (L.I.) for all solenoidal vector fields $u$ with symmetry. See also Morimoto [12].

5. For solvability of (N-S’) itself in the case $n = 2, 3$, the Leray inequality (L.I.) can be relaxed to the following weaker condition (E.C.).

$$
-(u \cdot \nabla b_\varepsilon, u) \leq \varepsilon \|\nabla u\|_{L^2(\Omega)}^2 \quad \text{for all } u \in H^1_{0,\sigma}(\Omega).
$$

Galdi[6, page 21] called it extension condition on $\beta$, and proved that in the 2D annular region (E.C.) necessarily implies (R.F.) for $j = 0, 1$ under the inflow condition that $\gamma_j \leq 0$. It is also possible to prove in Theorem 1 that (E.C.) yields the same conclusion as (1.1) provided $\gamma_1 + \cdots + \gamma_k \leq 0$.

In Theorem 1, it is sufficient to cover $\Gamma_1, \cdots, \Gamma_k$ by the sphere $S$ in $\Omega$ so that the rests of $\Gamma_{k+1}, \cdots, \Gamma_L$ and $\Gamma_0$ lie outside of $S$. Taking a slightly larger sphere $S'$ in $\Omega$ containing $S$ with
its same origin so that $\Gamma_{k+1}, \ldots, \Gamma_L$ and $\Gamma_0$ also lie outside of $S'$, we may reduce the problem to that in the annular domain $D$ between $S$ and $S'$. Indeed, if the given boundary data $\beta$ satisfies (G.F.) and (L.I.) with some solenoidal vector field $b_\varepsilon \in H^1(\Omega) \cap W^{1, \frac{2}{n}}(\Omega)$, then it holds

$$\int_S b_\varepsilon \cdot \nu dS = \sum_{i=1}^k \gamma_i \equiv \gamma, \quad \int_{S'} b_\varepsilon \cdot \nu dS = -\gamma.$$

In the similar manner to Takeshita [13], by introducing the mean $M(b_\varepsilon)$ of $b_\varepsilon$ with respect to the normalized Haar measure on the $SO(n)$-action, we see that each flux on $S$ and $S'$ of $M(b_\varepsilon)$ remains invariant, and that the inequality

$$(1.2) \quad \left| \int_D u \cdot \nabla M(b_\varepsilon) \cdot \nu dx \right| \leq \varepsilon \int_D |\nabla u|^2 dx$$

holds for all $u \in C_0^\infty(D)$ with $\text{div} u = 0$. An appropriate choice of $u$ in (1.2) enables us to obtain $\gamma = 0$.

### 2 Proof of Theorem 1.

Suppose that the boundary data $\beta \in H^{1/2}(\partial \Omega) \cap W^{1-\frac{2}{n}, \frac{2}{n}}(\partial \Omega)$ satisfies the Leray inequality in $\Omega$ in the sense of Definition 1. Then, for every $\varepsilon > 0$ there exists $b_\varepsilon \in H^1(\Omega) \cap W^{1, \frac{2}{n}}(\Omega)$ with $\text{div} b_\varepsilon = 0$ in $\Omega$ and $b_\varepsilon = \beta$ such that (L.I.) holds. By the hypothesis on $\partial \Omega$, without loss of generality, we may take $0 < R < R'$ such that both spheres $S_R \equiv \{x \in \mathbb{R}^n; |x| = R\}$ and $S_{R'} \equiv \{x \in \mathbb{R}^n; |x| = R'\}$ are contained in $\Omega$, and such that $\Gamma_{k+1}, \ldots, \Gamma_L$ and $\Gamma_0$ lie outside of $S_{R'}$. Since $\sum_{i=0}^L \gamma_i = 0$, implied by (G.F.), and since $\text{div} b_\varepsilon = 0$ in $\Omega$ with $b_\varepsilon = \beta$ on $\partial \Omega$, it holds

$$(2.1) \quad \int_{S_R} b_\varepsilon \cdot \nu dS = \gamma \equiv \sum_{i=1}^k \gamma_i, \quad \int_{S_{R'}} b_\varepsilon \cdot \nu dS = -\gamma.$$

Now we reduce our problem to that in the concentric spherical domain $D \equiv \{x \in \mathbb{R}^n; R < |x| < R'\}$ and follow the argument given by Takeshita [13].

Let us take the mean $M(b_\varepsilon)$ of $b_\varepsilon$ with respect to the normalized Haar measure $dg$ on $SO(n)$-action. That is,

$$M(b_\varepsilon) = \int_{SO(n)} T_g b_\varepsilon dg,$$

$$T_g b_\varepsilon(x) = g b_\varepsilon(g^{-1}x), \quad x \in D, g \in SO(n).$$

By (2.1) it holds

$$\left\{ \begin{array}{l} \text{div} M(b_\varepsilon) = 0 \quad \text{in} \; D, \\ \int_{S_R} M(b_\varepsilon) \cdot \nu dS = \gamma, \quad \int_{S_{R'}} M(b_\varepsilon) \cdot \nu dS = -\gamma. \end{array} \right.$$
Furthermore, by (L.I.) we have

\( (2.3) \quad \left| \int_D v \cdot \nabla M(b_\varepsilon) \cdot vdx \right| \leq \varepsilon \int_D |\nabla v|^2 dx \quad \text{for all } v \in C^\infty_0(D), \)

where \( C^\infty_0(D) \) is the set of all solenoidal vector fields with compact support in \( D \). Indeed, since \( \det g = 1 \), by changing the variable \( x \in D \to y = g^{-1}x \in D \), we have

\[
\int_D v \cdot \nabla(T_g b_\varepsilon) \cdot vdx = \int_D T_g^{-1}v \cdot \nabla b_\varepsilon \cdot T_g^{-1}vdy
\]

for all \( g \in SO(n) \), which yields with the aid of the Fubini theorem that

\( (2.4) \quad \left| \int_D v \cdot \nabla(Mb_\varepsilon) \cdot vdx \right| = \left| \int_{SO(n)} \left( \int_D T_g^{-1}v \cdot \nabla b_\varepsilon \cdot T_g^{-1}vdy \right) dg \right| \leq \int_{SO(n)} \left| \int_D T_g^{-1}v \cdot \nabla b_\varepsilon \cdot T_g^{-1}vdy \right| dg. \)

Since \( T_g^{-1}v \in C^\infty_0(D) \) and since \( |\nabla T_g^{-1}v(y)|^2 = |\nabla v(gy)|^2 \) for all \( y \in D \), we have by (L.I.) and again by changing variable \( y \in D \to x = gy \in D \) with \( \det g^{-1} = 1 \) that

\[
\left| \int_D T_g^{-1}v \cdot \nabla b_\varepsilon \cdot T_g^{-1}vdy \right| = \left| \int_\Omega T_g^{-1}v \cdot \nabla b_\varepsilon \cdot T_g^{-1}vdy \right| \\
\leq \varepsilon \int_\Omega |\nabla T_g^{-1}v|^2 dy \\
= \varepsilon \int_D |\nabla T_g^{-1}v|^2 dy \\
= \varepsilon \int_D |\nabla v|^2 dx
\]

for all \( g \in SO(n) \). It follows from (2.4) and (2.5) that

\[
\left| \int_D v \cdot \nabla(Mb_\varepsilon) \cdot vdx \right| \leq \varepsilon \int_{SO(n)} \left( \int_D |\nabla v|^2 dx \right) dg = \varepsilon \int_D |\nabla v|^2 dx,
\]

which implies (2.3).

In the next step, we test (2.3) by an appropriate \( v \in C^\infty_0(D) \). First, it follows from (2.2) that \( M(b_\varepsilon) \) has the representation as

\( (2.6) \quad M(b_\varepsilon) = \frac{\gamma}{\omega_n r^n} x, \quad x \in D, \)

where \( r = |x| \) and \( \omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \) is the surface area of \( S^{n-1} \). Now, we choose a test vector function \( v \) of (2.3) as

\[
v(x) = (-\rho(r)x_2, \rho(r)x_1, 0, \ldots, 0), \quad x = (x_1, \ldots, x_n) \in D
\]
with $\rho \in C_0^\infty((R, R'))$. It is easy to see that $v \in C_{0,\sigma}^\infty(D)$ with the property that $v(x) \cdot x = 0$ for all $x \in D$. Since

$$\frac{\partial}{\partial x_j} M(b_\varepsilon)_k = \frac{\gamma}{\omega_n r^n} \left( \delta_{jk} - n \frac{x_j x_k}{r} \right), \quad j, k = 1, \ldots, n$$

and since $v(x) \cdot x = 0$ for all $x \in D$, it holds that

$$v \cdot \nabla M(b_\varepsilon) \cdot v = \sum_{j,k=1}^n v_j \frac{\partial}{\partial x_j} M(b_\varepsilon)_k v_k = \frac{\gamma}{\omega_n r^n} \left( |v|^2 - n \left( \frac{v \cdot x}{r} \right)^2 \right)$$

(2.7)

in $D$. Hence it follows from (2.3) and (2.7) that

$$\frac{\gamma}{\omega_n} \int_{D} |v|^2 \, dx \leq \varepsilon \int_{D} |\nabla v|^2 \, dx$$

(2.8)

for all $\varepsilon > 0$. Since the left and side of (2.8) is independent of $\varepsilon$ and since $\nabla v \neq 0$, by letting $\varepsilon \to 0$ we conclude from (2.8) that

$$\gamma = 0.$$

This proves Theorem 1.

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**References**


