Extensions of Serrin's uniqueness and regularity conditions for the Navier-Stokes equations

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Abstract

Consider a smooth bounded domain $\Omega \subseteq \mathbb{R}^3$, a time interval $[0, T)$, $0 < T \le \infty$, and a weak solution u of the Navier-Stokes system. Our aim is to develop several new sufficient conditions on u yielding uniqueness and/or regularity. Based on semigroup properties of the Stokes operator we obtain that the local left-hand Serrin condition for each $t \in (0, T)$ is sufficient for the regularity of u. Somehow optimal conditions are obtained in terms of Besov spaces. In particular we obtain such properties under the limiting Serrin condition $u \in L^\infty_{\text{loc}}([0, T); L^3(\Omega))$. The complete regularity under this condition has been shown recently for bounded domains using some additional assumptions in particular on the pressure. Our result avoids such assumptions but yields global uniqueness and the right-hand regularity at each time when $u \in L^\infty_{\text{loc}}([0, T); L^3(\Omega))$ or when $u(t) \in L^3(\Omega)$ pointwise and u satisfies the energy equality. In the last section we obtain uniqueness and right-hand regularity for completely general domains.

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1 Introduction and Preliminaries

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$ and let $0 < T \leq \infty$. Then we consider in $[0,T) \times \Omega$ the Navier-Stokes system

$$\begin{aligned} u_t - \Delta u + u \cdot \nabla u + \nabla p &= 0, \quad \operatorname{div} u = 0 \\ u\big|_{\partial\Omega} &= 0, \quad u\big|_{t=0} = u_0\,. \end{aligned} \tag{1.1}$$

In particular, we are interested in weak solutions u of this system for initial values $u_0 \in L^2_\sigma(\Omega)$; here p means the associated pressure.

Definition 1.1: *Let $u_0 \in L^2_\sigma(\Omega)$. Then a vector field*

$$u \in L^\infty(0,T; L^2_\sigma(\Omega)) \cap L^2(0,T; W^{1,2}_0(\Omega)) \tag{1.2}$$

is called a (Leray-Hopf type) weak solution of the system (1.1)*, if the relation*

$$-\langle u, w_t\rangle_{\Omega,T} + \langle \nabla u, \nabla w\rangle_{\Omega,T} - \langle uu, \nabla w\rangle_{\Omega,T} = \langle u_0, w(0)\rangle_\Omega \tag{1.3}$$

holds for each $w \in C_0^\infty([0,T); C^\infty_{0,\sigma}(\Omega))$, and if the strong energy inequality

$$\frac{1}{2}||u(t)||_2^2 + \int_{t_0}^t ||\nabla u||_2^2\, d\tau \leq \frac{1}{2}||u(t_0)||_2^2 \tag{1.4}$$

holds for almost all $t_0 \in [0,T)$, including for $t_0 = 0$, and all $t \in [t_0, T)$.

Usually, the energy inequality (1.4) is supposed for weak solutions u only for $t_0 = 0$. However, since Ω is bounded, we can prove the existence of weak solutions u satisfying (1.2) - (1.4), see [10, Theorem V.3.6.2].

Another interesting problem concerns the *energy equality*

$$\frac{1}{2}\,||u(t)||_2^2 + \int_{t_0}^t ||\nabla u||_2^2\; d\tau = \frac{1}{2}\,||u(t_0)||_2^2 \quad \text{for all } 0 \leq t_0 \leq t < T, \tag{1.5}$$

describing the precise energy balance between the kinetic energy $\frac{1}{2}\,||u(t)||_2^2$ and the dissipation energy $\int\limits_{t_0}^{t} ||\nabla u||_2^2\, d\tau$ in the interval $[t_0, t]$.

To prove (1.5) we need an additional condition on the given weak solution u. Assume that u satisfies one of the following conditions:

a) $uu \in L^2_{\mathrm{loc}}([0,T); L^2(\Omega))$, (1.6)

b) $u \in L^4_{\mathrm{loc}}([0,T); L^4(\Omega))$, (1.7)

c) $u \in L^s_{\mathrm{loc}}([0,T); L^q(\Omega)),\ 2 < s \leq \infty,\ 3 \leq q < \infty,\ \dfrac{2}{s} + \dfrac{3}{q} = 1.$ (1.8)

Then the energy equality (1.5) is satisfied for all $0 \le t_0 \le t < T$.

Concerning the proof for a), see [10, Theorem V.3.6.1], obviously b) implies a), and to prove the assertion in c) we observe that "$c) \Rightarrow a)$" follows from the embedding inequality

$$\begin{aligned} \|uu\|_{L^2(0,T';L^2(\Omega))} &\le C\,\|u\|_{L^s(0,T';L^q(\Omega))}\,\|u\|_{L^{s_1}(0,T';L^{q_1}(\Omega))} \\ &\le C\,\|u\|_{L^s(0,T';L^q(\Omega))}\,\|\nabla u\|^{3/q}_{L^2(0,T';L^2(\Omega))}\,\|u\|^{2/s}_{L^\infty(0,T';L^2(\Omega))}\,, \end{aligned} \tag{1.9}$$

$0 < T' \le T, T' < \infty$, where $C = C(\Omega) > 0$ is a constant and $s_1 = (\frac{1}{2} - \frac{1}{s})^{-1}, q_1 = (\frac{1}{2} - \frac{1}{q})^{-1}$; see [10, Lemma V.1.2.1, b)]. Note that the case $s = \infty, q = 3$ is included in (1.8).

It is important in Definition 1.1 that, after redefinition on a null set of $[0,T)$,

$$u : [0,T) \to L^2_\sigma(\Omega) \qquad \text{is weakly continuous,} \tag{1.10}$$

see [10, Theorem V.1.3.1]. Therefore, each value $u(t) \in L^2_\sigma(\Omega)$, $t \in [0,T)$, and, in particular, the condition $u\big|_{t=0} = u(0) = u_0$ are well defined.

A weak solution u as in Definition 1.1 is called a *strong solution* of (1.1) if Serrin's condition

$$u \in L^s_{\text{loc}}([0,T); L^q(\Omega)),\ 2 < s < \infty,\ 3 < q < \infty,\ \frac{2}{s} + \frac{3}{q} = 1 \tag{1.11}$$

is satisfied. It is well known, see e.g. [10, Theorem V.1.8.2], that a strong solution u is regular in $(0,T) \times \Omega$ and uniquely determined within the class of Leray-Hopf weak solutions.

In this context we also consider the following restricted Serrin condition. The weak solution u satisfies the local right-hand $L^s(L^q)$-Serrin condition in $[0,T)$ if

$$u \in L^s(t, t+\delta; L^q(\Omega)),\ 2 < s < \infty,\ 3 < q < \infty,\ \frac{2}{s} + \frac{3}{q} = 1 \tag{1.12}$$

holds for each $0 \le t \le T$ with some $\delta = \delta(t) > 0,\ t + \delta < T$. Obviously, in this case u is regular in each right-hand interval $(t, t+\delta) \subseteq [0,T)$.

Next we explain some notations. By $\langle \cdot, \cdot \rangle_\Omega$ we denote the pairing of vector fields in Ω, and $\langle \cdot, \cdot \rangle_{\Omega,T}$ means the corresponding pairing in $[0,T) \times \Omega$. Given a vector field $u = (u_1, u_2, u_3)$ in Ω, let $u \cdot \nabla u = (u \cdot \nabla)u = u_1 D_1 u + u_2 D_2 u + u_3 D_3$ where $D_j = \frac{\partial}{\partial x_j}, j = 1,2,3,\ x = (x_1, x_2, x_3) \in \Omega,\ \nabla = (D_1, D_2, D_3)$. Further let $uu = (u_i u_j)^3_{i,j=1}$ such that $u \cdot \nabla u = \operatorname{div} uu = (D_1(u_1 u_j) + D_2(u_2 u_j) + D_3(u_3 u_j))^3_{j=1}$ if $\operatorname{div} u = \nabla \cdot u = D_1 u_1 + D_2 u_2 + D_3 u_3$ vanishes. Finally, we set $u_t = \frac{\partial u}{\partial t}$.

With $C^\infty_{0,\sigma}(\Omega) = \{v \in C^\infty_0(\Omega) :\ \operatorname{div} v = 0\}$ we define $L^q_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}$, $1 < q < \infty$, where $\|\cdot\|_q$ denotes the norm of the Lebesgue space $L^q(\Omega)$. Further, $W^{k,q}(\Omega),\ k \in \mathbb{N}$, and $W^{k,q}_0(\Omega) = \overline{C^\infty_0(\Omega)}^{\|\cdot\|_{W^{k,q}(\Omega)}}$ denote the usual Sobolev spaces.

Further we need the following spaces of solenoidal vector fields, see also [2, Theorem 3.2]. Let $L^{r,s}(\Omega)$, $1 \le r < \infty$, $1 \le s \le \infty$, with norm $||\cdot||_{L^{r,s}}$ denote the usual Lorentz space, see [12, 1.18.6]. In particular, for $r = 3$, define

$$L^{3,s}_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{||\cdot||_{L^{3,s}}}, \quad 1 \le s < \infty,$$

cf. [1, (0.16)]. See [1, (0.17)] concerning $L^{r,\infty}_\sigma(\Omega)$ with $r > q$. Special Besov spaces will be considered in §4.

We also need the Bochner spaces $L^s(0,T;L^q(\Omega))$, $1 < s,\ q < \infty$, with norm

$$||\cdot||_{L^s(0,T;L^q(\Omega))} = ||\cdot||_{q,s;T} = \left(\int_0^T ||\cdot||_q^s \, d\tau\right)^{1/s}, \tag{1.13}$$

and also the spaces $L^\infty(0,T,L^q(\Omega))$, $L^\infty_{\text{loc}}([0,T);L^q(\Omega))$, and $L^2(0,T;W^{1,2}_0(\Omega))$.

Let $P_q : L^q(\Omega) \to L^q_\sigma(\Omega)$ denote the Helmholtz projection, and let $A_q = -P_q\Delta : D(A_q) \to L^q_\sigma(\Omega)$ be the Stokes operator with domain $D(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$ and range $R(A_q) = L^q_\sigma(\Omega)$. Then $A^\alpha_q : D(A^\alpha_q) \to L^q_\sigma(\Omega)$, $-1 \le \alpha \le 1$, denote the fractional powers with $D(A_q) \subseteq D(A^\alpha_q) \subseteq L^q_\sigma(\Omega)$, $R(A^\alpha_q) = L^q_\sigma(\Omega)$ for $0 \le \alpha \le 1$. For a bounded smooth domain, the domain $D(A_2^{1/4}) \subseteq L^2_\sigma(\Omega)$ will be equipped with the norm $\left\|A_2^{1/4} v\right\|_2$, $v \in D(A^{1/4})$, a norm equivalent to the usual graph norm. Important is the embedding estimate

$$\|v\|_q \le c \left\|A^\alpha_\gamma v\right\|_\gamma, \quad v \in D(A^\alpha_\gamma),\ 1 < \gamma \le q,\ 2\alpha + \frac{3}{q} = \frac{3}{\gamma},\ 0 \le \alpha \le 1. \tag{1.14}$$

The operator $-A_q$ generates an exponentially decreasing analytic semigroup $e^{-tA_q} : L^q_\sigma(\Omega) \to L^q_\sigma(\Omega)$, $0 \le t < \infty$, such that for $v \in L^q_\sigma(\Omega)$ and $0 \le \alpha \le 1$

$$\left|\left|A^\alpha_q e^{-tA_q} v\right|\right|_q \le C t^{-\alpha} e^{-\delta t} \, ||v||_q \tag{1.15}$$

with $C = C(\Omega, q, \alpha) > 0$ and $\delta = \delta(\Omega) > 0$, see [5], [6], [7]. We may write $A_q = A$, $P_q = P$ if there is no misunderstanding.

If B_1, B_2 are two Banach spaces with norms $||\cdot||_{B_1}$, $||\cdot||_{B_2}$, then we write $B_1 \hookrightarrow B_2$ if B_1 is strictly contained in B_2 and if

$$||v||_{B_2} \le C \, ||v||_{B_1}, \quad v \in B_1,$$

holds with some constant $C > 0$ not depending on v.

Our results are based in particular on Proposition 1.2 and Corollary 1.3 below concerning initial value conditions for the existence of strong solutions at least in a certain (sufficiently) small initial interval $[0,T), T > 0$. These conditions are optimal in a certain sense, see [2, Theorem 1.1, Theorem 1.2]. Replacing $[0,T)$ by some interval $[t_0, t_0 + \delta)$, $\delta > 0$, and the initial value u_0 by any $u(t_0), 0 < t_0 < T$, we try to identify a given weak solution u locally in $[t_0, t_0 + \delta)$ by a strong solution. This method of local identification of u with strong solutions enables us to obtain several new uniqueness and regularity results for weak solutions (compare [2]) .

Proposition 1.2: *Let $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$, and let $u_0 \in L^2_\sigma(\Omega)$. Then there exists a constant $\varepsilon_* = \varepsilon_*(\Omega, q) > 0$ with the following property: If*

$$\left(\int_0^T \left|\left| e^{-tA} u_0 \right|\right|_q^s dt \right)^{1/s} \leq \varepsilon_*, \tag{1.16}$$

then the Navier-Stokes system (1.1) *has a unique strong solution $u \in L^s_{\text{loc}}([0,T); L^q(\Omega))$.*

Conversely, if $u \in L^s_{\text{loc}}([0,T); L^q_\sigma(\Omega))$ is a strong solution of the system (1.1), *then it holds* (1.16) *with T replaced by some (sufficiently small) T' with $0 < T' \leq T$.*

Corollary 1.3: *Let u be a weak solution as in Definition 1.1, let $t_0 \in [0,T)$, let s, q, ε_* be as in Lemma 1.2, and let* (1.4) *be valid for t_0 and $t \in [t_0, T)$. Suppose*

$$\left(\int_0^\delta \left|\left| e^{-\tau A} u(t_0) \right|\right|_q^s d\tau \right)^{1/s} \leq \varepsilon_* \quad \textit{with} \quad \delta > 0,\ t_0 + \delta < T. \tag{1.17}$$

Then

$$u \in L^s(t_0, t_0 + \delta; L^q(\Omega)). \tag{1.18}$$

In particular, u is regular in $(t_0, t_0 + \delta)$.

Conversely, if u satisfies (1.18) *then* (1.17) *is satisfied with δ replaced by some $\delta' \in (0, \delta)$.*

In §2 we describe left-hand and right-hand side conditions for local and also global regularity. Optimal initial value conditions in the $L^s(L^q)$-framework are given in terms of Besov spaces, see §3. Conditions in the limit space $L^\infty(L^3)$ to get uniqueness and the local right-hand regularity (4.1) are found in §4; e.g., if $u(t) \in L^3(\Omega)$ for all $t > 0$ and u satisfies the energy equality rather than the energy inequality, then u satisfies the local right-hand Serrin condition. Finally §5 deals with general unbounded domains where only the L^2-theory of the Stokes operator can be used to get results similar to those of §4.

2 New regularity conditions for weak solutions

Applying (1.17) for a.a. $t_0 \in [0,T)$ we obtain the following regularity results. In the following u is always a weak solution of the system (1.1) with initial value $u_0 \in L^2_\sigma(\Omega)$ in the sense of Definition 1.1, and q, s, ε_* are given as in Proposition 1.2.

In the first result we suppose (1.17) for a.a. $t_0 \in [0,T)$ with a $\delta > 0$ independent of t_0.

Theorem 2.1: *Suppose it holds*

$$\left(\int_0^\delta \left|\left|e^{-\tau A}u(t_0)\right|\right|_q^s d\tau\right)^{1/s} \leq \varepsilon_* \quad \textit{with fixed } \delta > 0 \tag{2.1}$$

for almost all $t_0 \in [0,T)$ including $t_0 = 0$. Then it holds $u \in L^s_{\mathrm{loc}}([0,T); L^q(\Omega))$; hence u is regular in $(0,T)$.

Proof: Applying (1.17) for a.a. $t_0 \in [0,T)$, including $t_0 = 0$, we are able to cover $[0,T)$ by intervals $[t_0, t_0+\delta)$ as in Corollary 1.3 for a.a. $t_0 \in [0,T)$. Then the result follows from Corollary 1.3 □

Corollary 2.2: *Suppose it holds*

$$\lim_{\delta \downarrow 0} \left(\int_0^\delta \left|\left|e^{-\tau A}u(t_0)\right|\right|_q^s d\tau\right)^{1/s} = 0 \tag{2.2}$$

for $t_0 = 0$ and uniformly for almost all $t_0 \in [0,T)$. Then $u \in L^s_{\mathrm{loc}}([0,T); L^q(\Omega))$, and u is regular in $(0,T)$.

Proof: Using (2.2) and the uniform condition we find for the given ε_*, some fixed $\delta_0 > 0$ such that (2.1) is satisfied with δ replaced by given δ_0. This proves the corollary. □

Next we obtain so-called local regularity results.

Theorem 2.3: *Suppose that at $a \in (0,T)$ the left-hand condition*

$$\delta^{-\alpha}\left(\int_{a-\delta}^a \left|\left|A^{-\alpha}u(t)\right|\right|_q^s dt\right)^{1/s} \leq \varepsilon_* \tag{2.3}$$

holds with $0 < \delta < a$, and some $0 \leq \alpha < \frac{1}{s}$. Then there exists $0 < \delta' < \delta$ such that $u \in L^s(a-\delta', a+\delta'; L^q(\Omega))$. Thus u is regular in $(a-\delta', a+\delta')$ and a is a so-called regular point.

Proof: Using (1.15) we obtain that

$$\frac{1}{\delta}\int_{a-\delta}^a \left(\int_0^\delta \left|\left|e^{-\tau A}u(t)\right|\right|_q^s d\tau\right) dt = \frac{1}{\delta}\int_{a-\delta}^a \left(\int_0^\delta \left|\left|A^\alpha e^{-\tau A}A^{-\alpha}u(t)\right|\right|_q^s d\tau\right) dt$$

$$\leq \frac{C}{\delta}\int_{a-\delta}^a \left(\int_0^\delta \frac{1}{\tau^{\alpha s}}\, d\tau\right) \left|\left|A^{-\alpha}u(t)\right|\right|_q^s dt \leq C\left(\delta^{-\alpha s}\int_{a-\delta}^a \left|\left|A^{-\alpha}u(t)\right|\right|_q^s dt\right) \leq C\varepsilon_*^s$$

with $C = C(\Omega, q, \alpha) > 0$. We conclude that there is at least one $t_0 \in (a-\delta, a)$ such that

$$\left(\int_0^\delta \left|\left|e^{-\tau A}u(t_0)\right|\right|_q^s d\tau\right)^{1/s} \leq C^{\frac{1}{s}}\varepsilon_* \tag{2.4}$$

and that (1.4) is satisfied for this t_0. Otherwise we would obtain that

$$C\varepsilon_*^s = \frac{1}{\delta}\int_{a-\delta}^{a} C\varepsilon_*^s\,dt < \frac{1}{\delta}\int_{a-\delta}^{a}\left(\int_0^{\delta} \left|\left|e^{-\tau A}u(t)\right|\right|_q^s d\tau\right) dt \le C\varepsilon_*^s$$

which is a contradiction. We may replace $C^{1/s}\varepsilon_*$ by ε_* in (2.4), and use (1.17). This shows that $u \in L^s(t_0, t_0+\delta; L^q(\Omega))$, and setting $\delta' = \min(a-t_0, t_0+\delta-a)$ we see that $[a-\delta', a+\delta') \subseteq [t_0, t_0+\delta)$. This proves Theorem 2.3. $\square$

Setting $\alpha = 0$ in (2.3) we obtain as a consequence the next corollary which has been shown in [4].

Corollary 2.4: *Suppose that the conditions*

$$u \in L^s(0, \delta_0; L^q(\Omega)) \quad \textit{with } 0 < \delta_0 < T \textit{ and} \tag{2.5}$$

$$u \in L^s(t-\delta; t; L^q(\Omega)) \quad \textit{for each } t \in [\delta_0, T) \textit{ with } 0 < \delta = \delta(t) < \delta_0 \tag{2.6}$$

are satisfied. Then Serrin's condition

$$u \in L^s_{\mathrm{loc}}([0,T); L^q(\Omega)) \tag{2.7}$$

is satisfied, and u is regular in $(0,T)$.

Remark 2.5: *Obviously* (2.5) *and* (2.6) *yield a strictly weaker condition than* (2.7) *for general vector fields. Hence Corollary 2.4 gives a strict extension of Serrin's regularity class* (2.7) *for weak solutions.*

The following corollary yields a local regularity result.

Corollary 2.6: *Suppose that at $a \in (0,T)$*

$$\delta^{-2/3}\,||u_0||_2^{2/3}\int_{a-\delta}^{a} ||\nabla u(t)||_2^2\,dt \le \varepsilon_* \tag{2.8}$$

holds with $0 < \delta < a$. Then there is some $\delta' \in (0,\delta)$ such that $u \in L^s(a-\delta', a+\delta'; L^q(\Omega))$. Thus u is regular in $(a-\delta', a+\delta')$ and a is a regular point.

Proof: Using (1.14) with $q = 12$, $\gamma = 4$, $\alpha = \frac{1}{4}$, the multiplicative estimate $\|v\|_4 \le c\|\nabla v\|_2^{3/4}\|v\|_2^{1/4}$ for $v \in W_0^{1,2}(\Omega)$, see [10, Lemma II.1.3.1 a)], and the energy inequality (1.4) with $t_0 = 0$, we obtain with $s = \frac{8}{3}$ and for $t \in (a-\delta, a)$ that

$$\left|\left|A^{-\alpha}u(t)\right|\right|_q^s \le C\,||u(t)||_4^{8/3} \le C\,||\nabla u(t)||_2^2\,||u(t)||_2^{2/3} \le C\,||\nabla u(t)||_2^2\,||u_0||_2^{2/3}$$

with some absolute constant $C > 0$. Now the result follows from Theorem 2.3. $\square$

We can treat (2.8) as an energy smallness condition. Thus a is a regular point if the dissipation energy $\int_{a-\delta}^{a} ||\nabla u(t)||_2^2\, dt$ in a left-hand neighborhood of a is sufficiently small. Using the energy inequality and some sufficiently large $\delta > 0$ we conclude from (2.8) in the case $T = \infty$ the well-known result that u is regular for $t > T_0$ where $T_0 = T_0(||u_0||_2) > 0$ is sufficiently large. See [4] for further local regularity results.

3 Optimal initial value conditions with norms in Besov spaces

Consider a weak solution u of the system (1.1) with initial value $u(0) = u_0 \in L^2_\sigma(\Omega)$ as in Definition 1.1, and let q, s be as in Proposition 1.2.

We need the Besov spaces

$$\mathbb{B}^{-2/s}_{q,s}(\Omega) = \big(\mathbb{B}^{2/s}_{q',s'}(\Omega)\big)', \quad \frac{1}{q} + \frac{1}{q'} = 1, \quad \frac{1}{s} + \frac{1}{s'} = 1, \tag{3.1}$$

which are defined as follows: Let $B^{2/s}_{q',s'}(\Omega)$ be the usual Besov space, see [12, 4.2.1 (1)], here to be considered for vector fields with values in $\mathbb{R}^3$. Then the Besov space $\mathbb{B}^{2/s}_{q',s'}(\Omega)$ of solenoidal vector fields in $B^{2/s}_{q',q'}(\Omega)$ is defined as the closed subspace

$$\mathbb{B}^{2/s}_{q',s'}(\Omega) = B^{2/s}_{q',s'}(\Omega) \cap L^{q'}_\sigma(\Omega) = \big\{v \in B^{2/s}_{q',s'}(\Omega) :\ \operatorname{div} v = 0, N \cdot v\big|_{\partial\Omega} = 0\big\}$$

where $N \cdot v\big|_{\partial\Omega}$ means the (well-defined) normal component of v at $\partial\Omega$; see [1, (0.5),(0.6)] concerning this space. The space $\mathbb{B}^{-2/s}_{q,s}(\Omega)$ is defined in (3.1) as the dual space of $\mathbb{B}^{2/s}_{q',s'}(\Omega)$. Further we need, among others, the interpolation space $(L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{s},s}$, cf. [12, 1.14.5 (2)].

Let $u_0 \in L^2_\sigma(\Omega)$ and $q > 2$. Then $e^{-tA_q}u_0 = e^{-tA_2}u_0$ is well-defined for $t > 0$, but need not be bounded in $L^q_\sigma(\Omega)$ as $t \to 0+$. However, if (1.16) holds, then

$$\int_0^\infty \big|\big|e^{-tA}u_0\big|\big|_q^s\, dt \le C \int_0^T \big|\big|e^{-tA}u_0\big|\big|_q^s\, dt < \infty \tag{3.2}$$

by (1.15). Conversely, if $\int_0^\infty \big|\big|e^{-tA}u_0\big|\big|_q^s\, dt < \infty$, then there is some $0 < T \le \infty$ such that the optimal initial value condition (1.16) is satisfied.

Using several well-known arguments we can prove the following equivalence result.

Lemma 3.1: *Let $u_0 \in L^2_\sigma(\Omega)$, $2 < s < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$. Then the norms*

$$\left(\int_0^\infty \big|\big|e^{-tA}u_0\big|\big|_q^s\, dt\right)^{1/s} \quad \text{and} \quad ||u_0||_{\mathbb{B}^{-2/s}_{q,s}(\Omega)} \tag{3.3}$$

are equivalent. Therefore, if one of these norms is finite, then also the other one is finite.

Proof: We use step by step the following arguments: First [12, 1.14.5 (2)] together with (1.15), then the identity $\langle A^{-1}u_0, A\varphi\rangle_\Omega = \langle u_0, \varphi\rangle_\Omega$, $\varphi \in D(A)$, then [12, 1.11.2 (3a)], then [12, 1.3.3 (1)], then [1, Prop. 3.4, (3.18)], and finally the definition [1, (0.6)]. This proves the lemma. Note that this calculation slightly improves the arguments in [2, Sect. 3]. □

Lemma 3.2: *Let $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$. Then there hold the following embedding properties:*

$$a)\quad L^3_\sigma(\Omega) \hookrightarrow L^{3,s}_\sigma(\Omega), \quad \text{if } s \ge 3, \tag{3.4}$$

$$b)\quad L^{3,s}_\sigma(\Omega) \hookrightarrow \mathbb{B}^{-2/s}_{q,s}(\Omega), \quad \text{if } s \ge q, \tag{3.5}$$

$$c)\quad D(A^{1/4}) \hookrightarrow \mathbb{B}^{-2/s}_{q,s}(\Omega), \tag{3.6}$$

$$d)\quad L^{r,\infty}_\sigma(\Omega) \hookrightarrow \mathbb{B}^{-2/s}_{q,s}(\Omega), \quad \text{if } r \ge 3. \tag{3.7}$$

Proof: See [1, (0.16)] concerning (3.4), (3.5), and [1, (0.17)] concerning (3.7). To prove (3.6) we use the embedding estimate (1.14) with $\alpha = \frac{1}{s} + \frac{1}{4}$ and [10, Lemma IV.1.5.3] to get that

$$\left(\int_0^\infty \left|\left|e^{-tA}v\right|\right|_q^s dt\right)^{1/s} \le C\left(\int_0^\infty \left|\left|A^{1/s}e^{-tA}A^{1/4}v\right|\right|_2^s dt\right)^{1/s} \le C\left|\left|A^{1/4}v\right|\right|_2,$$

$v \in D(A^{\frac{1}{4}})$, $C = C(\Omega, q) > 0$. This proves Lemma 3.2. □

The next result follows from Proposition 1.2 using the norm equivalence of Lemma 3.1, see [2, Theorem 1.2].

Proposition 3.3: *Let $u_0 \in L^2_\sigma(\Omega)$, let u be a weak solution of the system (1.1) in $[0,T) \times \Omega$ as in Definition 1.1, and let $2 < s < \infty$, $3 < q < \infty$ with $\frac{2}{s} + \frac{3}{q} = 1$. Then the condition*

$$u_0 \in \mathbb{B}^{-2/s}_{q,s}(\Omega) \tag{3.8}$$

is sufficient and necessary that

$$u \in L^s_{\text{loc}}([0,T'); L^q(\Omega)) \tag{3.9}$$

is a strong solution in some interval $[0,T')$ with $0 < T' \le T$.

In particular, if $u_0 \notin \mathbb{B}^{-2/s}_{q,s}(\Omega)$ then (3.9) does not hold for each $0 < T' \le T$.

By Lemma 3.2 we obtain the following local regularity properties.

Lemma 3.4: *Let u_0, u, s, q be as in Proposition 3.3. Then each of the following conditions is sufficient for the Serrin condition* (3.9) *in some interval* $[0, T'), 0 < T' \le \infty$*:*

$$a)\quad u_0 \in L^3_\sigma(\Omega), \tag{3.10}$$

$$b)\quad u_0 \in L^{3,s}_\sigma(\Omega), \quad s \ge q, \tag{3.11}$$

$$c)\quad u_0 \in D(A^{1/4}), \tag{3.12}$$

$$d)\quad u_0 \in L^{r,\infty}_\sigma(\Omega), \quad r > 3. \tag{3.13}$$

4 Uniqueness and local right-hand regularity

The class $L^\infty_{\text{loc}}([0,T); L^3(\Omega))$ is the limit case $s = \infty, q = 3$ of the usual Serrin class $L^s_{\text{loc}}([0,T); L^q(\Omega))$, $2 < s < \infty$, $3 < q < \infty$. Therefore, it is interesting to develop uniqueness and regularity properties of weak solutions u in this class. Seregin [11] and Mikhailov-Shilkin [9] proved the complete regularity of a weak solution $u \in L^\infty_{\text{loc}}([0,T); L^3(\Omega))$ under some additional assumptions in particular on the pressure p. Our result below does not contain such assumptions but yields, on the other hand, only the uniqueness and a local right-hand regularity property.

In the following u is a weak solution of the system (1.1) in $[0,T) \times \Omega$ with initial value $u(0) = u_0 \in L^2_\sigma(\Omega)$ in the sense of Definition 1.1, and $2 < s < \infty$, $3 < q < \infty$ are given satisfying $\frac{2}{s} + \frac{3}{q} = 1$.

We say that u *satisfies the local right-hand $L^s(L^q)$-Serrin condition in* $[0,T)$ if

$$u \in L^s(t, t+\delta; L^q(\Omega)) \ \text{ for each } t \in [0,T) \text{ with } \delta = \delta(t) > 0,\ t + \delta < T. \tag{4.1}$$

Theorem 4.1: *Suppose that the given weak solution u satisfies*

$$u \in L^\infty_{\text{loc}}([0,T); L^3(\Omega)). \tag{4.2}$$

a) Then u is unique in the sense that there is no other weak solution of the system (1.1) *with initial value u_0, and*

b) u satisfies in $[0,T)$ the local right-hand $L^s(L^q)$-Serrin condition (4.1) *with $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$.*

Proof: By (1.8) with $s = \infty, q = 3$ we conclude that u satisfies the energy equality (1.5) for all $0 \le t_0 \le t < T$. Using (4.2), (1.10) and weak convergence arguments we see that $u_0 \in L^3_\sigma(\Omega)$. Hence (3.10) in Lemma 3.4 allows to conclude that u satisfies Serrin's condition (3.9) at least in some initial interval $[0,T'), 0 < T' \le T$.

Suppose there is another weak solution v with $v(0) = u_0$ in the sense of Definition 1.1. Then Serrin's uniqueness result shows that $u = v$ for $t \in [0, T')$. To prove the same result on $(0, T]$ let $t_* \in (0, T]$ be defined by

$$t_* = \sup\{t \in (0,T) : u(\tau) = v(\tau) \text{ for all } \tau \in [0,t]\}; \tag{4.3}$$

note that $u(t) - v(t)$ is well-defined in $L^2_\sigma(\Omega)$ for each $t \in [0, T)$ because of (1.10). If $t_* = T$, the theorem is proved. Thus we assume that $0 < t_* < T$ and we get from (1.10) that $u = v$ holds in $[0, t_*]$.

Since (1.4) holds for v and for a.a. $t_0 \in [0, T)$, we obtain a sequence

$$0 < t_1 < t_2 < \ldots < t_j < \ldots < t_*, \quad j \in \mathbb{N}, \tag{4.4}$$

with $\lim\limits_{j\to\infty} t_j = t_*$ such that

$$\frac{1}{2}\,||v(t)||_2^2 + \int_{t_j}^{t} ||\nabla v||_2^2 \; d\tau \le \frac{1}{2}\,||v(t_j)||_2^2\,, \quad t \ge t_*, \tag{4.5}$$

holds for each $j \in \mathbb{N}$. Since $t_j < t_*$ we obtain using (1.5) for $u = v$ in $[0, t_*]$ that

$$\frac{1}{2}\,||v(t_*)||_2^2 + \int_{t_j}^{t_*} ||\nabla v||_2^2 \; d\tau = \frac{1}{2}\,||v(t_j)||_2^2\,, \quad j \in \mathbb{N},$$

which shows that $\lim\limits_{j\to\infty} ||v(t_j)||_2^2 = ||v(t_*)||_2^2$, and from (4.5) we obtain that

$$\frac{1}{2}\,||v(t)||_2^2 + \int_{t_*}^{t} ||\nabla v||_2^2 \; d\tau \le \frac{1}{2}\,||v(t_*)||_2^2\,, \quad t \ge t_*. \tag{4.6}$$

Now Serrin's uniqueness result implies that $u = v$ also holds in some interval $[t_*, T'')$ with $t_* < T'' \le T$, which is a contradiction to (4.3). Therefore, u is uniquely determined in $[0, T)$.

Consider any $t_0 \in [0, T)$. Then $u(t_0) \in L^3_\sigma(\Omega)$ is well defined and since $L^3_\sigma(\Omega) \subseteq \mathbb{B}^{-2/s}_{q,s}(\Omega)$, $s \ge q$, we obtain from Proposition 3.3 - with $[0, T)$ replaced by $[t_0, T)$ and u_0 replaced by $u(t_0) \in \mathbb{B}^{-2/s}_{q,s}(\Omega)$ - a local strong solution $u^* \in L^s(t_0, t_0 + \delta; L^q(\Omega))$ in some interval $[t_0, t_0 + \delta)$, $\delta = \delta(t_0) > 0$, $t_0 + \delta < T$, which can be identified with u by Serrin's uniqueness result using (1.5). This proves the right-hand $L^s(L^q)$-Serrin condition. The proof of Theorem 4.1 is complete. □

Remark 4.2: *a) The result of Theorem 4.1 is "very close" to the complete regularity of the given weak solution u. Indeed, if the right-hand local regularity condition* (4.1) *holds for u with some fixed $0 < \delta_0 = \delta(t)$ for each $t \in [0, T)$, then we conclude from the proof of Theorem 4.1 that $u \in L^s_{\text{loc}}([0, T); L^q(\Omega))$.*
b) More general, if the local right-hand condition (4.1) *holds for u with fixed*

$\delta(t) = \delta_0 > 0$ only for almost all $t \in [0,T)$, then the proof above implies that $u \in L^s_{\rm loc}([0,T); L^q(\Omega))$.
c) Consider a general weak solution u as in Theorem 4.1. Then, omitting the condition (4.2), we can use the arguments in its proof because of (1.2) at least for almost all $t_0 \in [0,T)$, and we obtain that $u^* = u \in L^s(t_0, t_0+\delta; L^q(\Omega))$ holds with $\delta = \delta(t_0) > 0$, $t_0 + \delta < T$. In this case these local regularity intervals need not cover the whole interval $(0,T)$. The union $\tau \subseteq (0,T)$ of such intervals yields a dense open subset of regular points of $(0,T)$. The complement $S = (0,T)\backslash\tau$ is the null set of singular points in $(0,T)$, which is (in the case $t = \infty$) always bounded because of the regularity property in (2.8).

To extend Theorem 4.1 to an even larger class than $L^\infty_{\rm loc}([0,T); L^3(\Omega))$ recall that each of the conditions $uu \in L^2_{\rm loc}([0,T); L^2(\Omega))$ or $u \in L^4_{\rm loc}([0,T); L^4(\Omega))$ is sufficient for the energy equality (1.5), see (1.6), (1.7). Note that in Theorem 4.3 below u need not satisfy any Serrin condition; actually, it holds $\frac{2}{4} + \frac{3}{4} = 1 + \frac{1}{4}$.

Theorem 4.3: *Consider a weak solution u with initial value $u_0 \in L^2_\sigma(\Omega)$, and suppose that*

$$a)\quad u(t) \in L^3_\sigma(\Omega) \quad \text{for each } t \in [0,T), \tag{4.7}$$

$$b)\quad \frac{1}{2}\,||u(t)||_2^2 + \int_{t_0}^t ||\nabla u||_2^2 \; d\tau = \frac{1}{2}\,||u(t_0)||_2^2 \quad \text{for all } \; 0 \le t_0 \le t < T. \tag{4.8}$$

Then u is uniquely determined in the class of weak solutions with initial value u_0 and it holds the local right-hand $L^s(L^q)$-Serrin condition with some $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$.

Proof: Suppose there is another weak solution v of (1.1) with $v(0) = u_0$. Since $u_0 = u(0) \in L^3_\sigma(\Omega)$ we argue as in the proof of Theorem 4.1 that u satisfies (3.9) and that $u = v$ holds in some initial interval $[0, T'), T' > 0$.

Using t_* as in (4.3)-(4.6) we obtain in the same way that $u = v$ holds in $[0,T)$, and the same argument as in the proof of Theorem 4.1 also yields the property (4.1). This completes the proof. □

The same arguments are valid if $L^3(\Omega)$ (or equivalently $L^3_\sigma(\Omega)$) in (4.7) is replaced by one of the spaces (3.11)-(3.13), or in the most general case by the space in (3.8). This finally yields the following result.

Theorem 4.4: *Consider a weak solution u with initial value $u_0 \in L^2_\sigma(\Omega)$ and suppose that*

$$a)\quad u(t) \in \mathbb{B}^{-2/s}_{q,s}(\Omega),\;\; 2 < s < \infty,\;\; 3 < q < \infty,\;\; \frac{2}{s} + \frac{3}{q} = 1 \text{ for all } \; t \in [0,T), \tag{4.9}$$

$$b)\quad \frac{1}{2}\,||u(t)||_2^2 + \int_{t_0}^t ||\nabla u||_2^2 \; d\tau = \frac{1}{2}\,||u(t_0)||_2^2 \quad \text{for all } \; 0 \le t_0 \le t < T. \tag{4.10}$$

Then u is uniquely determined and it holds the local right-hand $L^s(L^q)$-Serrin condition as in Theorem 4.3.

Obviously, see Lemma 3.2, Theorem 4.4 remains valid if $\mathbb{B}_{q,s}^{-2/s}(\Omega)$ in (4.9) is replaced by each of the following spaces: either

$$L_\sigma^{3,s}(\Omega),\ s \geq q, \quad \text{or} \quad D(A_2^{1/4}) \subseteq L_\sigma^2(\Omega) \quad \text{or} \quad L_\sigma^{r,\infty}(\Omega),\ r > 3. \tag{4.11}$$

5 A uniqueness and regularity result for general domains

In this section let $\Omega \subseteq \mathbb{R}^3$ be a completely general domain, i.e. an open and connected subset of $\mathbb{R}^3$. In this case we have to modify slightly the definition of a weak solution u for the system

$$u_t - \Delta u + u \cdot \nabla u + \nabla p = 0, \quad \operatorname{div} u = 0,\ u\big|_{t=0} = u_0 \tag{5.1}$$

in $[0,T) \times \Omega$, $0 < T \leq \infty$.

Let $u_0 \in L_\sigma^2(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)}^{||\cdot||_2}$ and $W_{0,\sigma}^{1,2}(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)}^{||\cdot||_2 + ||\nabla \cdot||_2}$. Then

$$u \in L^\infty(0,T; L_\sigma^2(\Omega)) \cap L_{loc}^2([0,T); W_{0,\sigma}^{1,2}(\Omega)) \tag{5.2}$$

is called a (Leray-Hopf type) weak solution of the system (5.1) if the relation

$$-\langle u, w_t\rangle_{\Omega,T} + \langle \nabla u, \nabla w\rangle_{\Omega,T} - \langle uu, \nabla w\rangle_{\Omega,T} = \langle u_0, w(0)\rangle_\Omega \tag{5.3}$$

holds for each $w \in C_0^\infty([0,T); C_{0,\sigma}^\infty(\Omega))$, and if the (simple) energy inequality

$$\frac{1}{2}\,||u(t)||_2^2 + \int_0^t ||\nabla u||_2^2 \; d\tau \leq \frac{1}{2}\,||u_0||_2^2 \tag{5.4}$$

holds for all $t \in [0,T)$. A weak solution u is called a strong solution if Serrin's condition (1.11) is satisfied.

Without loss of generality we may assume that u in this definition is weakly continuous as in (1.10); see [10, Theorem V.3.1.1] concerning the existence of a weak solution. It is an open problem whether each weak solution u satisfies the strong energy inequality (1.4). However, if Ω is of uniform C^2-type, see [3], there exist weak solutions satisfying (1.4). If a weak solution satisfies additionally one of the conditions (1.6), (1.7) or (1.8), then the energy equality (1.5) holds for all $0 \leq t_0 \leq t < T$, see [10, Theorem V.1.4.1]. The proof is the same as for bounded domains.

Our result for general domains rests on [2, Theorem 4.1] which shows that Proposition 1.2 can be extended to the general domain Ω for the special exponents $s = 8$, $q = 4$. Thus it holds for this domain the following result even with some absolute constant $\varepsilon_* > 0$ (independent of the domain):

Lemma 5.1: [2, Theorem 4.1] *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain and let $u_0 \in L^2_\sigma(\Omega)$, $0 < T \le \infty$. There exists some absolute constant $\varepsilon_* > 0$ such that if*

$$\left(\int_0^T \left|\left| e^{-tA} u_0 \right|\right|_4^8 \, dt \right)^{1/8} \le \varepsilon_*, \tag{5.5}$$

then system (5.1) *has a uniquely determined strong solution $u \in L^8_{\mathrm{loc}}([0,T); L^4(\Omega))$ with $u|_{t=0} = u_0$.*

In this case we essentially use the L^2-approach to the Stokes operator $A = A_2$. Further, using (1.14) and [10, Lemma IV.1.5.3], we obtain with $u_0 \in D(A^{1/4})$ the estimate

$$\left(\int_0^\infty \left|\left| e^{-tA} u_0 \right|\right|_4^8 \, dt \right)^{1/8} \le C \left(\int_0^\infty \left|\left| A^{1/8} e^{-tA} A^{1/4} u_0 \right|\right|_2^8 \, dt \right)^{1/8} \le C \left|\left| A^{1/4} u_0 \right|\right|_2, \tag{5.6}$$

see [2, p.109], with some absolute constant $C > 0$. Moreover, using the embedding estimate (1.14) with $\alpha = \frac{1}{4}$, $q = 3$, $\gamma = 2$, we obtain for $u_0 \in D(A^{1/4}) \subseteq L^2_\sigma(\Omega)$ that

$$||u_0||_3 = \left\| A^{-\frac{1}{4}} A^{1/4} u_0 \right\|_3 \le C \left\| A^{1/4} u_0 \right\|_2 \tag{5.7}$$

with some absolute constant $C > 0$. This shows that

$$D(A^{1/4}) \subseteq L^3_\sigma(\Omega) \text{ with } ||\cdot||_{D(A^{1/4})} = \left\| A_2^{1/4} \cdot \right\|_2. \tag{5.8}$$

The properties (5.5)-(5.8) enable us to carry out the proof of Theorem 4.1 with $L^3_\sigma(\Omega)$ replaced by the space $D(A^{1/4})$ which means a certain restriction. Another restriction is given by the fact that a weak solution u in (5.1)-(5.4) need not satisfy the strong energy inequality (1.4).

Suppose $u_0 \in D(A^{1/4})$. Then (5.6) shows that $\int_0^\infty \left|\left| e^{-tA} u_0 \right|\right|_4^8 \, dt < \infty$, and we find some $0 < T \le \infty$ such that (5.5) is satisfied. This yields a strong solution u as above with $u(0) = u_0$. If additionally $u \in L^\infty_{\mathrm{loc}}([0,T); D(A^{1/4}))$ is satisfied, we conclude using (5.8) and (1.8) that u satisfies the energy equality (1.5).

Using these arguments, modifying slightly the proof of Theorem 4.1, we obtain the following result.

Theorem 5.2: *Consider a weak solution u of the system* (5.1) *in $[0,T) \times \Omega$ with $u(0) = u_0 \in L^2_\sigma(\Omega)$ satisfying* (5.2)-(5.4), *and suppose that*

$$u \in L^\infty_{\mathrm{loc}}([0,T); D(A_2^{1/4})).$$

a) Then u is uniquely determined in the sense that if there is another weak solution $v \in L^\infty_{\mathrm{loc}}([0,T); D(A_2^{1/4}))$ with $v(0) = u_0$, then $u = v$.

b) u satisfies in $[0,T)$ the local right-hand $L^s(L^q)$-Serrin condition (4.1) *with $s = 8$, $q = 4$.*

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