CONCENTRATION-DIFFUSION PHENOMENA OF HEAT CONVECTION IN AN INCOMPRESSIBLE FLUID

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We study in the whole space $\mathbb{R}^n$ the behaviour of solutions to the Boussinesq equations at large distances. Therefore, we investigate the solvability of these equations in weighted L^∞-spaces and determine the asymptotic profile for sufficiently fast decaying initial data. For $n = 2, 3$ we are able to construct initial data such that the velocity exhibits an interesting concentration-diffusion phenomenon.

Keywords: Instationary Boussinesq equations, rate of decay in space, mild and strong solutions, weighted spaces, concentration-diffusion
MSC 2000: Primary: 76D05; Secondary: 35K05; 35Q30; 35Q35

1. Introduction

The Boussinesq equations describe the heat convection in a viscous incompressible fluid under the influence of gravity:

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla)u + \nabla p &= g\theta \quad \text{in} \quad \mathbb{R}^n \times [0,T), \\ \theta_t - \Delta\theta + (u \cdot \nabla)\theta &= 0 \quad \text{in} \quad \mathbb{R}^n \times [0,T), \\ \operatorname{div} u &= 0 \quad \text{in} \quad \mathbb{R}^n \times [0,T), \\ u(0) &= u_0 \quad \text{in} \quad \mathbb{R}^n, \\ \theta(0) &= \theta_0 \quad \text{in} \quad \mathbb{R}^n, \end{cases}$$

where $u = (u^1(x,t), ..., u^n(x,t))$, $\theta = \theta(x,t)$ and $p = p(x,t)$ denote the velocity vector field, the temperature and the pressure of the fluid at the point $(x,t) \in \mathbb{R}^n \times [0,T)$, respectively. Here u_0 and θ_0 are the given initial data. Usually the Boussinesq equations are considered under the influence of a constant gravity g making sense for small spatial scales in bounded domains.

However, in this paper we will study the Boussinesq equations in the whole space $\mathbb{R}^n$. In these cases it is expedient for $n = 3$ to deal with a gravitational force g which satisfies the well-known law of Newton, i.e., by classical theory g depends on the distance like $\sim \frac{1}{|x|^2}$. At first sight it seems to be a purely academic problem to extend this result to the general n-dimensional case, $n \geq 2$. But current research in theoretical physics gives cogent justifications to investigate our problem also in higher dimensions, especially within very tiny scales, cf. [1]. So we assume the gravity $g = (g_1, ..., g_n)$ to decay as $\frac{1}{|x|^{n-1}}$ for $|x| \to \infty$, modeling the gravitation field of a compact mass in $\mathbb{R}^n$.

This research was supported by the International Research Training Group (IRTG 1529) Darmstadt-Tokyo on Mathematical Fluid Mechanics of the funding organisation DFG.

To study the spatial behaviour of solutions to the Boussinesq equations it will be helpful to consider the solvability of these equations in weighted L^∞-spaces. In the case of slow decay the solution decreases in the same way as the initial velocity. But already Brandolese, Vigneron and Bae, see [2] and [7], proved in the case of the Navier-Stokes equations that in general we cannot expect a faster decay behaviour than $\frac{1}{|x|^n}$.

The Boussinesq system has been investigated by numerous authors and in various domains, see e.g. [6], [8], [11], [12], [16], [18], [19]. In our case of the whole space more tools especially from harmonic analysis are available leading to more sophisticated results. Using the Riesz transforms $\mathcal{R}_j = \partial_j(-\Delta)^{-\frac{1}{2}}$, $1 \le j \le n$, the Helmholtz projection is given by $\mathbb{P} = (\delta_{j,h} + \mathcal{R}_j\mathcal{R}_h)_{j,h=1}^n$. Applying $\mathbb{P}$ to the first equation of the Boussinesq system we get

$$(BE)\begin{cases} u_t - \Delta u + \mathbb{P}(u\cdot\nabla)u &= \mathbb{P}\,(g\theta) \quad \text{in} \quad \mathbb{R}^n \times [0,T),\\ \theta_t - \Delta\theta + (u\cdot\nabla)\theta &= 0 \quad \text{in} \quad \mathbb{R}^n \times [0,T),\\ \operatorname{div} u &= 0 \quad \text{in} \quad \mathbb{R}^n \times [0,T),\\ u(0) &= u_0 \quad \text{in} \quad \mathbb{R}^n,\\ \theta(0) &= \theta_0 \quad \text{in} \quad \mathbb{R}^n. \end{cases}$$

Furthermore, it will be helpful to consider an integral equation instead of the differential equation (BE). For the Boussinesq equations we get the system of integral equations

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}(u\cdot\nabla u)(\tau)\,d\tau + \int_0^t e^{(t-\tau)\Delta}\mathbb{P}(g\theta)(\tau)\,d\tau \tag{1.1}$$

$$\theta(t) = e^{t\Delta}\theta_0 - \int_0^t e^{(t-\tau)\Delta}(u\cdot\nabla\theta)(\tau)\,d\tau, \tag{1.2}$$

where $e^{t\Delta}$ denotes the semigroup of heat conduction. In the whole space $\mathbb{R}^n$ $e^{t\Delta}$ is nothing but the convolution with the heat kernel: for $f \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$,

$$e^{t\Delta}f = \mathcal{G}_t * f, \quad \mathcal{G}_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{\frac{-|x|^2}{4t}} \quad \text{for} \quad t > 0,\ x \in \mathbb{R}^n.$$

A solution (u,θ) of (1.1), (1.2) is called a *mild solution.* Since the operator $\mathbb{P}$ is not bounded on L^∞, we will handle u, θ in some proofs in homogeneous Besov spaces.

The main open question of mathematical fluid dynamics is whether a non-stationary Navier-Stokes fluid with finite energy and smooth initial data stays regular or blow-up will occur. Recently, Brandolese introduced a new idea to better understand this question, see [4]. He constructed an example of a smooth solution of the Navier-Stokes equations such that for a given finite sequence of instants $0 < t_1 < ... < t_N$ the velocity has some concentration-diffusion effects close to each moment t_i, $i = 1, ..., N$, i.e., the solution concentrates by approaching t_i such that it becomes better localized and spreads out again afterwards.

Our aim is to extend this result to the Boussinesq equations by a procedure similar to [4].

2. Main Results

In this paper we assume that the initial data belong to weighted L^∞-spaces. The Banach space $L^\infty_\mu(\mathbb{R}^n)$, $\mu > 0$, is defined as the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^\infty_\mu} := \operatorname{ess}\sup_{x\in\mathbb{R}^n} (1+|x|)^\mu |f(x)| < \infty.$$

Using Banach's fixed point theorem we get the following existence theorem of mild solutions in spaces of weakly-* continuous functions in time with values in weighted L^∞-spaces.

Theorem 2.1: (Existence and Uniqueness of Mild Solutions) *For initial data* $(u_0, \theta_0) \in L^\infty_\mu(\mathbb{R}^n)^n \times L^\infty_\nu(\mathbb{R}^n)$ *with* $\operatorname{div} u_0 = 0$, $\mu \in (0, n]$, $\nu > \max\{0,\ \mu - n + 1\}$, *and* $g \in L^\infty_{n-1}(\mathbb{R}^n)^n$ *there exists a constant* $T > 0$ *and a unique mild solution*

$$(u, \theta) \in C_\omega\left([0,T]; L^\infty_\mu(\mathbb{R}^n)^n\right) \times C_\omega\left([0,T]; L^\infty_\nu(\mathbb{R}^n)\right)$$

to the Boussinesq equations (1.1), (1.2). In particular, with the bound C_0 *for the operator norms in Lemma 4.1 below, any* $T > 0$ *satisfying*

$$8C_0(\sqrt{T} + T^{1+\kappa})\big(\|u_0\|_{L^\infty_\mu(\mathbb{R}^n)} + \|\theta_0\|_{L^\infty_\nu(\mathbb{R}^n)} + \|g\|_{L^\infty_{n-1}(\mathbb{R}^n)}\big) < 1$$

is possible with $\kappa := \frac{1}{2}\max\{\mu + \nu - n, 0\}$.

The space $C_\omega\left([0,T]; L^\infty_\mu\right)$ denotes all L^∞_μ-valued weakly-* continuous functions $v(t)$ defined in $[0,T]$. The necessity for working in the space C_ω lies in the fact that in general $e^{t\Delta} f$, with $f \in L^\infty_\mu$, does not converge to f in L^∞_μ as $t \searrow 0$, but only weakly-*. Therefore, we just get weak-* continuity for u and θ.

Let us now study the strong solvability of solutions of the Boussinesq equations (BE) in weighted L^∞-spaces assuming more regularity on the gravity. We will obtain that the solution (u, θ) depends continuously on time t. At this point we introduce the space

$$W^{m,\infty}_\mu(\mathbb{R}^n) = \left\{f \in W^{m,\infty}(\mathbb{R}^n) : \partial^\alpha f \in L^\infty_\mu(\mathbb{R}^n) \text{ for all } \alpha, |\alpha| \le m\right\},\ m \in \mathbb{N}.$$

Theorem 2.2: (Existence of Strong Solutions) *Let* $g \in W^{1,\infty}_{n-1}(\mathbb{R}^n)^n$, $u_0 \in L^\infty_\mu(\mathbb{R}^n)^n$ *with* $\operatorname{div} u_0 = 0$, $\mu \in (0, n]$, *and let* $\theta_0 \in L^\infty_\nu(\mathbb{R}^n)$ *where* $\nu > \max\{0,\ \mu + 1 - n\}$. *Then the mild solution* (u, θ) *of (1.1), (1.2) given in Theorem 2.1 solves (BE) in* $L^\infty(\mathbb{R}^n)$ *and satisfies*

$$u \in C_\omega\left([0,T]; L^\infty_\mu\right) \cap C^1\left((0,T]; \mathrm{BUC}\right) \cap C\left((0,T]; W^{2,\infty}\right),$$
$$\theta \in C_\omega\left([0,T]; L^\infty_\nu\right) \cap C^1\left((0,T]; \mathrm{BUC}\right) \cap C\left((0,T]; W^{2,\infty}\right).$$

Remark: In the proof of this theorem, see §5 and also (5.2), we will see how the regularity of the solution (u, θ) depends on the regularity of the gravity g. In general, $u, \theta \in C\left((0,T]; W^{m+1,\infty}\right)$ if $g \in W^{m,\infty}_{n-1}$, $m \in \mathbb{N}$. So a smooth gravity yields a smooth solution. However, the initial data u_0 and θ_0 have no contribution to the regularity of the solution reflecting the smoothing property of parabolic differential equations.

In view of the result $(u,\theta)(t) \in L^\infty_\mu(\mathbb{R}^n)^n \times L^\infty_\nu(\mathbb{R}^n)$ with $\mu \in (0,n]$ and $\nu > \max\{0, \mu+1-n\}$ for mild as well as strong solutions in Theorems 2.1 and 2.2 the question occurs whether the upper bound n for μ is optimal in some sense. Actually, the decay $|x|^{-(n+1)}$ is optimal for generic solutions to the Navier-Stokes equations, see [7, Theorem 1.2, Proposition 1.6]. In general the solution $(u,\theta)(t)$ will not belong to $L^\infty_\mu(\mathbb{R}^n)^n \times L^\infty_\nu(\mathbb{R}^n)$ if $\mu > n$: a decay of u like $\frac{1}{|x|^\mu}$, $\mu > n$, will imply some properties of the integrals

$$\int_0^t \int_{\mathbb{R}^n} (g\theta)(y,s)\,dy\,ds \quad \text{and} \quad \int_0^t \int_{\mathbb{R}^n} \big((u\otimes u)(y,s) + y \otimes (g\theta)(y,s)\big)\,dy\,ds,$$

see Theorem 2.3 below.

Theorem 2.3: (Spatial Asymptotic Behaviour) *Let $\varepsilon > 0$. For $\mu > \frac{n+2}{2}, \nu > 3$, $g \in W^{1,\infty}_{n-1}(\mathbb{R}^n)^n$ and initial data $(u_0,\theta_0) \in L^\infty_\mu(\mathbb{R}^n)^n \times L^\infty_\nu(\mathbb{R}^n)$ with $\operatorname{div} u_0 = 0$, let (u,θ) be the strong solution of Theorem 2.2. Then the following profile holds for $|x| \gg \sqrt{t}$:*

$$\begin{aligned} u(x,t) &= e^{t\Delta}u_0(x) - \nabla\left[\frac{\gamma_n}{n}\frac{x}{|x|^n}\cdot\int_0^t\int_{\mathbb{R}^n}(g\theta)\,dy\,ds\right] \\ &\quad - \nabla\left[\gamma_n \sum_{h,k=1}^n \left(\frac{x_h x_k}{|x|^{n+2}} - \frac{\delta_{h,k}}{n|x|^n}\right)\cdot\int_0^t\int_{\mathbb{R}^n}(u_h u_k + y_k g_h \theta)\,dy\,ds\right] \\ &\quad + \mathcal{O}_t(|x|^{-n-2+\varepsilon}), \\ \theta(x,t) &= e^{t\Delta}\theta_0(x) + \mathcal{O}_t(|x|^{-\mu-\nu}). \end{aligned}$$

Here $\gamma_n = \frac{n}{2}\pi^{-\frac{n}{2}}\Gamma\big(\frac{n}{2}\big)$.

As long as the inital data u_0 belongs to L^∞_μ, with $\mu > n$, but $g\theta$ has non-zero mean this theorem shows that in general we expect an $|x|^{-n}$-decay of the velocity. In particular, this implies no matter how small and well localized, e.g. compactly supported, the gravity g is, it has a significant effect at large distances. Thus the force $g\theta$ causes the velocity of the fluid to decrease less fast in the far-field.

This conclusion is the starting point to construct solutions of the Boussinesq equations (BE) with a concentration-diffusion property. For this we define the orthogonal transformation $\tilde{} : \mathbb{R}^n \to \mathbb{R}^n$, by

$$\tilde{x} := (x_2, ..., x_n, x_1),$$

cf. [4]. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *B-symmetric* if $f(\tilde{x}) = f(x)$ for all $x \in \mathbb{R}^n$, and a vector-valued function $h : \mathbb{R}^n \to \mathbb{R}^n$ is called *B-symmetric* if $h(\tilde{x}) = \tilde{h}(x)$ for all $x \in \mathbb{R}^n$. This B-symmetry is compatible with the Fourier transform as well as with the Laplace operator. Furthermore, we require the regularity assumptions

$$g \in W^{2,\infty}_{n-1}(\mathbb{R}^n)^n \setminus \{0\} \quad \text{and } \Delta g \in L^\infty_{n+\delta}(\mathbb{R}^n)^n \tag{2.1}$$

for some $\delta > 0$. This assumption on the decay of Δg is physically justified.

Theorem 2.4: *Let $n = 2, 3$, $\kappa > 0$, let g satisfying (2.1) be either an odd or an even function with $g(\tilde{x}) = \tilde{g}(x)$, and let $0 =: t_0 < t_1 < ... < t_N < t_{N+1} := T$, $N \in \mathbb{N}$, be a finite sequence and $0 < \varepsilon < \min\{\frac{1}{2}(t_{k+1} - t_k) : k = 0, ..., N\}$. Further we assume that the initial velocity $u_0 \in L^\infty_{n+2}(\mathbb{R}^n)^n$ satisfies $\tilde{u}_0(x) = u_0(\tilde{x})$ and the symmetry properties*

$$u_{0,1}(-x_1, x_2, ..., x_n) = -u_{0,1}(x_1, x_2, ..., x_n)$$
$$u_{0,1}(x_1, ..., -x_j, ..., x_n) = u_{0,1}(x_1, ..., x_j, ..., x_n) \tag{2.2}$$

for all $j = 2, ..., n$.

Then there exists an initial temperature $\theta_0 \in \mathcal{S}(\mathbb{R}^n)$ and for each $i = 1, \dots, N$ there are instants $t_i', t_i^*, \in (t_i - \varepsilon, t_i + \varepsilon)$ such that the corresponding unique strong solution $u, \theta \in C\left((0, T]; W^{2,\infty}\right)$, see Theorem 2.2, of the Boussinesq equations (BE) with initial data $(\eta u_0, \eta\theta_0)$ and $\eta > 0$ sufficiently small satisfies, for all $i = 1, ..., N$ and all $|x|$ large enough, the pointwise estimate

$$|u(x, t_i^*)| \le C|x|^{-n-2+\kappa},$$

and with $\omega = \frac{x}{|x|}$ there holds for almost all $|x|$ large enough

$$|u(x, t_i')| \ge c_\omega |x|^{-n}.$$

3. Preliminaries

Let us recall the definition of the homogeneous Besov space $\dot{B}^s_{p,q}$ on $\mathbb{R}^n$; for details see e.g. [3] or [15]. Let the family of functions $\{\varphi_j\}_{j\in\mathbb{Z}}$ define a Littlewood-Paley decomposition. For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, we write

$$\|f\|_{\dot{B}^s_{p,q}} := \begin{cases} \left[\sum_{j=-\infty}^{\infty} (2^{js} \|\varphi_j * f\|_p)^q\right]^{\frac{1}{q}} & \text{for } q < \infty, \\ \sup_{-\infty<j<\infty} 2^{js} \|\varphi_j * f\|_p & \text{for } q = \infty. \end{cases}$$

The *homogeneous Besov space* $\dot{B}^s_{p,q}$ is defined by

$$\dot{B}^s_{p,q} := \{f \in \mathcal{Z}' \mid \|f\|_{\dot{B}^s_{p,q}} < \infty\}.$$

Here $\mathcal{Z}'$ is the topological dual space of the space

$$\mathcal{Z} := \{f \in \mathcal{S}(\mathbb{R}^n) \mid \partial^\alpha \hat{f}(0) = 0 \text{ for all } \alpha \in \mathbb{N}^n\}.$$

The above definition implies that all polynomials vanish in $\dot{B}^s_{p,q}$. However, it is well known that

$$\dot{B}^s_{p,q} \cong \{f \in \mathcal{S}' \mid \|f\|_{\dot{B}^s_{p,q}} < \infty \text{ and } f = \sum_{j=-\infty}^{\infty} \varphi_j * f \text{ in } \mathcal{S}'\}$$

if $s < \frac{n}{p}$ or $s = \frac{n}{p}$ and $q = 1$.

We first describe some elementary properties of these spaces.

Lemma 3.1: *(i) There exists a constant $C = C(n) > 0$ such that for all $f \in \dot{B}^{s+1}_{\infty,1}$, $s \in \mathbb{R}$, the gradient belongs to $\dot{B}^s_{\infty,1}$ and satisfies the estimate*

$$\|\nabla f\|_{\dot{B}^s_{\infty,1}} \le C\|f\|_{\dot{B}^{s+1}_{\infty,1}}. \tag{3.1}$$

(ii) [13] *Let $s>0$. Then there exists a constant $C(n,s)>0$ such that for all $f,g\in L^\infty\cap \dot{B}^s_{\infty,1}$ there holds $fg\in \dot{B}^s_{\infty,1}$ and the Hölder type inequality*

$$\|fg\|_{\dot{B}^s_{\infty,1}}\le C(n,s)\left(\|f\|_\infty\|g\|_{\dot{B}^s_{\infty,1}}+\|g\|_\infty\|f\|_{\dot{B}^s_{\infty,1}}\right). \tag{3.2}$$

(iii) [9] *There holds for all $f\in L^\infty$ and $\alpha\in\mathbb{R}$ the inequality*

$$\|(-\Delta)^\alpha\varphi_j*f\|_\infty\le 2^{2j\alpha}\,\|\varphi_j*f\|_\infty,\quad j\in\mathbb{Z}, \tag{3.3}$$

That means if $f\in\dot{B}^s_{\infty,1}$ then $(-\Delta)^\alpha f\in\dot{B}^{s-2\alpha}_{\infty,1}$.

Lemma 3.2: [17] *(i) Let $s>0$. There exists a constant $C(n,s)>0$ such that for all $f\in L^\infty$ there holds*

$$\|e^{t\Delta}f\|_{\dot{B}^s_{\infty,1}}\le C(n,s)t^{-\frac{s}{2}}\|f\|_\infty,\quad t>0\,. \tag{3.4}$$

(ii) Let $\alpha\ge 0$, $s\in\mathbb{R}$ and $1\le p,q\le\infty$. There exists a constant $C(\alpha,n,s)>0$ such that for all $f\in\dot{B}^s_{p,q}$

$$\|e^{t\Delta}f\|_{\dot{B}^{s+\alpha}_{p,q}}\le C(\alpha,n,s)t^{-\frac{\alpha}{2}}\|f\|_{\dot{B}^s_{p,q}},\quad t>0\,. \tag{3.5}$$

(iii) There exists a constant $C(\alpha,n)$ independent of $f\in L^\infty$ such that

$$\|e^{t'\Delta}f-e^{t\Delta}f\|_{\dot{B}^s_{\infty,1}}\le C(\alpha,n)(t'-t)^\alpha\|e^{t\Delta}f\|_{\dot{B}^{s+2\alpha}_{\infty,1}} \tag{3.6}$$

holds for all $0<t<t'<\infty$, $\alpha>0$ and $s\in\mathbb{R}$.

Lemma 3.3: *(i)* [14, Prop. 11.1] *The operator*

$$O_{j,h;t}:=\Delta^{-1}\partial_j\partial_h e^{t\Delta},\quad 1\le j,h\le n,$$

is a convolution operator with kernel $K_{j,h;t}(x)=t^{-\frac{n}{2}}K_{j,h}\left(\frac{x}{\sqrt{t}}\right)$, also called Oseen kernel, where the smooth function $K=(K_{j,h})$ satisfies

$$(1+|x|)^{n+|\alpha|}\partial^\alpha K\in L^\infty(\mathbb{R}^n)\qquad \text{for all } \alpha\in\mathbb{N}^n. \tag{3.7}$$

(ii) The operator family $e^{t\Delta}\mathbb{P}=e^{-tA}\mathbb{P}$, $t>0$, where $A=-\mathbb{P}\Delta$ denotes the Stokes operator on $\mathbb{R}^n$, has the following properties: $e^{t\Delta}\mathbb{P}$ is defined by a convolution kernel $E=\left(E_{j,h}\right)^n_{j,h=1}$,

$$E(x,t):=\int_{\mathbb{R}^n}e^{-4\pi^2t|\xi|^2+2\pi ix\cdot\xi}\left(I-\frac{\xi\otimes\xi}{|\xi|^2}\right)\,d\xi\,.$$

Moreover, [2], *E has the asymptotic structure*

$$E(x,t)=\gamma_n\left(\frac{x\otimes x}{|x|^{n+2}}-\frac{1}{n|x|^n}I\right)+|x|^{-n}\Psi\left(\frac{x}{\sqrt{t}}\right) \tag{3.8}$$

for $|x|\gg\sqrt{t}$, where the matrix field Ψ and its gradient have an exponential decay and $\gamma_n:=\frac{n}{2}\pi^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)$.

(iii) [17] *The Riesz transforms are well-defined bounded operators on the Besov space $\dot{B}^0_{\infty,1}$. In particular, for all $s\ge 0$ and $\alpha\ge 0$*

$$\|e^{t\Delta}\mathbb{P}f\|_{\dot{B}^{s+\alpha}_{\infty,1}}\lesssim t^{-\frac{\alpha}{2}}\|f\|_{\dot{B}^s_{\infty,1}}\,. \tag{3.9}$$

Proof of (3.9): By (3.3), (3.5)

$$\begin{aligned}\|e^{t\Delta}\mathbb{P}f\|_{\dot B^{s+\alpha}_{\infty,1}} &= \|(-\Delta)^{-\frac{s}{2}}e^{t\Delta}\mathbb{P}((-\Delta)^{\frac{s}{2}}f)\|_{\dot B^{s+\alpha}_{\infty,1}}\\ &\lesssim \|e^{t\Delta}\mathbb{P}((-\Delta)^{\frac{s}{2}}f)\|_{\dot B^{\alpha}_{\infty,1}} \lesssim t^{-\frac{\alpha}{2}}\|\mathbb{P}((-\Delta)^{\frac{s}{2}}f)\|_{\dot B^{0}_{\infty,1}}\\ &\lesssim t^{-\frac{\alpha}{2}}\|(-\Delta)^{\frac{s}{2}}f\|_{\dot B^{0}_{\infty,1}} \lesssim t^{-\frac{\alpha}{2}}\|f\|_{\dot B^{s}_{\infty,1}},\end{aligned}$$

where we exploited also the boundedness of the Helmholtz projection $\mathbb{P}$ on $\dot B^{0}_{\infty,1}$. □

Note that in Lemma 3.3 (ii) we used the Fourier transform, e.g. of a Schwartz function $\phi \in \mathcal{S}(\mathbb{R}^n)$, in the form

$$\mathcal{F}\phi(\xi) = \hat\phi(\xi) := \int_{\mathbb{R}^n} \phi(x)e^{-2\pi i x\cdot\xi}\,dx.$$

4. Proof of Theorems 2.1 and 2.4

To construct a unique mild solution of (1.1), (1.2) for given initial data $(u_0, \theta_0) \in L^\infty_\mu(\mathbb{R}^n) \times L^\infty_\nu(\mathbb{R}^n)$ we introduce the bilinear integral operators

$$\mathcal{B}\left(u_1, u_2\right) := -\int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot\left(u_1\otimes u_2\right)(s)\,ds, \tag{4.1}$$

$$\mathcal{D}\left(u, \theta\right) := -\int_0^t e^{(t-s)\Delta}\nabla\cdot\left(\theta u\right)(s)\,ds. \tag{4.2}$$

We also define a linear operator which handles the buoyancy term, namely

$$\mathcal{C}\left(\theta\right) := \int_0^t e^{(t-s)\Delta}\mathbb{P}\left(g\theta\right)(s)\,ds \tag{4.3}$$

depending on the given gravity field g.

Sketch of the proof of Theorem 2.1: The existence and uniqueness of mild solutions to (1.1), (1.2) base on the abstract formulation of a solution (u, θ) as a fixed point of the coupled system

$$\begin{aligned}u(t) &= e^{t\Delta}u_0 + \mathcal{B}\left(u,u\right)(t) + \mathcal{C}\left(\theta\right)(t),\\ \theta(t) &= e^{t\Delta}\theta_0 + \mathcal{D}\left(u,\theta\right)(t)\end{aligned}$$

in the Banach space $C_\omega\left([0,T]; L^\infty_\mu\right) \times C_\omega\left([0,T]; L^\infty_\nu\right)$. With the help of Lemma 4.1 below the result is proved by Banach's fixed point theorem. □

Lemma 4.1: *Let $T > 0$, $g \in L^\infty_{n-1}$, $\mu \in (0, n]$, and $\nu > \max\{0,\ \mu - n + 1\}$. Then the operators*

$$\begin{aligned}&\mathcal{B} : C_\omega\left([0,T]; L^\infty_\mu\right) \times C_\omega\left([0,T]; L^\infty_\mu\right) \to C_\omega\left([0,T]; L^\infty_\mu\right),\\ &\mathcal{C} : C_\omega\left([0,T]; L^\infty_\nu\right) \to C_\omega\left([0,T]; L^\infty_\mu\right),\\ &\mathcal{D} : C_\omega\left([0,T]; L^\infty_\mu\right) \times C_\omega\left([0,T]; L^\infty_\nu\right) \to C_\omega\left([0,T]; L^\infty_\nu\right),\end{aligned}$$

see (4.1), (4.2), (4.3), are continuous with operator norms $\mathcal{O}(\sqrt{T}+T^{1+\kappa})$ *where* $\kappa := \frac{1}{2}\max\{\mu+\nu-n, 0\}$.

Proof: The estimate for $\mathcal{B}$ is proved in [14, Prop. 25.1]. The other assertions follow the same lines. □

For the proof of Theorem 2.3 we anticipate the results of Theorem 2.2 to be proved in Sect. 5.

Sketch of the proof of Theorem 2.3: Besides the result of Lemma 3.3 (i) on the Oseen kernel we note that the operators $e^{t\Delta}\mathbb{P}\mathrm{div}$, $e^{t\Delta}\mathbb{P}$ and $e^{t\Delta}\mathrm{div}$ are matrices of convolution operators with bounded kernels.

Similarly to [2], [7] we proceed to get an asymptotic profile of solutions of the Boussinesq equations and have to deal mainly with the terms $\mathcal{B}(u,u)$, $\mathcal{C}(\theta)$ and $\mathcal{D}(u,\theta)$ in the integral equations (1.1), (1.2). E.g., looking at $\mathcal{B}(u,u)$, we write $e^{t\Delta}\mathbb{P}\nabla$ as a convolution operator the kernel of which has the asymptotic profile

$$\gamma_n\partial_j\left(\frac{x\otimes x}{|x|^{n+2}}-\frac{1}{n|x|^n}I\right)+|x|^{-n-1}\Psi_j\left(\frac{x}{\sqrt{t}}\right), \quad |x|\gg\sqrt{t}\,,$$

cf. (3.8) Further, we define remainder terms $v_{h,k}$ such that

$$(u_h u_k)(x,t)=\mathcal{G}_1(x)\int_{\mathbb{R}^n}(u_h u_k)(y,t)\,dy+v_{h,k}(x,t).$$

Finally, we have to combine both and to estimate the remainder terms, for further details see [7]. The operator $\mathcal{D}$ is treated in an analogous way. But the convolution operator $e^{t\Delta}\mathbb{P}$ corresponding to the term $\mathcal{C}(\theta)$ has a worse decay, see [2]. Therefore, we study this term more carefully by a Taylor type formula of convolutions, see [5]:

Lemma 4.2: [5] *Let* $n\geq 2$, $m\in\mathbb{N}$, $0\leq\tau<n$. *Let* $f\in C^m(\mathbb{R}^n\setminus\{0\})$ *such that*

$$|x|^{\tau+|\alpha|}\partial^\alpha f\in L^\infty(\mathbb{R}^n)\quad \textit{for all } \alpha\in\mathbb{N}^n, |\alpha|\leq m,$$

and $h\in C(\mathbb{R}^n\setminus\{0\})\cap L^1(\mathbb{R}^n,(1+|x|)^m dx)\cap L^\infty_{n+m}(\mathbb{R}^n)$. *Then the convolution product* $f*h$ *satisfies*

$$f*h(x)=\sum_{0\leq|\beta|\leq m-1}\frac{(-1)^{|\beta|}}{\beta!}\left(\int_{\mathbb{R}^n}y^\beta h(y)dy\right)\partial^\beta f(x)+R(x),$$

where $R(x)$ *can be estimated for all* $x\neq 0$ *by*

$$C|x|^{-m-\tau}\max_{|\alpha|\leq m}\sup_{y\neq 0}|y|^{\tau+|\alpha|}|\partial^\alpha f(y)|\Big(\|h\|_{L^1(|y|^m)}+\sup_{y\neq 0}|y|^{n+m}|h(y)|\Big).$$

Assuming a sufficiently fast decaying data $\theta_0\in L^\infty_\nu$, $\nu>3$, we can replace the function h by $g\theta$, since due to Theorem 2.2 $g\theta(t)$ is continuous and $g\theta(t)\in L^1\left((1+|x|)^2\right)\cap L^\infty_{n+2}$ for all $t>0$. Applying Lemma 4.2 with $m=2$ and the functions $f=E_{j,h}$ which satisfy (3.7) we obtain for all $j=1,\ldots,n$

$$E_{j,h}*g_h\theta(x)=E_{j,h}(x)\int_{\mathbb{R}^n}(g_h\theta)(y)\,dy-\nabla E_{j,h}(x)\cdot\int_{\mathbb{R}^n}y\,(g_h\theta)(y)\,dy+R_j(x),$$

where $R_j(x) = \mathcal{O}\left(|x|^{-n-2+\varepsilon}\right)$ with an arbitrary small $\varepsilon > 0$. Thus we obtain

$$\begin{aligned}
&\left(\int_0^t e^{(t-s)\Delta}\mathbb{P}(g\theta)(s)ds\right)_j(x) = \sum_{h=1}^n \int_0^t \left(E_{j,h}(t-s) * (g_h\theta)(s)\right)(x)\,ds \\
&\quad = \sum_{h=1}^n \int_0^t E_{j,h}(x,t-s)\int_{\mathbb{R}^n}(g_h\theta)(y,s)\,dy\,ds \\
&\quad\quad - \sum_{h=1}^n \int_0^t \nabla E_{j,h}(x,t-s)\cdot \int_{\mathbb{R}^n} y(g_h\theta)(y,s)\,dy\,ds + \int_0^t R_j(x,t,s)\,ds \\
&\quad = \gamma_n \sum_{h=1}^n \left(\frac{x_h x_j}{|x|^{n+2}} - \frac{\delta_{j,h}}{n|x|^n}\right)\int_0^t\int_{\mathbb{R}^n}(g_h\theta)(y,s)\,dy\,ds \\
&\quad\quad - \gamma_n \sum_{h,l=1}^n \left(\frac{\sigma_{j,h,l}(x)}{|x|^{n+2}} - (n+2)\frac{x_j x_h x_l}{|x|^{n+4}}\right)\int_0^t\int_{\mathbb{R}^n} y_l(g_h\theta)(y,s)\,dy\,ds \\
&\quad\quad + R_1^{(j)}(x,t) + R_2^{(j)}(x,t) + R_3^{(j)}(x,t),
\end{aligned}$$

where $\sigma_{j,h,l}(x) := \delta_{j,h}x_l + \delta_{h,l}x_j + \delta_{j,l}x_h$. The remainder terms $R_1^{(j)}$ and $R_2^{(j)}$, $j = 1, ..., n$, are decaying exponentially:

$$\begin{aligned}
R_1^{(j)}(x,t) &:= \sum_{h=1}^n \int_0^t |x|^{-n}\Psi_{j,h}\left(\frac{x}{\sqrt{s}}\right)\int_{\mathbb{R}^n}(g_h\theta)(y,t-s)\,dy\,ds \\
R_2^{(j)}(x,t) &:= \sum_{h=1}^n \int_0^t \nabla_x\left[|x|^{-n}\Psi_{j,h}\left(\frac{x}{\sqrt{s}}\right)\right]\cdot\int_{\mathbb{R}^n} y(g_h\theta)(y,t-s)\,dy\,ds\,,
\end{aligned}$$

and for all $\varepsilon > 0$ we have

$$\begin{aligned}
R_3^{(j)}(x,t) &:= \int_0^t R_j(x,t,s)\,ds \\
&\lesssim t\sum_{h=1}^n |x|^{-2-\tau}\sup_{0<s<t}\max_{|\alpha|\le 2}\sup_{y\ne 0}|y|^{\tau+|\alpha|}|\partial^\alpha E_{j,h}(y,s)| \\
&\quad\times\sup_{0<s<t}\left(\|g\theta(s)\|_{L^1(|y|^2)} + \sup_{y\ne 0}|y|^{n+2}|(g\theta)(y,s)|\right) \\
&= \mathcal{O}_t\left(|y|^{-n-2+\varepsilon}\right).
\end{aligned}$$

Altogether, this completes the proof of Theorem 2.3. □

5. Proof of Theorem 2.2

At first we deal with first order spatial derivatives. Taking the partial derivative ∂_i in (1.1) and (1.2) we are led to the fixed point problem

$$\begin{aligned}
\partial_i u &= \Theta(\partial_i u, \partial_i\theta), \\
\partial_i\theta &= \tilde{\Theta}(\partial_i u, \partial_i\theta),
\end{aligned}$$

where

$$\begin{aligned}\Theta(w,\tilde{w}) &:= \partial_i e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(w\otimes u + u\otimes w)(s)\,ds \\ &\quad + \int_0^t e^{(t-s)\Delta}\mathbb{P}\left((\partial_i g)\theta + g\tilde{w}\right)(s)\,ds, \\ \tilde{\Theta}(w,\tilde{w}) &:= \partial_i e^{t\Delta}\theta_0 - \int_0^t e^{(t-s)\Delta}\nabla\cdot(\theta w + \tilde{w}u)(s)\,ds.\end{aligned}$$

From the properties of the heat kernel we obtain

$$\begin{aligned}|\partial_i e^{t\Delta}u_0(x)| &\le \int_{\mathbb{R}^n} |\partial_i \mathcal{G}_t(x-y)\cdot u_0(y)|\,dy \\ &\lesssim \int_{\mathbb{R}^n} \frac{|u_0(y)|}{(|x-y|+\sqrt{t})^{n+1}}\,dy \lesssim \left(t^{-\frac{1}{2}}+1\right)(1+|x|)^{-\mu}\|u_0\|_{L^\infty_\mu}\end{aligned}$$

and similarly

$$|\partial_i e^{t\Delta}\theta_0(x)| \lesssim \left(t^{-\frac{1}{2}}+1\right)(1+|x|)^{-\nu}\|\theta_0\|_{L^\infty_\nu}.$$

Thus we easily see, with the space

$$Y := \{w:\ t^{\frac{1}{2}}w \in C_\omega([0,T_0];L^\infty_\mu(\mathbb{R}^n)^n)\} \times \{\tilde{w}:\ t^{\frac{1}{2}}\tilde{w} \in C_\omega([0,T_0];L^\infty_\nu(\mathbb{R}^n))\}$$

constituting a Banach space Y with the norm

$$\|(w,\tilde{w})\|_Y := \sup_{t\in(0,T_0]} t^{\frac{1}{2}}\|w(t)\|_{L^\infty_\mu} + \sup_{t\in(0,T_0]} t^{\frac{1}{2}}\|\tilde{w}(t)\|_{L^\infty_\nu},$$

and Lemma 4.1 that $\left(\Theta,\tilde{\Theta}\right): Y \to Y$. Actually, given $(w,\tilde{w}) \in Y$ it is straightforward to show the weak-* continuity of $t^{\frac{1}{2}}\Theta(w,\tilde{w})$ and $t^{\frac{1}{2}}\tilde{\Theta}(w,\tilde{w})$ in $[0,T_0]$. Furthermore, by Lemma 4.1, the continuity of the operator $\left(\Theta,\tilde{\Theta}\right)$ on Y for all $0<T_0<T$ is achieved:

$$\begin{aligned}&\|\left(\Theta,\tilde{\Theta}\right)(w_1,\tilde{w}_1) - \left(\Theta,\tilde{\Theta}\right)(w_2,\tilde{w}_2)\|_Y \\ &\quad\lesssim \left(\sqrt{T_0}+T_0\right)\cdot\left(\sup_{t\in[0,T]}\|u(t)\|_{L^\infty_\mu} + \|g\|_{L^\infty_{n-1}}\right)\|(w_1,\tilde{w}_1)-(w_2,\tilde{w}_2)\|_Y.\end{aligned}$$

Choosing $T_0>0$ sufficiently small such that the operator $\left(\Theta,\tilde{\Theta}\right)$ is a contraction on Y, we get a unique fixed point $(w_0,\tilde{w}_0)$. By construction of the mappings Θ and $\tilde{\Theta}$ the fixed point $(w_0,\tilde{w}_0)$ is just the derivative ∂_i of u and θ, respectively. The same argument also holds on $[T_0,2T_0]$, etc., and finally leads to

$$t^{\frac{1}{2}}\partial_i u \in C_\omega\left([0,T];L^\infty_\mu(\mathbb{R}^n)^n\right), \quad t^{\frac{1}{2}}\partial_i\theta \in C_\omega\left([0,T];L^\infty_\nu(\mathbb{R}^n)\right).$$

Hence by the previous Theorem 2.1 $u, t^{\frac{1}{2}}\partial_i u$ belong to $C_\omega\left([0,T];L^\infty_\mu(\mathbb{R}^n)^n\right)$ and $\theta, t^{\frac{1}{2}}\partial_i\theta$ belong to $C_\omega\left([0,T];L^\infty_\nu(\mathbb{R}^n)\right)$, $i=1,...,n$, and thus $u,\theta \in C_\omega\left((0,T];W^{1,\infty}\right)$. Moreover, there holds the embedding $W^{1,\infty} \subseteq \mathrm{BUC}$, see [20, Lemma 9.2]. Since, in contrast to L^∞_μ, the operators $\{e^{t\Delta}\}_{t\ge 0}$ define in the space BUC a strongly continuous and even analytic semigroup, $e^{t\Delta}f$ converges to f in BUC as $t \searrow 0$. With this and Lemma 4.1 we get

$$\|u(t')-u(t)\|_\infty + \|\theta(t')-\theta(t)\|_\infty \longrightarrow 0 \qquad \text{as } t' \searrow t$$

for all $0 < t < t' \leq T$. Thus we have

$$u, \theta \in C\left((0, T]; \mathrm{BUC}\right),$$

i.e. continuous dependence on time. We notice that for all $0 < \varepsilon < T$ the solution (u, θ) belongs additionally to $L^\infty\left([\varepsilon, T]; W^{1,\infty}\right)$ and satisfies

$$u(t) = e^{(t-\varepsilon)\Delta}u(\varepsilon) - \int_\varepsilon^t e^{(t-\tau)\Delta}\mathbb{P}(u \cdot \nabla u)(\tau)\, d\tau + \int_\varepsilon^t e^{(t-\tau)\Delta}\mathbb{P}(g\theta)(\tau)\, d\tau,$$
$$\theta(t) = e^{(t-\varepsilon)\Delta}\theta(\varepsilon) - \int_\varepsilon^t e^{(t-\tau)\Delta}(u \cdot \nabla\theta)(\tau)\, d\tau.$$

Moreover, since there holds the embedding $W^{1,\infty}/\mathbb{R} \subseteq \dot{B}^s_{\infty,1}$ for all $s \in (0,1)$, see [13], we even have

$$u, \theta \in C\big([\varepsilon, T]; \dot{B}^s_{\infty,1}\big), \quad s \in (0,1). \tag{5.1}$$

In the following we will show that u and θ belong to

$$C\big((0, T]; \dot{B}^s_{\infty,1}\big), \quad s \in (0,3).$$

Using (3.1) and (3.9) we get

$$\|e^{(t-\tau)\Delta}\mathbb{P}(u \cdot \nabla u)\|_{\dot{B}^{s+\frac{1}{2}}_{\infty,1}} \lesssim \|e^{(t-\tau)\Delta}\mathbb{P}(u \otimes u)\|_{\dot{B}^{s+\frac{3}{2}}_{\infty,1}} \lesssim (t-\tau)^{-\frac{3}{4}}\|u \otimes u\|_{\dot{B}^s_{\infty,1}}.$$

Furthermore, choosing $\alpha > 0$ such that $\max\{0, s - \frac{3}{2}\} < \alpha < \min\{1, s + \frac{1}{2}\}$, i.e. $s < \frac{5}{2}$, we see from (3.3), (3.5) that

$$\begin{aligned}
&\|e^{(t-\tau)\Delta}\mathbb{P}(g\theta)\|_{\dot{B}^{s+\frac{1}{2}}_{\infty,1}} \\
&\quad \leq \|(-\Delta)^{-\frac{\alpha}{2}}e^{(t-\tau)\Delta}\mathbb{P}((-\Delta)^{\frac{\alpha}{2}}(g\theta))\|_{\dot{B}^{s+\frac{1}{2}}_{\infty,1}} \lesssim (t-\tau)^{-\frac{1}{2}(s-\alpha+\frac{1}{2})}\|g\theta\|_{\dot{B}^\alpha_{\infty,1}}.
\end{aligned}$$

For example, we can set $\alpha := \frac{s}{3}$. The previous estimates and (3.5) as well as (3.2) yield

$$\begin{aligned}
\|u(t)\|_{\dot{B}^{s+\frac{1}{2}}_{\infty,1}} &\lesssim (t-\varepsilon)^{-\frac{1}{4}}\|u(\varepsilon)\|_{\dot{B}^s_{\infty,1}} + \int_\varepsilon^t (t-\tau)^{-\frac{3}{4}}\|(u \otimes u)(\tau)\|_{\dot{B}^s_{\infty,1}}\, d\tau \\
&\quad + \int_\varepsilon^t (t-\tau)^{-\frac{1}{2}(s-\alpha+\frac{1}{2})}\|g\theta(\tau)\|_{\dot{B}^\alpha_{\infty,1}}\, d\tau \\
&\lesssim (t-\varepsilon)^{-\frac{1}{4}}\|u(\varepsilon)\|_{\dot{B}^s_{\infty,1}} + t^{\frac{1}{4}} \sup_{\varepsilon \leq \tau \leq T}\|u(\tau)\|_\infty \sup_{\varepsilon \leq \tau \leq T}\|u(\tau)\|_{\dot{B}^s_{\infty,1}} \\
&\quad + t^{\frac{3}{4}-\frac{s}{2}+\frac{\alpha}{2}} \sup_{\varepsilon \leq \tau \leq T}\Big(\|g\|_\infty\|\theta(\tau)\|_{\dot{B}^\alpha_{\infty,1}} + \|g\|_{\dot{B}^\alpha_{\infty,1}}\|\theta(\tau)\|_\infty\Big).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\|\theta(t)\|_{\dot{B}^{s+\frac{1}{2}}_{\infty,1}} &\lesssim (t-\varepsilon)^{-\frac{1}{4}}\|\theta(\varepsilon)\|_{\dot{B}^s_{\infty,1}} + \int_\varepsilon^t (t-\tau)^{-\frac{3}{4}}\|(\theta u)(\tau)\|_{\dot{B}^s_{\infty,1}}\, d\tau \\
&\lesssim (t-\varepsilon)^{-\frac{1}{4}}\|\theta(\varepsilon)\|_{\dot{B}^s_{\infty,1}} \\
&\quad + t^{\frac{1}{4}} \sup_{\varepsilon \leq \tau \leq T}\Big(\|u(\tau)\|_\infty\|\theta(\tau)\|_{\dot{B}^s_{\infty,1}} + \|\theta(\tau)\|_\infty\|u(\tau)\|_{\dot{B}^s_{\infty,1}}\Big).
\end{aligned}$$

Immediately, from (5.1), we get that $u, \theta \in L^\infty\big([\varepsilon, T]; \dot B^s_{\infty,1}\big)$, $s \in [1, \frac{3}{2})$. So we can conclude by iteration that

$$\sup_{\varepsilon \le \tau < T} \|u(\tau)\|_{\dot B^s_{\infty,1}} + \sup_{\varepsilon \le \tau < T} \|\theta(\tau)\|_{\dot B^s_{\infty,1}} < \infty$$

holds for all $0 < \varepsilon < T$ and all $s \in (0,3)$. Thus

$$u, \theta \in L^\infty\big([\varepsilon, T]; \dot B^s_{\infty,1}\big), \quad s \in (0,3).$$

Now we show that

$$u, \theta \in C\big((0, T]; \dot B^s_{\infty,1}\big), \quad s \in (0,3). \tag{5.2}$$

To this aim we choose $\beta \in (0, \frac{1}{2})$ such that $-\frac{s}{3} < \beta < 1 - \frac{s}{3}$ with $s \in (0,3)$. Since for all $0 < \varepsilon < t < t' < T$ the function u satisfies

$$\begin{aligned} u(t') - u(t) = (e^{t'\Delta} - e^{t\Delta})u(\varepsilon) - \int_t^{t'} e^{(t'-s)\Delta}\mathbb{P}(\nabla \cdot (u \otimes u) - g\theta)(s)\, ds \\ - \int_\varepsilon^t \big(e^{(t'-s)\Delta} - e^{(t-s)\Delta}\big)\mathbb{P}(\nabla \cdot (u \otimes u) - g\theta)(s)\, ds, \end{aligned}$$

we get the following estimate by Lemmata 3.1 and 3.2 as well as (3.9):

$$\begin{aligned} \|u(t') - u(t)\|_{\dot B^s_{\infty,1}} &\lesssim (t'-t)^{\frac{1}{2}} \|e^{t\Delta} u(\varepsilon)\|_{\dot B^{s+1}_{\infty,1}} \\ &+ \int_\varepsilon^t (t'-t)^\beta \|\nabla e^{(t-\tau)\Delta}\mathbb{P}(u \otimes u)(\tau)\|_{\dot B^{s+2\beta}_{\infty,1}}\, d\tau \\ &+ \int_\varepsilon^t (t'-t)^\beta \|e^{(t-\tau)\Delta}\mathbb{P}(g\theta)(\tau)\|_{\dot B^{s+2\beta}_{\infty,1}}\, d\tau \\ &+ \int_t^{t'} \Big(\|\nabla e^{(t'-\tau)\Delta}\mathbb{P}(u \otimes u)(\tau)\|_{\dot B^s_{\infty,1}} + \|e^{(t'-\tau)\Delta}\mathbb{P}(g\theta)(\tau)\|_{\dot B^s_{\infty,1}}\Big)\, d\tau \\ &\lesssim (t'-t)^{\frac{1}{2}} \|e^{t\Delta} u(\varepsilon)\|_{\dot B^{s+1}_{\infty,1}} \\ &+ \int_\varepsilon^t (t'-t)^\beta (t-\tau)^{-\frac{1}{2}-\beta} \|u \otimes u(\tau)\|_{\dot B^s_{\infty,1}}\, d\tau \\ &+ \int_\varepsilon^t (t'-t)^\beta (t-\tau)^{-\frac{s}{3}-\beta} \|g\theta(\tau)\|_{\dot B^{\frac{s}{3}}_{\infty,1}}\, d\tau \\ &+ \int_t^{t'} \Big((t'-\tau)^{-\frac{1}{2}} \|u \otimes u(\tau)\|_{\dot B^s_{\infty,1}} + (t'-\tau)^{-\frac{s}{3}} \|g\theta(\tau)\|_{\dot B^{\frac{s}{3}}_{\infty,1}}\Big)\, d\tau. \end{aligned}$$

Finally, (3.2), (3.4) yield

$$\begin{aligned}\|u(t')-u(t)\|_{\dot B^s_{\infty,1}} &\lesssim (t'-t)^{\frac12}t^{-\frac{s+1}{2}}\|u(\varepsilon)\|_\infty \\ &+(t'-t)^\beta t^{\frac12-\beta}\left(\sup_{\varepsilon\le\tau\le t'}\|u(\tau)\|_{\dot B^s_{\infty,1}\cap L^\infty}\right)^2 \\ &+(t'-t)^\beta t^{1-\frac{s}{3}-\beta}\left(\|g\|_\infty\sup_{\varepsilon\le\tau\le t'}\|\theta(\tau)\|_{\dot B^{\frac{s}{3}}_{\infty,1}}+\|g\|_{\dot B^{\frac{s}{3}}_{\infty,1}}\sup_{\varepsilon\le\tau\le t'}\|\theta(\tau)\|_\infty\right) \\ &+(t'-t)^{\frac12}\left(\sup_{\varepsilon\le\tau\le t'}\|u(\tau)\|_{\dot B^s_{\infty,1}\cap L^\infty}\right)^2 \\ &+(t'-t)^{1-\frac{s}{3}}\left(\|g\|_\infty\sup_{\varepsilon\le\tau\le t'}\|\theta(\tau)\|_{\dot B^{\frac{s}{3}}_{\infty,1}}+\|g\|_{\dot B^{\frac{s}{3}}_{\infty,1}}\sup_{\varepsilon\le\tau\le t'}\|\theta(\tau)\|_\infty\right).\end{aligned}$$

Therefore, we get $u\in C\big((0,T];\dot B^s_{\infty,1}\big)$ for all $s\in(0,3)$.

Moreover, for θ we have by (3.6)

$$\begin{aligned}\|\theta(t')-\theta(t)\|_{\dot B^s_{\infty,1}} &\lesssim (t'-t)^{\frac12}\|e^{t\Delta}\theta(\varepsilon)\|_{\dot B^{s+1}_{\infty,1}} \\ &+\int_\varepsilon^t (t'-t)^\beta\|\nabla\cdot e^{(t-\tau)\Delta}(u\theta)(\tau)\|_{\dot B^{s+2\beta}_{\infty,1}}\,d\tau \\ &+\int_t^{t'}\|\nabla\cdot e^{(t'-\tau)\Delta}(u\theta)(\tau)\|_{\dot B^s_{\infty,1}}\,d\tau\end{aligned}$$

and further by Lemmata 3.1 and 3.2

$$\begin{aligned}\|\theta(t')-\theta(t)\|_{\dot B^s_{\infty,1}} \lesssim (t'-t)t^{-\frac{s+1}{2}}\|\theta(\varepsilon)\|_\infty &+\Big[(t'-t)^\beta t^{\frac12-\beta}+(t'-t)^{\frac12}\Big] \\ &\times\sup_{\varepsilon\le\tau\le t'}\Big(\|u(\tau)\|_\infty\|\theta(\tau)\|_{\dot B^s_{\infty,1}}+\|u(\tau)\|_{\dot B^s_{\infty,1}}\|\theta(\tau)\|_\infty\Big),\end{aligned}$$

and thus $\theta\in C\big((0,T];\dot B^s_{\infty,1}\big)$ for all $s\in(0,3)$. Altogether, this estimate, the same result for u and (3.1) imply that

$$\partial_i u,\ \partial_i\theta\ ,\partial_i\partial_j u\ ,\partial_i\partial_j\theta\ \in\ C\big((0,T];\dot B^0_{\infty,1}\big)\ \subseteq\ C\big((0,T];\mathrm{BUC}\big)$$

for all $i,j=1,...,n$ and hence

$$u,\theta\in C\left((0,T];W^{2,\infty}\right).$$

In the final step of the proof we show that (u,θ) is a solution to (BE) in the strong sense. Using the boundedness of the Helmholtz projection on $\dot B^0_{\infty,1}$ and Lemmata 3.1 and 3.2 we get

$$\begin{aligned}\|\mathbb P(u\cdot\nabla u)\|_{\dot B^0_{\infty,1}}\lesssim\|\nabla(u\otimes u)\|_{\dot B^0_{\infty,1}}\lesssim\|u\otimes u\|_{\dot B^1_{\infty,1}}\lesssim\|u\|^2_{\dot B^1_{\infty,1}\cap L^\infty},\\ \|u\cdot\nabla\theta\|_{\dot B^0_{\infty,1}}=\|\nabla\cdot(u\theta)\|_{\dot B^0_{\infty,1}}\lesssim\|u\|_\infty\|\theta\|_{\dot B^1_{\infty,1}}+\|u\|_{\dot B^1_{\infty,1}}\|\theta\|_\infty.\end{aligned}$$

Since $g\theta$ and $\nabla(g\theta)$ belong to $L^\infty_{n-1}\subseteq L^p$, $p>\frac{n}{n-1}$, we get $\mathbb P(g\theta)\in W^{1,p}$. But in the case $p>n$ this Sobolev space is embedded into the Hölder space $C^{0,\gamma}$ with $\gamma=1-\frac{n}{p}$, see [20, Lemma 9.2]. That means $\mathbb P(g\theta)$ is uniformly continuous. Moreover, $\mathbb{P}(g\theta) \in L^p \cap C^{0,1-\frac{n}{p}}$, $n < p < \infty$, is bounded. Using the inclusion $\dot{B}^0_{\infty,1} \subseteq \mathrm{BUC}/\mathbb{R}$ we have

$$\mathbb{P}(u \cdot \nabla u),\ \mathbb{P}(g\theta),\ u \cdot \nabla \theta \ \in\ C\left((0,T]; \mathrm{BUC}\right).$$

Since, for $0 \le t < t' \le T$,

$$\begin{aligned} &u(t') - u(t) \\ &\quad = (e^{(t'-t)\Delta} - I)u(t) - \int_t^{t'} e^{(t'-\tau)\Delta}\left[\mathbb{P}(u \cdot \nabla u - g\theta)\right](\tau)\, d\tau \end{aligned}$$

and for each $h \in \mathrm{BUC}^2(\mathbb{R}^n)$

$$\lim_{t' \searrow t} \frac{e^{(t'-t)\Delta} - I}{t' - t} h = \Delta h \quad \text{in BUC},$$

we obtain

$$u_t = \lim_{t' \searrow t} \frac{u(t') - u(t)}{t' - t} = \Delta u - \mathbb{P}(u \cdot \nabla u) + \mathbb{P}(g\theta) \ \in\ C\left((0,T]; \mathrm{BUC}\right).$$

Similarly, with

$$\theta(t') - \theta(t) = \big(e^{(t'-t)\Delta} - I\big)\theta(t) - \int_t^{t'} e^{(t'-\tau)\Delta}(u \cdot \nabla \theta)(\tau)\, d\tau,$$

we get in BUC

$$\theta_t = \lim_{t' \searrow t} \frac{\theta(t') - \theta(t)}{t' - t} = \Delta\theta - u \cdot \nabla\theta \ \in\ C((0,T]; \mathrm{BUC}).$$

Now the proof of Theorem 2.2 is complete. □

6. Proof of Theorem 2.4

To prove this quantitative result of the solution we need a representation of (u, θ), as the limit of an iteration, following ideas from [4, §2.1]:

$$\begin{aligned} &T_1(u_0,\theta_0) := e^{t\Delta}u_0, \qquad \tilde{T}_1(u_0,\theta_0) := e^{t\Delta}\theta_0, \\ &T_k(u_0,\theta_0) := \sum_{l=1}^{k-1} \mathcal{B}\big(T_l(u_0,\theta_0), T_{k-l}(u_0,\theta_0)\big) + \mathcal{C}\big(\tilde{T}_{k-1}(u_0,\theta_0)\big), \\ &\tilde{T}_k(u_0,\theta_0) := \sum_{l=1}^{k-1} \mathcal{D}\big(T_l(u_0,\theta_0), \tilde{T}_{k-l}(u_0,\theta_0)\big), \quad k \ge 2. \end{aligned} \tag{6.1}$$

Under smallness assumptions on the initial data the series

$$\phi(u_0,\theta_0) := \sum_{k=1}^{\infty} T_k(u_0,\theta_0) \quad \text{and} \quad \psi(u_0,\theta_0) := \sum_{k=1}^{\infty} \tilde{T}_k(u_0,\theta_0) \tag{6.2}$$

will be shown to be absolutely convergent. Then $(u,\theta) = (\phi,\psi)(u_0,\theta_0)$ is a solution of the equations

$$u = e^{t\Delta}u_0 + \mathcal{B}(u,u) + \mathcal{C}(\theta), \quad \theta = e^{t\Delta}\theta_0 + \mathcal{D}(u,\theta)$$

in the space $Y \times \tilde{Y}$ where

$$Y = C_\omega\left([0,T]; L^\infty_\mu(\mathbb{R}^n)\right), \quad \tilde{Y} = C_\omega\left([0,T]; L^\infty_\nu(\mathbb{R}^n)\right). \tag{6.3}$$

Assume that

$$c := \max\{\|e^{t\Delta}u_0\|_Y, \|e^{t\Delta}\theta_0\|_{\tilde{Y}}\} < 1.$$

Then mathematical induction and Lemma 4.1 applied to (6.1) yield a sequence of estimates of $\|T_k\|_Y$, $\|\tilde{T}_k\|_{\tilde{Y}}$ in terms of $\|T_l\|_Y$, $\|\tilde{T}_l\|_{\tilde{Y}}$, $1 \le l \le k-1$, which finally leads to the bound

$$\|T_k\|_Y + \|\tilde{T}_k\|_{\tilde{Y}} \le k^{-\frac{3}{2}}(12C_0\sqrt{c})^{k-1}(\sqrt{c}+c), \quad k \ge 2,$$

where the constant $C_0 = C_0(T)$ is a bound of the norms in Lemma 4.1. If the initial data (u_0, θ_0) is small enough, such that $\max\{12C_0\sqrt{c}, c\} < 1$ the series

$$\sum_{k=1}^{\infty} \|\big(T_k, \tilde{T}_k\big)(u_0,\theta_0)\|_{Y\times\tilde{Y}} \le (\sqrt{c}+c)\sum_{k=1}^{\infty} k^{-\frac{3}{2}}(12C_0\sqrt{c})^{k-1} < \infty$$

converges, i.e., the series $\sum_{k=1}^{\infty}\big(T_k, \tilde{T}_k\big)(u_0,\theta_0)$ converges in the Banach space $Y \times \tilde{Y}$. Finally, the limit $\phi(u_0,\theta_0)$, $\psi(u_0,\theta_0)$, see (6.2), solves the Boussinesq integral equations (1.1) and (1.2). We notice that this representation of a solution is unique on $[0,T]$ due to Theorem 2.1.

Lemma 6.1: *Let $n \in \{2,3\}$, $0 =: t_0 < t_1 < ... < t_N$ with $N \in \mathbb{N}$ and $0 < \varepsilon < \min\{\frac{1}{2}(t_{k+1}-t_k) : k = 0,...,N-1\}$. Let g belong to (2.1) and be either an odd or an even B-symmetric vector field. Then there exists a real-valued B-symmetric function $\underline{\theta}_0 \in \mathcal{S}(\mathbb{R}^n)$ such that the function*

$$\mathcal{E}(\underline{\theta}_0) : \mathbb{R}^+ \to \mathbb{R}, \quad \mathcal{E}(\underline{\theta}_0)(t) := \int_0^t \int_{\mathbb{R}^n} g_1(x)\left(e^{s\Delta}\underline{\theta}_0\right)(x)\,dx\,ds,$$

changes sign inside $(t_i - \varepsilon, t_i + \varepsilon)$, $i = 1, ..., N$.

Proof: At first we treat the two-dimensional case. Without lost of generality we prove this assertion by assuming that $g = (g_1, g_2)$ is odd. By our assumption on the gravity $g \in W_1^{2,\infty}(\mathbb{R}^2)$ we do not expect that $g \in L^2(\mathbb{R}^2)$. So we cannot use Fourier methods like the Parseval relation directly. But the Laplacian $\Delta g \in L^\infty_{n+\delta}(\mathbb{R}^2)$ lies in $L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. In particular the Fourier transform $\mathcal{F}(\Delta g)$ is odd, continuous and vanishes at infinity. Hence there is a vector $0 \ne \alpha_0 \in \mathbb{R}^2$ such that

$$\left(\mathcal{T}g\right)(\alpha_0) := \mathcal{F}\left(\Delta(g_1+g_2)\right)(\alpha_0) - \mathcal{F}\left(\Delta(g_1+g_2)\right)(-\alpha_0) \ne 0.$$

Otherwise, since $\mathcal{F}(\Delta(g_j)$ is odd and $\mathcal{T}g$ is continuous, $\mathcal{T}g$ would vanish identically. Since

$$|\mathcal{T}g(\alpha_0)| = |\mathcal{T}g(\tilde{\alpha_0})| = |\mathcal{T}g(-\tilde{\alpha_0})| = |\mathcal{T}g(-\alpha_0)|$$

we can assume that α_0 belongs to the open sector $\{\xi \in \mathbb{R}^2 | \; \xi_1 > |\xi_2| > 0\}$. Furthermore, due to the continuity of $\mathcal{T}g$, there exists a constant $\sigma_1 > 0$ such that $\left(\mathcal{T}g\right)((1+\sigma)\alpha_0) \ne 0$ for all $0 \le \sigma < \sigma_1$. Note that $\mathcal{T}g(\cdot) \in i\mathbb{R}$.

Continuing, for $0 < \delta < \frac{\sigma_1}{N+1}$, we regard with

$$\alpha_j = \sqrt{1+\delta(j-1)}\alpha_0 \in \mathbb{R}^2 \tag{6.4}$$

and $\lambda_j \in \mathbb{R}$, $j = 1, \ldots, N+1$, to be determined below, the function

$$E(t) := \sum_{j=1}^{N+1} \lambda_j \frac{(1 - e^{-4\pi^2 t|\alpha_j|^2})}{(2\pi)^4 i |\alpha_j|^4} \left(\mathcal{T} g\right)(\alpha_j) = \sum_{j=1}^{N+1} b_j (1 - e^{-4\pi^2 t|\alpha_j|^2})$$

where $b_j := \lambda_j \frac{1}{(2\pi)^4 i|\alpha_j|^4} \left(\mathcal{T} g\right)(\alpha_j)$. With $T_i := e^{-4\pi^2|\alpha_0|^2 t_i}$ we have

$$E(t_i) = \sum_{j=1}^{N+1} b_j \big(1 - T_i^{1+\delta(j-1)}\big).$$

We want to determine $\lambda_j, ..., \lambda_{N+1}$ in such a way that $E(t)$ vanishes at $t_1, ..., t_N$ and changes sign at these points. In particular there has to hold

$$0 \neq E'(t_i) = 4\pi^2|\alpha_0|^2 \sum_{j=1}^{N+1} (1 + \delta(j-1)) b_j T_i^{1+\delta(j-1)}. \tag{6.5}$$

To satisfy these conditions we consider a corresponding linear system with the unknowns $b = (b_1, ..., b_{N+1})^T \in \mathbb{R}^{N+1}$. To be more precise, we define the $(N+1) \times (N+1)$-matrix

$$M(\delta) := \begin{pmatrix} 1 - T_1^1 & 1 - T_1^{1+\delta} & \ldots & 1 - T_1^{1+\delta N} \\ \vdots & \vdots & \ddots & \vdots \\ 1 - T_N^1 & 1 - T_N^{1+\delta} & \ldots & 1 - T_N^{1+\delta N} \\ 1 \cdot T_1^1 & (1+\delta) T_1^{1+\delta} & \ldots & (1+\delta N) T_1^{1+\delta N} \end{pmatrix}.$$

Note that

$$M(1) = \begin{pmatrix} 1 - T_1 & 1 - T_1^2 & \ldots & 1 - T_1^{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 - T_N & 1 - T_N^2 & \ldots & 1 - T_N^{N+1} \\ T_1 & 2T_1^2 & \ldots & (N+1) T_1^{N+1} \end{pmatrix},$$

where an explicit computation, see [4], yields

$$\det M(1) = -T_1(1 - T_1) \prod_{i=1}^{N} (1 - T_i) \prod_{i=2}^{N} (T_1 - T_i) \prod_{1 \le i < j \le N} (T_j - T_i) \neq 0$$

since $T_i \in (0, 1)$ and $T_i \neq T_j$ for $i \neq j$, $i, j = 1, ..., N$. Now $\det M(\delta)$ can be considered as an analytic function on $\mathbb{C}$, and we conclude that there exists $0 < \delta < \frac{\sigma_1}{N+1}$ such that $\det M(\delta) \neq 0$.

The equations

$$E(t_i) = 0, \quad i = 1, ..., N, \quad \text{and} \quad E'(t_1) = \gamma$$

are fulfilled with $b = (b_1, ..., b_{N+1})^T \in \mathbb{R}^{N+1}$ if and only if

$$M(\delta) b = 4\pi^2 |\alpha_0|^2 \, e_{N+1}, \quad e_{N+1} = (0, ..., 0, 1)^T. \tag{6.6}$$

Since $\det M(\delta) \neq 0$, we obtain a unique vector $0 \neq b \in \mathbb{R}^{N+1}$ such that E vanishes at $t_1, ..., t_N$ and changes sign at t_1. The conditions $E'(t_i) \neq 0$, $i = 2, ..., N$, are then automatically fulfilled. Indeed, if we had $E'(t_i) = 0$ for some $i = 2, ..., N$, then the matrix $M(\delta)$ obtained when replacing the last row by

$$T_i \quad (1+\delta) T_i^{1+\delta} \quad \ldots \quad (1+\delta N) T_i^{1+\delta N}$$

would have a vanishing determinant in contradiction with the general formula for $\det M$; to use this argument for $i = 1, \dots, N$ we possibly have to choose $\delta > 0$ smaller as before. Finally this solution determines the desired coefficients

$$\lambda_j = \frac{(2\pi)^4 i b_j |\alpha_j|^4}{(\mathcal{T} g)\,(\alpha_j)}.$$

We note that $\mathcal{T}g(\alpha_j) \neq 0$, $j = 1, ..., N+1$, by construction.

To construct the initial temperature θ_0 we choose a real-valued radially symmetric function $\phi \in \mathcal{S}(\mathbb{R}^2)$ such that $\hat{\phi} \in C_0^\infty(\mathbb{R}^2)$ satisfying $\operatorname{supp} \hat{\phi} \subseteq \overline{B_1(0)}$ and $\int_{\mathbb{R}^2} \hat{\phi} = 1$. Moreover, for $\rho > 0$, we define $\hat{\phi}^\rho(\xi) := \rho^{-n} \hat{\phi}(\rho^{-1}\xi)$. Then for each $\alpha \in \mathbb{R}^2$ let

$$\hat{\theta}_\alpha(\xi) := i\big(\hat{\phi}(\xi - \alpha) - \hat{\phi}(\xi + \alpha) + \hat{\phi}(\xi - \tilde{\alpha}) - \hat{\phi}(\xi + \tilde{\alpha})\big)$$

satisfying $\hat{\theta}_\alpha(\tilde{\xi}) = \hat{\theta}_\alpha(\xi)$ and $\hat{\theta}_\alpha(-\xi) = -\hat{\theta}_\alpha(\xi)$. Thus θ_α is real-valued, odd and B-symmetric, i.e. $\theta_\alpha(\tilde{x}) = \theta_\alpha(x)$. We define $\hat{\theta}_\alpha^\rho$ as before, by replacing $\hat{\phi}$ with $\hat{\phi}^\rho$ in the corresponding definition.

Using the theorem of Parseval we get

$$\begin{aligned} \mathcal{E}(\theta_{\alpha_j}^\rho)(t) &= \int_0^t \int_{\mathbb{R}^n} g_1(x) \left(e^{s\Delta} \theta_{\alpha_j}^\rho\right)(x)\, dx\, ds \\ &= -\int_{\mathbb{R}^n} \left(1 - e^{-4\pi^2 t|\xi|^2}\right) \frac{(\Delta g_1)\,\hat{}\,(\xi)}{(2\pi)^4 |\xi|^4} \hat{\theta}_{\alpha_j}^\rho(\xi)\, d\xi. \end{aligned}$$

Furthermore, due to the symmetry properties of $\hat{\theta}_{\alpha_j}^\rho$, we obtain

$$\mathcal{E}(\theta_{\alpha_j}^\rho)(t) = -\int_{\xi_1 > |\xi_2|} \left(1 - e^{-4\pi^2 t|\xi|^2}\right) \frac{(\mathcal{T} g)\,(\xi)}{(2\pi)^4 |\xi|^4} \hat{\theta}_{\alpha_j}^\rho(\xi)\, d\xi.$$

Let us now choose $\rho_0 > 0$ sufficiently small, such that

$$\begin{aligned} \operatorname{supp} \hat{\theta}_{\alpha_j}^{\rho_0} \cap \operatorname{supp} \hat{\theta}_{\alpha_k}^{\rho_0} &= \emptyset \qquad \text{and} \\ \operatorname{supp} \hat{\phi}^{\rho_0}(\cdot - \beta_j) \cap \operatorname{supp} \hat{\phi}^{\rho_0}(\cdot - \beta_j') &= \emptyset \end{aligned}$$

for all $j \neq k$ with $\beta_j, \beta_j' \in \{\alpha_j, -\alpha_j, \tilde{\alpha}_j, -\tilde{\alpha}_j\}$, $\beta_j \neq \beta_j'$. Therewith, for all $0 < \rho \leq \rho_0$, there holds

$$\mathcal{E}(\theta_{\alpha_j}^\rho)(t) = -i \int_{\xi_1 > |\xi_2|} \left(1 - e^{-4\pi^2 t|\xi|^2}\right) \frac{(\mathcal{T} g)\,(\xi)}{(2\pi)^4 |\xi|^4} \hat{\phi}^\rho(\xi - \alpha_j)\, d\xi.$$

Since $\mathcal{T}g$ is continuous and $\{\hat{\phi}^\rho : \rho > 0\}$ is an approximation of identity, $\mathcal{E}(\theta_{\alpha_j}^\rho)(t)$ converges (uniformly with respect to $t \geq 0$) to

$$E_{\alpha_j}(t) := \left(1 - e^{-4\pi^2 t|\alpha_j|^2}\right) \frac{(\mathcal{T} g)\,(\alpha_j)}{(2\pi)^4 i |\alpha_j|^4}$$

as $\rho \to 0$. We observe that $E_{\alpha_j}(t)$ is real-valued.

Eventually, we consider

$$\theta_0^\rho := \sum_{j=1}^{N+1} \lambda_j \theta_{\alpha_j}^\rho.$$

Since supp $\hat{\theta}^{\rho_0}_{\alpha_j} \cap \operatorname{supp} \hat{\theta}^{\rho_0}_{\alpha_k} = \emptyset$, $j \neq k$, there holds

$$\mathcal{E}\big(\theta^{\rho}_0\big)(t) = \sum_{j=1}^{N+1} \lambda_j \mathcal{E}\big(\theta^{\rho}_{\alpha_j}\big)(t)$$

for all $0 < \rho \leq \rho_0$. As $\rho \to 0$, this term converges uniformly to $E(t) = \sum_{j=1}^{N+1} \lambda_j E_{\alpha_j}(t)$. Finally, we see that if $\rho' > 0$ is sufficiently small then $\mathcal{E}\big(\theta^{\rho'}_0\big)$ changes sign in the interval $(t_i - \varepsilon, t_i + \varepsilon)$, for $i = 1, ..., N$. Hence we choose $\underline{\theta}_0 := \theta^{\rho'}_0$ as initial temperature.

The case $n = 3$. Firstly we assume g to be odd and choose

$$\alpha_0 \in \Omega := \{\xi \in \mathbb{R}^3 \mid \min\{\xi_2, \xi_3\} > \max\{\xi_1, 0\}\ \}$$

to be a vector such that

$$\begin{aligned}(\mathcal{T} g)(\alpha_0) &:= \mathcal{F}\left(\Delta(g_1 + g_2 + g_3)\right)(\alpha_0) - \mathcal{F}\left(\Delta(g_1 + g_2 + g_3)\right)(-\alpha_0) \\ &= 2 \sum_{k=1}^{3} \mathcal{F}\left(\Delta g_k\right)(\alpha_0) \neq 0.\end{aligned}$$

Moreover, let $\sigma_1 > 0$ be a constant such that $(\mathcal{T} g)\left((1 + \sigma)\alpha_0\right) \neq 0$ for all $0 \leq \sigma < \sigma_1$. In contrast to the two-dimensional case the gravity $g \in L^{\infty}_2(\mathbb{R}^3)^3$ belongs now to $L^2(\mathbb{R}^3)$. However, we will again use Δg since we need a continuous and decaying Fourier transform.

We build the initial temperature analogously as above and define $\theta_\alpha \in \mathcal{S}(\mathbb{R}^3)$ through

$$\hat{\theta}_\alpha(\xi) = i\big(\hat{\phi}(\xi - \alpha) - \hat{\phi}(\xi + \alpha) + \hat{\phi}(\xi - \tilde{\alpha}) - \hat{\phi}(\xi + \tilde{\alpha}) + \hat{\phi}(\xi - \tilde{\tilde{\alpha}}) - \hat{\phi}(\xi + \tilde{\tilde{\alpha}})\big).$$

Once again we have $\hat{\theta}_\alpha(\tilde{\xi}) = \hat{\theta}_\alpha(\xi)$ and $\hat{\theta}_\alpha(-\xi) = -\hat{\theta}_\alpha(\xi)$. Thus θ_α is real-valued, odd and B-symmetric. The definition (6.4) for α_j,

$$E(t) := \sum_{j=1}^{N+1} \lambda_j \frac{(1 - e^{-4\pi^2 t|\alpha_j|^2})}{(2\pi)^4 i |\alpha_j|^4} \left(\mathcal{T} g\right)(\alpha_j)$$

and the conditions on E at $t_1, \ldots, t_N$ yield the same linear system (6.6) as above. Hence we obtain a vector of coefficients $(\lambda_1, ..., \lambda_N) \neq 0$, such that $E(t)$ vanishes at $t_1, ..., t_N$ and changes sign at these points. Imposing the condition $\alpha_j \in \Omega$ (which is satisfied if we set α_j as in (6.4)) we get

$$\begin{aligned}\mathcal{E}(\theta^{\rho}_{\alpha_j})(t) &= \int_0^t \int_{\mathbb{R}^n} g_1(x) e^{s\Delta} \theta^{\rho}_{\alpha_j}(x)\, dx\, ds \\ &= -\int_{\tilde{\Omega}} \left(1 - e^{-4\pi^2 t|\xi|^2}\right) \frac{(\Delta g_1)\hat{}(\xi) + (\Delta g_1)\hat{}(\tilde{\xi}) + (\Delta g_1)\hat{}(\tilde{\tilde{\xi}})}{(2\pi)^4 |\xi|^4} \hat{\theta}^{\rho}_{\alpha_j}(\xi)\, d\xi \\ &= -\int_{\Omega} \left(1 - e^{-4\pi^2 t|\xi|^2}\right) \frac{(\mathcal{T} g)(\xi)}{(2\pi)^4 |\xi|^4} \hat{\theta}^{\rho}_{\alpha_j}(\xi)\, d\xi\end{aligned}$$

with $\tilde{\Omega} := \{\xi \in \mathbb{R}^3 |\ \min(\xi_2, \xi_3) > \max(\xi_1, 0) \text{ or } \max(\xi_2, \xi_3) < \min(\xi_1, 0)\}$. Geometrically, the condition $\alpha_j \in \Omega$ corresponds to cutting $\mathbb{R}^3$ into six congruent regions that can be obtained from each other through the orthogonal transforms $\xi \mapsto \tilde{\xi}$ and $\xi \mapsto -\xi$. If we choose again $\rho_0 > 0$ small enough then $\mathcal{E}(\theta^\rho_{\alpha_j})(t)$ equals

$$-i\int_\Omega \left(1-e^{-4\pi^2 t|\xi|^2}\right)\frac{(\mathcal{T}g)(\xi)}{(2\pi)^4|\xi|^4}\hat{\phi}^\rho(\xi-\alpha_j)\,d\xi.$$

As $\rho \to 0$, the function $\mathcal{E}(\theta^\rho_{\alpha_j})(t)$ converges uniformly in t to

$$E_{\alpha_j}(t) := \left(1-e^{-4\pi^2 t|\alpha_j|^2}\right)\frac{(\mathcal{T}g)(\alpha_j)}{(2\pi)^4 i|\alpha_j|^4}$$

and thus

$$\mathcal{E}\big(\theta^\rho_0\big)(t) = \sum_{j=1}^{N+1}\lambda_j \mathcal{E}\big(\theta^\rho_{\alpha_j}\big)(t) \quad \to \quad E(t).$$

Finally, we choose $\rho' > 0$ such that $\mathcal{E}\big(\theta^{\rho'}_0\big)$ changes sign inside $(t_i-\varepsilon, t_i+\varepsilon)$, for $i = 1, ..., N$, and set $\underline{\theta}_0 := \theta^{\rho'}_0$.

If g is an even function we can show this lemma in the same way by defining

$$\mathcal{T}g := \sum_{k=1}^n \mathcal{F}(\Delta g_k)(\,\cdot\,) + \mathcal{F}(\Delta g_k)(-\cdot) = 2\sum_{k=1}^n \mathcal{F}(\Delta g_k)$$

and

$$\hat{\theta}_\alpha(\xi) := \hat{\phi}(\xi-\alpha)+\hat{\phi}(\xi+\alpha)+\hat{\phi}(\xi-\tilde{\alpha})+\hat{\phi}(\xi+\tilde{\alpha}) \qquad \text{or,}$$

$$\hat{\theta}_\alpha(\xi) := \hat{\phi}(\xi-\alpha)+\hat{\phi}(\xi+\alpha)+\hat{\phi}(\xi-\tilde{\alpha})+\hat{\phi}(\xi+\tilde{\alpha})+\hat{\phi}(\xi-\tilde{\tilde{\alpha}})+\hat{\phi}(\xi+\tilde{\tilde{\alpha}})$$

for $n = 2$ or $n = 3$, respectively. □

Proof of Theorem 2.4. At first we will construct a solution such that $t \mapsto \int_0^t \int_{\mathbb{R}^n} (g_1\theta)(x,s)\,dx\,ds$ changes sign inside $(t_i-\varepsilon, t_i+\varepsilon)$ for $i = 1, ..., N$. Maybe we have to modify the initial data θ_0 constructed in Lemma 6.1 by multiplying it by a sufficiently small constant $\eta_0 > 0$ to ensure that the corresponding solution (u,θ) is defined on $(0, t_N+\varepsilon)$. By our representation of $(u,\theta) \in Y \times \tilde{Y}$ with initial data $(\eta u_0, \eta\theta_0)$, $0 < \eta \le \eta_0$, introduced in (6.2), (6.3), we obtain

$$\begin{aligned}\int_0^t \int_{\mathbb{R}^n} (g\theta)(x,s)\,dx\,ds &= \int_0^t \int_{\mathbb{R}^n} g(x)\sum_{k=1}^\infty \tilde{T}_k(\eta u_0, \eta\theta_0)(x,s)\,dx\,ds \\ &= \sum_{k=1}^\infty \eta^k S_k(u_0,\theta_0)(t)\end{aligned}$$

where $S_k : Y \times \tilde{Y} \to C([0,T])$.

Remembering the notation of Lemma 6.1 we see that

$$(S_1(u_0,\theta_0))_1(t) = \int_0^t \int_{\mathbb{R}^n} \left(g_1 e^{s\Delta}\theta_0\right)(x)\,dx\,ds = \mathcal{E}(\theta_0)(t).$$

Hence for small $\eta > 0$ the series $\sum_{k=1}^\infty \eta^k (S_k(u_0,\theta_0))_1(t)$ behaves like $\eta\mathcal{E}(\theta_0)(t)$. By Lemma 6.1 $\mathcal{E}(\theta_0)$ changes sign in the interval $(t_i-\varepsilon, t_i+\varepsilon)$ for $i = 1, ..., N$. Let $t_i^+, t_i^- \in (t_i-\varepsilon, t_i+\varepsilon)$ for $i = 1, ..., N$ such that $\mathcal{E}(\theta_0)(t_i^+) > 0$

and $\mathcal{E}(\theta_0)(t_i^-) < 0$. At each such instant t_i^+ or t_i^-, $i = 1, ..., N$, we can find a small $0 < \eta_i^+ \leq \eta_0$ or $0 < \eta_i^- \leq \eta_0$, respectively, such that

$$\int_0^{t_i^+} \int_{\mathbb{R}^n} (g_1\theta)(x,s)\,dx\,ds > 0 \qquad \text{and} \qquad \int_0^{t_i^-} \int_{\mathbb{R}^n} (g_1\theta)(x,s)\,dx\,ds < 0.$$

With $\eta := \min_{i=1,...,N}\{\eta_i^+, \eta_i^-\}$ we see that the term $\int_0^t \int_{\mathbb{R}^n} (g_1\theta)(x,s)\,dx\,ds$ changes sign inside $(t_i - \varepsilon, t_i + \varepsilon)$, $i = 1, ..., N$, too. In particular, due to the continuity of $t \mapsto \int_0^t \int_{\mathbb{R}^n} (g_1\theta)(x,s)\,dx\,ds$ this map has a zero $t_i^* \in (t_i - \varepsilon, t_i + \varepsilon)$, $i = 1, ..., N$.

The assumption on the symmetry of the gravity and the initial data, i.e. $g(\tilde{x}) = \tilde{g}(x)$ and $\theta_0(\tilde{x}) = \theta_0(x)$, respectively, are obviously preserved during the evolution in the sense that $\theta(\tilde{x},t) = \theta(x,t)$. Furthermore, we get in the case $n = 3$ that

$$\int_0^t \int_{\mathbb{R}^3} g_1\theta = \int_0^t \int_{\mathbb{R}^3} g_2\theta = \int_0^t \int_{\mathbb{R}^3} g_3\theta.$$

Thus all these terms vanish at $t_i^* \in (t_i - \varepsilon, t_i + \varepsilon)$, $i = 1, ..., N$.

By Theorem 2.3 and the assumption $u_0 \in L^\infty_{n+2}(\mathbb{R}^n)^n$ we know that

$$u(x,t) = \frac{\gamma_n}{n}\nabla\left[\frac{x}{|x|^n}\cdot\int_0^t\int_{\mathbb{R}^n}(g\theta)(y,s)\,dy\,ds\right] + \mathcal{O}_t(|x|^{-n-1}), \tag{6.7}$$

and $\theta \in C\left((0,T); L^\infty_\nu(\mathbb{R}^n)\right)$ for all $0 < t < T$ and all $\nu > 0$. Consider the gradient on the right-hand side of (6.7). The map

$$x \mapsto \nabla\left[\frac{x}{|x|^n}\cdot\int_0^t\int_{\mathbb{R}^n}(g\theta)(y,s)\,dy\,ds\right] \tag{6.8}$$

is identically zero if and only if the term $\int_0^t \int_{\mathbb{R}^n} g_1\theta$ and with it the terms $\int_0^t \int_{\mathbb{R}^n} g_2\theta$ and conditionally $\int_0^t \int_{\mathbb{R}^n} g_3\theta$ vanish, like this is the case at the instants t_i^*, $i = 1, ..., N$. Hence for some constant $C' > 0$ we obtain the upper bound

$$|u(x,t_i^*)| \leq C'|x|^{-n-1}$$

for all x sufficiently large and $i = 1, ..., N$.

Otherwise, if the map (6.8) is not identically zero, it is homogeneous of degree $-n$. Thus we can reduce our consideration to the sphere $\mathbb{S}^{n-1}$. Unless

$$\frac{\partial}{\partial x_j}\left[\frac{x}{|x|^n}\cdot\int_0^t\int_{\mathbb{R}^n}(g\theta)(y,s)\,dy\,ds\right] \tag{6.9}$$

has a zero at some point of $\mathbb{S}^{n-1}$, we find $t_i' \in (t_i - \varepsilon, t_i + \varepsilon)$, $i = 1, ..., N$, and a constant $c_\omega^{(j)}$, $\omega := \frac{x}{|x|}$, such that

$$|u_j(x,t)| \geq c_\omega^{(j)}|x|^{-n}$$

for all x large enough and $i = 1, ..., N$, $j = 1, ..., n$. But since the zeros of the map (6.9) are the zeros on the unit sphere of a homogeneous polynomial of degree two, $c_\omega > 0$ for almost every $\omega \in \mathbb{S}^{n-1}$.

Finally, due to Theorem 2.3 we know that the term of order $|x|^{-n-1}$ in (6.7) equals

$$Q(x) := \nabla \left[\gamma_n \sum_{h,k=1}^{n} \left(\frac{x_h x_k}{|x|^{n+2}} - \frac{\delta_{h,k}}{n|x|^n} \right) \cdot \int_0^t \int_{\mathbb{R}^n} (u_h u_k + y_k g_h \theta) \, dy \, ds \right].$$

Let us define the matrix $\mathcal{K} = (\mathcal{K}_{h,k})_{h,k=1}^n$ by

$$\mathcal{K}_{h,k}(t) := \int_0^t \int_{\mathbb{R}^n} (u_h u_k + y_k g_h \theta) \, dy \, ds.$$

In the case of a symmetric matrix $\mathcal{K}$ Brandolese and Vigneron [7, Prop. 2.9] showed that $\mathcal{K}_{h,k} = \alpha \delta_{h,k}$ for any $\alpha \in \mathbb{R}$ if and only if $Q(x) \equiv 0$. Apparently, in our case the matrix $\mathcal{K}$ is not symmetric in general. But we can prove in the same manner as in [7] that

$$Q \equiv 0 \quad \text{if and only if} \quad \mathcal{K}_{h,k} = -\mathcal{K}_{k,h} \ \text{ and } \ \mathcal{K}_{h,h} = \mathcal{K}_{k,k} \ \text{ for all } \ h \neq k.$$

Due to our symmetry assumptions on the initial velocity u_0, see (2.2), the k-th component of the initial data u_0 is odd in the k-th variable and even in the j-th variable, $j, k = 1, ..., n$ and $j \neq k$. Due to the invariance of the Boussinesq equations under the transformations of this symmetry group, these symmetries are preserved during the evolution and are thus satisfied at each moment $t \in [0, T]$ by the solution $u(t)$. Under these symmetry assumptions we finally get

$$\int_0^t \int_{\mathbb{R}^n} (u_h u_k)(y, s) \, dy \, ds = 0,$$
$$\int_0^t \int_{\mathbb{R}^n} u_i^2(y, s) \, dy \, ds = \int_0^t \int_{\mathbb{R}^n} u_1^2(y, s) \, dy \, ds$$

for all $i, k, h = 1, ..., n$ and $h \neq k$. Furthermore, since due to our construction of the initial temperature θ_0 in Lemma 6.1 $g\theta_0$ is an even function we obtain

$$\int_{\mathbb{R}^n} (y_1 g_2 \theta_0 + y_2 g_1 \theta_0)(y) \, dy = 0.$$

Also this property preserves during the evolution such that the $|x|^{-n-1}$-term of the asymptotic profile of u vanishes at all moments $t \in [0, T]$. Hence for some constant $C > 0$ we obtain the upper bound

$$|u(x, t_i^*)| \leq C|x|^{-n-2+\varepsilon}$$

for all x with sufficiently large norm and $i = 1, ..., N$.

Now Theorem 2.4 is completely proved. □

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