

Global existence and uniqueness result for the diffusive Peterlin viscoelastic model

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Abstract

The aim of this paper is to present the existence and uniqueness result for the diffusive Peterlin viscoelastic model describing the unsteady behaviour of some incompressible polymeric fluids. The polymers are treated as two beads connected by a nonlinear spring. The Peterlin approximation of the spring force is used to derive the equation for the conformation tensor. The latter is the time evolution equation with spatial diffusion of the conformation tensor. Using the energy estimates we prove global in time existence of a weak solution in two space dimensions. We are also able to show the regularity and consequently the uniqueness of the weak solution.

1 Introduction

The long chain molecules, typical for polymeric liquids, are modelled as chains of beads and springs or beads and rods. The spring forces, stochastic forces and forces exerted by the surrounding fluid are responsible for the movement of molecules. There are basically three different approaches how to model the environment with which a polymer molecule interacts. Firstly, the dilute solution theories, which consider the polymer molecule to be surrounded by a Newtonian fluid. In this case, the hydrodynamic drag forces are resulting in the interaction between the molecules and the flow. On the contrary, there are the network theories motivated by theories of rubber elasticity. Molecules are linked together at junction points in a network. The interaction between the polymer molecule and the flow results from the motion of the junctions. Finally, the middle ground of the above two theories are the reptation theories, which visualize the molecule as slithering inside a tube formed by the other polymer molecules. For details see e.g. [33], [34]. The history of molecular modelling can be found in [9] and the references therein.

The simplest model representing the dilute solution theories is the so called dumbbell model consisting of two beads connected by a spring. Considering the linear force law for the spring force: $\mathbf{F}(\mathbf{R}) = H\mathbf{R}$, where \mathbf{R} is the vector connecting the beads, we obtain the upper convected Maxwell model, cf. [33]. The well-known *Oldroyd-B* model has the stress that is a linear superposition of the upper convected Maxwell model and the Newtonian model.

For the nonlinear force, $\mathbf{F}(\mathbf{R}) = \gamma(|\mathbf{R}|^2)\mathbf{R}$ it is not possible to obtain a closed system of equations for the conformation tensor, except by approximating the force law. The Peterlin approximation replaces this law by $\mathbf{F}(\mathbf{R}) = \gamma(\langle |\mathbf{R}|^2 \rangle)\mathbf{R}$. That means, the length of the spring in the spring function γ is replaced by the length of the average spring $\langle |\mathbf{R}|^2 \rangle = \text{tr } \mathbf{C}$. Consequently, we can derive the evolution equation for the conformation tensor \mathbf{C} , which is in a closed form, see [33].

Mathematical analysis of complex viscoelastic fluids is an active research area. In the literature we can find already various mathematical results dealing with the questions of well-posedness of the viscoelastic flows and in particular with the Oldroyd-B model. Concerning the local in time existence results and global in time results for small data let us mention the classical results of Fernández-Cara, Guillén and Ortega [17] and of Guillopé and Saut [18]. Theoretical results for stationary generalized Oldroyd-B, power-law flows were published by Arada and Sequeira [1], see also [19] for further related results on the existence of strong solutions in exterior domains obtained by Hieber, Naito and Shibata.

Recently, the global existence result for fully two- and three-dimensional flow has been obtained by Lions and Masmoudi [25] for the case of the so-called corotational Oldroyd-B model, where the gradient of velocity $\nabla \mathbf{v}$ in the evolution equation for the elastic stress tensor is replaced by its anti-symmetric part $\frac{1}{2}(\nabla \mathbf{v} - \nabla \mathbf{v}^T)$. The goal is to obtain strong convergence of elastic stress tensor. To this end, the authors introduce a new quantity that measures losses of compactness in nonlinear terms and apply DiPerna, Lions theory of renormalized solutions. Once the strong convergence for elastic stress tensor is obtained one can clearly pass to limit in all nonlinear terms involving and deal with other terms as in the Navier-Stokes theory. Unfortunately, the proof cannot be extended easily to other Oldroyd-type fluids since a specific structure of corotational model has been used here. In the viscoelastic models the transport equation for the elastic stress tensor plays an important role. Bahouri and Chemin [2] proved a losing a priori estimate for the transport equation. Based on this theory Chemin and Masmoudi [27] showed the blowup criterion in two-dimensional situation. Recently this result was improved by Lei, Masmoudi and Zhou [23]. Global existence of weak solutions for small data can be found, e.g., in [15]. Local existence of solutions and global existence of small solutions of some rate type fluids have been shown by Lin, Liu and Zhang in [24]. In the recent work [3] Barrett and Boyaval studied the so-called diffusive Oldroyd-B model both from numerical as well as analytical point of view. For two space dimensions they were able to prove the global existence of weak solutions.

On the other hand, as already pointed up above, complex viscoelastic fluids can be also modelled using the molecular description of the complex fluids, which yields the so-called micro-macro models. Here we couple the macroscopic equation for the conservation of mass and momentum (time evolution of fluid velocity and divergence freedom of the velocity for incompressible fluid) with the Fokker-Planck equation arising from the kinetic approach. The Fokker-Planck equation is a nonlinear equation describing time evolution for the particle distribution. The (macroscopic) elastic stress tensor, appearing on the right hand side of the momentum equation, is then obtained by an averaging process by means of the particle distribution function, cf. the Kramers expression. Indeed, the Oldroyd-B model can be obtained as an exact closure of the linear Fokker-Planck equation, see, e.g., [13]. Mathematical literature dealing with the analysis of such micro-macro

viscoelastic models is growing quite rapidly, see, e.g., [6, 7, 10–12, 14–16, 20, 21, 24, 28–30, 32] and the references therein. For example, in [28–30] Masmoudi and collaborators combine the macroscopic fluid model with the so-called FENE (finitely extensible nonlinear elastic) model, which assumes that the interaction potential can be infinite at finite extension length. In [29] the global existence of weak solution for FENE dumbbell polymeric flows is proved. The proof is based on the control of the propagation of strong convergence of some well chosen quantity by studying a transport equation for its defect measure. Furthermore, in [30] the existence of global smooth solutions for a coupled micro-macro model for polymeric fluid in two space dimensions under the co-rotational assumption is obtained.

For the dilute polymers using the kinetic model having a diffusive term the global existence of weak solutions has been proved by Barrett and Süli in [4]. In this paper the authors work with the FENE model in order to represent viscoelastic effects. Thus, the spring force $\mathbf{F}(\mathbf{R})$ is no more linear but given by such a nonlinear potential, see [4] for more details. In [5] analogous existence result for the Hookean-type kinetic model with a diffusive term has been presented. The diffusive Oldroyd-B model has been also studied by Constantin and Kliegl in [13] and the global regularity in two space dimensions has been proven.

Let us point out that in standard derivations of bead-spring models the diffusive term in the equation for the elastic stress tensor is routinely omitted. As pointed out in [4, 16, 37] in the case of heterogeneous fluid velocity this diffusive term indeed appears in the Fokker-Planck equation and, consequently, also in the corresponding macroscopic equation for the elastic stress.

The main aim of the present paper is to analyse a model for complex viscoelastic fluids, where the Peterlin approximation is used in order to derive the evolution equation for the elastic conformation tensor. Following Barrett and Süli [4] we would like to emphasize that the diffusive term appearing in the evolution equation for the conformation tensor is *not a regularizing term* but rather an outcome of physical modelling. The paper is organized in the following way. In the next section we present a mathematical model for our complex viscoelastic fluid and formal energy estimates. Further, in the Section 3 we show the global existence of the weak solutions in two space dimensions by studying the Galerkin approximation, a priori estimates, compactness results and passage to the limit. Unfortunately the functional spaces obtained for the conformation tensor do not allow to obtain the uniqueness of the weak solution. Therefore we show that this model indeed enjoys higher regularity, provided data are more regular. Consequently, we are able to show the uniqueness of global more regular weak solution.

2 Model

In this section we will firstly describe the mathematical model and show formal energy estimates. Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain and let $T > 0$. We consider the system of equations on $\Omega \times (0, T)$ describing the unsteady motion of an incompressible viscoelastic fluid

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v} \cdot \nabla) \mathbf{v} = \nu \Delta \mathbf{v} + \operatorname{div} \mathbf{T} - \nabla p \quad (1a)$$

$$\operatorname{div} \mathbf{v} = 0. \quad (1b)$$

Here $\mathbf{v}(\mathbf{x}, t) \in \mathbb{R}^2$ denotes the velocity of fluid, $p(\mathbf{x}, t) \in \mathbb{R}$ is the pressure for all $(\mathbf{x}, t) \in \Omega \times (0, T)$. The elastic stress tensor \mathbf{T} can be expressed by the conformation tensor \mathbf{C} in the following way

$$\mathbf{T} = \operatorname{tr} \mathbf{C} \mathbf{C}, \quad (1c)$$

where $\mathbf{C}(\mathbf{x}, t) \in \mathbb{R}^{2 \times 2}$ is a symmetric positive definite tensor for all $(\mathbf{x}, t) \in \Omega \times (0, T)$, which satisfies the equation of the form

$$\frac{\partial \mathbf{C}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{C} - (\nabla \mathbf{v}) \mathbf{C} - \mathbf{C} (\nabla \mathbf{v})^T = \operatorname{tr} \mathbf{C} \mathbf{I} - (\operatorname{tr} \mathbf{C})^2 \mathbf{C} + \varepsilon \Delta \mathbf{C}. \quad (1d)$$

We impose the homogeneous Dirichlet boundary condition on \mathbf{v} and the no-flux condition on \mathbf{C} on the boundary $\partial\Omega$

$$\left(\mathbf{v}, \frac{\partial \mathbf{C}}{\partial \mathbf{n}} \right) = (\mathbf{0}, \mathbf{0}), \quad (1e)$$

and consider \mathbf{v}_0 and \mathbf{C}_0 to be the enough smooth initial data

$$(\mathbf{v}(0), \mathbf{C}(0)) = (\mathbf{v}_0, \mathbf{C}_0). \quad (1f)$$

ρ , ν and ε are given constants describing the density, fluid viscosity and elastic stress viscosity, respectively.

Remark 1. *The above model is a variant of the so-called Peterlin model, cf. [33]. As pointed out in the introduction we replace a general nonlinear spring force $\mathbf{F}(\mathbf{R}) = \gamma(|\mathbf{R}|^2)\mathbf{R}$ by its suitable approximation $\mathbf{F}(\mathbf{R}) = \gamma(\langle |\mathbf{R}|^2 \rangle)\mathbf{R}$ taking into account the length of an average spring, represented by $\langle |\mathbf{R}|^2 \rangle = \operatorname{tr} \mathbf{C}$. In our model (1) we consider a particular nonlinear Peterlin approximation with γ being a linear function of $\operatorname{tr} \mathbf{C}$ and allow a nonlinear (quadratic) dependence on $\operatorname{tr} \mathbf{C}$ at the right hand side of (1d).*

Let us point out that in the recent papers of Renardy [35, 36] the yield stress behaviour of non-Newtonian fluids has been investigated. In particular it has been shown how complex yield stress behaviour can arise as a limiting case of nonlinear viscoelastic constitutive theories. To this goal a general “dumbbell inspired” model for dilute polymer solutions is proposed and studied in [35, 36]

$$\begin{aligned} \mathbf{T} &= \psi(\operatorname{tr} \mathbf{C}) \mathbf{C} \\ \frac{\partial \mathbf{C}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{C} - (\nabla \mathbf{v}) \mathbf{C} - \mathbf{C} (\nabla \mathbf{v})^T &= \chi(\operatorname{tr} \mathbf{C}) \mathbf{I} - \phi(\operatorname{tr} \mathbf{C}) \mathbf{C}. \end{aligned} \quad (2)$$

Here ψ, χ, ϕ are in principle arbitrary functions of $\operatorname{tr} \mathbf{C}$ yielding constitutive relations for a particular viscoelastic fluid. As pointed out in [36] we can view this model as a modification of the upper convected Maxwell model, in which the relaxation time and viscosity depend on $\operatorname{tr} \mathbf{C}$. For example, for the well-known PEC model we would have that ϕ, χ are linear functions of $\operatorname{tr} \mathbf{C}$ and ψ is a function of $(\operatorname{tr} \mathbf{C})^{-1}$. In the viscoelastic model studied in the present paper we assume that χ, ψ are linear functions of $\operatorname{tr} \mathbf{C}$, and allow nonlinear behaviour through a quadratic function ϕ .

Moreover, we would like to note that the results presented in our paper can be also extended to a slightly more general situation. In principle we can assume that ϕ, χ are \mathcal{C}^1 functions satisfying polynomial growth conditions of the order α and β , respectively. Further, ϕ is a nonnegative function. Using the same mathematical techniques and arguments as we will present below the existence, uniqueness and regularity results can be extended for the cases $0 \leq \alpha \leq 2$, $1 \leq \beta < \alpha + 1$, if $\alpha = 0$ we have $\beta = 1$.

For better readability we have decided to present in this paper theoretical proofs only for quadratic/linear polynomial dependence (i.e. $\alpha = 2, \beta = 1$), which is the most difficult and limiting case. The reader is referred to [31] for theoretical proofs in the cases $0 \leq \alpha < 2$, $1 \leq \beta < \alpha + 1$.

Remark 2. In the constitutive equation (1d) we have moreover taken into account the diffusive effects arising due to a heterogeneous velocity field. As pointed out by Barrett and Süli [5], see also [16, 37], this fact yields x -dissipative terms in the microscopic model, i.e. in the Fokker-Planck equation, and consequently in the corresponding macroscopic model, i.e. in the constitutive equation describing time evolution of the conformation stress tensor. In [16] a careful justification of the presence of the diffusive terms in the Fokker-Planck equations through the asymptotic analysis is presented. The diffusion coefficient ε is proportional to $(\ell/L)^2/We$, where L and ℓ are characteristic macroscopic and microscopic length scales and We is the so-called Weissenberg number. It is a reference number characterizing viscoelastic property of the material. Estimates for $(\ell/L)^2$ presented in [8] show that $(\ell/L)^2$ is in the range of about 10^{-9} and 10^{-7} . As emphasized in [5] the model reduction by neglecting these small diffusive effects is mathematically counterproductive leading to a degenerate parabolic-hyperbolic system (1) with $\varepsilon = 0$. On the other hand when the diffusive terms are taken into account the resulting system (1) remains parabolic. We would like to point out that in the analysis presented below we only require $\varepsilon > 0$ and there is no assumption on the size of ε .

In order to show the formal energy estimates we first need the following result.

Proposition 1.

Let $\mathbf{C} \in \mathbb{R}^{2 \times 2}$ be a symmetric tensor and let $\mathbf{v} \in \mathbb{R}^2$ be a solenoidal vector field. Then the following identity holds true¹

$$\text{tr } \mathbf{C} \mathbf{C} : \nabla \mathbf{v} = \frac{1}{2} [(\nabla \mathbf{v}) \mathbf{C} + \mathbf{C} (\nabla \mathbf{v})^T] : \mathbf{C}. \quad (3)$$

Proof. Let us rewrite the left hand side of (3) using the symmetry of \mathbf{C} in the following way

$$\text{tr } \mathbf{C} \mathbf{C} : \nabla \mathbf{v} = \text{tr } \mathbf{C} \sum_{i,j=1}^2 C_{ij} \frac{\partial v_i}{\partial x_j} = \sum_{i=1}^2 C_{ii}^2 \frac{\partial v_i}{\partial x_i} + \text{tr } \mathbf{C} C_{12} \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) + C_{11} C_{22} \text{div } \mathbf{v}.$$

The right hand side of (3) can be rewritten in an analogous way

$$\frac{1}{2} [(\nabla \mathbf{v}) \mathbf{C} + \mathbf{C} (\nabla \mathbf{v})^T] : \mathbf{C} = \sum_{i,j,k=1}^2 \frac{\partial v_i}{\partial x_k} C_{kj} C_{ij} + C_{ik} \frac{\partial v_j}{\partial x_k} C_{ij} =$$

¹Note that for square matrices of the same size we use the notation $\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^2 A_{ij} B_{ij}$. Moreover, $\mathbf{A} : \mathbf{A} = |\mathbf{A}|^2$.

$$= \sum_{i=1}^2 C_{ii}^2 \frac{\partial v_i}{\partial x_i} + \text{tr } \mathbf{C} C_{12} \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) + C_{12}^2 \text{div } \mathbf{v}.$$

Since $\text{div } \mathbf{v} = 0$, we can conclude that the identity (3) holds true. \square

Remark 3. *Let us point out that the above result is crucial in order to control nonlinear term $(\nabla \mathbf{v})\mathbf{C} + \mathbf{C}(\nabla \mathbf{v})^T$ arising from the objective derivative in the equation for the conformation tensor. Indeed, due to (3) in the energy estimates this term will be cancelled by the $\text{div } \mathbf{T}$ appearing in the momentum equation (1a). Unfortunately, the property (3) does not hold in three space dimensions.*

Now, we proceed with the formal energy estimates for our model. We multiply the momentum equation (1a) by \mathbf{v} and integrate using the Gauss theorem

$$\begin{aligned} & \frac{\rho}{2} \int_{\Omega} |\mathbf{v}|^2 d\mathbf{x} - \frac{\rho}{2} \int_{\Omega} |\mathbf{v}_0|^2 d\mathbf{x} - \underbrace{\frac{\rho}{2} \int_0^t \int_{\Omega} \text{div } \mathbf{v} |\mathbf{v}|^2 d\mathbf{x} dt}_{=0} + \underbrace{\frac{\rho}{2} \int_0^t \int_{\partial\Omega} (\mathbf{v} \cdot \mathbf{n}) |\mathbf{v}|^2 dS dt}_{=0} = \\ & = -\nu \int_0^t \int_{\Omega} |\nabla \mathbf{v}|^2 d\mathbf{x} dt + \nu \underbrace{\int_0^t \int_{\partial\Omega} (\mathbf{n} \cdot \nabla) \mathbf{v} \cdot \mathbf{v} dS dt}_{=0} - \int_0^t \int_{\Omega} \mathbf{T} : \nabla \mathbf{v} d\mathbf{x} dt + \\ & + \underbrace{\int_0^t \int_{\partial\Omega} (\mathbf{n} \cdot \mathbf{T}) \cdot \mathbf{v}^T dS dt}_{=0} + \underbrace{\int_0^t \int_{\Omega} \text{div } \mathbf{v} p d\mathbf{x} dt}_{=0} - \underbrace{\int_0^t \int_{\partial\Omega} (\mathbf{v} \cdot \mathbf{n}) p dS dt}_{=0}. \end{aligned}$$

The divergence freedom of velocity and the boundary conditions yield the following equality

$$\frac{\rho}{2} \int_{\Omega} |\mathbf{v}|^2 d\mathbf{x} - \frac{\rho}{2} \int_{\Omega} |\mathbf{v}_0|^2 d\mathbf{x} = -\nu \int_0^t \int_{\Omega} |\nabla \mathbf{v}|^2 d\mathbf{x} dt - \int_0^t \int_{\Omega} \mathbf{T} : \nabla \mathbf{v} d\mathbf{x} dt. \quad (4)$$

Now, we multiply (1d) by $\frac{1}{2}\mathbf{C}$ and integrate this equation using the Gauss theorem. Thus we obtain

$$\begin{aligned} & \frac{1}{4} \int_{\Omega} |\mathbf{C}|^2 d\mathbf{x} - \frac{1}{4} \int_{\Omega} |\mathbf{C}_0|^2 d\mathbf{x} - \underbrace{\frac{1}{2} \int_0^t \int_{\Omega} \text{div } \mathbf{v} \mathbf{C} : \mathbf{C} d\mathbf{x} dt}_{=0} + \underbrace{\frac{1}{2} \int_0^t \int_{\partial\Omega} (\mathbf{v} \cdot \mathbf{n}) \mathbf{C} : \mathbf{C} dS dt}_{=0} - \\ & - \frac{1}{2} \int_0^t \int_{\Omega} [(\nabla \mathbf{v})\mathbf{C} + \mathbf{C}(\nabla \mathbf{v})^T] : \mathbf{C} d\mathbf{x} dt = \frac{1}{2} \int_0^t \int_{\Omega} (\text{tr } \mathbf{C})^2 d\mathbf{x} dt - \\ & - \frac{1}{2} \int_0^t \int_{\Omega} (\text{tr } \mathbf{C})^2 \mathbf{C} : \mathbf{C} d\mathbf{x} dt - \frac{\varepsilon}{2} \int_0^t \int_{\Omega} |\nabla \mathbf{C}|^2 d\mathbf{x} dt + \underbrace{\frac{\varepsilon}{2} \int_0^t \int_{\partial\Omega} \mathbf{C}(\nabla \mathbf{C} \cdot \mathbf{n}) dS dt}_{=0}. \end{aligned}$$

Again, by the divergence freedom of velocity and the boundary conditions, we have

$$\begin{aligned} & \frac{1}{4} \int_{\Omega} |\mathbf{C}|^2 d\mathbf{x} + \frac{\varepsilon}{2} \int_0^t \int_{\Omega} |\nabla \mathbf{C}|^2 d\mathbf{x} dt + \frac{1}{2} \int_0^t \int_{\Omega} (\text{tr } \mathbf{C})^2 \mathbf{C} : \mathbf{C} d\mathbf{x} dt - \\ & - \frac{1}{2} \int_0^t \int_{\Omega} [(\nabla \mathbf{v})\mathbf{C} + \mathbf{C}(\nabla \mathbf{v})^T] : \mathbf{C} d\mathbf{x} dt = \frac{1}{4} \int_{\Omega} |\mathbf{C}_0|^2 d\mathbf{x} + \frac{1}{2} \int_0^t \int_{\Omega} (\text{tr } \mathbf{C})^2 d\mathbf{x} dt. \end{aligned} \quad (5)$$

Thus by adding the equations (4) and (5) together, and using the identity (3), we get the following energy equality

$$\begin{aligned} & \frac{\rho}{2} \int_{\Omega} |\mathbf{v}|^2 \, d\mathbf{x} + \frac{1}{4} \int_{\Omega} |\mathbf{C}|^2 \, d\mathbf{x} + \nu \int_0^t \int_{\Omega} |\nabla \mathbf{v}|^2 \, d\mathbf{x} \, dt + \frac{\varepsilon}{2} \int_0^t \int_{\Omega} |\nabla \mathbf{C}|^2 \, d\mathbf{x} \, dt + \\ & + \frac{1}{2} \int_0^t \int_{\Omega} (\operatorname{tr} \mathbf{C})^2 \mathbf{C} : \mathbf{C} \, d\mathbf{x} \, dt = \frac{\rho}{2} \int_{\Omega} |\mathbf{v}_0|^2 \, d\mathbf{x} + \frac{1}{4} \int_{\Omega} |\mathbf{C}_0|^2 \, d\mathbf{x} + \frac{1}{2} \int_0^t \int_{\Omega} |\operatorname{tr} \mathbf{C}|^2 \, d\mathbf{x} \, dt. \end{aligned} \quad (6)$$

Since $(\operatorname{tr} \mathbf{C})^2 \leq 2\mathbf{C} : \mathbf{C}$, equation (6) indicates, by the Gronwall inequality, that the following functional spaces are appropriate for \mathbf{v} and \mathbf{C}

$$\mathbf{v} \in \mathbf{L}^{\infty}(0, T; \mathbf{L}^2(\Omega)) \cap \mathbf{L}^2(0, T; \mathbf{H}^1(\Omega)), \quad \mathbf{C} \in \mathbf{L}^{\infty}(0, T; \mathbf{L}^2(\Omega)) \cap \mathbf{L}^2(0, T; \mathbf{H}^1(\Omega)).$$

Moreover, $\mathbf{T} \in \mathbf{L}^2((0, T) \times \Omega)$. In what follows we will need the following interpolation inequalities. For proof see e.g. [22].

Proposition 2. (*interpolation inequalities*)

Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain. Then the following inequalities hold true

$$\|\mathbf{v}\|_{\mathbf{L}^4(\Omega)} \leq c \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}^{1/2} \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)}^{1/2}, \quad \mathbf{v} \in \mathbf{H}_0^1(\Omega) \quad (7a)$$

$$\|\mathbf{C}\|_{\mathbf{L}^4(\Omega)} \leq c(\Omega) \left(\|\mathbf{C}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{C}\|_{\mathbf{L}^2(\Omega)}^{1/2} \|\nabla \mathbf{C}\|_{\mathbf{L}^2(\Omega)}^{1/2} \right), \quad \mathbf{C} \in \mathbf{H}^1(\Omega). \quad (7b)$$

Let us point out that the energy equality (6) together with (7b) yields $\mathbf{C} \in \mathbf{L}^4((0, T) \times \Omega)$.

3 Existence of weak solution

The goal of this section is to show the existence of a weak solution. In the next part, we will use the following notation:

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \mid \operatorname{div} \mathbf{v} = 0\}, \quad \text{equipped with the norm } \|\mathbf{v}\| := \|\nabla \mathbf{v}\|_{\mathbf{L}^2(\Omega)}$$

$$\mathbf{H} = \{\mathbf{v} \in \mathbf{L}^2(\Omega) \mid \operatorname{div} \mathbf{v} = 0\}$$

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} \, d\mathbf{x} \quad \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V}$$

$$((\mathbf{v}, \mathbf{w})) = \int_{\Omega} \nabla \mathbf{v} : \nabla \mathbf{w} \, d\mathbf{x} \quad \mathbf{v}, \mathbf{w} \in \mathbf{V}$$

$$B(\mathbf{v}, \mathbf{C}, \mathbf{D}) = \int_{\Omega} (\mathbf{v} \cdot \nabla) \mathbf{C} : \mathbf{D} \, d\mathbf{x} \quad \mathbf{v} \in \mathbf{V}, \mathbf{C}, \mathbf{D} \in \mathbf{H}^1(\Omega)$$

$$((\mathbf{C}, \mathbf{D})) = \int_{\Omega} \nabla \mathbf{C} : \nabla \mathbf{D} \, d\mathbf{x} \quad \mathbf{C}, \mathbf{D} \in \mathbf{H}^1(\Omega).$$

Analogously as in [38] one can easily show that

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}) \quad \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{V} \quad (8)$$

$$B(\mathbf{v}, \mathbf{C}, \mathbf{D}) = -B(\mathbf{v}, \mathbf{D}, \mathbf{C}) \quad \mathbf{v} \in \mathbf{V}, \mathbf{C}, \mathbf{D} \in \mathbf{H}^1(\Omega). \quad (9)$$

Definition 1. (*weak solution*)

Let $(\mathbf{v}_0, \mathbf{C}_0) \in \mathbf{H} \times \mathbf{L}^2(\Omega)$. The couple

$$(\mathbf{v}, \mathbf{C}) \in [\mathbf{L}^\infty(0, T; \mathbf{H}) \cap \mathbf{L}^2(0, T; \mathbf{V})] \times [\mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega)) \cap \mathbf{L}^2(0, T; \mathbf{H}^1(\Omega))] \quad (10a)$$

is called a weak solution to (1) if

$$(\mathbf{v}(0), \mathbf{C}(0)) = (\mathbf{v}_0, \mathbf{C}_0) \quad (10b)$$

and if it satisfies

$$\rho \int_{\Omega} \frac{\partial \mathbf{v}}{\partial t} \cdot \mathbf{w} + (\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \mathbf{w} \, d\mathbf{x} + \nu \int_{\Omega} \nabla \mathbf{v} : \nabla \mathbf{w} \, d\mathbf{x} = - \int_{\Omega} \text{tr} \, \mathbf{C} \mathbf{C} : \nabla \mathbf{w} \, d\mathbf{x} \quad (10c)$$

$$\begin{aligned} & \int_{\Omega} \frac{\partial \mathbf{C}}{\partial t} : \mathbf{D} \, d\mathbf{x} + \int_{\Omega} (\mathbf{v} \cdot \nabla) \mathbf{C} : \mathbf{D} \, d\mathbf{x} + \varepsilon \int_{\Omega} \nabla \mathbf{C} : \nabla \mathbf{D} \, d\mathbf{x} - \\ & - \int_{\Omega} [(\nabla \mathbf{v}) \mathbf{C} + \mathbf{C} (\nabla \mathbf{v})^T] : \mathbf{D} \, d\mathbf{x} = \int_{\Omega} [\text{tr} \, \mathbf{C} \mathbf{I} - (\text{tr} \, \mathbf{C})^2 \mathbf{C}] : \mathbf{D} \, d\mathbf{x} \end{aligned} \quad (10d)$$

$$\forall \mathbf{w} \in \mathbf{V}, \forall \mathbf{D} \in \mathbf{H}^1(\Omega), \text{ a.e. } t \in (0, T).$$

Theorem 1. (*existence of weak solution*)

There exists a weak solution to the problem (1) such that (10) is satisfied.

Proof. The proof of the existence of a weak solution to (1) will be based on the Galerkin approximation.

Galerkin approximation

Let us take the orthonormal countable bases of the spaces \mathbf{V} and $\mathbf{H}^1(\Omega)$

$$\mathbf{V} = \overline{\text{span}\{\mathbf{w}_i\}_{i=1}^{\infty}} \quad \text{and} \quad \mathbf{H}^1(\Omega) = \overline{\text{span}\{\mathbf{D}_i\}_{i=1}^{\infty}}.$$

The m -th approximate Galerkin solution can be expressed as follows

$$\begin{aligned} & \left(\mathbf{v}_m(t) = \sum_{i=1}^m g_{im}(t) \mathbf{w}_i, \mathbf{C}_m(t) = \sum_{i=1}^m G_{im}(t) \mathbf{D}_i \right) \\ & \rho (\mathbf{v}'_m(t), \mathbf{w}_j) + \rho b(\mathbf{v}_m(t), \mathbf{v}_m(t), \mathbf{w}_j) + \nu ((\mathbf{v}_m(t), \mathbf{w}_j)) = -(\text{tr} \, \mathbf{C}_m(t) \mathbf{C}_m(t), \nabla \mathbf{w}_j) \end{aligned} \quad (11a)$$

$$\begin{aligned} & (\mathbf{C}'_m(t), \mathbf{D}_j) + B(\mathbf{v}_m(t), \mathbf{C}_m(t), \mathbf{D}_j) + \varepsilon ((\mathbf{C}_m(t), \mathbf{D}_j)) = \\ & = ((\nabla \mathbf{v}_m(t)) \mathbf{C}_m(t) + \mathbf{C}_m(t) (\nabla \mathbf{v}_m(t))^T, \mathbf{D}_j) + (\text{tr} \, \mathbf{C}_m(t) \mathbf{I} - (\text{tr} \, \mathbf{C}_m(t))^2 \mathbf{C}_m(t), \mathbf{D}_j) \end{aligned} \quad (11b)$$

$$(\mathbf{v}_m(0), \mathbf{C}_m(0)) = (\mathbf{v}_{0m}, \mathbf{C}_{0m})$$

$$\text{for } j = 1, \dots, m, t \in [0, T].$$

Functions \mathbf{v}_{0m} and \mathbf{C}_{0m} are the orthogonal projections in \mathbf{H} of \mathbf{v}_0 and in $\mathbf{L}^2(\Omega)$ of \mathbf{C}_0 on the spaces spanned by \mathbf{w}_j and \mathbf{D}_j . The nonlinear system of differential equations together with the initial conditions (and the following a priori bounds) gives us the solution $(\mathbf{v}_m, \mathbf{C}_m)$ defined in interval $[0, T]$.

A priori estimates

Repeating the energy estimates from Section 2 for the m -th Galerkin approximation we obtain

$$(\mathbf{v}_m, \mathbf{C}_m) \in [\mathbf{L}^\infty(0, T; \mathbf{H}) \cap \mathbf{L}^2(0, T; \mathbf{V})] \times [\mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega)) \cap \mathbf{L}^2(0, T; \mathbf{H}^1(\Omega))].$$

Finally, we have shown that there exists a positive constant $k = k(\Omega, T, 1/\rho, 1/\nu, 1/\varepsilon, \mathbf{v}_0, \mathbf{C}_0)$ such that

$$\|\mathbf{v}_m\|_{\mathbf{L}^2(0, T; \mathbf{V})} + \|\mathbf{v}_m\|_{\mathbf{L}^\infty(0, T; \mathbf{H})} + \|\mathbf{C}_m\|_{\mathbf{L}^2(0, T; \mathbf{H}^1(\Omega))} + \|\mathbf{C}_m\|_{\mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega))} \leq k.$$

In order to get the strong convergences of our approximate Galerkin sequence we first define the following operators

$$\begin{aligned} \mathcal{A} : \mathbf{V} &\rightarrow \mathbf{V}^* & \langle \mathcal{A}\mathbf{v}, \mathbf{w} \rangle &= ((\mathbf{v}, \mathbf{w})) \\ \mathcal{B} : \mathbf{V} &\rightarrow \mathbf{V}^* & \langle \mathcal{B}\mathbf{v}, \mathbf{w} \rangle &= b(\mathbf{v}, \mathbf{v}, \mathbf{w}) \\ \mathcal{E} : \mathbf{L}^2(\Omega) &\rightarrow \mathbf{V}^* & \langle \mathcal{E}\mathbf{T}, \mathbf{w} \rangle &= (\mathbf{T}, \nabla \mathbf{w}). \end{aligned}$$

Then (10c) can be rewritten in the following operator form

$$\mathbf{v}' = -\mathcal{E}\mathbf{T} - \rho\mathcal{B}\mathbf{v} - \nu\mathcal{A}\mathbf{v}, \quad \mathbf{v} \in \mathbf{V}, \mathbf{T} \in \mathbf{L}^2(\Omega).$$

First, we have the standard estimate of the operator $\mathcal{A}\mathbf{v}$

$$\int_0^T \|\mathcal{A}\mathbf{v}\|_{\mathbf{V}^*}^2 dt \leq \int_0^T \|\mathbf{v}\|^2 dt,$$

and by (7a), of the operator $\mathcal{B}\mathbf{v}$

$$\int_0^T \|\mathcal{B}\mathbf{v}\|_{\mathbf{V}^*}^2 dt \leq c \int_0^T \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}^2 \|\mathbf{v}\|^2 dt \leq c \|\mathbf{v}\|_{\mathbf{L}^\infty(0, T; \mathbf{H})}^2 \int_0^T \|\mathbf{v}\|^2 dt.$$

Moreover, we have also

$$\int_0^T \|\mathcal{E}\mathbf{T}\|_{\mathbf{V}^*}^2 dt \leq \int_0^T \|\mathbf{T}\|_{\mathbf{L}^2(\Omega)}^2 dt.$$

Consequently, we obtain $\mathbf{v}' \in \mathbf{L}^2(0, T; \mathbf{V}^*)$. Since $\mathbf{V} \hookrightarrow \mathbf{L}_{div}^4(\Omega) \hookrightarrow \mathbf{H} \hookrightarrow \mathbf{V}^*$, we can apply the Lions - Aubin lemma and thus we have a compact embedding of $\{\mathbf{v}_m\}_{m=1}^\infty$ into the space $\mathbf{L}^2(0, T; \mathbf{L}_{div}^4(\Omega))$.

We have a similar result for the conformation tensor \mathbf{C} . Let us define the operators

$$\begin{aligned} \tilde{\mathcal{A}} : \mathbf{H}^1(\Omega) &\rightarrow \mathbf{H}^{-1}(\Omega) & \langle \tilde{\mathcal{A}}\mathbf{C}, \mathbf{D} \rangle &= ((\mathbf{C}, \mathbf{D})) \\ \tilde{\mathcal{B}} : \mathbf{V} \times \mathbf{H}^1(\Omega) &\rightarrow \mathbf{H}^{-1}(\Omega) & \langle \tilde{\mathcal{B}}(\mathbf{v}, \mathbf{C}), \mathbf{D} \rangle &= B(\mathbf{v}, \mathbf{C}, \mathbf{D}) \\ \mathcal{O} : \mathbf{V} \times \mathbf{H}^1(\Omega) &\rightarrow \mathbf{H}^{-1}(\Omega) & \langle \mathcal{O}(\mathbf{v}, \mathbf{C}), \mathbf{D} \rangle &= ((\nabla \mathbf{v})\mathbf{C} + \mathbf{C}(\nabla \mathbf{v})^T, \mathbf{D}) \\ \mathcal{T} : \mathbf{H}^1(\Omega) &\rightarrow \mathbf{H}^{-1}(\Omega) & \langle \mathcal{T}\mathbf{C}, \mathbf{D} \rangle &= (\text{tr } \mathbf{C} \mathbf{I} - (\text{tr } \mathbf{C})^2 \mathbf{C}, \mathbf{D}). \end{aligned}$$

Then (10d) can be rewritten in the following operator form

$$\mathbf{C}' = \mathcal{O}(\mathbf{v}, \mathbf{C}) - \mathcal{T}\mathbf{C} - \tilde{\mathcal{B}}(\mathbf{v}, \mathbf{C}) - \varepsilon\tilde{\mathcal{A}}\mathbf{v}, \quad \mathbf{v} \in \mathbf{V}, \mathbf{C} \in \mathbf{H}^1(\Omega).$$

We have the analogous estimates of the operators $\tilde{\mathcal{A}}\mathbf{C}$, $\mathcal{O}(\mathbf{v}, \mathbf{C})$ and $\mathcal{T}\mathbf{C}$ defined above

$$\begin{aligned} \int_0^T \|\tilde{\mathcal{A}}\mathbf{v}\|_{\mathbf{H}^{-1}(\Omega)}^2 dt &\leq c \int_0^T \|\nabla \mathbf{C}(t)\|_{\mathbf{L}^2(\Omega)}^2 dt \\ \int_0^T \|\mathcal{O}(\mathbf{v}, \mathbf{C})\|_{\mathbf{H}^{-1}(\Omega)}^{4/3} dt &\leq c \int_0^T \|\mathbf{v}(t)\|^{4/3} \|\mathbf{C}(t)\|_{\mathbf{L}^4(\Omega)}^{4/3} dt \leq c \|\mathbf{v}\|_{\mathbf{L}^2(0,T;\mathbf{V})}^{4/3} \|\mathbf{C}\|_{\mathbf{L}^4((0,T)\times\Omega)}^{4/3} \\ \int_0^T \|\mathcal{T}\mathbf{C}\|_{\mathbf{H}^{-1}(\Omega)}^{4/3} dt &\leq c \int_0^T \|\operatorname{tr} \mathbf{C}(t)\|_{\mathbf{L}^2(\Omega)}^{4/3} + \|\operatorname{tr} \mathbf{C}(t)\|_{\mathbf{L}^4(\Omega)}^{8/3} \|\mathbf{C}(t)\|_{\mathbf{L}^4(\Omega)}^{4/3} dt \leq \\ &\leq c \|\operatorname{tr} \mathbf{C}\|_{\mathbf{L}^2((0,T)\times\Omega)}^{4/3} + c \|\operatorname{tr} \mathbf{C}\|_{\mathbf{L}^4((0,T)\times\Omega)}^{8/3} \|\mathbf{C}\|_{\mathbf{L}^4((0,T)\times\Omega)}^{4/3}. \end{aligned}$$

By using (7a) and (9) we obtain the following estimates of the operator $\tilde{\mathcal{B}}(\mathbf{v}, \mathbf{C})$

$$\begin{aligned} \int_0^T \|\tilde{\mathcal{B}}(\mathbf{v}, \mathbf{C})\|_{\mathbf{H}^{-1}(\Omega)}^2 dt &\leq c \int_0^T \|\mathbf{v}(t)\| \|\mathbf{v}(t)\|_{\mathbf{L}^2(\Omega)} \|\mathbf{C}(t)\|_{\mathbf{L}^4(\Omega)}^2 dt \leq \\ &\leq c \|\mathbf{v}\|_{\mathbf{L}^\infty(0,T;\mathbf{H})} \int_0^T \|\mathbf{v}\| \|\mathbf{C}\|_{\mathbf{L}^4(\Omega)}^2 dt. \end{aligned}$$

Since $\tilde{\mathcal{A}}\mathbf{C}, \tilde{\mathcal{B}}(\mathbf{v}, \mathbf{C}) \in \mathbf{L}^2(0, T; \mathbf{H}^{-1}(\Omega))$ and $\mathcal{O}(\mathbf{v}, \mathbf{C}), \mathcal{T}\mathbf{C} \in \mathbf{L}^{4/3}(0, T; \mathbf{H}^{-1}(\Omega))$ we get $\mathbf{C}' \in \mathbf{L}^{4/3}(0, T; \mathbf{H}^{-1}(\Omega))$. The following embeddings $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^4(\Omega) \hookrightarrow \mathbf{H}^{-1}(\Omega)$ yield due to the Lions - Aubin lemma the compact embedding of $\{\mathbf{C}_m\}_{m=1}^\infty$ into the space $\mathbf{L}^2(0, T; \mathbf{L}^4(\Omega))$. Consequently, there exists a subsequence, denoted again by $(\mathbf{v}_m, \mathbf{C}_m)$, of the Galerkin approximate sequence such that

$$\begin{aligned} \mathbf{v}_m &\rightharpoonup^* \mathbf{v} \text{ in } \mathbf{L}^\infty(0, T; \mathbf{H}) \\ \mathbf{v}_m &\rightharpoonup \mathbf{v} \text{ in } \mathbf{L}^2(0, T; \mathbf{V}) \\ \mathbf{v}_m &\rightarrow \mathbf{v} \text{ in } \mathbf{L}^2(0, T; \mathbf{L}^4(\Omega)) \\ \mathbf{C}_m &\rightharpoonup^* \mathbf{C} \text{ in } \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega)) \\ \mathbf{C}_m &\rightharpoonup \mathbf{C} \text{ in } \mathbf{L}^2(0, T; \mathbf{H}^1(\Omega)) \\ \mathbf{C}_m &\rightarrow \mathbf{C} \text{ in } \mathbf{L}^2(0, T; \mathbf{L}^4(\Omega)). \end{aligned}$$

Passage to the limit

Now we are able to pass to the limit in (11) for $m \rightarrow \infty$. Let us take $\varphi \in \mathcal{C}^1([0, T])$, $\varphi(T) = 0$. Multiply (11a) and (11b) by $\varphi(t)$ and integrate per partes over $[0, T]$. In what follows we only concentrate on the limiting process in some nonlinear terms. The limiting process in other terms can be easily done. In the velocity equation (11a) we need to control the elastic term

$$\begin{aligned} &\left| \int_0^T \int_\Omega (\operatorname{tr} \mathbf{C}(t) \mathbf{C}(t) - \operatorname{tr} \mathbf{C}_m(t) \mathbf{C}_m(t)) : \nabla(\varphi(t) \mathbf{w}_j) \, d\mathbf{x} \, dt \right| \leq \\ &\leq \max_{t \in [0, T]} |\varphi(t)| \|\nabla \mathbf{w}_j\|_{\mathbf{L}^2(\Omega)} \left(\int_0^T \left(\int_\Omega |\operatorname{tr} \mathbf{C}(t) - \operatorname{tr} \mathbf{C}_m(t)|^4 \, d\mathbf{x} \right)^{1/4} \left(\int_\Omega |\mathbf{C}_m(t)|^4 \, d\mathbf{x} \right)^{1/4} dt + \right. \\ &\quad \left. + \int_0^T \left(\int_\Omega |\operatorname{tr} \mathbf{C}(t)|^4 \, d\mathbf{x} \right)^{1/4} \left(\int_\Omega |\mathbf{C}(t) - \mathbf{C}_m(t)|^4 \, d\mathbf{x} \right)^{1/4} dt \right) \leq \end{aligned}$$

$$\leq c \|\nabla \mathbf{w}_j\|_{\mathbf{L}^2(\Omega)} \left(\|\operatorname{tr} \mathbf{C} - \operatorname{tr} \mathbf{C}_m\|_{L^2(0,T;L^4(\Omega))} \|\mathbf{C}_m\|_{L^2(0,T;L^4(\Omega))} + \|\operatorname{tr} \mathbf{C}\|_{L^2(0,T;L^4(\Omega))} \|\mathbf{C} - \mathbf{C}_m\|_{L^2(0,T;L^4(\Omega))} \right),$$

which goes to zero letting m to ∞ . The nonlinear term in the equation for the conformation tensor (11b) can be estimated in the following way

$$\begin{aligned} & \left| \int_0^T ((\operatorname{tr} \mathbf{C}_m(t))^2 \mathbf{C}_m(t) - (\operatorname{tr} \mathbf{C}(t))^2 \mathbf{C}(t), \mathbf{D}_j \varphi(t)) dt \right| \leq \\ & \leq \max_{t \in [0,T]} |\varphi(t)| \|\mathbf{D}_j\|_{\mathbf{L}^4(\Omega)} \left(\int_0^T \left(\int_{\Omega} |\operatorname{tr} \mathbf{C}_m(t) - \operatorname{tr} \mathbf{C}(t)|^4 dx \right)^{1/4} \left(\int_{\Omega} |\mathbf{C}_m(t)|^4 dx \right)^{1/2} dt + \right. \\ & \quad \left. + \int_0^T \left(\int_{\Omega} |\operatorname{tr} \mathbf{C}_m - \operatorname{tr} \mathbf{C}(t)|^4 dx \right)^{1/4} \left(\int_{\Omega} |\operatorname{tr} \mathbf{C}(t)|^4 dx \right)^{1/4} \left(\int_{\Omega} |\mathbf{C}_m(t)|^4 dx \right)^{1/4} dt \leq \right. \\ & \quad \left. + \int_0^T \left(\int_{\Omega} |\mathbf{C}_m(t) - \mathbf{C}(t)|^4 dx \right)^{1/4} \left(\int_{\Omega} |\operatorname{tr} \mathbf{C}(t)|^4 dx \right)^{1/2} dt \right) \leq \\ & \leq c \|\mathbf{D}_j\|_{\mathbf{L}^4(\Omega)} \left(\|\operatorname{tr} \mathbf{C}_m - \operatorname{tr} \mathbf{C}\|_{L^2(0,T;L^4(\Omega))} \|\mathbf{C}_m\|_{\mathbf{L}^4(0,T;L^4(\Omega))}^2 + \right. \\ & \quad \left. + \|\operatorname{tr} \mathbf{C}_m - \operatorname{tr} \mathbf{C}\|_{L^2(0,T;L^4(\Omega))} \|\operatorname{tr} \mathbf{C}\|_{\mathbf{L}^4(0,T;L^4(\Omega))} \|\mathbf{C}_m\|_{\mathbf{L}^4(0,T;L^4(\Omega))} + \right. \\ & \quad \left. + \|\mathbf{C} - \mathbf{C}_m\|_{L^2(0,T;L^4(\Omega))} \|\operatorname{tr} \mathbf{C}\|_{\mathbf{L}^4(0,T;L^4(\Omega))}^2 \right), \end{aligned}$$

which goes to zero letting m to ∞ . Let us point out that the limiting process in the convective terms $\int_0^T b(\mathbf{v}_m(t), \mathbf{v}_m(t), \mathbf{w}_j \varphi(t)) dt$ and $\int_0^T B(\mathbf{v}_m(t), \mathbf{C}_m(t), \mathbf{D}_j \varphi(t)) dt$ is straightforward and can be done in an analogous way as, e.g. in [38]. Finally, after the limiting process, we obtain that the limit of the Galerkin approximate solution satisfies the weak formulation for any $\mathbf{w} \in \mathbf{V}$, any $\mathbf{D} \in \mathbf{H}^1(\Omega)$ and any $\varphi \in \mathcal{C}^1([0, T])$, $\varphi(T) = 0$, i.e.

$$\begin{aligned} & -\rho \int_0^T (\mathbf{v}(t), \mathbf{w} \varphi'(t)) dt + \rho(\mathbf{v}_0, \mathbf{w}) \varphi(0) + \nu \int_0^T ((\mathbf{v}(t), \mathbf{w} \varphi(t))) dt + \\ & + \rho \int_0^T b(\mathbf{v}(t), \mathbf{v}(t), \mathbf{w} \varphi(t)) dt = - \int_0^T \int_{\Omega} \operatorname{tr} \mathbf{C}(t) \mathbf{C}(t) : \nabla \mathbf{w} \varphi(t) dx dt \tag{12a} \\ & - \int_0^T (\mathbf{C}(t), \mathbf{D} \varphi'(t)) dt + (\mathbf{C}_0, \mathbf{D}) \varphi(0) + \varepsilon \int_0^T ((\mathbf{C}(t), \mathbf{D} \varphi(t))) dt + \\ & + \int_0^T B(\mathbf{v}(t), \mathbf{C}(t), \mathbf{D} \varphi(t)) dt - \int_0^T ((\nabla \mathbf{v}(t)) \mathbf{C}(t) + \mathbf{C}(t) (\nabla \mathbf{v}(t))^T, \mathbf{D} \varphi(t)) dt = \\ & = \int_0^T (\operatorname{tr} \mathbf{C}(t) \mathbf{I} - (\operatorname{tr} \mathbf{C}(t))^2 \mathbf{C}(t), \mathbf{D} \varphi(t)) dt. \tag{12b} \end{aligned}$$

To show that the initial condition is satisfied we first realise that since $(\mathbf{v}, \mathbf{C}) \in \mathbf{L}^2(0, T; \mathbf{V}) \times \mathbf{L}^2(0, T; \mathbf{H}^1(\Omega))$ satisfies the weak formulation (10) we have that \mathbf{v} is almost everywhere

equal to a continuous function from $[0, T]$ to \mathbf{V}^* and \mathbf{C} is almost everywhere equal to a continuous function from $[0, T]$ to \mathbf{H}^{-1} . Thus the initial condition (10b) makes sense. Multiplying (10c) and (10d) by $\varphi(t)$ and integrating per partes in time we get

$$\rho(\mathbf{v}_0 - \mathbf{v}(0), \mathbf{w})\varphi(0) = 0 \quad \text{and} \quad (\mathbf{C}_0 - \mathbf{C}(0), \mathbf{D})\varphi(0) = 0$$

for each $\mathbf{w} \in \mathbf{V}$, $\mathbf{D} \in \mathbf{H}^1(\Omega)$. We can choose φ such that $\varphi(0) = 1$, and therefore

$$\rho(\mathbf{v}_0 - \mathbf{v}(0), \mathbf{w}) = 0 \quad \forall \mathbf{w} \in \mathbf{V} \quad \text{and} \quad (\mathbf{C}_0 - \mathbf{C}(0), \mathbf{D}) = 0 \quad \forall \mathbf{D} \in \mathbf{H}^1(\Omega).$$

This implies $(\mathbf{v}(0), \mathbf{C}(0)) = (\mathbf{v}_0, \mathbf{C}_0)$. We have proven the existence of the weak solution to (1). \square

4 Uniqueness and regularity

We would like to point out that in the above existence result we only obtain that $\mathbf{C}' \in \mathbf{L}^{4/3}(0, T; \mathbf{H}^{-1}(\Omega))$, which implies that the following property needed for the uniqueness study

$$\mathbf{C} \in \mathcal{C}([0, T]; \mathbf{L}^2(\Omega)), \quad 2\langle \mathbf{C}', \mathbf{C} \rangle = \frac{d}{dt} \|\mathbf{C}\|_{\mathbf{L}^2(\Omega)}^2$$

is missing even in two space dimensions. In order to obtain the uniqueness of the weak solution we firstly investigate possible higher regularity of our weak solution.

4.1 More regular solutions

Theorem 2. (*regularity result*)

Let the domain Ω be of class \mathcal{C}^2 and $(\mathbf{v}_0, \mathbf{C}_0) \in [\mathbf{H}^2(\Omega) \cap \mathbf{V}] \times \mathbf{H}^2(\Omega)$. Then the weak solution (10) satisfies additionally

$$\begin{aligned} \mathbf{v}' &\in \mathbf{L}^2(0, T; \mathbf{V}) \cap \mathbf{L}^\infty(0, T; \mathbf{H}), & \mathbf{C}' &\in \mathbf{L}^2(0, T; \mathbf{H}^1(\Omega)) \cap \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega)), \\ \mathbf{v} &\in \mathbf{L}^\infty(0, T; \mathbf{H}^2(\Omega)), & \mathbf{C} &\in \mathbf{L}^\infty(0, T; \mathbf{H}^2(\Omega)). \end{aligned}$$

Proof. We return to the Galerkin approximation used in the proof of existence of the weak solution. We need to show that this approximate solution also satisfies the second a priori estimates, i.e. there exists a positive constant K independent of m such that

$$\|\mathbf{v}'_m\|_{\mathbf{L}^2(0, T; \mathbf{V})} + \|\mathbf{v}'_m\|_{\mathbf{L}^\infty(0, T; \mathbf{H})} + \|\mathbf{C}'_m\|_{\mathbf{L}^2(0, T; \mathbf{H}^1(\Omega))} + \|\mathbf{C}'_m\|_{\mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega))} \leq K.$$

In order to obtain a regularity result we now assume that our basis functions $\{\mathbf{w}_i\}_{i=1}^\infty$, $\{\mathbf{D}_i\}_{i=1}^\infty$, are the eigenfunctions of the Stokes and the Laplace operator, respectively. Since $\mathbf{v}_0 \in \mathbf{H}^2(\Omega) \cap \mathbf{V}$, $\mathbf{C}_0 \in \mathbf{H}^2(\Omega)$, we can choose \mathbf{v}_{0m} and \mathbf{C}_{0m} as the orthogonal projections in $\mathbf{V} \cap \mathbf{H}^2(\Omega)$ of \mathbf{v}_0 onto the space spanned by $\mathbf{w}_1, \dots, \mathbf{w}_m$ and in $\mathbf{H}^2(\Omega)$ of \mathbf{C}_0 onto the space spanned by $\mathbf{D}_1, \dots, \mathbf{D}_m$, respectively. Then, for $\tilde{c} = \tilde{c}(\Omega) > 0$,

$$\|\mathbf{v}_{0m}\|_{\mathbf{H}^2(\Omega)} \leq \tilde{c} \|\mathbf{v}_0\|_{\mathbf{H}^2(\Omega)} \quad \text{and} \quad \mathbf{v}_{0m} \rightarrow \mathbf{v}_0 \quad \text{in} \quad \mathbf{H}^2(\Omega), \quad \text{as} \quad m \rightarrow \infty, \quad (13a)$$

$$\|\mathbf{C}_{0m}\|_{\mathbf{H}^2(\Omega)} \leq \tilde{c}\|\mathbf{C}_0\|_{\mathbf{H}^2(\Omega)} \text{ and } \mathbf{C}_{0m} \rightarrow \mathbf{C}_0 \text{ in } \mathbf{H}^2(\Omega), \text{ as } m \rightarrow \infty. \quad (13b)$$

For proof of (13) see e.g. [26]. We multiply the equation for the Galerkin approximation of \mathbf{v} by $g'_{jm}(t)$ and of \mathbf{C} by $G'_{jm}(t)$, respectively. We add the resulting equations for $j = 1, \dots, m$. This gives us

$$\begin{aligned} \rho\|\mathbf{v}'_m(t)\|_{\mathbf{L}^2(\Omega)}^2 + \nu((\mathbf{v}_m(t), \mathbf{v}'_m(t))) + \rho b(\mathbf{v}_m(t), \mathbf{v}_m(t), \mathbf{v}'_m(t)) &= -(\text{tr } \mathbf{C}_m(t) \mathbf{C}_m(t), \nabla \mathbf{v}'_m(t)) \\ \|\mathbf{C}'_m(t)\|_{\mathbf{L}^2(\Omega)}^2 + \varepsilon((\mathbf{C}_m(t), \mathbf{C}'_m(t))) + B(\mathbf{v}_m(t), \mathbf{C}_m(t), \mathbf{C}'_m(t)) &= \\ &= ((\nabla \mathbf{v}_m(t)) \mathbf{C}_m(t) + \mathbf{C}_m(t) (\nabla \mathbf{v}_m(t))^T, \mathbf{C}'_m(t)) + \\ &+ (\text{tr } \mathbf{C}_m(t) \mathbf{I} - (\text{tr } \mathbf{C}_m(t))^2 \mathbf{C}_m(t), \mathbf{C}'_m(t)). \end{aligned}$$

In particular, at time $t = 0$

$$\begin{aligned} \|\mathbf{v}'_m(0)\|_{\mathbf{L}^2(\Omega)}^2 &= \frac{\nu}{\rho}(\Delta \mathbf{v}_{0m}, \mathbf{v}_{0m}) - b(\mathbf{v}_{0m}, \mathbf{v}_{0m}, \mathbf{v}'_m(0)) + \frac{1}{\rho}(\text{div } (\text{tr } \mathbf{C}_{0m} \cdot \mathbf{C}_{0m}), \mathbf{v}'_m(0)) \\ \|\mathbf{v}'_m(0)\|_{\mathbf{L}^2(\Omega)} &\leq \frac{\nu}{\rho}\|\Delta \mathbf{v}_{0m}\|_{\mathbf{L}^2(\Omega)} + \|\mathcal{B}\mathbf{v}_{0m}\|_{\mathbf{L}^2(\Omega)} + \frac{1}{\rho}\|\text{div } (\text{tr } \mathbf{C}_{0m} \cdot \mathbf{C}_{0m})\|_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

Using (13) we get

$$\begin{aligned} \|\Delta \mathbf{v}_{0m}\|_{\mathbf{L}^2(\Omega)} &\leq c\|\mathbf{v}_{0m}\|_{\mathbf{H}^2(\Omega)} \leq c_0\|\mathbf{v}_0\|_{\mathbf{H}^2(\Omega)} \\ \|\text{div } (\text{tr } \mathbf{C}_{0m} \cdot \mathbf{C}_{0m})\|_{\mathbf{L}^2(\Omega)} &= \|\nabla \text{tr } \mathbf{C}_{0m} \cdot \mathbf{C}_{0m} + \text{tr } \mathbf{C}_{0m} \cdot \text{div } \mathbf{C}_{0m}\|_{\mathbf{L}^2(\Omega)} \leq \\ &\leq c\|\text{tr } \mathbf{C}_{0m}\|_{\mathbf{H}^2(\Omega)}\|\mathbf{C}_{0m}\|_{\mathbf{H}^2(\Omega)} \leq c\|\mathbf{C}_{0m}\|_{\mathbf{H}^2(\Omega)}^2 \leq c_1\|\mathbf{C}_0\|_{\mathbf{H}^2(\Omega)}^2. \end{aligned}$$

For $\mathcal{B}\mathbf{v}_{0m}$ we have, by the Hölder inequality,

$$b(\mathbf{v}, \mathbf{v}, \mathbf{w}) \leq c\|\mathbf{v}\|_{\mathbf{L}^4(\Omega)}\|\nabla \mathbf{v}\|_{\mathbf{L}^4(\Omega)}\|\mathbf{w}\|_{\mathbf{L}^2(\Omega)} \leq c\|\mathbf{v}\|_{\mathbf{H}^2(\Omega)}\|\mathbf{v}\|_{\mathbf{H}^2(\Omega)}\|\mathbf{w}\|_{\mathbf{L}^2(\Omega)}, \quad \mathbf{v} \in \mathbf{H}^2(\Omega), \quad \mathbf{w} \in \mathbf{L}^2(\Omega),$$

and hence

$$\|\mathcal{B}\mathbf{v}_{0m}\|_{\mathbf{L}^2(\Omega)} \leq c\|\mathbf{v}_{0m}\|_{\mathbf{H}^2(\Omega)}\|\mathbf{v}_{0m}\|_{\mathbf{H}^2(\Omega)} \leq c\|\mathbf{v}_{0m}\|_{\mathbf{H}^2(\Omega)}^2 \leq c_2\|\mathbf{v}_0\|_{\mathbf{H}^2(\Omega)}.$$

Thus we get

$$\|\mathbf{v}'_m(0)\|_{\mathbf{L}^2(\Omega)} \leq \left(\frac{\nu c_0}{\rho} + c_2\right)\|\mathbf{v}_0\|_{\mathbf{H}^2(\Omega)} + \frac{c_1}{\rho}\|\mathbf{C}_0\|_{\mathbf{H}^2(\Omega)}^2 =: a_1, \quad (14)$$

which implies that $\mathbf{v}'_m(0)$ belongs to a bounded set in \mathbf{H} . Further we have

$$\begin{aligned} \|\mathbf{C}'_m(0)\|_{\mathbf{L}^2(\Omega)}^2 &= \varepsilon(\Delta \mathbf{C}_{0m}, \mathbf{C}'_m(0)) - B(\mathbf{v}_{0m}, \mathbf{C}_{0m}, \mathbf{C}'_m(0)) + \\ &+ ((\nabla \mathbf{v}_{0m}) \mathbf{C}_{0m} + \mathbf{C}_{0m} (\nabla \mathbf{v}_{0m})^T, \mathbf{C}'_m(0)) + (\text{tr } \mathbf{C}_{0m} \mathbf{I} - (\text{tr } \mathbf{C}_{0m})^2 \mathbf{C}_{0m}, \mathbf{C}'_m(0)) \\ \|\mathbf{C}'_m(0)\|_{\mathbf{L}^2(\Omega)} &\leq \varepsilon\|\Delta \mathbf{C}_{0m}\|_{\mathbf{L}^2(\Omega)} + \|\mathcal{B}(\mathbf{v}_{0m}, \mathbf{C}_{0m})\|_{\mathbf{L}^2(\Omega)} + \\ &+ \|((\nabla \mathbf{v}_{0m}) \mathbf{C}_{0m} + \mathbf{C}_{0m} (\nabla \mathbf{v}_{0m})^T)\|_{\mathbf{L}^2(\Omega)} + \|\text{tr } \mathbf{C}_{0m} \mathbf{I} - (\text{tr } \mathbf{C}_{0m})^2 \mathbf{C}_{0m}\|_{\mathbf{L}^2(\Omega)}. \end{aligned}$$

Again, using (13) we get

$$\begin{aligned} \|\Delta \mathbf{C}_{0m}\|_{\mathbf{L}^2(\Omega)} &\leq c\|\mathbf{C}_{0m}\|_{\mathbf{H}^2(\Omega)} \leq c_3\|\mathbf{C}_0\|_{\mathbf{H}^2(\Omega)} \\ \|((\nabla \mathbf{v}_{0m}) \mathbf{C}_{0m} + \mathbf{C}_{0m} (\nabla \mathbf{v}_{0m})^T)\|_{\mathbf{L}^2(\Omega)} &\leq 2\|\nabla \mathbf{v}_{0m}\|_{\mathbf{L}^4(\Omega)}\|\mathbf{C}_{0m}\|_{\mathbf{L}^4(\Omega)} \leq c\|\mathbf{v}_{0m}\|_{\mathbf{H}^2(\Omega)}\|\mathbf{C}_{0m}\|_{\mathbf{H}^2(\Omega)} \leq \\ &\leq c_4\|\mathbf{v}_0\|_{\mathbf{H}^2(\Omega)}\|\mathbf{C}_0\|_{\mathbf{H}^2(\Omega)} \end{aligned}$$

$$\begin{aligned} \|\operatorname{tr} \mathbf{C}_{0m} \mathbf{I} - (\operatorname{tr} \mathbf{C}_{0m})^2 \mathbf{C}_{0m}\|_{\mathbf{L}^2(\Omega)} &\leq n^{1/2} \|\operatorname{tr} \mathbf{C}_{0m}\|_{L^2(\Omega)} + \|\operatorname{tr} \mathbf{C}_{0m}\|_{L^6(\Omega)}^3 \leq \\ &\leq c \|\mathbf{C}_{0m}\|_{\mathbf{H}^2(\Omega)} + c \|\mathbf{C}_{0m}\|_{\mathbf{H}^2(\Omega)}^3 \leq c_5 \|\mathbf{C}_0\|_{\mathbf{H}^2(\Omega)} + c_6 \|\mathbf{C}_0\|_{\mathbf{H}^2(\Omega)}^2. \end{aligned}$$

By the Hölder inequality we have

$$\begin{aligned} B(\mathbf{v}, \mathbf{C}, \mathbf{D}) &\leq c \|\mathbf{v}\|_{\mathbf{L}^4(\Omega)} \|\nabla \mathbf{C}\|_{\mathbf{L}^4(\Omega)} \|\mathbf{w}\|_{\mathbf{L}^2(\Omega)} \leq c \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)} \|\mathbf{C}\|_{\mathbf{H}^2(\Omega)} \|\mathbf{D}\|_{\mathbf{L}^2(\Omega)}, \\ &\mathbf{v} \in \mathbf{H}^2(\Omega), \mathbf{C} \in \mathbf{H}^2(\Omega), \mathbf{D} \in \mathbf{L}^2(\Omega), \end{aligned}$$

and thus

$$\|\mathcal{B}(\mathbf{v}_{0m}, \mathbf{C}_{0m})\|_{\mathbf{L}^2(\Omega)} \leq c \|\mathbf{v}_{0m}\|_{\mathbf{H}^2(\Omega)} \|\mathbf{C}_{0m}\|_{\mathbf{H}^2(\Omega)} \leq c_7 \|\mathbf{v}_0\|_{\mathbf{H}^2(\Omega)} \|\mathbf{C}_0\|_{\mathbf{H}^2(\Omega)}.$$

Finally we obtain

$$\|\mathbf{C}'_m(0)\|_{\mathbf{L}^2(\Omega)} \leq (\varepsilon c_3 + c_5) \|\mathbf{C}_0\|_{\mathbf{H}^2(\Omega)} + (c_4 + c_7) \|\mathbf{v}_0\|_{\mathbf{H}^2(\Omega)} \|\mathbf{C}_0\|_{\mathbf{H}^2(\Omega)} + c_6 \|\mathbf{C}_0\|_{\mathbf{H}^2(\Omega)}^3 =: a_2. \quad (15)$$

This implies that $\mathbf{C}'_m(0)$ belongs to a bounded set in $\mathbf{L}^2(\Omega)$.

Differentiating the equations for the Galerkin approximation in time we get

$$\begin{aligned} \rho(\mathbf{v}''_m(t), \mathbf{w}_j) + \nu((\mathbf{v}'_m(t), \mathbf{w}_j)) + \rho b(\mathbf{v}'_m(t), \mathbf{v}_m(t), \mathbf{w}_j) + \rho b(\mathbf{v}_m(t), \mathbf{v}'_m(t), \mathbf{w}_j) = \\ = -(\operatorname{tr} \mathbf{C}'_m(t) \mathbf{C}_m(t), \nabla \mathbf{w}_j) - (\operatorname{tr} \mathbf{C}_m(t) \mathbf{C}'_m(t), \nabla \mathbf{w}_j) \end{aligned} \quad (16a)$$

$$\begin{aligned} (\mathbf{C}''_m(t), \mathbf{D}_j) + \varepsilon((\mathbf{C}'_m(t), \mathbf{D}_j)) + B(\mathbf{v}'_m(t), \mathbf{C}_m(t), \mathbf{D}_j) + B(\mathbf{v}_m(t), \mathbf{C}'_m(t), \mathbf{D}_j) = \\ = ((\nabla \mathbf{v}'_m(t)) \mathbf{C}_m(t) + \mathbf{C}_m(t) (\nabla \mathbf{v}'_m(t))^T, \mathbf{D}_j) + \\ + ((\nabla \mathbf{v}_m(t)) \mathbf{C}'_m(t) + \mathbf{C}'_m(t) (\nabla \mathbf{v}_m(t))^T, \mathbf{D}_j) + \\ + (\operatorname{tr} \mathbf{C}'_m(t) \mathbf{I} - (\operatorname{tr} \mathbf{C}_m(t))^2 \mathbf{C}'_m(t) - 2 \operatorname{tr} \mathbf{C}_m(t) \operatorname{tr} \mathbf{C}'_m(t) \mathbf{C}_m(t), \mathbf{D}_j) \end{aligned} \quad (16b)$$

$$\text{for } j = 1, \dots, m, t \in [0, T].$$

Now, we will show the a priori estimates for \mathbf{v}'_m and \mathbf{C}'_m . We multiply the equation (16a) for the velocity and the equation (16b) for the conformation tensor by $g'_{jm}(t)$ and $G'_{jm}(t)$, respectively. Summing the resulting equations for $j = 1, \dots, m$ we obtain

$$\begin{aligned} \frac{\rho}{2} \frac{d}{dt} \|\mathbf{v}'_m(t)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{1}{2} \frac{d}{dt} \|\mathbf{C}'_m(t)\|_{\mathbf{L}^2(\Omega)}^2 + \nu \|\mathbf{v}'_m(t)\|^2 + \varepsilon \|\nabla \mathbf{C}'_m(t)\|_{\mathbf{L}^2(\Omega)}^2 = \\ = -\rho b(\mathbf{v}'_m(t), \mathbf{v}_m(t), \mathbf{v}'_m(t)) - B(\mathbf{v}'_m(t), \mathbf{C}_m(t), \mathbf{C}'_m(t)) - \\ - (\operatorname{tr} \mathbf{C}'_m(t) \mathbf{C}_m(t), \nabla \mathbf{v}'_m(t)) - (\operatorname{tr} \mathbf{C}_m(t) \mathbf{C}'_m(t), \nabla \mathbf{v}'_m(t)) + \\ + ((\nabla \mathbf{v}'_m(t)) \mathbf{C}_m(t) + \mathbf{C}_m(t) (\nabla \mathbf{v}'_m(t))^T, \mathbf{C}'_m(t)) + \\ + ((\nabla \mathbf{v}_m(t)) \mathbf{C}'_m(t) + \mathbf{C}'_m(t) (\nabla \mathbf{v}_m(t))^T, \mathbf{C}'_m(t)) + (\operatorname{tr} \mathbf{C}'_m(t), \operatorname{tr} \mathbf{C}'_m(t)) - \\ - ((\operatorname{tr} \mathbf{C}_m(t))^2 \mathbf{C}'_m(t), \mathbf{C}'_m(t)) - 2(\operatorname{tr} \mathbf{C}_m(t) \operatorname{tr} \mathbf{C}'_m(t) \mathbf{C}_m(t), \mathbf{C}'_m(t)). \end{aligned} \quad (17)$$

Further, we shall estimate the integrals on the right hand side of (17). Using the Hölder, the Young inequalities, the interpolation inequality (7a) and the property (9) for the convective terms we have

$$\rho b(\mathbf{v}'_m(t), \mathbf{v}_m(t), \mathbf{v}'_m(t)) \leq \frac{\nu}{6} \|\mathbf{v}'_m(t)\|^2 + \frac{c\rho^2}{\nu} \|\mathbf{v}'_m(t)\|_{\mathbf{L}^2(\Omega)}^2 \|\mathbf{v}_m(t)\|^2$$

$$B(\mathbf{v}'_m(t), \mathbf{C}_m(t), \mathbf{C}'_m(t)) \leq \frac{\varepsilon}{6} \|\nabla \mathbf{C}'_m(t)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\nu}{6} \|\mathbf{v}'_m(t)\|^2 + \frac{c}{\nu \varepsilon^2} \|\mathbf{v}'_m(t)\|_{\mathbf{L}^2(\Omega)}^2 \|\mathbf{C}_m(t)\|_{\mathbf{L}^4(\Omega)}^4.$$

For the nonlinear viscoelastic term, by the Hölder, the Young and the interpolation inequality (7b), we have

$$\begin{aligned} (\operatorname{tr} \mathbf{C}_m(t) \mathbf{C}'_m(t), \nabla \mathbf{v}'_m(t)) &\leq c \|\operatorname{tr} \mathbf{C}_m(t)\|_{\mathbf{L}^4(\Omega)} \|\mathbf{C}'_m(t)\|_{\mathbf{L}^4(\Omega)} \|\mathbf{v}'_m(t)\| \leq \\ &\leq c \|\mathbf{C}_m(t)\|_{\mathbf{L}^4(\Omega)} \left(\|\mathbf{C}'_m(t)\|_{\mathbf{L}^2(\Omega)}^{1/2} \|\nabla \mathbf{C}'_m(t)\|_{\mathbf{L}^2(\Omega)}^{1/2} + \|\mathbf{C}'_m(t)\|_{\mathbf{L}^2(\Omega)} \right) \|\mathbf{v}'_m(t)\| \leq \\ &\leq \frac{\nu}{12} \|\mathbf{v}'_m(t)\|^2 + \frac{\varepsilon}{12} \|\nabla \mathbf{C}'_m(t)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{c}{\nu} \|\mathbf{C}'_m(t)\|_{\mathbf{L}^2(\Omega)}^2 \|\mathbf{C}_m(t)\|_{\mathbf{L}^4(\Omega)}^2 + \\ &+ \frac{c}{\nu^2 \varepsilon} \|\mathbf{C}'_m(t)\|_{\mathbf{L}^2(\Omega)}^2 \|\mathbf{C}_m(t)\|_{\mathbf{L}^4(\Omega)}^4. \end{aligned}$$

The remaining elastic terms can be estimated in an analogous way. Using these estimates we obtain from (17) that

$$\begin{aligned} \frac{\rho}{2} \frac{d}{dt} \|\mathbf{v}'_m(t)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{1}{2} \frac{d}{dt} \|\mathbf{C}'_m(t)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\nu}{2} \|\mathbf{v}'_m(t)\|^2 + \frac{\varepsilon}{2} \|\nabla \mathbf{C}'_m(t)\|_{\mathbf{L}^2(\Omega)}^2 &\leq \\ &\leq \left(\frac{\rho}{2} \|\mathbf{v}'_m(t)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{1}{2} \|\mathbf{C}'_m(t)\|_{\mathbf{L}^2(\Omega)}^2 \right) \beta(t), \end{aligned} \quad (18)$$

where

$$\beta(t) = c(\rho, 1/\nu, 1/\varepsilon) \left(1 + \|\mathbf{C}_m(t)\|_{\mathbf{L}^4(\Omega)}^4 + \|\mathbf{C}_m(t)\|_{\mathbf{L}^4(\Omega)}^2 + \|\mathbf{v}_m(t)\|^2 + \|\mathbf{v}_m(t)\| \right)$$

is an integrable function. The Gronwall inequality, using (14) and (15), yields

$$\begin{aligned} \frac{\rho}{2} \|\mathbf{v}'_m(s)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{1}{2} \|\mathbf{C}'_m(s)\|_{\mathbf{L}^2(\Omega)}^2 &\leq \left(\frac{\rho}{2} \|\mathbf{v}'_m(0)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{1}{2} \|\mathbf{C}'_m(0)\|_{\mathbf{L}^2(\Omega)}^2 \right) \exp \left\{ \int_0^T \beta(t) \right\} \leq \\ &\leq \left(\frac{\rho}{2} a_1^2 + \frac{1}{2} a_2^2 \right) c, \end{aligned}$$

where $c = c(\rho, 1/\nu, 1/\varepsilon, \|\mathbf{v}_m\|_{\mathbf{L}^2(0,T;\mathbf{V})}, \|\mathbf{C}_m\|_{\mathbf{L}^4((0,T)\times\Omega)})$ is a positive constant, Consequently we have the second a priori estimate

$$\|\mathbf{v}'_m\|_{\mathbf{L}^\infty(0,T;\mathbf{H})} + \|\mathbf{C}'_m\|_{L^\infty(0,T;L^2(\Omega))} \leq K, \quad (19)$$

and by using (18) we have

$$\|\mathbf{v}'_m\|_{\mathbf{L}^2(0,T;\mathbf{V})} + \|\mathbf{C}'_m\|_{L^2(0,T;H^1(\Omega))} \leq K.$$

From now on $K = K(\Omega, T, \rho, 1/\nu, 1/\varepsilon, \mathbf{v}_0, \mathbf{C}_0)$ is a positive constant depending only on the data. We have shown the uniform a priori estimates for $\mathbf{v}'_m, \mathbf{C}'_m$, so we finally get that also the limit

$$(\mathbf{v}', \mathbf{C}') \in [\mathbf{L}^\infty(0, T; \mathbf{H}) \cap \mathbf{L}^2(0, T; \mathbf{V})] \times [\mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega)) \cap \mathbf{L}^2(0, T; \mathbf{H}^1(\Omega))].$$

Now, in order to show that $\mathbf{v} \in \mathbf{L}^\infty(0, T; \mathbf{H}^2(\Omega))$ we consider the velocity equation (10c) in the following form

$$\nu((\mathbf{v}(t), \mathbf{w})) = (g(t), \mathbf{w}), \quad \mathbf{w} \in \mathbf{V},$$

where $g(t) = -\mathbf{v}'(t) - \mathcal{B}\mathbf{v}(t) - \mathcal{E}\mathbf{T}(t)$. Since

$$|\langle \mathcal{E}\mathbf{T}(t), \mathbf{w} \rangle| = |(\operatorname{tr} \mathbf{C}(t) \mathbf{C}(t), \nabla \mathbf{w})| \leq c \|\mathbf{w}\| \|\mathbf{C}(t)\|_{\mathbf{H}^1(\Omega)}^2$$

and $\mathbf{C} \in \mathbf{L}^\infty(0, T; \mathbf{H}^1(\Omega))$ we know that $\mathcal{E}\mathbf{T} \in \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega))$. We already know that $\mathbf{v}' \in \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega))$. Further,

$$|b(\mathbf{v}(t), \mathbf{v}(t), \mathbf{w})| \leq c \|\mathbf{v}(t)\|_{\mathbf{L}^4(\Omega)} \|\mathbf{v}(t)\| \|\mathbf{w}\|_{\mathbf{L}^4(\Omega)} \leq c \|\mathbf{v}(t)\|^2 \|\mathbf{w}\|_{\mathbf{L}^4(\Omega)}, \quad (20)$$

thus $\mathcal{B}\mathbf{v} \in \mathbf{L}^\infty(0, T; \mathbf{L}^{4/3}(\Omega))$. Consequently

$$g \in \mathbf{L}^\infty(0, T; \mathbf{L}^{4/3}(\Omega)). \quad (21)$$

Using the regularity result for the Stokes equation, cf. [38], we have $\mathbf{v} \in \mathbf{L}^\infty(0, T; \mathbf{W}^{2,4/3})$. By the Sobolev theorem in two dimensions we have $\mathbf{W}^{2,4/3}(\Omega) \hookrightarrow \mathcal{C}^0(\Omega)$. Hence $\mathbf{v} \in \mathbf{L}^\infty(\Omega \times (0, T))$. Now we can improve (21). We replace (20) by the inequality

$$|b(\mathbf{v}(t), \mathbf{v}(t), \mathbf{w})| \leq c_4 \|\mathbf{v}(t)\|_{\mathbf{L}^\infty(\Omega \times (0, T))} \|\mathbf{v}(t)\| \|\mathbf{w}\|_{\mathbf{L}^2(\Omega)},$$

and thus $g \in \mathbf{L}^\infty(0, T; \mathbf{H})$. Again, the regularity of the solution to the Stokes equation yields $\mathbf{v} \in \mathbf{L}^\infty(0, T; \mathbf{H}^2(\Omega))$.

Now, let us consider the equation for the conformation tensor to show that $\mathbf{C} \in \mathbf{L}^\infty(0, T; \mathbf{H}^2(\Omega))$. We can rewrite the weak formulation for the conformation tensor (10d) in the following operator form

$$-\varepsilon((\mathbf{C}(t), \mathbf{D})) = (G(t), \mathbf{D}), \quad \mathbf{D} \in \mathbf{H}^1(\Omega),$$

where $G(t) = -\mathcal{C}'(t) - \tilde{\mathcal{B}}(\mathbf{v}(t), \mathbf{C}(t)) + \mathcal{O}\mathbf{C}(t) + \mathcal{T}\mathbf{C}(t)$. The following inequality

$$\begin{aligned} |\langle \mathcal{T}\mathbf{C}(t), \mathbf{D} \rangle| &= |(\operatorname{tr} \mathbf{C}(t) \mathbf{I} - (\operatorname{tr} \mathbf{C}(t))^2 \mathbf{C}(t), \mathbf{D})| \leq \\ &\leq c \left(\|\operatorname{tr} \mathbf{C}(t)\|_{L^2(\Omega)} \|\operatorname{tr} \mathbf{D}\|_{L^2(\Omega)} + \|\operatorname{tr} \mathbf{C}(t)\|_{L^6(\Omega)}^3 \|\mathbf{D}\|_{L^2(\Omega)} \right) \leq \\ &\leq c \left(\|\operatorname{tr} \mathbf{C}(t)\|_{\mathbf{H}^1(\Omega)} + \|\operatorname{tr} \mathbf{C}(t)\|_{\mathbf{H}^1(\Omega)}^3 \right) \|\mathbf{D}\|_{L^2(\Omega)} \end{aligned}$$

implies $\mathcal{T}\mathbf{C} \in \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega))$, since we know $\mathbf{C} \in \mathbf{L}^\infty(0, T; \mathbf{H}^1(\Omega))$. Further, we can write

$$|B(\mathbf{v}(t), \mathbf{C}(t), \mathbf{D})| \leq c \|\mathbf{v}(t)\|_{\mathbf{L}^\infty(\Omega \times (0, T))} \|\mathbf{C}(t)\|_{\mathbf{H}^1(\Omega)} \|\mathbf{D}\|_{L^2(\Omega)}, \quad (22)$$

which leads to $\mathcal{B}(\mathbf{v}, \mathbf{C}) \in \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega))$. Similarly we get $\mathcal{O}(\mathbf{v}, \mathbf{C}) \in \mathbf{L}^\infty(0, T; \mathbf{L}^{4/3}(\Omega))$ by the estimate

$$\begin{aligned} |\langle \mathcal{O}(\mathbf{v}(t), \mathbf{C}(t)), \mathbf{D} \rangle| &= |((\nabla \mathbf{v}(t))\mathbf{C}(t) + \mathbf{C}(t)(\nabla \mathbf{v}(t))^T, \mathbf{D})| \leq c \|\mathbf{C}(t)\|_{\mathbf{L}^4(\Omega)} \|\mathbf{v}(t)\| \|\mathbf{D}\|_{\mathbf{L}^4(\Omega)} \leq \\ &\leq c \|\mathbf{C}(t)\|_{\mathbf{H}^1(\Omega)} \|\mathbf{v}(t)\| \|\mathbf{D}\|_{\mathbf{L}^4(\Omega)}. \end{aligned} \quad (23)$$

Thus, we have

$$G(t) \in \mathbf{L}^\infty(0, T; \mathbf{L}^{4/3}(\Omega)). \quad (24)$$

Using the regularity result for the solution to the Laplace equation we get $\mathbf{C} \in \mathbf{L}^\infty(0, T; \mathbf{W}^{2,4/3})$. The embedding $\mathbf{W}^{2,4/3}(\Omega) \hookrightarrow \mathcal{C}^0(\Omega)$ yields $\mathbf{C} \in \mathbf{L}^\infty(\Omega \times (0, T))$. Realizing that $\mathbf{v} \in \mathbf{L}^\infty(\Omega \times (0, T))$, we can improve (24) and replace (23) by the inequality

$$|\langle \mathcal{O}(\mathbf{v}(t), \mathbf{C}(t)), \mathbf{D} \rangle| \leq c_3 \|\mathbf{C}(t)\|_{\mathbf{L}(\Omega \times (0, T))} \|\mathbf{v}(t)\| \|\mathbf{D}\|_{L^2(\Omega)}.$$

Finally we get $\mathcal{O}(\mathbf{v}, \mathbf{C}) \in \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega))$ and $G(t) \in \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega))$. Hence, the regularity of the solution to the Laplace equation gives us $\mathbf{C} \in \mathbf{L}^\infty(0, T; \mathbf{H}^2(\Omega))$. \square

4.2 Uniqueness of regular solutions

In this section we will study the question of uniqueness of regular solutions.

Theorem 3. (*uniqueness*)

Let the domain Ω be of class \mathcal{C}^2 and $(\mathbf{v}_0, \mathbf{C}_0) \in [\mathbf{H}^2(\Omega) \cap \mathbf{V}] \times \mathbf{H}^2(\Omega)$. Then the weak solution of (1) is unique.

Proof. Let us recall that for more regular data satisfying the assumptions of Theorem 3 we have also more regular weak solution, i.e.

$$\begin{aligned} \mathbf{v}' &\in \mathbf{L}^2(0, T; \mathbf{V}) \cap \mathbf{L}^\infty(0, T; \mathbf{H}), & \mathbf{C}' &\in \mathbf{L}^2(0, T; \mathbf{H}^1(\Omega)) \cap \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega)), \\ \mathbf{v} &\in \mathbf{L}^\infty(0, T; \mathbf{H}^2(\Omega)), & \mathbf{C} &\in \mathbf{L}^\infty(0, T; \mathbf{H}^2(\Omega)). \end{aligned} \quad (25)$$

Let us assume $(\mathbf{v}_1, \mathbf{C}_1), (\mathbf{v}_2, \mathbf{C}_2)$ are two different solutions satisfying the same initial data and let us denote $(\mathbf{v}, \mathbf{C}) = (\mathbf{v}_1 - \mathbf{v}_2, \mathbf{C}_1 - \mathbf{C}_2)$. We test the weak solution at a.e. t with $\mathbf{v}(t)$ and $\mathbf{C}(t)$, respectively. Then the difference (\mathbf{v}, \mathbf{C}) satisfies the following equality

$$\begin{aligned} &\frac{\rho}{2} \frac{d}{dt} \|\mathbf{v}(t)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{1}{2} \frac{d}{dt} \|\mathbf{C}(t)\|_{\mathbf{L}^2(\Omega)}^2 + \nu \|\mathbf{v}(t)\|^2 + \varepsilon \|\nabla \mathbf{C}(t)\|_{\mathbf{L}^2(\Omega)}^2 = \\ &= -\rho b(\mathbf{v}(t), \mathbf{v}_1(t), \mathbf{v}(t)) - B(\mathbf{v}(t), \mathbf{C}_1(t), \mathbf{C}(t)) + \\ &+ ((\nabla \mathbf{v}_2(t)) \mathbf{C}(t) + \mathbf{C}(t) (\nabla \mathbf{v}_2(t))^T, \mathbf{C}(t)) + ((\nabla \mathbf{v}(t)) \mathbf{C}_1(t) + \mathbf{C}_1(t) (\nabla \mathbf{v}(t))^T, \mathbf{C}(t)) - \\ &- (\text{tr } \mathbf{C}(t) \mathbf{C}_1(t), \nabla \mathbf{v}(t)) - (\text{tr } \mathbf{C}_2(t) \mathbf{C}(t), \nabla \mathbf{v}(t)) + (\text{tr } \mathbf{C}(t), \text{tr } \mathbf{C}(t)) - \\ &- (\text{tr } \mathbf{C}(t) \text{tr } \mathbf{C}_1(t) \mathbf{C}_1(t), \mathbf{C}(t)) - (\text{tr } \mathbf{C}_2(t) \text{tr } \mathbf{C}(t) \mathbf{C}_1(t), \mathbf{C}(t)) - \\ &- ((\text{tr } \mathbf{C}_2(t))^2 \mathbf{C}(t), \mathbf{C}(t)). \end{aligned} \quad (26)$$

In what follows we need to estimate each term on the right hand side of (26). Let us firstly show the estimates of trilinear terms of the velocity and the conformation tensor. Applying the Hölder, the Young inequalities, the interpolation inequality (7a) and (9) we get

$$\rho b(\mathbf{v}(t), \mathbf{v}_1(t), \mathbf{v}(t)) \leq c\rho \|\mathbf{v}(t)\|_{\mathbf{L}^2(\Omega)} \|\mathbf{v}(t)\| \|\mathbf{v}_1(t)\| \leq \frac{\nu}{14} \|\mathbf{v}(t)\|^2 + \frac{c\rho^2}{\nu} \|\mathbf{v}(t)\|_{\mathbf{L}^2(\Omega)}^2 \|\mathbf{v}_1(t)\|^2$$

$$\begin{aligned} B(\mathbf{v}(t), \mathbf{C}_1(t), \mathbf{C}(t)) &\leq c \|\mathbf{v}(t)\|_{\mathbf{L}^2(\Omega)}^{1/2} \|\mathbf{v}(t)\|^{1/2} \|\nabla \mathbf{C}(t)\|_{\mathbf{L}^2(\Omega)} \|\mathbf{C}_1(t)\|_{\mathbf{L}^4(\Omega)} \leq \\ &\leq \frac{\varepsilon}{16} \|\nabla \mathbf{C}(t)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\nu}{14} \|\mathbf{v}(t)\|^2 + \frac{c}{\nu\varepsilon^2} \|\mathbf{v}(t)\|_{\mathbf{L}^2(\Omega)}^2 \|\mathbf{C}_1(t)\|_{\mathbf{L}^4(\Omega)}^4. \end{aligned}$$

Further, nonlinear terms from (26) can be estimated by using the Hölder, the Young and the interpolation inequalities (7), i.e. in an analogous way as for this term

$$\begin{aligned} ((\nabla \mathbf{v}_2(t)) \mathbf{C}(t) + \mathbf{C}(t) (\nabla \mathbf{v}_2(t))^T, \mathbf{C}(t)) &\leq \frac{\varepsilon}{16} \|\nabla \mathbf{C}(t)\|_2^2 + \frac{c}{\varepsilon} \|\mathbf{C}(t)\|_{\mathbf{L}^2(\Omega)}^2 \|\mathbf{v}_2(t)\|^2 + \\ &+ c \|\mathbf{C}(t)\|_{\mathbf{L}^2(\Omega)}^2 \|\mathbf{v}_2(t)\|. \end{aligned}$$

Putting the above estimates together we obtain

$$\begin{aligned} &\frac{\rho}{2} \frac{d}{dt} \|\mathbf{v}(t)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{1}{2} \frac{d}{dt} \|\mathbf{C}(t)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\nu}{2} \|\mathbf{v}(t)\|^2 + \frac{\varepsilon}{2} \|\nabla \mathbf{C}(t)\|_{\mathbf{L}^2(\Omega)}^2 \leq \\ &\leq \left(\frac{\rho}{2} \|\mathbf{v}(t)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{1}{2} \|\mathbf{C}(t)\|_{\mathbf{L}^2(\Omega)}^2 \right) G(t), \end{aligned} \quad (27)$$

where

$$G(t) = c(\Omega, \rho, 1/\nu, 1/\varepsilon) \left(\|\mathbf{v}_1(t)\|^2 + \|\mathbf{v}_2(t)\|^2 + \|\mathbf{v}_2(t)\| + \|\operatorname{tr} \mathbf{C}_2(t)\|_{L^4(\Omega)}^4 + \|\operatorname{tr} \mathbf{C}_2(t)\|_{L^4(\Omega)}^2 + \|\mathbf{C}_1(t)\|_{L^4(\Omega)}^4 + \|\mathbf{C}_1(t)\|_{L^4(\Omega)}^2 \right).$$

The Gronwall inequality then yields

$$\frac{\rho}{2} \|\mathbf{v}(s)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{1}{2} \|\mathbf{C}(s)\|_{\mathbf{L}^2(\Omega)}^2 \leq \left(\frac{\rho}{2} \|\mathbf{v}(0)\|_{\mathbf{L}^2(\Omega)}^2 + \frac{1}{2} \|\mathbf{C}(0)\|_{\mathbf{L}^2(\Omega)}^2 \right) \underbrace{\exp \left\{ \int_0^T G(s) \right\}}_{\bar{G}},$$

where \bar{G} is a positive constant depending on Ω , ρ , $1/\nu$, $1/\varepsilon$, $\|\mathbf{v}_1\|_{\mathbf{L}^2(0,T;\mathbf{V})}$, $\|\mathbf{v}_2\|_{\mathbf{L}^2(0,T;\mathbf{V})}$, $\|\operatorname{tr} \mathbf{C}_2\|_{L^4((0,T)\times\Omega)}$ and $\|\mathbf{C}_1\|_{L^4((0,T)\times\Omega)}$. Consequently, we have the uniqueness of velocity \mathbf{v} and of the conformation tensor \mathbf{C} . □

5 Conclusions

We have proven global in time existence of weak solutions to the diffusive Peterlin model describing time evolution of complex viscoelastic fluids in two space dimensions, see Theorem 1. Our weak solutions belong to the following spaces

$$(\mathbf{v}, \mathbf{C}) \in [\mathbf{L}^\infty(0, T; \mathbf{H}) \cap \mathbf{L}^2(0, T; \mathbf{V})] \times [\mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega)) \cap \mathbf{L}^2(0, T; \mathbf{H}^1(\Omega))].$$

For the time derivative of the conformation tensor we only have $\mathbf{C}' \in \mathbf{L}^{4/3}(0, T; \mathbf{H}^{-1}(\Omega))$. Thus, in order to prove the uniqueness of the weak solution, we show in Theorem 2 the higher regularity of the weak solution for more regular data; more precisely we obtain that

$$\begin{aligned} \mathbf{v}' &\in \mathbf{L}^2(0, T; \mathbf{V}) \cap \mathbf{L}^\infty(0, T; \mathbf{H}), & \mathbf{C}' &\in \mathbf{L}^2(0, T; \mathbf{H}^1(\Omega)) \cap \mathbf{L}^\infty(0, T; \mathbf{L}^2(\Omega)), \\ \mathbf{v} &\in \mathbf{L}^\infty(0, T; \mathbf{H}^2(\Omega)), & \mathbf{C} &\in \mathbf{L}^\infty(0, T; \mathbf{H}^2(\Omega)). \end{aligned}$$

The above regularity results finally allow to show the uniqueness of this more regular solution, cf. Theorem 3. The analytical results on global in time existence and uniqueness of (more regular) weak solution is an important prerequisite in order to study convergence and error estimates of a suitable finite element approximation of the diffusive Peterlin model. This task is our future goal. Another interesting question is the generalization of analytical results presented in this paper for a general constitutive model (2).

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