Global existence result for the generalized Peterlin viscoelastic model

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Abstract

We consider a class of differential models of viscoelastic fluids with diffusive stress. These constitutive models are motivated by Peterlin dumbbell theories with a nonlinear spring law for an infinitely extensible spring. A diffusion term is included in the constitutive model. Under appropriate assumptions on the nonlinear constitutive functions, we prove global existence of weak solutions for large data. For creeping flows and two-dimensional flows, we prove global existence of a classical solution under stronger assumptions.

1 Introduction

Modelling of polymeric fluids is a very challenging problem. They can be described by long chain molecules, represented in simplified models as chains of beads and springs or beads and rods, surrounded by a Newtonian fluid. Hereby, the spring forces, stochastic forces and forces exerted by the surrounding fluid are responsible for the movement of molecules. There are basically three different approaches how to model the environment with which a polymer molecule interacts: dilute theories, network theories and reptation theories. The simplest model representing the dilute solution theories is the so-called dumbbell model consisting of two beads connected by a spring. Considering the linear force law for the spring force: $\mathbf{F}(\mathbf{R}) = H\mathbf{R}$, where $\mathbf{R}$ is the vector connecting the beads, we obtain the upper convected Maxwell model, cf. [31]. The well-known *Oldroyd-B* model has the stress that is a linear superposition of the upper convected Maxwell model and the Newtonian model. For the nonlinear force, $\mathbf{F}(\mathbf{R}) = \gamma(|\mathbf{R}|^2)\mathbf{R}$, it is not possible to obtain a closed system of equations for the conformation tensor, except by approximating the force law. The Peterlin approximation replaces this law by $\mathbf{F}(\mathbf{R}) = \gamma(\langle|\mathbf{R}|^2\rangle)\mathbf{R}$. That means, the length of the spring in the spring function γ is replaced by the length of the average spring $\langle|\mathbf{R}|^2\rangle = \mathrm{tr}\ \mathbf{C}$. Consequently, we can derive the evolution equation for the conformation tensor $\mathbf{C}$, which is in a closed form, see [31].

In standard derivations of bead-spring models the diffusive term in the equation for the elastic stress tensor is routinely omitted, and it is generally believed to be very small.

However, as pointed out in [2, 8, 33] there is indeed a physical rationale for this diffusive term, which appears in the Fokker-Planck equation and, consequently, also in the corresponding macroscopic equation for the elastic stress.

Mathematical literature dealing with the analysis of micro-macro viscoelastic models is growing quite rapidly, see, e.g., [3, 4, 6–8, 13–18, 25, 27, 28, 30] and the references therein. In the following, we focus in particular on the state of the art regarding the global well-posedness of initial value problems.

Concerning existence results local in time and global in time for small data let us mention the classical results of Fernández-Cara, Guillén and Ortega [10], Guillopé and Saut [12], Engler [9]. Recently, Geissert et al. [11] proved results of this kind for models covering a wide range of nonlinear fluids including generalized Newtonian fluids, generalized Oldroyd-B fluids or Peterlin dumbbell models.

The global existence result for fully two- and three-dimensional flow has been obtained by Lions and Masmoudi [19] for the case of the so-called corotational Oldroyd-B model, where the gradient of velocity $\nabla \mathbf{v}$ in the evolution equation for the elastic stress tensor is replaced by its anti-symmetric part $\frac{1}{2}(\nabla \mathbf{v} - \nabla \mathbf{v}^T)$. Unfortunately, the proof cannot be extended easily to other Oldroyd-type fluids since a specific structure of corotational model has been used here. Another classical model, the FENE (finitely extensible nonlinear elastic) model, assumes that the interaction potential can be infinite at finite extension length. Taking into account the Peterlin approximation we can close the microscopic model and arrive at the macroscopic FENE-P model. Recently, Masmoudi [26] has proved global existence of weak solutions for the FENE-P model. The proof is based on the propagation of some defect measures that control the lack of strong convergence in an approximating sequence. Masmoudi also considers the PTT and Giesekus models.

A global existence result for one-dimensional shear flows of a class of differential models of viscoelastic fluids with retardation time can be found in Renardy [32].

In the recent work [1] Barrett and Boyaval studied the diffusive Oldroyd-B model both from the numerical as well as the analytical point of view. For two space dimensions they were able to prove the global existence of weak solutions. Constantin and Kliegl [5] establish global regularity in two space dimensions for the diffusive Odroyd-B model.

For finitely extensible dumbbell models having a diffusive term the global existence of weak solutions has been proved by Barrett and Süli in [2].

The main aim of the present paper is to study a model for complex viscoelastic fluids, where the Peterlin approximation is used in order to derive the evolution equation for the elastic conformation tensor. Note that we also allow diffusive effects for the evolution of elastic stress. This paper is a generalization of our recent results [20], where polynomial (linear or quadratic) functions ψ, χ, ϕ have been considered in the definition of the elastic stress tensor and in the evolution equation of the conformation tensor. Numerical approximation of this model by means of the stabilized Lagrange-Galerkin method has been analysed in our recent papers [21, 22]. Let us note that in [20] we have shown that for more regular initial data $\mathbf{v}_0 \in H^2(\Omega) \cap H^1_0(\Omega)$, $\operatorname{div} \mathbf{v}_0 = 0$, $\mathbf{C}_0 \in H^2(\Omega)$, $\Omega \subset \mathbb{R}^2$ there is a unique weak solution that is more regular; in particular $\mathbf{v} \in L^\infty(0, T; H^2(\Omega))$, $\mathbf{C} \in L^\infty(0, T; H^2(\Omega))$. In the present paper we will extend the results from [20] and allow more general behavior of nonlinear functions ψ, χ, ϕ arising in the viscoelastic constitutive law. For a new general model we will be now able to show the existence of a weak solution in two or three space dimensions, such that $\mathbf{v} \in L^\infty(0,T;H) \cap L^2(0,T;H_0^1(\Omega))$, div $\mathbf{v} = 0$, $\mathbf{C} \in L^p(\Omega \times (0,T)) \cap L^{1+\delta}(0,T;W^{1,1+\delta})$, for some $\delta > 0$.

Furthermore, in the case of three-dimensional creeping flow or for two-dimensional flows we will show the existence of a global classical solution under stronger assumptions.

The paper is organized in the following way. In the next section we present a mathematical model for our complex viscoelastic fluid. Further, in Section 3 we show formal energy estimates. Section 4 is devoted to the proof of global existence. We solve our problem by combining the Galerkin approximation in velocity with the theory of quasi-linear parabolic equations for the stress tensor. The existence of a global classical solution is studied in Section 5.

2 Governing equations

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded smooth domain and let $T > 0$. We consider the equation of motion of an incompressible viscoelastic fluid,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \nu \Delta \mathbf{v} + \operatorname{div} \mathbf{T} - \nabla p, \tag{1a}$$

$$\operatorname{div} \mathbf{v} = 0, \tag{1b}$$

on $\Omega \times (0,T)$. Here $\mathbf{v}(x,t) \in \mathbb{R}^d$ and $p(x,t) \in \mathbb{R}$ denote, for all $(x,t) \in \Omega \times (0,T)$, the velocity of the fluid and the pressure, respectively. The elastic stress tensor $\mathbf{T}$ is related to the conformation tensor $\mathbf{C}$ in the following way:

$$\mathbf{T} = \psi(\operatorname{tr} \mathbf{C})\, \mathbf{C}, \tag{1c}$$

where $\mathbf{C}(x,t) \in \mathbb{R}^{d\times d}$ is a symmetric positive definite tensor for all $(x,t) \in \Omega \times (0,T)$ and satisfies an equation of the form

$$\frac{\partial \mathbf{C}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{C} - (\nabla \mathbf{v})\mathbf{C} - \mathbf{C}(\nabla \mathbf{v})^T = \chi(\operatorname{tr} \mathbf{C})\, \mathbf{I} - \phi(\operatorname{tr} \mathbf{C})\mathbf{C} + \varepsilon \Delta \mathbf{C}. \tag{1d}$$

We prescribe the homogeneous Dirichlet boundary condition on $\mathbf{v}$ and the no-flux boundary condition on $\mathbf{C}$, i.e.

$$\left(\mathbf{v}, \frac{\partial \mathbf{C}}{\partial \mathbf{n}}\right) = (\mathbf{0}, \mathbf{0})$$

on $\partial\Omega \times (0,T)$. We impose the initial condition

$$(\mathbf{v}(0), \mathbf{C}(0)) = (\mathbf{v}_0, \mathbf{C}_0) \tag{1e}$$

on Ω for sufficiently smooth initial data $(\mathbf{v}_0, \mathbf{C}_0)$. The given constants ν and ε describe the fluid viscosity and elastic stress diffusivity, respectively.

3 Assumptions and a priori bounds

We assume that ψ, χ and ϕ are continuous positive functions defined on $[0,\infty)$ and ψ is moreover continuously differentiable and non-decreasing. Further, we suppose that for some positive constants A_i, B_i, C_i, $i = 1,2$ the following polynomial growth conditions are satisfied for large s:

$$A_1 s^\alpha \le \phi(s) \le A_2 s^\alpha, \qquad B_1 s^\beta \le \psi(s) \le B_2 s^\beta, \quad C_1 s^\gamma \le \chi(s) \le C_2 s^\gamma, \tag{2a}$$

where

$$\alpha+\beta+1 > 2,\ \alpha > 0,\ \beta \ge 0,\ \text{and } \gamma < \alpha+1 \text{ or } \gamma = \alpha+1 \text{ with } dB_2C_2 < A_1B_1. \tag{2b}$$

Recall that d is the space dimension.

Remark 1. *The growth conditions* (2a) *for sufficiently large $s >> 1$ hold, in particular, if*

$$\lim_{s\to\infty} \frac{\phi(s)}{s^\alpha} = A, \qquad \lim_{s\to\infty} \frac{\psi(s)}{s^\beta} = B, \qquad \lim_{s\to\infty} \frac{\chi(s)}{s^\gamma} = C$$

for some positive constants A, B, C.

In what follows we shall use the following notation

$$\begin{aligned}
V &:= \{\mathbf{v} \in H_0^1(\Omega)^d |\ \mathrm{div}\ \mathbf{v} = 0\},\ \text{equipped with the norm } \|\mathbf{v}\| := \|\nabla \mathbf{v}\|_{L^2(\Omega)},\\
H &:= \{\mathbf{v} \in L^2(\Omega)^d |\ \mathrm{div}\ \mathbf{v} = 0, \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\},\\
b\big(\mathbf{u},\mathbf{v},\mathbf{w}\big) &:= \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\,dx\,, \quad \mathbf{u},\mathbf{v},\mathbf{w} \in V,\\
\big((\mathbf{v},\mathbf{w})\big) &:= \int_\Omega \nabla\mathbf{v} : \nabla\mathbf{w}\,dx\,, \quad \mathbf{v},\mathbf{w} \in V,\\
B\big(\mathbf{v},\mathbf{C},\mathbf{D}\big) &:= \int_\Omega (\mathbf{v}\cdot\nabla)\mathbf{C} : \mathbf{D}\,dx\,, \quad \mathbf{v} \in V,\ \mathbf{C},\mathbf{D} \in H^1(\Omega)^{d\times d},\\
\big((\mathbf{C},\mathbf{D})\big) &:= \int_\Omega \nabla\mathbf{C} : \nabla\mathbf{D}\,dx\,, \quad \mathbf{C},\mathbf{D} \in H^1(\Omega)^{d\times d}.
\end{aligned}$$

Analogously as in [34] one can easily show that

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = -b(\mathbf{u},\mathbf{w},\mathbf{v}) \quad \mathbf{u},\mathbf{v},\mathbf{w} \in V \tag{3a}$$

$$B(\mathbf{v},\mathbf{C},\mathbf{D}) = -B(\mathbf{v},\mathbf{D},\mathbf{C}) \quad \mathbf{v} \in V,\ \mathbf{C},\mathbf{D} \in H^1(\Omega)^{d\times d}. \tag{3b}$$

3.1 Formal energy estimates

We proceed with the formal energy estimates for our model. We multiply the momentum equation (1a) by $\mathbf{v}$ and integrate using the Gauss theorem. The solenoidality of velocity and the boundary conditions yield the following equality

$$\frac{1}{2}\int_\Omega |\mathbf{v}|^2\,dx - \frac{1}{2}\int_\Omega |\mathbf{v}_0|^2\,dx = -\nu\int_0^t\int_\Omega |\nabla\mathbf{v}|^2\,dx\,dt - \int_0^t\int_\Omega \mathbf{T} : \nabla\mathbf{v}\,dx\,dt\,. \tag{4}$$

Now, we multiply (1d) by $\psi(\mathrm{tr}\,\mathbf{C})$, take half the trace and integrate this equation using the Gauss theorem. Again, by the divergence freedom of velocity and the boundary conditions, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \Psi(\mathrm{tr}\,\mathbf{C})\,dx + \frac{\varepsilon}{2}\int_0^t\int_\Omega \psi'(\mathrm{tr}\,\mathbf{C})|\nabla \mathrm{tr}\,\mathbf{C}|^2\,dx\,dt + \\
&+\frac{1}{2}\int_0^t\int_\Omega \phi(\mathrm{tr}\,\mathbf{C})\psi(\mathrm{tr}\,\mathbf{C})\mathrm{tr}\,\mathbf{C}\,dx\,dt - \int_0^t\int_\Omega (\nabla\mathbf{v}):\mathbf{C}\,\psi(\mathrm{tr}\,\mathbf{C})\,dx\,dt = \\
&=\frac{1}{2}\int_\Omega \Psi(\mathrm{tr}\,\mathbf{C}_0)\,dx + \frac{d}{2}\int_0^t\int_\Omega \chi(\mathrm{tr}\,\mathbf{C})\psi(\mathrm{tr}\,\mathbf{C})\,dx\,dt\,,
\end{aligned} \tag{5}$$

where Ψ denotes the primitive function of ψ. Thus, the sum of equations (4) and (5) yields the following energy equality

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\mathbf{v}|^2\,dx + \frac{1}{2}\int_\Omega \Psi(\mathrm{tr}\,\mathbf{C})\,dx + \nu\int_0^t\int_\Omega |\nabla\mathbf{v}|^2\,dx\,dt + \\
&+\frac{\varepsilon}{2}\int_0^t\int_\Omega \psi'(\mathrm{tr}\,\mathbf{C})|\nabla \mathrm{tr}\,\mathbf{C}|^2\,dx\,dt + \frac{1}{2}\int_0^t\int_\Omega \phi(\mathrm{tr}\,\mathbf{C})\psi(\mathrm{tr}\,\mathbf{C})\mathrm{tr}\,\mathbf{C}\,dx\,dt = \\
&=\frac{1}{2}\int_\Omega |\mathbf{v}_0|^2\,dx + \frac{1}{2}\int_\Omega \Psi(\mathrm{tr}\,\mathbf{C}_0)\,dx + \frac{d}{2}\int_0^t\int_\Omega \chi(\mathrm{tr}\,\mathbf{C})\psi(\mathrm{tr}\,\mathbf{C})\,dx\,dt\,.
\end{aligned} \tag{6}$$

Let us note that we have used the property of the trace of the product of square matrices that $\mathrm{tr}\,\mathbf{A}\mathbf{B}^T = \mathrm{tr}\,\mathbf{B}\mathbf{A}^T = \mathbf{A}:\mathbf{B}$ and consequently the identity

$$\int_0^t\int_\Omega \mathbf{T}:\nabla\mathbf{v}\,dx\,dt - \int_0^t\int_\Omega (\nabla\mathbf{v}):\mathbf{C}\,\psi(\mathrm{tr}\,\mathbf{C})\,dx\,dt = 0.$$

Employing the assumptions (2a) we get

$$\frac{1}{2}\int_0^t\int_\Omega \phi(\mathrm{tr}\,\mathbf{C})\psi(\mathrm{tr}\,\mathbf{C})\mathrm{tr}\,\mathbf{C}\,dx\,dt \geq \frac{A_1B_1}{2}\int_0^t\int_\Omega (\mathrm{tr}\,\mathbf{C})^{\alpha+\beta+1}\,dx\,dt$$

and

$$\frac{d}{2}\int_0^t\int_\Omega \chi(\mathrm{tr}\,\mathbf{C})\psi(\mathrm{tr}\,\mathbf{C})\,dx\,dt \leq \frac{dB_2C_2}{2}\int_0^t\int_\Omega (\mathrm{tr}\,\mathbf{C})^{\beta+\gamma}\,dx\,dt\,.$$

Hence, equality (6) together with (2b) implies

$$\mathrm{tr}\,\mathbf{C} \in L^p(\Omega\times(0,T)) \text{ for } p := \alpha+\beta+1 > 2 \tag{7a}$$

and

$$\mathbf{v} \in L^\infty(0,T;H)\cap L^2(0,T;V). \tag{7b}$$

It can be proved that (1d) preserves the positive definiteness of the conformation tensor at least for enough smooth $\mathbf{C}$, cf. [29] and the references therein. For a symmetric positive definite matrix $\mathbf{D}$ there exists an equivalent norm given by its trace, i.e. for some positive constants c_1, c_2 it holds that

$$c_1\|\mathrm{tr}\,\mathbf{C}\|_{L^p(\Omega\times(0,T))} \leq \|\mathbf{C}\|_{L^p(\Omega\times(0,T))} \leq c_2\|\mathrm{tr}\,\mathbf{C}\|_{L^p(\Omega\times(0,T))}. \tag{8}$$

See, e.g., [29] for the details. Therefore (7a) implies $\mathbf{C} \in L^p(\Omega \times (0,T))$. Consequently, we get bounds on $\phi(\text{tr } \mathbf{C})\mathbf{C}$, $\psi(\text{tr } \mathbf{C})\mathbf{C}$ and $\chi(\text{tr } \mathbf{C})$ in $L^q(\Omega \times (0,T))$ for

$$q = \min \left\{ \frac{p}{\alpha+1}, \frac{p}{\beta+1}, \frac{p}{\gamma} \right\}. \tag{9}$$

If $\beta > 0$, our assumptions imply that $q > 1$.

However, if $\beta = 0$ we only have $p = \alpha + 1 > 2$ (which implies $\alpha > 1$), and hence q as defined above equals 1, which is not enough for our further needs. Therefore we proceed with the second a priori estimate for the conformation tensor. We multiply equation (1d) by $(\text{tr } \mathbf{C})^{\alpha-1}$. Taking the trace of the resulting equation and integrating it using the Gauss theorem yields

$$\begin{aligned} \frac{1}{\alpha} \int_\Omega (\text{tr } \mathbf{C})^\alpha \, dx \, + \varepsilon(\alpha-1) \int_0^t \int_\Omega (\text{tr } \mathbf{C})^{\alpha-2} |\nabla \text{tr } \mathbf{C}|^2 \, dx \, dt \, + \\ - \int_0^t \int_\Omega (\nabla \mathbf{v}) : \mathbf{C} \, (\text{tr } \mathbf{C})^{\alpha-1} \, dx \, dt \, + \int_0^t \int_\Omega \phi(\text{tr } \mathbf{C})(\text{tr } \mathbf{C})^\alpha \, dx \, dt \, = \\ = \frac{1}{\alpha} \int_\Omega (\text{tr } \mathbf{C}_0)^\alpha \, dx \, + d \int_0^t \int_\Omega \chi(\text{tr } \mathbf{C})(\text{tr } \mathbf{C})^{\alpha-1} \, dx \, dt \, . \end{aligned} \tag{10}$$

Employing assumptions (2) we can write

$$\int_0^t \int_\Omega \phi(\text{tr } \mathbf{C})(\text{tr } \mathbf{C})^\alpha \, dx \, dt \, \geq A_1 \int_0^t \int_\Omega (\text{tr } \mathbf{C})^{2\alpha} \, dx \, dt \, = A_1 \|\text{tr } \mathbf{C}\|^{2\alpha}_{L^{2\alpha}(\Omega \times (0,T))}$$

and, by the Hölder and the Young inequalities, we have

$$\begin{aligned} d \int_0^t \int_\Omega \chi(\text{tr } \mathbf{C})(\text{tr } \mathbf{C})^{\alpha-1} \, dx \, dt \, &\leq dC_2 \int_0^t \int_\Omega (\text{tr } \mathbf{C})^{\alpha+\gamma-1} \, dx \, dt \\ &\leq dC_2 |\Omega|^{(\alpha+1-\gamma)/(2\alpha)} \int_0^T \|\text{tr } \mathbf{C}\|^{\alpha+\gamma-1}_{L^{2\alpha}} \, dt \, . \end{aligned}$$

For the following, let $\|\cdot\|_q$ abbreviates the notation of the norm in $L^q(\Omega)$. If $\gamma < \alpha + 1$, we can estimate that right hand side by

$$\epsilon_1 \|\text{tr } \mathbf{C}\|^{2\alpha}_{L^{2\alpha}(\Omega \times (0,T))} + c(T, |\Omega|, 1/\epsilon_1)$$

for sufficiently small $\epsilon_1 > 0$. If, on the other hand $\gamma = \alpha + 1$, we note that, by our assumptions, $A_1 > dC_2$. Similarly, using the first a priori estimates for velocity (7b), we get

$$\begin{aligned} \int_0^t \int_\Omega (\nabla \mathbf{v}) : \mathbf{C} \, (\text{tr } \mathbf{C})^{\alpha-1} \, dx \, dt \, &\leq c \int_0^T \|\nabla \mathbf{v}\|_{L^2} \|\text{tr } \mathbf{C}\|^\alpha_{L^{2\alpha}} \, dt \\ &\leq \epsilon_2 \|\text{tr } \mathbf{C}\|^{2\alpha}_{L^{2\alpha}(\Omega \times (0,T))} + \frac{c}{\epsilon_2} \|\nabla \mathbf{v}\|^2_{L^2(\Omega \times (0,T))}, \end{aligned}$$

for sufficiently small $\epsilon_2 > 0$. Thus, the energy equality (10) yields $\text{tr } \mathbf{C} \in L^{2\alpha}(\Omega \times (0,T))$ and the norm equivalence (8) gives us also $\mathbf{C} \in L^{2\alpha}(\Omega \times (0,T))$, $\alpha > 1$. From this, we can now also conclude that $\phi(\text{tr } \mathbf{C})\mathbf{C}$, $\psi(\text{tr } \mathbf{C})\mathbf{C}$ and $\chi(\text{tr } \mathbf{C})$ are in $L^q(\Omega \times (0,T))$ for a $q > 1$.

4 Main result

In what follows we explain in which sense a weak solution to the generalized Peterlin model (1) will be considered.

Definition 1. *(weak solution)*
Let the initial data $(\mathbf{v}_0, \mathbf{C}_0) \in H \times L^2(\Omega)^{d\times d}$ *and let the couple* $(\mathbf{v}, \mathbf{C})$ *be such that*

$$\mathbf{v} \in L^\infty(0,T;H) \cap L^2(0,T;V), \qquad \frac{\partial \mathbf{v}}{\partial t} \in L^q(0,T;W^{-3,2}(\Omega)),$$
$$\mathbf{C} \in L^p(\Omega\times(0,T)) \cap L^{1+\delta}(0,T;W^{1,1+\delta}), \qquad \frac{\partial \mathbf{C}}{\partial t} \in L^{1+\delta}(0,T;W^{-1,1+\delta}(\Omega))$$

for some $q > 1$, $p > 2$ *and* $0 < \delta << 1$. *Then* $(\mathbf{v}, \mathbf{C})$ *is a weak solution to the generalized Peterlin model* (1) *if it satisfies*

$$\int_\Omega \frac{\partial \mathbf{v}}{\partial t}\cdot \mathbf{w}\,dx + \int_\Omega (\mathbf{v}\cdot\nabla)\,\mathbf{v}\cdot\mathbf{w}\,dx + \nu\int_\Omega \nabla\mathbf{v} : \nabla\mathbf{w}\,dx = -\int_\Omega \psi(\mathrm{tr}\ \mathbf{C})\mathbf{C} : \nabla\mathbf{w}\,dx$$
$$\int_\Omega \frac{\partial \mathbf{C}}{\partial t} : \mathbf{D}\,dx + \int_\Omega (\mathbf{v}\cdot\nabla)\,\mathbf{C} : \mathbf{D}\,dx - \int_\Omega \left((\nabla\mathbf{v})\mathbf{C} + \mathbf{C}(\nabla\mathbf{v})^T\right) : \mathbf{D}\,dx +$$
$$+\varepsilon\int_\Omega \nabla\mathbf{C} : \nabla\mathbf{D}\,dx = \int_\Omega \chi(\mathrm{tr}\ \mathbf{C})\mathbf{I} : \mathbf{D}\,dx - \int_\Omega \phi(\mathrm{tr}\ \mathbf{C})\mathbf{C} : \mathbf{D}\,dx$$

for all smooth functions $(\mathbf{w}, \mathbf{D})$ *such that* $\mathbf{w}$ *is divergence free and zero on the boundary, and* $(\mathbf{v}(0), \mathbf{C}(0)) = (\mathbf{v}_0, \mathbf{C}_0)$.

Theorem 1. *(existence of weak solutions)*
There exists a weak solution to model (1) *in the sense of Definition 1.*

In what follows we present the proof of the existence of a weak solution to the generalized Peterlin model. To this end we introduce an approximation scheme, find energy estimates and, based on a compactness argument, we pass to the limit with the approximate solution.

4.1 Preliminaries

First of all let us list some preliminary results that shall be used later in the proof of existence of a weak solution. In order to pass to the limit in some nonlinear terms we shall employ the following consequence of Vitali's convergence theorem, see e.g., [24, 29].

Lemma 1.
Let $M \subset \mathbb{R}^n$ *be measurable and bounded. Let the sequence* $\{g_m\}_{m=1}^\infty$ *be uniformly bounded in* $L^q(M)$ *for a* $q > 1$. *Finally, let* $g_m \to g$ *a.e. in* M *for some* $g \in L^q(M)$. *Then*

$$\int_M g_m \to \int_M g.$$

The interpolation inequalities in the Sobolev spaces, see e.g., [23], are useful to ensure the strong convergence of a sequence of approximations in an appropriate functional space, cf. (17).

Lemma 2. *(interpolation inequality)*
Let $g \in W^{k,r_1}(\Omega)$. Then, for $r_1 \geq r \geq r_2$ it holds that

$$\|g\|_{W^{k,r}} \leq c\|g\|^{\alpha}_{W^{k,r_1}}\|g\|^{1-\alpha}_{W^{k,r_2}}, \tag{11}$$

where $\alpha \in (0,1)$ satisfies the equality

$$\frac{1}{r} = \frac{\alpha}{r_1} + \frac{1-\alpha}{r_2}.$$

4.2 Approximation scheme

The aim of this subsection is to define an approximation scheme and to find uniform a priori bounds following the formal energy estimates from Subsection 3.1.

Let $\{\mathbf{w}_i\}_{i=1}^{\infty}$ denote the orthonormal countable base of space V, i.e.

$$V = \overline{\text{span}\{\mathbf{w}_i\}_{i=1}^{\infty}}.$$

The m-th approximate solution $(\mathbf{v}_m, \mathbf{C}_m)$ satisfies

$$\begin{aligned}
&\mathbf{v}_m(t) = \sum_{i=1}^{m} g_{im}(t)\mathbf{w}_i,\\
&\big(\mathbf{v}'_m(t), \mathbf{w}_j\big) + b\big(\mathbf{v}_m(t), \mathbf{v}_m(t), \mathbf{w}_j\big) + \nu\big(\big(\mathbf{v}_m(t), \mathbf{w}_j\big)\big) = -\big(\text{tr } \mathbf{C}_m(t)\, \mathbf{C}_m(t), \nabla \mathbf{w}_j\big),\\
&\mathbf{v}_m(0) = \mathbf{v}_{0m},
\end{aligned} \tag{12a}$$

$$\frac{\partial \mathbf{C}_m}{\partial t} + (\mathbf{v}_m \cdot \nabla)\mathbf{C}_m - (\nabla \mathbf{v}_m)\mathbf{C}_m - \mathbf{C}_m(\nabla \mathbf{v}_m)^T = \chi(\text{tr } \mathbf{C}_m)\,\mathbf{I} - \phi(\text{tr } \mathbf{C}_m)\mathbf{C}_m + \varepsilon\Delta\mathbf{C}_m, \tag{12b}$$

$$\mathbf{C}_m(0) = \mathbf{C}_0$$

$$\text{for } j = 1, \ldots, m,\ t \in [0,T].$$

Obviously, $\mathbf{v}_m$ is the Galerkin approximation of the velocity. The function $\mathbf{v}_{0m}$ is the orthogonal projection in H of $\mathbf{v}_0$ on the space spanned by $\mathbf{w}_j$. For any m there exists a maximal solution $\mathbf{v}_m$ defined in the interval $[0, t_m]$, since (12a) can be rewritten as a nonlinear system of differential equations equipped with the initial conditions. Uniform a priori bounds, see (13) below, imply $t_m = T$ for all m. Due to parabolic regularity and a priori bounds on finitely dimensional velocity (13) there exists an approximation of the conformation tensor $\mathbf{C}_m = \mathbf{C}(\mathbf{v}_m)$ satisfying (12b) such that $\mathbf{C}_m \in C^1([0,T], C^2(\Omega))$. It can be shown that the positive definiteness of the conformation tensor is preserved for a smooth solution, see, e.g., [5, 29].

To find a priori bounds for $(\mathbf{v}_m, \mathbf{C}_m)$ we repeat the formal energy estimates. Thus, we multiply (12a) by g_{jm} and take the sum for $j = 1, \ldots, m$. Further, we multiply equation (12b) by $\psi(\text{tr } \mathbf{C}_m)$, take half the trace and integrate. Adding up the resulting equations we obtain the energy equality that indicates the appropriate functional spaces for the approximate solution, i.e.

$$\begin{aligned}
\mathbf{v}_m &\in L^{\infty}(0,T;H) \cap L^2(0,T;V),\\
\text{tr } \mathbf{C}_m &\in L^p(\Omega \times (0,T)),
\end{aligned} \tag{13a}$$

and by (8) we also obtain $\mathbf{C}_m \in L^p(\Omega \times (0,T))$, $p = \alpha + \beta + 1 > 2$. Consequently, by (2) there exists $q > 1$ of (9) such that

$$\phi(\text{tr } \mathbf{C}_m)\mathbf{C}_m,\ \psi(\text{tr } \mathbf{C}_m)\mathbf{C}_m,\ \chi(\text{tr } \mathbf{C}_m) \in L^q(\Omega \times (0,T)). \tag{13b}$$

Moreover, for $\beta = 0$ we are able to show that $\mathbf{C}_m \in L^{2\alpha}(\Omega \times (0,T))$ for $\alpha > 1$.

4.3 Compact imbeddings

In order to pass to the limit as m goes to infinity in our approximation scheme (12) we need compact imbeddings. To this end, let $-\mathcal{A}$ denote the Stokes operator, let X^k denote the domain of $(\mathcal{A})^{k/2}$ and let X^{-k} denote its dual. Note that $V = X^1$ and $H = X^0$. We rewrite (12a) in the operator form

$$\mathbf{v}_m' = -\mathcal{A}\mathbf{v}_m + \mathcal{B}\mathbf{v}_m + \mathcal{E}\mathbf{C}_m,$$

where the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{E}$ are defined as

$$\begin{aligned}
&\mathcal{A} : V \to X^{-3} && \langle \mathcal{A}\mathbf{v}, \mathbf{w}\rangle := ((\mathbf{v}, \mathbf{w}))\,,\ \mathbf{w} \in X^3,\\
&\mathcal{B} : V \to X^{-3} && \langle \mathcal{B}\mathbf{v}, \mathbf{w}\rangle := -b\,(\mathbf{v}, \mathbf{v}, \mathbf{w})\,,\ \mathbf{w} \in X^3,\\
&\mathcal{E} : L^p(\Omega)^{d\times d} \to X^{-3} && \langle \mathcal{E}\mathbf{C}, \mathbf{v}\rangle := -\left(\psi(\text{tr } \mathbf{C})\mathbf{C}, \nabla\mathbf{w}\right),\ \mathbf{w} \in X^3.
\end{aligned}$$

From the theory of the Navier-Stokes equations, cf. e.g., [29, 34] it follows immediately that $\mathcal{A}\mathbf{v} \in L^2(0,T;X^{-3})$ and $\mathcal{B}\mathbf{v} \in L^2(0,T;X^{-3})$. Further, the Sobolev imbeddings theorem yields the continuous imbedding of $W^{2,2}(\Omega)$ into $C(\Omega)$ for both $d = 2, 3$, see e.g., [34]. Thus, we get

$$\langle \mathcal{E}\mathbf{C}, \mathbf{w}\rangle \le \|\nabla\mathbf{w}\|_C \|\psi(\text{tr } \mathbf{C})\mathbf{C}\|_{L^1} \le \|\mathbf{w}\|_{X^3} \|\psi(\text{tr } \mathbf{C})\mathbf{C}\|_1\|_{L^1},$$

$$\int_0^T \|\mathcal{E}\mathbf{C}\|_{X^{-3}}^q\, dt \ \le c \int_0^T \int_\Omega |\psi(\text{tr } \mathbf{C})\mathbf{C}|^q \ dx\, dt\,.$$

This yields $\mathcal{E}\mathbf{C} \in L^q(0,T;X^{-3})$, cf. (13b). We conclude $\mathbf{v}_m' \in L^q(0,T;X^{-3})$. The Lions-Aubin lemma and (13a) directly implies that the sequence $\{\mathbf{v}_m\}_{m=1}^\infty$ is compactly embedded into $L^2(0,T;L^2(\Omega))$, since $V \hookrightarrow\hookrightarrow H \hookrightarrow X^{-3}$.

Let us now rewrite equation (12b) in the following form

$$\frac{\partial \mathbf{C}_m}{\partial t} - \varepsilon\Delta\mathbf{C}_m = F_m,$$

where

$$F_m := -(\mathbf{v}_m \cdot \nabla)\mathbf{C}_m + (\nabla\mathbf{v}_m)\mathbf{C}_m + \mathbf{C}_m(\nabla\mathbf{v}_m)^T + \chi(\text{tr } \mathbf{C}_m)\mathbf{I} - \phi(\text{tr } \mathbf{C}_m)\mathbf{C}_m.$$

Employing a priori bounds (13) we are able to show that $F_m \in L^{1+\delta}(0,T;W^{-1,1+\delta}(\Omega))$ for a sufficiently small $\delta > 0$. Then, using the bootstrapping algorithm and the standard parabolic estimates, we obtain

$$\frac{\partial \mathbf{C}_m}{\partial t} \in L^{1+\delta}(0,T;W^{-1,1+\delta}(\Omega)), \qquad \mathbf{C}_m \in L^{1+\delta}(0,T;W^{1,1+\delta}(\Omega)).$$

This suffices for compact imbedding of $\{\mathbf{C}_m\}_{m=1}^\infty$ into $L^{1+\delta}(\Omega \times (0,T))$. Indeed, since

$$W^{1,1+\delta}(\Omega) \hookrightarrow\hookrightarrow L^{1+\delta} \hookrightarrow W^{-1,1+\delta}(\Omega),$$

the Lions-Aubin lemma yields the desired imbedding. This is good enough to pass to the limit in all terms which appear in the equations satisfied by the approximate solution.

4.4 Passage to the limit

A priori estimates (13) and the compact imbeddings of $\{\mathbf{v}_m\}_{m=1}^{\infty}$ into $L^2(\Omega\times(0,T))$ and $\{\mathbf{C}_m\}_{m=1}^{\infty}$ into $L^{1+\delta}(\Omega\times(0,T))$ imply the *-weak, weak and strong convergences. More precisely,

$$
\begin{aligned}
&\mathbf{v}_m \rightharpoonup^* \mathbf{v} && \text{in } L^\infty(0,T;H), && \text{(14a)}\\
&\mathbf{v}_m \rightharpoonup \mathbf{v} && \text{in } L^2(0,T;V), && \text{(14b)}\\
&\mathbf{v}_m \to \mathbf{v} && \text{in } L^2(\Omega\times(0,T)), && \text{(14c)}\\
&\mathbf{C}_m \rightharpoonup \mathbf{C} && \text{in } L^p(\Omega\times(0,T)), && \text{(14d)}\\
&\mathbf{C}_m \rightharpoonup \mathbf{C} && \text{in } L^{1+\delta}(0,T;W^{1,1+\delta}(\Omega)), && \text{(14e)}\\
&\mathbf{C}_m \to \mathbf{C} && \text{in } L^{1+\delta}(\Omega\times(0,T)). && \text{(14f)}
\end{aligned}
$$

We can now pass to the limit in the approximation scheme (12) letting $m\to\infty$. Let us multiply (12a) and (12b) by $\varphi(t)$ such that $\varphi\in C^1([0,T])$, $\varphi(T)=0$ and integrate by parts over $[0,T]$.

We first pass to the limit in the equation for the velocity. Let us only concentrate on the nonlinear term arising due to the divergence of the elastic stress tensor, since the limiting process in (12a) is then straightforward due to (14a) - (14c). In order to pass to the limit in the term

$$\int_0^T\int_\Omega \psi(\mathrm{tr}\,\mathbf{C}_m)\mathbf{C}_m : \nabla\mathbf{w}\,\varphi(t)\;dx\,dt \tag{15}$$

we employ the consequence of Vitali's convergence theorem, see Lemma 1. There exists $l>1$ such that $g_m := \psi(\mathrm{tr}\,\mathbf{C}_m)\mathbf{C}_m : \nabla\mathbf{w}\,\varphi(t)$ is uniformly bounded in $L^l(\Omega\times(0,T))$. Indeed,

$$\int_0^T\int_\Omega |\psi(\mathrm{tr}\,\mathbf{C}_m)\mathbf{C}_m : \nabla\mathbf{w}\,\varphi(t)|^l\;dx\,dt\;\le \|\varphi\|_C^l\|\nabla\mathbf{w}\|_C^l\int_0^T|\psi(\mathrm{tr}\,\mathbf{C}_m)\mathbf{C}_m|^l\,dt\,.$$

It suffices to take $l=q$, cf. (9), and employ a priori bound (13b).

In what follows we only concentrate on the limiting process in the difficult terms in equation (12b). The limiting process in other terms can be easily done using (14d) - (14f). It holds that

$$\big((\nabla\mathbf{v}_m)\mathbf{C}_m-(\nabla\mathbf{v})\mathbf{C}\big) = (\nabla\mathbf{v}_m-\nabla\mathbf{v})\,\mathbf{C}-(\nabla\mathbf{v}_m)\,(\mathbf{C}-\mathbf{C}_m)\,. \tag{16}$$

Suppose that $\mathbf{C}\mathbf{D}\in L^2(\Omega)$ (which holds for smooth $\mathbf{D}$). Then, by (14b) we get

$$\int_0^T\int_\Omega(\nabla\mathbf{v}_m-\nabla\mathbf{v})\,\mathbf{C}\mathbf{D}\varphi(t)\;dx\,dt\;\to 0 \text{ as } m\to\infty.$$

To pass to the limit in the second term on the righ-hand side of (16) we recall Lemma 2. By the interpolation inequality (11) for $k=0$, $r_1=p$, $r_2=1+\delta$ and $\alpha\in(0,1)$ satisfying

$$\frac{1}{r}=\frac{\alpha}{p}+\frac{1-\alpha}{1+\delta}$$

there exists r such that $p \geq r \geq 1+\delta$ and

$$\|\mathbf{C}-\mathbf{C}_m\|_{L^r} \leq c\|\mathbf{C}-\mathbf{C}_m\|_{L^p}^{\alpha}\|\mathbf{C}-\mathbf{C}_m\|_{L^{1+\delta}}^{1-\alpha}. \tag{17}$$

Since we know that $\mathbf{C}_m$ is bounded in $L^p(\Omega \times (0,T))$, $p > 2$ and (14f) holds, we can conclude that $\mathbf{C}_m$ converges to $\mathbf{C}$ strongly in $L^r(\Omega \times (0,T))$, $r > 2$. Hence, we write

$$\begin{aligned}\int_0^T \int_\Omega (\nabla \mathbf{v}_m)\,(\mathbf{C}-\mathbf{C}_m)\,\mathbf{D}\varphi(t)\; dx\, dt \; &\leq c\|\varphi\|_C \int_0^T \|\nabla \mathbf{v}_m\|_{L^2}\|\mathbf{C}-\mathbf{C}_m\|_{L^r}\, dt \\ &\leq c\|\varphi\|_C\|\nabla \mathbf{v}_m\|_{L^2(\Omega\times(0,T))}\|\mathbf{C}-\mathbf{C}_m\|_{L^r(\Omega\times(0,T))} \to 0\end{aligned}$$

as $m \to \infty$. Passing to the limit in the terms containing the functions ϕ and χ is done analogously as for the term (15), i.e. employing Lemma 1.

5 Strong solutions

More many applications of viscoelastic flows, the Reynolds number is very small. It is hence of interest to consider the case where inertial terms, or at least the inertial nonlinearity, can be neglected. We shall therefore consider the case where (1a) is simplified to

$$\frac{\partial \mathbf{v}}{\partial t} = \nu \Delta \mathbf{v} + \operatorname{div}\, \mathbf{T} - \nabla p. \tag{18}$$

Our goal in this section is the following theorem.

Theorem 2. *In addition to our assumptions above, assume that* $\alpha > \beta + 1$ *and that* $|\psi'(s)| \leq K s^{\beta-1}$ *for large* s*. We also assume that the functions* ϕ*,* ψ *and* χ *are smooth. If either the inertial nonlinearity* $(\mathbf{v}\cdot\nabla)\mathbf{v}$ *is neglected or the problem is two-dimensional, then there exists a global classical solution.*

We begin with the case where the inertial nonlinearity is neglected. Using maximal regularity for the Stokes system, we can obtain the estimate

$$\|\nabla \mathbf{v}\|_{L^{p/(\beta+1)}(\Omega\times(0,T))} \leq C(1+\|\mathbf{T}\|_{L^{p/(\beta+1)}(\Omega\times(0,T))}) \leq C(1+\|\mathbf{C}\|_{L^p(\Omega\times(0,T))}^{\beta+1}), \tag{19}$$

for any $p < \infty$. We now choose p large, multiply (1d) by $(\operatorname{tr}\mathbf{C})^{p-\alpha-1}$ and proceed as in (5). To estimate the term arising from the integral of $(\nabla \mathbf{v}) : \mathbf{C}\,(\operatorname{tr}\mathbf{C})^{p-\alpha-1}$, we use the bound

$$\|(\nabla \mathbf{v}) : \mathbf{C}(\operatorname{tr}\mathbf{C})^{p-\alpha-1}\|_{L^{p/(p+\beta-\alpha+1)}(\Omega\times(0,T))} \leq C\|\mathbf{C}\|_{L^p(\Omega\times(0,T))}^{p+\beta-\alpha+1}. \tag{20}$$

On the other hand, the integral of $\phi(\mathbf{C})\mathbf{C}(\operatorname{tr}\mathbf{C})^{p-\alpha-1}$ is bounded below by a constant times $\|\mathbf{C}\|_{L^p(\Omega\times(0,T))}^p$. In this fashion, we therefore obtain an a priori bound for $\|\mathbf{C}\|_{L^p(\Omega\times(0,T))}$. If we choose p large enough, we can use this as the starting point of a bootstrap argument to obtain higher regularity of the solution. We note that the energy estimate shown so far has been formal. However, for sufficiently regular initial data, existence of a smooth solution is guaranteed at least on some time interval, and while a smooth solution exists, our estimates hold. We can then use this as a basis for a continuation argument in the usual way, which guarantees global existence of a smooth solution.

In the two dimensional case, we can adapt this argument even if the inertial nonlinearity is included. To do this, we need to consider bounds for the term $(\mathbf{v}\cdot\nabla)\mathbf{v} = \mathrm{div}\,(\mathbf{v}\mathbf{v}^T)$. As a preparation for this, we need to obtain some additional a priori bounds. We revisit the equation

$$\frac{\partial \mathbf{C}}{\partial t} - \varepsilon\Delta\mathbf{C} = \\ -(\mathbf{v}\cdot\nabla)\mathbf{C} + (\nabla\mathbf{v})\mathbf{C} + \mathbf{C}(\nabla\mathbf{v})^T + \chi(\mathrm{tr}\ \mathbf{C})\mathbf{I} - \phi(\mathrm{tr}\ \mathbf{C})\mathbf{C}. \tag{21}$$

Note that we have a priori bounds for $\mathbf{v}$ in $L^\infty(0,T;L^2(\Omega))$ and for $\mathbf{C}$ in $L^{2\alpha}(\Omega\times(0,T))$. This yields a bound for the product $\mathbf{C}\mathbf{v}^T$ in $L^{2\alpha}(0,T;L^q(\Omega))$, where $q = 2\alpha/(1+\alpha) > 1$. It follows that $(\mathbf{v}\cdot\nabla)\mathbf{C} = \mathrm{div}(\mathbf{C}\mathbf{v}^T)$ is bounded in $L^{2\alpha}(0,T;W^{-1,s}(\Omega))$ for some $s > 2$. Moreover, the term $\phi(\mathrm{tr}\ \mathbf{C})\mathbf{C}$ is in $L^{(2\alpha)/(\alpha+1)}(\Omega\times(0,T))$, and the remaining terms are only better behaved. It follows that all terms are in $L^2(0,T;H^{-1}(\Omega))$, and consequently we obtain an a priori bound for $\nabla\mathbf{C}$ in $L^2(\Omega\times(0,T))$. This yields a bound for $\nabla\mathbf{T}$ in $L^q(\Omega\times(0,T))$ for some $q > 1$. Now we also note that $\mathbf{T}\in L^{2\alpha/(\beta+1)}(\Omega\times(0,T))$, and $2\alpha/(\beta+1) > 2$. Thus $\mathbf{T}\in L^{2\alpha/(\beta+1)}(0,T;L^2(\Omega))\cap L^1(0,T;W^{1,q}(\Omega))$ for some $q > 1$. We note that in two dimensions $W^{1,q}(\Omega)$ embeds into $H^\delta(\Omega)$ for some $\delta > 0$. By interpolation, we find that $\mathbf{T}\in L^2(0,T;H^\epsilon(\Omega))$ for some $\epsilon > 0$. Consequently, div $\mathbf{T}\in L^2(0,T;H^{-1+\epsilon}(\Omega))$.

We now consider the Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f}, \quad \mathrm{div}\ \mathbf{v} = 0, \tag{22}$$

with $\mathbf{f}\in L^2(0,T;H^{1-\epsilon}(\Omega))$. Our goal is to obtain an a priori bound on $\mathbf{v}\in L^2(0,T;H^\epsilon(\Omega))$. With $\mathcal{A}$ denoting the Stokes operator as above, we multiply by $\mathcal{A}^\epsilon\mathbf{v}$ and integrate. The result is the identity

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{A}^{\epsilon/2}\mathbf{v}\|_{L^2}^2 + \|\mathcal{A}^{(1+\epsilon)/2}\mathbf{v}\|_{L^2}^2 = (\mathbf{f},\mathcal{A}^\epsilon\mathbf{v}) - \int_\Omega((\mathbf{v}\cdot\nabla)\mathbf{v})\mathcal{A}^\epsilon\mathbf{v}\,d\mathbf{x}. \tag{23}$$

On the right hand side, we have

$$\begin{aligned} |(\mathbf{f},\mathcal{A}^\epsilon\mathbf{v})| &\le C\|f\|_{H^{-1+\epsilon}}\|\mathbf{v}\|_{H^{1+\epsilon}}, \\ \left|\int_\Omega((\mathbf{v}\cdot\nabla)\mathbf{v})\mathcal{A}^\epsilon\mathbf{v}\,d\mathbf{x}\right| &\le C\|(\mathbf{v}\cdot\nabla)\mathbf{v}\|_{H^{-1+\epsilon}}\|\mathbf{v}\|_{H^{1+\epsilon}} \le C\|(\mathbf{v}\cdot\nabla)\mathbf{v}\|_{L^{2/(2-\epsilon)}}\|\mathbf{v}\|_{H^{1+\epsilon}} \\ &\le C\|\mathbf{v}\|_{L^{2/(1-\epsilon)}}\|\nabla\mathbf{v}\|_{L^2}\|\mathbf{v}\|_{H^{1+\epsilon}} \le C\|\mathbf{v}\|_{H^\epsilon}\|\mathbf{v}\|_{H^1}\|\mathbf{v}\|_{H^{1+\epsilon}} \\ &\le \delta\|\mathbf{v}\|_{H^{1+\epsilon}}^2 + C(\delta)\|\mathbf{v}\|_{H^1}^2\|\mathbf{v}\|_{H^\epsilon}^2. \end{aligned} \tag{24}$$

We can now use the existing a priori estimate on $\|\mathbf{v}\|_{H^1}^2$ in $L^1(0,T)$ and a standard Gronwall argument to obtain the a priori bound on $\mathbf{v}\in L^\infty(0,T;H^\epsilon(\Omega))$.

We now turn to getting bounds on $(\mathbf{v}\cdot\nabla)\mathbf{v} = \mathrm{div}\,(\mathbf{v}\mathbf{v}^T)$. We first note that, from the energy inequality, we already have an a priori bound for the norm of $\mathbf{v}$ in $L^\infty(0,T;L^2(\Omega))$ and hence for $\mathrm{div}\,(\mathbf{v}\mathbf{v}^T)$ in $L^\infty(0,T;W^{-1,1}(\Omega))$. Next, we choose p large, and set $s = p/(\beta+1)$, $s' = s+\delta$ and $r = 2s'/(s'+2)$. Then we note that $L^r(\Omega)\subset W^{-1,s'}(\Omega)$ by the Sobolev embedding theorem, and moreover, if we choose δ sufficiently small relative to ϵ, then the product of a function in $L^s(\Omega)$ and a function in $H^\epsilon(\Omega)$ is in $L^r(\Omega)$. We find

$$\|(\mathbf{v}\cdot\nabla)\mathbf{v}\|_{L^s(0,T;W^{-1,s'}(\Omega))} \le C\|(\mathbf{v}\cdot\nabla)\mathbf{v}\|_{L^s0,T;L^r(\Omega)} \le C\|\mathbf{v}\|_{L^\infty(0,T;H^\epsilon(\Omega))}\|\nabla\mathbf{v}\|_{L^s(\Omega\times(0,T))}. \tag{25}$$

Finally, we note that, by interpolation, we have

$$\|(\mathbf{v}\cdot\nabla)\mathbf{v}\|_{L^s(0,T;W^{-1,s}(\Omega))} \leq \nu\|(\mathbf{v}\cdot\nabla)\mathbf{v})\|_{L^s(0,T;W^{-1,s'}(\Omega))} + C(\nu)\|(\mathbf{v}\cdot\nabla)\mathbf{v})\|_{L^\infty(0,T;W^{-1,1}(\Omega))}, \tag{26}$$

where ν can be chosen arbitrarily small. Combining all these estimates, we finally have the bound

$$\|(\mathbf{v}\cdot\nabla)\mathbf{v}\|_{L^s(0,T;W^{-1,s}(\Omega))} \leq \nu\|\nabla\mathbf{v}\|_{L^s(\Omega\times(0,T))} + C(\nu), \tag{27}$$

where ν can be chosen arbitrarily small and $C(\nu)$ is bounded. Consequently, we can treat the inertial nonlinearity as a perturbation and follow the same argument as above for the case of creeping flow.

6 Conclusions

We have proved global in time existence of weak solutions to the diffusive Peterlin model describing time evolution of complex viscoelastic fluids, see Theorem 1. The governing equations are as given in Section 2 and we need to assume growth conditions on the nonlinear constitutive functions as given in Section 3. The method of proof is based on a priori estimates which combine energy estimates and standard estimates for linear parabolic equations. These a priori estimates are combined with a Galerkin discretization of the momentum equation and compactness estimates which allow passing to the limit. Under a strengthend growth condition, we can also prove global existence of smooth solutions for creeping flows, as well as in the two-dimensional case, see Theorem 2.

Acknowledgements

The present research has been supported by the German Science Agency (DFG) under the grant "Mathematical Fluid Dynamics" IRTG 1529 and the grant TRR 146 "Multiscale simulation methods for soft matter systems". Š.N. acknowledges the support of the GAČR (Czech Science Foundation) project P201-13-00522S in the framework of RVO: 67985840. M.L.-M. and H.M. would like to thank A. Świerczewska-Gwiazda and P. Gwiazda (Warsaw) for fruitful discussions on the topic. Michael Renardy's research was supported by the National Science Foundation under Grant DMS-1514576.

References

[1] J. W. Barrett and S. Boyaval, *Existence and approximation of a (regularized) Oldroyd-B model*, Math. Models Methods Appl. Sci., 21 (2011), pp. 1783–1837.

[2] J. W. Barrett and E. Süli, *Existence and equilibration of global weak solutions to kinetic models for dilute polymers I: finitely extensible nonlinear bead-spring chains*, Math. Models Methods Appl. Sci., 21 (2011), pp. 1211–1289.

[3] J. W. Barrett and E. Süli, *Existence of global weak solutions to finitely extensible nonlinear bead-spring chain models for dilute polymers with variable density and viscosity*, J. Differ. Equations, 253 (2012), pp. 3610–3677.

[4] J. W. Barrett and E. Süli, *Finite element approximation of finitely extensible nonlinear elastic dumbbell models for dilute polymers*, ESAIM: M2AN, 46 (2012), pp. 949–978.

[5] P. Constantin and M. Kliegl, *Note on global regularity for two-dimensional Oldroyd-B fluids with diffusive stress*, Arch. Ration. Mech. An., 206 (2012), pp. 725–740.

[6] P. Constantin and N. Masmoudi, *Global well-posedness for a Smoluchowski equation coupled with Navier-Stokes equations in 2D*, Commun. Math. Phys., 278 (2008), pp. 179–191.

[7] P. Constantin and W. Sun, *Remarks on Oldroyd-B and related complex fluid models*, Commun. Math. Sci., 10 (2012), pp. 33–73.

[8] P. Degond and H. Liu, *Kinetic models for polymers with inertial effects*, Netw. Heterog. Media, 4 (2009), pp. 625–647.

[9] H. Engler, *On the dynamic shear flow problem for viscoelastic liquids*, SIAM J. Math. Anal., 18 (1987), pp. 972–990.

[10] E. Fernández-Cara, E. Guillén, and R. R. Ortega, *Mathematical modeling and analysis of viscoelastic fluids of the Oldroyd kind*, in Handbook of numerical analysis, Vol. VIII, North-Holland, Amsterdam, 2002, pp. 543–661.

[11] M. Geissert, D. Götz, and M. Nesensohn, *L_p-theory for a generalized nonlinear viscoelastic fluid model of differential type in various domains*, Nonlinear Anal.-Theor., 75 (2012), pp. 5015–5026.

[12] C. Guillope and J. C. Saut, *Global existence and one-dimensional nonlinear stability of shearing motions of viscoelastic fluids of Oldroyd type*, RAIRO Modélisation Math. Anal. Numér., 24 (1990), pp. 369–401.

[13] B. Jourdain, T. Lelièvre, and C. Le Bris, *Existence of solution for a micro-macro model of polymeric fluid: the FENE model*, J. Funct. Anal., 209 (2004), pp. 162–193.

[14] B. Jourdain, T. Lelièvre, C. Le Bris, and F. Otto, *Long-time asymptotics of a multiscale model for polymeric fluid flows*, Arch. Ration. Mech. An., 181 (2006), pp. 97–148.

[15] C Le Bris and T. Lelièvre, *Multiscale modelling of complex fluids: a mathematical initiation*, in Multiscale modeling and simulation in science, vol. 66 of Lect. Notes Comput. Sci. Eng., Springer, Berlin, 2009, pp. 49–137.

[16] C. Le Bris and T. Lelièvre, *Micro-macro models for viscoelastic fluids: modelling, mathematics and numerics*, Sci. China Math., 55 (2012), pp. 353–384.

[17] C. Le Bris and P. L. Lions, *Existence and uniqueness of solutions to Fokker–Planck type equations with irregular coefficients*, Commun. Part. Diff. Eq., 33 (2008), pp. 1272–1317.

[18] F. Lin, C. Liu, and P. Zhang, *On hydrodynamics of viscoelastic fluids*, Commun. Pur. Appl. Math., 58 (2005), pp. 1437–1471.

[19] P. L. Lions and N. Masmoudi, *Global solutions for some Oldroyd models of non-Newtonian flows*, Chinese Ann. Math. B, 21 (2000), pp. 131–146.

[20] M. Lukáčová-Medviďová, H. Mizerová, and Š. Nečasová, *Global existence and uniqueness result for the diffusive Peterlin viscoelastic model*, Nonlinear Anal.-Theor., 120 (2015), pp. 154–170.

[21] M. Lukáčová-Medviďová, H. Mizerová, H. Notsu, and M. Tabata, *Numerical analysis of the Oseen-type Peterlin viscoelastic model by the stabilized Lagrange–Galerkin method Part I: A linear scheme*, submitted, arXiv:1603.01074, (2016), pp. 1–15.

[22] M. Lukáčová-Medviďová, H. Mizerová, H. Notsu, and M. Tabata, *Numerical analysis of the Oseen-type Peterlin viscoelastic model by the stabilized Lagrange–Galerkin method Part II: A nonlinear scheme*, submitted, arXiv:1603.01339, (2016), pp. 1–15.

[23] J. Málek, J. Nečas, M. Růžička, and M. Rokyta, *Weak and measure-valued solutions for evolutionary PDE's*, vol. 13, Chapman & Hall, London, 1996.

[24] J. Málek, D. Pražák, and M. Steinhauer, *On the existence and regularity of solutions for degenerate power-law fluids*, Differential Integral Equations, 19 (2006), pp. 449–462.

[25] N. Masmoudi, *Well-posedness for the FENE dumbbell model of polymeric flows*, Commun. Pur. Appl. Math., 61 (2008), pp. 1685–1714.

[26] Nader Masmoudi, *Global existence of weak solutions to macroscopic models of polymeric flows*, J. Math. Pures Appl. (9), 96 (2011), pp. 502–520.

[27] N. Masmoudi, *Global existence of weak solutions to the FENE dumbbell model of polymeric flows*, Invent. Math., 191 (2013), pp. 427–500.

[28] N. Masmoudi, P. Zhang, and Z. Zhang, *Global well-posedness for 2D polymeric fluid models and growth estimate*, Physica D, 237 (2008), pp. 1663–1675.

[29] H. Mizerová, *Analysis and numerical solution of the Peterlin viscoelastic model*, PhD thesis, University of Mainz, Germany, 2015.

[30] M. Renardy, *An existence theorem for model equations resulting from kinetic theories of polymer solutions*, SIAM J. Math. Anal., 22 (1991), pp. 313–327.

[31] M. Renardy, *Mathematical analysis of viscoelastic flows*, CBMS-NSF Conference Series in Applied Mathematics 73, Society for Industrial and Applied Mathematics, 2000.

[32] M. Renardy, *Global existence of solutions for shear flow of certain viscoelastic fluids*, J. Math. Fluid Mech., 11 (2007), pp. 91–99.

[33] J. D. Schieber, *Generalized Brownian configuration field for Fokker–Planck equations including center-of-mass diffusion*, J. Non-Newton. Fluid, 135 (2006), pp. 179–181.

[34] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, North-Holland Publishing Company, Amsterdam, New York, Oxford, 1977.