Restricted Arithmetic Quantum Unique Ergodicity

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The Eigenvalue Problem for the Laplacian

(M, g) is a compact n-dimensional Riemannian manifold, such as the n-sphere

$$S^n = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = n\right\}.$$

We study Laplacian eigenfunctions:
L^2-normalised $f \in L^2(M)$ satisfying

$$\Delta f = \lambda f,$$

$$\Delta := -\frac{1}{\sqrt{|\det g|}} \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} g^{jk} \sqrt{|\det g|} \frac{\partial}{\partial x_k}.$$

The Laplacian eigenvalue of f is $\lambda \in [0, \infty)$.

On $\mathbb{R}^n$,

$$\Delta = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

Example: Modular Surface

Interesting setting for number theorists:
Riemannian locally symmetric spaces $M = \Gamma\backslash G/K$;

- G a Lie group,
- K a maximal compact subgroup of G,
- Γ a lattice in G.

Simplest interesting case: $G = \mathrm{SL}_2(\mathbb{R})$, $K = \mathrm{SO}(2)$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

- $G/K \cong \mathbb{H}$, the upper half-plane
$$\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\},$$
- $\Gamma\backslash G/K \cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, the modular surface
$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} = \left\{z = x + iy \in \mathbb{H} : -\frac{1}{2} < x < \frac{1}{2},\ x^2 + y^2 > 1\right\},$$
- Laplacian eigenfunctions are automorphic forms.

Example: Modular Surface

$\mathbb{H}$ is a negatively curved hyperbolic surface.

$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ inherits a hyperbolic metric from $\mathbb{H}$.

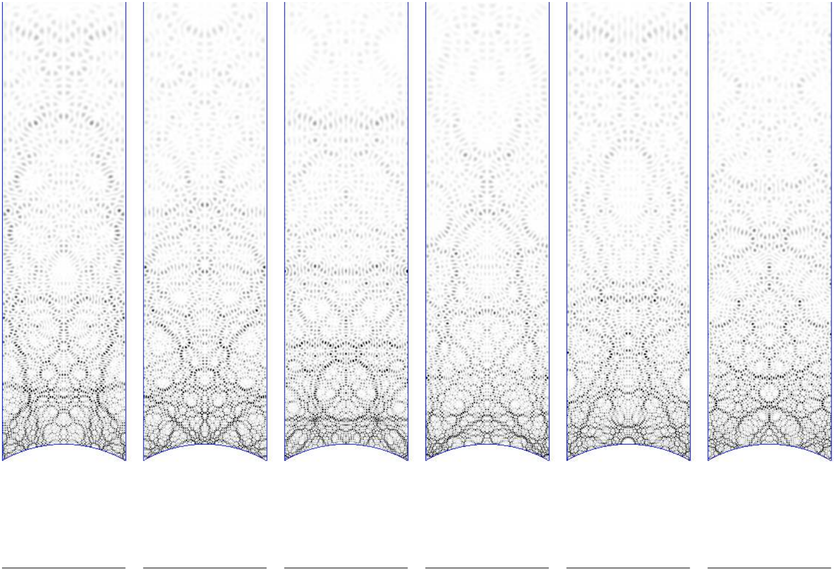
The Laplacian is $\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$.

The volume measure on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ is $d\mu(z) = \frac{dx\,dy}{y^2}$; $\mathrm{vol}(\Gamma\backslash\mathbb{H}) = \frac{\pi}{3}$.

Nonconstant eigenfunctions of Δ on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ are *Maaß forms g* with Laplacian eigenvalue $\lambda_g = 1/4 + t_g^2$.

The space of Maaß forms has an orthonormal basis $\mathcal{B}_0$ consisting of Hecke–Maaß cusp forms.

Example: Modular Surface



Quantum Unique Ergodicity

Conjecture (Quantum Unique Ergodicity)

For all $f \in C_b(M)$,

$$\lim_{\lambda_g \to \infty} \int_M |g(x)|^2 f(x) \, d\mathrm{vol}(x) = \int_M f(x) \, d\mathrm{vol}(x).$$

Equivalently,

$$\lim_{\lambda_g \to \infty} \int_B |g(x)|^2 \, d\mathrm{vol}(x) = \mathrm{vol}(B)$$

for every continuity set $B \subset M$.

Quantum Unique Ergodicity

Known results on QUE:

- False without negative curvature of M, even if geodesic flow on M is ergodic (Hassell);
- True for *almost all* eigenfunctions (Shnirelman, Colin de Verdière, Zelditch);
- Any weak-* limit has positive entropy and hence cannot completely concentrate on a geodesic (Anantharaman);
- Any weak-* limit must give positive measure to nonempty open sets (Dyatlov–Jin).

Quantum Unique Ergodicity for $\Gamma\backslash\mathbb{H}$

Theorem (Lindenstrauss (2006), Soundararajan (2010))

For $g \in \mathcal{B}_0$ with Laplacian eigenvalue $\lambda_g = 1/4 + t_g^2$,

$$\lim_{t_g \to \infty} \int_{\Gamma\backslash\mathbb{H}} |g(z)|^2 f(z)\, d\mu(z) = \frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} \int_{\Gamma\backslash\mathbb{H}} f(z)\, d\mu(z)$$

for all $f \in C_b(\Gamma\backslash\mathbb{H})$.

Theorem (Luo–Sarnak (1995))

For $g(z) = E(z, 1/2 + it_g)$,

$$\lim_{t_g \to \infty} \frac{1}{\log \lambda_g} \int_{\Gamma\backslash\mathbb{H}} |g(z)|^2 f(z)\, d\mu(z) = \frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} \int_{\Gamma\backslash\mathbb{H}} f(z)\, d\mu(z)$$

for all $f \in C_c(\Gamma\backslash\mathbb{H})$.

Quantum Unique Ergodicity for $\Gamma\backslash\mathbb{H}$

Theorem (Lindenstrauss (2006), Soundararajan (2010))

For $g \in \mathcal{B}_0$ with Laplacian eigenvalue $\lambda_g = 1/4 + t_g^2$,

$$\lim_{t_g \to \infty} \int_B |g(z)|^2 \, d\mu(z) = \frac{\mathrm{vol}(B)}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}$$

for every continuity set $B \subset \Gamma\backslash\mathbb{H}$.

Theorem (Luo–Sarnak (1995))

For $g(z) = E(z, 1/2 + it_g)$,

$$\lim_{t_g \to \infty} \frac{1}{\log \lambda_g} \int_B |g(z)|^2 \, d\mu(z) = \frac{\mathrm{vol}(B)}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}$$

for every compact continuity set $B \subset \Gamma\backslash\mathbb{H}$.

Effective QUE

Conjecture (Luo–Sarnak (1995))

We have that

$$\sup_{B_R(w)\subset\Gamma\backslash\mathbb{H}} \left| \int_{B_R(w)} |g(z)|^2 \, d\mu(z) - \frac{\mathrm{vol}(B_R(w))}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} \right| \ll_\varepsilon t_g^{-\frac{1}{2}+\varepsilon}.$$

Effective form of QUE: bounds for the discrepancy.

Conjectured bound is (essentially) optimal.

Theorem (Watson (2002), Young (2016))

Conjecture is true assuming GLH.

Spectral Decomposition of $\Gamma\backslash\mathbb{H}$

L^2-spectral decomposition of $\phi \in L^2(\Gamma\backslash\mathbb{H})$ is

$$\phi(z) = \langle \phi, f_0 \rangle f_0(z) + \sum_{f \in \mathcal{B}_0} \langle \phi, f \rangle f(z) + \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle \phi, E\left(\cdot, \frac{1}{2} + it\right)\right\rangle E\left(z, \frac{1}{2} + it\right) \, dt.$$

Converges uniformly for $\phi \in C_c^\infty(\Gamma\backslash\mathbb{H})$.

- $f_0(z) = \mathrm{vol}(\Gamma\backslash\mathbb{H})^{-1/2}$,
- $\mathcal{B}_0$ orthonormal basis of Hecke–Maaß cusp forms,
- $E(z, s)$ Eisenstein series,
- $\langle g_1, g_2 \rangle := \int_{\Gamma\backslash\mathbb{H}} g_1(z)\overline{g_2(z)} \, d\mu(z)$.

Idea of Proof of Effective QUE

Take $\phi = 1_{B_R(w))}$:

$$\sup_{B_R(w) \subset \Gamma \backslash \mathbb{H}} \left| \int_{B_R(w)} |g(z)|^2 \, d\mu(z) - \frac{\mathrm{vol}(B_R(w))}{\mathrm{vol}(\Gamma \backslash \mathbb{H})} \right|$$

$$= \sup_{B_R(w) \subset \Gamma \backslash \mathbb{H}} \left| \sum_{f \in \mathcal{B}_0} \langle |g|^2, f \rangle \langle f, 1_{B_R(w)} \rangle \right.$$

$$\left. + \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle |g|^2, E\left(\cdot, \frac{1}{2} + it\right) \right\rangle \left\langle E\left(\cdot, \frac{1}{2} + it\right), 1_{B_R(w)} \right\rangle dt \right|.$$

Need to bound RHS.

- Good averaged bounds for $\langle f, 1_{B_R(w)} \rangle$ via Selberg–Harish-Chandra transform:

$$\sum_{T \leq t_f \leq 2T} \left| \left\langle f, 1_{B_R(w)} \right\rangle \right|^2 \ll \begin{cases} R^2 T^2 & \text{if } T \leq \frac{1}{R}, \\ R^{1/2} T^{1/2} & \text{if } T \geq \frac{1}{R}. \end{cases}$$

- Relate triple product $\langle |g|^2, f \rangle$ to L-functions;
  - need good bounds for L-functions.

Triple Product Formula

Proposition (Watson (2002), Ichino (2008))

We have that

$$\left|\left\langle |g|^2, f\right\rangle\right|^2 \approx \frac{L\left(\frac{1}{2}, \operatorname{ad} g \otimes f\right) L\left(\frac{1}{2}, f\right)}{L(1, \operatorname{ad} g)^2 L(1, \operatorname{ad} f)}$$
$$\times \frac{1}{t_f (1 + 2t_g + t_f)^{1/2} (1 + |2t_g - t_f|)^{1/2}} \times \begin{cases} 1 & \text{if } t_f \leq 2t_g, \\ e^{-\pi(t_f - 2t_g)} & \text{if } t_f \geq 2t_g. \end{cases}$$

Assuming GLH, first line on RHS is $\ll_\varepsilon t_f^\varepsilon t_g^\varepsilon$.

Yields effective QUE with optimal error term (i.e. Luo–Sarnak conjecture).

Refinements of QUE

Refinements of QUE beyond effective bounds for the discrepancy:

- Small-scale QUE: how fast can the radius R of $B_R(w)$ shrink as t_g grows for equidistribution to still hold?
  - Young (2016): $R \gg t_g^{-1/3+\delta}$ under GLH.
  - H. (2018): $R \gg t_g^{-1+\delta}$ for almost every $w \in \Gamma\backslash\mathbb{H}$ under GLH.
- Restricted QUE: does QUE hold when restricted to a submanifold?
  - Toth–Zelditch (2013), Dyatlov–Zworski (2013): RQE holds for negatively curved manifolds M and generic hypersurfaces Σ for a density one sequence of Laplacian eigenfunctions.
  - Young (2016, 2018): RQUE holds for Eisenstein series restricted to vertical geodesics.
  - Hu (2020): a version of RQUE holds in the level (depth) aspect for Hecke–Maaß cusp forms.

Theorem (H. (2023+))

Fix a closed geodesic $\mathcal{C} \subset \Gamma\backslash\mathbb{H}$. Assume GLH. For $g \in \mathcal{B}_0$,

$$\lim_{t_g \to \infty} \int_{\mathcal{C}} |g(z)|^2 \psi(z)\, ds = \frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} \int_{\mathcal{C}} \psi(z)\, ds$$

for all $\psi \in C(\mathcal{C})$.

Proof is effective and gives bounds for the discrepancy:

$$\sup_{I \subseteq \mathcal{C}} \left| \int_I |g(z)|^2\, ds - \frac{\ell(I)}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} \right| \ll t_g^{-\delta}.$$

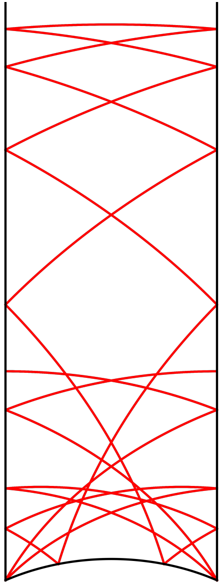
Maybe $\delta = 1/4$ is plausible.

Optimal bound is $\delta = 1/2$; currently out of reach.

Remarks on the Method

- Proof does not use ergodic theory.
- Need to assume Laplacian eigenfunctions are Hecke eigenfunctions.
- Requirement of GLH could be weakened;
  - need strong bounds for certain fractional moments of L-functions that imply hybrid subconvexity.
- Method works for Hecke–Maaß cusp forms on other congruence subgroups, including compact quotients.
- Method works for vertical geodesics from a *rational* point $x \in \mathbb{Q}$ to $i\infty$
  - conditional partial resolution of a conjecture of Young (2018).
- Method might (?) work for other arithmetic submanifolds:
  - horocycles;
  - geodesic circles centred at Heegner points.

Closed Geodesics on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$



Closed Geodesics on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$

Key properties of closed geodesics:

- Bijective correspondence with narrow ideal classes of real quadratic number fields. $\mathbb{Q}(\sqrt{D})$ (*arithmetic* submanifold)
- Length is $2\log\epsilon$, where ϵ is the fundamental unit of $\mathbb{Q}(\sqrt{D})$.
- Infinitely many closed geodesics.
- Union of all closed geodesics is dense in $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.
- Topologically equivalent to a circle.

For the proof, we assume for simplicity that $h_D^+ = 1$.

Reduction of RQUE

Since $\mathcal{C}$ is topologically a circle, by the Weyl equidistribution criterion, suffices to show that for each $m \in \mathbb{Z}$,

$$\lim_{t_g \to \infty} \int_{\mathcal{C}} |g(z)|^2 \psi_m(z)\, ds = \frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} \int_{\mathcal{C}} \psi_m(z)\, ds = \begin{cases} \dfrac{\ell(\mathcal{C})}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} & \text{if } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\psi_m(\theta) = e^{2\pi i m\theta}$.

Additionally need explicit rate of decay in both t_g and m for discrepancy bounds.

First Approach to RQUE

Idea 1 of Proof.

Insert spectral expansion of $|g|^2 \in L^2(\Gamma\backslash\mathbb{H})$:

$$\int_{\mathcal{C}} |g(z)|^2 \psi_m(z)\, ds = \frac{1}{\operatorname{vol}(\Gamma\backslash\mathbb{H})} \int_{\mathcal{C}} \psi_m(z)\, ds + \sum_{f \in \mathcal{B}_0} \langle |g|^2, f \rangle \int_{\mathcal{C}} f(z) \psi_m(z)\, ds + \cdots .$$

First term is desired main term. Need to show second term is small.

Watson–Ichino and GLH imply $\langle |g|^2, f \rangle \ll t_f^{-1/2} t_g^{-1/2}$ unless $t_f \geq 2t_g$, in which case $\langle |g|^2, f \rangle$ is negligibly small.

For $\int_{\mathcal{C}} f \psi_m\, ds$, apply Waldspurger's formula to relate to L-functions. □

Waldspurger's Formula

Proposition (Waldspurger (1985))

We have that

$$\left|\int_{\mathcal{C}} f(z)\psi_m(z)\,ds\right|^2 \approx \frac{L\left(\frac{1}{2}, f \otimes \Theta_{\psi_m}\right)}{L(1, \mathrm{ad}\, f)}$$
$$\times \frac{1}{\left(1 + \left|t_f + \frac{2\pi m}{\ell(\mathcal{C})}\right|\right)^{1/2}\left(1 + \left|t_f - \frac{2\pi m}{\ell(\mathcal{C})}\right|\right)^{1/2}}$$
$$\times \begin{cases} 1 & \text{if } \frac{2\pi|m|}{\ell(\mathcal{C})} \leq t_f, \\ e^{-\pi\left(\frac{2\pi|m|}{\ell(\mathcal{C})} - t_f\right)} & \text{if } \frac{2\pi|m|}{\ell(\mathcal{C})} \geq t_f. \end{cases}$$

Θ_{ψ_m} is a dihedral Maaß form of spectral parameter $\frac{2\pi|m|}{\ell(\mathcal{C})}$: automorphic induction of ψ_m.

Assuming GLH, first line on RHS is $\ll_\varepsilon t_f^\varepsilon(1 + |m|)^\varepsilon$.

First Approach to RQUE

Idea 1 of Proof (cont'd).

Want to show that as $t_g \to \infty$,

$$\sum_{f \in \mathcal{B}_0} \langle |g|^2, f \rangle \int_{\mathcal{C}} f(z)\psi_m(z)\, ds + \cdots = o(1).$$

Take absolute values, apply Watson–Ichino and Waldspurger, and assume GLH: get the upper bound

$$\sum_{t_f \leq 2t_g} \frac{1}{t_f\, t_g^{1/2}} \approx t_g^{1/2}.$$

□

Much too big!

Lossy since taking absolute values wastes oscillations of sign of $\langle |g|^2, f \rangle$ and $\int_{\mathcal{C}} f\psi_m\, ds$.

After taking absolute values, spectral sum is too long; need instead $\sum_{t_f \leq t_g^\delta}$ for some $\delta < 1/2$.

Idea 2 of Proof.

Use Parseval for $L^2(\mathcal{C})$:

$$\int_{\mathcal{C}} |g(z)|^2 \psi_m(z)\, ds = \sum_{n=-\infty}^{\infty} \int_{\mathcal{C}} g(z)\psi_{m+n}(z)\, ds \overline{\int_{\mathcal{C}} g(z)\psi_n(z)\, ds}.$$

Used the fact that $\psi_m(z)\psi_n(z) = \psi_{m+n}(z)$ as $e^{2\pi i m\theta} e^{2\pi i n\theta} = e^{2\pi i(m+n)\theta}$.

Next, take absolute values and apply Waldspurger.

Terms are negligibly small unless $\frac{2\pi|n|}{\ell(\mathcal{C})} \leq t_g$.

Assuming GLH, looks like

$$\sum_{|n| \leq \frac{t_g \ell(\mathcal{C})}{2\pi}} \frac{1}{\left(1 + \left|t_g + \frac{2\pi n}{\ell(\mathcal{C})}\right|\right)^{1/2} \left(1 + \left|t_g - \frac{2\pi n}{\ell(\mathcal{C})}\right|\right)^{1/2}} \approx 1. \qquad \square$$

Better, but still not quite good enough.

Method cannot even extract a main term when $m = 0$!

Second Approach to RQUE

Second approach can be used to prove good bounds for L^2-restriction problem.

Theorem (Ali (2022))

Unconditionally,

$$\int_{\mathcal{C}} |g(z)|^2 \, ds \ll_\varepsilon t_g^{2\vartheta+\varepsilon},$$

where $\vartheta = 7/64$ is the best known exponent towards the Ramanujan conjecture.

Assuming GRH, one can get $\ll 1$ via Harper's method for upper bounds for (fractional) moments of L-functions.

Method below gives correct asymptotic for $\int_{\mathcal{C}} |g|^2 \, ds$ under GLH.

Proof of RQUE

Second approach barely fails. Need to extract a main term from

$$\int_{\mathcal{C}} |g(z)|^2 \psi_m(z)\, ds = \sum_{n=-\infty}^{\infty} \int_{\mathcal{C}} g(z)\psi_{m+n}(z)\, ds \overline{\int_{\mathcal{C}} g(z)\psi_n(z)\, ds}.$$

Step 1 of proof.

Break up above sum into ranges. By Waldspurger and GLH, important range is when $|n|$ is quite close to t_g:

$$\sum_{t_g^{1-\delta} \leq |n| \leq \frac{t_g \ell(\mathcal{C})}{2\pi} - t_g^{1-2\delta}} \int_{\mathcal{C}} g(z)\psi_{m+n}(z)\, ds \overline{\int_{\mathcal{C}} g(z)\psi_n(z)\, ds}.$$

Remaining terms contribute $O_\varepsilon(t_g^{-\delta+\varepsilon})$. □

Proof of RQUE

Key new idea: modify the test vector g in this period integral of automorphic forms.

Step 2 of proof.

Construct an automorphic form $\widetilde{g} : \Gamma\backslash\mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ that closely approximates g along $\mathcal{C}$ (but not necessarily elsewhere in $\Gamma\backslash\mathbb{H}$);

- $\widetilde{g}$ *not* of weight 0: infinite linear combination of raised and lowered automorphic forms from g of weight $2k$, $k \in \mathbb{Z}$.
- $\widetilde{g}$ constructed such that if $t_g^{1-\delta} \leq |n| \leq \frac{t_g \ell(\mathcal{C})}{2\pi} - t_g^{1-2\delta}$, then

$$\int_{\mathcal{C}} g(z)\psi_n(z)\, ds \sim \int_{\mathcal{C}} \widetilde{g}(z)\psi_n(z)\, ds.$$

- If n is *not* in this range, $\int_{\mathcal{C}} \widetilde{g}\psi_n\, ds$ is exponentially small (rather than just polynomially small like $\int_{\mathcal{C}} g\psi_n\, ds$).

□

Proof of RQUE

Step 3 of proof.

Use Parseval for $L^2(\mathcal{C})$ to write

$$\int_{\mathcal{C}} |g(z)|^2 \psi_m(z)\,ds = \sum_{n=-\infty}^{\infty} \int_{\mathcal{C}} g(z)\psi_{m+n}(z)\,ds \overline{\int_{\mathcal{C}} g(z)\psi_n(z)\,ds},$$

$$\int_{\mathcal{C}} g(z)\overline{\widetilde{g}(z)}\psi_m(z)\,ds = \sum_{n=-\infty}^{\infty} \int_{\mathcal{C}} g(z)\psi_{m+n}(z)\,ds \overline{\int_{\mathcal{C}} \widetilde{g}(z)\psi_n(z)\,ds}.$$

Expansions essentially equal for $t_g^{1-\delta} \leq |n| \leq \frac{t_g \ell(\mathcal{C})}{2\pi} - t_g^{1-2\delta}$. Both expansions are negligibly small otherwise.

Upshot:

$$\int_{\mathcal{C}} |g(z)|^2 \psi_m(z)\,ds = \int_{\mathcal{C}} g(z)\overline{\widetilde{g}(z)}\psi_m(z)\,ds + o(1). \qquad \square$$

Proof of RQUE

Step 4 of proof.

Insert spectral expansion of $g\overline{\widetilde{g}} \in L^2(\Gamma\backslash \mathrm{SL}_2(\mathbb{R}))$:

$$\int_{\mathcal{C}} g(z)\overline{\widetilde{g}(z)}\psi_m(z)\, ds$$
$$= \frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} \int_{\Gamma\backslash \mathrm{SL}_2(\mathbb{R})} g(z)\overline{\widetilde{g}(z)}\, d\mu(z) \int_{\mathcal{C}} \psi_m(z)\, ds$$
$$+ \sum_{f\in\mathcal{B}} \langle g\widetilde{g}, f\rangle \int_{\mathcal{C}} f(z)\psi_m(z)\, ds + \cdots .$$

From construction of $\widetilde{g}$, $\int g\overline{\widetilde{g}}\, d\mu \sim 1$.

Spectral expansion is more complicated;
involves automorphic forms of *all* weights $2k$, $k \in \mathbb{Z}$:

- raised and lowered Hecke–Maaß forms,
- raised and lowered holomorphic Hecke cusp forms,
- raised and lowered Eisenstein series.

□

Proof of RQUE

Step 5 of proof.

Need to show that

$$\sum_{f \in \mathcal{B}} \langle g\widetilde{g}, f\rangle \int_{\mathcal{C}} f(z)\psi_m(z)\, ds + \cdots = o(1).$$

Again apply period formulæ to relate to L-functions:

- Watson–Ichino for $\langle g\widetilde{g}, f\rangle$;
- Waldspurger for $\int_{\mathcal{C}} f\psi_m\, ds$.

Replacing g with $\widetilde{g}$ gives the same L-functions but different archimedean weight; more complicated than gamma functions. □

Proof of RQUE

Step 5 of proof (cont'd).

Need to show that

$$\sum_{f \in \mathcal{B}} \langle g\widetilde{g}, f \rangle \int_{\mathcal{C}} f(z)\psi_m(z)\, ds + \cdots = o(1).$$

Key input to control size of archimedean weight: inversion formula for test functions involved (following Nelson's work on Motohashi's formula), then delicate multivariable stationary phase analysis.

Delicate analysis shows that archimedean weight is

- essentially the same as previous gamma factors for $t_f \leq t_g^{\delta}$;
- exponentially small for $t_f \geq t_g^{\delta}$.

Upshot: still lossy since taking absolute values wastes oscillations of sign of $\langle |g|^2, f \rangle$ and $\int_{\mathcal{C}} f\psi_m\, ds$. However, after taking absolute values, spectral sum is sufficiently short. □

Generalisations

Question

Does this method work for curves other than closed geodesics?

- Key idea is spectral expansion on $L^2(\mathcal{C})$ can be related to L-functions via Waldspurger's formula.
- Need similar period integral identities for automorphic forms along curves:
  - Vertical geodesics from a rational point $x \in \mathbb{Q}$ to $i\infty$ (Hecke integral)
  - Geodesic circles centred at Heegner points (Waldspurger's formula)
  - Horocycles (Fourier coefficients).
- Challenge in each case is threefold:
  - construct $\widetilde{g}$ (different in each case!);
  - show that it closely approximates g along the given curve;
  - show that the spectral expansion via $L^2(\Gamma\backslash\mathbb{H})$ is sufficiently short.

Thank you!