Periods of modular functions and diophantine approximation

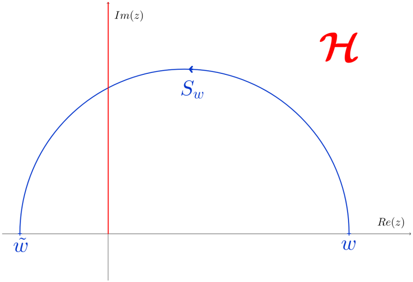
Paloma Bengoechea

ETH Zurich

April 14, 2020

Periods of modular functions

$\Gamma = \mathrm{SL}(2,\mathbb{Z})$



$$Q(x,y) = ax^2 + bxy + cy^2 \in \mathbb{Z}[x,y], \qquad (a,b,c) = 1$$

$$b^2 - 4ac = D > 0, \qquad D \neq \square$$

$$w = \frac{-b+\sqrt{D}}{2a}, \qquad \tilde{w} = \frac{-b-\sqrt{D}}{2a}$$

$$S_w : a|z|^2 + b\mathrm{Re}(z) + c = 0 \qquad (z \in \mathcal{H})$$

Stabilizer of Q in Γ: $\Gamma_w = \langle A_w \rangle$ where $A_w = \begin{pmatrix} \frac{1}{2}(t-bu) & -cu \\ au & \frac{1}{2}(t+bu) \end{pmatrix}$, with (t,u) the smallest primitive solution to Pell's equation $t^2 - Du^2 = 4$.

$$\begin{array}{rcl} \Gamma_w & \xrightarrow{\simeq} & \left\{ \text{units of norm 1 of } \mathbb{Q}(\sqrt{D}) \right\}^2 \\ A_w & \mapsto & \varepsilon_D^2 = ((t+u\sqrt{D})/2)^2 \end{array}$$

$$j(w) := \frac{1}{\mathrm{length}(S_w)} \int_{\Gamma_w \backslash S_w} j(z) \frac{\sqrt{D}}{Q(z,1)} dz$$

$$\mathrm{length}(S_w) = \int_{\Gamma_w \backslash S_w} \frac{\sqrt{D}}{Q(z,1)} dz = 2 \log \varepsilon_D$$

Choose any point z_0 in $\mathcal{H}$,

$$\int_{\Gamma_w \backslash S_w} = \int_{z_0}^{A_w z_0}$$

Distribution of periods of j

$$Tr_D(j) = \sum_{Q \in \mathcal{Q}_D/\Gamma} j(w_Q), \qquad \mathcal{Q}_D = \{\text{bin. quad. forms of discr. } D\}$$

Duke-Friedlander-Iwaniec and independently Masri:

$$\frac{Tr_D(j)}{h^+(D)} \to 720 \text{ as } D \to +\infty.$$

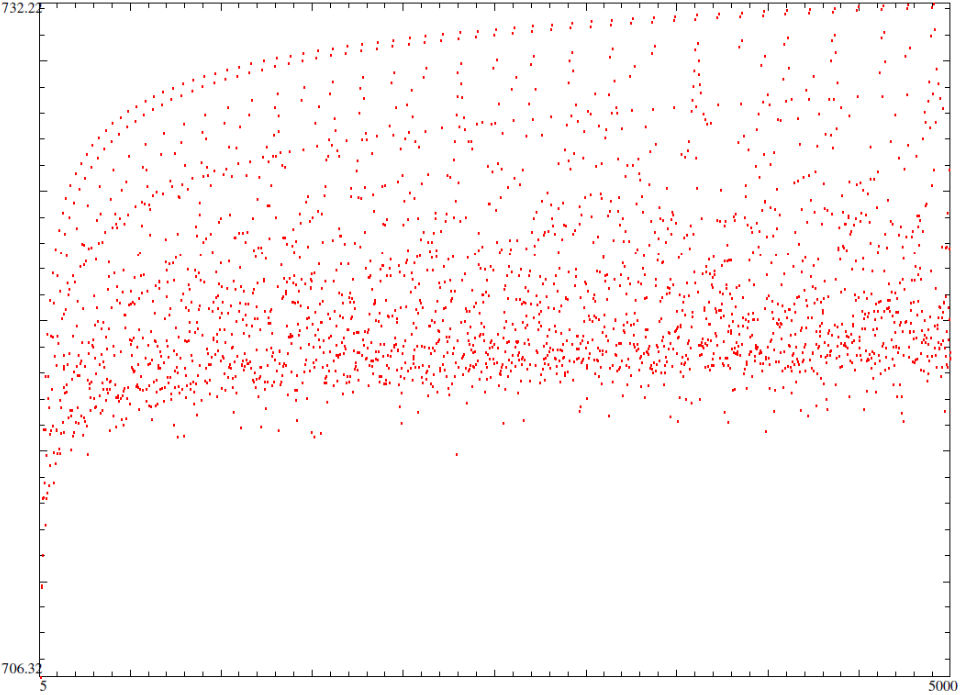
Theorem (B-Imamoglu)

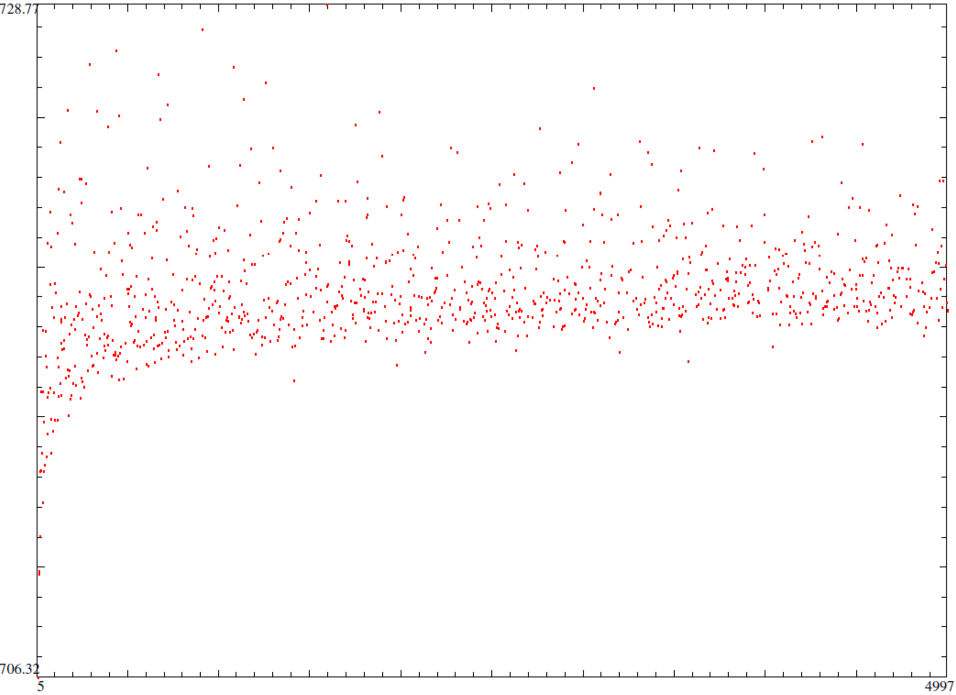
[Conjectures of Kaneko]

(i) $\mathrm{Re}(j(w)) \leq 744, \qquad (j(z) = q^{-1} + 744 + \dots)$
and the bound is optimal;

(ii) *If all the partial quotients in the period of the (negative) continued fraction expansion of w are $\geq 3e^{55}$, then*

$$\mathrm{Re}(j(w)) \geq j\Big(\frac{1+\sqrt{5}}{2}\Big) \approx 706.3$$


732.22
706.32
5
5000



728.77
706.32
5
4997

For $N > 2$, consider

$$\frac{N+\sqrt{N^2-4}}{2} = (N, N, N\ldots) = (\overline{N}),$$

where

$$(a_0, a_1, a_2, \ldots) = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots}}}$$

with $(a_0 \in \mathbb{Z}, a_i \geq 2$ for $i \geq 1)$.

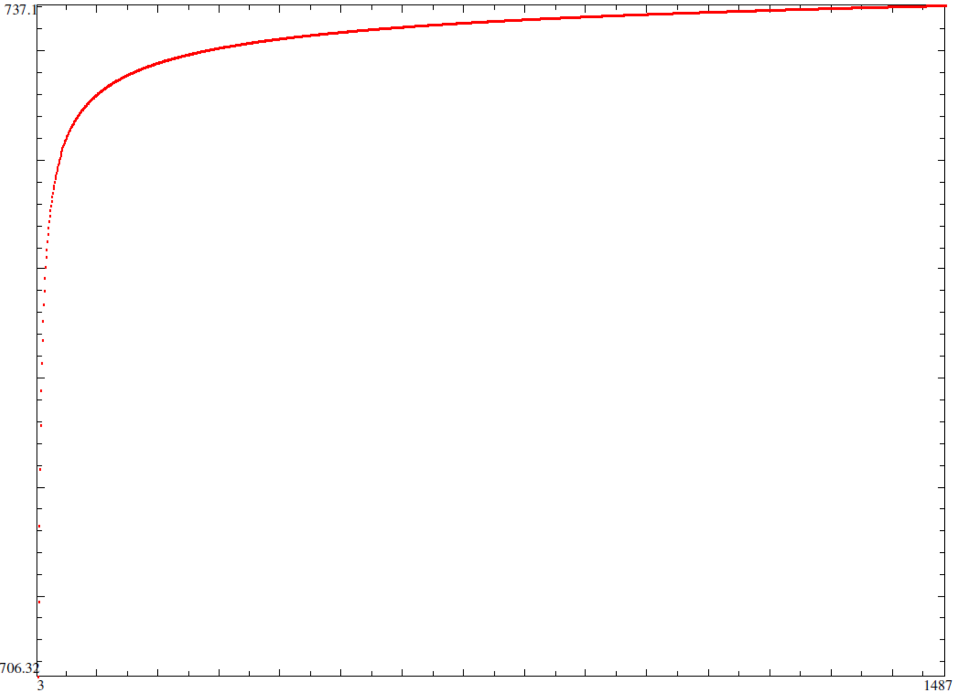
Theorem (B-I)

For all $N > 2$, $j(\frac{N+\sqrt{N^2-4}}{2}) \in \mathbb{R}$ and $\lim_{N\to\infty} j\left(\frac{N+\sqrt{N^2-4}}{2}\right) = 744$.

Theorem (B-I)

Let v and w be two quadratic irrationalities with respective periods in their c.f. $\overline{a_1, \ldots, a_n}$ and $\overline{b_1, \ldots, b_m}$ with $m|n$.
If $b_r \geq e^{55} a_r$ for all $r = 1, \ldots, n$, then

$$\mathrm{Re}(j(w)) > \mathrm{Re}(j(v)).$$


737.1
706.32
3
1487

Periods of j at Markov quadratics

Hurwitz: For all $x \in \mathbb{R}$,

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{\sqrt{5}} \frac{1}{q^2}$$

for infinitely many $\frac{p}{q}$ with $(p, q) = 1$.

$$\nu(x) = \lim \inf_{q \to \infty} q \| qx \|, \qquad L = \{\nu(x)\}_{x \in \mathbb{R}} \subseteq \left[0, \frac{1}{\sqrt{5}} \right]$$

- $\nu(x) = 0$ for almost all $x \in \mathbb{R}$
- if $\nu(x) > 0$, x is called badly approximable. $\nu(x) > 0 \Leftrightarrow$ all partial quotients of x are bounded. Example: quadratic irrationalities (their c.f. is eventually periodic)
- if x and x' are $\mathrm{PGL}(2, \mathbb{Z})$-equivalent, then $\nu(x) = \nu(x')$

$L \cap [0, F]$ is continuous, where $F \approx 0.220856$ is Freiman's constant, $L \cap (F, \frac{1}{3}]$ has a fractal structure, $L \cap (\frac{1}{3}, \frac{1}{\sqrt{5}}]$ is discrete.

w is a Markov quadratic if $\nu(w) \in L \cap (\frac{1}{3}, \frac{1}{\sqrt{5}}]$ and any other x with $\nu(x) = \nu(w)$ is $\mathrm{PGL}(2, \mathbb{Z})$ equivalent to w.

Markov quadratics:

$$(2, \overline{3}) \quad (3, \overline{2, 4})$$

$$(3, \overline{2, 3, 4})$$

$$(3, \overline{2, 3_2, 4}) \qquad (3, \overline{2, 4, 2, 3, 4})$$

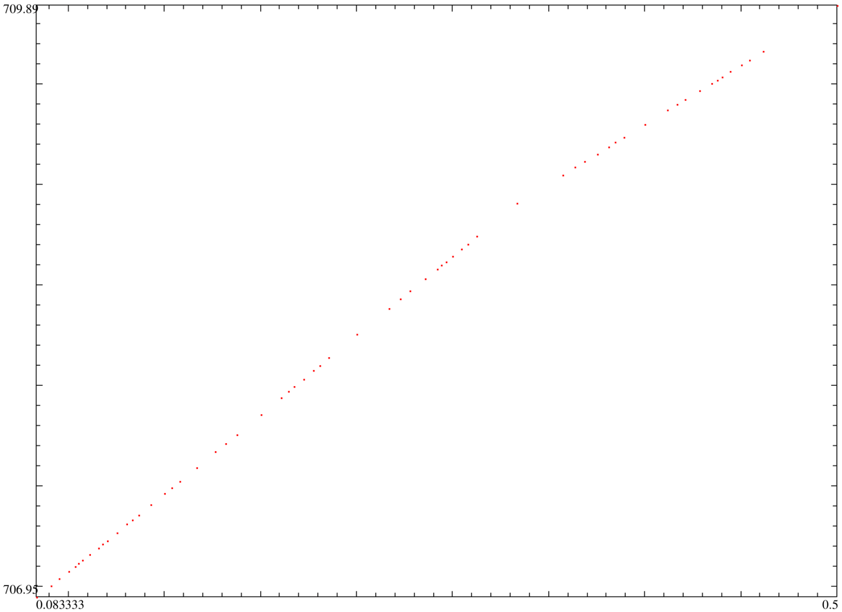
$$(3, \overline{2, 3_3, 4}) \quad (3, \overline{2, 3, 4, 2, 3_2, 4}) \quad (3, \overline{2, 4, (2, 3, 4)_2}) \quad (3, \overline{(2, 4)_2, 2, 3, 4})$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

Theorem (B-I)

(i) *The convergence property is preserved for the periods of j at the Markov quadratics.*

(ii) *The sandwich property is preserved for the periods of j at the Markov quadratics that are located below some level in the tree.*


709.89
706.95
0.083333
0.5

Farey parametrization of Markov tree

$(2, \overline{3})$ $(3, \overline{2, 4})$
$0/1$ $1/2$

$(3, \overline{2, 3, 4})$
$1/3$

$(3, \overline{2, 3_2, 4})$ $(3, \overline{2, 4, 2, 3, 4})$
$1/4$ $2/5$

$(3, \overline{2, 3_3, 4})$ $(3, \overline{2, 3, 4, 2, 3_2, 4})$ $(3, \overline{2, 4, (2, 3, 4)_2})$ $(3, \overline{(2, 4)_2, 2, 3, 4})$
$1/5$ $2/7$ $3/8$ $3/7$

Why does Diophantine approximation come in?

Let w have a purely periodic c.f.

$$v_1 = w, \quad a_1 = \lceil v_1 \rceil; \qquad v_{i+1} = \frac{1}{a_i - v_i}, \quad a_{i+1} = \lceil v_{i+1} \rceil$$

Then $w = (\overline{a_1, \dots, a_n})$ and $v_i = (\overline{a_i, a_{i+1}, \dots, a_n, \dots, a_{i-1}})$.

Another algorithm:

$$w_1 = w - 1, \quad w_{i+1} = \begin{cases} w_i - 1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}(w_i) & \text{if } w_i \geq 1, \\[2ex] \dfrac{w_i}{1 - w_i} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}(w_i) & \text{if } 0 < w_i < 1. \end{cases}$$

This algorithm is cyclic: we get $w_1, \dots, w_\ell$.

Lemma

$$j(w) = \frac{1}{2\log \varepsilon_D} \int_{e^{\pi i/3}}^{e^{2\pi i/3}} j(z) K(z, w) dz,$$

where

$$K(z, w) = \sum_{i=1}^{\ell} \frac{1}{z - w_i} - \frac{1}{z - \tilde{w}_i}$$

and $\tilde{w}_i$ *is the Galois conjugate of* w_i*.*

Example: proof of Th. 2.

$$w = (\overline{N}), \qquad D = N^2 - 4, \qquad \varepsilon_D = \frac{N + \sqrt{N^2 - 4}}{2}$$

$$w{-}1 = (N{-}1, \overline{N}) \xrightarrow{T^{-1}} (N{-}2, \overline{N}) \xrightarrow{T^{-1}} \ldots \xrightarrow{T^{-1}} (1, \overline{N}) \xrightarrow{V^{-1}} (N{-}1, \overline{N})$$

$$K(z, w) = K_N(z) = \sum_{k=1}^{N-1} \frac{1}{z - (k, \overline{N})} - \frac{1}{z + (k, \overline{N})}$$

For $\theta \in [\pi/3, 2\pi/3]$,

- $\overline{K_N(e^{i\theta})} = K_N(e^{i(\pi-\theta)}) \Rightarrow \mathrm{Im}(j(w)) = 0$,
- $K_N(e^{i\theta}) \to 2\log N$ as $N \to \infty \Rightarrow \frac{K_N(e^{i\theta})}{2\log \varepsilon_D} \to 1$ as $N \to \infty \Rightarrow$

$$\lim_{N\to\infty} j((\overline{N})) = \int_{\pi/3}^{2\pi/3} j(e^{i\theta}) i e^{i\theta} d\theta = 744.$$