Herbrand analyses in geometry: a case study*

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Abstract

This paper provides a case study for the extraction of computational content of proofs in geometry using Herbrand's theorem. More specifically, we show how a valid Herbrand disjunction for the Outer Pasch Theorem can be extracted in a modular way from its proof by Schwabhäuser, Szmielew and Tarski.

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1 Introduction

We present a case study of an application of Herbrand's theorem to the field of Euclidean geometry. That Herbrand's theorem can be utilized in connection with Tarski's axioms for geometry was first observed in [2], where it is used to show the underivability of the parallel axiom from (a first-order version of) the other axioms. There Herbrand's theorem is applied negatively to show that a certain proof cannot exist as the Herbrand terms extractable from such a proof would have a property which cannot hold for the conclusion (as is pointed out in [2, p.120], a related negative use of Herbrand's theorem is implicit already in [15]). In this paper, we indicate that Herbrand's theorem can also be used in a positive way suggested by the 'proof mining' paradigm, namely to extract computational content from proofs in geometry. Our case study concerns the proof of the 'Outer Pasch Theorem' given in [13, Satz I.9.6] from which we extract a valid Herbrand disjunction whose terms are built up only from the input variables and the function symbols used to skolemize the axiomatization of geometry from [13]. It turns out that the extraction of a Herbrand disjunction for that theorem can be done in a modular way by combining Herbrand disjunctions of the various lemmas used in its proof. This is in contrast to the fact that in general Herbrand's theorem has a bad behavior w.r.t. the modus ponens rule (see [9]) and, as a result of this, requires nonmodular techniques such as cut-elimination (see e.g. [1]). While one always can obtain a high-level description of Herbrand terms (involving λ-abstraction in higher types) using the modular (Shoenfield-variant of) Gödel's functional interpretation, to obtain the actual Herbrand terms then requires a normalization procedure (see [7]).

Our modular approach is possible in the case at hand since in the lemmas φ used which are of the logical form[1] $\forall\exists\forall$, and so require a reformulation to their Herbrand normal form φ^H, we obtain Herbrand terms which do not involve the Herbrand index functions used to built φ^H. As a consequence of this, these terms actually satisfy (disjunctively) the original lemma φ and not just φ^H. This feature is due to the fact that the proofs of these lemmas φ use the law-of-excluded-middle principle LEM only for quantifier-free formulas in the parameters of the statement which in turn is a consequence of the fact that the innermost universally quantified subformulas of φ allow for different formulations which are in $\exists$-form.

Our case study may indicate that such a modular approach might typically apply to proofs in geometry and that Herbrand-techniques allow one to convert ordinary textbook proofs in geometry which make use of (even nested) quantifiers into quantifier-free proofs, i.e. proofs in constructive geometry in the sense of [10]. Here

*This paper grew out of a Bachelor thesis [5] of the first author written under the supervision of the 2nd author.

[1]Lemmas of the form $\forall\exists$ do not create any problems w.r.t. the modus ponens rule even from the perspective of Herbrand disjunctions.

'constructive geometry' refers to a quantifier-free axiomatization which takes certain geometric operations as primitives (see also [14, 11, 12] for further developments in this direction). Similarly to [10, p.144] one can also define in our setting the relations T and E in terms of the function symbol *ext* and equality.

2 Herbrand's theorem and Euclidean geometry

2.1 Herbrand's theorem

Theorem 2.1 (Herbrand's theorem for theories **T** with purely universal axioms ('open theories')). *Let φ be a formula in prenex normal form and φ^H its Herbrand normal form (see e.g. [8]). Then $\mathbf{T} \vdash \varphi$ if and only if there exist terms $\underline{t}_1, \ldots, \underline{t}_n, \underline{s}_1, \ldots, \underline{s}_k, \underline{r}_1, \ldots, \underline{r}_m$ (which are built up from the free or outmost universally quantified variables $\underline{y}_0$ and the constant and function symbols occurring $\varphi^H \wedge \mathbf{T}_{qf}(\underline{v})$, possibly with a default constant symbol c if there is no other constant occurring) such that*

$$\bigwedge_{i=1}^{k} \mathbf{E}_{qf}(\underline{s}_i) \wedge \bigwedge_{i=1}^{m} \mathbf{T}_{qf}(\underline{r}_i) \to \bigvee_{i=1}^{n} \varphi^H_{qf}(\underline{y}_0, \underline{t}_i) \in TAUT.$$

Here $\forall \underline{v} \mathbf{T}_{qf}(\underline{v})$ is the universal closure of the conjunction of the $\mathbf{T}$-axioms used in the proof and $\forall \underline{u} \mathbf{E}_{qf}(\underline{u})$ is the purely universal prenex normal form of the conjunction of the equality axioms for all the function and predicate symbols occurring in φ and the $\mathbf{T}$-axioms used in the proof. In particular

$$\mathbf{T} \vdash \varphi \;\Rightarrow\; \mathbf{T}' \vdash \bigvee_{i=1}^{n} \varphi^H_{qf}(\underline{y}_0, \underline{t}_i)$$

where $\mathbf{T}'$ results from $\mathbf{T}$ by adding the new Herbrand index functions to the language but no non-logical axioms in which they occur.

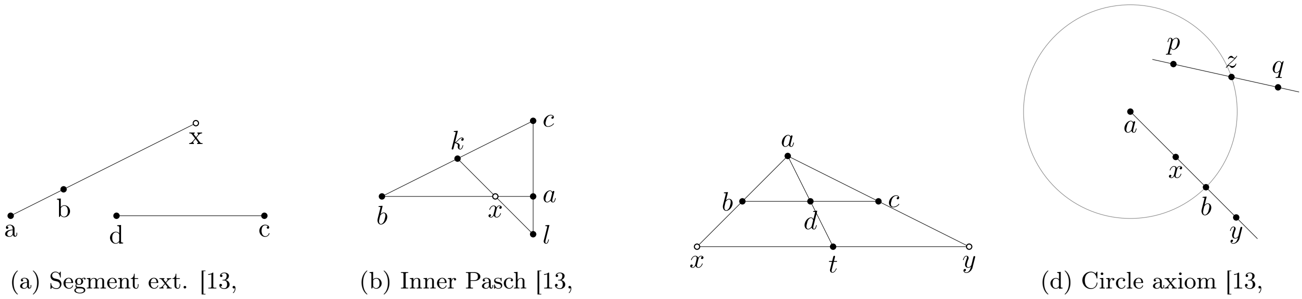
Remark 2.2. Let $\mathbf{T}$ be a theory. Via replacing $\mathbf{T}$ by its Skolem normal form $\mathbf{T}^S$, the theorem also holds for $\mathbf{T}$ via the purely universal $\mathbf{T}^S$. In cases where $\mathbf{T}$ is not purely universal, the Herbrand terms will in general involve the Skolem functions used to define $\mathbf{T}^S$ from $\mathbf{T}$.

In this paper, we call the terms t_i 'Herbrand terms' or 'realizers' and $\bigvee_{i=1}^{n} \varphi^H_{qf}(\underline{y}_0, \underline{t}_i)$ a 'Herbrand disjunction'. We say that we realize a formula φ or its existential variables if we provide realizers for the existential variables in its Herbrand normal form. We will further refer to the function symbols introduced in the process of Herbrandization as 'Herbrand index functions'.

2.2 Tarski's axioms

We now give the axiomatization of geometry from [13] using the notation from [2] as well as its Skolemized form introduced in [2]. This axiom system only has variables for points, which we will denote by small Latin letters, and two primitive relations T and E, which are three- and four-ary, respectively. We call T 'betweenness relation', and write $\mathrm{T}(a, b, c)$ to express that b lies between a and c. We call E 'equidistance relation', and write $\mathrm{E}(a, b, c, d)$ to express that the segment ab is congruent to the segment cd [2].

[2]for a formal definition of 'segment', see Definition 3.3.



(a) Segment ext. [13, Abb. 1]

(b) Inner Pasch [13, Abb. 3]

(c) Parallel [13, Abb. 6]

(d) Circle axiom [13, Abb. 8]

Figure 1: Depiction of (from left to right) Segment extension axiom $A4$, Inner Pasch axiom A7, Parallel axiom $A10$, Circle axiom CA.

Table 1: Tarski's axioms for geometry. [2, TABLE 1.]

A1	$\forall a, b\ \mathrm{E}(a, b, b, a)$ (Symmetry)
A2	$\forall a, b, c, d, e, f[\mathrm{E}(a, b, c, d) \wedge \mathrm{E}(a, b, e, f) \rightarrow \mathrm{E}(c, d, e, f)]$ (Pseudo-Transitivity)
A3	$\forall a, b, c[\mathrm{E}(a, b, c, c) \rightarrow a = b]$ (Cong Identity)
A4	$\forall a, b, c, d\exists x[\mathrm{T}(a, b, x) \wedge \mathrm{E}(b, x, c, d)]$ (Segment extension)
A5	$\forall a, a', b, b', c, c', d, d'[\mathrm{E}(a, b, a', b') \wedge \mathrm{E}(b, c, b', c') \wedge \mathrm{E}(a, d, a', d') \wedge \mathrm{E}(b, d, b', d') \wedge a \neq b \wedge$ $\mathrm{T}(a, b, c) \wedge \mathrm{T}(a', b', c') \rightarrow \mathrm{E}(c, d, c', d')]$ (Five segments)
A6	$\forall a, b[\mathrm{T}(a, b, a) \rightarrow a = b]$ (Between Identity)
A7	$\forall a, b, c, k, l[\mathrm{T}(b, k, c) \wedge \mathrm{T}(l, a, c) \rightarrow \exists x(\mathrm{T}(a, x, b) \wedge \mathrm{T}(l, x, k))]$ (Inner Pasch)
A8	$\exists abc[\neg \mathrm{T}(a, b, c) \wedge \neg \mathrm{T}(b, c, a) \wedge \neg \mathrm{T}(c, a, b)]$ (Lower Dimension)
A9	$\forall a, b, c, p, q[\mathrm{E}(a, p, a, q) \wedge \mathrm{E}(b, p, b, q) \wedge \mathrm{E}(c, p, c, q) \wedge p \neq q \rightarrow \mathrm{T}(a, b, c) \vee \mathrm{T}(b, c, a) \vee \mathrm{T}(c, a, b)]$ (Upper Dim.)
A10	$\forall a, b, c, d, t\exists xy[\mathrm{T}(a, d, t) \wedge \mathrm{T}(b, d, c) \wedge a \neq d \rightarrow \mathrm{T}(a, b, x) \wedge \mathrm{T}(a, c, y) \wedge \mathrm{T}(x, t, y)]$ (Parallel)
A11	$\forall XY\left[\exists a\forall x, y[x \in X \wedge y \in Y \rightarrow \mathrm{T}(a, x, y)] \rightarrow \exists b\forall x, y[x \in X \wedge y \in Y \rightarrow \mathrm{T}(x, b, y)]\right]$ (Continuity)
A11'	$\exists a\forall x, y[\Phi(x) \wedge \Psi(y) \rightarrow \mathrm{T}(a, x, y)] \rightarrow \exists b\forall x, y[\Phi(x) \wedge \Psi(y) \rightarrow \mathrm{T}(x, b, y)]$ (FoContinuity)
CA	$\forall a, b, p, q, x, y[\mathrm{T}(a, x, b) \wedge \mathrm{T}(a, b, y) \wedge \mathrm{E}(a, x, a, p) \wedge \mathrm{E}(a, q, a, y) \rightarrow \exists z(\mathrm{E}(a, z, a, b) \wedge \mathrm{T}(p, z, q))]$ (Circle)

The universal axioms $\mathrm{A1} - \mathrm{A3}, \mathrm{A5}, \mathrm{A6}$ and A9 are unproblematic from the perspective of Herbrand's theorem. The axioms $A4$, A7, A8, A10 and CA, however, are $\forall\exists$-axioms or purely existential. Under certain conditions, they assert the existence of new points with defined properties.

The segment extension axiom $A4$ (Figure 1a) states that there exists a point that extends a segment by the length of some other given segment. The inner Pasch axiom A7 (Figure 1b) 'intuitively says that if a line meets one side of a triangle and does not pass through the endpoints of that side, then it must meet one of the other sides of the triangle' [2, p.112]. The lower dimension axiom A8 states that there are three points that are not collinear, i.e. that the geometry under consideration is at least *plane*. Axiom A9 (not used in our case study) ensures that the dimension of the space is ≤ 2. The parallel axiom A10 (Figure 1c) is an equivalent formulation of Euclid's famous fifth postulate about parallel lines and states that 'through a point t inside an angle $\angle bac$, there always exists a line that intersects both sides of this angle' [13, p.13]. The continuity axiom expresses that 'first-order Dedekind cuts are filled' ([2, p.116]). It is the only axiom in this axiom system which cannot be formulated in terms of first-order logic and so needs to be replaced by an axiom schema as in [13, p.14]). An even weaker formulation is the circle axiom CA (Figure 1d) which states that if we have a circle and points inside and outside of that circle, then there exists a point *on* that circle. Since the continuity axiom and the circle axiom are not used in our case study we will not discuss them any further. We, therefore, consider the following first-order theory with language $\mathscr{L}(\mathrm{T}, \mathrm{E})$:

$$\mathbf{T} \coloneqq \mathrm{A1} - \mathrm{A10}.$$

As we are interested in discussing applications of Herbrand's theorems in the setting of (elementary) plane Euclidean geometry, we continue to define the purely universal Skolem normal form of $\mathbf{T}$ and give some intuition towards this: When considering axioms A4, A7 and A8, we can not only think of them as asserting the existence of some new points if certain conditions are met but, as Beeson, Boutry and Narboux describe it, as asserting 'the existence of *new* points that are *constructed* from other *given* points in various ways' [2, p. 111]. Intuitively,

this is what we think of when e.g. looking at axiom $A4$. It enables us to use our ruler to construct (or 'draw') a new point x that extends a segment in a certain way. It is thus also intuitive from a construction point of view to replace existential quantifiers in our axioms by function symbols. The terms in our extended language $\mathscr{L}(\mathbf{T}^S)$ now 'correspond to ruler and compass constructions' [2, p. 112]. Prima facie, we argue by the axiom of choice that this Skolemized version of $\mathbf{T}$ is equivalent to $\mathbf{T}$ with respect to satisfiability. However, if we can show that the points asserted to exist by our axioms are unique, we do not need the axiom of choice to argue for the introduction of a function symbol as, in every model of our theory, we are always able to pick one unique element without choice. Since in each case where the point asserted to exist is not unique it is possible to, nevertheless, declare it in a spefic way by which it gets uniquely determined (see below), the axiom of choice in fact is not needed at all.
We will now consider the Skolemized version of our axioms (which are not already purely universal). We adapt the notation of [2] and define

$$\mathrm{A4}^S := \forall a, b, c, d\big[\,\mathrm{T}(a, b, ext(a, b, c, d)) \wedge \mathrm{E}(b, ext(a, b, c, d), c, d)\big]$$

for a new (Skolem) function symbol $ext(a, b, c, d)$. That is, $ext(a, b, c, d)$ maps the points a, b, c, d to a point that extends the segment ab by the 'length' of the segment cd. Here already, as discussed above, we can show that if $a \neq b$ then $\forall a, b, c, d\exists!x\ [\mathrm{T}(a, b, x) \wedge \mathrm{E}(b, x, c, d)]$ and we don't need to rely on choice to argue for the introduction of ext. If however $a = b$, there are many ways to extend the segment ab and we need choice to argue for the introduction of our function symbol. We can think of $\mathrm{A4}^S$ as a simple ruler construction. We take our ruler, measure the distance of the segment cd and extend the segment ab by that distance. If $a \neq b$, this ruler construction is unique. If $a = b$, there are many ways to extend the segment ab, the axiom of choice 'selects' one. We further set

$$\mathrm{A7}^S := \forall a, b, c, k, l\big[\,\mathrm{T}(b, k, c) \wedge \mathrm{T}(l, a, c) \rightarrow \mathrm{T}(a, ip(b, l, c, k, a), b) \wedge \mathrm{T}(l, ip(b, l, c, k, a), k)\big]$$

for a new (Skolem) function symbol $ip(b, l, c, k, a)$. That is, $ip(b, l, c, k, a)$ maps the points a, b, c, k, l to a point that lies on the intersection of the segments ba and kl, we can think of this as being able to find the intersection point of two line segments if certain conditions are met. Again, the point claimed to exist by A7 can be shown to be unique if we are not in the degenerate case. In the degenerate case, i.e. if all the points we consider lie on a line, x is not unique. We choose to argue by the axiom of choice for the introduction of our function symbol in this case and will discuss this again later.
The lower dimension axiom A8 is purely existential. We hence consider:

$$\mathrm{A8}^S := \neg\,\mathrm{T}(ld_1, ld_2, ld_3) \wedge \neg\,\mathrm{T}(ld_2, ld_3, ld_1) \wedge \neg\,\mathrm{T}(ld_3, ld_1, ld_2)$$

for three new constant symbols ld_1, ld_2 and ld_3. This can be understood as always having 'access' to three points that are not collinear, namely ld_1, ld_2, ld_3. Here, we do not need to argue by the axiom of choice.
We further define

$$\mathrm{A10}^S := \forall a, b, c, d, t\big[\,\mathrm{T}(a, d, t) \wedge \mathrm{T}(b, d, c) \wedge a \neq d \rightarrow \\ \mathrm{T}(a, b, pa_1(a, b, c, d, t)) \wedge \mathrm{T}(a, c, pa_2(a, b, c, d, t)) \wedge \mathrm{T}(pa_1(a, b, c, d, t), t, pa_2(a, b, c, d, t))\big]$$

for two new function symbols $pa_1(...)$ and $pa_2(...)$. Again, we can interpret this as being able to use our ruler in yet another way. The points x and y of axiom A10 are not unique. This can be visualized in Figure 1c. Here one could extend segment bx and shorten segment cy such that still $T(a, b, x) \wedge T(a, c, y) \wedge T(x, t, y)$. We thus argue by choice here.
For the circle axiom we introduce a new Skolem function symbol $ilc(a, b, p, q, x, y)$:

$$\mathrm{CA}^S := \mathrm{T}(a, x, b) \wedge \mathrm{T}(a, b, y) \wedge \mathrm{E}(a, x, a, p) \wedge \mathrm{E}(a, q, a, y) \rightarrow \mathrm{E}(a, il(a, b, p, q, x, y), a, b) \wedge \mathrm{T}(p, il(a, b, p, q, x, y), q))$$

Just as for A4, it can be shown that z in CA is unique. We hence do not need the axiom of choice to argue for the existence of a function symbol with the desired properties and can think of our function as a simple compass construction where we have a compass with radius ab, draw a circle around a and find the point at which it intersects with pq.
The axiom A10 is only used in our case study to derive some purely universal facts which could be treated as axioms in the process of the extraction of Herbrand terms which explains why the Skolem functions pa_1, pa_2 do not occur in our extracted terms. As mentioned already, CA (and hence CA^S) are not used at all in our case study. We, hence, define

Definition 2.3. $\mathbf{T}^S := \mathrm{A1} - \mathrm{A3} \wedge \mathrm{A4}^S \wedge \mathrm{A5} - \mathrm{A6} \wedge \mathrm{A7}^S \wedge \mathrm{A8}^S \wedge \mathrm{A9\text{-}A10}$, with Ai and Ai^S as above.

Note that $\mathbf{T}^S$ is only a partial Skolemization of $\mathbf{T}$ since we did not skolemize A10 for the reason given above. By $(\mathbf{T}^S)'$ we denote extensions of $\mathbf{T}^S$ by Herbrand index functions needed to define the Herbrand normal form φ^H of φ when needed.

As mentioned already, it is possible to strengthen the axioms A4 and A7, whose Skolem normal forms are included in $\mathbf{T}^S$, in such a way that the claimed existence statement is unique and so the corresponding Skolem functions are uniquely determined and hence are not choice functions (for $\mathrm{A8}^S$, which only states the existence of Skolem constants, the issue of the axiom of choice does not arise). This can be achieved by declaring in the degenerate cases the then non-unique object of existence in a specific way. We indicate this for A4: consider

$$\mathrm{A4}_* := \forall a, b, c, d\, \exists! x \begin{bmatrix} (a \neq b \to \mathrm{T}(a, b, x) \wedge \mathrm{E}(b, x, c, d)) \\ \wedge(a = b \wedge ld_1 \neq b \to \mathrm{T}(ld_1, b, x) \wedge \mathrm{E}(b, x, c, d) \\ \wedge(a = b \wedge ld_1 = b \wedge ld_2 \neq b \to \mathrm{T}(ld_2, b, x) \wedge \mathrm{E}(b, x, c, d)) \end{bmatrix}.$$

By $\mathrm{A8}^S$ exactly one of the cases considered above holds and in each case x is uniquely determined. Hence the Skolem function ext in

$$\mathrm{A4}_*^S := \forall a, b, c, d \begin{bmatrix} (a \neq b \to \mathrm{T}(a, b, ext(a, b, c, d)) \wedge \mathrm{E}(b, ext(a, b, c, d), c, d)) \\ \wedge(a = b \wedge ld_1 \neq b \to \mathrm{T}(ld_1, b, ext(a, b, c, d)) \wedge \mathrm{E}(b, ext(a, b, c, d), c, d) \\ \wedge(a = b \wedge ld_1 = b \wedge ld_2 \neq b \to \mathrm{T}(ld_2, b, ext(a, b, c, d)) \wedge \mathrm{E}(b, ext(a, b, c, d), c, d)) \end{bmatrix}$$

is uniquely defined.
As a consequence of this, the equality axiom

$$\forall a, a', b, b', c, c', d, d' \, (a = a' \wedge b = b' \wedge c = c' \wedge d = d' \to ext(a, b, c, d) = ext(a', b', c', d'))$$

for ext becomes provable and so is not an extra requirement of ext.
Using $\mathrm{A4}_*^S$ instead of $\mathrm{A4}^S$ (and similarly for $\mathrm{A7}^S$) would allow one in some case to shorten the extracted Herbrand disjunction. However, as we aim at extracting the Herbrand disjunction which corresponds to the original proof as given in [13], we are not making use of such consequences of having fixed the meaning of the Skolem functions also in the degenerate cases and, correspondingly, are not applying equality axioms for the Skolem functions realizing the axioms. This is also the case for the index functions used to convert φ into φ^H since - as mentioned already in the introduction - our Herbrand terms do not involve these function symbols.

3 Herbrand analysis of the outer Pasch theorem

In section 2.2, we have given a geometrical intuition towards the definition of $\mathbf{T}^S$. With Herbrand's theorem in mind, but also from a geometrical perspective, it now makes sense to ask if we can *construct* any point that is proven to *exist* in $\mathbf{T}$ from these axioms. Suppose for instance that we have proven an existential statement in $\mathbf{T}$, e.g: on a line, there is always exactly one perpendicular from a point outside the line ([13, I.8.18. Lotsatz]). We now ask, how, given our four different ways (via $\mathrm{A4}^S, \mathrm{A7}^S$ and $\mathrm{A8}^S$) of using a ruler to construct new points, this perpendicular (however 'perpendicular' or 'line' is defined from our points) is constructed from these. That is, we ask for a list of finitely many possible 'chains' of constructions for such a point (accounting for different situations) such that one of these chains of constructions will yield in our desired point. This leads us to considering Herbrand's theorem, stating that we can extract from a proof of our statement realizers for our point in a (potentially slightly weaker) Herbrandized form of the statement.

In the case study presented in this paper, our goal is to provide a Herbrand disjunction for the so-called outer Pasch theorem (Figure 5):

Theorem 3.1 (outer Pasch)**.** *[13, Satz I.9.6]*

$\mathrm{T} \vdash \varphi := \exists x \big[\mathrm{T}(a, c, l) \wedge \mathrm{T}(b, k, c) \to \mathrm{T}(a, x, b) \wedge \mathrm{T}(l, k, x) \big]$.

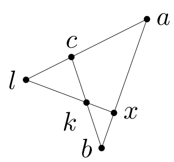


Figure 2: Outer Pasch [13, Abb.4]

In earlier versions of his axiom system, instead of A7 (Figure 6), Tarski included the outer Pasch theorem as an axiom. Taken as such, a particularly nice result about its role in the characterization of the models of the respective theory was found: In a suitable axiomatization, the outer Pasch axiom corresponds to the monotonicity law for the multiplication ([13, p.448f]).

In later versions, the outer Pasch axiom was replaced by the inner Pasch axiom A7 ([13, p.21f]), following results obtained by Gupta in [6], where it is shown that outer Pasch can be derived from inner Pasch. It is this result that makes the constructive axiomatization of absolute geoemetry given in [11] possible ([11, p.129f]).

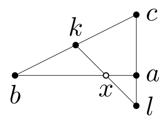


Figure 3: Inner Pasch [13, Abb.3]

The inner Pasch axiom states if $\mathrm{T}(b,k,c)$ and $\mathrm{T}(l,a,c)$, then there exists a point x that fulfills $\mathrm{T}(a,x,b)$ and lies *between* the points k and l. The outer Pasch theorem on the other hand states that if $\mathrm{T}(b,k,c)$ and if $\mathrm{T}(a,c,l)$ (i.e. 'a lies on the opposite extension of segment cl' [13, p.12]), then there exists a point x that fulfills $\mathrm{T}(b,x,a)$ and lies *outside* of the segment kl. In order to provide Herbrand terms for the existential quantifier in Theorem 3.1, we analyze its proof. The proof is based on multiple other propositions and lemmas, we thus start by considering those.

Remark 3.2. In the course of this section, we will often use without explicit mention that the equidistance relation is an equivalence relation and independent of the order of its points (i.e. $\mathrm{E}(a,b,c,d) \to \mathrm{E}(b,a,c,d)$ and $\mathrm{E}(a,b,c,d) \to \mathrm{E}(a,b,d,c)$). These properties of E can be shown from axioms A1 and A2 (see [13, p.27f]). We further use without explicit mention or proof that T is symmetric (i.e. $\mathrm{T}(a,x,c) \to \mathrm{T}(c,x,a)$). This can be shown by axioms A7 and A6 (see [13, Satz I.3.2]).

The next definition will be used later.

Definition 3.3. [13, Definition I.2.6] By a *segment* we mean an unordered pair $\{a,b\}$ of points, which we also denote as ab or ba; we call a and b the *endpoints* of the segment ab and ab the *connecting segment* of the points a and b.

3.1 The ruler and a disjunction due to choice

Before constructing terms for more involved theorems like outer Pasch, we want to expand our 'ruler-abilities' a little, e.g. if we can extend a segment by some existing length, we should be able to argue that we can perform a *reflection of a point through another point.* We hence start by considering two propositions about the possibility of *reflection of a point through another point.* To this end, we define a new relation symbol (or: 'abbreviation'), stating that the point m 'lies in the middle of' the segment ab:

Definition 3.4. [13, Definition I.7.1] $\mathrm{M}(a,m,b) :\leftrightarrow \mathrm{T}(a,m,b) \wedge \mathrm{E}(m,a,m,b)$.

We also define the notion of three points a,b,c being collinear. To this end, we want to say that they all lie on the same 'line'. Using our betweenness relation, we can say that either b must lie between a and c or a must lie between c and b and so on. Hence:

Definition 3.5. [13, Definition I.4.10] $\mathrm{Col}(a,b,c) :\leftrightarrow \mathrm{T}(a,b,c) \vee \mathrm{T}(b,c,a) \vee \mathrm{T}(c,a,b)$.

We can now show in $\mathbf{T}$ that there exists exactly one point x that is the reflection of a point b through a point a:

Proposition 3.6. *[13, Satz I.7.4]* $\mathbf{T} \vdash \varphi := \exists! x\, \mathrm{M}(b,a,x) = \exists x \forall x' \big[\mathrm{M}(b,a,x) \wedge [\mathrm{M}(b,a,x') \to x = x']\big]$.

The formula φ in Proposition 3.6 is in Σ^0_2 and so seemingly seems to require to be weakened to its Herbrand normal form

$$\varphi^H = \exists x \varphi^H_{qf}(a,b,x) = \exists x \big[\mathrm{M}(b,a,x) \wedge [\mathrm{M}(b,a,g(x)) \to x = g(x)]\big]$$

to allow for a Herbrand disjunction. However, φ can be decomposed into its purely existential part (1) $\exists x\, M(b,a,x)$ and the purely universal uniqueness part (2) $\forall a,b,x,x'[\mathrm{M}(b,a,x) \wedge \mathrm{M}(b,a,x') \to x = x']$) and so we only have to realize (1) as this - together with (2) - also realizes φ. To do so we recall the

Proof of Proposition 3.6. (see [13, p.49]) <u>Case 1</u>: Suppose $b \neq a$. Then the existence of a point x with $\mathrm{M}(b,a,x)$ follows from axiom A4. Suppose there is another x' with $\mathrm{M}(b,a,x')$. But then, using pseudo-transitivity (A2) of $\mathrm{E}(\ldots)$, from $\mathrm{E}(a,b,a,x)$ and $\mathrm{E}(a,b,a,x')$ we get that $\mathrm{E}(a,x,a,x')$. Hence
$\mathrm{T}(b,a,x) \wedge \mathrm{T}(b,a,x') \wedge \mathrm{E}(b,a,b,a) \wedge \mathrm{E}(a,x,a,x') \wedge \mathrm{E}(b,x,b,x) \wedge \mathrm{E}(a,x,a,x) \wedge b \neq a$ and again by A5 we get that $\mathrm{E}(x,x,x',x)$ holds. Hence, by A3 we conclude $x = x'$.
<u>Case 2</u>: Suppose $b = a$. Then $x = a$ satisfies $\mathrm{M}(b,a,x)$ and is with A3 the only such point. □

The case distinction made in this proof suggests a realizing disjunction with the two Herbrand terms $t_1 = ext(b,a,a,b), t_2 = a$. The 2nd term t_2, however, is only needed in the case $b = a$. But in this situation it is provable via A3 that $t_1 = t_2$. Put together have have shown that

Proposition 3.7. *Let* φ *be as above then* $\mathbf{T}^S \vdash \varphi(t_1)$ *for* $t_1 = ext(b,a,a,b)$.

As we will be referring to this term quite often in this paper, we give it a special name:

Definition 3.8. Let φ be as above. Set $S_a(b) := ext(b, a, a, b)$.

The next lemma we consider states that the segment ab has a midpoint under the assumption that there already is a point c from which they have the same distance (Figure 4).

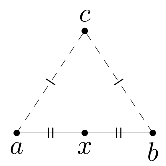


Figure 4

Lemma 3.9. *[13, Lemma I.7.25]* $\mathbf{T} \vdash \varphi := \exists x \big[\mathrm{E}(c, a, c, b) \to \mathrm{M}(a, x, b) \big]$

The proof of this lemma can be found in [13, p.55]. It distinguishes two cases. Here, we cannot eliminate the case distinction in the proof of Lemma 3.9 in the same way as we did above. The proof of Lemma 3.9 distinguishes the cases $\mathrm{Col}(a, b, c)$ and $\neg\,\mathrm{Col}(a, b, c)$. In case $\mathrm{Col}(a, b, c)$, it deduces that either $a = b$ or $\mathrm{M}(a, c, b)$ and hence $\mathrm{M}(a, x, b)$ for $x = b$ or $x = c$. If $\neg\,\mathrm{Col}(a, b, c)$, it proceeds to do a construction of a point x with $\mathrm{M}(a, x, b)$ that is not as trivial. We can show that the construction for $\neg\,\mathrm{Col}(a, b, c)$ works for *almost* all cases *in* $\mathrm{Col}(a, b, c)$ (what is meant here will become clear below), but not for all of them. We hence, with Herbrand, will get a disjunction that is of length 2.

Lemma 3.10. *Let* φ *be as above. Set* $\varphi^H = \exists x \varphi_{qf}(a, b, c, x) = \exists x (\mathrm{E}(c, a, c, b) \to \mathrm{M}(a, x, b))$*. Then* $\mathbf{T}^S \vdash \bigvee_{i=1}^{2} \varphi_{qf}(a, b, c, t_i(a, b, c))$ *for* $t_1 = ip(c, b, p, a, r),\ t_2 = c$*, where* $r := ip(p, q, c, a, b),\ q := ext(c, b, a, p),\ p := ext(c, a, ld_2, ld_3)$*.*

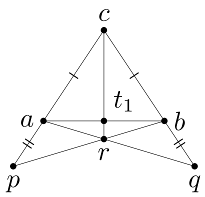


Figure 5: Construction of t_1 [13, Abb. 21]

Proof. (Consider Figure 5 for a visualization of this proof) We distinguish two cases
Case 1: $\mathrm{T}(a, c, b) \wedge a \neq b \neq c \neq a$.
From $\mathrm{T}(a, c, b)$ and $\mathrm{E}(c, a, c, b)$ we immediately get that $\mathrm{M}(a, c, b)$ hence $\varphi_{qf}(t_2)$.
Case 2: $\neg$ case 1, i.e. $\neg\,\mathrm{T}(a, c, b) \vee a = b \vee c = b \vee c = a$. Here, we have to construct an 'outer framework' (terms p,q and later r) to then use axiom $\mathrm{A7}^S$ to construct our desired term 'inside' that framework: By $\mathrm{A4}^S$, we get a term $p := ext(c, a, ld_2, ld_3)$ such that $\mathrm{T}(c, a, p) \wedge \mathrm{E}(a, p, ld_2, ld_3)$. By $\mathrm{A8}^S$ (using A3 and A4) we know that $ld_2 \neq ld_3$ (see [13, Proof of 1.3.13]) and hence deduce $a \neq p$ by A3. Again by $\mathrm{A4}^S$, we construct a term $q := ext(c, b, a, p)$ such that $\mathrm{T}(c, b, q) \wedge \mathrm{E}(b, q, a, p)$. As we now have $\mathrm{T}(p, a, c) \wedge \mathrm{T}(q, b, c)$ by $\mathrm{A7}^S$ we get a term $r := ip(p, q, c, a, b)$ such that $\mathrm{T}(a, r, q) \wedge \mathrm{T}(b, r, p)$. As we now have $\mathrm{T}(c, a, p) \wedge \mathrm{T}(b, r, p)$ by $\mathrm{A7}^S$ we get a term $t_1 := ip(c, b, p, a, r)$ such that $\mathrm{T}(a, t_1, b) \wedge \mathrm{T}(r, t_1, c)$. It remains to show that $\mathrm{E}(t_1, a, t_1, b)$ which is done as in [13, pp.55-56] (see also [5]), where one distinguish the cases 2a: '$a = b$ or $c = b$ or $c = a$' (and uses that then by $\mathrm{E}(c, a, c, b)$ one has $t_1 = a$) and 2b: 'a, b, c are pairwise distinct'. □

Remark 3.11. The only case which cannot be treated with t_1 is when $\mathrm{T}(a, c, b) \wedge a \neq b \neq c \neq a$ (Figure 6). We can neither show trivially that $t_1 = c$ (as we did above) nor carry this with us into case 2b, as we then wouldn't be able to deduce $L(aq) \neq L(bp)$. We hence have to make a case distinction. That we cannot show $t_1 = c$ is due to the fact that we are now in the degenerate case and a, b, c, p, q are all collinear where the value of $ip(p, q, c, a, b)$ is no longer uniquely determined by $\mathrm{A7}^S$. Of course, we could strengthen $\mathrm{A7}^S$ by stipulating that in this situation, the value should be c. Then, indeed the term t_2 would be sufficient in our Herbrand disjunction. In the light of these 'discontinuities', M. Beeson [3] discusses an alternate formulation of Tarski's theory, which he calls 'Continous Tarski geometry' that in particular formulates $A7$ (and $A4$) in a way that is strict and thus does not allow for degenerate cases, but reintroduces symmetry and transitivity of betweenness as axioms. For further reading on this, consider [3, §5.2.-§6.1.] as well as the subsequent [4].

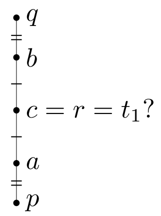


Figure 6: 'Case 2a'

Remark 3.12. Let φ be as above. From the Herbrand disjunction given in Lemma 3.10, one can obtain an explicit construction in the following sense: Suppose that betweeness, equidistance and equality of points are decidable (as discussed in section 1, it is even sufficient to ask only for the decidability of equality of points). Suppose further that it is possible to use a ruler to extend a segment by a given length (ext), to find an intersection point of two lines in the sense of inner Pasch (ip), and that we have access to three non-collinear points ld_1, ld_2, ld_3 in the sense of $A8^S$. Assume that there are three points a, b, c with $\mathrm{E}(a, c, b, c)$ from which we want to construct a point x such that $\mathrm{M}(a, x, b)$ by considering the respective Herbrand disjunction. As the disjunction is twofold, we do two constructions, which are "encoded" by the realizers. For the first one, we simply set $x := c$ (i.e., by extending the segment cc by its length). For the second construction, we consider term t_1 "from the inside out", thus first use the ruler to extend the segment ac by the length of ld_2ld_3. We call this point p. After constructing a further point q in a similar way, we find the (unique) intersection of segments pb and aq and call this point r. We then intersect ab and rc and complete the construction. We can now claim that one of the two constructed points satisfies $\mathrm{M}(a, x, b)$ and, since we assumed decidability, we can even check which one it is.

3.2 The square and a disjunction due to A8

The next proposition for which we will provide (just) one Herbrand term states, that 'on a line, from a point outside of that line, there always exists exactly one perpendicular' ([13, Satz I.8.18]). Or, in other words: given a line and a point outside of that line, we can always construct the 'foot' of this point on the line. To this end, we again define some new relational symbols (or: abbreviations).

We want to define a relation that states that a segment ab is perpendicular to a segment cd, where ab and cd intersect in point x. To this end, we define the notion of three points forming a right angle (Figure 7). As per usual, we describe a right angle as an angle that is congruent to it's adjacent angle.

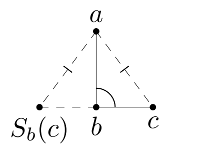


Figure 7: [13, Abb. 22]

Definition 3.13. [13, Definition I.8.1] $\mathrm{R}(a, b, c) :\leftrightarrow \mathrm{E}(a, c, a, S_b(c))$, where $S_b(c) := \iota c'\, \mathrm{M}(c, b, c')$, i.e. that point c' for which $\mathrm{M}(c, b, c')$.

For a formalization of this new operator, see [13, p.195f]. It can in particular be shown that ι-terms can be equivalently eliminated, i.e. for every formula there exists a logically equivalent formula that does not entail ι-terms [13, p.197, p.238f]. In our case, as we know that $\exists! c' M(a, b, c')$ (Proposition 3.6), it can be shown ([13, Satz II.3.38]) that the following are equivalent:
(i) $\mathrm{E}(a, c, a, \iota c'\, \mathrm{M}(c, b, c'))$, (ii) $\exists c'[\mathrm{M}(c, b, c') \wedge \mathrm{E}(a, c, a, c')]$, (iii) $\forall c'[\mathrm{M}(c, b, c') \to \mathrm{E}(a, c, a, c')]$.
Making use of this equivalence, the following lemma can be shown to hold true:

Lemma 3.14. *[13, Satz I.8.3]* $\mathbf{T} \vdash \mathrm{R}(a, b, c) \wedge a \neq b \wedge \mathrm{Col}(b, a, a') \to \mathrm{R}(a', b, c)$.

We further remark that, by Herbrand, we can in particular write (in $\mathbf{T}^S$) that $\mathrm{R}(a, b, c) \leftrightarrow \mathrm{E}(a, c, a, ext(c, b, b, c))$ as $\mathbf{T}^S \vdash \mathrm{M}(c, b, ext(c, b, b, c))$ (Proposition 3.7), where $ext(c, b, b, c) = S_b(c)$ (see Definition 3.8).
The line that is determined by two distinct points l and k is exactly the set of those points that are collinear to l and k, hence the following definition:

Definition 3.15. [13, Definition I.6.14] $L(lk) := \{x \mid \mathrm{Col}(l, k, x)\}$ defined for $l \neq k$.

We can now introduce the relation $\perp$ stating that ab is perpendicular to cd via x (Figure 8) iff $L(ab)$ is perpendicular to $L(cd)$ and $L(ab)$ and $L(cd)$ intersect in the point x. In our formal definition of this new relation, we want to avoid to talk about sets of points, as we mean to stay in a first order setting. We hence give the following definition which is equivalent to the intuitive set-formulation (see [13, Anmerkung I.6.26]). With $(\mathrm{Col}(a, b, x) \wedge \mathrm{Col}(c, d, x))$ we express that 'x is a point on the lines $L(ab)$ and $L(cd)$'. With $\forall u, v(\mathrm{Col}(a, b, u) \wedge \mathrm{Col}(c, d, v) \to \mathrm{R}(u, x, v))$ we express that the line $L(ab)$ is perpendicular to the line $L(cd)$ via x, i.e. if we find any point u on the line $L(ab)$ and any point v on the line $L(cd)$, then the points u, x, v form a right angle.

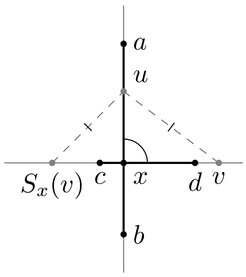


Figure 8: $ab \underset{x}{\perp} cd$

Definition 3.16. [13, Definition I.8.11 and Anmerkung I.6.26]
$ab \underset{x}{\perp} cd \leftrightarrow a \neq b \wedge c \neq d \wedge (\mathrm{Col}(a, b, x) \wedge \mathrm{Col}(c, d, x)) \wedge \forall u, v[\mathrm{Col}(a, b, u) \wedge \mathrm{Col}(c, d, v) \to \mathrm{R}(u, x, v)]$.

We will consider below statements of the form $(\forall a, b, c, d)\exists x \ldots ab \underset{x}{\perp} cd \ldots$, where $ab \underset{x}{\perp} cd$ occurs positively in a quantier-free context. As this is of the form $(\forall)\exists\forall$ (notice the 'hidden' $\forall$-quantifiers for u, v in the definition), when considering the Herbrandization of our statement, we will introduce function symbols for u and v. We hence state a 'Herbrandized' version of Definition 3.16 to the end of still being able to use the above shorthand in a Herbrand setting.

Definition 3.17. $ab \overset{g,h}{\underset{x}{\perp}} cd \leftrightarrow a \neq b \wedge c \neq d \wedge (\mathrm{Col}(a, b, x) \wedge \mathrm{Col}(c, d, x)) \wedge [\mathrm{Col}(a, b, g(x)) \wedge \mathrm{Col}(c, d, h(x)) \to \mathrm{R}(g(x), x, h(x))]$ for g, h the function symbols that will be introduced when showing in a Herbrand-sense a statement involving sentences of the form $(\forall a, b, c, d)\exists x\, ab \underset{x}{\perp} cd$.

We can now show a proposition that states that given a line and a point outside the line, we can always find/construct the foot of that point on the line (Figure 9).

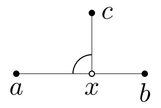


Figure 9

Proposition 3.18. *[13, Satz I.8.18]*

$$\mathbf{T} \vdash \varphi := \exists x \big[\neg \operatorname{Col}(a,b,c) \to \operatorname{Col}(a,b,x) \wedge ab \underset{x}{\perp} cx \big]^{3}$$
$$= \exists x \forall u, v \big[\neg \operatorname{Col}(a,b,c) \to \operatorname{Col}(a,b,x) \wedge a \neq b \wedge c \neq x \wedge [\operatorname{Col}(a,b,u) \wedge \operatorname{Col}(c,x,v) \to \operatorname{R}(u,x,v)] \big].$$

(On a line, from a point outside of that line, there always exists exactly one perpendicular).

The proof of this proposition can be found in [13, p.60]. There, it is also shown that x is unique. As discussed above, this can be shown as a separate universal statement, we thus only consider existence here.
Although, as discussed above, we first have to convert the statement into its Herbrand normal form, it turns out that we even can extract terms (in fact a single term t_1) such that t_1 realizes φ as the term does not depend on the index functions g, h :

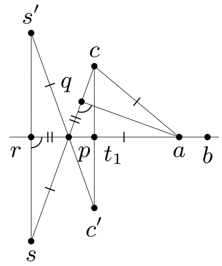


Figure 10: Construction of t_1 [13, p.61 Abb. 25]

Proposition 3.19. *Let φ be as above. Then* $\mathbf{T}^S \vdash \neg \operatorname{Col}(a,b,c) \to \operatorname{Col}(a,b,t_1) \wedge ab \underset{t_1}{\perp} ct_1,$

i.e. $\mathbf{T}^S \vdash \forall u, v \big[\neg \operatorname{Col}(a,b,c) \to \operatorname{Col}(a,b,t_1) \wedge a \neq b \wedge c \neq t_1 \wedge$
$[\operatorname{Col}(a,b,u) \wedge \operatorname{Col}(c,t_1,v) \to \operatorname{R}(u,t_1,v)] \big]$, *where*
$t_1 = ip(p, c', ext(p,c,ld_2,ld_3), c, ip(ext(p,c,ld_2,ld_3), ext(p,c',c,ext(p,c,ld_2,ld_3)), p, c, c'))$,
with $c' = ext(s',p,p,c),\ s' = S_r(s),\ s = ext(q,p,p,a),\ r = ext(a,p,p,q)$,
$q = ip(a,c,ext(a,p,ld_2,ld_3), p, ip(ext(a,p,ld_2,ld_3), ext(a,c,p,ext(a,p,ld_2,ld_3)), a, p, c))$,
$p = ext(b,a,a,c)$.

As t_1 is a very long term that we want to refer to later, we define the following:

Definition 3.20. For $t_1(a,b,c)$ as above, we define $foot(a,b,c) := t_1(a,b,c)$.

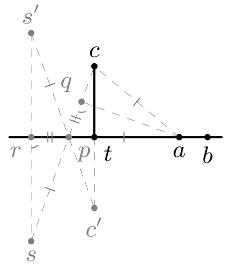


Figure 11: Using square to construct foot t

We can interpret this as having a new 'ability', besides the constructions we can do via $\mathrm{A4}^S, \mathrm{A7}^S, \ldots$ we can now also use some functions of a square, namely finding the foot of a point on a line (Figure 11). We of course argue that this special use of a square is merely a shortcut and can be replaced by a series of ruler (and compass) constructions at any time. We base the proof of this proposition on the proof of Proposition 3.18 in [13, p.60f].

Proof of Proposition 3.19. Consider first the Herbrand normal form of φ :

$\varphi^H := \exists x \varphi^H_{qf}(a,b,c,x) :=$
$\exists x \big[\neg \operatorname{Col}(a,b,c) \to \operatorname{Col}(a,b,x) \wedge a \neq b \wedge c \neq x \wedge [\operatorname{Col}(a,b,g(x)) \wedge \operatorname{Col}(c,x,h(x)) \to \operatorname{R}(g(x),x,h(x))] \big] =$
$\exists x \big[\neg \operatorname{Col}(a,b,c) \to \operatorname{Col}(a,b,x) \wedge ab \overset{g,h}{\underset{x}{\perp}} cx \big]$

for new function symbols g, h. It is helpful to consider Figure 10 in the course of this proof. The idea of the proof is to 'construct' a term c' that lies 'on the opposite side' of the line determined by ab with respect to c in such a way that the midpoint of the segment cc' will be the foot of c on the line determined by ab. To this end, we begin by invoking $\mathrm{A4}^S$ and get a term $p := ext(b,a,a,c)$ with $\operatorname{T}(b,a,p) \wedge \operatorname{E}(a,p,a,c)$. We further know that $\neg \operatorname{Col}(a,p,c)$: Suppose not, i.e. $\operatorname{Col}(a,p,c)$. Then with $\operatorname{T}(b,a,p)$ and $a \neq b$, also $\operatorname{Col}(a,b,c)$ which contradicts our assumption. We now want to construct the midpoint of the segment cp. We hence want to (sub)use Herbrand's theorem to get terms $q_i(c,p,a)$ and an n such that

$$(\mathbf{T}^S)' \vdash \left(\bigvee_{i=1}^{n} \operatorname{E}(a,c,a,p) \wedge \neg \operatorname{Col}(a,p,c) \to \operatorname{M}(c,q_i,p) \right).$$

[3] to be precise, in [13, 8.18 Satz, p.60] it is only shown that $\mathrm{T} \vdash \varphi := \exists x[\neg \operatorname{Col}(a,b,c) \to (\operatorname{Col}(a,b,x) \wedge ab \perp cx)]$ where $ab \perp cx := \exists y(ab \underset{y}{\perp} cx)$ but it is immediate from the proof that x is the point for which $ab \underset{x}{\perp} cx$, hence our formulation.

But now we are in the situation of Lemma 3.10, case 2. We have already provided such a term:

$$q := ip(a, c, ext(a, p, ld_2, ld_3), p, ip(ext(a, p, ld_2, ld_3), ext(a, c, p, ext(a, p, ld_2, ld_3)), a, p, c))$$

with $\mathrm{M}(c, q, p)$. We can now show the following equality between terms: via considering Proposition 3.7 for terms p and q, we can deduce that $c = S_q(p)$ (here, we need uniqueness). But then we have that $\mathrm{E}(a, p, a, S_q(p))$ and hence by Definition 3.13 $\mathrm{R}(a, q, p)$. Again via $\mathrm{A4}^S$ and falling back on what we have already shown in Proposition 3.7, we get terms $r := ext(a, p, p, q)$ with $\mathrm{T}(a, p, r) \wedge \mathrm{E}(p, r, p, q)$, $s := ext(q, p, p, a)$ with $\mathrm{T}(q, p, s) \wedge \mathrm{E}(p, s, p, a)$, $s' := S_r(s)$ with $\mathrm{M}(s, r, s')$ and $c' := ext(s', p, p, c)$ with $\mathrm{T}(s', p, c') \wedge \mathrm{E}(p, c', p, c)$. As we have $\mathrm{E}(p, c, p, c')$ we can now again by Lemma 3.10 ask for terms $t_i(c, c', p)$ such that

$$(\mathbf{T}^S)' \vdash \bigvee_{i=1}^{2} \mathrm{E}(p, c, p, c') \rightarrow \mathrm{M}(c, t_i, c').$$

Again we want to argue that we are in one of two cases of the proof of Lemma 3.10, namely 'case 2' and hence only get one term t with the desired properties. Assume to this end that $\mathrm{T}(c, p, c') \wedge c \neq c' \neq p \neq c$ (i.e that we are in case 1 of the proof of Lemma 3.10). But then by properties about $\mathrm{T}(\ldots)$ (see [13, Satz I.5.1]) we can show the following: $\mathrm{T}(c, p, c') \wedge \mathrm{T}(s', p, c') \wedge p \neq c' \Rightarrow \mathrm{T}(s', c, c') \vee \mathrm{T}(c, s', c')$, thus in particular $\mathrm{Col}(s', c, c')$. $\mathrm{T}(c, p, c') \wedge \mathrm{T}(c, p, s) \wedge c \neq p \Rightarrow \mathrm{T}(c, c', s) \vee \mathrm{T}(c, s, c')$, thus in particular $\mathrm{Col}(c, c', s)$. But now $\mathrm{T}(s', r, s) \wedge \mathrm{Col}(s', c, c') \wedge \mathrm{Col}(c, c', s) \wedge c \neq c' \Rightarrow \mathrm{Col}(c, r, c')$. From $\mathrm{T}(s, p, q)$ and $\mathrm{T}(p, q, c)$ as well as $p \neq q$ (otherwise $\mathrm{M}(c, q, p)$ implies $c = p$) we can infer that $\mathrm{T}(s, p, c)$ (this is rather intuitive, for reference see [13, Satz I.3.7]). But then again $\mathrm{T}(s, p, c) \wedge \mathrm{Col}(c, c', s) \wedge c \neq s \Rightarrow \mathrm{Col}(c, p, c')$. Finally, $\mathrm{Col}(c, r, c') \wedge \mathrm{Col}(c, p, c') \wedge \mathrm{Col}(b, a, r) \wedge \mathrm{Col}(b, a, p) \wedge \neg\mathrm{Col}(a, b, c) \wedge c \neq c' \wedge b \neq a \Rightarrow p = r$. (This can be proven with the same propositions referenced to above, we refrain from giving the proof here). Now $\mathrm{E}(p, r, p, q) \wedge p = r \Rightarrow p = q$. Contradiction. We hence, just as above, argue that (from Lemma 3.10) we get a term

$$t := ip(p, c', ext(p, c, ld_2, ld_3), c, ip(ext(p, c, ld_2, ld_3), ext(p, c', c, ext(p, c, ld_2, ld_3)), p, c, c'))$$

for which $\mathrm{M}(c, t, c')$. Again, in the same way as we demonstrated for q above, we can argue that $c' = S_t(c)$ and hence $\mathrm{R}(p, t, c)$. In order to show that t is the desired term, we have to show that $\mathrm{Col}(a, b, t)$ and that $ab \overset{g,h}{\underset{t}{\perp}} ct$. This is done as in [13, p.61] (which in turn uses Lemma 3.14).
Finally, since the index functions g, h do not occur in t, we may in the proof above replace all g- and h-terms by the variables u and v respectively and so obtain the claim of the proposition. □

Remark 3.21. As hinted already in the introduction, the reason why the index functions play no role in the above proof is that only quantifier-free case distinctions are made. This in turn is related to the fact that the universal formula $ab \underset{x}{\perp} cx$ can be written equivalently as an $\exists$-formula (see [13, Satz 1.8.13]) so that it can be treated essentially as being quantifier-free.
Next, we provide a Herbrand term for a slightly technical lemma, which we will also need later.

Lemma 3.22. *[13, Satz I.3.17]*

$$\mathbf{T} \vdash \varphi :=\exists x \varphi_{qf}(a, a', b, b', c, d, x)$$
$$:=\exists x \big[\mathrm{T}(a, b, c) \wedge \mathrm{T}(a', b', c) \wedge \mathrm{T}(a, d, a') \rightarrow \mathrm{T}(d, x, c) \wedge \mathrm{T}(b, x, b') \big].$$

The proof of this lemma can be found in [13, p.33].

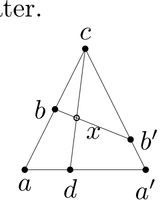


Figure 12: [13, Abb. 9]

With Herbrand, we get the following:

Lemma 3.23. *Let φ be as above. As $\varphi^H = \varphi$,*

$$\mathbf{T}^S \vdash \varphi_{qf}(a, b, c, a', b', d, t_1(a, b, c, a', b', d)) \, for \, t_1 = ip(c, b', a, b, ip(c, a, a', b', d)).$$

The proof is based on the proof of Lemma 3.22 in [13, p.33].

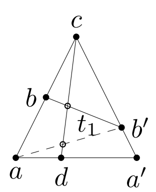


Figure 13: Construction of t_1 [13, Abb. 10]

Proof. It is helpful to consider Figure 13 in the course of this proof. As $\mathrm{T}(c, b', a') \wedge \mathrm{T}(a, d, a')$, $\mathrm{A7}^S$ implies $\mathrm{T}(b', ip(c, a, a', b', d), a) \wedge \mathrm{T}(d, ip(c, a, a', b', d), c)$. But then $\mathrm{T}(c, b, a) \wedge \mathrm{T}(b', ip(c, a, a', b', d), a)$ and again by $\mathrm{A7}^S$, $\mathrm{T}(b, t_1, b') \wedge \mathrm{T}(ip(c, a, a', b', d), t_1, c)$ for $t_1 := ip(c, b', a, b, ip(c, a, a', b', d))$. It can now be shown that since $\mathrm{T}(d, ip(c, a, a', b', d), c) \wedge \mathrm{T}(ip(c, a, a', b', d), t_1, c)$ also $\mathrm{T}(c, t_1, d)$ as desired (see e.g. [13, Satz I.3.5]). □

With Proposition 3.19, we can construct the foot of a point outside of a line on that line. We now show a proposition that provides us with means of constructing a perpendicular of a point on a line in a given half plane (Figure 14). We can think of this as being able to argue that we can extend the functionality of our square. If we show a proposition of this kind, we are not only able to connect a point and a line via a segment that is perpendicular to said line but also can draw a perpendicular from any point *on* that line (in a given half plane). To this end, we consider the following proposition.

Proposition 3.24. *[13, Satz I.8.21]*

$$\mathbf{T} \vdash \varphi := \exists x, y\big[a \neq b \to ab \underset{a}{\perp} xa \wedge \mathrm{Col}(a,b,y) \wedge \mathrm{T}(c,y,x)\big] =$$
$$\exists x, y \forall u, v\big[a \neq b \to a \neq b \wedge x \neq a \wedge [\mathrm{Col}(a,b,u) \wedge \mathrm{Col}(x,a,v) \to \mathrm{R}(u,a,v)] \wedge \mathrm{Col}(a,b,y) \wedge \mathrm{T}(c,y,x)]\big].$$

(On a line, from a point of that line, there is a perpendicular in a given half-plane).

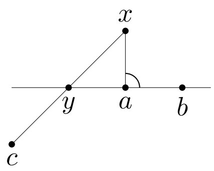


Figure 14: [13, Abb. 28]

The proof of Proposition 3.24 in [13, p.64] distinguishes two cases $\neg\,\mathrm{Col}(a,b,c)$ and $\mathrm{Col}(a,b,c)$. If $\neg\,\mathrm{Col}(a,b,c)$ they construct the foot f of the point c on $L(a,b)$ (Figure 15). They then 'mirror' (via what we call Proposition 3.6) the point c at its foot f and at the point a, we hence get two more points $S_f(c)$ and $S_a(c)$ respectively. It turns out that the center of the segment $S_f(c)S_a(c)$ is our desired point. The point y that witnesses that we are 'on the other side' of $L(ab)$ with respect to c is shown to exist due to Lemma 3.22. If $\mathrm{Col}(a,b,c)$, it is argued that by A8, we find a point c' for which $\neg\,\mathrm{Col}(a,b,c')$. The same construction as above can hence be carried out with c replaced by c'. The point y can now be chosen to be c, which trivially fulfills $\mathrm{Col}(a,b,y)$ and $\mathrm{T}(c,y,x)$. It is not needed as a witness, as $\mathrm{Col}(a,b,c)$ implies that the point f can live on any side of $L(ab)$.
We can now show the following:

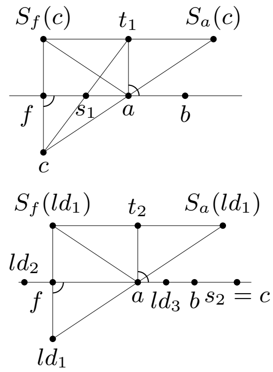


Figure 15: constr. for terms with index 1 and 2 (see proof) [13, Abb. 29]

Proposition 3.25. $\mathbf{T}^S \vdash \bigvee_{i=1}^{4} [a \neq b \to ab \underset{a}{\perp} t_i a \wedge \mathrm{Col}(a,b,s_i) \wedge \mathrm{T}(c,s_i,t_i)$ *for*

$$t_1 = ip(a, S_a(c), ext(a, S_f(c), ld_2, ld_3), S_f(c),$$
$$ip(ext(a, S_f(c), ld_2, ld_3), ext(a, S_a(c), S_f(c), ext(a, S_f(c), ld_2, ld_3)), a, s_f(c), S_a(c)))$$
$$s_1 = ip(c, f, S_a(c), a, ip(c, S_a(c),\ S_f(c), f, t_1)),\ \textit{where}$$
$f = foot(a,b,c)$ *is the term that we get from Proposition 3.19 and*
$$S_f(c) = ext(c,f,f,c),\ S_a(c) = ext(c,a,a,c),$$
$$t_2 = t_1[ld_1/c],\ s_2 = c,\ t_3 = t_1[ld_2/c],\ s_3 = c,\ t_4 = t_1[ld_3/c],\ s_4 = c.$$

As for Proposition 3.19, we define the following:

Definition 3.26. Let t_i, s_i be as above. Then we set
$perp(a,b,c) := t_1$, $wit(a,b,c) := s_1$. Then $perp(a,b,ld_{i-1}) = t_i$ for $i = 2,3,4$.

We can now officially use our square to draw a perpendicular from any given point on a line. For the proof, we argue in the same way as in the outline of the proof of Proposition 3.24.

Proof. Let

$$\varphi^H = \exists x, y\ \varphi^H_{qf}(a,b,c,x,y)$$
$$= \exists x, y\big[a \neq b \to a \neq b \wedge x \neq a \wedge [\mathrm{Col}(a,b,g(x,y)) \wedge \mathrm{Col}(x,a,h(x,y)) \to \mathrm{R}(g(x,y),a,h(x,y))] \wedge \mathrm{Col}(a,b,y) \wedge \mathrm{T}(c,y,x)\big]$$

for new function symbols $g(\ldots), h(\ldots)$.

We distinguish two cases. Case 1: $\neg\,\mathrm{Col}(a,b,c)$ (Figure 15, above). In the proof of Proposition 3.24, we set x to be the foot of the perpendicular from c to $L(a,b)$. We thus use Proposition 3.19, i.e.
$(*)$ $\mathbf{T}^S \vdash \forall u, v\big[\neg\,\mathrm{Col}(a,b,c) \to \mathrm{Col}(a,b,f) \wedge ab \overset{u,v}{\underset{f}{\perp}} cf\big]$ where $f = foot(a,b,c)$ and conclude that, as we have

$\neg Col(a,b,c)$ by assumption of our case, it follows (in particular) that $\mathrm{Col}(a,b,f) \wedge ab \overset{a,c}{\underset{f}{\perp}} cf$. By Proposition 3.7, we get terms $S_f(c) := ext(c,f,f,c)$ with $\mathrm{M}(c,f,S_f(c))$ and $S_a(c) := ext(c,a,a,c)$ with $\mathrm{M}(c,a,S_a(c))$. Lemma 3.10 provides terms t_i such that $(\mathbf{T}^S)' \vdash \bigvee_{i=1}^{2} \mathrm{E}(a,S_f(c),a,S_a(c)) \rightarrow \mathrm{M}(S_f(c),t_i,S_a(c))$. We show $\mathrm{E}(a,S_f(c),a,S_a(c))$, and thus know $\bigvee_{i=1}^{2} \mathrm{M}(S_f(c),t_i,S_a(c))$: $ab \overset{a,c}{\underset{f}{\perp}} cf$ implies $\mathrm{R}(a,f,c)$. Hence by definition of R, $\mathrm{E}(a,c,a,S_f(c))$. With $\mathrm{E}(a,c,a,S_f(c))$ and $\mathrm{E}(a,c,a,S_a(c))$ we get $\mathrm{E}(a,S_f(c),a,S_a(c))$. We further argue that already $\mathrm{M}(S_f(c),t_1,S_a(c))$ for

$$t_1 := ip(a,S_a(c),ext(a,S_f(c),ld_2,ld_3),S_f(c),ip(ext(a,S_f(c),ld_2,ld_3),\\ ext(a,S_a(c),S_f(c),ext(a,S_f(c),ld_2,ld_3)),a,S_f(c),S_a(c)))$$

as it can be shown that we are in 'case 2' of the proof of Lemma 3.10, i.e. that $\neg(\mathrm{T}(S_f(c),a,S_a(c)) \wedge S_f(c) \neq S_a(c) \neq a \neq S_f(c))$: Suppose that $\mathrm{T}(S_f(c),a,S_a(c)) \wedge S_f(c) \neq S_a(c) \neq a \neq S_f(c)$. Then, from $\mathrm{T}(S_a(c),a,c) \wedge \mathrm{T}(S_f(c),a,S_a(c)) \wedge a \neq S_a(c)$ we infer that $\mathrm{Col}(a,c,S_f(c))$. We further have $\mathrm{Col}(c,f,S_f(c)) \wedge S_f(c) \neq c$ and thus $\mathrm{Col}(a,c,f)$. But now with $\mathrm{Col}(a,b,f) \wedge a \neq f$ [4] it follows that $\mathrm{Col}(a,b,c)$. Contradiction.
As we have $\mathrm{T}(S_a(c),a,c) \wedge \mathrm{T}(S_f(c),f,c) \wedge \mathrm{T}(S_a(c),t_1,S_f(c))$, Lemma 3.23 provides us with a term $s_1 := ip(c,f,S_a(c),a,ip(c,S_a(c),S_f(c),f,t_1))$ such that $\mathrm{T}(c,s_1,t_1) \wedge \mathrm{T}(f,s_1,a)$. But then, in particular $\mathrm{Col}(a,b,s_1)$. We have now shown that s_1 is one of our realizers, i.e. $\mathrm{Col}(a,b,s_1)$ and $\mathrm{T}(c,s_1,t_1)$.
It remains to show that also t_1 is as desired, i.e $t_1 \neq a$ and $(\mathrm{Col}(a,b,g(t_1,s_1)) \wedge \mathrm{Col}(t_1,a,h(t_1,s_1)) \rightarrow \mathrm{R}(g(t_1,s_1),a,h(t_1,s_t)))$. To show $t_1 \neq a$, suppose for the sake of contradiction that $t_1 = a$. But then from $\mathrm{T}(S_f(c),t_1,S_a(c))$ we get that $\mathrm{T}(S_f(c),a,S_a(c))$. With $\mathrm{T}(S_a(c),a,c)$, $S_a(c) \neq a$ and $\mathrm{E}(a,c,a,S_f(c))$, this implies $c = S_f(c)$. Contradiction.
We show $(\mathrm{Col}(a,b,g(t_1,s_1)) \wedge \mathrm{Col}(t_1,a,h(t_1,s_t)) \rightarrow \mathrm{R}(g(t_1,s_1),a,h(t_1,s_1)))$ by a case distinction. Assume $\mathrm{Col}(a,b,g(t_1,s_1)) \wedge \mathrm{Col}(t_1,a,h(t_1,s_1))$. Suppose $f \neq a$. We have shown that $\mathrm{R}(a,f,c)$ and further know that $f \neq c$. In the same way as in the proof of Proposition 3.19 (i.e. using Lemma 3.14) it now follows that $\mathrm{R}(g(t_1,s_1),a,h(t_1,s_1)))$. Suppose $f = a$. Then since $\mathrm{T}(f,s_1,a)$ also $s_1 = a$, hence $L(t_1a) = L(t_1s_1) = L(cf)$. Thus $\mathrm{Col}(t_1,a,h(t_1,s_1))$ implies $\mathrm{Col}(c,f,h(t_1,s_1))$. By $(*)$ and reasoning as above, we know that in particular $ab \overset{g,h}{\underset{f}{\perp}} cf$ and as we have $Col(a,b,g(t_1,s_1))$ and $Col(c,f,h(t_1,s_1))$ it follows that $R(g(t_1,s_1),f,h(t_1,s_1))$. As $f = a$ it follows that $R(g(t_1,s_1),a,h(t_1,s_1))$ which was to show. We have shown that in case 1, $\varphi^H(t_1,s_1)$.
Case 2: $\mathrm{Col}(a,b,c)$. By A8 it holds that $\neg\mathrm{Col}(a,b,ld_1) \vee \neg\mathrm{Col}(a,b,ld_2) \vee \neg\mathrm{Col}(a,b,ld_3)$. We can thus set $s_i = c$ $(i = 2,3,4)$, for which trivially $\mathrm{Col}(a,b,s_i)$ and $\mathrm{T}(c,s_i,t_i)$ and do the same construction for t_i as we did in case 1 with c substituted by ld_1, ld_2, ld_3. Hence in case 2, $\varphi^H(t_2,s_2) \vee \varphi^H(t_3,s_3) \vee \varphi^H(t_4,s_4)$.
As our Herbrand terms do not involve the Herbrand index functions g, h, the proposition follows. ☐

Remark 3.27. Note that here, the case distinction that is made in the proof actually translates into different Herbrand disjunctions. This is due to the fact that we actually *have to* consider at least two different ways of realizing our points x,y as we can only find the foot of a perpendicular from c on $L(ab)$ if $\neg\mathrm{Col}(a,b,c)$. Hence, if $\mathrm{Col}(a,b,c)$, we first have to construct a point c' for which $\neg\mathrm{Col}(a,b,c')$. As axiom A8 that provides us with such a term is of disjunctive character itself (i.e. can only say that c' must be one of three options), this also translates into different Herbrand disjunctions. Of course, this does not imply that there is no Herbrand disjunction for φ^H for which $n < 4$. However, one would have to find a very different way of constructing points x and y, e.g. one wouldn't be allowed to use Proposition 3.18.

We have already shown that two points that have the same distance from a third point have a midpoint. With what we have considered above, we can now show that we can construct a midpoint for any two given points. We can prove the following statement:

Proposition 3.28. *[13, Satz I.8.22]* $\mathbf{T} \vdash \varphi := \exists x\, \mathrm{M}(a,x,b)$.

Remark 3.29. Again, it can be shown that x is unique. We will only consider existence here but can prove uniqueness as a universal statement.

[4]suppose $a = f$, then $S_a(c) = S_f(c)$

Proposition 3.30. *We have the following Herbrand disjunction (Figure 16)*

$$\mathbf{T}^S \vdash \bigvee_{i=1}^{4} M(a,t_i,b) \quad \textit{where} \quad t_i := wit(a,b,p_i) \textit{ for } i=1,2,3,\ t_4 := a$$

with $p_i = perp(a,b,q_i)$ *for* $i=1,2,3$, $q_i = perp(b,a,ld_i)$ *for* $i=1,2,3$.

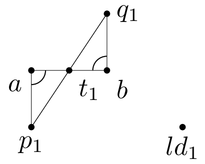


Figure 16: Constr. for t_1, using square

Definition 3.31. Set $m_i(a,b) := t_i$ for $i=1,\ldots,4$.

Proof. The proof is based on the proof of Proposition 3.28 in [13, p. 64f]. We distinguish two cases. Case 1: $a=b$. Then $\mathrm{M}(a,a,b)$, hence $t_4=a$. Case 2: $a \neq b$. We want to construct a perpendicular on ab in the point b, where we take a as our reference point (Figure 17). With the proof of Proposition 3.25 with (b,a,a) as (a,b,c) we, in particular have $\mathbf{T}^S \vdash b \neq a \to \bigvee_{i=1}^{3} ba \overset{a,q_i}{\underset{b}{\perp}} q_i b$ for $q_i := perp(b,a,ld_i)$ for $i=1,2,3$ (as by $\mathrm{Col}(b,a,a)$, we are in 'case 2' of the proof of Proposition 3.25). By assumption $b \neq a$, thus $\bigvee_{i=1}^{3} ba \overset{a,q_i}{\underset{b}{\perp}} q_i b$. Suppose that $ba \overset{a,q_i}{\underset{b}{\perp}} q_i b$. Then (as $\mathrm{Col}(b,a,a)$ and $\mathrm{Col}(q_i,b,q_i)$) also $\mathrm{R}(a,b,q_i)$.

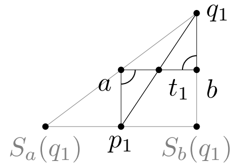


Figure 17: Constr. for t_1, only using square for q_1

We now again construct a perpendicular (Figure 17), however this time in a and take q_i to be our 'reference point' (i.e c in Proposition 3.25 is q_i). As we have that $\neg\,\mathrm{Col}(a,b,q_i)$, we are in 'case 1' of Proposition 3.25 and get terms $p_i = perp(a,b,q_i)$ and $t_i = wit(a,b,p_i)$ with properties as in Proposition 3.25 and in particular $\mathrm{R}(b,a,p_i)$. The proof of Proposition 3.28 in [13, p.64f] would now entice us to distinguish two subcases, '$ap_i \leq bq_i$' and '$ap_i \geq bq_i$' and construct a term with the desired properties for both of these cases. However, it can be shown that $\mathrm{E}(a,p_i,b,q_i)$, i.e. '$ap_i = bq_i$'. Yet, the argument is based on propositions that are themselves proved by Proposition 3.28 in [13]. Hence, in the proof of Proposition 3.28 in [13], the case distinction cannot be avoided by arguing via these propositions. We, however, can freely use true universal facts as axioms when extracting Herbrand terms. We thus now establish the truth of a suitable universal statement by proving it in $\mathbf{T}$.

But first, we extend our construction a little (Figure 18). We consider terms $S_a(p_i)$, $S_a(S_b(q_i))$ and $S_a(b)$ for which by Proposition 3.7 $\mathrm{M}(p_i,a,S_a(p_i))$, $\mathrm{M}(S_b(q_i),a,S_a(S_b(q_i)))$ and $\mathrm{M}(b,a,S_a(b))$ respectively. It can be shown that if we interpret $S_a(\ldots)$ as a function that reflects given points at a, it preserves congruence and betweenness (see [13, p.49-51]). Hence the marked sides of Figure 18 are congruent to each other, and our 'new' terms are still collinear. The same can be shown for mirroring points at a line ([13, p.89 f]) which is why from $\mathrm{T}(p_i,t_i,q_i)$ it follows that also $\mathrm{T}(S_a(p_i),t_i,S_b(q_i))$.

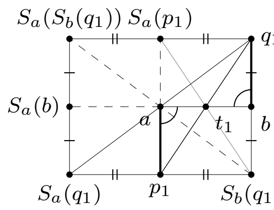


Figure 18: Ext. constr.

Now back to our purely universal statement. We will argue that (Figure 19)

$$\begin{aligned}\mathbf{T} \vdash \forall e,f,g,h,m,k,l,n\big[\,&\mathrm{E}(f,g,e,h) \wedge E(h,g,f,e) \wedge \neg\,\mathrm{Col}(e,f,g) \wedge f \neq h \wedge \\ &\mathrm{Col}(e,m,g) \wedge \mathrm{Col}(f,m,h) \wedge \mathrm{T}(e,k,f) \wedge \mathrm{T}(g,l,h) \wedge \\ &\mathrm{E}(k,f,l,g) \wedge \mathrm{T}(l,n,f) \wedge \mathrm{T}(k,n,g) \to \mathrm{E}(k,l,f,g)\big].\end{aligned}$$

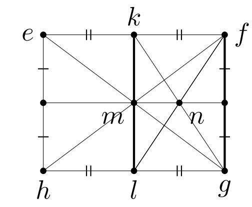


Figure 19

To show this, it can be shown that if $\mathrm{E}(f,g,e,h) \wedge E(h,g,f,e) \wedge \neg\,\mathrm{Col}(e,f,g) \wedge f \neq h \wedge \mathrm{Col}(e,m,g) \wedge \mathrm{Col}(f,m,h)$, then the segment ef is parallel to the segment gh (which is to be understood as: there is no point x that is both collinear to ef and gh, we refrain from giving a formal definition here but for reference see [13, Folgerung I.12.7(a)]). This is proven in [13, Satz I.12.18 (a)]. But then of course also for any point k between e and f, and l between g and h, i.e. with $\mathrm{T}(e,k,f)$ and $\mathrm{T}(g,l,h)$ it holds true that kf is parallel to gl. We can further show a sentence that states that if kf and gl are parallel and $\mathrm{E}(k,f,l,g)$ and if there exists a point n with $\mathrm{T}(l,n,f) \wedge \mathrm{T}(k,n,g)$ then $\mathrm{E}(k,l,f,g)$. This is proven in [13, Satz I.12.20], in particular via Proposition 3.28. We hence have shown our universal statement to hold in $\mathbf{T}$.

With the argument we have given above, now also

$$\begin{aligned}\mathbf{T}^S \vdash\ &\mathrm{E}(q_i,S_b(q_i),S_a(S_b(q_i)),S_a(q_i)) \wedge \mathrm{E}(S_a(q_i),S_b(q_i),q_i,S_a(S_b(q_i))) \wedge \neg\,\mathrm{Col}(S_a(S_b(q_i)),q_i,S_b(q_i)) \wedge q_i \neq S_a(q_i) \\ &\wedge \mathrm{Col}(S_a(S_b(q_i)),a,S_b(q_i)) \wedge \mathrm{Col}(q_i,a,S_a(q_i)) \wedge \mathrm{T}(S_a(S_b(q_i)),S_a(p_i),q_i) \wedge \mathrm{T}(S_b(q_i),p_i,S_a(q_i)) \\ &\wedge \mathrm{E}(S_a(p_i),q_i,p_i,S_b(q_i)) \wedge \mathrm{T}(p_i,t_i,q_i) \wedge \mathrm{T}(S_a(p_i),t_i,S_b(q_i)) \to \mathrm{E}(S_a(p_i),p_i,q_i,S_b(q_i)).\end{aligned}$$

We thus have $\mathrm{E}(S_a(p_i), p_i, q_i, S_b(q_i))$ which implies $\mathrm{E}(p_i, a, q_i, b)$ which was to be shown.
We can now proceed to show that already t_i is the term for which $\mathrm{M}(a, t_i, b)$. Here, we again proceed just as in the proof of Proposition 3.28 in [13, p. 64f]. By construction, $\mathrm{Col}(a, b, t_i)$. From above, we know that $\mathrm{R}(b, a, p_i)$. Further, $a \neq b$ and $a \neq p_i$, hence $\neg\,\mathrm{Col}(a, b, p_i)$. To see this (by contraposition), assume $\mathrm{Col}(a, b, p_i)$ and assume further $\mathrm{R}(b, a, p_i)$ and $b \neq a$. It then follows by Lemma 3.14 that $\mathrm{R}(p_i, a, p_i)$ and thus $p_i = a$ by [13, Satz I.8.8]. Analogously, $\neg\,\mathrm{Col}(a, b, q_i)$. It now is enough to show that $\mathrm{E}(b, p_i, a, q_i)$ (Figure 20), as then the points a, q_i, b, p_i form a non-degenerate quadrangle where the respective opposite sides are congruent. It then can be shown that the diagonals half each other and hence $\mathrm{M}(a, t_i, b) \wedge \mathrm{M}(p_i, t_i, q_i)$ (for a proof of this, see [13, Lemma I.7.21]). For the proof of $\mathrm{E}(b, p_i, a, q_i)$ we refer to [13, p.65]. In total, we have shown that in case 2, $M(a, t_i, b)$ holds for some of $i = 1, 2, 3$.
□

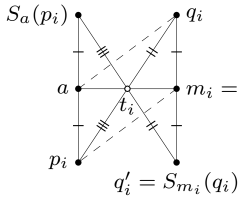


Figure 20

Remark 3.32. Note that we were only able to use Proposition 3.25 in the course of our proof because the terms provided there satisfy the original statement φ and not just its Herbrand normal form φ^H.

3.3 Opposed points and a disjunction due to the character of φ

If points a, b and c are pairwise distinct and in the relation $\mathrm{T}(a, b, c)$, we can say that points a and c *lie on opposite sides* of point b ([13, p.43, 6.1 Definition]). We now define a notion stating that points a and b lie *on the same side* of point c:

Definition 3.33. [13, Definition I.6.1] $a \underset{c}{\simeq} b :\leftrightarrow a \neq c \wedge b \neq c \wedge [\mathrm{T}(c, a, b) \vee \mathrm{T}(c, b, a)]$.

We now want to provide Herbrand disjunctions for three lemmas, all of which talk about points lying on different sides of a line in various ways. We therefore define a notion '$\mathrm{T}(a, L(lk), b)$' which is true iff there is a point $x \in L(lk)$ such that $\mathrm{T}(a, x, b)$, i.e. iff a and b lie on different sides of $L(lk)$.

Definition 3.34. [13, Definition I.9.1]
$\mathrm{T}(a, L(lk), b) :\leftrightarrow l \neq k \wedge \neg\,\mathrm{Col}(a, l, k) \wedge \neg\,\mathrm{Col}(b, l, k) \wedge \exists x[\mathrm{Col}(l, k, x) \wedge \mathrm{T}(a, x, b)]$.

We further define a notion that enables us to 'compare the length of two segments'. We say that ab is *less or equal* cd if there exists a point y between the points c and d in such a way that the segment ab is congruent (i.e. has the same length) to the segment cy:

Definition 3.35. [13, Definition I.5.4 and Definition I.5.14]
$ab \leq cd :\leftrightarrow \exists y[\mathrm{T}(c, y, d) \wedge \mathrm{E}(a, b, c, y)]$, $ab < cd :\leftrightarrow ab \leq cd \wedge \neg\,\mathrm{E}(a, b, c, d)$.

We also give an equivalent characterization:

Proposition 3.36. *[13, Satz I.5.5]* $ab \leq cd \leftrightarrow \exists y[\mathrm{T}(a, b, y) \wedge \mathrm{E}(a, y, c, d)]$.

For a proof, see [13, p.41f]

Corollary 3.37. *It holds that either* $ab \leq cd$ *or* $cd < ab$.

The first lemma (Figure 21) that we consider states that if points a and c lie on different sides of the line $L(lk)$ in such a way that they are each other's mirror image with respect to a point m on the line and if the point n is also on $L(lk)$, then every point b with $a \underset{n}{\simeq} b$ (i.e that lies on 'one side' of $L(lk)$ on the line determined by the points n and a) lies on the opposite side of $L(lk)$ with respect to c (see [13, description of Lemma I.9.3]).

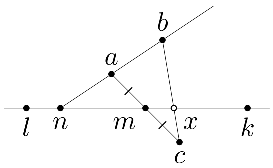


Figure 21: based on [13, Abb.33]

Lemma 3.38. *[13, Lemma I.9.3]*

$$\mathrm{T} \vdash \varphi := (\forall L(lk) \forall a, b, c, m, n)\big[\mathrm{T}(a, L(lk), c) \wedge \mathrm{Col}(m, l, k) \wedge \mathrm{M}(a, m, c) \wedge \mathrm{Col}(n, l, k) \rightarrow \forall b[a \underset{n}{\simeq} b \rightarrow \mathrm{T}(b, L(lk), c)]\big].$$

The above lemma is not formulated in our first order setting as we quantify over $L(lk)$. We further have hidden some quantifiers in expressions $\mathrm{T}(a, L(lk), c)$ and $\mathrm{T}(b, L(lk), c)$. We therefore state an equivalent, fully prenexed version. To this end, we introduce two new variables j and x and write $\mathrm{T}(a, L(lk), c)$ as
$\exists j[l \neq k \wedge \neg\,\mathrm{Col}(a, l, k) \wedge \neg\,\mathrm{Col}(c, l, k) \wedge \mathrm{Col}(l, k, j) \wedge \mathrm{T}(a, j, c)]$ and $\mathrm{T}(b, L(lk), c)$ as
$\exists x[l \neq k \wedge \neg\,\mathrm{Col}(b, l, k) \wedge \neg\,\mathrm{Col}(c, l, k) \wedge \mathrm{Col}(x, l, k) \wedge \mathrm{T}(b, x, c)]$:

Lemma 3.39.

$$\begin{aligned}\mathrm{T} \vdash \varphi :=& \exists x \varphi_{qf}(l,k,a,b,c,m,j,n,x)\\ :=& \exists x \big[l \neq k \wedge \neg\,\mathrm{Col}(a,l,k) \wedge \neg\,\mathrm{Col}(c,l,k) \wedge \mathrm{Col}(l,k,j) \wedge \mathrm{T}(a,j,c) \wedge \mathrm{Col}(m,l,k) \wedge \mathrm{M}(a,m,c) \wedge \mathrm{Col}(n,l,k)\\ &\rightarrow [a \underset{n}{\backsimeq} b \rightarrow l \neq k \wedge \neg\,\mathrm{Col}(b,l,k) \wedge \neg\,\mathrm{Col}(c,l,k) \wedge \mathrm{Col}(x,l,k) \wedge \mathrm{T}(b,x,c)]\big].\end{aligned}$$

We can now prove the following:

Lemma 3.40. *Let φ be as in Lemma 3.39. As $\varphi^H = \varphi$,*

$$\mathbf{T}^S \vdash \bigvee_{i=1}^{2} \varphi_{qf}(l,k,a,b,c,m,j,n,t_i(l,k,a,b,c,m,j,n)) \ \ \textit{for} \ \ t_1 := ip(n,c,a,b,m), \ \ t_2 := ip(S_m(n),b,S_m(b),c,m).$$[5]

We refrain from giving a detailed proof here but only give an outline. The proof is based on the proof of Lemma 3.39 in [13, p.68].

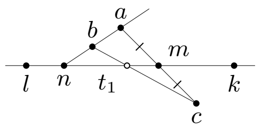


Figure 22: Case 1: Constr. for t_1, based on [13, Abb.33]

Outline of the proof. Let $a \underset{n}{\backsimeq} b$. Then either $\mathrm{T}(n,b,a)$ or $\mathrm{T}(n,a,b)$.
We distinguish two cases. **Case 1** (Figure 22) considers $\mathrm{T}(n,b,a)$. As we also have $\mathrm{T}(c,m,a)$, by $\mathrm{A7}^S$ we get a term $t_1 := ip(n,c,a,b,m)$ with $\mathrm{T}(b,t_1,c) \wedge \mathrm{T}(m,t_1,n)$.
From $n \neq b$ it follows that $\neg\,\mathrm{Col}(l,k,b)$ as else $\mathrm{Col}(l,k,a)$. From $\mathrm{T}(m,t_1,n)$ it follows that $\mathrm{Col}(t_1,l,k)$. Hence $\varphi^H_{qf}(t_1)$.
For **case 2** (Figure 23), $\mathrm{T}(n,a,b)$, we cannot do the same construction as we cannot invoke $\mathrm{A7}^S$ on our points in the same way. However, we can retreat ourselves to a situation as in case 1 but for different points. To this end, we consider terms $S_m(b)$ and $S_m(n)$ and deduce $\mathrm{T}(S_m(n),c,S_m(b))$ from $\mathrm{T}(n,a,b)$. But now $\mathrm{T}(S_m(b),L(lk),b)$ via m and $\mathrm{T}(S_m(n),c,S_m(b))$, i.e. case 1 for our new points. Hence again from $\mathrm{T}(S_m(n),c,S_m(b)) \wedge \mathrm{T}(b,m,S_m(b))$ and $\mathrm{A7}^S$ we can show that for t_2 as defined above, $\varphi^H_{qf}(t_2)$. □

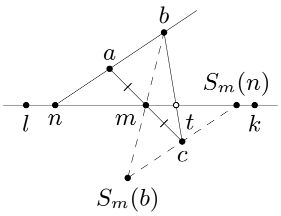


Figure 23: Case 2: Constr. for t_2, [13, Abb.33]

Remark 3.41. The case distinction that is made in the proof here, resulting in multiple Herbrand terms, originates in the disjunctive character of the sentence itself. By definition, $a \underset{n}{\backsimeq} b \leftrightarrow a \neq n \wedge b \neq n \wedge [\mathrm{T}(n,b,a) \vee \mathrm{T}(n,a,b)]$. We thus distinguish cases $\mathrm{T}(n,b,a)$ and $\mathrm{T}(n,a,b)$ and get two different terms.

The next lemma (Figure 24) that we will consider states that if the points a and c lie on opposite sides of a line and if the points n and o are the feet of the plumbs of a and c on that line respectively, then every point on the half-line that starts at the point n and passes through a lies opposite, with respect to our line, to every point
that lies on the half-line that starts at point o and passes through c ([13, description of Lemma I.9.4]).

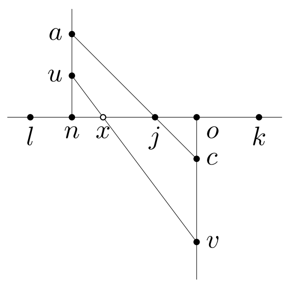


Figure 24: based on [13, Abb. 34]

Lemma 3.42. *[13, (subcase of) Lemma I.9.4]*

$$\begin{aligned}\mathbf{T} \vdash \varphi := (\forall L(lk),a,c,n,o,u,v)\big[\,&\mathrm{T}(a,L(lk),c) \wedge \mathrm{Col}(n,l,k) \wedge lk \underset{n}{\perp} an \wedge \mathrm{Col}(o,l,k)\\ &\wedge lk \underset{o}{\perp} co \rightarrow [u \underset{n}{\backsimeq} a \wedge v \underset{o}{\backsimeq} c \rightarrow \mathrm{T}(v,L(lk),u)]\big].\end{aligned}$$[6]

Again, we can formulate a version that is slightly more spelled out where we, just as above, introduce j and x to be the points witnessing $\mathrm{T}(a,L(lk),c)$ and $\mathrm{T}(v,L(lk),u)$ respectively.

[5] Note that j,l,k do not occur in t_1 or t_2 but j implicitly occurs as $j = m$.

[6] To be precise, in [13, 9.4 Lemma, p.68] it is only shown that $\mathbf{T} \vdash \varphi := (\forall L(lk),a,c,n,o,u,v)\big[\,\mathrm{T}(a,L(lk),c) \wedge \mathrm{Col}(n,l,k) \wedge lk \perp an \wedge \mathrm{Col}(o,l,k) \wedge lk \perp co \rightarrow [u \underset{n}{\backsimeq} a \wedge v \underset{o}{\backsimeq} c \rightarrow \mathrm{T}(v,L(lk),u)]\big]$ but it is immediate from the proof that n resp. o are the points for which $lk \underset{n}{\perp} an$ resp. $lk \underset{o}{\perp} co$, hence our formulation.

Lemma 3.43.

$$\mathbf{T} \vdash \varphi := \exists x\big[l \neq k \wedge \neg\,\mathrm{Col}(a,l,k) \wedge \neg\,\mathrm{Col}(c,l,k) \wedge \mathrm{Col}(j,l,k) \wedge \mathrm{T}(a,j,c) \wedge \mathrm{Col}(n,l,k) \wedge lk \underset{n}{\perp} an \wedge \mathrm{Col}(o,l,k)$$
$$\wedge\, lk \underset{o}{\perp} co \rightarrow [u \underset{n}{\asymp} a \wedge v \underset{o}{\asymp} c \rightarrow l \neq k \wedge \neg\,\mathrm{Col}(v,l,k) \wedge \neg\,\mathrm{Col}(u,l,k) \wedge \mathrm{Col}(x,l,k) \wedge \mathrm{T}(v,x,u)]\big].$$

Still, we are not in a prenexed setting as we find 4 more universally quantified variables in the expressions $lk \underset{n}{\perp} an$ and $lk \underset{o}{\perp} co$ (see Definition 3.16) which we will denote by $\underline{z} := (z_1, z_2, z_3, z_4)$. If we pull out these quantifiers, we can write φ in a form where $\varphi = (\forall \underline{y})\exists \underline{z}, x\ \psi_{qf}(\underline{y}, \underline{z}, x)$. If we applied Herbrand to this, we would hence realize five different points. In our further investigations however, we are only interested in realizing $\mathrm{T}(u, L(lk), v)$, i.e that point x for which $\mathrm{Col}(x,l,k) \wedge \mathrm{T}(v,x,u)$. We thus only formulate the following lemma:

Lemma 3.44. *Let $\varphi = \exists x \psi(l,k,a,c,n,o,u,v,j,x)$ be as above, where ψ denotes the formula $[\ldots]$. Then*

$$\mathbf{T}^S \vdash \bigvee_{i=1}^{4} \psi(l,k,a,c,n,o,u,v,j,t_i(l,k,a,c,n,o,u,v,j)) \quad \text{with}$$

$t_i = ip(o,u,S_{s_i}(u),v,s_i)$, *for $i = 1,2$, and* $t_i = ip(S_{s_{i-2}}(o), v, S_{s_{i-2}}(v), u, s_{i-2})$ *for $i = 3,4$, where*
$s_1 = ip(n,c,a,r_1,j)$ *with* $r_1 = ext(S_n(a),n,o,c)$ *and* $s_2 = ip(o,a,c,r_2,j)$ *with* $r_2 = ext(S_o(c),o,n,a)$.

Again, we only give an outline here. The proof is based on the proof of Lemma 3.42 in [13, p. 69].

Outline of the proof. For the proof, we distinguish two different cases and, after a few construction steps, argue that we are in the situation of Lemma 3.40. With Corollary 3.37, we know that $\mathbf{T} \vdash oc \leq na \vee na \leq oc$, i.e. $\mathbf{T} \vdash \exists x[\mathrm{T}(n,x,a) \wedge \mathrm{E}(n,x,o,c)] \vee \exists y[\mathrm{T}(o,y,c) \wedge \mathrm{E}(o,y,n,a)]$.

We show that $\mathbf{T}^S \vdash oc \overset{r_1}{\leq} na \vee na \overset{r_2}{\leq} oc$ where $oc \overset{r_1}{\leq} na := \mathrm{T}(n,r_1,a) \wedge \mathrm{E}(n,r_1,o,c)$. That is r_1 and r_2 realize x and y, respectively, in the above expression: If $oc \leq na$ (Figure 25), we argue via Proposition 3.7 and $\mathrm{A4}^S$ that for $r_1 := ext(S_n(a),n,o,c)$ we have $\mathrm{E}(n,r_1,o,c)$ i.e. '$nr_1 = oc$'. Further $\mathrm{T}(n,r_1,a)$. For, if not, then $\mathrm{T}(n,a,r_1) \wedge a \neq r_1$. But then Proposition 3.36 and $\neg\,\mathrm{E}(n,a,o,c)$ imply $na \overset{r_1}{<} oc$ and hence $na < oc$, contradiction. We hence have $oc \overset{r_1}{\leq} na$ and in particular $oc \overset{r_1}{\leq} na \vee na \overset{r_2}{\leq} oc$. An analogous argument for the case that $na \leq oc$ results in the same realizing terms r_1 and r_2. We can now distinguish cases $oc \overset{r_1}{\leq} na$ and $na \overset{r_2}{\leq} oc$. Just as above, this case distinction is necessary to be able to argue with $\mathrm{A7}^S$. If $oc \overset{r_1}{\leq} na$ (Figure 25), argue that from $\mathrm{T}(c,j,a) \wedge \mathrm{T}(n,r_1,a)$ and $\mathrm{A7}^S$ it follows that $\mathrm{T}(n,s_1,o) \wedge \mathrm{T}(r_1,s_1,c)$ for $s_1 = ip(n,c,a,r_1,j)$. In our situation, it even holds true that $\mathrm{M}(n,s_1,o) \wedge \mathrm{M}(r_1,s_1,c)$ (for a proof, consider e.g. realizers for [13, Lemma I.8.24]). From properties about $S_{s_1}(\ldots)$ and the assumptions $u \underset{n}{\asymp} a$ and $v \underset{o}{\asymp} c$ it can be shown that $S_{s_1}(u) \underset{o}{\asymp} v$. But now $\mathrm{T}(S_{s_1}(u), L(lk), u)$ via s_1 and further $\mathrm{Col}(l,k,s_1) \wedge \mathrm{M}(S_{s_1}(u), s_1, u) \wedge \mathrm{Col}(o,l,k) \wedge S_{s_1}(u) \underset{o}{\asymp} v$. We thus are in the situation of Lemma 3.40 and deduce that $\mathrm{Col}(t_1,l,k) \wedge \mathrm{T}(v,t_1,u)$ or $\mathrm{Col}(t_3,l,k) \wedge \mathrm{T}(v,t_3,u)$ for t_1 and t_3 as defined above. Hence $\psi_{qf}(\tau,t_1) \vee \psi_{qf}(\tau,t_3)$. An analogous procedure for case 2, where we proceed in the exact same way but 'on the other side', yields $\psi_{qf}(\tau,t_2) \vee \psi_{qf}(\tau,t_4)$. □

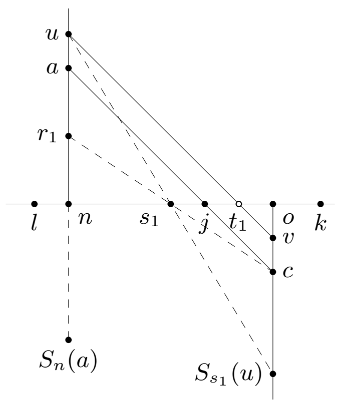


Figure 25: Case 1: Constr. for t_1 based on [13, Abb. 34]

The last lemma (Figure 26) of this kind we consider states, that if the points a and c lie on opposite sides of a line and if the point n lies on that same line, then every point b of the half-line that starts at n and passes through a lies on the opposite side of our line with respect to c ([13, description of Satz I.9.5]).

Lemma 3.45. *[13, Satz I.9.5]*

$$\mathrm{T} \vdash \varphi := (\forall L(lk) \forall a,b,c,n)\big[\,\mathrm{T}(a,L(lk),c) \wedge \mathrm{Col}(n,l,k) \rightarrow [a \underset{n}{\asymp} b \rightarrow \mathrm{T}(b,L(lk),c)]\big].$$

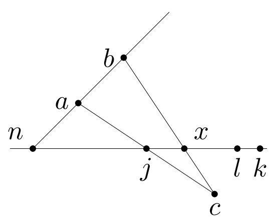


Figure 26

Again, equivalently

Lemma 3.46.

$$\mathrm{T} \vdash \varphi := \exists x \big[l \neq k \wedge \neg \operatorname{Col}(a,l,k) \wedge \neg \operatorname{Col}(c,l,k) \wedge \operatorname{Col}(j,l,k) \wedge \mathrm{T}(a,j,c) \wedge \operatorname{Col}(l,k,n) \rightarrow [a \underset{n}{\backsimeq} b \rightarrow l \neq k \wedge \neg \operatorname{Col}(b,l,k) \wedge \neg \operatorname{Col}(c,l,k) \wedge \operatorname{Col}(l,k,x) \wedge \mathrm{T}(b,x,c)]\big].$$

With Herbrand, we can show

Lemma 3.47. *Let φ be as above. As $\varphi^H = \varphi$, we can show*

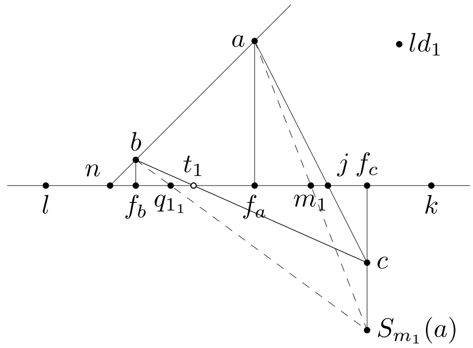


Figure 27: (based on [13, Abb. 35]) Construction for t_1, where q_{1_1} results from applying Lemma 3.40 and t_1 from applying Lemma 3.44.

$$\mathbf{T}^S \vdash \bigvee_{i=1}^{32} \varphi_{qf}(a,b,c,n,j,l,k,t_i(a,b,c,n,j,l,k)) \quad \textit{with}$$

$t_i = ip(f_c, b, S_{s_{1_{1_i}}}(b), c, n_{1_{1_i}})$ *for* $i = 1, \ldots, 4$,

$t_i = ip(f_c, b, S_{s_{1_{2_{i-4}}}}(b), c, s_{1_{2_{i-4}}})$ *for* $i = 5, \ldots, 8$,

$t_i = ip(f_c, b, S_{s_{2_{1_{i-8}}}}(b), c, n_{2_{1_{i-8}}})$ *for* $i = 9, \ldots, 12$,

$t_i = ip(f_c, b, S_{s_{2_{2_{i-12}}}}(b), c, s_{2_{2_{i-12}}})$ *for* $i = 13, \ldots, 16$,

$t_i = ip(S_{s_{1_{1_{i-16}}}}(f_c), c, S_{s_{1_{1_{i-16}}}}(c), b, s_{1_{1_{i-16}}})$ *for* $i = 17, \ldots, 20$,

$t_i = ip(S_{s_{1_{2_{i-20}}}}(f_c), c, S_{s_{1_{2_{i-20}}}}(c), b, s_{1_{2_{i-20}}})$ *for* $i = 21, \ldots, 24$,

$t_i = ip(S_{s_{2_{1_{i-24}}}}(f_c), c, S_{s_{2_{1_{i-24}}}}(c), b, s_{2_{1_{i-24}}})$ *for* $i = 24, \ldots, 28$,

$t_i = ip(S_{s_{2_{2_{i-28}}}}(f_c), c, S_{s_{2_{2_{i-28}}}}(c), b, s_{2_{2_{i-28}}})$ *for* $i = 29, \ldots, 32$,

where

$s_{1_{1_i}} = ip(f_b, S_{m_i(f_a,f_c)}(a), r_{1_i}, q_{1_i})$, $\quad s_{1_{2_i}} = ip(f_b, S_{m_i(f_a,f_c)}(a), r_{1_i}, q_{2_i})$, $\quad r_{1_i} = ext(S_{f_b}(b), y, f_c, S_{m_i(f_a,f_c)}(a))$,

$s_{2_{1_i}} = ip(f_c, b, S_{m_i(f_a,f_c)}(a), r_{2_i}, q_{1_i})$, $s_{2_{2_i}} = ip(f_c, b, S_{m_i(f_a,f_c)}(a), r_{2_i}, q_{2_i})$, $r_{2_i} = ext(S_z(S_{m_i(f_a,f_c)}(a)), f_c, y, b)$,

$q_{1_i} = ip(n, S_{m_i(f_a,f_c)}(a), a, b, m_i(f_a,f_c))$, $q_{2_i} = ip(S_{m_i(f_a,f_c)}(n), b, S_{m_i(f_a,f_c)}(b), S_{m_i(f_a,f_c)}(a), m_i(f_a,f_c))$,

$f_a := foot(l,k,a)$, $f_b := foot(l,k,b)$, $f_c := foot(l,k,c)$.

Just as above, we give an outline of the proof. It is based on the proof of Lemma 3.45 in [13, p.70].

Outline of the proof. It is helpful to consider Figure 27 in the course of this proof. By Proposition 3.19 we get terms $f_a := foot(l,k,a)$ with $lk \underset{f_a}{\perp} af_a$, $f_b := foot(l,k,b)$ with $lk \underset{f_b}{\perp} bf_b$, $f_c := foot(l,k,c)$ with $lk \underset{f_c}{\perp} cf_c$. By Proposition 3.30, we know that $\bigvee_{i=1}^{4} \mathrm{M}(f_a, m_i(f_a,f_c), f_c)$. Suppose that $\mathrm{M}(f_a, m_i(f_a,f_c), f_c)$ for some $i \in \{1, \ldots, 4\}$. Additionally, we know that $\mathrm{M}(a, m_i(f_a,f_c), S_{m_i(f_a,f_c)}(a))$. Further, by the uniqueness in Proposition 3.7 and $\mathrm{M}(f_a, m_i(f_a,f_c), f_c)$, it holds that $f_c = S_{m_i(f_a,f_c)}(f_a)$. Similar to the proof of Lemma 3.44 it can thus be shown that $S_{m_i(f_a,f_c)}(a) \underset{f_c}{\backsimeq} c$. From $\mathrm{T}(a, L(lk), S_{m_i(f_a,f_c)}(a))$ via $m_i(f_a,f_c)$ and $\operatorname{Col}(l,k,m_i(f_a,f_c)) \wedge \mathrm{M}(a, m_i(f_a,f_c), S_{m_i(f_a,f_c)}(a)) \wedge \operatorname{Col}(n,l,k) \wedge a \underset{n}{\backsimeq} b$ and Lemma 3.40 we know that $q_{1_i} := ip(n, S_{m_i(f_a,f_c)}(a), a, b, m_i(f_a,f_c))$ or $q_{2_i} := ip(S_{m_i(f_a,f_c)}(n), b, S_{m_i(f_a,f_c)}(b), S_{m_i(f_a,f_c)}(a), m_i(f_a,f_c))$ witnesses $\mathrm{T}(b, L(lk), S_{m_i(f_a,f_c)}(a))$. But now we can invoke Lemma 3.44 on the following: $\mathrm{T}(b, L(lk), S_{m_i(f_a,f_c)}(a))$ via q_{1_i} or q_{2_i} and $\operatorname{Col}(f_b,l,k) \wedge lk \underset{f_b}{\perp} bf_b \wedge \operatorname{Col}(f_c,l,k) \wedge lk \underset{f_c}{\perp} S_{m_i(f_a,f_c)}(a)f_c \wedge b \underset{f_b}{\backsimeq} b \wedge c \underset{f_c}{\backsimeq} S_{m_i(f_a,f_c)}(a)$. We thus get terms as above witnessing $\mathrm{T}(b, L(lk), c)$. □

With these Herbrand disjunctions, we can now show our final result, a Herbrand disjunction for the outer Pasch theorem (Figure 28). We formulate it again here.

Theorem 3.48 (outer Pasch)**.** *[13, Satz I.9.6]*

$$\mathbf{T} \vdash \varphi := \exists x \,\big[\, \mathrm{T}(a,c,l) \wedge \mathrm{T}(b,k,c) \rightarrow \mathrm{T}(a,x,b) \wedge \mathrm{T}(l,k,x)\big].$$

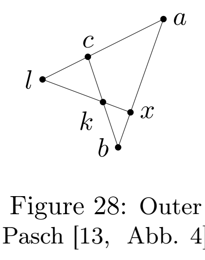


Figure 28: Outer Pasch [13, Abb. 4]

Our final Herbrand analysis of the ‘Outer Pasch Theorem’ is now as follows:

Theorem 3.49 (Herbrand for outer Pasch)**.** *Let* φ *be as above and* $\varphi_{qf} := [\ldots]$. *Then*

$$\mathbf{T}^S \vdash \bigvee_{i=1}^{10} \varphi_{qf}(a,b,c,l,k,t_i(a,b,c,l,k)) \text{ for}$$

$t_i = ip(f_b, a, S_{s_i}(a), b, s_i)$ *for* $i = 1, \ldots, 4$,

$t_i = ip(f_a, b, S_{s_{i-4}}(b), a, s_{i-4})$ *for* $i = 5, \ldots, 8$

$t_9 = a, \quad t_{10} = b \quad$ *where*

$s_i = ip(f_a, S_{m_i(f_c,f_b)}(c), a, r_i, q_i), \;\; r_i = ext(S_{f_a}(a), f_a, f_b, S_{m_i(f_c,f_b)}(c))$

$q_i = ip(S_{m_i(f_c,f_b)}(l), a, S_{m_i(f_c,f_b)}(a), S_{m_i(f_c,f_b)}(c), m_i(f_c,f_b)),$

$f_c = foot(l,k,c), \quad f_a = foot(l,k,a), \quad f_b = foot(l,k,b).$

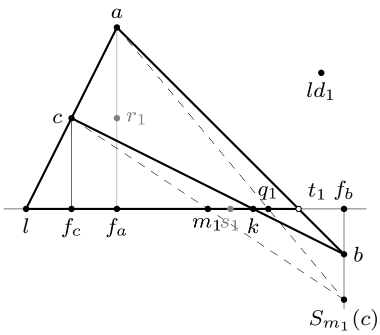


Figure 29: (based on [13, Abb. 35])
Construction of t_1.

The proof is based on the proof of Theorem 3.48 in [13, p.70f].

Proof. We distinguish two cases.

Case 1: $\mathrm{Col}(l,k,c)$. Here, we further distinguish $\mathrm{T}(l,k,c) \vee \neg\, \mathrm{T}(l,k,c)$.
If $\mathrm{T}(l,k,c)$, with $\mathrm{T}(l,c,a)$ we can show $\mathrm{T}(l,k,a)$. Hence $\varphi_{qf}(a)$. If $\neg\, \mathrm{T}(l,k,c)$, by Definition 3.33 we know that $l \underset{k}{\backsimeq} c$. Further, as $\mathrm{T}(c,k,b)$, it follows that $\mathrm{T}(l,k,b)$. Hence, $\varphi_{qf}(b)$.

Case 2: $\neg\, \mathrm{Col}(l,k,c)$. Then $L(lk) \neq L(ck)$. Again, we distinguish two subcases.

Case 2a: $b \in L(lk)$, i.e. $\mathrm{Col}(b,k,l)$. Then $b = k$ as, otherwise, $L(lk) = L(ck)$. Hence again $\varphi_{qf}(b)$.

Case 2b: $b \notin L(lk)$, i.e. $\neg\, \mathrm{Col}(b,k,l)$ (consider Figure 29). We now want to find the point witnessing that $\mathrm{T}(a, L(lk), b)$. To this end, we want to use Lemma 3.47. If we went by the proof of Theorem 3.48 in [13, p.70f], we would now state that $\mathrm{T}(c, L(lk), b)$ via k and $c \underset{l}{\backsimeq} a \wedge \mathrm{Col}(l,l,k)$ and could thus directly argue via Lemma 3.47 that $\varphi_{qf}(t_1) \vee \ldots \vee \varphi_{qf}(t_{32})$, where the t_i are defined as in Lemma 3.47.[7] But in this way, we are forgetting some of our information as we only use that $c \underset{l}{\backsimeq} a$ even though we know that in fact $\mathrm{T}(a,c,l)$ (and of course still $c \neq l$). We can thus show that we only need 8 terms to realize φ in this case 2b. To this end, we consider the proof of Lemma 3.47. Here, after some constructions, we invoke Lemma 3.40 on $\mathrm{T}(c, L(lk), S_{m_i(f_c,f_b)}(c))$ via $m_i(f_c,f_b)$ and $\mathrm{Col}(m_i(f_c,f_b), l, k) \wedge \mathrm{M}(c, m_i(f_c,f_b), S_{m_i(f_c,f_b)}(c)) \wedge \mathrm{Col}(l,l,k) \wedge \mathbf{c} \neq \mathbf{l} \wedge \mathrm{T}(\mathbf{a},\mathbf{c},\mathbf{l})$ (instead of $\wedge\; c \underset{l}{\backsimeq} a$). We can thus argue that we are in case 2 of the proof of Lemma 3.40 and only get one term q_i as above that witnesses $\mathrm{T}(a, L(lk), S_{m_i(f_c,f_b)}(c))$. Now, just as in the proof of Lemma 3.47, we invoke Lemma 3.44 on $\mathrm{T}(a, L(lk), S_{m_i(f_c,f_b)}(c))$ via q_i and $\mathrm{Col}(f_a,l,k) \wedge lk \underset{f_a}{\perp} af_a \wedge \mathrm{Col}(f_b,l,k) \wedge lk \underset{f_b}{\perp} bf_b \wedge a \underset{f_a}{\backsimeq} a \wedge S_{m_i(f_c,f_b)}(c) \underset{f_b}{\backsimeq} b$. But again, we can show that $S_{m_i(f_c,f_b)}(c) f_b \overset{r_i}{\leq} af_a$, thus avoiding the outer case distinction that is made in the proof of Lemma 3.44. We cannot however avoid the ‘inner’ case distinction in the proof of Lemma 3.44: Here (in our case), the proof of Lemma 3.44 invokes Lemma 3.40 on $\mathrm{T}(S_{s_i}(a), L(lk), a)$ via s_i and $\mathrm{Col}(l,k,s_i) \wedge \mathrm{M}(S_{s_i}(a), s_i, a) \wedge \mathrm{Col}(f_b,k,l) \wedge S_{s_i}(a) \underset{f_b}{\backsimeq} b$ for r_i and s_i as defined above, whose proof distinguishes cases $\mathrm{T}(f_b, b, S_{s_i}(a))$ and $\mathrm{T}(f_b, S_{s_i}(a), b)$. As both can be the case, we have now argued that terms $t_1, \ldots, t_8$ as above witness $\mathrm{T}(a, L(lk), b)$ as desired. Let now $\mathrm{T}(a, t_i, b)$ for $i \in \{1, \ldots, 8\}$. It remains to show that also $\mathrm{T}(l,k,t_i)$. To this end, we invoke $\mathrm{A7}^S$ on $\mathrm{T}(l,c,a) \wedge \mathrm{T}(b,t_i,a)$ such that $\mathrm{T}(l,r,t_i) \wedge \mathrm{T}(b,r,c)$ for $r := ip(l,b,a,c,t_i)$. But now r is the intersection point of lines $L(lk)$ and $L(cb) = L(ck)$. Hence $r = k$ and $\mathrm{T}(l,k,t_i)$ as desired. We conclude the proof by giving an outline as to why $S_{m_i(f_c,f_b)}(c) f_b \overset{r_i}{\leq} af_a$ can be shown in case 2b. To this end, we prove (in $\mathbf{T}$) the following existential formula (see Figure 30):

$$\begin{aligned}\varphi := \;& \big[\mathrm{Col}(b,l,k) \wedge \mathrm{Col}(d,l,k) \wedge \mathrm{T}(l,c,e) \wedge \mathrm{R}(l,b,c) \wedge \mathrm{R}(l,d,e) \wedge l \neq c \wedge \\ & \neg\, \mathrm{Col}(l,k,c) \wedge \mathrm{E}(c'',b'',c,b) \to c''b'' \leq ed\big] \\ =& \exists h \big[\mathrm{Col}(b,l,k) \wedge \mathrm{Col}(d,l,k) \wedge \mathrm{T}(l,c,e) \wedge \mathrm{R}(l,b,c) \wedge \mathrm{R}(l,d,e) \wedge p \neq c \wedge \\ & \neg\, \mathrm{Col}(l,k,c) \wedge \mathrm{E}(c'',b'',c,b) \to \mathrm{T}(e,h,d) \wedge \mathrm{E}(c'',b'',e,h)\big]\end{aligned}$$

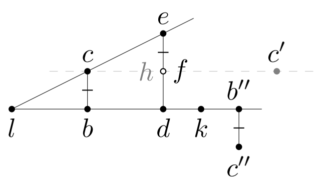


Figure 30: $c''b'' \leq ed$

and then show that - for the case at hand - r_i is a witnessing term for $\exists$. To

[7] To be precise, Lemma 3.47 only provides terms t_i so that $\mathbf{T}^S \vdash \bigvee_{i=1}^{32} \mathrm{T}(a,t_i,b)$. We show below that also $\mathrm{T}(l,k,t_i)$ and thus $\mathbf{T}^S \vdash \varphi_{qf}(t_1) \vee \ldots \vee \varphi_{qf}(t_{32})$.

establish the existence h, we first argue that $cb \leq ed$, i.e. $\exists f[\mathrm{T}(e,f,d) \wedge \mathrm{E}(c,d,e,f)]$. To this end, consider the following:

- If $c = e$ (by the uniqueness of the foot point) we are done. Assume hence that $c \neq e$.
- It can be shown that for every line $L(lk)$ and every point c there exists exactly one line that is parallel to $L(lk)$ and entails c. Let hence c' be a point such that $L(cc')$ is (strictly) parallel to $L(lk)$. This point exists by [13, Satz I.12.13] and (for strictness) [13, Definition I.12.3] with $\neg\,\mathrm{Col}(c,l,k)$.
- As the line $L(ed)$ intersects $L(lk)$ at the point d, it can be shown ([13, Satz I.12.16]) that it also intersects any line that is parallel to $L(lk)$. Let hence in particular f be the intersection point of lines $L(ed)$ and $L(cc')$.
- We can further show that 'two lines are parallel to each other if and only if they form congruent alternate angles with another line' [13, Satz I.12.21]. As we know that $L(bd)$ and $L(cf)$ are parallel and that $\mathrm{R}(l,d,e)$ and $\mathrm{R}(c,b,l)$ it thus follows that $\mathrm{R}(c,f,d)$ and $\mathrm{R}(b,c,f)$.
- As now lines $L(bc)$ and $L(df)$ form congruent angles with $L(cf)$, by the same argument (but in the other direction), they are parallel.
- We have now shown that segment cf is parallel to segment bd and that segment cb is parallel to segment fd. We further know that $\neg\,\mathrm{Col}(b,c,f)$ and thus the points c,b,d,f form a parallelogram. It can be shown ([13, Satz I.12.19]) that now the opposite sides of this parallelogram are congruent, in particular $\mathrm{E}(b,c,d,f)$.
- It remains to show that also $\mathrm{T}(e,f,d)$. We first establish $\mathrm{T}(e,L(cc'),d)$.
  - We know that $\mathrm{T}(e,L(cc'),l)$ as $\mathrm{T}(e,c,l)$ and $\neg\,\mathrm{Col}(l,c,c')$ and $\neg\,\mathrm{Col}(e,c,c')$ by assumption of our case and the choice of c': $\neg\,\mathrm{Col}(l,c,c')$ holds as $L(cc')$ and $L(lk)$ are (strictly) parallel (see e.g. [13, Definition I.12.2]), $\neg\,\mathrm{Col}(e,c,c')$ holds as otherwise $\mathrm{Col}(e,c,c') \wedge \mathrm{T}(l,c,e) \wedge c \neq e$ would imply $\mathrm{Col}(l,c,c')$ contrary to what was just shown.
  - It is intuitively clear that if segments cc' and ld are strictly parallel, then the points l,d must 'lie on the same side' of the line determined by the points c and c' (see [13, Definition I.9.7] for a definition) and [13, Folgerung I.12.7 (a)] for a proof).
  - We now have that $\mathrm{T}(e,L(cc'),l)$, i.e the points e and l lie on opposite sides of $L(cc')$ and that l,d lie on the same side of $L(cc')$. It is thus again intuitive that also $\mathrm{T}(e,L(cc'),d)$ (see [13, Satz I.9.8] for a proof).
- As $\mathrm{Col}(e,d,f)$ and $\mathrm{Col}(f,c,c')$ we further know that f is that point for which $\mathrm{T}(e,L(cc'),d)$ ([13, Satz I.6.21]) and hence deduce $\mathrm{T}(e,f,l)$ as desired.

Now, from $cb \leq ed$ and $\mathrm{E}(c,b,c'',b'')$ it follows that $c''b'' \leq ed$ ([13, Satz I.5.6]), i.e. $\exists h[\mathrm{T}(e,h,d) \wedge \mathrm{E}(c'',b'',e,h)]$ as desired which finishes the proof of φ. By instantiating the variables in φ by suitable terms and argueing as in the proof of Lemma 3.44 we get

$$\mathbf{T}^S \vdash \mathrm{Col}(f_c,l,k) \wedge \mathrm{Col}(f_a,l,k) \wedge \mathrm{T}(l,c,a) \wedge \mathrm{R}(l,f_c,c) \wedge \mathrm{R}(l,f_a,a) \wedge l \neq c \wedge \neg\,\mathrm{Col}(l,k,c) \wedge \mathrm{E}(S_{m_i(f_c,f_b)}(c),f_b,c,f_c) \\ \to S_{m_i(f_c,f_b)}(c)f_b \overset{r_i}{\leq} af_a$$

where r_i is as defined above. Thus $S_{m_i(f_c,f_b)}(c)f_b \overset{r_i}{\leq} af_a$ as all the premises are satisfied, notice that $\mathrm{E}(S_{m_i(f_c,f_b)}(c),f_b,c,f_c)$ follows from [13, Satz I.7.13] as $f_b = S_{m_i(f_c,f_b)}(f_c)$. □

Note that although A10 is used in the proof of [13, Satz I.12.13] which we refrence above, we do not need the Skolemization $\mathrm{A10}^S$ since it is only used to show the uniqueness of the constructed parallel. However, even if A10 had been used to construct c', we still would not need $\mathrm{A10}^S$ since c' is only used to verify the quantifier-free formula $T(e,f,d)$ and so does not need to be constructed.

If one allows for decision functions for 'T' and '=' and hence for Boolean combinations of these relations, then one can contract the Herbrand disjunction in Theorem 3.49 into a single program.

4 Conclusion

In this paper we showed in a case study how Herbrand's theorem can be used to extract realizers for the existential quantifier in the outer Pasch theorem by analyzing its proof in [13]. We thereby made two interesting observations: In order to provide realizers for the outer Pasch theorem, a modular analysis of the proof is *possible*, i.e. we e.g. did not have to consider a cut-free proof of the theorem but were able to analyze the existing proof. This will likely be the case for most of the theorems in [13], even if their proofs use lemmas more complex than themselves, as it *seems* inherent to proofs in plane elementary Euclidean geometry to be 'constructive enough'. We also observed that the plurality of terms realizing existential statements is due to case distinctions which in general cannot be avoided as e.g. $A8$ itself expresses an alternative and we have to rule out trivial cases resulting in degenerate situations when e.g. invoking $A7^S$, as we loose uniqueness here.

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