Rates of convergence and metastability for Chidume's algorithm for the approximation of zeros of accretive operators in Banach spaces*

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Abstract

In this paper we give a quantitative analysis of an explicit iteration method due to C.E. Chidume for the approximation of a zero of an m-accretive operator $A : X \to 2^X$ in Banach spaces which does not involve the computation of the resolvent of A.

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1 Introduction

The approximation of zeros of monotone set-valued operators $A : X \to 2^X$ in Hilbert spaces X and - more generally - of accretive operators in Banach spaces is a central theme in both nonsmooth optimization as well as in the study of abstract Cauchy problems. The connection to optimization stems from the fact that the set of minimizers of proper l.s.c. convex functions $f : X \to (-\infty, +\infty]$ coincides with the zeros of $A := \partial f$, the so-called subdifferential of f.
The famous Proximal Point Algorithm (PPA) approximates zeros of A using the resolvent function (for $\gamma > 0$)

$$J_{\gamma A} : R(I + \gamma A) \to D(A),\ J_{\gamma A}(x) := (I + \gamma A)^{-1}(x)$$

which is a single-valued firmly nonexpansive function. If A is maximal monotone (such as ∂f) or m-accretive, then $R(I + \gamma A) = X$ and so $J_{\gamma A}$ is defined on the whole space X and it makes sense to consider for a sequence $(\gamma_n) \subset (0, \infty)$ the iteration

$$(\text{PPA}) :\ x_{n+1} := J_{\gamma_n A}(x_n),\quad x_1 \in X.$$

Under suitable conditions on (γ_n), this algorithm (x_n) converges weakly to a zero of A (provided that A has one, [22, 28, 4]) but - already in the case of Hilbert spaces - it in general fails to converge

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strongly (see [11, 2]).
Subsequently, so-called Halpern-type variants (HPPA)

$$x_{n+1} = \alpha_n u + (1-\alpha_n)(I+\gamma_n A)^{-1} x_n \quad (\gamma_n > 0, \alpha_n \in [0,1])$$

of (PPA) have been considered (see e.g. [12, 34, 1]) which - under suitable conditions on $(\alpha_n), (\gamma_n)$ - do converge strongly even in Banach spaces which e.g. are uniformly smooth and at the same time uniformly convex (so e.g. for L^p with $1 < p < \infty$).
One problem with both (PPA) and (HPPA) is that they involve the computation of $J_{\gamma A}$ and so the solution of an inverse problem. Hence the existence of strongly convergence algorithms based on explicit iterations of A itself instead of its resolvent is of interest. Such an algorithm

$$(*)\ x_{n+1} = x_n - \lambda_n u_n - \lambda_n \theta_n (x_n - x_1), u \in A(x_n), \lambda_n, \theta_n \in (0,1)$$

was given in [3] (in Hilbert space) and studied in certain Banach spaces in the case of single-valued A in [25] and in 2-uniformly smooth Banach spaces in [8] and for set-valued A and general uniformly smooth Banach spaces in [7]. The strong convergence of (x_n) to a zero of A (under suitable conditions on $(\lambda_n), (\theta_n)$) is shown in [7] by reducing the situation to the following seminal result of Reich (see also [5] for another proof of an extension of this result):

Theorem 1.1 ([26])**.** *Let X be a real uniformly smooth Banach space and $A : X \to 2^X$ be m-accretive with $A^{-1}(0) \neq \emptyset$. Then $\lim\limits_{t\to\infty} J_{tA}(x)$ exists and belongs to $A^{-1}(0)$.*

More specifically, Chidume shows that

$$\|x_{n+1} - y_n\| \to 0, \text{ where } y_n := J_{\theta_n^{-1}A}(x_1).$$

In this paper we first extract from Chidume's proof an explicit rate of convergence for $\|x_{n+1} - y_n\| \to 0$ (Theorem 3.1). It follows from general results in computability that even for $X := \mathbb{R}$ there is in general no computable rate of convergence for neither (x_n) nor (y_n). However, what recently has been obtained is the next best thing, namely a rate of metastability in the sense of Tao [31, 32] for (y_n) (see [29] which in turn builds on [17]):

$$(+)\ \forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n \leq \Psi(k,g)\, \forall m,l \in [n, n+g(n)]\ (\|y_m - y_l\| \leq 2^{-k}).$$

Note that (+), noneffectively, implies the Cauchy property and hence the convergence of (y_n) but does not allow for an effective transformation of Ψ into a rate of convergence for (y_n).

Our rate of convergence $\|x_{n+1} - y_n\| \to 0$ together with the rate of metastability Ψ in (+) can be combined into a rate Φ satisfying

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n \leq \Phi(k,g)\, \forall l,m \in [n, n+g(n)]\ (\|x_l - x_m\|, \|x_l - y_{l-1}\| \leq 2^{-k})$$

(Theorem 3.3).
This quantitative result can be seen as a finitization (in the sense of Tao [31]) of Chidume's theorem as it mathematically trivially (though noneffectively) implies not only that (x_n) is Cauchy and hence strongly convergent but also that (x_n) converges to the same limit as (y_n) converges to.

Definition 1.2 ([30], p.99). *Let X be a real Banach space. $J : X \to 2^{X^*}$ with*

$$J(x) = \{f \in X^* : \langle x, f \rangle = \|x\|^2, \|f\| = \|x\|\}$$

is called the normalized duality mapping of X.

More information on (normalized) duality mappings can be found in [9, 27].

Definition 1.3 ([30], p.128 and [7]). *Let X be a Banach space and $A : X \to 2^X$ be a set-valued operator. The domain $D(A)$, the range $R(A)$ and the graph $G(A)$ of A are defined as follows:*

$$D(A) = \{x \in X : A(x) \neq \emptyset\},\ A(S) = \bigcup_{x \in S} A(x),$$

$$R(A) = A(X),\ G(A) = \{(x, u) : x \in D(A), u \in A(x)\}.$$

A is accretive if for all $\forall x, y \in X, u \in A(x), v \in A(y)\, \exists j(x-y) \in J(x-y)$ with $\langle u - v, j(x-y)\rangle \geq 0$. By Kato [13] this is equivalent to the statement that for all $s > 0, x, y \in D(A), u \in A(x), v \in A(y)$

$$\|x - y\| \leq \|x - y + s(u - v)\|.$$

A is m-accretive if for all $t > 0$ (or - equivalently - for some $t > 0$) $R(I + tA) = X$.

$x \in X$ is a zero of A if $0 \in A(x)$. $zer\, A$ denotes the set of all zeros of A.

Definition 1.4 ([6], p.13). *A Banach space X is called uniformly smooth if*

$$\forall \varepsilon > 0\, \exists \delta > 0\, \forall x, y \in X\ (\|x\| = 1 \wedge \|y\| \leq \delta \Rightarrow \|x + y\| + \|x - y\| \leq 2 + \varepsilon \|y\|).$$

Remark 1.5. *It is well-known (see e.g. [6], p.13) that the definition above is equivalent to*

$$\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0,$$

where

$$\rho_X(t) = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| \leq t\right\}.$$

Definition 1.6. *X is called uniformly convex if*

$$\forall \varepsilon \in (0, 2]\, \exists \delta \in (0, 1]\, \forall x, y \in X \left(\|x\|, \|y\| \leq 1 \wedge \|x - y\| \geq \varepsilon \Rightarrow \left\|\frac{1}{2}(x + y)\right\| \leq 1 - \delta\right).$$

As in [15], we call functions $\tau : (0, \infty) \to (0, \infty), \eta : (0, 2] \to (0, 1]$ which provide a $\delta(\varepsilon)$ in the definitions of uniform smoothness and uniform convexity moduli of uniform smoothness and uniform convexity respectively. Note that this differs from the terminology used in functional analysis where ρ_X is called the modulus of smoothness of X while it is not a modulus of smoothness in the sense of τ[1] and what is called the modulus of uniform convexity δ_X is a particular, namely the optimal, modulus of uniform convexity η in our sense.

[1]The existence of a modulus τ is equivalent to the uniform smoothness of X while ρ_X is also defined for non-smooth Banach spaces and only the aforementioned limit statement expresses uniform smoothness.

Chidume assumes that A is bounded which is meant as 'bounded on bounded sets'. As discussed in [24] this is equivalent to A possessing a uniform majorant $A^* : \mathbb{N} \to \mathbb{N}$ satisfying

$$\forall x \in X \, \forall n \in \mathbb{N} \, (\|x\| \leq n \Rightarrow \forall y \in A(x) \, (\|y\| \leq A^*(n))) \, .$$

By a majorant for a sequence $(x_n)_{n\in\mathbb{N}}$ in X we mean a sequence $(p_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $p_n \geq \|x_n\|$ for all $n \in \mathbb{N}$.

Notation: Following [7], all our sequences start with the index $n \geq 1$ and we, therefore, use $\mathbb{N} := \{1, 2, 3, \ldots\}$.

2 Technical Lemmas

In this section we collect some technical estimates for uniformly smooth Banach spaces which are essentially known but in some cases the values of certain constants had to be extracted from the literature.

Lemma 2.1 (see [21], p.64-65, [35], p.208)**.** *Let X be a uniformly smooth Banach space and ρ_X be the function defined in Remark 1.5. Then for all $s, t \in \mathbb{R}$ with $s \geq t > 0$*

$$\frac{\rho_X(s)}{s^2} \leq C \frac{\rho_X(t)}{t^2} \text{ and } \frac{\rho_X(t)}{t} \leq \frac{\rho_X(s)}{s},$$

where $C = \frac{4\tau_0}{\sqrt{1+\tau_0^2}-1} \prod_{j=1}^{\infty} \left(1 + \frac{15\tau_0}{4\cdot 2^j}\right)$, $\tau_0 = \frac{\sqrt{339}-18}{30}$.

Lemma 2.2 (see [7], p.36 and [35])**.** *Let X be uniformly smooth. Then for all $x, y \in X$ and $j(x) \in J(x)$*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + D \max(\|x\| + \|y\|, C)\rho_X(\|y\|),$$

where

$$D = 2\max(8, (40 - 16\sqrt{3})C), \; C = \frac{4\tau_0}{\sqrt{1+\tau_0^2} - 1} \prod_{j=1}^{\infty} \left(1 + \frac{15\tau_0}{4 \cdot 2^j}\right), \; \tau_0 = \frac{\sqrt{339} - 18}{30} \, .$$

Proof. In [35](p.208) it is shown that for all $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$

$$\|x + y\|^p \leq \|x\|^p + p\langle y, j(x)\rangle + \sigma_p(x, y)$$

with

$$\sigma_p(x, y) = p \cdot l \int_0^1 \frac{\max(\|x + ty\|, \|x\|)^2}{t} \rho_X \left(\frac{t\|y\|}{\max(\|x + ty\|, \|x\|)}\right) dt,$$

where

$$l = \max\left(8, 64C\frac{1}{K_q}\right),$$

$$K_q = 4(2 + \sqrt{3}) \min \left\{ \begin{array}{l} \min(1, \frac{1}{2}q(q-1)), (q-1)\min(1, \frac{1}{2}q), \\ (q-1)(1 - (\sqrt{3}-1)^{\frac{q}{q-1}}), 1 - (1 + \frac{(2-\sqrt{3})q}{q-1})^{1-q} \end{array} \right\}.$$

For $p = q = 2$ this yields

$$\sigma_2(x, y) = 2\max(8, (40 - 16\sqrt{3})C) \int_0^1 \frac{\max(\|x + ty\|, \|x\|)^2}{t} \rho_X\left(\frac{t\|y\|}{\max(\|x + ty\|, \|x\|)}\right) dt.$$

Distinguishing the cases $\frac{t}{\max(\|x+ty\|,\|x\|)} \geq 1$ and $\frac{t}{\max(\|x+ty\|,\|x\|)} \leq 1$, the respective inequalities from Lemma 2.1 imply that

$$\begin{aligned} \|x + y\|^2 &\leq \|x\|^2 + 2\langle y, j(x)\rangle + 2\max(8, (40 - 16\sqrt{3})C) \int_0^1 \max(\|x + ty\|, \|x\|, Ct)\, dt \rho_X(\|y\|) \\ &\leq \|x\|^2 + 2\langle y, j(x)\rangle + 2\max(8, (40 - 16\sqrt{3})C) \max(\|x\| + \|y\|, C)\rho_X(\|y\|). \end{aligned}$$

□

Lemma 2.3 (see [23], p.284)**.** *Let X be a Banach space. Then for all $x, y \in X$ and for all $j(x+y) \in J(x + y)$*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle.$$

Lemma 2.4 ([34], p.243)**.** *Let $(\rho_n)_{n\in\mathbb{N}}, (\gamma_n)_{n\in\mathbb{N}}$ be sequences of nonnegative real numbers, $(\sigma_n)_{n\in\mathbb{N}}$ a sequence of real numbers and $(\alpha_n)_{n\in\mathbb{N}}$ a sequence in $[0, 1)$ such that for all $n \in \mathbb{N}$*

$$\rho_{n+1} \leq (1 - \alpha_n)\rho_n + \alpha_n\sigma_n + \gamma_n.$$

If

$$\sum_{n=1}^{\infty} \alpha_n = \infty \text{ (or - equivalently - } \prod_{n=1}^{\infty}(1 - \alpha_n) = 0), \ \limsup_{n\to\infty} \sigma_n \leq 0 \text{ and } \sum_{n=1}^{\infty} \gamma_n < \infty,$$

then it follows that

$$\lim_{n\to\infty} \rho_n = 0.$$

We now give a quantitative version of Lemma 2.4. Similar versions have been used repeatedly in the context of proof mining e.g. in [16, 18]. For completeness, however, we give the proof for the particular formulation we need.

Lemma 2.5. *Let $(\rho_n), (\gamma_n), (\sigma_n), (\alpha_n)$ be as in the previous lemma and let $(p_n) \subset \mathbb{N}$ be a majorant for (ρ_n). Let Φ_1, Φ_2, Φ_3 be rates witnessing quantitatively the conditions on $(\alpha_n), (\sigma_n), (\gamma_n)$, i.e.*

$$\forall k \in \mathbb{N}\, \forall N \in \mathbb{N} \left(\prod_{n=N}^{\Phi_1(k,N)} (1 - \alpha_n) \leq 2^{-k} \right),$$

$$\forall k \in \mathbb{N}\, \forall n \geq \Phi_2(k) \left(\sigma_n \leq 2^{-k}\right),$$

$$\forall k \in \mathbb{N} \left(\sum_{n=\Phi_3(k)}^{\infty} \gamma_n \leq 2^{-k} \right).$$

Then

$$\forall k \in \mathbb{N}\, \forall n \geq \Phi^*(k)\, (\rho_n \leq 2^{-k}),$$

where $\Phi^*(k) = \max(\Phi_1(k + \lceil\log_2 p_N\rceil + 1, N), N) + 1$ and $N = \max(\Phi_2(k + 2), \Phi_3(k + 2))$.

Proof. Let $k \in \mathbb{N}$ be arbitrary and let $N := \max\left(\Phi_2(k+2), \Phi_3(k+2)\right)$.
We prove by induction on n that for all $n \geq N$:

$$\rho_{n+1} \leq \left(\prod_{i=N}^{n}(1-\alpha_i)\right)\rho_N + \left(1 - \prod_{i=N}^{n}(1-\alpha_i)\right) \cdot 2^{-k-2} + \sum_{i=N}^{n}\gamma_i.$$

The case $n = N$ holds by assumption. Assume that the claim holds for $n \geq N$. Then

$$\rho_{n+2} \leq (1-\alpha_{n+1})\left(\left(\prod_{i=N}^{n}(1-\alpha_i)\right)\rho_N + \left(1 - \prod_{i=N}^{n}(1-\alpha_i)\right) \cdot 2^{-k-2} + \sum_{i=N}^{n}\gamma_i\right) + \alpha_{n+1}\sigma_{n+1} + \gamma_{n+1}.$$

Moving $(1-\alpha_{n+1})$ inside, using $\alpha_{n+1} \in [0,1]$ as well as $\sigma_{n+1} \leq 2^{-k-2}$, the induction step follows.
Let now $n \geq \max\left(\Phi_1(k + \lceil \log_2 p_N \rceil + 1, N), N\right) + 1$. Then - using $\rho_N \leq p_N$ - $\prod\limits_{i=N}^{n-1}(1-\alpha_i) \leq \frac{2^{-k-1}}{p_N}$, $(1 - \prod\limits_{i=N}^{n-1}(1-\alpha_{i-1})) \leq 1$ and $\sum\limits_{i=N}^{n-1}\gamma_i \leq 2^{-k-2}$ imply $\rho_n \leq 2^{-k}$. □

The following bound on the iterative sequence (x_n) of Chidume's algorithm is crucially used:

Lemma 2.6 (see [7], p.37)**.** *Let X be a uniformly smooth Banach space, $A : X \to 2^X$ be a bounded set-valued accretive operator with $D(A) = X$ and $x^* \in zer\, A$. Let $(\lambda_n)_{n\in\mathbb{N}}$ and $(\theta_n)_{n\in\mathbb{N}}$ be sequences in $(0,1)$ and $x_1 \in X$ be arbitrary. Let C, D be as in Lemma 2.2 and let $L \in \mathbb{N}$ be such that*

$$\|x^*\|, \|x^* - x_1\| \leq L.$$

Let $\gamma_0, M_0, (x_n)_{n\in\mathbb{N}}$ be such that

$$x_{n+1} = x_n - \lambda_n u_n - \lambda_n\theta_n(x_n - x_1), \quad u_n \in A(x_n),$$

$$M_0 = \max\left\{1, \sup\{\|u + \theta(x - x_1)\| : \theta \in (0,1), u \in A(x), x \in X, \|x - x^*\| \leq 2L\}\right\},$$

$$M^* = \sup\{D\max(\|x\| + \lambda M_0, C) : \lambda \in (0,1), x \in X, \|x - x^*\| \leq 2L\},\ \gamma_0 = \frac{1}{2}\min\left(1, \frac{L^2}{M^* M_0}\right).$$

If for all $n \in \mathbb{N}$

$$\frac{\rho_X(\lambda_n M_0)}{\lambda_n M_0} \leq \gamma_0\theta_n,$$

then $(x_n)_{n\in\mathbb{N}}$ is bounded by $3L$.

Proof. [7][p.37] shows that under the conditions given, one has $\|x^* - x_n\| \leq 2L$, which implies the claim. □

In the following we give a more explicit and effective description of the bound on (x_n) which avoids the use of sup's.

Corollary 2.7. *Let $A^* : \mathbb{N} \to \mathbb{N}$ be a uniform majorant of A witnessing that A is bounded on bounded sets, i.e.*

$$\forall n \in \mathbb{N}\, \forall (x,y) \in G(A)\, (\|x\| \leq n \to \|y\| \leq A^*(n)).$$

Then the condition

$$\frac{\rho_X(\lambda_n M_0)}{\lambda_n M_0} \le \gamma_0 \theta_n \text{ with } \gamma_0 = \frac{1}{2}\min\left(1, \frac{L^2}{M^* M_0}\right)$$

in Lemma 2.6 can be replaced by

$$\frac{\rho_X(\lambda_n)}{\lambda_n} \le \gamma_0 \theta_n,$$

with

$$\gamma_0 = \frac{1}{2}\min\left(1, \frac{L^2}{D(A^*(3L)+5L)\max(A^*(3L)+8L, C)}\right)\frac{1}{C(A^*(3L)+5L)}.$$

Proof. With Lemma 2.1 we get

$$\frac{\rho_X(\lambda_n M_0)}{\lambda_n M_0} \le C M_0 \frac{\rho_X(\lambda_n)}{\lambda_n}.$$

Together with the easy estimates

$$M_0 \le A^*(3L) + 5L \text{ and } M^* \le D\max(A^*(3L)+8L, C),$$

the corollary follows. □

3 Main Results

The next theorem gives an explicit and effective rate for the convergence of $\|x_{n+1} - y_n\| \to 0$, where (x_n) is the sequence generated by Chidume's algorithm $(*)$ and $y_n := J_{t_n A}(x_1)$ with $t_n := \theta_n^{-1}$:

Theorem 3.1. *Let X be a uniformly smooth Banach space, $A : X \to 2^X$ be a bounded set-valued m-accretive operator with $D(A) = X$ with $zer\, A \ne \emptyset$. Let $x^* \in zer\, A$ and let $A^* : \mathbb{N} \to \mathbb{N}$ be a uniform majorant of A. Let $(\lambda_n)_{n\in\mathbb{N}}$ and $(\theta_n)_{n\in\mathbb{N}}$ be sequences in $(0,1)$ and $x_1 \in X$ arbitrary. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in A satisfying*

$$x_{n+1} = x_n - \lambda_n u_n - \lambda_n\theta_n(x_n - x_1), u_n \in A(x_n) \text{ and } t_n = \theta_n^{-1}, y_n = J_{t_n A}(x_1).$$

Let $L \in \mathbb{N}$ be such that $\|x^*\|, \|x^* - x_1\| \le L$, C and D as in Lemma 2.2, $M_1 = A^*(3L) + 5L$, $M_2 = D\max(M_1 + 5L, C)$, $C^* = 40L^2$, $\sigma_n = C^* \frac{\frac{\theta_{n-1}}{\theta_n} - 1}{\lambda_n \theta_n}$ and

$$\gamma_0 = \frac{1}{2}\min\left(1, \frac{L^2}{D(A^*(3L)+5L)\max(A^*(3L)+8L, C)}\right)\frac{1}{C(A^*(3L)+5L)}.$$

If then

$$\prod_{n=1}^{\infty}(1 - \lambda_n\theta_n) = 0, \ \lim_{n\to\infty}\sigma_{n+1} = 0, \ \sum_{n=1}^{\infty}(M_2\rho_X(\lambda_n M_1)) < \infty,$$

and

$$\forall n \in \mathbb{N} \left(\theta_n \ge \theta_{n+1}, \frac{\rho_X(\lambda_n)}{\lambda_n} \le \gamma_0\theta_n\right)$$

and the first three properties are quantitatively witnessed by $\Phi_1 : \mathbb{N}^2 \to \mathbb{N}, \Phi_2, \Phi_3 : \mathbb{N} \to \mathbb{N}$, i.e.

$$\forall k \in \mathbb{N}\, \forall N \in \mathbb{N} \left(\prod_{n=N}^{\Phi_1(k,N)} (1-\lambda_{n+1}\theta_{n+1}) \leq 2^{-k} \right),$$

$$\forall k \in \mathbb{N}\, \forall n \geq \Phi_2(k)\, (\sigma_{n+1} \leq 2^{-k}),$$

$$\forall k \in \mathbb{N} \left(\sum_{n=\Phi_3(k)}^{\infty} (M_2 \rho_X(\lambda_{n+1}M_1)) \leq 2^{-k} \right),$$

then

$$\forall k \in \mathbb{N}\, \forall n \geq \Phi^*(k, L, \Phi_1, \Phi_2, \Phi_3)\, \left(\|x_{n+1} - y_n\| \leq 2^{-k} \right),$$

where

$\Phi^*(k, L, \Phi_1, \Phi_2, \Phi_3) = \max(\Phi_1(2k + \lceil \log_2 25L^2 \rceil + 1, N), N) + 1$ *and* $N = \max(\Phi_2(2k+2), \Phi_3(2k+2))$.

Proof. We follow closely the proof of Theorem 3.2 in [7]. First we show the boundedness of $(y_n)_{n\in\mathbb{N}}$ using the nonexpansivity of J_{t_nA} and the fact that x^* being a zero of A is a common fixed point of the resolvents J_{t_nA} :

$$\|y_n\| \leq \|y_n - x^*\| + \|x^*\| = \|J_{t_nA}(x_1) - J_{t_nA}(x^*)\| + \|x^*\| \leq \|x_1 - x^*\| + \|x^*\| \leq 2L.$$

With Lemma 2.2, the boundedness of x_n (see Lemma 2.6 and Corollary 2.7) and y_n and the majorant A^* of A together with Lemma 2.1 and the definition of (x_n) it follows that

$$\begin{aligned}
\|x_{n+1} - y_n\|^2 &= \|x_n - y_n - \lambda_n(u_n + \theta_n(x_n - x_1))\|^2 \\
&\leq \|x_n - y_n\|^2 - 2\lambda_n \langle u_n + \theta_n(x_n - x_1), j(x_n - y_n)\rangle \\
&\quad + D\max(\|x_n - y_n\| + \lambda_n \|u_n + \theta_n(x_n - x_1)\|, C) \\
&\quad \cdot \rho_X(\lambda_n \|u_n + \theta_n(x_n - x_1)\|) \\
&\leq \|x_n - y_n\|^2 - 2\lambda_n \langle u_n + \theta_n(x_n - x_1), j(x_n - y_n)\rangle \\
&\quad + D\max(\|x_n\| + \|y_n\| + \|u_n\| + \|x_n\| + \|x_1\|, C) \\
&\quad \cdot \rho_X(\lambda_n(\|u_n\| + \|x_n\| + \|x_1\|)) \\
&\leq \|x_n - y_n\|^2 - 2\lambda_n \langle u_n + \theta_n(x_n - x_1), j(x_n - y_n)\rangle \\
&\quad + D\max(3L + 2L + A^*(3L) + 3L + 2L, C) \cdot \rho_X(\lambda_n(A^*(3L) + 3L + 2L)) \\
&= \|x_n - y_n\|^2 - 2\lambda_n \langle u_n + \theta_n(x_n - x_1), j(x_n - y_n)\rangle \\
&\quad + D\max(M_1 + 5L, C) \cdot \rho_X(\lambda_n M_1) \\
&= \|x_n - y_n\|^2 - 2\lambda_n \langle u_n + \theta_n(x_n - x_1), j(x_n - y_n)\rangle \\
&\quad + M_2 \rho_X(\lambda_n M_1).
\end{aligned}$$

As in [7][p.38-39] one shows that

$$\langle u_n + \theta_n(x_n - x_1), j(x_n - y_n)\rangle \geq \frac{\theta_n}{2} \|x_n - y_n\|^2$$

and (for $n \geq 2$)

$$\|y_{n-1} - y_n\| \leq \frac{\theta_{n-1} - \theta_n}{\theta_n} \|y_{n-1} - x_1\|,$$

as well as

$$\|x_n - y_n\|^2 \leq \|x_n - y_{n-1}\|^2 + 2\|y_{n-1} - y_n\| \|x_n - y_n\|.$$

Combining these four inequalities we get (reasoning as in [7]) for all $n \geq 2$

$$\begin{aligned}
\|x_{n+1} - y_n\|^2 &\leq \|x_n - y_n\|^2 - 2\lambda_n \langle u_n + \theta_n(x_n - x_1), j(x_n - y_n)\rangle \\
&\quad + M_2 \rho_X(\lambda_n M_1) \\
&\leq \|x_n - y_n\|^2 - \lambda_n\theta_n \|x_n - y_n\|^2 \\
&\quad + M_2 \rho_X(\lambda_n M_1) \\
&\leq (1 - \lambda_n\theta_n)(\|x_n - y_{n-1}\|^2 + 2\|y_{n-1} - y_n\| \|x_n - y_n\|) \\
&\quad + M_2 \rho_X(\lambda_n M_1) \\
&\leq (1 - \lambda_n\theta_n)\|x_n - y_{n-1}\|^2 + 2(1 - \lambda_n\theta_n)\frac{\theta_{n-1} - \theta_n}{\theta_n} \|y_{n-1} - x_1\| \|x_n - y_n\| \\
&\quad + M_2 \rho_X(\lambda_n M_1) \\
&\leq (1 - \lambda_n\theta_n)\|x_n - y_{n-1}\|^2 + 2(\frac{\theta_{n-1}}{\theta_n} - 1)(\|y_{n-1}\| + \|x_1\|)(\|x_n\| + \|y_n\|) \\
&\quad + M_2 \rho_X(\lambda_n M_1) \\
&\leq (1 - \lambda_n\theta_n)\|x_n - y_{n-1}\|^2 + 2\frac{\lambda_n\theta_n}{\lambda_n\theta_n}(\frac{\theta_{n-1}}{\theta_n} - 1)(2L + 2L)(3L + 2L) \\
&\quad + M_2 \rho_X(\lambda_n M_1) \\
&= (1 - \lambda_n\theta_n)\|x_n - y_{n-1}\|^2 + \lambda_n\theta_n \frac{\frac{\theta_{n-1}}{\theta_n} - 1}{\lambda_n\theta_n}(40L^2) \\
&\quad + M_2 \rho_X(\lambda_n M_1) \\
&= (1 - \lambda_n\theta_n)\|x_n - y_{n-1}\|^2 + \lambda_n\theta_n \frac{(\frac{\theta_{n-1}}{\theta_n} - 1)}{\lambda_n\theta_n} C^* \\
&\quad + M_2 \rho_X(\lambda_n M_1) \\
&= (1 - \lambda_n\theta_n)\|x_n - y_{n-1}\|^2 + \lambda_n\theta_n\sigma_n \\
&\quad + M_2 \rho_X(\lambda_n M_1).
\end{aligned}$$

With Lemma 2.5 applied to $\rho_n := \|x_{n+1} - y_n\|^2$ and $p_n := 25L^2 \geq \rho_n$ for all $n \geq 1$ and applying the square root we finally get

$$\forall k \in \mathbb{N}\, \forall n \geq \Phi^*(k) \ \left(\|x_{n+1} - y_n\| \leq 2^{-k}\right).$$

□

By Reich's theorem, mentioned already in the introduction, one can show under the additional condition that (t_n) diverges to $+\infty$, i.e. that (θ_n) tends to 0, that (y_n) strongly converges to a zero of A. An explicit rate of metastability witnessing this quantitatively has been computed (under the additional assumption that X is also uniformly convex) in the case of single-valued A in [17] and was recently generalized to the set-valued case in [29].

Theorem 3.2 ([29](Corollary 4.2))**.** *Let X be a Banach space which is both uniformly smooth and uniformly convex with respective moduli τ and η. Let $A : X \to 2^X$ be m-accretive and x^* such that $0 \in A(x^*)$. Let $x \in X$ be arbitrary and $L \in \mathbb{N}$ such that $\|x - x^*\| \leq L$ and $(t_n)_{n\in\mathbb{N}} \subseteq (0, \infty)$ with*

$$\lim_{n\to\infty} t_n = \infty$$

and functions $\alpha, \gamma : \mathbb{N} \to \mathbb{N}$ such that

$$\forall n \in \mathbb{N} \forall m \geq \alpha(n)(t_m \geq n + 1, t_n \leq \gamma(n)).$$

Then there exists an explicit and fully effective rate $\Psi_{\tau,\eta,2L,\alpha,\gamma}(k, g)$ of metastability for the sequence $y_n = J_{t_n A}(x)$ which only depends on $\tau, \eta, L, \alpha, \gamma$ and k, g :[2]

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists\, 1 \leq n \leq \Psi_{\tau,\eta,2L,\alpha,\gamma}(k, g)\, \forall l, m \in [n, n + g(n)]\, (\|y_l - y_m\| \leq 2^{-k})$$

As in [20][Theorem 2.8] we can now combine this with Theorem 3.1:

Theorem 3.3. *In addition to the assumptions made in Theorem 3.1 we assume that X is also uniformly convex with a modulus η and that $t_n := \theta_n^{-1}$ diverges to $+\infty$ and that we have functions $\alpha, \gamma : \mathbb{N} \to \mathbb{N}$ with*

$$\forall n \in \mathbb{N}\, \forall m \geq \alpha(n)\, (t_m \geq n + 1,\ t_n \leq \gamma(n)).$$

Let $\Psi_{\tau,\eta,2L,\alpha,\gamma}$ be the rate of metastability for $y_n := J_{t_n A}(x_1)$ as in Theorem 3.2. Then (x_n) converges to the limit of (y_n) and we have the following explicit rate of metastability witnessing this fact:

$$\begin{aligned}\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists\, 2 \leq n \leq \Psi_{\tau,\eta,2L,\alpha,\gamma}(k + 1, g_k) + \Phi^*(k + 2) + 1 \\ \forall l, m \in [n, n + g(n)]\ (\|x_l - x_m\|, \|x_l - y_{l-1}\| \leq 2^{-k}),\end{aligned}$$

where Φ^* is as in Theorem 3.1 and $g_k(n) := g(n + \Phi^*(k + 2) + 1) + \Phi^*(k + 2)$.

Proof. For (y_n) we know by Theorem 3.2 that

$$\forall k \in \mathbb{N}\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists\, 1 \leq n \leq \Psi_{\tau,\eta,2L,\alpha,\gamma}(k, g)\, \forall l, m \in [n, n + g(n)]\, (\|y_l - y_m\| \leq 2^{-k}).$$

Given k, g, we apply this to $k+1$ and $g_k(n) := g(n+\Phi^*(k+2)+1)+\Phi^*(k+2)$, where $\Phi^*(k)$ is the rate from Theorem 3.1:

$$\exists 1 \leq n \leq \Psi_{\tau,\eta,2L,\alpha,\gamma}(k+1, g_k)\, \forall l, m \in [n, n+g(n+\Phi^*(k+2)+1)+\Phi^*(k+2)]\, (\|y_l - y_m\| \leq 2^{-k-1}).$$

Let n be as in the formula above and define $n' := n + \Phi^*(k+2)$. Since $n \leq n'$, we can restrict things to the smaller interval $[n', n' + g(n' + 1)] = [n + \Phi^*(k + 2), n + \Phi^*(k + 2) + g(n + \Phi^*(k + 2) + 1)]$. By Theorem 3.1 we have for all l, m in this smaller interval that

$$\|x_{l+1} - x_{m+1}\| \leq \|y_l - y_m\| + \|x_{l+1} - y_l\| + \|x_{m+1} - y_m\| \leq \|y_l - y_m\| + 2^{-k-1} \leq 2^{-k}.$$

[2]In [29], $g : \mathbb{N}_0 \to \mathbb{N}_0$ and so for $g : \mathbb{N} \to \mathbb{N}$ we have to apply the rate given in [29] to $g'(n) := g(\max(1, n))$ and to replace 0 by 1 in the original bound.

This means in total that for $n'' := n' + 1$

$$\forall l, m \in [n'', n'' + g(n'')] \, (\|x_l - x_m\|, \|x_l - y_{l-1}\| \leq 2^{-k}).$$

By construction

$$2 \leq n'' \leq n + \Phi^*(k+2) + 1 \leq \Psi_{\tau,\eta,2L,\alpha,\gamma}(k+1, g_k) + \Phi^*(k+2) + 1$$

which finishes the proof. □

The condition

$$\sum_{n=1}^{\infty} \rho_X(\lambda_n M_1) < \infty$$

involves ρ_X which is defined as a supremum which may not be computable. The next proposition shows how we can replace this with a condition involving only the modulus τ instead of ρ_X :

Proposition 3.4. *Let X be a uniformly smooth Banach space with a modulus of uniform smoothness τ, $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $(0,1)$, $M_1 > 0$ some constant and C as in Lemma 2.1. Assume that τ is strictly increasing and $(0,1) \subseteq \tau(0,\infty)$ so that the inverse τ^{-1} of τ is defined on $(0,1)$. If for $K > 0$*

$$\sum_{n=1}^{\infty} \left(\frac{1}{2} C M_1^2 \tau^{-1}(\lambda_n)\lambda_n \right) \leq K,$$

then

$$\sum_{n=1}^{\infty} \rho_X(\lambda_n M_1) \leq K \ .$$

Proof. By Lemma 2.1

$$\rho_X(\lambda_n M_1) \leq C M_1^2 \rho_X(\lambda_n)$$

and by the definition of τ

$$\forall \varepsilon > 0 \, \forall x, y \in X \, (\|x\| = 1 \wedge \|y\| \leq \tau(\varepsilon) \Rightarrow \|x+y\| + \|x-y\| \leq 2 + \varepsilon \|y\|) \, .$$

Applying this to $\varepsilon := \tau^{-1}(\lambda_n)$ we see that

$$\forall x, y \in X \, \left(\|x\| = 1 \wedge \|y\| \leq \lambda_n \Rightarrow \|x+y\| + \|x-y\| \leq 2 + \tau^{-1}(\lambda_n)\lambda_n\right)$$

which in turn implies $\rho_X(\lambda_n) \leq \frac{1}{2}\tau^{-1}(\lambda_n)\lambda_n$ and so

$$\rho_X(\lambda_n M_1) \leq \frac{1}{2} C M_1^2 \tau^{-1}(\lambda_n)\lambda_n$$

from which the proposition follows. □

We now elaborate on an example from [7] for a special choice of the scalars λ_n, θ_n in the case of L^p with $1 < p < \infty$ which satisfy the conditions in Chidume's theorem and compute the corresponding rates Φ_1, Φ_2, Φ_3 as well as α, γ used in our bounds:

Example 3.5. *Let $X = L^p$, $p \in (1,\infty)$ and define*

$$\lambda_n = (n+1)^{-a} \text{ and } \theta_n = (n+1)^{-b}$$

with $a+b<1$ and

$$a \in (\frac{1}{2},1), b \in (0,a) \text{ for } p \geq 2 \text{ resp.}$$

$$a \in (\frac{1}{p},1), b \in (0,a(p-1)) \text{ for } p<2.$$

Let

$$N_0 \geq \begin{cases} \sqrt[a-b]{\frac{p-1}{2\gamma_0}} & p \geq 2 \\ \sqrt[a(p-1)-b]{\frac{1}{p\gamma_0}} & p<2. \end{cases}$$

Then $(\lambda_{N_0+n})_{n\in\mathbb{N}}, (\theta_{N_0+n})_{n\in\mathbb{N}}$ satisfy the conditions in our theorems 3.1 and (together with α,γ) 3.3 with

$$\Phi_1(k,N) = (N_0+N+1)2^k - N_0 - 2,$$

$$\Phi_2(k) = \max\left\{1, \left\lceil (80L^2 b 2^k)^{\frac{1}{1-a-b}} \right\rceil - N_0 - 2\right\},$$

$$\Phi_3(k) = \begin{cases} \max\left\{1, \left\lceil \left(\frac{2^{k-1}(p-1)M_1^2M_2}{2a-1}\right)^{\frac{1}{2a-1}} \right\rceil - N_0 - 1\right\} & \text{if } p \geq 2, \\ \max\left\{1, \left\lceil \left(\frac{2^k M_1^p M_2}{p(pa-1)}\right)^{\frac{1}{pa-1}} \right\rceil - N_0 - 1\right\} & \text{if } p<2, \end{cases}$$

$$\alpha(n) = \max\left\{1, (n+1)^{\frac{1}{b}} - N_0 - 1\right\},$$

$$\gamma(n) = (N_0+n+1)^b.$$

Proof. First observe that we get (using the estimates for ρ_X with $X = L^p$ from [21][p.63] or [35][p.193])

$$\begin{aligned} \frac{\rho_X(\lambda_{N_0+n})}{\lambda_{N_0+n}} &\leq \frac{p-1}{2}\lambda_{N_0+n} \\ &= \frac{p-1}{2}(N_0+n+1)^{-a} \\ &= \theta_{N_0+n}\frac{p-1}{2}(N_0+n+1)^{b-a} \\ &\leq \theta_{N_0+n}\frac{p-1}{2}N_0^{b-a} \\ &\leq \gamma_0\theta_{N_0+n}, \end{aligned}$$

if $p \geq 2$ and

$$\begin{aligned}\frac{\rho_X(\lambda_{N_0+n})}{\lambda_{N_0+n}} &\leq \frac{1}{p}\lambda_{N_0+n}^{p-1} \\ &= \frac{1}{p}(N_0+n+1)^{-a(p-1)} \\ &= \theta_{N_0+n}\frac{1}{p}(N_0+n+1)^{b-a(p-1)} \\ &\leq \theta_{N_0+n}\frac{1}{p}N_0^{b-a(p-1)} \\ &\leq \gamma_0\theta_{N_0+n},\end{aligned}$$

if $1 < p < 2$. Next we have

$$\begin{aligned}\prod_{n=N}^{\Phi_1(k,N)}(1-\lambda_{N_0+n+1}\theta_{N_0+n+1}) &= \prod_{n=N}^{\Phi_1(k,N)}(1-\frac{1}{(N_0+n+2)^{a+b}}) \\ &= \prod_{n=N}^{\Phi_1(k,N)}(1-\frac{(N_0+n+2)^{1-a-b}}{N_0+n+2}) \\ &\leq \prod_{n=N}^{\Phi_1(k,N)}\frac{N_0+n+1}{N_0+n+2} \\ &= \frac{N_0+N+1}{N_0+\Phi_1(k,N)+2} \\ &= 2^{-k}.\end{aligned}$$

Let $n \geq \Phi_2(k)$. Then (using the Bernoulli inequality $(1+x)^b \leq 1+bx$ with exponent $0 \leq b \leq 1$ and $x \geq -1$)

$$\begin{aligned}\sigma_n &= C^*\frac{\frac{\theta_{N_0+n}}{\theta_{N_0+n+1}}-1}{\lambda_{N_0+n+1}\theta_{N_0+n+1}} \\ &= 40L^2(N_0+n+2)^{a+b}((1+\frac{1}{N_0+n+1})^b-1) \\ &\leq 40L^2b\frac{(N_0+n+2)^{a+b}}{N_0+n+1} \\ &=40L^2b\frac{N_0+n+2}{N_0+n+1}(N_0+n+2)^{a+b-1} \\ &\leq 80L^2b(N_0+n+2)^{a+b-1} \\ &\leq 2^{-k}.\end{aligned}$$

Ad $\Phi_3(k)$: Using again the aforementioned estimates for ρ_X we have (for $2 \leq p < \infty$)

$$(*)\quad \sum_{n=\Phi_3(k)}^{\infty} M_2\rho_X(\lambda_{n+1}M_1) \leq \sum_{n=\Phi_3(k)}^{\infty}\frac{p-1}{2}M_1^2M_2(N_0+n+2)^{-2a} \leq \frac{p-1}{2}M_1^2M_2\sum_{n=\Phi_3(k)+N_0+2}^{\infty} n^{-2a}.$$

Let $s := 2a > 1$. Then for $M \in \mathbb{N}, M \geq 2$

$$\sum_{n=M}^{\infty} n^{-s} \leq \int_{M-1}^{\infty} \frac{1}{x^s} ds = \frac{1}{s-1} \cdot \frac{1}{(M-1)^{s-1}}.$$

Hence for

$$M \geq \left(\frac{2^{k-1}(p-1)M_1^2 M_2}{2a-1} \right)^{\frac{1}{2a-1}} + 1$$

we get

$$\sum_{n=M}^{\infty} n^{-2a} \leq 2^{-k} \cdot \frac{2}{(p-1)M_1^2 M_2}.$$

Together with $(*)$ this yields

$$\sum_{n=\Phi_3(k)}^{\infty} M_2 \rho_X(\lambda_{n+1} M_1) \leq 2^{-k}.$$

The case $1 < p < 2$ is treated analogously.
One easily verifies that α, γ satisfy the requirements from Theorem 3.3 for our choice of scalars. □

Remark 3.6. *A rate similar to* Φ_2 *above can be also obtained using* R_4 *from* [19][section 5]*, where a different argument is used.*

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