Quantitative analysis of a Halpern-type Proximal Point Algorithm for accretive operators in Banach spaces

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Abstract
Recently, Aoyama and Toyoda showed that a Halpern-type proximal point algorithm strongly converges under very general conditions on the scalars involved to a zero of an accretive operator in uniformly convex Banach spaces with a uniformly Gâteaux differentiable norm. We give a quantitative analysis of this result in the slightly more restricted context of Banach spaces which are uniformly convex and uniformly smooth.

Keywords: Accretive operators, proximal point algorithm, uniformly convex Banach spaces, uniformly smooth Banach spaces, rates of convergence, metastability, proof mining.
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1 Introduction
The fundamental Proximal Point Algorithm (PPA) is a method to approximate zeros of maximally monotone operators \( A \subseteq H \times H \) in Hilbert space ([18, 24]). While the algorithm converges weakly, the strong convergence in general fails ([5]). To obtain strongly convergent versions of (PPA), the definition of the iteration usually is modified in a way suggested by the so-called Halpern-type iteration ([6]) which uses a certain point \( u \in H \) as an anchor. The resulting Halpern-type form (HPPA) of (PPA) is given by:

\[
x_{n+1} := \alpha_n u + (1 - \alpha_n)J_{\lambda_n A}x_n,
\]

where \((\alpha_n) \subset (0, 1), (\lambda_n) \subset (0, \infty)\) and \(J_{\lambda_n A} := (I + \lambda_n A)^{-1}\) is the resolvent of \(A\) (see e.g. [7, 28, 4, 17]).

In [1], the strong convergence of this algorithm is shown even for the class of uniformly convex Banach spaces \(X\) whose norm is uniformly Gâteaux differentiable and for general accretive operators \(A\). As conditions on \((\alpha_n) \subset (0, 1)\) only

\[
\sum_{n=0}^{\infty} \alpha_n = \infty \text{ and } \lim_{n \to \infty} \alpha_n = 0,
\]

known to be necessary for Halpern’s classical strong convergence result, are needed and the only assumption on \((\lambda_n) \subset (0, \infty)\) is to be bounded away from 0, i.e. \(\inf \lambda_n > 0\).
The strong convergence of \((x_n)\) is established in [1] by reducing the situation to a famous result of Reich [23] on the strong convergence of the path \((z_t)\) where \(z_t = tu + (1 - t)J_{\lambda_t}z_t\) for \(t \in (0,1)\).

In this paper, we give a quantitative analysis of the main theorem in [1] in the slightly more restricted case where \(X\) is assumed to be uniformly smooth (in addition to being uniformly convex) as for this class of spaces logical bound-extraction metatheorems are available ([8, 12]).

It is known that even for trivial situations such as \(H = \mathbb{R}\) one in general does not have a computable rate of convergence for \((x_n)\) (see [19]) and so one has to aim at the next best thing which is an explicit so-called rate of metastability in the sense of Tao [26, 27], i.e. a function \(\Theta : \mathbb{N} \times \mathbb{N}^3 \to \mathbb{N}\) such that

\[
\forall k \in \mathbb{N} \forall g \in \mathbb{N}^g \exists N \leq \Theta(k, g) \forall n, m \in [N, N + g(N)] \left( \|x_n - x_m\| < \frac{1}{k + 1} \right),
\]

where \([N, N + g(N)] := \{N, N + 1, N + 2, \ldots, N + g(N)\}\), whose complexity reflects the computational content of the original convergence proof from which it is extractable by proof-theoretic methods (see [8]). Note that, noneffectively, the metastability of \((x_n)\) implies the ordinary Cauchy property of \((x_n)\).

General results from mathematical logic ([8, 12]) guarantee the extractability of a rate of metastability from the proof given in [1] which only depends on moduli \(\eta, \tau\) of uniform convexity and uniform smoothness of \(X\), rates of convergence for \(\prod_{i=0}^{\infty} (1 - \alpha_i) \to 0\) (which is equivalent to \(\sum_{i=0}^{\infty} \alpha_i = \infty\)) and \(\alpha_n \to 0\), a positive lower bound \(0 < \lambda \leq \lambda_n\) (for all \(n \in \mathbb{N}\)), sequences of positive lower bounds \(0 < \tilde{\alpha}_n \leq \alpha_n\) of \((\alpha_n)\) and of upper bounds \(\tilde{\lambda}_n \geq \lambda_n\) for \((\lambda_n)\), an upper bound \(b \geq \|u - p\|, \|x_0 - p\|\) for some zero \(p\) of \(A\), the error \(\varepsilon = 1/(k + 1), g\) and a given rate of metastability \(\xi\) for \((z_t)\), i.e. for Reich’s result. Such a \(\xi\) has recently been constructed for uniformly convex and uniformly smooth Banach spaces in [13]. In the case where \(X\) is a Hilbert space, a much simpler such \(\xi\) has been known already since [9]. For more information on the logic-based approach to the extraction of explicit bounds from prima facie noneffective proofs and the concept of metastability we refer to the recent survey [11].

While many explicit rates of metastability have been extracted in recent years for a number of algorithms in nonlinear analysis, for the Halpern-type Proximal Point Algorithm such rates were obtained only recently in [21, 15, 22] (also using a logic-based approach) which consider the HPPA in Hilbert spaces (also with error terms) where either \((\lambda_n)\) is assumed to diverge to \(\infty\) or is assumed to converge to some \(\lambda > 0\) (in the latter case an additional assumption on \((\alpha_n)\) is used) which are more restrictive than the situation in [1] which we study. Obviously, we have to pay a price for the greater generality namely that our rate is somewhat more complicated. Also, our rate depends on some sequence \((\tilde{\alpha}_n)\) with \(0 < \tilde{\alpha}_n \leq \alpha_n\) witnessing the strict positivity of \(\alpha_n\) which is used in the proof in [1], whereas in [21, 22] the special case where \(\sum \gamma_n = \infty\) is treated in a way which does not require this. In any case, the proof from [1] is rather different from the proofs analyzed in [21, 15, 22] and makes crucial use of the fact that \(J_{\lambda_nA}\) as a firmly nonexpansive mapping in a uniformly convex space is strongly nonexpansive. The class of strongly nonexpansive mappings has very nice quantitative properties which we exhibited in [10] and which are used in the present paper as well.
2 Preliminaries

**Definition 1.** A real Banach space \((X, \| \cdot \|)\) is uniformly convex with a modulus of convexity \(\eta : (0, 2] \to (0, 1]\) if
\[
\forall \varepsilon \in (0, 2] \forall x, y \in X \left( \|x\|, \|y\| \leq 1 \land \|x - y\| \geq \varepsilon \Rightarrow \left\| \frac{1}{2}(x + y) \right\| \leq 1 - \eta(\varepsilon) \right).\
\]

**Definition 2.** A real Banach space \((X, \| \cdot \|)\) is uniformly smooth if for all \(\varepsilon > 0\) there exists some \(\delta = \tau(\varepsilon) > 0\)
\[
\forall x, y \in X (\|x\| = 1 \land \|y\| \leq \delta \Rightarrow \|x + y\| + \|x - y\| \leq 2 + \varepsilon \|y\|)
\]
and a function \(\tau : (0, \infty) \to (0, \infty)\) producing such a \(\delta = \tau(\varepsilon)\) is called a modulus of uniform smoothness for \(X\).

Throughout this paper \((X, \| \cdot \|)\) is a uniformly convex and uniformly smooth real Banach space with respective moduli \(\eta\) and \(\tau\).

It is well known that in uniformly smooth spaces, the normalized duality mapping \(J\) is single-valued and uniformly norm-to-norm continuous on bounded sets. The next lemma gives a quantitative formulation of this fact:

**Lemma 3** ([12]). Let \(X\) be uniformly smooth with modulus \(\tau\). Define \(\omega_J : (0, \infty) \times (0, \infty) \to (0, \infty)\) by
\[
\omega_J(b, \varepsilon) := \frac{\varepsilon^2}{12b} \cdot \tau \left( \frac{\varepsilon}{2b} \right), \quad \varepsilon \in (0, 2], b \geq 1,
\]
with \(\omega_J(b, \varepsilon) := \omega_J(1, \varepsilon)\) for \(b < 1\) and \(\omega_J(b, \varepsilon) := \omega_J(b, 2)\) for \(\varepsilon > 2\). Then the single-valued duality map \(J : X \to X^*\) is norm-to-norm uniformly continuous on bounded subsets with modulus \(\omega_J\), that is, for all \(b, \varepsilon > 0\) and \(x, y \in X\) with \(\|x\|, \|y\| \leq b\) we have
\[
\|x - y\| \leq \omega_J(b, \varepsilon) \Rightarrow \|Jx - Jy\| \leq \varepsilon.
\]

If \(X\) is a Hilbert space, we may simply take \(\omega_J\) as the identity mapping.

Let \(A \subseteq X \times X\) be an accretive operator, i.e.
\[
\forall (x, u), (y, v) \in A \left( \langle u - v, J(x - y) \rangle \geq 0 \right).
\]

It is well known that for any \(\lambda > 0\)
\[
J_{\lambda A} : R(I + \lambda A) \to X, \quad x \mapsto (I + \lambda A)^{-1}(x)
\]
is a single valued firmly nonexpansive mapping with \(R(J_{\lambda A}) = D(A)\) and the fixed point set \(\text{Fix}(J_{\lambda A})\) of \(J_{\lambda A}\) coincides with the set \(\text{zer} A := A^{-1}0 = \{q \in X : 0 \in Aq\}\) of zeros of \(A\) (see [3], p.466, and [25], pp.130,135 as well as [2]). Since \(J_{\lambda A}\) is firmly nonexpansive it also is - using the uniform convexity of \(X\) - strongly nonexpansive (see [3]).

In [10], a quantitative form of this fact is established (for arbitrary firmly nonexpansive mappings but stated here in terms \(J_{\lambda A}\)):

**Lemma 4** ([10], Proposition 2.17). \(J_{\lambda A}\) is strongly nonexpansive with SNE-modulus
\[
\omega_\eta(c, \varepsilon) = \frac{1}{4} \eta(\varepsilon/c) \cdot \varepsilon
\]

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(for $\varepsilon > 2c$ the claim is trivial and we may simply put $\omega_\eta(c, \varepsilon) := 1$) which does not depend on $\lambda > 0$, i.e. for all $c, \lambda, \varepsilon > 0$, $x, y \in R(I + \lambda A)$

$$
\|x - y\| \leq c \wedge \|x - y\| - \|J_{\lambda A} x - J_{\lambda A} y\| < \omega_\eta(c, \varepsilon) \rightarrow \|x - y\| - (J_{\lambda A} x - J_{\lambda A} y) \| < \varepsilon.
$$

If $\eta$ can be written as $\eta(\varepsilon) = \varepsilon \cdot \tilde{\eta}(\varepsilon)$ with $\tilde{\eta}$ such that

$$
\varepsilon_1 \leq \varepsilon_2 \rightarrow \tilde{\eta}(\varepsilon_1) \leq \tilde{\eta}(\varepsilon_2), \text{ for all } \varepsilon_1, \varepsilon_2 \in (0, 2],
$$

then the modulus can be taken as $\omega_\eta(c, \varepsilon) := \frac{1}{2} \tilde{\eta}(\varepsilon/c) \cdot \varepsilon$.

This gives a modulus of order $p$ in $L^p$ with $2 \leq p < \infty$. In particular, for the case of Hilbert spaces we may take $\omega_\eta(c, \varepsilon) := \frac{1}{16} c^2 \varepsilon^2$.

As in [1], we always assume that the accretive operator $A$ satisfies the range condition

$$
\overline{D(A)} \subseteq C \subseteq R(I + \lambda A) \text{ for all } \lambda > 0,
$$

where $\overline{D(A)}$ is the closure of the domain $D(A)$ of $A$ and $C$ is a nonempty closed and convex subset of $X$ and that $\text{zer} A \neq \emptyset$.

For $(\lambda_n) \subseteq [\lambda, \infty)$ with $\lambda > 0$, [1] studies the Halpern-type variant of the Proximal Point Algorithm for an accretive operator $A$ satisfying the conditions above is given by the sequence $(x_n) \subseteq C$ defined by (for given $x_0, u \in C$)

$$
(*) \quad x_{n+1} := \alpha_n u + (1 - \alpha_n) J_{\lambda_n A} x_n.
$$

Here $(\alpha_n)$ is a sequence in $(0, 1)$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\alpha_n \rightarrow 0$.

The main result (proved even under the weaker assumption of a uniformly Gâteaux differentiable norm rather than uniform smoothness) in [1] is:

**Theorem 5** ([1], Theorem 3.1). Under the conditions stated above, $(x_n)$ converges strongly to $Qu$, where $Q$ is the unique sunny nonexpansive retraction of $C$ onto $\text{zer} A$.

## 3 Quantitative lemmas

$\mathbb{N} := \{0, 1, 2, \ldots\}, \mathbb{N}^* := \{1, 2, 3, \ldots\}$. Throughout this paper, for $f : \mathbb{N} \rightarrow \mathbb{N}$, $f^M : \mathbb{N} \rightarrow \mathbb{N}$ denotes the function $f^M(n) := \max\{f(i) : i \leq n\}$.

**Lemma 6** ([1]). Let $A \subseteq X \times X$ be accretive with the range condition and $\lambda, \mu > 0$. Then

$$
\|x - J_{\mu A} x\| \leq \left(2 + \frac{\mu}{\lambda}\right) \|x - J_{\lambda A} x\|
$$

for all $x \in R(I + \lambda A) \cap R(I + \mu A)$.

**Lemma 7** ([20]). For all $x, y \in X$ we have $\|x + y\|^2 \leq \|x\|^2 + 2(y, J(x + y))$.

**Lemma 8** (Quantitative version of Lemma 2.3 in [1]). Let $w \in C$ and let $(x_n)$ be any sequence in $C$ with $\|x_n - w\| \leq b$ for all $n \in \mathbb{N}$ and $(\lambda_n)$ be a sequence in $(0, \infty)$. Then for $\omega_\eta$ from Lemma 4, $J_{\lambda_n} := J_{\lambda_n A}$ and $\tilde{\omega}_\eta(b, \varepsilon) := \min\{\frac{1}{2} \omega_\eta(b, \varepsilon/2)\}$ :

$$
\forall \varepsilon > 0 \forall n \in \mathbb{N} \left(\|x_n - w\| - \|J_{\lambda_n} x_n - w\| \leq \tilde{\omega}_\eta(b, \varepsilon) \wedge \|w - J_{\lambda_n} w\| \leq \tilde{\omega}_\eta(b, \varepsilon) \rightarrow \|x_n - J_{\lambda_n} x_n\| \leq \varepsilon\right).
$$
Proof: Since \( \omega_{\eta} \) is an SNE-modulus for \( J_{\lambda_n} \),

\[
\|x_n - w\| - \|J_{\lambda_n} x_n - J_{\lambda_n} w\| \leq \|x_n - w\| - \|J_{\lambda_n} x_n - w\| + \|w - J_{\lambda_n} w\| \leq \tilde{\omega}_{\eta}(b, \varepsilon) + \tilde{\omega}_{\eta}(b, \varepsilon) \leq \omega_{\eta}(b, \frac{\varepsilon}{2})
\]

implies that \( \|x_n - J_{\lambda_n} x_n\| \leq (\|x_n - w\| - (J_{\lambda_n} x_n - J_{\lambda_n} w\|) + \|w - J_{\lambda_n} w\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \)

\[
\square
\]

Lemma 9 (Quantitative version of Lemma 2.7 in [1]). Let \( b > 0 \) and \((a_n)\) be a sequence in \([0,b]\).

1. Let \( \tau : \mathbb{N} \to \mathbb{N} \) be such that

\[
(+) \ \forall n, k \in \mathbb{N} (k \leq n \land a_k < a_{k+1} \to k \leq \tau(n)).
\]

Define for \( K \in \mathbb{N}, g \in \mathbb{N}^{\mathbb{N}}, \varepsilon > 0 \) and \( \bar{g}(n) := n + g(n) \)

\[
\psi(\varepsilon, g, K, b) := \bar{g}(\left\lceil \frac{\varepsilon}{b} \right\rceil)(K).
\]

Then

\[
\tau(\psi(\varepsilon, g, K, b)) < K \to \exists n \leq \psi(\varepsilon, g, K, b) (n \geq K \land \forall i, j \in [n, n + g(n)] (|a_i - a_j| \leq \varepsilon)).
\]

2. Let \( n_0 \in \mathbb{N} \) be such that \( \exists n \leq n_0(a_n < a_{n+1}) \). Define

\[
\tau(n) := \max\{k \leq \max\{n_0, n\} : a_k < a_{k+1}\}.
\]

Then \( \tau \) is well-defined and satisfies \((+). \) Moreover,

\[
(i) \ \forall n \in \mathbb{N}(a_{\tau(n)} \leq a_{\tau(n)+1}),
\]

\[
(ii) \ \forall n \in \mathbb{N}(\tau(n) \leq \tau(n+1)),
\]

\[
(iii) \ \forall n \geq n_0(a_n \leq a_{\tau(n)+1}).
\]

Proof: 1) Assume \( \tau(\psi(\varepsilon, g, K, b)) < K \). Then

\[
\forall k \in [K, \psi(\varepsilon, g, K, b)) (a_k \geq a_{k+1}),
\]

since, if \( k \in [K, \psi(\varepsilon, g, K, b)] \) with \( a_k < a_{k+1} \), then by \((+ \) \( k \leq \tau(\psi(\varepsilon, g, K, b)) < K \) which is a contradiction. Hence

\[
(++) \ \forall k \in [K, \bar{g}(\left\lceil \frac{\varepsilon}{b} \right\rceil)(K)] (0 \leq a_{k+1} \leq a_k \leq b).
\]

Suppose now that

\[
\forall i < \left\lceil \frac{b}{\varepsilon} \right\rceil (a_{\bar{g}(i)(K)} < a_{\bar{g}(i+1)(K)} - \varepsilon).
\]

Then \( a_K - a_{\bar{g}(i+1)(K)} \geq \left\lceil \frac{b}{\varepsilon} \right\rceil \cdot \varepsilon \geq b \) which contradicts \( a_K, a_{\bar{g}(i+1)(K)} \in [0, b] \). Hence

\[
\exists i_0 < \left\lceil \frac{b}{\varepsilon} \right\rceil (a_{\bar{g}(i)(K)} \geq a_{\bar{g}(i+1)(K)} - \varepsilon)
\]

and so for \( K \leq n := \bar{g}(i_0)(K) \leq \psi(\varepsilon, g, K, b) \) - using \((++ \) -

\[
\forall i, j \in [n, n + g(n)] (|a_i - a_j| \leq \varepsilon).
\]
2) (+), (i), (ii) are obvious from the definition of \( \tau \).

(iii) follows as in the proof of Lemma 3.1 in [16] which we repeat here for completeness: we assume
\( n \geq n_0 \) (so that \( \tau(n) \leq n \)) and hence only have to consider three cases:

Case 1: \( \tau(n) = n \). Then (iii) follows from (i).

Case 2: \( \tau(n) = n - 1 \). Then (iii) holds trivially.

Case 3: \( \tau(n) < n - 1 \), i.e. \( \tau(n) \leq n - 2 \). By definition of \( \tau \) we have
\[
a_{\tau(n)+1} \geq a_{\tau(n)+2} \geq \ldots \geq a_{n-1} \geq a_n.
\]

\[ \square \]

Lemma 10 (Quantitative version of Lemma 2.8 in [1]). Let \( b > 0 \) and \((a_n)\) be a sequence in \([0, b]\) with
\[
a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \beta_n + \gamma_n \quad (n \in \mathbb{N}),
\]
where \((\alpha_n) \subset (0, 1] \subset \mathbb{R} \) and \((\gamma_n) \subset \mathbb{R}^+ \) with \( \sum_{n=0}^{\infty} \alpha_n = \infty \) (i.e. \( \prod_{n=m}^{\infty} (1 - \alpha_n) = 0 \) for all \( m \in \mathbb{N} \)).

Let \( S : (0, \infty) \times \mathbb{N} \to \mathbb{N} \) be such that
\[
\forall m \in \mathbb{N} \forall \varepsilon > 0 \left( \prod_{k=m}^{\infty} (1 - \alpha_k) \leq \varepsilon \right).
\]

W.l.o.g. we may assume that \( S \) is nondecreasing in \( m \).

For \( \varepsilon > 0 \) and \( g \in \mathbb{N} \) define
\[
\tilde{g}(n) := g^M(n + S(\frac{\varepsilon}{4b}, n) + 1) + S(\frac{\varepsilon}{4b}, n).
\]

Suppose that \( N \in \mathbb{N} \) satisfies that
\[
\exists m \leq N \forall i \in [m, m + \tilde{g}(m)] (\beta_i \leq \frac{\varepsilon}{4}).
\]

Define
\[
\varphi(\varepsilon, S, N, b) := N + S(\frac{\varepsilon}{4b}, N) + 1.
\]

Then
\[
\varphi(\varepsilon, S, N, b) + g^M(\varphi(\varepsilon, S, N, b)) \sum_{i=0}^{\infty} \gamma_i \leq \frac{\varepsilon}{2} \implies \exists n \leq \varphi(\varepsilon, S, N, b) \forall i \in [n, n + g(n)] (a_i \leq \varepsilon).
\]

Proof: By the assumption on \( N \) we have
\[
\exists m \leq N \forall i \in [m, m + S(\frac{\varepsilon}{4b}, m) + g(m + S(\frac{\varepsilon}{4b}, m) + 1)] (\beta_i \leq \frac{\varepsilon}{4})
\]
and so for \( n := m + S(\frac{\varepsilon}{4b}, m) + 1 \)
\[
\forall i \in [m, n + g(n) - 1] (\beta_i \leq \frac{\varepsilon}{4}).
\]
From the proof of Lemma 2.3 in [14] it follows that for all \( i \in [n, n + g(n)] \) (using \( i \geq n \geq S(\varepsilon/4b, m) + 1 \))
\[
a_t \leq a_m \cdot \prod_{k=m}^{i-1} (1 - \alpha_k) + \max \{ \beta_k : m \leq k \leq i - 1 \} + \sum_{k=m}^{i-1} \gamma_k \leq b \frac{\varepsilon}{4b} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.
\]

\[\square\]

**Lemma 11** (Quantitative version of Lemma 2.9 in [1]). Let \( b > 0 \) and \( (x_n) \) be a sequence in \( C \), \( u \in C \) and for \( t \in (0, 1) \) let \( z_t \in C \) be the unique point with
\[
z_t = tu + (1 - t)J_{\lambda A} z_t
\]
for \( \lambda > 0 \) (which exists by Banach’s fixed point theorem). Assume that \( \| z_t - x_n \|, \| J_{\lambda A} x_n - x_n \| \leq b \) for all \( n \in \mathbb{N}, t \in (0, 1) \). Let \( (t_k) \) be a sequence in \( (0, 1) \) with \( t_k \to 0 \) and let \( \rho : (0, \infty) \to \mathbb{N} \) be a rate of convergence (i.e. \( t_k \leq \varepsilon \) for \( k \geq \rho(\varepsilon) \)) and \( \chi : \mathbb{N} \to \mathbb{N}^+ \) such that \( t_k \geq \frac{1}{\chi(k)} \) for all \( k \in \mathbb{N} \). Let \( k \geq \rho \left( \frac{\varepsilon}{\| x \|} \right) \) and for some \( n \in \mathbb{N} \) assume that \( \| J_{\lambda A} x_n - x_n \| \leq \eta_{k, \varepsilon} := \frac{\varepsilon}{3\chi(k)} \). Then for this \( n \) we get \( \langle u - z_{t_k}, J(x_n - z_{t_k}) \rangle \leq \varepsilon \).

**Proof:** Reasoning as in [1](p.808) one has
\[
(u - z_{t_k}, J(x_n - z_{t_k})) \\
\leq t_k^2 \| z_{t_k} - x_n \|^2 + \frac{(1 - t_k)^2}{2t_k} \| J_{\lambda A} x_n - x_n \| (\| J_{\lambda A} x_n - x_n \| + 2 \| z_{t_k} - x_n \|) \\
\leq \frac{t_k^2}{2} + \frac{2b^2}{3} \| J_{\lambda A} x_n - x_n \| \\
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

\[\square\]

The next Lemma addresses the specific form in which in the proof of the main result, a given rate of metastability for the sequence \( (z_{t_k}) \) will be used to construct a rate of metastability for the proximal sequence \( (x_n) \):

**Lemma 12.** Let \( (a_n) \) be a Cauchy sequence in \( C \) with a rate of metastability \( \xi \) in the form
\[
\forall \varepsilon > 0 \ \forall g \in \mathbb{N}^\mathbb{N} \exists n \leq \xi(\varepsilon, g) \forall i, j \in [n, g(n)] (\| a_i - a_j \| \leq \varepsilon).
\]

Let now \( \varepsilon > 0, c \in \mathbb{N} \) and \( f : \mathbb{N} \to \mathbb{N} \) and define \( f_c(l) := f(l + c) \). Then
\[
\exists k \leq \xi(\varepsilon, f_c) + c (k \geq c \land \forall i, j \in [k, f(k)] (\| a_i - a_j \| \leq \varepsilon)).
\]

**Proof:** By the definition of \( \xi \)
\[
\exists k \leq \xi(\varepsilon, f_c) \forall i, j \in [k, f(k) + c] (\| a_i - a_j \| \leq \varepsilon).
\]
Hence for \( k := \tilde{k} + c \) we have that \( k \geq c \) and
\[
\forall i, j \in [k, f(k)] (\| a_i - a_j \| \leq \varepsilon).
\]

\[\square\]

The next Lemma establishes a crucial bound on the various sequences involved in this paper:

\[\text{Correction Jan.22, 2023: in the next line replace max\{\ldots\} by } (1 - \prod_{k=m}^{i-1} (1 - \alpha_k)) \text{ max\{\ldots\}.}\]
Lemma 13. Let \((x_n), u\) as defined in (*) above and for \(t \in (0,1)\) let \(z_t \in C\) be the unique point with \(z_t = tu + (1-t)J_{\lambda_t}z_t\). Let \(p \in \text{zer } A\) and \(\mathbb{N}^* \ni b \geq 2 \max\{|u-p|,|x_0-p|\}\). Then
\[
\text{diam}\{u,x_n,J_{\lambda_n}A x_n,z_t,J_{\lambda_n}A z_t : n \in \mathbb{N}, t \in (0,1)\} \leq b.
\]

**Proof:** As in [1](p.809) one shows that (using \(\text{zer } A = \text{Fix}(J_{\lambda_n}A)\)) for all \(n \in \mathbb{N}\)
\[
\|J_{\lambda_n}A x_n - p\| \leq \|x_n - p\| \leq \max\{|u-p|,|x_0-p|\}.
\]
Also
\[
\|z_t - p\| = \|tu + (1-t)J_{\lambda_t}Az_t - p\| = \|(u-p) + (1-t)(J_{\lambda_t}A z_t - J_{\lambda_t}Ap)\|
\]
\[
\leq \ell\|u-p\| + (1-t)\|J_{\lambda_t}A z_t - J_{\lambda_t}Ap\|
\]
\[
\leq \ell\|u-p\| + (1-t)\|z_t - p\|.
\]
Hence
\[
\|J_{\lambda_t}A z_t - p\| \leq \|z_t - p\| \leq \|u-p\|.
\]
Thus \(u,x_n,J_{\lambda_n}A x_n,z_t,J_{\lambda_n}A z_t \in B_{b/2}(p) := \{x \in X : \|x-p\| \leq b/2\}\) which implies the lemma. \(\Box\)

Lemma 14. Let \(K \in \mathbb{N}, \varepsilon > 0\) and \((\lambda_n) \subset [\lambda, \infty)\) for \(\lambda > 0\). Let \(b\) be as in Lemma 13 and let \(z_{t_k} = t_k u + (1-t_k)J_{\lambda_t}A z_{t_k}\), where \((t_k) \subset (0,1)\) converges to 0 with rate of convergence \(\rho\). Let \(\tilde{\lambda}_i \geq \lambda_i\) for all \(i \in \mathbb{N}\). Define \(\tilde{\lambda}_n^{\varepsilon, K} := \max\{\tilde{\lambda}_i : i \leq n\}\). Then
\[
\forall k \geq \tilde{\rho}(\varepsilon, K) := \rho \left(\frac{\varepsilon}{2(\lambda^{\tilde{\lambda}^{\varepsilon,K}}/\lambda)}\right) \quad \forall n \leq K \quad (\|z_{t_k} - J_{\lambda_n}A z_{t_k}\| \leq \varepsilon).
\]

**Proof:** \(\|z_{t_k} - J_{\lambda_n}A z_{t_k}\| = \|t_k u + (1-t_k)J_{\lambda_n}A z_{t_k} - J_{\lambda_n}A z_{t_k}\| = t_k \|u - J_{\lambda_n}A z_{t_k}\| \leq t_k \cdot b\) and so by Lemma 6 for \(n \leq K\)
\[
\|z_{t_k} - J_{\lambda_n}A z_{t_k}\| \leq \left(2 + \frac{\lambda_n}{\lambda_1}\right) \|z_{t_k} - J_{\lambda_1}A z_{t_k}\| \leq \left(2 + \frac{\lambda_n}{\lambda_1}\right) t_k \cdot b \leq \left(2 + \frac{\tilde{\lambda}_n^{\varepsilon,K}}{\lambda}\right) t_k \cdot b
\]
which implies the claim. \(\Box\)

4 Proof of the main result

In this section we construct our rate of metastability for \((x_n)\):

In the following, let \((x_n),(z_t)_{t \in (0,1)}\) and \(b\) be as in Lemma 13. For \(k \in \mathbb{N}^*\) let \(t_k := 1/k\) so that \(\chi(k) := k\) and \(\rho(\varepsilon) := [1/\varepsilon]\) satisfy the requirements in Lemma 11. Let \(S_n := J_{\lambda_n}A\) and \(z_k := z_{t_k}\).

Instead of \(\tilde{\omega}_\varepsilon(b,\varepsilon)\) (from Lemma 8) and \(\omega_f(b,\varepsilon)\) we simply write \(\tilde{\omega}_\varepsilon(\varepsilon)\) and \(\omega_f(\varepsilon)\). Let \(\zeta\) be a rate of convergence for \(\alpha_n \to 0\) and \(S\) as be as in Lemma 10. Define \((\tilde{\lambda}_1 \geq \lambda_1) C := 2 + \frac{\lambda_1}{\lambda}\) and
\[
\tilde{\varepsilon} := \min\{\frac{\varepsilon^2}{128b},\omega_f(\varepsilon^2/128b)\}, \quad \eta_k := \frac{\varepsilon^2/64}{3b(\chi(k))} = \frac{\varepsilon^2}{192b \cdot k},
\]

Let now \(L, k \in \mathbb{N}\) be arbitrary and let \(n_k\) be so large that for all \(m \geq n_k\)
\[
\alpha_m b \leq M_1(k) := \min\left\{\frac{1}{2} \tilde{\omega}_\varepsilon(\eta_k/C), \tilde{\omega}_\varepsilon(\frac{1}{2} \omega_f\left(\frac{\varepsilon^2}{64}\right))\right\} \leq \frac{1}{2} \omega_f\left(\frac{1}{64b^2}\right),
\]
e.g. $n_k := \max\{\zeta(M_1(i)/b) : i \leq k\}$ (where we take the maximum to make the dependence on $k$ monotone which is used later).

Let $\hat{g}$ be as in Lemma 10 with $\epsilon^2/4 = (\epsilon/2)^2$ as $\epsilon$ and $b^2$ as $b$, i.e.

$$\hat{g}(n) = g^M\left(n + S\left(\frac{\epsilon^2}{16b^2}, n\right) + 1\right) + S\left(\frac{\epsilon^2}{16b^2}, n\right).$$

For $\psi$ as in Lemma 9 let

$$\psi(i) := \psi\left(\frac{1}{2}\tilde{\omega}_\eta(\eta_k/C), \hat{g} + 2, i, b\right) \geq i.$$

Define

$$K := \psi(n_k) + \hat{g}^M(\psi(n_k)) + 2$$

and

$$\hat{K} := K + S\left(\frac{\epsilon^2}{16b^2}, K\right) + g^M\left(K + S\left(\frac{\epsilon^2}{16b^2}, K\right) + 1\right) + 1.$$

Now let $k' \geq L$ be so large that $z_{k'}$ is a $\delta$-approximate fixed point for all $S_m$ for all $m \leq \hat{K}$, where

$$\delta \leq M_2(k) := \min\left\{\frac{\epsilon^2}{16b(K + 1)}, \tilde{\omega}_\eta\left(\frac{1}{2}\omega_j\left(\frac{1}{64b}\epsilon^2\right)\right), \tilde{\omega}_\eta(\eta_k/C), \frac{1}{16b}\epsilon^2, \min\{\tilde{\alpha}(i) : i \leq K\}\right\},$$

where $0 < \tilde{\alpha}_i \leq \alpha_i$ for all $i \in \mathbb{N}$. E.g. we may take $k' := \max\{\bar{\rho}(M_2(k), \hat{K})\} \geq L$ with $\bar{\rho}$ from Lemma 14.

Define now the function $f : \mathbb{N}^* \to \mathbb{N}$ by $f(k) := k'$.

For the function $f$ let $k \leq \xi(\epsilon, f_c) + c$ by Lemma 12 applied to $\epsilon$ as $\epsilon$ and

$$a_n := z_n, c := \rho\left(\frac{\epsilon^2}{64b^2}\right) = \left[\frac{64b^2}{\epsilon^2}\right]$$

be such that $k \geq c$ and

$$(+) \forall i, j \in [k, f(k)] (\|z_i - z_j\| \leq \epsilon).$$

**Theorem 15.** Define for given $\epsilon > 0, L \in \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ the quantities $\tilde{\epsilon}, f, c$ as above and take

$$k^* := \xi(\epsilon, f_c) + c$$

from Lemma 12 and $\xi$ being a rate of metastability for $(z_k)$ as in Lemma 12 and define

$$K^* := \psi^M(n_{k^*}) + \hat{g}^M(\psi^M(n_{k^*})) + 2.$$ Then

$$\begin{cases} (i) \exists n \leq K^* + S\left(\frac{\epsilon^2}{16b^2}, K^*\right) \exists k' \in [L, f^M(k^*)] \forall i \in [n, n + g(n)] (\|x_i - z_{k'}\| \leq \epsilon/2) \\
(ii) \exists n \leq K^* + S\left(\frac{\epsilon^2}{16b^2}, K^*\right) + 1 \forall i, j \in [n, n + g(n)] (\|x_i - x_j\| \leq \epsilon).
\end{cases}$$

and so, in particular (taking e.g. $L := 0$),

$$\begin{cases} (i) \exists n \leq K^* + S\left(\frac{\epsilon^2}{16b^2}, K^*\right) + 1 \exists k' \in [L, f^M(k^*)] \forall i \in [n, n + g(n)] (\|x_i - z_{k'}\| \leq \epsilon/2) \\
(ii) \exists n \leq K^* + S\left(\frac{\epsilon^2}{16b^2}, K^*\right) + 1 \forall i, j \in [n, n + g(n)] (\|x_i - x_j\| \leq \epsilon).
\end{cases}$$

**Remark 16.** 1. Note by inspection that the bounds only depend on $\epsilon, b, g, L, \chi, \tilde{\lambda}_n, (\bar{\alpha}_n)$ and the rates and moduli $\chi, S, \xi, \eta, \tau$.  

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2. In what follows we give a completely elementary proof of the theorem. Since (ii) trivially implies the Cauchy property of \((x_n)\) one obtains (using that \(C\) is closed and \(X\) is complete) that \((x_n)\) strongly converges. Moreover, by (i) it converges to the same limit as \((z_k)\) converges to (take e.g. \(g(n) := L\) so that by (i) we have \(\exists n, k' \geq L(\|x_n - z_{k'}\| \leq \varepsilon)\), i.e. to \(Q_u\), where \(Q_u\) is the sunny nonexpansive retraction of \(C\) onto \(zer\ A\) (for the latter statement an elementary proof is given in [13] where also an explicit rate of metastability \(\xi\) for \((z_k)\) is constructed). If \(X\) is a Hilbert space, we can simply take \(\xi(\varepsilon, g) := \tilde{g}(\xi(\varepsilon, g))\) (0) (see [9], Theorem 4.2). So in total our theorem gives an explicit quantitative account of Theorem 3.1 in [1].

**Proof:** Let \(a_m := \|x_m - z_{k'}\|\).

**Case I:** \(\forall i \leq \psi(n_k) (a_{i+1} \leq a_i)\). Then (reasoning as in the proof of Lemma 9.1)

\[
\exists n \leq \psi(n_k) (n \geq n_k \land \forall i, j \in [n, n + \tilde{g}(n) + 2] \left( |a_i - a_j| \leq \frac{1}{2}\tilde{\omega}_q(\eta_k/C) \right).
\]

Moreover, \(n + \tilde{g}(n) + 2 \leq K \leq \tilde{K}\).

**Case II:** \(\exists i \leq \psi(n_k) (a_i < a_{i+1})\). Define for \((a_m)\) and \(n_0 := \psi(n_k)\) the function \(\tau\) as in Lemma 9.2. Then

1. \(\forall n \in \mathbb{N} (a_{\tau(n)} \leq a_{\tau(n)+1}, \tau(n) \leq \tau(n+1))\);
2. \(\forall n \geq \psi(n_k) (a_n \leq a_{\tau(n)+1})\).

Let \(m \in [\psi(n_k), \psi(n_k) + \tilde{g}(\psi(n_k)) + 2] \left( \tau(m) \geq n_k \right)\).

Let \(m \in [\psi(n_k), \psi(n_k) + \tilde{g}(\psi(n_k)) + 2] \left( \tau(m) \geq n_k \right)\).

\[
\|x_{\tau(m)+1} - z_{k'}\| \leq \alpha_{\tau(m)} \|u - z_k\| + (1 - \alpha_{\tau(m)}))\|S_{\tau(m)}x_{\tau(m)} - z_{k'}\|
\]

implies (using Lemma 13)

\[
(3) \|x_{\tau(m)+1} - z_{k'}\| - \|S_{\tau(m)}x_{\tau(m)} - z_{k'}\| \leq \alpha_{\tau(m)} \|u - z_{k'}\| \leq \alpha_{\tau(m)} \|u - z_{k'}\| \leq \alpha_{\tau(m)} b.
\]

Hence by (1) and using that \(\tau(m) \geq n_k\)

\[
(4) \|x_{\tau(m)} - z_{k'}\| - \|S_{\tau(m)}x_{\tau(m)} - z_{k'}\|
\leq \|x_{\tau(m)} - z_{k'}\| - \|S_{\tau(m)}x_{\tau(m)} - z_{k'}\| \leq \alpha_{\tau(m)} b
\]

\[
\leq \min \left\{ \tilde{\omega}_q \left( \frac{1}{2} \omega_J \left( \frac{1}{64 b} \varepsilon^2 \right) \right), \tilde{\omega}_q(\eta_k/C) \right\}.
\]

Since

\[
\tau(m) \leq \max\{m, \psi(n_k)\} \leq \psi(n_k) + g^M(\psi(n_k)) + 2 = K \leq \tilde{K},
\]

we have

\[
\|z_{k'} - S_{\tau(m)}z_{k'}\| \leq \min \left\{ \tilde{\omega}_q \left( \frac{1}{2} \omega_J \left( \frac{1}{64 b} \varepsilon^2 \right) \right), \tilde{\omega}_q(\eta_k/C) \right\}.
\]

Hence by Lemma 8 (and Lemma 13)

\[
\|x_{\tau(m)} - S_{\tau(m)}x_{\tau(m)}\| \leq \min \left\{ \frac{1}{2} \omega_J \left( \frac{1}{64 b} \varepsilon^2 \right), \eta_k/C \right\}.
\]
By Lemma 6 and the definition of the constant $C$ this also gives

$$\| x_{\tau(m)} - S_1 x_{\tau(m)} \| \leq \eta_k.$$  

Using again that $\tau(m) \geq n_k$ we get (involving Lemma 13)

$$ (5) \quad \| x_{\tau(m)+1} - x_{\tau(m)} \| \leq \| x_{\tau(m)+1} - S_{\tau(m)} x_{\tau(m)} \| + \| S_{\tau(m)} x_{\tau(m)} - x_{\tau(m)} \|
\leq \alpha_{\tau(m)} \cdot b + \frac{1}{2} \omega_f \left( \frac{1}{64} \epsilon^2 \right) \leq \omega_f \left( \frac{1}{64} \epsilon^2 \right).$$

Since

$$\| x_{\tau(m)} - S_1 x_{\tau(m)} \| \leq \eta_k$$  

we get from Lemma 11 (using that $k \geq \rho \left( \frac{\epsilon^2}{64} \right)$)

$$\forall m \in [\psi(n_k), \psi(n_k) + \tilde{g}(\psi(n_k)) + 2] \left( \langle u - z_k, J(x_{\tau(m)} - z_k) \rangle \leq \frac{\epsilon^2}{64} \right).$$

Hence by (5)

$$ (6) \quad \forall m \in [\psi(n_k), \psi(n_k) + \tilde{g}(\psi(n_k)) + 2] \left( \langle u - z_k, J(x_{\tau(m)+1} - z_k) \rangle \leq \frac{\epsilon^2}{32} \right).$$

By (+) and the definition of $f$ we have

$$\| z_k - z_{k'} \| \leq \min \left\{ \omega_f \left( \frac{\epsilon^2}{128b} \right), \frac{\epsilon^2}{128b} \right\}$$

and so (6) implies

$$ (7) \quad \langle u - z_{k'}, J(x_{\tau(m)+1} - z_{k'}) \rangle \leq \frac{\epsilon^2}{32} + \frac{\epsilon^2}{64} < \frac{\epsilon^2}{16}.$$  

We, moreover, have using Lemma 7 and Lemma 13

$$\| x_{\tau(m)+1} - z_{k'} \|^2 = \| \alpha_{\tau(m)} (u - z_{k'}) + (1 - \alpha_{\tau(m)}) (S_{\tau(m)} x_{\tau(m)} - z_{k'}) \|^2
\leq (1 - \alpha_{\tau(m)})^2 \| S_{\tau(m)} x_{\tau(m)} - z_{k'} \|^2 + 2 \alpha_{\tau(m)} \langle u - z_{k'}, J(x_{\tau(m)+1} - z_{k'}) \rangle
\leq (1 - \alpha_{\tau(m)}) \| x_{\tau(m)} - z_{k'} \|^2 + 2 b \| S_{\tau(m)} z_{k'} - z_{k'} \|^2 + 2 \alpha_{\tau(m)} \langle u - z_{k'}, J(x_{\tau(m)+1} - z_{k'}) \rangle
\leq (1 - \alpha_{\tau(m)}) \| x_{\tau(m)} - z_{k'} \|^2 + 2 b \| S_{\tau(m)} z_{k'} - z_{k'} \|^2 + 2 \alpha_{\tau(m)} \langle u - z_{k'}, J(x_{\tau(m)+1} - z_{k'}) \rangle.$$  

By (1), we have $\| x_{\tau(m)} - z_{k'} \| \leq \| x_{\tau(m)+1} - z_{k'} \|$ and so

$$\begin{align*}
\| x_{\tau(m)+1} - z_{k'} \|^2 &\leq (1 - \alpha_{\tau(m)}) \| x_{\tau(m)+1} - z_{k'} \|^2 + 2 b \| S_{\tau(m)} z_{k'} - z_{k'} \|^2 + 2 \alpha_{\tau(m)} \langle u - z_{k'}, J(x_{\tau(m)+1} - z_{k'}) \rangle.
\end{align*}$$

Hence by (7) we get for all $m \in [\psi(n_k), \psi(n_k) + \tilde{g}(\psi(n_k)) + 2]$ (since $\tau(m) \leq K$):

$$\begin{align*}
\| x_{\tau(m)+1} - z_{k'} \|^2 &\leq 2 \langle u - z_{k'}, J(x_{\tau(m)+1} - z_{k'}) \rangle + \frac{2 b \| S_{\tau(m)} z_{k'} - z_{k'} \|^2}{\alpha_{\tau(m)}}
\leq \frac{1}{8} \epsilon^2 + \frac{1}{8} \epsilon^2 = \frac{1}{4} \epsilon^2
\end{align*}$$

and using that by (2) (since $m \geq \psi(n_k)$) we have $\| x_m - z_{k'} \| \leq \| x_{\tau(m)+1} - z_{k'} \|$ we obtain

$$\forall m \in [\psi(n_k), \psi(n_k) + \tilde{g}(\psi(n_k)) + 2] \left( \| x_m - z_{k'} \|^2 \leq \frac{1}{4} \epsilon^2 \right).$$  

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So
\[ \forall m \in [\psi(n_k), \psi(n_k) + g(\psi(n_k))] \subseteq [\psi(n_k), \psi(n_k) + \tilde{g}(\psi(n_k)) + 2] \left( \|x_m - z_{k'}\| \leq \frac{1}{2} \varepsilon \right), \]
i.e. we have established already the theorem in this case with \( n := \psi(n_k) \leq K \leq K^* \) (note also that \( L \leq k' = f(k) \leq f^M(k^*) \)).

Case II.2: \( \exists m \in [\psi(n_k), \psi(n_k) + \tilde{g}(\psi(n_k)) + 2] \) (\( \tau(m) < n_k \)). By (1) we have \( \tau(\psi(n_k)) \leq \tau(m) < n_k \).

Hence by Lemma 9 we get the existence of a \( \hat{n} \geq n_k \) with \( \hat{n} + \tilde{g}(\hat{n}) + 2 \leq K \leq \hat{K} \) (since \( \hat{n} \leq \psi(n_k) \)) such that
\[ \forall i, j \in [\hat{n}, \hat{n} + \tilde{g}(\hat{n}) + 2] \left( \|x_i - z_{k'}\| - \|x_j - z_{k'}\| \leq \frac{1}{2} \tilde{\omega}_\eta(\eta_k/C) \right). \]

So in both of the cases I and II.2 in which the theorem is not yet established we get an \( n \geq n_k \) with \( n + \tilde{g}(n) + 2 \leq K \leq \hat{K} \) such that
\[ \forall m \geq n \left( \alpha_m b \leq \frac{1}{2} \tilde{\omega}_\eta(\eta_k/C) \right) \]
and
\[ \forall i, j \in [n, n + \tilde{g}(n) + 1] \left( \|x_i - z_{k'}\| - \|x_j - z_{k'}\| \leq \frac{1}{2} \tilde{\omega}_\eta(\eta_k/C) \right) \]
and so for all \( m \in [n, n + \tilde{g}(n) + 1] \)
\[ \|x_m - z_{k'}\| - \|S_m x_m - z_{k'}\| \leq \|x_{m+1} - z_{k'}\| - \|S_m x_m - z_{k'}\| + \|x_{m+1} - z_{k'}\| - \|x_m - z_{k'}\| \]
\[ \leq \alpha_m \|u - z_{k'}\| + \frac{1}{2} \tilde{\omega}_\eta(\eta_k/C) \leq \tilde{\omega}_\eta(\eta_k/C) \]
since \( \|x_{m+1} - z_{k'}\| \leq \alpha_m \|u - z_{k'}\| + (1 - \alpha_m) \|S_m x_m - z_{k'}\| \).

Hence by Lemma 8 (using that \( m \leq K \leq \hat{K} \) and so \( \|S_m z_{k'} - z_{k'}\| \leq \tilde{\omega}_\eta(\eta_k/C) \))
\[ \forall m \in [n, n + \tilde{g}(n) + 1] \left( \|x_m - S_m x_m\| \leq \eta_k/C \right) \]
and so by Lemma 6
\[ \forall m \in [n, n + \tilde{g}(n) + 1] \left( \|x_m - S_1 x_m\| \leq \eta_k \right). \]

By Lemma 11 (using that \( k \geq \rho(\varepsilon^2/64b^2) \)) we get
\[ \forall m \in [n, n + \tilde{g}(n) + 1] \left( \langle u - z_{k'}, J(x_m - z_{k'}) \rangle \leq \frac{\varepsilon^2}{64} \right) \]
and so by \( \omega_J \), the definition of \( \hat{\varepsilon} \), \( f \) and (+)
\[ \forall m \in [n, n + \tilde{g}(n) + 1] \left( \langle u - z_{k'}, J(x_m - z_{k'}) \rangle \leq \frac{\varepsilon^2}{32} \right). \]

Moreover, for all \( i \in \mathbb{N} \) we have (using Lemma 7)
\[ \|x_{i+1} - z_{k'}\|^2 = \|\alpha_i (u - z_{k'}) + (1 - \alpha_i)(S_i x_i - z_{k'})\|^2 \]
\[ \leq (1 - \alpha_i)^2 \|S_i x_i - z_{k'}\|^2 + 2\alpha_i \langle u - z_{k'}, J(x_{i+1} - z_{k'}) \rangle \]
\[ \leq (1 - \alpha_i)^2 \|S_i x_i - z_{k'}\|^2 + 2b \|S_i z_{k'} - z_{k'}\| + 2\alpha_i \langle u - z_{k'}, J(x_{i+1} - z_{k'}) \rangle \]
\[ \leq (1 - \alpha_i)^2 \|x_i - z_{k'}\|^2 + 2b \|S_i z_{k'} - z_{k'}\| + 2\alpha_i \langle u - z_{k'}, J(x_{i+1} - z_{k'}) \rangle. \]
We can now apply Lemma 10 to $\varepsilon^2/4$ as $\varepsilon$ and $b^2$ as $b$ and

$$a_i := \|x_i - z_k\|^2, \quad N := K, \quad \gamma_i := 2b\|S_i z_k - z_k\| \quad \text{and} \quad \beta_i := 2\langle u - z_k, J(x_{i+1} - z_k) \rangle$$

since $n \leq K$ and

$$\forall i \in [n, n + \tilde{g}(n)] \left( \beta_i \leq \frac{\varepsilon^2}{16} = \frac{1}{4}(\varepsilon/2)^2 \right)$$

and

$$\begin{align*}
\varphi(\varepsilon^2/4, S, K, b^2) + g^M (\varphi(\varepsilon^2/4, S, K, b^2) + 1) \\
-2b\max\{\|S_i z_k - z_k\| : i \leq \varphi(\varepsilon^2/4, S, K, b^2) + g^M (\varphi(\varepsilon^2/4, S, K, b^2)) = \tilde{K}\}
\end{align*}$$

$$\leq \frac{1}{4}(\varepsilon/2)^2$$

to conclude the existence of an $\bar{n} \leq \varphi(\varepsilon^2/4, S, K, b^2) = K + S \left( \frac{\varepsilon^2}{16b^2}, K \right) + 1$ such that

$$\forall i \in [\bar{n}, \bar{n} + g(\bar{n})] \left( \|x_i - z_k\|^2 \leq (\varepsilon/2)^2 \right)$$

and so

$$\forall i \in [\bar{n}, \bar{n} + g(\bar{n})] \left( \|x_i - z_k\| \leq \varepsilon/2 \right).$$

Now with $k^* := \xi(\tilde{e}, f_c) + c$ from Lemma 12 with $\tilde{e}, f_c$ as above and $K^*$ being defined as in the theorem, we get

$$\bar{n} \leq K + S \left( \frac{\varepsilon^2}{16b^2}, K \right) + 1 \leq K^* + S \left( \frac{\varepsilon^2}{16b^2}, K^* \right) + 1.$$

Moreover, $L \leq k' = f(k) \leq f^M(k^*)$. \qed

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