Quantitative analysis of a Halpern-type Proximal Point Algorithm for accretive operators in Banach spaces

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Abstract

Recently, Aoyama and Toyoda showed that a Halpern-type proximal point algorithm strongly converges under very general conditions on the scalars involved to a zero of an accretive operator in uniformly convex Banach spaces with a uniformly Gâteaux differentiable norm. We give a quantitative analysis of this result in the slightly more restricted context of Banach spaces which are uniformly convex and uniformly smooth.

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1 Introduction

The fundamental Proximal Point Algorithm (PPA) is a method to approximate zeros of maximally monotone operators $A \subseteq H \times H$ in Hilbert space ([18, 24]). While the algorithm converges weakly, the strong convergence in general fails ([5]). To obtain strongly convergent versions of (PPA), the definition of the iteration usually is modified in a way suggested by the so-called Halpern-type iteration ([6]) which uses a certain point $u \in H$ as an anchor. The resulting Halpern-type form (HPPA) of (PPA) is given by:

$$x_{n+1} := \alpha_n u + (1-\alpha_n) J_{\lambda_n A} x_n,$$

where $(\alpha_n) \subset (0,1), (\lambda_n) \subset (0,\infty)$ and $J_{\lambda_n A} := (I + \lambda_n A)^{-1}$ is the resolvent of A (see e.g. [7, 28, 4, 17]).

In [1], the strong convergence of this algorithm is shown even for the class of uniformly convex Banach spaces X whose norm is uniformly Gâteaux differentiable and for general accretive operators A. As conditions on $(\alpha_n) \subset (0,1]$ only

$$\sum_{n=0}^{\infty} \alpha_n = \infty \text{ and } \lim_{n\to\infty} \alpha_n = 0,$$

known to be necessary for Halpern's classical strong convergence result, are needed and the only assumption on $(\lambda_n) \subset (0,\infty)$ is to be bounded away from 0, i.e. $\inf \lambda_n > 0$.

The strong convergence of (x_n) is established in [1] by reducing the situation to a famous result of Reich [23] on the strong convergence of the path (z_t) where $z_t = tu + (1-t)J_{\lambda_1} z_t$ for $t \in (0,1)$.

In this paper, we give a quantitative analysis of the main theorem in [1] in the slightly more restricted case where X is assumed to be uniformly smooth (in addition to being uniformly convex) as for this class of spaces logical bound-extraction metatheorems are available ([8, 12]).

It is known that even for trivial situations such as $H = \mathbb{R}$ one in general does not have a computable rate of convergence for (x_n) (see [19]) and so one has to aim at the next best thing which is an explicit so-called rate of metastability in the sense of Tao [26, 27], i.e. a function $\Theta : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ such that

$$\forall k \in \mathbb{N}\, \forall g \in \mathbb{N}^{\mathbb{N}}\, \exists N \leq \Theta(k,g)\, \forall n,m \in [N, N+g(N)] \ \left(\|x_n - x_m\| < \frac{1}{k+1} \right),$$

where $[N, N+g(N)] := \{N, N+1, N+2, \ldots, N+g(N)\}$, whose complexity reflects the computational content of the original convergence proof from which it is extractable by proof-theoretic methods (see [8]). Note that, noneffectively, the metastability of (x_n) implies the ordinary Cauchy property of (x_n).

General results from mathematical logic ([8, 12]) guarantee the extractability of a rate of metastability from the proof given in [1] which only depends on moduli η, τ of uniform convexity and uniform smoothness of X, rates of convergence for $\prod_{i=0}^{n}(1-\alpha_i) \to 0$ (which is equivalent to $\sum_{i=0}^{\infty} \alpha_i = \infty$) and $\alpha_n \to 0$, a positive lower bound $0 < \lambda \leq \lambda_n$ (for all $n \in \mathbb{N}$), sequences of positive lower bounds $0 < \tilde{\alpha}_n \leq \alpha_n$ of (α_n) and of upper bounds $\tilde{\lambda}_n \geq \lambda_n$ for (λ_n), an upper bound $b \geq \|u-p\|, \|x_0 - p\|$ for some zero p of A, the error $\varepsilon = 1/(k+1), g$ **and** a given rate of metastability ξ for (z_t), i.e. for Reich's result. Such a ξ has recently been constructed for uniformly convex and uniformly smooth Banach spaces in [13]. In the case where X is a Hilbert space, a much simpler such ξ has been known already since [9]. For more information on the logic-based approach to the extraction of explicit bounds from prima facie noneffective proofs and the concept of metastability we refer to the recent survey [11]. While many explicit rates of metastability have been extracted in recent years for a number of algorithms in nonlinear analysis, for the Halpern-type Proximal Point Algorithm such rates were obtained only recently in [21, 15, 22] (also using a logic-based approach) which consider the HPPA in Hilbert spaces (also with error terms) where either (λ_n) is assumed to diverge to ∞ or is assumed to converge to some $\lambda > 0$ (in the latter case an additional assumption on (α_n) is used) which are more restrictive then the situation in [1] which we study. Obviously, we have to pay a price for the greater generality namely that our rate is somewhat more complicated. Also, our rate depends on some sequence $(\tilde{\alpha}_n)$ with $0 < \tilde{\alpha}_n \leq \alpha_n$ witnessing the strict positivity of α_n which is used in the proof in [1], whereas in [21, 22] the special case where $\sum \gamma_n = \infty$ is treated in a way which does not require this. In any case, the proof from [1] is rather different from the proofs analyzed in [21, 15, 22] and makes crucial use of the fact that $J_{\lambda_n A}$ as a firmly nonexpansive mapping in a uniformly convex space is strongly nonexpansive. The class of strongly nonexpansive mappings has very nice quantitative properties which we exhibited in [10] and which are used in the present paper as well.

2 Preliminaries

Definition 1. *A real Banach space* $(X, \|\cdot\|)$ *is uniformly convex with a modulus of convexity* $\eta : (0,2] \to (0,1]$ *if*

$$\forall \varepsilon \in (0,2]\, \forall x, y \in X \,\left(\|x\|, \|y\| \leq 1 \wedge \|x-y\| \geq \varepsilon \to \left\| \frac{1}{2}(x+y) \right\| \leq 1 - \eta(\varepsilon) \right).$$

Definition 2. *A real Banach space* $(X, \|\cdot\|)$ *is uniformly smooth if for all* $\varepsilon > 0$ *there exists some* $\delta = \tau(\varepsilon) > 0$

$$\forall x, y \in X (\|x\| = 1 \wedge \|y\| \leq \delta \to \|x+y\| + \|x-y\| \leq 2 + \varepsilon\|y\|)$$

and a function $\tau : (0,\infty) \to (0,\infty)$ *producing such a* $\delta = \tau(\varepsilon)$ *is called a modulus of uniform smoothness for* X.

Throughout this paper $(X, \|\cdot\|)$ is a uniformly convex and uniformly smooth real Banach space with respective moduli η and τ.

It is well known that in uniformly smooth spaces, the normalized duality mapping J is single-valued and uniformly norm-to-norm continuous on bounded sets. The next lemma gives a quantitative formulation of this fact:

Lemma 3 ([12])**.** *Let* X *be uniformly smooth with modulus* τ*. Define* $\omega_J : (0,\infty) \times (0,\infty) \to (0,\infty)$ *by*

$$\omega_J(b, \varepsilon) := \frac{\varepsilon^2}{12b} \cdot \tau\left(\frac{\varepsilon}{2b}\right), \quad \varepsilon \in (0,2], b \geq 1,$$

with $\omega_J(b,\varepsilon) := \omega_J(1,\varepsilon)$ *for* $b < 1$ *and* $\omega_J(b,\varepsilon) := \omega_J(b,2)$ *for* $\varepsilon > 2$*. Then the single-valued duality map* $J : X \to X^*$ *is norm-to-norm uniformly continuous on bounded subsets with modulus* ω_J*, that is, for all* $b, \varepsilon > 0$ *and* $x, y \in X$ *with* $\|x\|, \|y\| \leq b$ *we have*

$$\|x-y\| \leq \omega_J(b,\varepsilon) \to \|Jx - Jy\| \leq \varepsilon.$$

If X is a Hilbert space, we may simply take ω_J as the identity mapping.

Let $A \subseteq X \times X$ be an accretive operator, i.e.

$$\forall (x,u), (y,v) \in A \ (\langle u - v, J(x-y) \rangle \geq 0).$$

It is well known that for any $\lambda > 0$

$$J_{\lambda A} : R(I + \lambda A) \to X, \ x \mapsto (I + \lambda A)^{-1}(x)$$

is a single valued firmly nonexpansive mapping with $R(J_{\lambda A}) = D(A)$ and the fixed point set $Fix(J_{\lambda A})$ of $J_{\lambda A}$ coincides with the set $zer\, A := A^{-1}0 = \{q \in X : 0 \in Aq\}$ of zeros of A (see [3], p.466, and [25], pp.130,135 as well as [2]). Since $J_{\lambda A}$ is firmly nonexpansive it also is - using the uniform convexity of X - strongly nonexpansive (see [3]).

In [10], a quantitative form of this fact is established (for arbitrary firmly nonexpansive mappings but stated here in terms $J_{\lambda A}$):

Lemma 4 ([10], Proposition 2.17)**.** $J_{\lambda A}$ *is strongly nonexpansive with SNE-modulus*

$$\omega_\eta(c, \varepsilon) = \frac{1}{4}\eta(\varepsilon/c) \cdot \varepsilon$$

(for $\varepsilon > 2c$ the claim is trivial and we may simply put $\omega_\eta(c,\varepsilon) := 1$) which does not depend on $\lambda > 0$, i.e. for all $c, \lambda, \varepsilon > 0$, $x, y \in R(I + \lambda A)$

$$\|x - y\| \le c \wedge \|x - y\| - \|J_{\lambda A}x - J_{\lambda A}y\| < \omega_\eta(c,\varepsilon) \to \|(x - y) - (J_{\lambda A}x - J_{\lambda A}y)\| < \varepsilon.$$

If η can be written as $\eta(\varepsilon) = \varepsilon \cdot \tilde{\eta}(\varepsilon)$ with $\tilde{\eta}$ such that

$$\varepsilon_1 \le \varepsilon_2 \to \tilde{\eta}(\varepsilon_1) \le \tilde{\eta}(\varepsilon_2), \text{ for all } \varepsilon_1, \varepsilon_2 \in (0,2],$$

then the modulus can be taken as $\omega_\eta(c,\varepsilon) := \frac{1}{2}\tilde{\eta}(\varepsilon/c) \cdot \varepsilon$.

This gives a modulus of order p in ε for L^p with $2 \le p < \infty$. In particular, for the case of Hilbert spaces we may take $\omega_\eta(c,\varepsilon) := \frac{1}{16c}\varepsilon^2$.

As in [1], we always assume that the accretive operator A satisfies the range condition

$$\overline{D(A)} \subseteq C \subseteq R(I + \lambda A) \text{ for all } \lambda > 0,$$

where $\overline{D(A)}$ is the closure of the domain $D(A)$ of A and C is a nonempty closed and convex subset of X and that $\operatorname{zer} A \neq \emptyset$.

For $(\lambda_n) \subset [\lambda, \infty)$ with $\lambda > 0$, [1] studies the Halpern-type variant of the Proximal Point Algorithm for an accretive operator A satisfying the conditions above is given by the sequence $(x_n) \subseteq C$ defined by (for given $x_0, u \in C$)

$$(*)\ x_{n+1} := \alpha_n u + (1 - \alpha_n)J_{\lambda_n A}x_n.$$

Here (α_n) is a sequence in $(0,1]$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\alpha_n \to 0$.

The main result (proved even under the weaker assumption of a uniformly Gâteaux differentiable norm rather than uniform smoothness) in [1] is:

Theorem 5 ([1], Theorem 3.1)**.** *Under the conditions stated above, (x_n) converges strongly to Qu, where Q is the unique sunny nonexpansive retraction of C onto zer A.*

3 Quantitative lemmas

$\mathbb{N} := \{0, 1, 2, \ldots\}, \mathbb{N}^* := \{1, 2, 3, \ldots\}$. Throughout this paper, for $f : \mathbb{N} \to \mathbb{N}$, $f^M : \mathbb{N} \to \mathbb{N}$ denotes the function $f^M(n) := \max\{f(i) : i \le n\}$.

Lemma 6 ([1])**.** *Let $A \subseteq X \times X$ be accretive with the range condition and $\lambda, \mu > 0$. Then*

$$\|x - J_{\mu A}x\| \le \left(2 + \frac{\mu}{\lambda}\right) \|x - J_{\lambda A}x\|$$

for all $x \in R(I + \lambda A) \cap R(I + \mu A)$.

Lemma 7 ([20])**.** *For all $x, y \in X$ we have $\|x + y\|^2 \le \|x\|^2 + 2\langle y, J(x + y)\rangle$.*

Lemma 8 (Quantitative version of Lemma 2.3 in [1])**.** *Let $w \in C$ and let (x_n) be any sequence in C with $\|x_n - w\| \le b$ for all $n \in \mathbb{N}$ and (λ_n) be a sequence in $(0,\infty)$. Then for ω_η from Lemma 4, $J_{\lambda_n} := J_{\lambda_n A}$ and $\tilde{\omega}_\eta(b,\varepsilon) := \min\left\{\frac{\varepsilon}{2}, \frac{1}{2}\omega_\eta(b,\varepsilon/2)\right\}$:*

$$\forall \varepsilon > 0\, \forall n \in \mathbb{N}\ \left(\|x_n - w\| - \|J_{\lambda_n}x_n - w\| \le \tilde{\omega}_\eta(b,\varepsilon) \wedge \|w - J_{\lambda_n}w\| \le \tilde{\omega}_\eta(b,\varepsilon) \to \|x_n - J_{\lambda_n}x_n\| \le \varepsilon\right).$$

Proof: Since ω_η is an SNE-modulus for J_{λ_n},

$$\|x_n - w\| - \|J_{\lambda_n}x_n - J_{\lambda_n}w\| \le \|x_n - w\| - \|J_{\lambda_n}x_n - w\| + \|w - J_{\lambda_n}w\| \le \tilde{\omega}_\eta(b,\varepsilon) + \tilde{\omega}_\eta(b,\varepsilon) \le \omega_\eta(b, \frac{\varepsilon}{2})$$

implies that $\|x_n - J_{\lambda_n}x_n\| \le \|(x_n - w) - (J_{\lambda_n}x_n - J_{\lambda_n}w)\| + \|w - J_{\lambda_n}w\| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$ □

Lemma 9 (Quantitative version of Lemma 2.7 in [1]). *Let $b > 0$ and (a_n) be a sequence in $[0, b]$.*

1. *Let $\tau : \mathbb{N} \to \mathbb{N}$ be such that*

$$(+)\ \forall n, k \in \mathbb{N}\,(k \le n \wedge a_k < a_{k+1} \to k \le \tau(n)).$$

Define for $K \in \mathbb{N}, g \in \mathbb{N}^{\mathbb{N}}, \varepsilon > 0$ and $\tilde{g}(n) := n + g(n)$

$$\psi(\varepsilon, g, K, b) := \tilde{g}^{(\lceil \frac{b}{\varepsilon} \rceil)}(K).$$

Then

$$\tau(\psi(\varepsilon, g, K, b)) < K \to \exists n \le \psi(\varepsilon, g, K, b)\,(n \ge K \wedge \forall i, j \in [n, n + g(n)]\ (|a_i - a_j| \le \varepsilon)).$$

2. *Let $n_0 \in \mathbb{N}$ be such that $\exists n \le n_0(a_n < a_{n+1})$. Define*

$$\tau(n) := \max\{k \le \max\{n_0, n\} : a_k < a_{k+1}\}.$$

Then τ is well-defined and satisfies $(+)$. Moreover,

(i) $\forall n \in \mathbb{N}(a_{\tau(n)} \le a_{\tau(n)+1})$,

(ii) $\forall n \in \mathbb{N}(\tau(n) \le \tau(n+1))$,

(iii) $\forall n \ge n_0(a_n \le a_{\tau(n)+1})$.

Proof: 1) Assume $\tau(\psi(\varepsilon, g, K, b)) < K$. Then

$$\forall k \in [K, \overbrace{\psi(\varepsilon, g, K, b))}^{\ge K}]\,(a_k \ge a_{k+1}),$$

since, if $k \in [K, \psi(\varepsilon, g, K, b)]$ with $a_k < a_{k+1}$, then by $(+)$ $k \le \tau(\psi(\varepsilon, g, K, b)) < K$ which is a contradiction. Hence

$$(++)\ \forall k \in [K, \tilde{g}^{(\lceil \frac{b}{\varepsilon} \rceil)}(K)]\,(0 \le a_{k+1} \le a_k \le b).$$

Suppose now that

$$\forall i < \left\lceil \frac{b}{\varepsilon} \right\rceil\ \left(a_{\tilde{g}^{(i+1)}(K)} < a_{\tilde{g}^{(i)}(K)} - \varepsilon\right).$$

Then $a_K - a_{\tilde{g}^{\lceil b/\varepsilon \rceil}(K)} > \lceil \frac{b}{\varepsilon} \rceil \cdot \varepsilon \ge b$ which contradicts $a_K, a_{\tilde{g}^{\lceil b/\varepsilon \rceil}(K)} \in [0, b]$. Hence

$$\exists i_0 < \left\lceil \frac{b}{\varepsilon} \right\rceil\ \Big(a_{\underbrace{\tilde{g}^{(i_0+1)}(K)}_{=\tilde{g}^{(i_0)}(K) + g(\tilde{g}^{(i_0)}(K))}} \ge a_{\tilde{g}^{(i_0)}(K)} - \varepsilon\Big)$$

and so for $K \le n := \tilde{g}^{(i_0)}(K) \le \psi(\varepsilon, g, K, b)$ - using $(++)$ -

$$\forall i, j \in [n, n + g(n)]\ (|a_i - a_j| \le \varepsilon).$$

2) $(+), (i), (ii)$ are obvious from the definition of τ.
(iii) follows as in the proof of Lemma 3.1 in [16] which we repeat here for completeness: we assume $n \geq n_0$ (so that $\tau(n) \leq n$) and hence only have to consider three cases:
Case 1: $\tau(n) = n$. Then (iii) follows from (i).
Case 2: $\tau(n) = n - 1$. Then (iii) holds trivially.
Case 3: $\tau(n) < n - 1$, i.e. $\tau(n) \leq n - 2$. By definition of τ we have

$$a_{\tau(n)+1} \geq a_{\tau(n)+2} \geq \ldots \geq a_{n-1} \geq a_n.$$

$\square$

Lemma 10 (Quantitative version of Lemma 2.8 in [1])**.** *Let $b > 0$ and (a_n) be a sequence in $[0, b]$ with*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\beta_n + \gamma_n \quad (n \in \mathbb{N}),$$

where $(\alpha_n) \subset (0, 1]$, $(\beta_n) \subset \mathbb{R}$ and $(\gamma_n) \subset \mathbb{R}^+$ with $\sum_{n=0}^{\infty} \alpha_n = \infty$ (i.e. $\prod_{n=m}^{\infty}(1 - \alpha_n) = 0$ for all $m \in \mathbb{N}$).
Let $S : (0, \infty) \times \mathbb{N} \to \mathbb{N}$ be such that

$$\forall m \in \mathbb{N}\, \forall \varepsilon > 0 \left(\prod_{k=m}^{S(\varepsilon,m)} (1 - \alpha_k) \leq \varepsilon \right).$$

W.l.o.g. we may assume that S is nondecreasing in m.
For $\varepsilon > 0$ and $g \in \mathbb{N}^{\mathbb{N}}$ define

$$\widehat{g}(n) := g^M(n + S(\frac{\varepsilon}{4b}, n) + 1) + S(\frac{\varepsilon}{4b}, n).$$

Suppose that $N \in \mathbb{N}$ satisfies that

$$\exists m \leq N\, \forall i \in [m, m + \widehat{g}(m)]\, (\beta_i \leq \frac{\varepsilon}{4}).$$

Define

$$\varphi(\varepsilon, S, N, b) := N + S(\frac{\varepsilon}{4b}, N) + 1.$$

Then

$$\sum_{i=0}^{\varphi(\varepsilon,S,N,b)+g^M(\varphi(\varepsilon,S,N,b))} \gamma_i \leq \frac{\varepsilon}{2} \to \exists n \leq \varphi(\varepsilon, S, N, b)\, \forall i \in [n, n + g(n)]\, (a_i \leq \varepsilon).$$

Proof: By the assumption on N we have

$$\exists m \leq N\, \forall i \in [m, m + S(\frac{\varepsilon}{4b}, m) + g(m + S(\frac{\varepsilon}{4b}, m) + 1)]\, (\beta_i \leq \frac{\varepsilon}{4})$$

and so for $n := m + S(\frac{\varepsilon}{4b}, m) + 1$

$$\forall i \in [m, n + g(n) - 1]\, (\beta_i \leq \frac{\varepsilon}{4}).$$

From the proof of Lemma 2.3 in [14] it follows that for all $i \in [n, n+g(n)]$
(using $i \geq n \geq S(\varepsilon/4b, m) + 1$)

$$a_i \leq a_m \cdot \prod_{k=m}^{i-1}(1-\alpha_k) + \max\{\beta_k : m \leq k \leq i-1\} + \sum_{k=m}^{i-1}\gamma_k \leq b\frac{\varepsilon}{4b} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Lemma 11 (Quantitative version of Lemma 2.9 in [1])**.** *Let $b > 0$ and (x_n) be a sequence in C, $u \in C$ and for $t \in (0,1)$ let $z_t \in C$ be the unique point with*

$$z_t = tu + (1-t)J_{\lambda A} z_t$$

for $\lambda > 0$ (which exists by Banach's fixed point theorem). Assume that $\|z_t - x_n\|, \|J_{\lambda A}x_n - x_n\| \leq b$ for all $n \in \mathbb{N}, t \in (0,1)$. Let (t_k) be a sequence in $(0,1)$ with $t_k \to 0$ and let $\rho : (0,\infty) \to \mathbb{N}$ be a rate of convergence (i.e. $t_k \leq \varepsilon$ for $k \geq \rho(\varepsilon)$) and $\chi : \mathbb{N} \to \mathbb{N}^*$ such that $t_k \geq \frac{1}{\chi(k)}$ for all $k \in \mathbb{N}$. Let $k \geq \rho\left(\frac{\varepsilon}{b^2}\right)$ and for some $n \in \mathbb{N}$ assume that $\|J_{\lambda A}x_n - x_n\| \leq \eta_{k,\varepsilon} := \frac{\varepsilon}{3b\chi(k)}$. Then for this n we get $\langle u - z_{t_k}, J(x_n - z_{t_k})\rangle \leq \varepsilon$.

Proof: Reasoning as in [1](p.808) one has

$$\begin{aligned}
&\langle u - z_{t_k}, J(x_n - z_{t_k})\rangle \\
&\leq \tfrac{t_k}{2}\|z_{t_k} - x_n\|^2 + \tfrac{(1-t_k)^2}{2t_k}\|J_{\lambda A}x_n - x_n\|\left(\|J_{\lambda A}x_n - x_n\| + 2\|z_{t_k} - x_n\|\right) \\
&\leq \tfrac{t_k}{2}b^2 + \tfrac{3b}{2t_k}\|J_{\lambda A}x_n - x_n\| \\
&\leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

□

The next Lemma addresses the specific form in which in the proof of the main result, a given rate of metastability for the sequence (z_{t_k}) will be used to construct a rate of metastability for the proximal sequence (x_n) :

Lemma 12. *Let (a_n) be a Cauchy sequence in C with a rate of metastability ξ in the form*

$$\forall \varepsilon > 0\, \forall g \in \mathbb{N}^{\mathbb{N}}\, \exists n \leq \xi(\varepsilon, g)\, \forall i,j \in [n, g(n)]\, (\|a_i - a_j\| \leq \varepsilon).$$

Let now $\varepsilon > 0, c \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$ and define $f_c(l) := f(l+c)$. Then

$$\exists k \leq \xi(\varepsilon, f_c) + c\ (k \geq c \wedge \forall i,j \in [k, f(k)]\, (\|a_i - a_j\| \leq \varepsilon)).$$

Proof: By the definition of ξ

$$\exists \tilde{k} \leq \xi(\varepsilon, f_c)\ \forall i,j \in [\tilde{k}, f(\tilde{k}+c)]\, (\|a_i - a_j\| \leq \varepsilon).$$

Hence for $k := \tilde{k} + c$ we have that $k \geq c$ and

$$\forall i,j \in [k, f(k)]\, (\|a_i - a_j\| \leq \varepsilon).$$

□

The next Lemma establishes a crucial bound on the various sequences involved in this paper:

Lemma 13. *Let $(x_n), u$ as defined in $(*)$ above and for $t \in (0,1)$ let $z_t \in C$ be the unique point with $z_t = tu + (1-t)J_{\lambda_1 A}z_t$. Let $p \in \mathrm{zer}\, A$ and $\mathbb{N}^* \ni b \geq 2\max\{\|u-p\|, \|x_0 - p\|\}$. Then*

$$\mathrm{diam}\{u, x_n, J_{\lambda_n A}x_n, z_t, J_{\lambda_n A}z_t : n \in \mathbb{N}, t \in (0,1)\} \leq b.$$

Proof: As in [1](p.809) one shows that (using $zer\, A = Fix(J_{\lambda_n A})$) for all $n \in \mathbb{N}$

$$\|J_{\lambda_n A}x_n - p\| \leq \|x_n - p\| \leq \max\{\|u-p\|, \|x_0 - p\|\}.$$

Also

$$\begin{aligned}
&\|z_t - p\| = \|tu + (1-t)J_{\lambda_1 A}z_t - p\| = \|t(u-p) + (1-t)(J_{\lambda_1 A}z_t - J_{\lambda_1 A}p)\| \\
&\leq t\|u-p\| + (1-t)\|J_{\lambda_1 A}z_t - J_{\lambda_1 A}p\| \\
&\leq t\|u-p\| + (1-t)\|z_t - p\|.
\end{aligned}$$

Hence

$$\|J_{\lambda_n A}z_t - p\| \leq \|z_t - p\| \leq \|u - p\|.$$

Thus $u, x_n, J_{\lambda_n A}x_n, z_t, J_{\lambda_n A}z_t \in B_{b/2}(p) := \{x \in X : \|x - p\| \leq b/2\}$ which implies the lemma. $\square$

Lemma 14. *Let $K \in \mathbb{N}, \varepsilon > 0$ and $(\lambda_n) \subset [\lambda, \infty)$ for $\lambda > 0$. Let b be as in Lemma 13 and let $z_{t_k} = t_k u + (1-t_k)J_{\lambda_1 A}z_{t_k}$, where $(t_k) \subset (0,1)$ converges to 0 with rate of convergence ρ. Let $\tilde{\lambda}_i \geq \lambda_i$ for all $i \in \mathbb{N}$. Define $\tilde{\lambda}_n^M := \max\{\tilde{\lambda}_i : i \leq n\}$. Then*

$$\forall k \geq \tilde{\rho}(\varepsilon, K) := \rho\left(\frac{\varepsilon}{\left(2 + (\tilde{\lambda}_K^M/\lambda)\right)\cdot b}\right) \forall n \leq K\, (\|z_{t_k} - J_{\lambda_n A}z_{t_k}\| \leq \varepsilon).$$

Proof: $\|z_{t_k} - J_{\lambda_1 A}z_{t_k}\| = \|t_k u + (1-t_k)J_{\lambda_1 A}z_{t_k} - J_{\lambda_1 A}z_{t_k}\| = t_k\|u - J_{\lambda_1 A}z_{t_k}\| \leq t_k \cdot b$ and so by Lemma 6 for $n \leq K$

$$\|z_{t_k} - J_{\lambda_n A}z_{t_k}\| \leq \left(2 + \frac{\lambda_n}{\lambda_1}\right)\|z_{t_k} - J_{\lambda_1 A}z_{t_k}\| \leq \left(2 + \frac{\lambda_n}{\lambda_1}\right)\cdot t_k \cdot b \leq \left(2 + \frac{\tilde{\lambda}_K^M}{\lambda}\right)\cdot t_k \cdot b$$

which implies the claim. $\square$

4 Proof of the main result

In this section we construct our rate of metastability for (x_n) :

In the following, let $(x_n), (z_t)_{t\in(0,1)}$ and b be as in Lemma 13. For $k \in \mathbb{N}^*$ let $t_k := 1/k$ so that $\chi(k) := k$ and $\rho(\varepsilon) := \lceil 1/\varepsilon \rceil$ satisfy the requirements in Lemma 11. Let $S_n := J_{\lambda_n A}$ and $z_k := z_{t_k}$. Instead of $\tilde{\omega}_\eta(b, \varepsilon)$ (from Lemma 8) and $\omega_J(b,\varepsilon)$ we simply write $\tilde{\omega}_\eta(\varepsilon)$ and $\omega_J(\varepsilon)$. Let ζ be a rate of convergence for $\alpha_n \to 0$ and S be as in Lemma 10. Define for $(\tilde{\lambda}_1 \geq \lambda_1)$ $C := 2 + \frac{\tilde{\lambda}_1}{\lambda}$ and

$$\widehat{\varepsilon} := \min\{\frac{\varepsilon^2}{128b}, \omega_J(\varepsilon^2/128b)\}, \quad \eta_k := \frac{\varepsilon^2/64}{3b\chi(k)} = \frac{\varepsilon^2}{192b\cdot k}.$$

Let now $L, k \in \mathbb{N}$ be arbitrary and let n_k be so large that for all $m \geq n_k$

$$\alpha_m b \leq M_1(k) := \min\left\{\frac{1}{2}\tilde{\omega}_\eta(\eta_k/C), \tilde{\omega}_\eta\left(\frac{1}{2}\omega_J\left(\frac{1}{64b}\varepsilon^2\right)\right)\right\}\ \left(\leq \frac{1}{2}\omega_J\left(\frac{1}{64b}\varepsilon^2\right)\right),$$

e.g. $n_k := \max\{\zeta(M_1(i)/b) : i \leq k\}$ (where we take the maximum to make the dependence on k monotone which is used later).
Let $\widehat{g}$ be as in Lemma 10 with $\varepsilon^2/4 = (\varepsilon/2)^2$ as ε and b^2 as b, i.e.

$$\widehat{g}(n) = g^M\left(n + S\left(\frac{\varepsilon^2}{16b^2}, n\right) + 1\right) + S\left(\frac{\varepsilon^2}{16b^2}, n\right).$$

For ψ as in Lemma 9 let

$$\psi(i) := \psi\left(\frac{1}{2}\tilde{\omega}_\eta(\eta_k/C), \widehat{g} + 2, i, b\right) \geq i.$$

Define

$$K := \psi(n_k) + \widehat{g}^M(\psi(n_k)) + 2$$

and

$$\widehat{K} := K + S\left(\frac{\varepsilon^2}{16b^2}, K\right) + g^M\left(K + S\left(\frac{\varepsilon^2}{16b^2}, K\right) + 1\right) + 1.$$

Now let $k' \geq L$ be so large that $z_{k'}$ is a δ-approximate fixed point for all S_m for all $m \leq \widehat{K}$, where

$$\delta \leq M_2(k) := \min\left\{\frac{\varepsilon^2}{16b(\widehat{K}+1)}, \tilde{\omega}_\eta\left(\frac{1}{2}\omega_J\left(\frac{1}{64b}\varepsilon^2\right)\right), \tilde{\omega}_\eta(\eta_k/C), \frac{1}{16b}\varepsilon^2 \cdot \min\{\tilde{\alpha}(i) : i \leq K\}\right\},$$

where $0 < \tilde{\alpha}_i \leq \alpha_i$ for all $i \in \mathbb{N}$. E.g. we may take $k' := \max\{L, \tilde{\rho}(M_2(k), \widehat{K})\} \geq L$ with $\tilde{\rho}$ from Lemma 14.
Define now the function $f : \mathbb{N}^* \to \mathbb{N}$ by $f(k) := k'$.
For the function f let $k \leq \xi(\widehat{\varepsilon}, f_c) + c$ by Lemma 12 applied to $\widehat{\varepsilon}$ as ε and

$$a_n := z_n, c := \rho\left(\varepsilon^2/64b^2\right) = \lceil 64b^2/\varepsilon^2 \rceil$$

be such that $k \geq c$ and

$$(+)\ \forall i, j \in [k, f(k)]\ (\|z_i - z_j\| \leq \widehat{\varepsilon}).$$

Theorem 15. *Define for given $\varepsilon > 0, L \in \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ the quantities $\widehat{\varepsilon}, f, c$ as above and take*

$$k^* := \xi(\widehat{\varepsilon}, f_c) + c$$

from Lemma 12 and ξ being a rate of metastability for (z_k) as in Lemma 12 and define $K^* := \psi^M(n_{k^*}) + \widehat{g}^M(\psi^M(n_{k^*})) + 2$. Then

$$(i)\ \exists n \leq K^* + S\left(\frac{\varepsilon^2}{16b^2}, K^*\right) + 1\, \exists k' \in [L, f^M(k^*)] \forall i \in [n, n+g(n)]\, (\|x_i - z_{k'}\| \leq \varepsilon/2)$$

and so, in particular (taking e.g. $L := 0$),

$$(ii)\ \exists n \leq K^* + S\left(\frac{\varepsilon^2}{16b^2}, K^*\right) + 1\, \forall i, j \in [n, n+g(n)]\, (\|x_i - x_j\| \leq \varepsilon).$$

Remark 16. *1. Note by inspection that the bounds only depend on $\varepsilon, b, g, L, \lambda, (\tilde{\lambda}_n), (\tilde{a}_n)$ and the rates and moduli χ, S, ξ, η, τ.*

2. In what follows we give a completely elementary proof of the theorem. Since (ii) trivially implies the Cauchy property of (x_n) one obtains (using that C is closed and X is complete) that (x_n) strongly converges. Moreover, by (i) it converges to the same limit as (z_k) converges to (take e.g. $g(n) := L$ so that by (i) we have $\exists n, k' \geq L(\|x_n - z_{k'}\| \leq \varepsilon)$, i.e. to Qu, where Qu is the sunny nonexpansive retraction of C onto zer A (for the latter statement an elementary proof is given in [13] *where also an explicit rate of metastability ξ for (z_k) is constructed). If X is a Hilbert space, we can simply take $\xi(\varepsilon, g) := \tilde{g}^{(\lceil b^2/\varepsilon^2 \rceil)}(0)$ (see* [9]*, Theorem 4.2). So in total our theorem gives an explicit quantitative account of Theorem 3.1 in* [1].

Proof: Let $a_m := \|x_m - z_{k'}\|$.
Case I: $\forall i \leq \psi(n_k)\,(a_{i+1} \leq a_i)$. Then (reasoning as in the proof of Lemma 9.1)

$$\exists n \leq \psi(n_k)\,(n \geq n_k \wedge \forall i, j \in [n, n + \widehat{g}(n) + 2]\,(|a_i - a_j| \leq \frac{1}{2}\tilde{\omega}_\eta(\eta_k/C)).$$

Moreover, $n + \widehat{g}(n) + 2 \leq K \leq \widehat{K}$.

Case II: $\exists i \leq \psi(n_k)\,(a_i < a_{i+1})$. Define for (a_m) and $n_0 := \psi(n_k)$ the function τ as in Lemma 9.2. Then

(1) $\forall n \in \mathbb{N}\,(a_{\tau(n)} \leq a_{\tau(n)+1},\ \tau(n) \leq \tau(n+1))$;

(2) $\forall n \geq \psi(n_k)\,(a_n \leq a_{\tau(n)+1})$.

Case II.1: $\forall m \in [\psi(n_k), \psi(n_k) + \widehat{g}(\psi(n_k)) + 2]\,(\tau(m) \geq n_k)$.
Let $m \in [\psi(n_k), \psi(n_k) + \widehat{g}(\psi(n_k)) + 2]$:

$$\|x_{\tau(m)+1} - z_{k'}\| \leq \alpha_{\tau(m)}\|u - z_{k'}\| + (1 - \alpha_{\tau(m)})\|S_{\tau(m)}x_{\tau(m)} - z_{k'}\|$$

implies (using Lemma 13)

$$(3)\ \|x_{\tau(m)+1} - z_{k'}\| - \|S_{\tau(m)}x_{\tau(m)} - z_{k'}\| \leq \alpha_{\tau(m)}\|u - z_{k'}\| \leq \alpha_{\tau(m)}b.$$

Hence by (1) and using that $\tau(m) \geq n_k$

$$\begin{aligned}(4)\ &\|x_{\tau(m)} - z_{k'}\| - \|S_{\tau(m)}x_{\tau(m)} - z_{k'}\| \\ &\leq \|x_{\tau(m)+1} - z_{k'}\| - \|S_{\tau(m)}x_{\tau(m)} - z_{k'}\| \leq \alpha_{\tau(m)}b \\ &\leq \min\left\{\tilde{\omega}_\eta\left(\tfrac{1}{2}\omega_J\left(\tfrac{1}{64b}\varepsilon^2\right)\right), \tilde{\omega}_\eta(\eta_k/C)\right\}.\end{aligned}$$

Since

$$\tau(m) \leq \max\{m, \psi(n_k)\} \leq \psi(n_k) + \widehat{g}^M(\psi(n_k)) + 2 = K \leq \widehat{K},$$

we have

$$\|z_{k'} - S_{\tau(m)}z_{k'}\| \leq \min\left\{\tilde{\omega}_\eta\left(\frac{1}{2}\omega_J\left(\frac{1}{64b}\varepsilon^2\right)\right), \tilde{\omega}_\eta(\eta_k/C)\right\}.$$

Hence by Lemma 8 (and Lemma 13)

$$\|x_{\tau(m)} - S_{\tau(m)}x_{\tau(m)}\| \leq \min\left\{\frac{1}{2}\omega_J\left(\frac{1}{64b}\varepsilon^2\right), \eta_k/C\right\}.$$

By Lemma 6 and the definition of the constant C this also gives

$$\|x_{\tau(m)} - S_1 x_{\tau(m)}\| \leq \eta_k.$$

Using again that $\tau(m) \geq n_k$ we get (involving Lemma 13)

$$(5)\ \|x_{\tau(m)+1} - x_{\tau(m)}\| \leq \|x_{\tau(m)+1} - S_{\tau(m)} x_{\tau(m)}\| + \|S_{\tau(m)} x_{\tau(m)} - x_{\tau(m)}\| \\ \overset{(3.3),[1]}{\leq} \alpha_{\tau(m)} \cdot b + \tfrac{1}{2}\omega_J\left(\tfrac{1}{64b}\varepsilon^2\right) \leq \omega_J\left(\tfrac{1}{64b}\varepsilon^2\right).$$

Since

$$\|x_{\tau(m)} - S_1 x_{\tau(m)}\| \leq \eta_k$$

we get from Lemma 11 (using that $k \geq \rho\left(\frac{\varepsilon^2}{64b^2}\right)$)

$$\forall m \in [\psi(n_k), \psi(n_k) + \widehat{g}(\psi(n_k)) + 2]\ \left(\langle u - z_k, J(x_{\tau(m)} - z_k)\rangle \leq \frac{\varepsilon^2}{64}\right).$$

Hence by (5)

$$(6)\ \forall m \in [\psi(n_k), \psi(n_k) + \widehat{g}(\psi(n_k)) + 2]\ \left(\langle u - z_k, J(x_{\tau(m)+1} - z_k)\rangle \leq \frac{\varepsilon^2}{32}\right).$$

By (+) and the definition of f we have

$$\|z_k - z_{k'}\| \leq \min\left\{\omega_J\left(\frac{\varepsilon^2}{128b}\right), \frac{\varepsilon^2}{128b}\right\}$$

and so (6) implies

$$(7)\ \langle u - z_{k'}, J(x_{\tau(m)+1} - z_{k'})\rangle \leq \frac{\varepsilon^2}{32} + \frac{\varepsilon^2}{64} < \frac{\varepsilon^2}{16}.$$

We, moreover, have using Lemma 7 and Lemma 13

$$\begin{aligned}
&\|x_{\tau(m)+1} - z_{k'}\|^2 = \|\alpha_{\tau(m)}(u - z_{k'}) + (1 - \alpha_{\tau(m)})(S_{\tau(m)} x_{\tau(m)} - z_{k'})\|^2 \\
&\quad \leq (1 - \alpha_{\tau(m)})^2 \|S_{\tau(m)} x_{\tau(m)} - z_{k'}\|^2 + 2\alpha_{\tau(m)}\langle u - z_{k'}, J(x_{\tau(m)+1} - z_{k'})\rangle \\
&\quad \leq (1 - \alpha_{\tau(m)})^2 \|S_{\tau(m)} x_{\tau(m)} - S_{\tau(m)} z_{k'}\|^2 + 2b\|S_{\tau(m)} z_{k'} - z_{k'}\| + 2\alpha_{\tau(m)}\langle u - z_{k'}, J(x_{\tau(m)+1} - z_{k'})\rangle \\
&\quad \leq (1 - \alpha_{\tau(m)})\|x_{\tau(m)} - z_{k'}\|^2 + 2b\|S_{\tau(m)} z_{k'} - z_{k'}\| + 2\alpha_{\tau(m)}\langle u - z_{k'}, J(x_{\tau(m)+1} - z_{k'})\rangle.
\end{aligned}$$

By (1), we have $\|x_{\tau(m)} - z_{k'}\| \leq \|x_{\tau(m)+1} - z_{k'}\|$ and so

$$\|x_{\tau(m)+1} - z_{k'}\|^2 \leq (1 - \alpha_{\tau(m)})\|x_{\tau(m)+1} - z_{k'}\|^2 + 2b\|S_{\tau(m)} z_{k'} - z_{k'}\| + 2\alpha_{\tau(m)}\langle u - z_{k'}, J(x_{\tau(m)+1} - z_{k'})\rangle.$$

Hence by (7) we get for all $m \in [\psi(n_k), \psi(n_k) + \widehat{g}(\psi(n_k)) + 2]$ (since $\tau(m) \leq K \leq \widehat{K}$):

$$\|x_{\tau(m)+1} - z_{k'}\|^2 \leq 2\langle u - z_{k'}, J(x_{\tau(m)+1} - z_{k'})\rangle + \tfrac{2b\|S_{\tau(m)} z_{k'} - z_{k'}\|}{\alpha_{\tau(m)}} \\ \leq \tfrac{1}{8}\varepsilon^2 + \tfrac{1}{8}\varepsilon^2 = \tfrac{1}{4}\varepsilon^2$$

and using that by (2) (since $m \geq \psi(n_k)$) we have $\|x_m - z_{k'}\| \leq \|x_{\tau(m)+1} - z_{k'}\|$ we obtain

$$\forall m \in [\psi(n_k), \psi(n_k) + \widehat{g}(\psi(n_k)) + 2]\ \left(\|x_m - z_{k'}\|^2 \leq \frac{1}{4}\varepsilon^2\right).$$

So

$$\forall m \in [\psi(n_k), \psi(n_k) + g(\psi(n_k))] \subseteq [\psi(n_k), \psi(n_k) + \widehat{g}(\psi(n_k)) + 2] \left(\|x_m - z_{k'}\| \le \frac{1}{2}\varepsilon \right),$$

i.e. we have established already the theorem in this case with $n := \psi(n_k) \le K \le K^*$ (note also that $L \le k' = f(k) \le f^M(k^*)$).

Case II.2: $\exists m \in [\psi(n_k), \psi(n_k) + \widehat{g}(\psi(n_k)) + 2]\ (\tau(m) < n_k)$. By (1) we have $\tau(\psi(n_k)) \le \tau(m) < n_k$. Hence by Lemma 9 we get the existence of a $\check{n} \ge n_k$ with $\check{n} + \widehat{g}(\check{n}) + 2 \le K \le \widehat{K}$ (since $\check{n} \le \psi(n_k)$) such that

$$\forall i, j \in [\check{n}, \check{n} + \widehat{g}(\check{n}) + 2] \left(|\|x_i - z_{k'}\| - \|x_j - z_{k'}\|| \le \frac{1}{2}\tilde{\omega}_\eta(\eta_k/C) \right).$$

So in both of the cases I and II.2 in which the theorem is not yet established we get an $n \ge n_k$ with $n + \widehat{g}(n) + 2 \le K \le \widehat{K}$ such that

$$\forall m \ge n \left(\alpha_m b \le \frac{1}{2}\tilde{\omega}_\eta(\eta_k/C) \right)$$

and

$$\forall i, j \in [n, n + \widehat{g}(n) + 2] \left(|\|x_i - z_{k'}\| - \|x_j - z_{k'}\|| \le \frac{1}{2}\tilde{\omega}_\eta(\eta_k/C) \right)$$

and so for all $m \in [n, n + \widehat{g}(n) + 1]$

$$\|x_m - z_{k'}\| - \|S_m x_m - z_{k'}\| \le \|x_{m+1} - z_{k'}\| - \|S_m x_m - z_{k'}\| + |\|x_{m+1} - z_{k'}\| - \|x_m - z_{k'}\||$$
$$\le \alpha_m \overbrace{\|u - z_{k'}\|}^{\le b} + \tfrac{1}{2}\tilde{\omega}_\eta(\eta_k/C) \le \tilde{\omega}_\eta(\eta_k/C)$$

since $\|x_{m+1} - z_{k'}\| \le \alpha_m\|u - z_{k'}\| + (1 - \alpha_m)\|S_m x_m - z_{k'}\|$.

Hence by Lemma 8 (using that $m \le K \le \widehat{K}$ and so $\|S_m z_{k'} - z_{k'}\| \le \tilde{\omega}_\eta(\eta_k/C)$)

$$\forall m \in [n, n + \widehat{g}(n) + 1]\ (\|x_m - S_m x_m\| \le \eta_k/C)$$

and so by Lemma 6

$$\forall m \in [n, n + \widehat{g}(n) + 1]\ (\|x_m - S_1 x_m\| \le \eta_k).$$

By Lemma 11 (using that $k \ge \rho(\varepsilon^2/64b^2)$) we get

$$\forall m \in [n, n + \widehat{g}(n) + 1] \left(\langle u - z_k, J(x_m - z_k) \rangle \le \frac{\varepsilon^2}{64} \right)$$

and so by ω_J, the definition of $\widehat{\varepsilon}, f$ and $(+)$

$$\forall m \in [n, n + \widehat{g}(n) + 1] \left(\langle u - z_{k'}, J(x_m - z_{k'}) \rangle \le \frac{\varepsilon^2}{32} \right).$$

Moreover, for all $i \in \mathbb{N}$ we have (using Lemma 7)

$$\begin{aligned}
&\|x_{i+1} - z_{k'}\|^2 = \|\alpha_i(u - z_{k'}) + (1 - \alpha_i)(S_i x_i - z_{k'})\|^2 \\
&\le (1 - \alpha_i)^2\|S_i x_i - z_{k'}\|^2 + 2\alpha_i\langle u - z_{k'}, J(x_{i+1} - z_{k'})\rangle \\
&\le (1 - \alpha_i)^2\|S_i x_i - S_i z_{k'}\|^2 + 2b\|S_i z_{k'} - z_{k'}\| + 2\alpha_i\langle u - z_{k'}, J(x_{i+1} - z_{k'})\rangle \\
&\le (1 - \alpha_i)\|x_i - z_{k'}\|^2 + 2b\|S_i z_{k'} - z_{k'}\| + 2\alpha_i\langle u - z_{k'}, J(x_{i+1} - z_{k'})\rangle.
\end{aligned}$$

We can now apply Lemma 10 to $\varepsilon^2/4$ as ε and b^2 as b and

$$a_i := \|x_i - z_{k'}\|^2,\ N := K,\ \gamma_i := 2b\|S_i z_{k'} - z_{k'}\| \text{ and } \beta_i := 2\langle u - z_{k'}, J(x_{i+1} - z_{k'})\rangle$$

since $n \le K$ and

$$\forall i \in [n, n + \widehat{g}(n)] \left(\beta_i \le \frac{\varepsilon^2}{16} = \frac{1}{4}(\varepsilon/2)^2\right)$$

and

$$\begin{aligned}&(\varphi(\varepsilon^2/4, S, K, b^2) + g^M(\varphi(\varepsilon^2/4, S, K, b^2)) + 1)\\ &\qquad \cdot 2b \max\{\|S_i z_{k'} - z_{k'}\| : i \le \varphi(\varepsilon^2/4, S, K, b^2) + g^M(\varphi(\varepsilon^2/4, S, K, b^2)) = \widehat{K}\}\\ &\le \tfrac{1}{2}(\varepsilon/2)^2\end{aligned}$$

to conclude the existence of an $\tilde{n} \le \varphi(\varepsilon^2/4, S, K, b^2) = K + S\left(\frac{\varepsilon^2}{16b^2}, K\right) + 1$ such that

$$\forall i \in [\tilde{n}, \tilde{n} + g(\tilde{n})] \left(\|x_i - z_{k'}\|^2 \le (\varepsilon/2)^2\right)$$

and so

$$\forall i \in [\tilde{n}, \tilde{n} + g(\tilde{n})] \left(\|x_i - z_{k'}\| \le \varepsilon/2\right).$$

Now with $k^* := \xi(\widehat{\varepsilon}, f_c) + c$ from Lemma 12 with $\widehat{\varepsilon}, f_c$ as above and K^* being defined as in the theorem, we get

$$\tilde{n} \le K + S\left(\frac{\varepsilon^2}{16b^2}, K\right) + 1 \le K^* + S\left(\frac{\varepsilon^2}{16b^2}, K^*\right) + 1.$$

Moreover, $L \le k' = f(k) \le f^M(k^*)$. □

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