Quantitative results on the Proximal Point Algorithm in uniformly convex Banach spaces

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Abstract

We give rates of strong convergence for the proximal point algorithm PPA computing the unique zero z of operators A in uniformly convex Banach spaces which are uniformly accretive at z. We also get a rate of convergence to some zero of A if A has a modulus of regularity. In the boundedly compact case, we obtain a rate of metastability of PPA in the sense of Tao for arbitrary accretive operators A (satisfying a range condition so that the PPA is well-defined).

Keywords: Accretive operators, proximal point algorithm, uniformly convex Banach spaces, rates of convergence, metastability, proof mining.
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1 Introduction

The famous Proximal Point Algorithm PPA is used to approximate zeros of monotone operators A in Hilbert spaces ([13, 15]; for an adaptation to the metric context of CAT(0)-spaces see [1]). While the PPA converges weakly, strong convergence (in the infinite dimensional case) only holds when A is strongly monotone or at least satisfies some strong metric regularity conditions. In such cases explicit rates of convergence have been obtained e.g. in [12] and [9]. In the boundedly compact case one has strong convergence but (already for $\mathbb{R}$) effective rates of convergence in general are not possible due to results in [14, 6]. In this case, the next best thing to hope for are effective rates of metastability in the sense of Tao [18, 19]. Such rates are established in [8, 12].

In this paper we give for the first time effective rates of convergence for uniformly accretive at zero (in the sense of [7]) operators and for metrically regular operators (in the sense of [9]) in the context of uniformly convex Banach spaces. We also provide a rate metastability (in the boundedly compact case) for arbitrary accretive operators in uniformly convex Banach spaces. We crucially use, that the class of firmly nonexpansive operators, and hence the class of all resolvents $J_{\lambda A}$ of accretive operators A and positive scalars $\lambda > 0$, is strongly nonexpansive with a **common** modulus for being strongly nonexpansive in the sense of [5] which only depends on a given modulus of uniform convexity of X. Rates of convergence for other algorithms, e.g. of Ishikawa type, computing unique zeroes of uniformly accretive (at zero) operators in Banach spaces have recently been obtained in [10].

2 Main Results

In this paper $(X, \|\cdot\|)$ always is a real uniformly convex normed space with a modulus of convexity $\eta : (0,2] \to (0,1]$, i.e.

$$\forall \varepsilon \in (0,2]\, \forall x, y \in X \, \left(\|x\|, \|y\| \leq 1 \, \wedge \, \|x-y\| \geq \varepsilon \to \left\| \frac{1}{2}(x+y) \right\| \leq 1 - \eta(\varepsilon) \right).$$

Let $A \subseteq X \times X$ be an accretive operator, i.e.

$$\forall (x,u), (y,v) \in A \, \exists j(x-y) \in J(x-y) \; (\langle u-v, j(x-y) \rangle \geq 0),$$

where J is the normalized duality mapping of X. It is well known that for any $\lambda > 0$

$$J_{\lambda A} : R(I + \lambda A) \to X, \; x \mapsto (I + \lambda A)^{-1}(x)$$

is a single valued firmly nonexpansive mapping with $R(J_{\lambda A}) = D(A)$ and the fixed point set $Fix(J_{\lambda A})$ of $J_{\lambda A}$ coincides with the set $zer\, A := \{q \in X : 0 \in Aq\}$ of zeros of A (see [4], p.466, and [17], pp.130,135). From Proposition 2.17 in [5] it follows that $J_{\lambda A}$ is strongly nonexpansive with modulus

$$\omega_\eta(c, \varepsilon) = \frac{1}{4}\eta(\varepsilon/c) \cdot \varepsilon$$

(for $\varepsilon > 2c$ the claim is trivial and we may simply put $\omega_\eta(c,\varepsilon) := 1$) which does not depend on $\lambda > 0$, i.e. for all $c, \lambda, \varepsilon > 0$, $x, y \in R(I + \lambda A)$

$$\|x-y\| \leq c \wedge \|x-y\| - \|J_{\lambda A}x - J_{\lambda A}y\| < \omega_\eta(c,\varepsilon) \to \|(x-y) - (J_{\lambda A}x - J_{\lambda A}y)\| < \varepsilon.$$

If η can be written as $\eta(\varepsilon) = \varepsilon \cdot \tilde{\eta}(\varepsilon)$ with $\tilde{\eta}$ such that

$$\varepsilon_1 \leq \varepsilon_2 \to \tilde{\eta}(\varepsilon_1) \leq \tilde{\eta}(\varepsilon_2), \text{ for all } \varepsilon_1, \varepsilon_2 \in (0,2],$$

then the modulus can be taken as $\omega_\eta(c,\varepsilon) := \frac{1}{2}\tilde{\eta}(\varepsilon/c) \cdot \varepsilon$. This gives a modulus of order p in ε for L^p with $2 \leq p < \infty$.

We say that an accretive operator A satisfies the *range condition* if $D(A) \subseteq R(I + \lambda A)$ for all $\lambda > 0$. Usually, the stronger condition $\overline{D(A)} \subseteq R(I + \lambda A)$ is imposed but we often don't need this here. In particular, if A is m-accretive, i.e. $R(I + \lambda A) = X$ for all $\lambda > 0$, then A satisfies the range condition. For $(\gamma_n) \subset (0, \infty)$ the Proximal Point Algorithm for an accretive operator A satisfying the range condition is given by the sequence

$$(*) \; x_{n+1} := J_{\gamma_n A} x_n, \quad x_0 \in R(I + \gamma_0 A).$$

We assume that $zer\, A \neq \emptyset$.

Proposition 1. $\liminf_{n\to\infty} \|x_n - x_{n+1}\| = 0$ *with modulus of* $\liminf$ *(in the sense of* [8]*)*

$$\Delta(\varepsilon, L, b, \eta) := \lceil b/\omega_\eta(b,\varepsilon) \rceil + L + 1,$$

where $b \geq \|x_0 - p\|$ *for some* $p \in zer\, A$, *i.e.*

$$\forall L \in \mathbb{N}, \, \varepsilon > 0 \, \exists n \, (L \leq n \leq \Delta(\varepsilon, L, b, \eta) \; \wedge \; \|x_n - x_{n+1}\| < \varepsilon).$$

Proof: p is a fixed point for $J_{\lambda A}$ for all $\lambda > 0$ and so for all $n \in \mathbb{N}$

$$\|x_{n+1} - p\| \leq \|x_n - p\| \leq \|x_0 - p\| \leq b.$$

Given $\delta > 0$ there exists an $n \in \mathbb{N}$ with $L \leq n \leq L + \lceil b/\delta \rceil + 1$ such that

$$\|x_n - p\| - \|J_{\gamma_n A} x_n - J_{\gamma_n A} p\| = \|x_n - p\| - \|x_{n+1} - p\| < \delta$$

since, otherwise,

$$b \geq \|x_L - p\| \geq \|x_L - p\| - \|x_{L+\lceil b/\delta \rceil + 1} - p\| \geq (\lceil b/\delta \rceil + 1) \cdot \delta > b.$$

Applied to $\delta := \omega_\eta(b, \varepsilon)$ we get the existence of an n with $L \leq n \leq \Delta(\varepsilon, L, b, \eta)$ such that

$$\|x_n - x_{n+1}\| = \|(x_n - p) - (J_{\gamma_n A} x_n - J_{\gamma_n A} p)\| < \varepsilon.$$

□

Proposition 2. *Define*

$$u_n := \frac{x_n - x_{n+1}}{\gamma_n}.$$

Then $(\|u_n\|)_{n \in \mathbb{N}}$ *is nonincreasing.*

Proof: The proof in [3] (p.346) for the Hilbert space case can be adapted: since $u_n \in A x_{n+1}, u_{n+1} \in A x_{n+2}$, the accretivity of A implies that $\exists j(x_{n+1} - x_{n+2}) \in J(x_{n+1} - x_{n+2})$ and (using e.g. [17], p.99) $\exists j((x_{n+1} - x_{n+2})/\gamma_{n+1}) \in J((x_{n+1} - x_{n+2})/\gamma_{n+1})$ with

$$\begin{aligned}
0 &\leq \tfrac{1}{\gamma_{n+1}} \langle u_n - u_{n+1}, j(x_{n+1} - x_{n+2}) \rangle \\
&= \langle u_n - u_{n+1}, j((x_{n+1} - x_{n+2})/\gamma_{n+1}) \rangle \\
&= \langle u_n, j(u_{n+1}) \rangle - \langle u_{n+1}, j(u_{n+1}) \rangle \\
&\leq \|u_n\| \cdot \|u_{n+1}\| - \|u_{n+1}\|^2 = \|u_{n+1}\| \left(\|u_n\| - \|u_{n+1}\| \right)
\end{aligned}$$

and so

$$\|u_n\| \geq \|u_{n+1}\|.$$

□

Proposition 3. *(i)* ([16]) $\|J_{\gamma_n A} x_n - J_{\gamma_i A} x_n\| \leq |\gamma_n - \gamma_i| \cdot \frac{\|x_n - x_{n+1}\|}{\gamma_n}$.

(ii) $\|x_n - J_{\gamma_i A} x_n\| \leq \|x_n - x_{n+1}\| + |\gamma_n - \gamma_i| \cdot \frac{\|x_n - x_{n+1}\|}{\gamma_n}$.

Proof: (i) By the the resolvent equation (see e.g. [2], p.105)

$$J_{\lambda A} x = J_{\rho A} \left(\frac{\rho}{\lambda} x + \left(1 - \frac{\rho}{\lambda} \right) J_{\lambda A} x \right), \quad \lambda, \rho > 0, \ x \in D(J_{\lambda A}),$$

we get

$$\begin{aligned}
&\|J_{\gamma_n A} x_n - J_{\gamma_i A} x_n\| = \\
&\left\| J_{\gamma_i A} \left(\tfrac{\gamma_i}{\gamma_n} x_n + (1 - \tfrac{\gamma_i}{\gamma_n}) J_{\gamma_n A} x_n \right) - J_{\gamma_i A} x_n \right\| \leq \\
&\left\| \tfrac{\gamma_i}{\gamma_n} x_n + (1 - \tfrac{\gamma_i}{\gamma_n}) J_{\gamma_n A} x_n - x_n \right\| = \\
&\left| 1 - \tfrac{\gamma_i}{\gamma_n} \right| \|x_n - x_{n+1}\| = |\gamma_n - \gamma_i| \tfrac{\|x_n - x_{n+1}\|}{\gamma_n}.
\end{aligned}$$

(ii) is an immediate consequence of (i). □

Corollary 4. *If $\gamma_n \geq \gamma > 0$ for all $n \in \mathbb{N}$, then $\|u_n\| \to 0$ with rate of convergence*

$$\forall \varepsilon > 0 \, \forall n \geq \rho(\varepsilon, b, \gamma, \eta) := \Delta(\varepsilon \cdot \gamma, 0, b, \eta) \ (\|u_n\| < \varepsilon).$$

Proof: By Proposition 1 there exists an $n \leq \rho(\varepsilon, b, \gamma, \eta)$ with $\|x_n - x_{n+1}\| < \varepsilon \cdot \gamma$ and so

$$\|u_n\| = \frac{\|x_n - x_{n+1}\|}{\gamma_n} \leq \frac{\|x_n - x_{n+1}\|}{\gamma} < \varepsilon.$$

The claim now follows since $(\|u_n\|)$ is nonincreasing. □

In the following $\overline{B}(q, r) := \{x \in X : \|x - q\| \leq r\}$, where $q \in X$ and $r > 0$.

Definition 5 ([9]). *Let $A \subseteq X \times X$ be a set-valued operator with $p \in zer\, A$ and define $F(x) := \mathrm{dist}(0_X, A(x))$ (with $F(x) := \infty$ for $x \notin D(A)$). Then $\phi : (0, \infty) \to (0, \infty)$ is a modulus of regularity for A w.r.t. $zer\, A$ and $\overline{B}(p, r)$ with $r > 0$ if for all $\varepsilon > 0$ and $x \in \overline{B}(p, r)$ the following implication holds*

$$F(x) < \phi(\varepsilon) \to \mathrm{dist}(x, zer\, F) < \varepsilon.$$

Lemma 6. *Let $A \subseteq X \times X$ be an accretive operator satisfying the range condition and (x_n) be defined by the proximal point algorithm for some sequence $(\gamma_n) \subset (0, \infty)$. Then (x_n) is Fejér monotone w.r.t. $zer\, F := \{x \in X : F(x) = 0\}$, i.e.*

$$\forall p \in zer\, F \, \forall n \in \mathbb{N} \ (\|x_{n+1} - p\| \leq \|x_n - p\|).$$

Proof: Obviously, the claim holds for $p \in zer\, A$ and so it remains to show that $zer\, F = zer\, A$: Let $F(p) = 0$ and $\varepsilon > 0$. Clearly, $\exists v_\varepsilon \in Ap$ with $\|v_\varepsilon\| < \varepsilon$. Since $p + v_\varepsilon \in (I + A)p$ and $(I + A)^{-1}$ is single valued we get

$$p = (I + A)^{-1}(p + v_\varepsilon) = J_A(p + v_\varepsilon).$$

Thus

$$\|J_A p - p\| = \|J_A p - J_A(p + v_\varepsilon)\| \leq \|v_\varepsilon\| \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we conclude that $p \in Fix(J_A) = zer\, A$. □

Remark 7. Similarly to the argument in the proof of Lemma 6 one can slightly improve Theorem 3.10(ii) and Proposition 4.21 in [9]: in Theorem 3.10(ii) we can allow that $r' = r$ by using in the proof of this claim x instead of $x + \gamma$ to show (as in the proof above for p) that $\mathrm{dist}(x, zer \partial f) < \varepsilon$ for $x \in \bar{B}(z, r)$ with $\mathrm{dist}(0, \partial f(x)) < \phi^*(\varepsilon)$, where now

$$\phi^*(\varepsilon) = \frac{1}{\gamma} \phi(\varepsilon).$$

As a consequence of this, in Proposition 4.21 it suffices to have the modulus of uniform continuity for f restricted to $\bar{B}(z, b + 1)$ instead of $\bar{B}(z, b + 2)$ and in ϕ one can drop the minimum with $\varepsilon/2$ and replace $\varepsilon/2$ by ε.

Theorem 8. *Let X be complete and $A \subseteq X \times X$ be an accretive operator with $0 \in Ap$ satisfying $\overline{D(A)} \subseteq R(I + \lambda A)$. Let $\gamma_n \geq \gamma > 0$ for all $n \in \mathbb{N}$ and $b \geq \|x_0 - p\|$. If A has a modulus ϕ of regularity w.r.t $zer\, A$ and $\overline{B}(p, b)$, then (x_n) (given by $(*)$) converges to a zero $z := \lim x_n$ of A with rate of convergence $\rho(\phi(\varepsilon/2, b), b, \gamma, \eta) + 1$.*

Proof: As shown already in the proof of Lemma 6 we have that $zer\, F = zer\, A$ which is closed as this set coincides with the fixed point set of the nonexpansive mapping J_A (viewed as a mapping $\overline{D(A)} \to \overline{D(A)}$). The claim now follows from Theorem 4.1 in [9] using Lemma 6 and the fact that by Corollary 4 above and $u_n \in A(x_{n+1})$

$$\forall n \geq \rho(\varepsilon, b, \gamma, \eta) \ \ (|F(x_{n+1})| \leq \|u_n\| \leq \varepsilon) .$$

□

The concept of a *modulus of regularity* generalizes that of a *modulus of uniqueness* (see [9]) and so one, in particular, gets a rate of convergence under conditions which provide a modulus of uniqueness:

Definition 9 ([7])**.** *An accretive operator $A \subseteq X \times X$ with $0 \in Ap$ is called uniformly accretive at zero if*

$$\forall \varepsilon > 0\, \forall K > 0\, \exists \delta > 0\, \forall (x, u) \in A$$
$$(\|x - p\| \in [\varepsilon, K] \to \exists j(x - p) \in J(x - p) \ (\langle u, j(x - p)\rangle \geq \delta)) .$$

Any function $\Theta_{(\cdot)}(\cdot) : (0, \infty) \times (0, \infty) \to (0, \infty)$ such that $\delta := \Theta_K(\varepsilon)$ satisfies the above formula is called a modulus of accretivity at zero for A.

Note that A is in particular uniformly accretive at zero if

$$\forall (x, u) \in A\, \exists j(x - p) \in J(x - p) \ \ (\langle u, j(x - p)\rangle \geq \Phi(\|x - p\|)) ,$$

where $\Phi : [0, \infty) \to [0, \infty)$ is continuous with $\Phi(0) = 0$ and $\Phi(x) > 0$ for $x > 0$: take

$$\Theta_K(\varepsilon) := \inf\{\Phi(x) \ : \ x \in [\varepsilon, \max\{\varepsilon, K\}]\}.$$

Theorem 10. *Let $A \subseteq X \times X$ be an accretive operator with $0 \in Ap$ satisfying the range condition which is uniformly accretive at zero with modulus Θ. Let $\gamma > 0$ and (γ_n) be a sequence of real numbers with $\gamma_n \geq \gamma$ for all $n \in \mathbb{N}$. Define for $x_0 \in R(I + \gamma_0 A)$*

$$x_{n+1} := J_{\gamma_n A} x_n.$$

Then (x_n) strongly converges to the unique zero p of A with rate of convergence

$$\forall \varepsilon > 0\, \forall n \geq \rho\,(\Theta_b(\varepsilon)/b, b, \gamma, \eta) + 1 \ \ (\|x_n - p\| \leq \varepsilon)$$

with ρ from Corollary 4 and $b \geq \|x_0 - p\|$.

Proof: If $0 \in Ap$, then p is a fixed point of each $J_{\gamma_n A}$. By Corollary 4, $\|u_n\| < \frac{\Theta_b(\varepsilon)}{b}$ for all $n \geq \rho\,(\Theta_b(\varepsilon)/b, b, \gamma, \eta)$. Now assume that $\|x_{n+1} - p\| \geq \varepsilon$ for some $n \geq \rho\,(\Theta_b(\varepsilon)/b, b, \gamma, \eta)$. Then $\|x_{n+1} - p\| \in [\varepsilon, b]$ and so $\exists j(x_{n+1} - p) \in J(x_{n+1} - p)$ with (using that $u_n \in Ax_{n+1}$)

$$\|u_n\| \cdot \|x_{n+1} - p\| \geq \langle u_n, j(x_{n+1} - p)\rangle \geq \Theta_b(\varepsilon).$$

Hence $\|u_n\| \geq \frac{\Theta_b(\varepsilon)}{b}$ which is a contradiction. □

For constant $\gamma_n := \gamma > 0$ a related result can be found as Theorem 2 in [11].

If A is only accretive but not uniformly accretive at zero, the proximal point algorithm is known to converge only weakly (even in Hilbert spaces). In the boundedly compact case one has strong convergence but in general there is no computable rate of convergence (even for $X := \mathbb{R}$, see e.g. [14]). So the next best thing to hope for is an effective rate of metastability in the sense of Tao ([18, 19]), i.e. a bound $\Psi : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ with

$$\forall k \in \mathbb{N} \, \forall g \in \mathbb{N}^{\mathbb{N}} \, \exists n \leq \Psi(k,g) \, \forall i,j \in [n, n+g(n)] \, \left(\|x_i - x_j\| \leq \frac{1}{k+1} \right).$$

Such a rate has been established in the context of Hilbert spaces in [8]. We now construct a rate of metastability for uniformly convex normed spaces. For this we need one further result:

Proposition 11. *Let $A \subseteq X \times X$ be an accretive operator with $\overline{D(A)} \subseteq \bigcap_{\lambda > 0} R(I + \lambda A)$ satisfying $0 \in Ap$. Let $\gamma_n \geq \gamma > 0$ for all $n \in \mathbb{N}$ and $C_k \geq 2 + \frac{\gamma_i}{\gamma}$ for all $i \leq k$. Let (x_n) be defined as in $(*)$ with $x_0 \in \overline{D(A)}$. Then (for $b \geq \|x_0 - p\|$) $\Phi(k,b,\gamma,\eta) := \Delta(((k+1)C_k)^{-1}, 0, b, \eta)$ is an approximate F-bound (in the sense of* [8]*) for*

$$F := \bigcap_{k \in \mathbb{N}} \tilde{F}_k, \quad \tilde{F}_k := \bigcap_{i \leq k} \left\{ x \in \overline{D(A)} \, : \, \|x - J_{\gamma_i A} x\| \leq \frac{1}{k+1} \right\},$$

i.e.

$$\exists n \leq \Phi(k,b,\gamma,\eta) \, \forall i \leq k \, \left(\|x_n - J_{\gamma_i A} x_n\| \leq \frac{1}{k+1} \right).$$

Proof: Since

$$\frac{|\gamma_n - \gamma_i|}{\gamma_n} = \left| 1 - \frac{\gamma_i}{\gamma_n} \right| \leq 1 + \frac{\gamma_i}{\gamma_n} \leq 1 + \frac{\gamma_i}{\gamma},$$

Proposition 3(ii) implies that for all $i \leq k$

$$\|x_n - J_{\gamma_i A} x_n\| \leq C_k \cdot \|x_n - x_{n+1}\|$$

and so by Proposition 1

$$\exists n \leq \Phi(k,b,\gamma,\eta) \, \forall i \leq k \, \left(\|x_n - J_{\gamma_i A} x_n\| \leq C_k \cdot \frac{1}{C_k(k+1)} \leq \frac{1}{k+1} \right).$$

□

We can now apply Theorem 5.3 from [8] to obtain the following metastability result for the strong convergence of (x_n) in the finite dimensional case:

Theorem 12. *Let A be as in Proposition 11 and assume additionally that $\overline{D(A)}$ is boundedly compact. Let α be a II-modulus of total boundedness (in the sense of* [8]*) for $\overline{D(A)} \cap \overline{B}(0,M)$, where $M \geq b + \|p\|$ so that $\|x_n\| \leq \|x_n - p\| + \|p\| \leq M$ for all $n \in \mathbb{N}$, where $b \geq \|x_0 - p\|$, for the sequence (x_n) produced by the proximal point algorithm with $x_0 \in \overline{D(A)}$ and $\gamma_n \geq \gamma > 0$. Let $\chi(n,m,r) := \max\{n+m-1, m(r+1)\}$ and Φ from Proposition 11 above. Then $\Psi(k,g,\alpha) := \Psi_0(P, k_0, g)$, with*

$$\begin{cases} \Psi_0(0,k_0,g) := 0 \\ \Psi_0(n+1,k_0,g) := \Phi\left(\chi_{k,g}^M \left(\Psi_0(n,k_0,g), 4k_0 + 3 \right) \right), \end{cases}$$

and

$\chi_{k,g}(n,r) := \max\{2k+1, \chi(n,g(n),r)\}$, $\chi^M_{k,g}(n,r) := \max_{i\leq n}\{\chi_{k,g}(i,r)\}$, $P := \alpha\,(4k_0+3)$, $k_0 = 2k+1$

is a rate of metastability for (x_n). *In fact*

$$\forall k \in \mathbb{N}\, \forall g \in \mathbb{N}^{\mathbb{N}}\, \exists n \leq \Psi(k,g,\alpha)\, \forall i,j \in [n, n+g(n)] \left(\|x_i - x_j\| \leq \frac{1}{k+1} \text{ and } x_i \in \tilde{F}_k \right).$$

Proof: As in Section 8 from [8] (using that Lemma 8.1 in [8] holds with the same proof in our context) we can apply Theorem 5.3 and Remark 5.4 in [8] to $X := \overline{D(A)}$ with $\omega_F(k) := 4k+3, \delta_F(k) := 2k+1, \chi(n,m,r) := \max\{n+m-1, m(r+1)\}, \alpha_G := \beta_H := I$, $\gamma := \alpha$ as in the theorem and Φ from Proposition 11 above as approximate fixed point bound. □

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