EFFECTIVE RATES FOR NONLINEAR ERGODIC AVERAGES

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Abstract. In this paper, we obtain explicit rates of asymptotic regularity for ergodic averages under various circumstances, and apply them to extracting explicit and metastable rates of convergence for Cesàro means. Moreover, by obtaining a rate of asymptotic regularity for an averaged Mann type iteration, we extract an effective rate of convergence depending on a modulus of regularity, and a rate of metastability by computing a modulus of uniform Fejér monotonicity. In the presence of a modulus of uniqueness, we compute a rate of metastability for averaged Mann type iterations without any condition on coefficients. Our approach in this paper proposes a procedure that unifies methods for developing nonlinear ergodic theorems and facilitating further research.

1. Introduction

The first nonlinear mean ergodic theorem for nonexpansive mappings on bounded closed convex (not necessarily compact) subsets of a Hilbert space was established by Baillon [5], based on Zarantonello's inequality which is valid in Hilbert spaces. In [9, 10], Bruck simplified Baillon's method and studied the mean ergodic theorem in Banach spaces, based on an inequality for nonexpansive mappings called of type (γ) (see (2.1)). By [9, Lemma 1.1], if C is a bounded closed convex nonempty subset of a uniformly convex Banach space E, there exists a γ depending on the diameter of C and a modulus of uniform convexity for E such that every nonexpansive mapping on C is of type (γ). Baillon's theorem is extended in [9, Theorem 2.1] to Banach spaces as follows:

Suppose C is a weakly compact and convex subset of a Banach space E with a Fréchet differentiable norm, and $T : C \to C$ is a nonexpansive mapping such that T^n is of type (γ) for all n, then for each $x \in C$ the Cesàro means $\frac{1}{n}\sum_{j=0}^{n-1} T^{j+k}x$ converge weakly to a fixed point of T uniformly in $k \in \mathbb{N}$.

In [10], for a mapping T of type (γ) on a bounded, closed, and convex subset C of a B-convex space, the asymptotic regularity of Cesàro means of T was established, which relied on the following crucial property of the approximate fixed point sets:

$$\forall \varepsilon > 0 \exists \delta > 0 \big(coF_\delta(T) \subseteq F_\varepsilon(T) \big). \tag{1.1}$$

To prove (1.1) (and also asymptotic regularity), Bruck [10] employed recursive constructions based on γ and the convex approximation property of a B-convex space. Bruck's approach subsequently became the cornerstone of research on nonlinear mean ergodic theorems; for examples, see [19, 18, 17, 32, 14, 13, 1]. In particular, following Bruck's

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methodology, Freund and the first author derived in [14] a rate of asymptotic regularity for Cesàro means of T in uniformly convex Banach spaces, which depends on a bound for C and a modulus of uniform convexity, and in [13] a rate of metastability for a strong nonlinear ergodic theorem due to [19].

On the other hand, in 1964, Edelstein [12] studied first the convergence of Cesàro means of nonexpansive mappings for finding fixed points on compact domains in strictly convex Banach spaces. Atsushiba and Takahashi showed in [2] that for a compact, convex subset C of a strictly convex Banach space, there exists a $\gamma \in \Gamma$ such that every nonexpansive mapping $T : C \to C$ is of type (γ). Furthermore, by establishing (1.1), utilizing the techniques of [9, 10], and using the compactness of C, they provided a uniform version of [12, Theorem II] as follows:

Let C be a nonempty compact convex subset of a strictly convex Banach space and let T be a nonexpansive mapping of C into itself. Then for each $x \in C$, the Cesàro means $S_n(T^k x) = \frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x$ converge strongly to a fixed point of T (which does not depend on k), uniformly in $k \in \mathbb{N}$.

The latter strong convergence is equivalent to the following:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n, m \geq N \forall k \in \mathbb{N}\big(\|S_n(T^k x) - S_m(T^k x)\| \leq \varepsilon\big). \tag{1.2}$$

For a quantitative version of (1.2), we naturally look for a Cauchy rate. It is known that such a rate cannot be computed in general, even for a linear T (see [3, Theorem 5.1]). However, we will be able to construct a Cauchy rate (Theorem 6.2), depending on rates of convergence for the nonincreasing sequences $(\|T^{n+k}x - T^n x\|)_{n\geq 0}$ and a modulus of regularity of T w.r.t. $Fix(T)$. Moreover, in a general situation, we construct a rate of metastability (Theorem 6.1), which is a map $(\varepsilon, g, h) \mapsto \Phi(\varepsilon, g, h)$ ensuring

$$\begin{gathered}\forall \varepsilon > 0 \forall x \in C \forall g, h : \mathbb{N} \to \mathbb{N} \exists N \leq \Phi(\varepsilon, g, h) \forall m, n \in [N; N + g(N)] \forall k \leq h(N) \\ \big(\|S_n(T^k x) - S_m(T^k x)\| \leq \varepsilon\big).\end{gathered} \tag{1.3}$$

Note that a rate for N in (1.2) trivially implies (1.3). Moreover, (1.3) is as strong as (1.2). Indeed, (1.2) is equivalent to

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall M, K \in \mathbb{N} \forall n, m \in [N; N+M] \forall k \leq K\big(\|S_n(T^k x) - S_m(T^k x)\| \leq \varepsilon\big). \tag{1.4}$$

If (1.2) fails, then for some $\varepsilon > 0$, we have

$$\forall N \in \mathbb{N} \exists M, K \in \mathbb{N} \exists n, m \in [N; N + M] \exists k \leq K\big(\|S_n(T^k x) - S_m(T^k x)\| > \varepsilon\big). \tag{1.5}$$

Therefore, if we set $g(N) := M$ and $h(N) := K$ for such numbers, (1.3) must fail. That is, (1.3) is a quantitative version of (1.2).

Ignoring the bound Φ, the $\forall\exists$-sentence (1.3) is, in fact, the so-called Herbrand normal form of the $\forall\exists\forall$-sentence (1.4). General theorems from logic enable the extraction of effective bounds from ineffective proofs of $\forall\exists$-theorems. Obtaining effective bounds for equivalent but constructively weakened reformulations started in Gödel's functional interpretation and the so-called no-counterexample interpretation due to Kreisel [30, 31].

Kreisel [31] also launched a program to analyze specific prima facie nonconstructive proofs, aiming to extract hidden constructive information. This initiative laid the conceptual groundwork for proof mining [21], which is the project of applying proof-theoretic transformations to produce (specific) quantitative and qualitative information from existing proofs in areas of core mathematics such as nonlinear analysis. The first applications of the proof mining methodology in analysis involved the extraction of explicit moduli of uniqueness in the context of Chebycheff approximation [22]. However, the scope of proof mining program has since broadened to include numerous other quantitative moduli, existing in mathematics or formulated as quantitative proof-theoretic counterparts of qualitative notions; notably, the program has been used to explicitly transform moduli between different settings.

Computable rates of convergence are in general unattainable even for computable bounded monotone sequence of rationals $\mathbb{R}$ by a classical result of [36]. Given this state of affairs, the first author suggested in [24] the following (noneffectively) equivalent but constructively weakened reformulation (of the form $\forall\exists$) of the Cauchy property of a sequence (x_n) in normed (and hyperbolic) spaces:

$$\forall k \in \mathbb{N} \forall g : \mathbb{N} \to \mathbb{N} \exists N \in \mathbb{N} \forall m, n \in [N; N + g(N)] \big(\|x_n - x_m\| \leq 2^{-k}\big), \tag{1.6}$$

in order to effectively convert other bounds in the premises of a concrete case study into a bound on $\exists N \in \mathbb{N}$. Such a bound, which indeed is a bound for the Kreisel's no-counterexample interpretation of the Cauchy property, is called a rate of metastability, since Tao [37, 38] calls an interval $[N; N + g(N)]$ with the property in (1.6) an interval of metastability. As an example, $\Phi(g, k) := \tilde{g}^{(2^k+1)}(0)$, where $\tilde{g}(n) := n + g(n)$, is a rate of metastability for monotone sequences in $[0, 1] \subset \mathbb{R}$. See, e.g., [21, Proposition 2.27]. The concept of metastability has been studied within the proof mining program, based on variants of Gödel's functional interpretation and monotone extensions (see [21]).

Regarding quantitative results on Baillon's mean ergodic theorem and related problems, we refer the reader to [20, 14, 13, 35], while the case of linear T was treated already in [25] and [3].

This paper begins with a quantitative version of the ε-δ sentence (1.1) under various circumstances. Then, in Section 5, we present a general result that yields explicit rates of asymptotic regularity for ergodic averages, applicable e.g., to any of the following cases: E is uniformly convex, refining the main result in [14]; E is strictly convex and C is compact; and T possesses a modulus of regularity in the sense of [28]. These sections reveal that Bruck's method consists of two steps: first, constructing a modulus of convexity for approximate fixed points (1.1), and then, recursively deriving a rate of asymptotic regularity for ergodic averages using that modulus. In Section 6, we extract an explicit rate of convergence for Cesàro means of a mapping T with a modulus of regularity ϕ, in terms of ϕ and rates for the nonincreasing sequences $(\|T^{n+k}x - T^n x\|)_{n\geq 0}$. Here, even the qualitative version of this result is new. Without such rates, we also extract a highly uniform rate of metastability ensuring (1.3) (Theorem 6.1). In Section 7, we apply our results to obtain rates of asymptotic regularity for averaged Mann type iterations $x_{n+1} = \alpha_n x_n + (1-\alpha_n)S_n(x_n)$, where $(\alpha_n) \subseteq [0,1)$ is such that $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ with a rate of divergence D. Based on [28], we obtain a rate of convergence when T has a modulus of regularity w.r.t. $Fix(T)$. Moreover, by computing a modulus of (x_n) being uniformly Fejér monotone (w.r.t $Fix(T)$) in the sense of [27], we extract a rate of metastability in view of [27, Theorem 5.1]. In the presence of a modulus of uniqueness (see [21] for this concept), we also obtain a rate of metastability for averaged Mann type iterations without any condition on $(\alpha_n) \subseteq [0,1)$. Our approach in this paper proposes a procedure that unifies methods, which appear to be different at first glance, for developing nonlinear ergodic theorems and averaged iterations, thereby facilitating further research.

2. Preliminaries

Throughout this paper, let E denote a (real) Banach space, and let C be a nonempty, bounded, closed, and convex subset of E. B_E denotes the closed unit ball of E. We assume that $b > 0$ is a constant such that $C \subset B_{b/2}(0)$. For a mapping $T : C \to E$ and a given $\varepsilon > 0$, let $F_\varepsilon(T)$ denote the set of ε-approximate fixed points defined as: $F_\varepsilon(T) := \{x \in C \mid \|x - Tx\| \leq \varepsilon\}$. If T is nonexpansive (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$) and $T(C) \subseteq C$, then $F_\varepsilon(T) \neq \emptyset$. For a subset M of E, co$M$ denotes the convex hull of M, and co_pM denotes the set

$$\{\sum_{i=1}^{p} \lambda_i x_i \mid \ \sum_{i=1}^{p} \lambda_i = 1,\ \lambda_i \geq 0,\ x_i \in M\}.$$

A Banach space E is said to be strictly convex if for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$, we have $\|\frac{x+y}{2}\| < 1$. E is called uniformly convex, if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|(x+y)/2\| \leq 1 - \delta$, for each $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$. In this case, a function $\eta : (0,2] \to (0,1]$ is a modulus of uniform convexity for E if for all $\varepsilon \in (0,2]$ and $x, y \in E$,

$$\|x\|, \|y\| \leq 1 \text{ and } \|x - y\| \geq \varepsilon \quad \Rightarrow \|\frac{x+y}{2}\| \leq 1 - \eta(\varepsilon).$$

Notation. We denote by Γ the set of continuous, strictly increasing, convex functions $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ with $\gamma(t) \leq t$ and $\gamma(0) = 0$.

Definition 2.1. Let $\gamma \in \Gamma$. A mapping $T : C \to E$ is of type (γ), if $\forall x, y \in C$ and $\forall \lambda \in [0,1]$,

$$\gamma(\|\lambda Tx + (1-\lambda)Ty - T(\lambda x + (1-\lambda)y)\|) \leq \|x - y\| - \|Tx - Ty\|. \tag{2.1}$$

The fixed point set of a mapping $T : C \to E$ which is of type (γ) is convex.

Remark 2.2. (cf. [13, 14]) Suppose that E is uniformly convex with a modulus $\eta : (0,2] \to (0,1]$. Then $\eta_1 : [0,\infty) \to [0,1]$,

$$\eta_1(\varepsilon) := \sup\{\eta(\varepsilon\prime) | 0 < \varepsilon\prime \leq \min\{2, \varepsilon\}\}, \quad \eta_1(0) := 0,$$

is increasing. Moreover, defining $\tilde{\eta}(\varepsilon) := \frac{1}{2}\int_0^\varepsilon \eta_1(t)dt$, we have $\tilde{\eta} \in \Gamma$. Bruck [1] proved that, defining $\gamma \in \Gamma$ by $\gamma(\varepsilon) := \frac{b}{2}\tilde{\eta}(\frac{4\varepsilon}{b})$, every nonexpansive mapping $T : C \to E$ is of type (γ). Since $\eta_1(t) \leq 1$, we have $\tilde{\eta}(\varepsilon) \leq \varepsilon/2$ and thus $\gamma(\varepsilon) \leq \varepsilon$. We also know that

$$\tilde{\eta}(\varepsilon) \geq \frac{1}{2}\int_{\frac{\varepsilon}{3}}^{\varepsilon} \eta_1(t)dt \geq \frac{\varepsilon}{3}\eta_1(\frac{\varepsilon}{3}) \geq \frac{\varepsilon}{3}\eta(\min\{2, \frac{\varepsilon}{3}\}).$$

Thus, we have for all $\varepsilon > 0$,

$$\gamma(\varepsilon) = \frac{b}{2}\tilde{\eta}(\frac{4}{b}\varepsilon) \geq \frac{b}{2}\tilde{\eta}(\frac{3}{b}\varepsilon) \geq \frac{\varepsilon}{2}\eta(\min\{2, \frac{\varepsilon}{b}\}). \tag{2.2}$$

Lemma 2.3. *Let C be a nonempty, compact, and convex subset of a strictly convex Banach space E. Then, there exists a $\gamma \in \Gamma$ such that every nonexpansive mapping $T : C \to C$ is of type (γ).*

Proof. Let $\operatorname{diam}C = 2r > 0$. Define $D := \frac{1}{2r}(C - C)$. Obviously, D is compact and convex, $0 \in D \subseteq B_1(0)$, and $\operatorname{diam}D \geq 1$. We define $\delta : [0, 2] \to [0, 1]$ by $\delta(0) = 0$ and for $0 < \varepsilon \leq 2$,

$$\delta(\varepsilon) := \inf\{\max\{\|u\|, \|v\|\} - \|\frac{u+v}{2}\|;\ u, v \in D \text{ and } \|u - v\| \geq \varepsilon/2\}. \tag{2.3}$$

Since $\operatorname{diam}D \geq 1$, the above set is nonempty and thus $\delta(\varepsilon) \leq 1$. Moreover, since D is compact and E is strictly convex, we have $\delta(\varepsilon) > 0$, for $\varepsilon \in (0, 2]$.

It is known that for all $0 \leq \lambda \leq 1$ and $x, y \in E$,

$$2\min\{\lambda, 1-\lambda\}(\max\{\|x\|, \|y\|\} - \frac{1}{2}\|x + y\|) \leq \max\{\|x\|, \|y\|\} - \|\lambda x + (1 - \lambda)y\|. \tag{2.4}$$

(Assume $0 \leq \lambda \leq \frac{1}{2}$. Then, taking $k := \max\{\|x\|, \|y\|\}$, we have

$$\begin{aligned} 2\lambda k + \|\lambda x + (1-\lambda)y\| &= 2\lambda k + \|\lambda(x + y) + (1 - 2\lambda)y\| \\ &\leq 2\lambda k + \lambda\|x + y\| + (1 - 2\lambda)\|y\| \leq 2\lambda k + \lambda\|x + y\| + (1 - 2\lambda)k = k + \lambda\|x + y\|. \end{aligned}$$

Thus $2\lambda(k - \frac{1}{2}\|x + y\|) \leq k - \|\lambda x + (1 - \lambda)y\|$, which is equivalent to (2.4).)

Note that by the definition of δ,

$$\delta(2\|u - v\|) \leq \max\{\|u\|, \|v\|\} - \|\frac{u+v}{2}\|,$$

for all $u, v \in D$. Thus, using (2.4), we have

$$\begin{aligned} 2\lambda(1-\lambda)\delta(2\|u - v\|) &\leq 2\min\{\lambda, 1-\lambda\}\delta(2\|u - v\|) \\ &\leq 2\min\{\lambda, 1-\lambda\}(\max\{\|u\|, \|v\|\} - \|\frac{u+v}{2}\|) \\ &\leq \max\{\|u\|, \|v\|\} - \|\lambda u + (1-\lambda)v\|. \end{aligned} \tag{2.5}$$

Define

$$\alpha(\varepsilon) := \begin{cases} \delta(\varepsilon), & \text{if } 0 \leq \varepsilon \leq 2 \\ \delta(2), & \text{if } 2 < \varepsilon, \end{cases}$$

and $d(\varepsilon) := 1/2 \int_0^\varepsilon \alpha(t)dt$. Then, for all $0 \le \varepsilon \le 2$, we have $d(\varepsilon) \le \alpha(\varepsilon)$ and $d \in \Gamma$. By (2.5), we have

$$2\lambda(1-\lambda)d(2\|u-v\|) \le \max\{\|u\|, \|v\|\} - \|\lambda u + (1-\lambda)v\|. \tag{2.6}$$

We take

$$\begin{cases} u := & \frac{(1-\lambda)}{2r}(Ty - T(\lambda x + (1-\lambda)y)), \\ v := & \frac{\lambda}{2r}(T(\lambda x + (1-\lambda)y) - Tx), \end{cases}$$

where $x, y \in C$ and $0 \le \lambda \le 1$. Since $0 \in D$ and D is convex, we have $u, v \in D$. Now, using (2.6), we have

$$\begin{aligned} 2\lambda(1-\lambda)d(\frac{2}{2r}\|\lambda Tx + (1-\lambda)Ty - T(\lambda x + (1-\lambda)y)\|) &= 2\lambda(1-\lambda)d(2\|u-v\|) \\ \le \max\{\frac{(1-\lambda)}{2r}\|Ty - T(\lambda x + (1-\lambda)y)\|, &\frac{\lambda}{2r}\|T(\lambda x + (1-\lambda)y) - Tx\|\} \\ - \|\frac{\lambda(1-\lambda)}{2r}(Ty - T(\lambda x + (1-\lambda)y)) &+ \frac{\lambda(1-\lambda)}{2r}(T(\lambda x + (1-\lambda)y) - Tx)\| \\ \le \frac{\lambda(1-\lambda)}{2r}\|x-y\| - \frac{\lambda(1-\lambda)}{2r}\|Tx - Ty\|. \end{aligned}$$

Consequently,

$$4rd(\frac{1}{r}\|\lambda Tx + (1-\lambda)Ty - T(\lambda x + (1-\lambda)y)\|) \le \|x-y\| - \|Tx - Ty\|.$$

Defining $\gamma(t) := 4rd(\frac{t}{r})$, the result follows. □

Remark 2.4. Lemma 2.3 was first proved by Atsushiba and Takahashi in [2]. Above, we have presented a different and easier proof compared to [2]. They proved the result of the lemma by defining a function $\gamma \in \Gamma$ as $\gamma(\varepsilon) := 2rd_r(\frac{1}{r}\varepsilon)$, where $d_r(\varepsilon) := \frac{1}{2}\int_0^\varepsilon \alpha_r(t)dt$,

$$\alpha_r(\varepsilon) := \begin{cases} \delta_r(\varepsilon), & \text{if } 0 \le \varepsilon \le 2 \\ 1, & \text{if } 2 < \varepsilon, \end{cases}$$

and $\delta_r : [0,2] \to [0,1]$ is defined as

$$\begin{aligned} \delta_r(\varepsilon) := \frac{1}{r}\inf\{\max\{\|z-x\|, \|z-y\|\} - \|z - \frac{x+y}{2}\| : \\ \|z-x\| \le r,\ \|z-y\| \le r,\ \|x-y\| \ge r\varepsilon,\ x, y, z \in C\}. \end{aligned}$$

Lemma 2.5. [10, Lemma 2.1] *Suppose $\gamma \in \Gamma$. Then for each positive integer n there exists $\gamma_n \in \Gamma$ such that for any $T : C \to E$ of type (γ), any $\lambda_1, \dots \lambda_n \in [0,1]$ with $\sum_{i=1}^n \lambda_i = 1$, and any $x_1, \dots, x_n \in C$,*

$$\gamma_n\big(\|T(\sum_{i=1}^n \lambda_i x_i) - \sum_{i=1}^n \lambda_i T(x_i)\|\big) \le \max_{1\le i,j\le n}\big(\|x_i - x_j\| - \|Tx_i - Tx_j\|\big). \tag{2.7}$$

In [10, Lemma 2.1], γ_n is defined recursively by $\gamma_2 := \gamma$ and γ_n to be any function in Γ such that

$$\gamma_{n+1}^{-1}(t) \ge \gamma_2^{-1}(t) + \gamma_n^{-1}(t + 2\gamma_2^{-1}(t). \tag{2.8}$$

Remark 2.6. (2.8) is a recursive construction of γ_n^{-1}. Since a function and its inverse do not generally belong to the same complexity class, the following construction suggested in [13]: for $n \geq 2$, define

$$\gamma_{n+1}(t) := \min\{\gamma_n(t), \gamma_2(\frac{\gamma_n(t/2)}{3})\}. \tag{2.9}$$

The mappings γ_n in preceding structure satisfy property (2.8) (see [13, Lemma 2.5]). Furthermore, the mappings γ_n in (2.9) are continuous strictly increasing, though not necessarily convex. Notably, Bruck's proof of [10, Lemma 2.1] does not require γ and γ_n to be convex.

3. A modulus for convex approximation property

Definition 3.1. E is said to be B-convex if for some natural number $k \geq 2$ and $\varepsilon > 0$, there holds for each choice of $x_1, x_2, \ldots, x_k$ from B_E, $\| \pm x_1 \pm x_2 \pm \cdots \pm x_k\| \leq k(1-\varepsilon)$ for some choice of the $+$ and $-$ signs.

The significance of B-convexity stems from a result by Beck [5, 6] establishing that a Banach space E is B-convex iff a certain strong law of large numbers holds for E-valued random variables. Giesy [16, Lemma 6] showed that E is B-convex iff l_1 is not finitely representable in E. In other words, E is not B-convex iff

$$\forall k \geq 2 \forall \varepsilon > 0 \exists x_1, \cdots, x_k \in B_E \forall \alpha_1, \cdots, \alpha_k \in \mathbb{R}\big((1-\varepsilon)\sum_{i=1}^{k} |\alpha_i| \leq \| \sum_{i=1}^{k} \alpha_i x_i\|\big).$$

Finite dimensional normed spaces and uniformly convex Banach spaces are B-convex, but the Banach spaces ℓ_1, ℓ_∞ and c_0 are not B-convex (see [16]).

Definition 3.2. A Banach space E is said to have Rademacher type $q \in [1, 2]$ with constant C_q if all finite sequences $(x_1, \cdots, x_n)$ in E validate

$$\mathbb{E}(\| \sum_{i=1}^{n} \varepsilon_i x_i\|^q)^{\frac{1}{q}} \leq C_q(\sum_{i=1}^{n} \|x_i\|^q)^{\frac{1}{q}},$$

where (ε_i) is a sequence of independent random variables that take values ± 1 with probability $\frac{1}{2}$, and $\mathbb{E}$ represents the expected value.

The Rademacher type of a Banach space is always less than or equal to 2. Pisier [33] showed that a Banach space E is B-convex iff it has Rademacher type $q > 1$. Based on [33], Bruck provided in [10, Theorem 1.1] a simple equivalent form for B-convexity that may be stated as follows: A Banach space E is B-convex if and only if it has the convex approximation property:

$$\forall \varepsilon > 0 \exists p \in \mathbb{N}^* \forall M \subseteq B_1(0)\Big(coM \subseteq co_p(M) + B_\varepsilon(0)\Big). \tag{3.1}$$

The proof of [10, Theorem 1.1] (see also [13, Lemma 2.6]) shows that in a B-convex space E, any $p \geq \lceil (\frac{2C_q}{\varepsilon})^{\frac{q}{q-1}} \rceil$ validates (3.1), where q is a Rademacher type with constant C_q for E. Taking

$$\varrho(\varepsilon) := \lceil (\frac{2C_q}{\varepsilon})^{\frac{q}{q-1}} \rceil, \tag{3.2}$$

it follows that (3.1) is equivalent to having

$$\forall \varepsilon > 0 \forall r > 0 \forall M \subseteq B_r(0) \Big(coM \subseteq co_{\varrho(\varepsilon/r)}(M) + B_\varepsilon(0) \Big). \tag{3.3}$$

Inspired by (3.3), we introduce the following definition.

Definition 3.3. A nonempty bounded subset C of E is said to have the convex approximation property if

$$\forall \varepsilon > 0 \exists p \in \mathbb{N}^* \forall M \subseteq C \Big(coM \subseteq co_p(M) + B_\varepsilon(0) \Big). \tag{3.4}$$

In this case, a function $\varrho_C : \mathbb{R}^*_+ \to \mathbb{N}^*$ is a modulus of convex approximation for C, if

$$\forall \varepsilon > 0 \forall M \subseteq C \Big(coM \subseteq co_{\varrho_C(\varepsilon)}(M) + B_\varepsilon(0) \Big). \tag{3.5}$$

Remark 3.4. Assume that C is a nonempty bounded subset of a B-convex space E such that $C \subset B_r(0)$ for some $r > 0$. From (3.2) and (3.3), $\varrho_C(\varepsilon) := \varrho(\varepsilon/r)$ is a modulus of convex approximation for C, depending on r, the Rademacher type q, and the constant C_q for E.

Remark 3.5. In [14, Theorem 3.1], q and C_q are explicitly determined for uniformly convex (and even for uniformly nonsquare) Banach spaces. Suppose η is a modulus of uniform convexity for E, and let $\delta := \eta(1)$ and $\lambda := 1 - \delta$. Assume that $\xi \in (0, 1)$ is sufficiently small and $p' \in [2, \infty)$ is sufficiently large such that:

$$\frac{1-\xi}{1+2\sqrt{2\xi}} \geq \frac{1}{2}\sqrt{2\lambda^2+2} \quad \text{and} \quad \frac{1}{2^{1/p'}} \geq 1 - \xi.$$

Let p satisfy $1 = \frac{1}{p} + \frac{1}{p'}$. Then, for any $q \in (1, p)$, the space E has Rademacher type q with constant

$$C_q = 3 \cdot \frac{2^{1/q}}{2^{(1/q)-(1/p)} - 1}.$$

Definition 3.6. Let A be a nonempty subset of a metric space (X, d).

(i) [28, Definition 2.1] A function $\alpha : \mathbb{N} \to \mathbb{N}$ is called a I-modulus of total boundedness for A if for every $k \in \mathbb{N}$, there exist elements $a_0, a_1, \ldots, a_{\alpha(k)}$ in X such that for all $x \in A$, there exists $0 \leq i \leq \alpha(k)$ satisfying $d(x, a_i) < \frac{1}{k+1}$.

(ii) [15, Definition 5.2](see also [28, Definition 2.2]) A function $\beta : \mathbb{N} \to \mathbb{N}$ is called a II-modulus of total boundedness for A if for any $k \in \mathbb{N}$ and for any sequence (x_n) in A, there exist $0 \leq i < j \leq \beta(k)$ such that $d(x_i, x_j) \leq \frac{1}{k+1}$.

Proposition 3.7. [27, Proposition 2.4] *(i) If α is a I-modulus of total boundedness for A, then $\beta(k) := \alpha(2k+1)+1$ is a II-modulus of total boundedness for A.*

(ii) If β is a II-modulus of total boundedness for A, then $\alpha(k) := \beta(k)-1$ is a I-modulus of total boundedness for A.

Lemma 3.8. *Let C be a nonempty, compact, convex set with a I-modulus of total boundedness $\alpha : \mathbb{N} \to \mathbb{N}$ in a Banach space E. Then C satisfies the following convex approximation property:*

$$\forall k \in \mathbb{N} \forall M \subseteq C \left(coM \subseteq co_{\alpha(2k)+1} M + B_{1/(k+1/2)}(0) \right). \tag{3.6}$$

In particular, $\varrho_C(\varepsilon) := \alpha(2\lceil 1/\varepsilon \rceil) + 1$ defines a modulus of convex approximation for C.

Proof. Let $k \in \mathbb{N}$ and $M \subseteq C$. Then there are $a_0, a_1, \ldots, a_{\alpha(2k)} \in C$ such that

$$\forall x \in C \; \exists 0 \leq i \leq \alpha(2k) \left(\|x - a_i\| < \frac{1}{2k+1} \right).$$

Let $J_M := \{0 \leq i \leq \alpha(2k) : \; M \cap B_{\frac{1}{2k+1}}(a_i) \neq \emptyset\}$. Then $|J_M| \leq \alpha(2k)+1$. For each $i \in J_M$, we pick some $y_i \in M \cap B_{\frac{1}{2k+1}}(a_i)$. Let $u \in coM$ and suppose $u = \sum_{j=1}^n \lambda_j u_j$, where $\lambda_j \geq 0$, $\sum_{j=1}^n \lambda_j = 1$ and $u_j \in M$. Thus for each $j \in \{1, 2, \ldots, n\}$, there exists some $i_j \in J_M$ such that

$$u_j \in B_{\frac{1}{2k+1}}(a_{i_j}) \subseteq B_{\frac{1}{k+1/2}}(y_{i_j}).$$

That is, for each $j \in \{1, 2, \ldots, n\}$, there exists $i_j \in J_M$ such that $u_j - y_{i_j} \in B_{\frac{1}{k+1/2}}(0)$. Consequently,

$$u = \sum_{j=1}^n \lambda_j u_j \in co_{\alpha(2k)+1} M + B_{\frac{1}{k+1/2}}(0).$$

□

4. Moduli of convex regularity

As previously indicated, establishing (1.1) is a crucial component in utilizing Bruck's approach for nonlinear ergodic theorems. To achieve our results, we must examine the circumstances under which (1.1) is valid and derive a quantitative form of (1.1).

We first recall the notion of the modulus of regularity, which was introduced in [28].

Definition 4.1. Let (X, d) be a metric space and $F : X \to \mathbb{R}$ be a mapping with $zerF = \{x \in X : F(x) = 0\} \neq \emptyset$. Fixing $z \in zerF$ and $r > 0$, we say that $\phi : (0, \infty) \to (0, \infty)$ is a modulus of regularity for F w.r.t. $zerF$ and $\overline{B}(z, r)$ if for all $\varepsilon > 0$ and $x \in \overline{B}(z, r)$ we have the following:

$$|F(x)| < \phi(\varepsilon) \Rightarrow \mathrm{dist}(x, zerF) < \varepsilon.$$

Let C be a closed convex subset of a Banach space E. Consider a mapping $T : C \to E$ with $Fix(T) \neq \emptyset$, and define $F : C \to \mathbb{R}$ by $F(x) = \|x - Tx\|$. Let $z \in Fix(T)$ and $r > 0$. A modulus of regularity for T with respect to $Fix(T)$ and $\overline{B}(z, r)$ is defined as a modulus of regularity for F with respect to $zerF$ and $\overline{B}(z, r)$. If C is locally compact, T

is continuous, $z \in Fix(T)$, and $r > 0$, then T has a modulus of regularity with respect to $Fix(T)$ and $\overline{B}(z,r)$ (see [28, Corollary 3.5]). In Example 6.6, a modulus of regularity is computed for a nonexpansive mapping on a non-compact set.

Given that C is assumed to be bounded throughout this paper, a modulus of regularity for $T : C \to E$ with $Fix(T) \neq \emptyset$ is defined as a function $\phi : (0,\infty) \to (0,\infty)$ satisfying

$$\|x - Tx\| < \phi(\varepsilon) \Rightarrow \mathrm{dist}(x, Fix(T)) < \varepsilon, \tag{4.1}$$

for each $\varepsilon > 0$ and $x \in C$.

Lemma 4.2. *Let $\phi : (0,\infty) \to (0,\infty)$ be a modulus of regularity for a nonexpansive mapping $T : C \to E$ such that $Fix(T)$ is nonempty and convex. Then, for all $\varepsilon > 0$,*

$$coF_{\phi(\frac{\varepsilon}{2})}(T) \subseteq F_\varepsilon(T).$$

Proof. Let $\varepsilon > 0$ be given. Consider $x_1, \ldots, x_n \in C$ such that $\|x_i - Tx_i\| < \phi(\frac{\varepsilon}{2})$, for $i = 1, \ldots, n$, and $\lambda_1, \ldots, \lambda_n \geq 0$ satisfying $\sum_{i=1}^n \lambda_i = 1$. By the property of ϕ in 4.1, we have $\mathrm{dist}(x_i, Fix(T)) < \varepsilon/2$, $i = 1, \ldots, n$. Thus we may choose $f_1, \ldots, f_n \in Fix(T)$ such that $\|x_i - f_i\| < \varepsilon/2$, $i = 1, \ldots, n$. Thus, $\| \sum_{i=1}^n \lambda_i x_i - \sum_{i=1}^n \lambda_i f_i \| < \varepsilon/2$. Given that $Fix(T)$ is convex, it follows that $\sum_{i=1}^n \lambda_i f_i \in Fix(T)$, and consequently $\sum_{i=1}^n \lambda_i x_i \in Fix(T) + B_{\varepsilon/2}(0) \subseteq F_\varepsilon(T)$, where the latter inclusion derives from the nonexpansivity of T. □

The above result shows that the existence of a modulus of regularity implies (1.1) and provides an explicit quantitative form thereof. In light of this, we introduce the following definitions for the sake of simplification.

Definition 4.3. (Convex regularity) Suppose that $\mathcal{F}$ is a family of nonexpansive mappings from C to E. We say that $\mathcal{F}$ is uniformly convex regular, if we have

$$\forall \varepsilon > 0 \exists \delta > 0 \forall T \in \mathcal{F} \left(coF_\delta(T) \subseteq F_\varepsilon(T) \right). \tag{4.2}$$

Definition 4.4. (Modulus of convex regularity) Suppose that $\mathcal{F}$ is a convex regular family of nonexpansive mappings from C to E. We say that $\theta : (0,\infty) \to (0,\infty)$ is a modulus of convex regularity for $\mathcal{F}$, if

$$\forall \varepsilon > 0 \forall T \in \mathcal{F} \left(coF_{\theta(\varepsilon)}(T) \subseteq F_\varepsilon(T) \right). \tag{4.3}$$

Remark 4.5. From Lemma 4.2, it follows that $\theta(t) := \phi(\frac{t}{2})$ is a modulus of convex regularity for $\mathcal{F} = \{T\}$, where $T : C \to E$ is a nonexpansive mapping such that $Fix(T)$ is nonempty and convex, and $\phi : (0,\infty) \to (0,\infty)$ is a modulus of regularity for T with respect to $Fix(T)$. It is worth noting that (4.2) (or (4.3)) implies that $Fix(T)$ is convex.

In the following, a modulus of convex regularity for mappings of type (γ) will be explicitly computed when either E is B-convex or C is compact. We recall that every uniformly convex Banach space is B-convex.

The subsequent result establishes a quantitative refinement of [10, Theorem 1.2]. Our proof differs slightly and provides explicit estimates.

Lemma 4.6. *Suppose that C is a nonempty, bounded, closed, and convex subset of a Banach space E possessing the convex approximation property with a modulus $\varrho_C : \mathbb{R}^*_+ \to \mathbb{N}^*$. Let $\gamma \in \Gamma$. Given $\varepsilon > 0$, define $\theta(\varepsilon) := \gamma^{\tilde{p}}(\varepsilon/3^{\tilde{p}+1})$, where $\tilde{p} \in \mathbb{N}$ is such that $2^{\tilde{p}} \geq \varrho_C(\frac{\varepsilon}{3})$. Then, $\theta : (0,\infty) \to (0,\infty)$ is a modulus of convex regularity for the family of mappings $T : C \to E$ of type (γ).*

Proof. Given $\varepsilon > 0$, define $\sigma(\varepsilon) := \frac{1}{2}\gamma(\frac{2\varepsilon}{3})$. Let $T : C \to E$ be a mapping of type (γ). First, we show that

$$co_2 F_{\sigma(\varepsilon)}(T) \subseteq F_\varepsilon(T). \tag{4.4}$$

Consider $x_1, x_2 \in F_{\sigma(\varepsilon)}(T)$, and $0 \leq \lambda \leq 1$. Then

$$\begin{aligned}
&\|\lambda x_1 + (1-\lambda)x_2 - T(\lambda x_1 + (1-\lambda)x_2)\| \\
&\quad \leq \lambda\|x_1 - Tx_1\| + (1-\lambda)\|x_2 - Tx_2\| + \|\lambda Tx_1 + (1-\lambda)Tx_2 - T(\lambda x_1 + (1-\lambda)x_2\| \\
&\quad \leq \sigma(\varepsilon) + \gamma^{-1}(\|x_1 - x_2\| - \|Tx_1 - Tx_2\|) \\
&\quad \leq \sigma(\varepsilon) + \gamma^{-1}(\|x_1 - Tx_1\| + \|x_2 - Tx_2\|) \\
&\quad \leq \sigma(\varepsilon) + \gamma^{-1}(2\sigma(\varepsilon)) = \frac{1}{2}\gamma(\frac{2\varepsilon}{3}) + \gamma^{-1}(\gamma(\frac{2\varepsilon}{3})) \leq \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon,
\end{aligned}$$

which proves (4.4). Now, by induction, we arrive at

$$co_{2^p} F_{\sigma^p(\varepsilon)}(T) \subseteq F_\varepsilon(T), \tag{4.5}$$

for all $p \in \mathbb{N}$. Besides, in light of (3.5), we have

$$\forall \varepsilon > 0 \forall M \subseteq C \Big(coM \subseteq co_{\varrho_C(\frac{\varepsilon}{3})}(M) + B_{\frac{\varepsilon}{3}}(0) \Big). \tag{4.6}$$

Let $\tilde{p}$ be such that $2^{\tilde{p}} \leq \varrho_C(\frac{\varepsilon}{3})$. Then, using (4.6) and (4.5), we arrive at

$$\begin{aligned}
coF_{\sigma^{\tilde{p}}(\varepsilon/3)}(T) &\subseteq co_{\varrho_C(\frac{\varepsilon}{3})} F_{\sigma^{\tilde{p}}(\varepsilon/3)}(T) + B_{\frac{\varepsilon}{3}}(0) \\
&\subseteq co_{2^{\tilde{p}}} F_{\sigma^{\tilde{p}}(\varepsilon/3)}(T) + B_{\frac{\varepsilon}{3}}(0) \\
&\subseteq F_{\frac{\varepsilon}{3}}(T) + B_{\frac{\varepsilon}{3}}(0) \subseteq F_\varepsilon(T).
\end{aligned} \tag{4.7}$$

We note that, due to the convexity of γ, $\gamma(\varepsilon/3) \leq \sigma(\varepsilon)$, and by induction, $\theta(\varepsilon) := \gamma^{\tilde{p}}(\varepsilon/3^{\tilde{p}+1}) \leq \sigma^{\tilde{p}}(\varepsilon/3)$. From (4.7), we deduce that $coF_{\theta(\varepsilon)}(T) \subseteq F_\varepsilon(T)$. ☐

Remark 4.7. From the above proof, we observe that we could also take $\theta(\varepsilon) := \sigma^{\tilde{p}}(\varepsilon/3)$ as a modulus of convex regularity, where $\sigma(\varepsilon) := \frac{1}{2}\gamma(\frac{2\varepsilon}{3})$.

Lemma 4.8. *Suppose that $C \subset B_{b/2}(0)$ is a nonempty, closed, and convex subset of a B-convex space E. Let ϱ be defined as in (3.2) and let $\gamma \in \Gamma$. Given $\varepsilon > 0$, define $\theta(\varepsilon) := \gamma^{\tilde{p}}(\varepsilon/3^{\tilde{p}+1})$, where $\tilde{p} \in \mathbb{N}$ is such that $2^{\tilde{p}} \geq \varrho(\frac{2\varepsilon}{3b})$. Then, $\theta : (0,\infty) \to (0,\infty)$ is a modulus of convex regularity for the family of mappings $T : C \to E$ of type (γ).*

Proof. Since E is B-convex, we can define $\varrho_C(\varepsilon) := \varrho(\frac{\varepsilon}{(b/2)})$. By Remark 3.4, it then follows that ϱ_C is a modulus of convex approximation for C, and it is enough to apply Lemma 4.6 to conclude the result. ☐

The following is a direct consequence of the preceding lemma.

Lemma 4.9. *Suppose that $C \subset B_{b/2}(0)$ is a nonempty, bounded, closed, and convex subset of a uniformly convex Banach space E with a modulus η. Given $\varepsilon > 0$, define $\gamma(\varepsilon) := \frac{b}{2}\tilde{\eta}(\frac{4\varepsilon}{b})$, and $\theta(\varepsilon) := \gamma^{\tilde{p}}(\varepsilon/3^{\tilde{p}+1})$, where $\tilde{p} \in \mathbb{N}$ is such that $(2^{\tilde{p}})^{\frac{1-q}{q}} C_q \leq \frac{\varepsilon}{3b}$, and q is a Rademacher type with constant C_q for E (q and C_q can be determined in terms of η, as noted in Remark 3.5). Then, θ is a modulus of convex regularity for the family of all nonexpansive mappings from C to E.*

Proof. Given that uniformly convex Banach spaces are B-convex, and since by Remark 2.2 every nonexpansive mapping from C to E is of type (γ), it suffices to use Lemma 4.8, considering $\varrho(\varepsilon) := \lceil (\frac{2C_q}{\varepsilon})^{\frac{q}{q-1}} \rceil$. □

Remark 4.10. In lemma 4.9, due to (2.2), we can replace $\gamma(\varepsilon)$ with its lower bound function $\frac{\varepsilon}{2}\eta(\min\{2, \frac{\varepsilon}{b}\})$, using the monotonicity of γ.

Lemma 4.11. *Let C be a nonempty, compact and convex set with a I-modulus of total boundedness α in a Banach space E, and let $\gamma \in \Gamma$. Given $\varepsilon > 0$, define $\theta(\varepsilon) := \gamma^{\tilde{p}}(\varepsilon/3^{\tilde{p}+1})$, where $\tilde{p} \in \mathbb{N}$ is such that $2^{\tilde{p}} \geq \alpha(2\lceil 3/\varepsilon \rceil) + 1$. Then, θ is a modulus of convex regularity for the family of mappings $T : C \to E$ of type (γ).*

Proof. We deduce from Lemma 3.8 that $\varrho_C(\varepsilon) := \alpha(2\lceil 1/\varepsilon \rceil) + 1$ defines a modulus of convex approximation for C. Therefore, by using Lemma 4.6, we conclude the desired result. □

5. Rates of asymptotic regularity for Cesàro means

Definition 5.1. Let (X, d) be a metric space and $F : X \to \mathbb{R}$ be a mapping. $(x_n) \subseteq X$ is said to be a sequence of approximate zeros of F if $\lim_n F(x_n) = 0$. In particular, if $F(x) = d(x, Tx)$, where $T : X \to X$, then a sequence (x_n) of approximate zeros of F is said to be asymptotically regular and a rate of convergence of $(d(x_n, Tx_n))$ to 0 is called a rate of asymptotic regularity for (x_n). See [21, p. 458].

The subsequent summation formula is well-known. However, we provide a concise proof for completeness.

Lemma 5.2. *For each sequence (y_n) in E, and $m, n \in \mathbb{N}^*$, we have*

$$\frac{1}{n}\sum_{i=0}^{n-1} y_i = \frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{m}\sum_{j=0}^{m-1} y_{j+i} + \frac{1}{nm}\sum_{i=1}^{m-1}(m-i)(y_{i-1} - y_{n+i-1}).$$

Proof. First note that for all $j \in \mathbb{N}$,

$$\sum_{i=0}^{n-1}(y_i - y_{i+j}) = \sum_{i=1}^{j}(y_{i-1} - y_{n+i-1}).$$

Then,

$$\frac{1}{n}\sum_{i=0}^{n-1}(y_i - \frac{1}{m}\sum_{j=0}^{m-1} y_{j+i}) = \frac{1}{n}\sum_{i=0}^{n-1}(\frac{1}{m}\sum_{j=0}^{m-1}(y_i - y_{i+j})) = \frac{1}{nm}\sum_{j=1}^{m-1}\sum_{i=0}^{n-1}(y_i - y_{i+j})$$

$$= \frac{1}{nm}\sum_{j=1}^{m-1}\sum_{i=1}^{j}(y_{i-1} - y_{i+n-1}) = (y_0 - y_n) + \big((y_0 - y_n) + (y_1 - y_{n+1})\big) + \cdots +$$

$$+ \big((y_0 - y_n) + (y_1 - y_{n+1}) + \cdots + (y_{j-1} - y_{n+j-1})\big) + \cdots$$

$$+ \big((y_0 - y_n) + (y_1 - y_{n+1}) + \cdots + (y_{m-2} - y_{n+m-2})\big) = \frac{1}{nm}\sum_{j=1}^{m-1}(m-j)(y_{j-1} - y_{n+j-1}),$$

which is the desired equality. □

We require the following lemma.

Lemma 5.3. *For any sequence* $\{w_i\}$ *in* $C \subseteq B_{\frac{b}{2}}(0)$*, any nonexpansive* $T : C \to E$*, any* $n \in \mathbb{N}^*$*, and every* $0 < \varepsilon \leq 1$*,*

$$\frac{1}{n}\sum_{i=0}^{n-1} \|w_i - Tw_i\| \leq \varepsilon^2$$

implies that

$$\frac{1}{n}\sum_{i=0}^{n-1} w_i \in coF_\varepsilon(T) + B_{\varepsilon b}(0).$$

Proof. Put $I := \{i|\ 0 \leq i \leq n-1,\ w_i \in F_\varepsilon(T)\}$ and $J := \{0, 1, 2, \ldots, n-1\} \setminus I$. Note that by assumption $I \neq \emptyset$ since $\varepsilon^2 \leq \varepsilon \leq 1$. On the other hand, if $J = \emptyset$ the proof is complete. So, let $J \neq \emptyset$ and note that

$$\frac{|J|\varepsilon}{n} < \frac{1}{n}\sum_{i=0}^{n-1}\|w_i - Tw_i\| \leq \varepsilon^2 \Rightarrow \frac{|J|}{n} < \varepsilon.$$

Fix $k_0 \in I$ and write

$$\begin{aligned}\frac{1}{n}\sum_{i=0}^{n-1} w_i &= \frac{1}{n}\sum_{i\in I} w_i + \frac{1}{n}\sum_{j\in J} w_j \\ &= \Bigg((\sum_{i\in I}\frac{1}{n}w_i) + \frac{|J|}{n}w_{k_0}\Bigg) + \Bigg(\frac{1}{n}\sum_{j\in J}(w_j - w_{k_0})\Bigg).\end{aligned}$$

But

$$\frac{1}{n}\sum_{j\in J}(w_j - w_{k_0}) \leq \frac{|J|}{n}b < \varepsilon b.$$

Thus

$$\frac{1}{n}\sum_{i=0}^{n-1} w_i \in coF_\varepsilon(T) + B_{\varepsilon b}(0).$$

□

Lemma 5.4. *(cf. [14, Lemma 4.1]) For $\gamma \in \Gamma$ and $\tilde{q} : [0,\infty) \to [0,\infty)$ with $\tilde{q}(t) = \gamma^{-1}(3t) + t$, we have*

$$0 < \delta \leq \gamma^p(\frac{t}{4^p}) \Rightarrow \tilde{q}^p(\delta) \leq t.$$

Proof. For $p = 1$:

$$\tilde{q}(\delta) \leq \tilde{q}(\gamma(\frac{t}{4})) = \gamma^{-1}(3\gamma(\frac{t}{4})) + \gamma(\frac{t}{4}) \leq \gamma^{-1}(\gamma(\frac{3t}{4})) + \gamma(\frac{t}{4}) \leq \frac{3t}{4} + \frac{t}{4} = t.$$

Now we may apply induction. □

Lemma 5.5. *([14, Lemma 4.2]) For $q_n(\delta) := \gamma^{-1}(2\delta + \frac{b}{n}) + \delta$, it holds*

$$n \geq \frac{b}{\delta} \Rightarrow q_n^p(\delta) \leq \tilde{q}^p(\delta),\ \forall p \in \mathbb{N}.$$

The following property is mentioned in the proof of [10, Theorem 3.1]; since the proof is not included explicitly there, we provide a proof here for completeness.

Lemma 5.6. *Suppose that $\gamma \in \Gamma$ and $C \subset B_{b/2}(0)$ is a nonempty and convex subset of a Banach space E. Let $T : C \to E$ be a mapping of type (γ), and let $\{y_i\}$ be a sequence in C. Put $w_i^p = \frac{1}{p}\sum_{j=0}^{p-1} y_{j+i}$, for $p \in \mathbb{N}^*$. Then, we have*

$$\forall \delta > 0 \forall n \in N \forall p \in \mathbb{N}^*$$
$$\left(\forall i \in [0, p+n-2](\|y_{i+1} - Ty_i\| \leq \delta) \to \frac{1}{n}\sum_{i=0}^{n-1} \|w_{i+1}^p - Tw_i^p\| \leq {q_n}^{p-1}(\delta)\right), \tag{5.1}$$

where $q_n(\delta) := \gamma^{-1}(2\delta + \frac{b}{n}) + \delta$.

Proof. Let $\delta > 0$ and $n \in \mathbb{N}$. The proof is by induction on p. The base case, $p = 1$, is readily verified. Assume that the assertion (5.1) is true for p. We will then demonstrate its validity for $p+1$. Let $\|y_{i+1} - Ty_i\| \leq \delta$ for any $i \in [0, (p+1)+n-2]$. Since $w_i^{p+1} = \frac{p}{p+1}w_i^p + \frac{1}{p+1}y_{p+i}$ for all i, we have by the induction hypothesis and the convexity and increasing property of γ,

$$\begin{aligned}
\frac{1}{n}\sum_{i=0}^{n-1}\|w_{i+1}^{p+1}-Tw_i^{p+1}\| &\leq \frac{1}{n}\sum_{i=0}^{n-1}\|\frac{p}{p+1}Tw_i^p+\frac{1}{p+1}Ty_{p+i}-Tw_i^{p+1}\| \\
&+\frac{1}{n}\sum_{i=0}^{n-1}\frac{p}{p+1}\|w_{i+1}^p-Tw_i^p\|+\frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{p+1}\|y_{p+i+1}-Ty_{p+i}\| \\
&\leq \frac{1}{n}\sum_{i=0}^{n-1}\|\frac{p}{p+1}Tw_i^p+\frac{1}{p+1}Ty_{p+i}-T(\frac{p}{p+1}w_i^p+\frac{1}{p+1}y_{p+i})\|+{q_n}^{p-1}(\delta) \\
&\leq \frac{1}{n}\sum_{i=0}^{n-1}\gamma^{-1}(\|w_i^p-y_{p+i}\|-\|Tw_i^p-Ty_{p+i}\|)+{q_n}^{p-1}(\delta) \\
&\leq \gamma^{-1}(\frac{1}{n}\sum_{i=0}^{n-1}(\|w_i^p-y_{p+i}\|-\|Tw_i^p-Ty_{p+i}\|))+{q_n}^{p-1}(\delta)\ \ (\text{since } \gamma^{-1} \text{ is concave}) \\
&\leq \gamma^{-1}(\frac{1}{n}\sum_{i=0}^{n-1}(\|w_i^p-y_{p+i}\|-\|w_{i+1}^p-y_{p+i+1}\|) \\
&+\frac{1}{n}\sum_{i=0}^{n-1}(\|w_{i+1}^p-Tw_i^p\|+\|y_{p+i+1}-Ty_{p+i}\|))+{q_n}^{p-1}(\delta) \\
&\leq \gamma^{-1}(\frac{1}{n}(\|w_0^p-y_p\|-\|w_n^p-y_{p+n}\|)+\frac{1}{n}\sum_{i=0}^{n-1}\|w_{i+1}^p-Tw_i^p\|+\delta)+{q_n}^{p-1}(\delta) \\
&\leq \gamma^{-1}(\frac{b}{n}+{q_n}^{p-1}(\delta)+\delta)+{q_n}^{p-1}(\delta)\leq \gamma^{-1}(\frac{b}{n}+2{q_n}^{p-1}(\delta))+{q_n}^{p-1}(\delta)={q_n}^{p}(\delta).
\end{aligned}$$

□

Theorem 5.7. *Suppose that $\gamma \in \Gamma$ and $C \subset B_{b/2}(0)$ is a nonempty, closed and convex subset of a Banach space E. Let $\mathcal{F}$ be a family of nonexpansive mappings from C to E of type (γ) and suppose that $\theta : (0,\infty) \to (0,\infty)$ is a modulus of convex regularity for $\mathcal{F}$. Then, we have*

$$\forall\varepsilon>0\ \forall T\in\mathcal{F}\ \forall(y_i)\subset C\forall n\geq\tilde{\varphi}(\varepsilon,\gamma,b,\theta)$$
$$\Big(\forall i\in[0;n+p-2]\big(\|y_{i+1}-Ty_i\|\leq\Delta(\varepsilon)\big)\to\|\frac{1}{n}\sum_{i=0}^{n-1}y_i-T(\frac{1}{n}\sum_{i=0}^{n-1}y_i)\|\leq\varepsilon\Big), \tag{5.2}$$

where $\Delta(\varepsilon) := \min\{\frac{\varepsilon}{3}, \gamma^{p-1}(\frac{2\tau(\varepsilon)^2}{4^p})\}$, $p := \lceil\frac{2b}{\tau(\varepsilon)^2}\rceil$, $\tau(\varepsilon) := \min\{\theta(\frac{\varepsilon}{3}), \frac{\varepsilon}{6b}, 1\}$, *and*

$$\varphi(\varepsilon,\gamma,b,\theta) := \max\{\lceil\frac{b}{\Delta(\varepsilon)}\rceil, \lceil\frac{p}{\tau(\varepsilon)}\rceil\}.$$

Proof. Given $\varepsilon > 0$, let $\delta := \Delta(\varepsilon)$, where $\Delta(\varepsilon)$, p and $\tau(\varepsilon)$ are defined as above. Since $\delta \leq \gamma^{p-1}(\frac{2\tau(\varepsilon)^2}{4^p})$, Lemma 5.4 implies that $\tilde{q}^{p-1}(\delta) \leq \frac{\tau(\varepsilon)^2}{2}$. Fix some $n \geq \varphi(\varepsilon,\gamma,b,\theta)$. Then $n \geq \frac{b}{\delta}$ and hence in view of Lemma 5.5, we have ${q_n}^{p-1}(\delta) \leq \frac{\tau(\varepsilon)^2}{2}$. Let $T : C \to E$ be a mapping of type (γ) in $\mathcal{F}$, and let $\{y_n\}$ be a sequence in C such that $\|y_{i+1} - Ty_i\| \leq \delta$

for any $i \in [0; n+p-2]$. It suffices to prove that $\|\frac{1}{n}\sum_{i=0}^{n-1} y_i - T(\frac{1}{n}\sum_{i=0}^{n-1} y_i)\| \leq \varepsilon$. Put $w_i = \frac{1}{p}\sum_{j=0}^{p-1} y_{j+i}$. Using Lemma 5.6, we get

$$\frac{1}{n}\sum_{i=0}^{n-1} \|w_{i+1} - Tw_i\| \leq {q_n}^{p-1}(\delta) \leq \frac{\tau(\varepsilon)^2}{2}.$$

Given that $\|w_{i+1} - w_i\| \leq \frac{b}{p} \leq \frac{\tau(\varepsilon)^2}{2}$, we obtain

$$\frac{1}{n}\sum_{i=0}^{n-1} \|w_i - Tw_i\| \leq \frac{b}{p} + {q_n}^{p-1}(\delta) \leq \tau(\varepsilon)^2.$$

Since $\tau(\varepsilon) \leq 1$, by Lemma 5.3, we have

$$\frac{1}{n}\sum_{i=0}^{n-1} w_i \in coF_{\tau(\varepsilon)}(T) + B_{\tau(\varepsilon)b}(0) \subseteq coF_{\tau(\varepsilon)}(T) + B_{\frac{\varepsilon}{6}}(0).$$

Now, by Lemma 5.2, we write

$$\frac{1}{n}\sum_{i=0}^{n-1} y_i = \frac{1}{n}\sum_{i=0}^{n-1} w_i + \frac{1}{np}\sum_{i=1}^{p-1}(p-i)(y_{i-1} - y_{n+i-1}). \tag{5.3}$$

Since

$$\|\frac{1}{np}\sum_{i=1}^{p-1}(p-i)(y_{i-1} - y_{n+i-1})\| \leq \frac{p-1}{2n}b \leq \frac{pb}{2\varphi(\varepsilon,\gamma,b,\theta)} \leq \frac{\tau(\varepsilon)b}{2} \leq \frac{\varepsilon}{12},$$

utilizing (5.3), we obtain

$$\begin{aligned}\frac{1}{n}\sum_{i=0}^{n-1} y_i &\in coF_{\tau(\varepsilon)}(T) + B_{\frac{\varepsilon}{6}}(0) + B_{\frac{\varepsilon}{12}}(0) \subseteq coF_{\tau(\varepsilon)}(T) + B_{\frac{\varepsilon}{3}}(0) \\ &\subseteq coF_{\theta(\frac{\varepsilon}{3})}(T) + B_{\frac{\varepsilon}{3}}(0) \subseteq F_{\frac{\varepsilon}{3}}(T) + B_{\frac{\varepsilon}{3}}(0) \subseteq F_\varepsilon(T).\end{aligned}$$

This completes the proof. □

Remark 5.8. In the above proof, our aim was to show $\|\frac{1}{n}\sum_{i=0}^{n-1} y_i - T(\frac{1}{n}\sum_{i=0}^{n-1} y_i)\| \leq \varepsilon$. Now, if $\mathcal{F}$ consists of self-mappings on C, then without lose of generality, we could assume by stipulating that for $i \geq n-1$, $y_i := T^{i-n+1}(y_{n-1})$, i.e. $y_{i+1} = Ty_i$. Therefore, in this case, we may replace "$\forall i \in [0; n+p-2]$" with "$\forall i \in [0; n-1]$" in (5.2).

Remark 5.9. With the same assumptions as in Theorem 5.7, if moreover the mappings in $\mathcal{F}$ are self-mappings on C, then we may assume that $\varepsilon \leq b$, since for $\varepsilon > b$ the conclusion of Theorem 5.7 trivially holds for any n. For $\varepsilon \leq b$, we find that

$$\tau(\varepsilon) = \min\{\theta(\frac{\varepsilon}{3}), \frac{\varepsilon}{6b}, 1\} = \min\{\theta(\frac{\varepsilon}{3}), \frac{\varepsilon}{6b}\}. \tag{5.4}$$

Lemma 5.10. *With the same assumptions as in Theorem 5.7, if moreover the mappings in $\mathcal{F}$ are self-mappings on C, then the same result holds with $\tau(\varepsilon)$ being replaced with $\tilde{\tau}(\varepsilon) := \min\{\theta(\frac{\varepsilon}{3}), \frac{\varepsilon}{6b}, \varepsilon\}$, and when this occurs, we have $\varphi(\varepsilon,\gamma,b,\theta)= \lceil \frac{b}{\Delta(\varepsilon)} \rceil$.*

Proof. We note that the conclusion of Theorem 5.7 holds with $\tau(\varepsilon)$ being replaced with any $0 < \delta \leq \tau(\varepsilon)$, since $\gamma(t) \leq t$ for all $t \geq 0$. Since we consider self-mappings on C, we have by (5.4), $\tilde{\tau}(\varepsilon) \leq \tau(\varepsilon)$, for $\varepsilon \leq b$. On the other hand, since $\tilde{\tau}(\varepsilon) \leq \varepsilon$, we have

$$\Delta(\varepsilon) \leq \gamma^{p-1}(\frac{2\tilde{\tau}(\varepsilon)^2}{4^p}) \leq \frac{2\tilde{\tau}(\varepsilon)^2}{4^p} \leq \frac{2\varepsilon\tilde{\tau}(\varepsilon)}{4^p} \leq \frac{\varepsilon\tilde{\tau}(\varepsilon)}{p},$$

and thus

$$\frac{b}{\Delta(\varepsilon)} \geq \frac{b}{\varepsilon}(\frac{p}{\tilde{\tau}(\varepsilon)}) \geq \frac{p}{\tilde{\tau}(\varepsilon)}.$$

Therefore, $\varphi(\varepsilon, \gamma, b, \theta) = \max\{\lceil\frac{b}{\Delta(\varepsilon)}\rceil, \lceil\frac{p}{\tilde{\tau}(\varepsilon)}\rceil\} = \lceil\frac{b}{\Delta(\varepsilon)}\rceil$. □

We now extend the main result of [14] as a corollary of Theorem 5.7.

Corollary 5.11. *(See* [14, Theorem 4.1]*) Let E be a uniformly convex Banach space with a modulus η, and define $\gamma(t) := \frac{b}{2}\tilde{\eta}(\frac{4t}{b})$. Given $t > 0$, define $\theta(t) := \gamma^{\tilde{p}}(t/3^{\tilde{p}+1})$, where $\tilde{p} \in \mathbb{N}$ is such that $(2^{\tilde{p}})^{\frac{1-q}{q}} C_q \leq \frac{t}{3b}$, and q is a Rademacher type with constant C_q for E.*

With these explicit θ and γ, define $\tilde{\tau}(\varepsilon) := \min\{\theta(\frac{\varepsilon}{3}), \frac{\varepsilon}{6b}\}$, $\Delta(\varepsilon) := \min\{\frac{\varepsilon}{3}, \gamma^{p-1}(\frac{2\tilde{\tau}(\varepsilon)^2}{4^p})\}$, $p := \lceil\frac{2b}{\tilde{\tau}(\varepsilon)^2}\rceil$, and $\varphi(\varepsilon, \gamma, b, \theta)= \lceil\frac{b}{\Delta(\varepsilon)}\rceil$. Then for the family $\mathcal{F}$ of nonexpansive mappings from C to C, (5.2) holds. We may replace $\gamma(\varepsilon)$ with its lower bound function $\frac{\varepsilon}{2}\eta(\min\{2, \frac{\varepsilon}{b}\})$ in definitions of θ and Δ.

Proof. Since E is uniformly convex, by Remark 2.2 every nonexpansisve mapping on C is of type (γ) for $\gamma(t) := \frac{b}{2}\tilde{\eta}(\frac{4t}{b})$. Moreover, Lemma 4.9 shows that the defined θ is a modulus of convex regularity for nonexpansive mappings on C. Since the mappings in $\mathcal{F}$ are from C into C, we can apply Lemma 5.10 to deduce (5.2) for $\tilde{\tau}(\varepsilon) = \min\{\theta(\frac{\varepsilon}{3}), \frac{\varepsilon}{6b}, \varepsilon\}$ instead of $\tau(\varepsilon)$, and $\varphi(\varepsilon, \gamma, b, \theta)= \lceil\frac{b}{\Delta(\varepsilon)}\rceil$. We note also that in this case the inequality $\theta(\varepsilon) = \gamma^{\tilde{p}}(\varepsilon/3^{\tilde{p}+1}) \leq \varepsilon$ implies that $\tilde{\tau}(\varepsilon) = \min\{\theta(\frac{\varepsilon}{3}), \frac{\varepsilon}{6b}\}$. □

Remark 5.12. In Corollary 5.11, if we choose $\varepsilon \in (0, 1]$, and enlarge the defined $\tilde{p}$ for $\theta(\varepsilon/3) = \gamma^{\tilde{p}}(\frac{\varepsilon/3}{3^{\tilde{p}+1}})$ sufficiently large such that $C_q\tilde{p}^{(1-q)/q} \leq \frac{\varepsilon}{6b}$, then we may write $\tilde{\tau}(\varepsilon) = \theta(\varepsilon/3)$. In fact, since the elements of $\mathcal{F}$ are self-mappings, we may assume that $\varepsilon \leq b$, and then

$$\theta(\varepsilon/3) = \gamma^{\tilde{p}}(\frac{\varepsilon/3}{3^{\tilde{p}+1}}) \leq \frac{\varepsilon/3}{3^{\tilde{p}+1}} \leq 3^{-\tilde{p}} \leq \tilde{p}^{-\frac{1}{2}} \leq \tilde{p}^{\frac{1-q}{q}} \leq C_q\tilde{p}^{\frac{1-q}{q}} \leq \frac{\varepsilon}{6b}.$$

That is, $\tilde{\tau}(\varepsilon) = \min\{\theta(\frac{\varepsilon}{3}), \frac{\varepsilon}{6b}\} = \theta(\frac{\varepsilon}{3})$.

We now state the main result of this section.

Theorem 5.13. *Suppose that $\gamma \in \Gamma$ and $C \subset B_{b/2}(0)$ is a nonempty, closed and convex subset of a Banach space E. Let $\mathcal{F}$ be a family of nonexpansive mappings from C to E of type (γ) and suppose that $\theta : (0, \infty) \to (0, \infty)$ is a modulus of convex regularity for $\mathcal{F}$.*

Then, we have

$$\forall\varepsilon>0\;\forall M\in\mathbb{N}\;\forall T\in\mathcal{F}\;\forall(x_i)\subseteq C\;\forall n\geq\tilde{\Phi}(\varepsilon,\gamma,b,\theta,M)$$
$$\Big(\forall i\in[M;M+n+\hat{p}-2]\big(\|x_{i+1}-Tx_i\|\leq\hat{\Delta}(\varepsilon)\big)\to\|\frac{1}{n}\sum_{i=1}^{n}x_i-T(\frac{1}{n}\sum_{i=1}^{n}x_i)\|\leq\varepsilon\Big), \tag{5.5}$$

where $\hat{\Delta}(\varepsilon):=\min\{\frac{\varepsilon}{6},\gamma^{\hat{p}-1}(\frac{2\hat{\tau}(\varepsilon)^2}{4^{\hat{p}}})\}$, $\hat{p}:=\lceil\frac{2b}{\hat{\tau}(\varepsilon)^2}\rceil$, $\hat{\tau}(\varepsilon):=\min\{\theta(\frac{\varepsilon}{6}),\frac{\varepsilon}{12b},1\}$, *and*

$$\tilde{\Phi}(\varepsilon,\gamma,b,\theta,M):=\max\{\lceil\frac{b}{\hat{\Delta}(\varepsilon)}\rceil,\lceil\frac{\hat{p}}{\hat{\tau}(\varepsilon)}\rceil,\lceil\frac{4(M-1)b}{\varepsilon}\rceil\}.$$

Remark 5.14. Theorem 5.13 includes the case where a sequences (x_n) satisfies $\|x_{n+1}-Tx_n\|\to 0$. In fact, if $A:(0,\infty)\to\mathbb{N}$ is a convergence rate for $\|x_{n+1}-Tx_n\|\to 0$, it just suffices to take $M:=A(\hat{\Delta}(\varepsilon))$.

Proof. Given $\varepsilon>0$, $M\in\mathbb{N}$, $T\in\mathcal{F}$ and $n\geq\tilde{\Phi}(\varepsilon,\gamma,b,\theta,M)$, let (x_i) be a sequence in C such that $\|x_{i+1}-Tx_i\|\leq\hat{\Delta}(\varepsilon)$ for each $i\in[M;M+n+\hat{p}-2]$. Our aim is to show $\|\frac{1}{n}\sum_{i=1}^{n}x_i-T(\frac{1}{n}\sum_{i=1}^{n}x_i)\|\leq\varepsilon$. Set $y_i:=x_{M+i}$. From Theorem 5.7, since $\hat{\Delta}(\varepsilon)=\Delta(\varepsilon/2)$, $\hat{\tau}(\varepsilon)=\tau(\varepsilon/2)$, $\tilde{\Phi}(\varepsilon,\gamma,b,\theta,M)\geq\varphi(\varepsilon/2,\gamma,b,\theta)$, and

$$\|y_{i+1}-Ty_i\|=\|x_{M+i+1}-Tx_{M+i}\|\leq\hat{\Delta}(\varepsilon),$$

for all $i\in[0;n+\hat{p}-2]$, we deduce

$$\|\frac{1}{n}\sum_{i=M}^{n+M-1}x_i-T(\frac{1}{n}\sum_{i=M}^{n+M-1}x_i)\|=\|\frac{1}{n}\sum_{i=0}^{n-1}y_i-T(\frac{1}{n}\sum_{i=0}^{n-1}y_i)\|\leq\frac{\varepsilon}{2}.$$

Using this, along with $n\geq\tilde{\Phi}(\varepsilon,\gamma,b,\theta,M)\geq\lceil\frac{4(M-1)b}{\varepsilon}\rceil$, we obtain

$$\begin{aligned}
&\|\frac{1}{n}\sum_{i=1}^{n}x_i-T(\frac{1}{n}\sum_{i=1}^{n}x_i)\|\\
&=\|(\frac{1}{n}\sum_{i=1}^{n+M-1}x_i-\frac{1}{n}\sum_{i=n+1}^{n+M-1}x_i)-T(\frac{1}{n}\sum_{i=M}^{n+M-1}x_i)+T(\frac{1}{n}\sum_{i=M}^{n+M-1}x_i)-T(\frac{1}{n}\sum_{i=1}^{n}x_i)\|\\
&\leq\|\frac{1}{n}\sum_{i=1}^{M-1}x_i-\frac{1}{n}\sum_{i=n+1}^{n+M-1}x_i\|+\|\frac{1}{n}\sum_{i=M}^{n+M-1}x_i-T(\frac{1}{n}\sum_{i=M}^{n+M-1}x_i)\|\\
&+\|T(\frac{1}{n}\sum_{i=M}^{n+M-1}x_i)-T(\frac{1}{n}\sum_{i=1}^{n}x_i)\|\leq\frac{(M-1)b}{n}+\frac{\varepsilon}{2}+\|\frac{1}{n}\sum_{i=M}^{n+M-1}x_i-\frac{1}{n}\sum_{i=1}^{n}x_i\|\\
&=\frac{(M-1)b}{n}+\frac{\varepsilon}{2}+\|\frac{1}{n}\sum_{i=n+1}^{n+M-1}x_i-\frac{1}{n}\sum_{i=1}^{M-1}x_i\|\leq\frac{(M-1)b}{n}+\frac{\varepsilon}{2}+\frac{(M-1)b}{n}\\
&=\frac{2(M-1)b}{n}+\frac{\varepsilon}{2}\leq\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}$$

□

Remark 5.15. If $\mathcal{F}$ consists of self-mappings on C, then in view of Remark 5.8, we may replace "$\forall i \in [M; M+n+\hat{p}-2]$" with "$\forall i \in [M; n-1]$" in (5.5).

As a direct corollary, we obtain the following qualitative result which generalizes [10, Corollary 1.2].

Corollary 5.16. *Assume that C is a bounded, closed, and convex subset of a Banach space E, $T : C \to E$ is of type (γ), and (x_n) is a sequence in C with $\|x_{n+1} - Tx_n\| \to 0$. If $\{T\}$ is convex regular, then $\lim_n \|\frac{1}{n}\sum_{i=1}^{n} x_{i+k} - T(\frac{1}{n}\sum_{i=1}^{n} x_{i+k})\| = 0$, uniformly in $k \in \mathbb{N}$.*

Remark 5.17. It is worth reiterating that the modulus $\theta : (0,\infty) \to (0,\infty)$ in Theorem 5.13 may be computed depending on the parameters and moduli specific to the situation at hand:

(i) If, additionally, E is B-convex, then, in view of Lemma 4.8, $\theta : (0,\infty) \to (0,\infty)$ can be explicitly defined by $\theta(\varepsilon) := \gamma^{\tilde{p}}(\varepsilon/3^{\tilde{p}+1})$, where $\tilde{p} \in \mathbb{N}$ is such that $2^{\tilde{p}} \geq \varrho(\frac{2\varepsilon}{3b})$, and ϱ is defined as in (3.2).

(ii) If E is a uniformly convex Banach space with a modulus η, then, by Lemma 4.9, defining $\gamma(\varepsilon) := \frac{b}{2}\tilde{\eta}(\frac{4\varepsilon}{b})$, and $\theta(\varepsilon) := \gamma^{\tilde{p}}(\varepsilon/3^{\tilde{p}+1})$, where $\tilde{p} \in \mathbb{N}$ is such that $(2^{\tilde{p}})^{\frac{1-q}{q}} C_q \leq \frac{\varepsilon}{3b}$, and q is a Rademacher type with constant C_q for E (see Remark 3.5), θ is a modulus of convex regularity for the family of all nonexpansive mappings from C to E.

(iii) If C is compact with a I-modulus of total boundedness α in a Banach space E, then by Lemma 4.11, a modulus θ can be computed as $\theta(\varepsilon) := \gamma^{\tilde{p}}(\varepsilon/3^{\tilde{p}+1})$, where $\tilde{p} \in \mathbb{N}$ is such that $2^{\tilde{p}} \geq \alpha(2\lceil 3/\varepsilon \rceil) + 1$.

(iv) For a nonexpansive $T : C \to E$ with $Fix(T) \neq \emptyset$ admitting a modulus of regularity ϕ with respect to $Fix(T)$, we can define $\theta(\varepsilon) = \phi(\frac{\varepsilon}{2})$ due to Lemma 4.2.

By the item (ii) of the above, we get the following result, where the mapping γ is computed in terms of the other data.

Corollary 5.18. *Let E be a uniformly convex Banach space with a modulus η, and define $\gamma(t) := \frac{b}{2}\tilde{\eta}(\frac{4t}{b})$. Define $\theta(t) := \gamma^{\tilde{p}}(t/3^{\tilde{p}+1})$, where $\tilde{p} \in \mathbb{N}$ is such that $(2^{\tilde{p}})^{\frac{1-q}{q}} C_q \leq \frac{t}{3b}$, and q is a Rademacher type with constant C_q for E (defined in Remark 3.5 in terms of $\eta(1)$). With these explicit θ and γ, (5.5) holds for the family $\mathcal{F}$ of all nonexpansive mappings from C to E.*

As mentioned in Remark 4.10, we can replace $\gamma(\varepsilon)$ in Corollary 5.18 with its lower bound function $\frac{\varepsilon}{2}\eta(\min\{2, \frac{\varepsilon}{b}\})$.

Remark 5.19. With the same assumptions as in Theorem 5.13, if moreover the mappings in $\mathcal{F}$ are from C into C, then, using the idea of Lemma 5.10, the assertion (5.5) holds for $\hat{\tau}(\varepsilon) := \min\{\theta(\frac{\varepsilon}{6}), \frac{\varepsilon}{12b}, \frac{\varepsilon}{2}\}$ and $\tilde{\Phi}(\varepsilon, \gamma, b, \theta, M) := \max\{\lceil \frac{b}{\hat{\Delta}(\varepsilon)} \rceil, \lceil \frac{4(M-1)b}{\varepsilon} \rceil\}$.

In view of the above remark, we deduce the following result:

Corollary 5.20. *Let E be a strictly convex Banach space, and let $C \subseteq B_{b/2}(0)$ be a compact, convex set, and let $\mathcal{F}$ be the family of nonexpansive mappings from C into C. Define $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ as in Lemma 2.3. Let α be a I-modulus of total boundedness for C, and define $\theta : (0, \infty) \to (0, \infty)$ by $\theta(t) := \gamma^{\tilde{p}}(t/3^{\tilde{p}+1})$, where $\tilde{p} \in \mathbb{N}$ is such that $2^{\tilde{p}} \geq \alpha(2\lceil 3/t \rceil) + 1$. Then, given $\varepsilon > 0$, (5.5) holds for $\hat{\Delta}(\varepsilon) := \min\{\frac{\varepsilon}{6}, \gamma^{\hat{p}-1}(\frac{2\hat{\tau}(\varepsilon)^2}{4^{\hat{p}}})\}$, $\hat{p} := \lceil \frac{2b}{\hat{\tau}(\varepsilon)^2} \rceil$, $\hat{\tau}(\varepsilon) := \min\{\theta(\frac{\varepsilon}{6}), \frac{\varepsilon}{12b}, \frac{\varepsilon}{2}\}$, and*

$$\tilde{\Phi}(\varepsilon, \gamma, b, \theta, M) := \max\{\lceil \frac{b}{\hat{\Delta}(\varepsilon)} \rceil, \lceil \frac{4(M-1)b}{\varepsilon} \rceil\}.$$

Proof. Since C is compact and E is strictly convex, by Lemma 2.3, every nonexpansive $T : C \to C$ is of type (γ) for a fixed $\gamma \in \Gamma$. Now it suffices to combine Theorem 5.13 with Remark 5.19 and Lemma 4.11. □

At this stage, we present some applications of the above ergodic results in studying the ergodic properties of sequences generated by known iterative methods. The following result involves both the Halpern and Mann iteration methods.

Proposition 5.21. *Let $b, b', b'' > 0$. Suppose that $\gamma \in \Gamma$, $C \subseteq B_{b/2}(0)$ is a nonempty closed convex subset of a Banach space E, and $\theta : (0, \infty) \to (0, \infty)$ is a modulus of convex regularity for a family $\mathcal{F}$ of mappings of type (γ) from C to E. Let $\{\alpha_n\} \subseteq [0, 1]$ converging to 0 with a rate A. Given $\varepsilon > 0$, let*

$$\tilde{\Phi}(\varepsilon, \gamma, b, \theta, A(\frac{\hat{\Delta}(\varepsilon)}{(b'' + b' + b)})) := \max\{\lceil \frac{b}{\hat{\Delta}(\varepsilon)} \rceil, \lceil \frac{\hat{p}}{\hat{\tau}(\varepsilon)} \rceil, \lceil \frac{4(A(\frac{\hat{\Delta}(\varepsilon)}{(b''+b'+b)}) - 1)b}{\varepsilon} \rceil\}, \tag{5.6}$$

where $\hat{\Delta}(\varepsilon) := \min\{\frac{\varepsilon}{6}, \gamma^{\hat{p}-1}(\frac{2\hat{\tau}(\varepsilon)^2}{4^{\hat{p}}})\}$, $\hat{p} := \lceil \frac{2b}{\hat{\tau}(\varepsilon)^2} \rceil$, and $\hat{\tau}(\varepsilon) := \min\{\theta(\frac{\varepsilon}{6}), \frac{\varepsilon}{12b}, 1\}$.

Then, for every T in $\mathcal{F}$ satisfying dist$(0, T(C)) < b'$, and every $(x_n) \subset C$ satisfying

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n y_n, \tag{5.7}$$

where (y_n) is an arbitrary sequence in $B_{b''}(0)$, we have

$$\forall n \geq \tilde{\Phi}(\varepsilon, \gamma, b, \theta, A(\frac{\hat{\Delta}(\varepsilon)}{(b'' + b' + b)}))\left(\|\frac{1}{n}\sum_{i=1}^{n} x_i - T(\frac{1}{n}\sum_{i=1}^{n} x_i)\| \leq \varepsilon\right). \tag{5.8}$$

Proof. Let $\mathrm{dist}(0, T(C)) < b'$. Then, we may choose some $p \in T(C)$ such that $\|T(p)\| < b'$. We note that for $n \geq A(\hat{\Delta}(\varepsilon)/(b'' + b' + b))$, we have

$$\begin{aligned} \|x_{n+1} - Tx_n\| &= \alpha_n\|y_n - Tx_n\| \leq \alpha_n(\|y_n\| + \|Tp\| + \|Tp - Tx_n\|) \\ &\leq \alpha_n(b'' + b' + b) \leq \frac{\hat{\Delta}(\varepsilon)}{(b'' + b' + b)}(b'' + b' + b) = \hat{\Delta}(\varepsilon). \end{aligned} \tag{5.9}$$

The result follows now from Theorem 5.13. □

Let E be uniformly convex Banach space with a modulus η. Let $T : E \to E$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$, and let $f : E \to E$ be a contraction with coefficient $0 < \ell < 1$. Let $\{\alpha_n\} \subseteq [0,1]$ converge to 0 with a rate A, and let (x_n) be generated under the Moudafi viscosity version of Halpern's iteration

$$x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n f(x_n), \tag{5.10}$$

$x_0 \in E$. We demonstrate that (5.10) constitutes a special case of (5.7). Fix some $p_0 \in Fix(T)$. We observe that

$$\begin{aligned}
\|x_{n+1} - p_0\| &\leq (1-\alpha_n)\|x_n - p_0\| + \alpha_n\|f(x_n) - f(p_0)\| + \alpha_n\|f(p_0) - p_0\| \\
&\leq (1-\alpha_n(1-\ell))\|x_n - p_0\| + \alpha_n(1-\ell)\frac{\|f(p_0)-p_0\|}{1-\ell} \\
&\leq \max\{\|x_n - p_0\|, \frac{\|f(p_0)-p_0\|}{1-\ell}\}.
\end{aligned}$$

Then, it is easy to show by induction that for all $n \in \mathbb{N}$,

$$\|x_n - p_0\| \leq \max\{\|x_0 - p_0\|, \frac{\|f(p_0)-p_0\|}{1-\ell}\} = L. \tag{5.11}$$

In particular,

$$\begin{aligned}
\|f(x_n)\| &\leq \|f(x_n) - f(p_0)\| + \|f(p_0)\| \\
&\leq \ell\|x_n - p_0\| + \|f(p_0)\| \leq \ell L + \|f(p_0)\| \\
&\leq \ell L + (1-\ell)L + \|p_0\| = L + \|p_0\|.
\end{aligned}$$

Thus, for $b \geq 2(L + \|p_0\|)$, the sequences $(f(x_n))$ and (x_n) are clearly contained in $B_{b/2}(0)$. We take $C := B_{b/2}(0)$. Since the fixed point p_0 belongs to C, we have also $\mathrm{dist}(0, T(C)) \leq b/2$. Now, considering $T|_C : C \to E$, we observe that (5.10) is a specific case of (5.7). Since E is uniformly convex, defining $\gamma(t) := \frac{b}{2}\tilde{\eta}(\frac{4t}{b})$, and $\theta(t) := \gamma^{\tilde{p}}(t/3^{\tilde{p}+1})$, where $\tilde{p} \in \mathbb{N}$ is such that $(2^{\tilde{p}})^{\frac{1-q}{q}} C_q \leq \frac{t}{3b}$, and q is a Rademacher type with constant C_q for E (see Remark 3.5), it follows by Lemma 4.9 that $T|_C : C \to E$ is of type (γ) and θ is a modulus of convex regularity for the family of all nonexpansive mappings from C to E. Therefore, applying Proposition 5.21, we conclude that the statement (5.8) holds for the Halpern iteration (5.10) with the bound defined in (5.6), where here $b' = b'' = b/2$.

Therefore, we have obtained a rate of asymptotic regularity for the ergodic averages of (x_n) generated by Halpern's method, under the sole condition $\alpha_n \to 0$ on (α_n). We recall that two necessary conditions on (α_n) for convergence of Halpern's method are $\alpha_n \to 0$ and $\sum \alpha_n = \infty$, and these conditions are not even sufficient to guarantee the convergence.

For mappings from C into C, in view of Remark 5.19, we may obtain the following corollary of Proposition 5.21:

Corollary 5.22. *Suppose that $\gamma \in \Gamma$, $C \subseteq B_{b/2}(0)$ is a nonempty closed convex subset of a Banach space E, and $\theta : (0,\infty) \to (0,\infty)$ is a modulus of convex regularity for a* *family $\mathcal{F}$ of self-mappings of type (γ) on C. Let $\{\alpha_n\} \subseteq [0,1]$ converging to 0 with a rate A. Given $\varepsilon > 0$, let*

$$\tilde{\Phi}(\varepsilon, \gamma, b, \theta, A(\frac{\hat{\Delta}(\varepsilon)}{b})) := \max\{\lceil \frac{b}{\hat{\Delta}(\varepsilon)} \rceil, \lceil \frac{4(A(\frac{\hat{\Delta}(\varepsilon)}{b}) - 1)b}{\varepsilon} \rceil\},$$

where $\hat{\Delta}(\varepsilon) := \min\{\frac{\varepsilon}{6}, \gamma^{\hat{p}-1}(\frac{2\hat{\tau}(\varepsilon)^2}{4^{\hat{p}}})\}$, $\hat{p} := \lceil \frac{2b}{\hat{\tau}(\varepsilon)^2} \rceil$, and $\hat{\tau}(\varepsilon) := \min\{\theta(\frac{\varepsilon}{6}), \frac{\varepsilon}{12b}, \frac{\varepsilon}{2}\}$.

Then, for every T in $\mathcal{F}$, each arbitrary sequence $(y_n) \subset C$, and $x_0 \in C$, defining (x_n) by

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n y_n, \tag{5.12}$$

we have

$$\forall n \geq \tilde{\Phi}(\varepsilon, \gamma, b, \theta, A(\frac{\hat{\Delta}(\varepsilon)}{b}))\Big(\|\frac{1}{n}\sum_{i=1}^{n} x_i - T(\frac{1}{n}\sum_{i=1}^{n} x_i)\| \leq \varepsilon\Big).$$

6. Cauchy Rates of Cesàro means

Theorem 6.1. *Suppose $\gamma \in \Gamma$, $C \subseteq B_{b/2}(0)$ is a nonempty, closed, and convex subset of a Banach space E, $T : C \to C$ is a nonexpansive mapping with $Fix(T) \neq \emptyset$, and T^n is of type (γ) for all n. Let $\phi : (0,\infty) \to (0,\infty)$ be a modulus of regularity for T with respect to $Fix(T)$. Then, we have*

$$\forall \varepsilon > 0 \forall x \in C \forall g, h : \mathbb{N} \to \mathbb{N} \exists N \leq \tilde{\Phi}(\varepsilon, b, h, g, \phi, \gamma) \forall m, n \in [N; N + g(N)] \forall k \leq h(N)$$
$$\big(\|\frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x - \frac{1}{m}\sum_{i=0}^{m-1} T^{i+k}x\| \leq \varepsilon\big),$$

where $\tilde{\Phi}(\varepsilon, b, h, g, \phi, \gamma) = \Phi(\tilde{f}, \tilde{k}, \tilde{c}) + \tilde{u}(\Phi(\tilde{f}, \tilde{k}, \tilde{c}))$, $\Phi(\tilde{f}, \tilde{k}, \tilde{c}) := \tilde{f}^{(\tilde{c}\cdot 2^{\tilde{k}}+1)}(0)$, $\tilde{u}(n) := \lceil \frac{4(n_0-1)b}{\varepsilon} \rceil + \lceil \frac{8nb}{\varepsilon} \rceil$, $n_0 := \varphi(\phi(\frac{\varepsilon}{9}), \gamma, b, \theta)$, $\theta(t) := \phi(\frac{t}{2})$, $f(n) := \max\{(g+h)(n), (g+h)(n+\tilde{u}(n))\} + \tilde{u}(n)$, $\tilde{k} = \max\{0, \lceil -\ln(\gamma_{n_0}(\frac{\varepsilon}{8}))/\ln(2) \rceil\}$, $\tilde{f}(n) := n + f(n)$, $\tilde{c} := \lceil \frac{bn_0(n_0-1)}{2} \rceil$, (γ_n) is defined by recursion as in (2.9) or (2.8), $\varphi(t, \gamma, b, \theta) = \lceil \frac{b}{\Delta(t)} \rceil$, $\Delta(t) := \min\{\frac{t}{3}, \gamma^{p-1}(\frac{2\tilde{\tau}(t)^2}{4^p})\}$ for $p := \lceil \frac{2b}{\tilde{\tau}(t)^2} \rceil$, and $\tilde{\tau}(t) := \min\{\theta(\frac{t}{3}), \frac{t}{6b}, t\}$.

In particular, the Cesàro means $\frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x$ converge strongly and uniformly in $k \in \mathbb{N}$. Moreover, this limit is a fixed point of T which does not depend on k.

Proof. Given $\varepsilon > 0$, in view of Theorem 5.7, Lemmas 5.10 and 4.2, for $n_0 := \varphi(\phi(\frac{\varepsilon}{9}), \gamma, b, \theta)$, where $\theta(t) := \phi(\frac{t}{2})$, we have for all $x \in C$ and $k \in \mathbb{N}$,

$$\|\frac{1}{n_0}\sum_{j=0}^{n_0-1} T^{j+k}x - T(\frac{1}{n_0}\sum_{j=0}^{n_0-1} T^{j+k}x)\| \leq \phi(\frac{\varepsilon}{9}). \tag{6.1}$$

Now, let $g, h : \mathbb{N} \to \mathbb{N}$ be given. Define

$$\tilde{u}(n) := \lceil \frac{4(n_0-1)b}{\varepsilon} \rceil + \lceil \frac{8nb}{\varepsilon} \rceil,$$

and $f(n) := \max\{(g+h)(n), (g+h)(n+\tilde{u}(n))\} + \tilde{u}(n)$. Choose $\tilde{k} \in \mathbb{N}$ with $2^{-\tilde{k}} \leq \gamma_{n_0}(\frac{\varepsilon}{8})$ (e.g., $\tilde{k} = \max\{0, \lceil -\ln(\gamma_{n_0}(\frac{\varepsilon}{8}))/\ln(2) \rceil\}$). Define $\tilde{f}(n) := n + f(n)$, $\tilde{c} := \lceil \frac{bn_0(n_0-1)}{2} \rceil$, and

$$\Phi(\tilde{f}, \tilde{k}, \tilde{c}) := \tilde{f}^{(\tilde{c}\cdot 2^{\tilde{k}}+1)}(0).$$

Now let $x \in C$ and define $\alpha_{j,\tilde{j},n} := \|T^{\tilde{j}+n}x - T^{j+n}x\|$ and $a_n := \sum_{j,\tilde{j}=0}^{n_0-1} \alpha_{j,\tilde{j},n}$. Obviously, (a_n) is nonincreasing and for all n, we have $a_n \leq \frac{bn_0(n_0-1)}{2} \leq \tilde{c}$. Moreover, it is easy to check that

$$\forall m, n\big(m \geq n \to \max_{0\leq j,\tilde{j}\leq n_0-1} (\alpha_{j,\tilde{j},n} - \alpha_{j,\tilde{j},m}) \leq a_n - a_m\big). \tag{6.2}$$

Using the metastability of bounded monotone sequences (see, [21, corollary 2.28]), we have

$$\exists \tilde{N} \leq \Phi(\tilde{f}, \tilde{k}, \tilde{c})\forall n, m \in [\tilde{N}; \tilde{N} + f(\tilde{N})]\big(|a_n - a_m| < 2^{-\tilde{k}} \leq \gamma_{n_0}(\frac{\varepsilon}{8})\big). \tag{6.3}$$

Taking $N := \tilde{N} + \tilde{u}(\tilde{N})$, we note that $\tilde{N} \leq N$ and

$$\tilde{N} + f(\tilde{N}) = \tilde{N} + \tilde{u}(\tilde{N}) + (h + g)(\tilde{N} + \tilde{u}(\tilde{N})) = N + (h + g)(N).$$

That is,

$$[N; N + h(N) + g(N)] \subseteq [\tilde{N}; \tilde{N} + f(\tilde{N})]. \tag{6.4}$$

Then, $N = \tilde{N}+\tilde{u}(\tilde{N}) \leq \tilde{\Phi}(\varepsilon, b, h, g, \phi, \gamma) := \Phi(\tilde{f}, \tilde{k}, \tilde{c})+\tilde{u}(\Phi(\tilde{f}, \tilde{k}, \tilde{c}))$, since $\tilde{u}$ is monotone. Using the definition of the modulus of regularity ϕ in (6.1) for $k = \tilde{N}$, we choose $f_0 \in Fix(T)$ such that

$$\|\frac{1}{n_0}\sum_{j=0}^{n_0-1} T^{j+\tilde{N}}x - f_0\| \leq \frac{\varepsilon}{8}. \tag{6.5}$$

From Lemma 5.2, we may write for all $k \in N$ and $n \in \mathbb{N}^*$,

$$\frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x = \frac{1}{n}\sum_{i=0}^{n-1} \frac{1}{n_0}\sum_{j=0}^{n_0-1} T^{j+i+k}x + \frac{1}{nn_0}\sum_{i=1}^{n_0-1}(n_0 - i)(T^{i+k-1}x - T^{n+i+k-1}x).$$

Then, for any $n \in [N; N+g(N)]$ and $k \in [0; h(N)]$, we have by (6.2) and (6.5), and Lemma 2.5 applied to $T^{i+k-\tilde{N}}$, $\lambda_i := 1/n_0$ and $x_j := T^{j+\tilde{N}}x$,

$$\begin{aligned}
&\|\frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x - f_0\| = \|\frac{1}{n}\sum_{i=0}^{\tilde{N}-1}\big(\frac{1}{n_0}\sum_{j=0}^{n_0-1} T^{j+i+k}x - f_0\big) \\
&+ \frac{1}{n}\sum_{i=\tilde{N}}^{n-1}\big(\frac{1}{n_0}\sum_{j=0}^{n_0-1} T^{j+i+k}x - T^{i+k-\tilde{N}}(\frac{1}{n_0}\sum_{j=0}^{n_0-1} T^{j+\tilde{N}}x)\big) \\
&+ \frac{1}{n}\sum_{i=\tilde{N}}^{n-1}\big(T^{i+k-\tilde{N}}(\frac{1}{n_0}\sum_{j=0}^{n_0-1} T^{j+\tilde{N}}x) - f_0\big) + \frac{1}{nn_0}\sum_{i=1}^{n_0-1}(n_0-i)(T^{i+k-1}x - T^{n+i+k-1}x)\| \\
&\le \frac{1}{n}\sum_{i=0}^{\tilde{N}-1} b + \frac{1}{n}\sum_{i=\tilde{N}}^{n-1}\gamma_{n_0}^{-1}\big(\max_{0\le j,\tilde{j}\le n_0-1}(\|T^{j+\tilde{N}}x - T^{\tilde{j}+\tilde{N}}x\| - \|T^{j+i+k}x - T^{\tilde{j}+i+k}x\|)\big) \\
&+ \frac{1}{n}\sum_{i=\tilde{N}}^{n-1}\|\big(T^{i+k-\tilde{N}}(\frac{1}{n_0}\sum_{j=0}^{n_0-1} T^{j+\tilde{N}}x) - f_0\big)\| + \frac{1}{nn_0}\sum_{i=1}^{n_0-1}(n_0-i)b \\
&\le \frac{\tilde{N}b}{n} + \frac{1}{n}\sum_{i=\tilde{N}}^{n-1}\gamma_{n_0}^{-1}\big(\max_{0\le j,\tilde{j}\le n_0-1}(\alpha_{j,\tilde{j},\tilde{N}} - \alpha_{j,\tilde{j},i+k})\big) + \|\frac{1}{n_0}\sum_{j=0}^{n_0-1} T^{j+\tilde{N}}x - f_0\| + \frac{(n_0-1)b}{2n} \\
&\le \frac{\tilde{N}b}{n} + \frac{1}{n}\sum_{i=\tilde{N}}^{n-1}\gamma_{n_0}^{-1}(a_{\tilde{N}} - a_{i+k}) + \frac{\varepsilon}{8} + \frac{(n_0-1)b}{2n}.
\end{aligned} \tag{6.6}$$

Moreover,

$$\frac{1}{n}\sum_{i=\tilde{N}}^{n-1}\gamma_{n_0}^{-1}(a_{\tilde{N}} - a_{i+k}) \le \gamma_{n_0}^{-1}(\gamma_{n_0}(\frac{\varepsilon}{8})) = \frac{\varepsilon}{8}, \tag{6.7}$$

by (6.3) and $\tilde{N} \le \tilde{N}+k \le i+k \le n-1+k \le N+g(N)+h(N) \le \tilde{N}+f(\tilde{N})$, in view of (6.4). Now, by (6.6) and (6.7), since

$$\frac{4(n_0-1)b}{\varepsilon} + \frac{8\tilde{N}b}{\varepsilon} \le \tilde{u}(\tilde{N}) \le N \le n,$$

we arrive at

$$\|\frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x - f_0\| \le \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{2},$$

and consequently, for any $m, n \in [N; N+g(N)]$ and $k \in [0; h(N)]$, we obtain

$$\|\frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x - \frac{1}{m}\sum_{i=0}^{m-1} T^{i+k}x\| \le \|\frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x - f_0\| + \|\frac{1}{m}\sum_{i=0}^{m-1} T^{i+k}x - f_0\| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This completes the proof of the first part. The claim that the limit of $(\frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x)$ is a fixed point of T follows from (6.1). The fact that this fixed point does not depend on

k is also clear: let $f_k \in Fix(T)$ be the limit of $(\frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x)$. From

$$\|\frac{1}{n}\sum_{i=0}^{n-1} T^{i+1}x - \frac{1}{n}\sum_{i=0}^{n-1} T^i x\| \leq \frac{b}{n} \overset{n\to\infty}{\to} 0$$

we obtain that $f_0 = f_1$ and so - by induction - $f_i = f_j$ for all i, j. $\square$

Theorem 6.2. *Suppose $\gamma \in \Gamma$, $C \subseteq B_{b/2}(0)$ is a nonempty, closed, and convex subset of a Banach space E, $T : C \to C$ is a nonexpansive mapping with $Fix(T) \neq \emptyset$, and T^n is of type (γ) for all n. Let $\phi : (0, \infty) \to (0, \infty)$ be a modulus of regularity for T with respect to $Fix(T)$. Let $x \in C$ and let for each $k \in N$, $\rho_k : \mathbb{R}^*_+ \to \mathbb{N}$ be a rate of convergence for the nonincreasing sequence $(\|T^{n+k}x - T^n x\|)_{n\geq 0}$. Then, we have*

$$\forall \varepsilon > 0 \forall n, m \geq \Omega(\varepsilon, (\rho_k), \varphi) \forall k \big(\|\frac{1}{n}\sum_{i=0}^{n-1} T^{i+k}x - \frac{1}{m}\sum_{i=0}^{m-1} T^{i+k}x\| \leq \varepsilon\big),$$

where $\Omega(\varepsilon, (\rho_k), \varphi) = \max\{\tilde{N}, \lceil\frac{8\tilde{N}b}{\varepsilon}\rceil, \lceil\frac{4(n_0-1)b}{\varepsilon}\rceil\}$, $n_0 := \varphi(\phi(\frac{\varepsilon}{9}), \gamma, b, \theta)$, $\theta(t) := \phi(\frac{t}{2})$, $\tilde{N} := \max\{\rho_1(\frac{1}{2}\gamma_{n_0}(\frac{\varepsilon}{8})), \cdots, \rho_{n_0}(\frac{1}{2}\gamma_{n_0}(\frac{\varepsilon}{8}))\}$, (γ_n) is defined by recursion as in (2.9) or (2.8), and $\varphi(t, \gamma, b, \theta)$ as in Theorem 6.1.

Proof. The proof is similar to the previous one. We only mention the differences. Given $\varepsilon > 0$, (6.1) holds similarly for $n_0 := \varphi(\phi(\frac{\varepsilon}{9}), \gamma, b, \theta)$, and for all $x \in C$ and $k \in \mathbb{N}$. Fix $x \in C$ and take

$$\tilde{N} := \max\{\rho_1(\frac{1}{2}\gamma_{n_0}(\frac{\varepsilon}{8})), \cdots, \rho_{n_0}(\frac{1}{2}\gamma_{n_0}(\frac{\varepsilon}{8}))\}. \tag{6.8}$$

Thus

$$\forall n, m \geq \tilde{N} \forall \tilde{k} \leq n_0 \big(\|T^{n+\tilde{k}}x - T^n x\| - \|T^{m+\tilde{k}}x - T^m x\| \leq \gamma_{n_0}(\frac{\varepsilon}{8})\big). \tag{6.9}$$

Using (6.1), we deduce the existence of some $f_0 \in Fix(T)$ such that

$$\|\frac{1}{n_0}\sum_{j=0}^{n_0-1} T^{j+\tilde{N}}x - f_0\| \leq \frac{\varepsilon}{8}, \tag{6.10}$$

where $\tilde{N}$ is as chosen in (6.8). Now, choose some

$$n \geq \max\{\tilde{N}, \lceil\frac{8\tilde{N}b}{\varepsilon}\rceil, \lceil\frac{4(n_0-1)b}{\varepsilon}\rceil\}.$$

Then, similarly to (6.6), using (6.9) and (6.10), we obtain

$$
\begin{aligned}
&\|\frac{1}{n}\sum_{i=0}^{n-1}T^{i+k}x-f_0\| \\
&\le \frac{1}{n}\sum_{i=0}^{\tilde{N}-1}b+\frac{1}{n}\sum_{i=\tilde{N}}^{n-1}\gamma_{n_0}^{-1}\big(\max_{0\le j,\tilde{j}\le n_0-1}(\|T^{j+\tilde{N}}x-T^{\tilde{j}+\tilde{N}}x\|-\|T^{j+i+\tilde{N}}x-T^{\tilde{j}+i+\tilde{N}}x\|)\big) \\
&+\frac{1}{n}\sum_{i=\tilde{N}}^{n-1}\|\big(T^{i+k-\tilde{N}}(\frac{1}{n_0}\sum_{j=0}^{n_0-1}T^{j+\tilde{N}}x)-f_0\big)\|+\frac{1}{nn_0}\sum_{i=1}^{n_0-1}(n_0-i)b \\
&\le \frac{\tilde{N}b}{n}+\frac{1}{n}\sum_{i=\tilde{N}}^{n-1}\gamma_{n_0}^{-1}(\gamma_{n_0}(\frac{\varepsilon}{8}))+\|\frac{1}{n_0}\sum_{j=0}^{n_0-1}T^{j+\tilde{N}}x-f_0\|+\frac{(n_0-1)b}{2n}\le\frac{\varepsilon}{8}+\frac{\varepsilon}{8}+\frac{\varepsilon}{8}+\frac{\varepsilon}{8}=\frac{\varepsilon}{2}.
\end{aligned}
\tag{6.11}
$$

This completes the proof. □

Theorem 6.3. *Suppose $\gamma\in\Gamma$, $C\subseteq B_{b/2}(0)$ is a nonempty, closed, and convex subset of a Banach space E, $T:C\to C$ is a nonexpansive mapping with $Fix(T)\ne\emptyset$, and T^n is of type (γ) for all n. Let $\phi:(0,\infty)\to(0,\infty)$ be a modulus of regularity for T with respect to $Fix(T)$. Let $x\in C$ and let $\rho:\mathbb{R}_+^*\to\mathbb{N}$ be a rate of asymptotic regularity for (T^nx). Then, (T^nx) is convergent with a Cauchy rate $\widetilde{\Omega}(\Omega,\rho)$, where*

$$\widetilde{\Omega}(\Omega,\rho):=\rho(\frac{\varepsilon}{2(\Omega(\varepsilon/4,(\rho_k),\varphi)+1)}),$$

$\rho_k(t):=\rho(t/k)$, and $\Omega(t,(\rho_k),\varphi)$ is defined as in Theorem 6.2.

Proof. Defining $\rho_k(t):=\rho(t/k)$, for each $k\in N$, we deduce that $\rho_k:\mathbb{R}_+^*\to\mathbb{N}$ be a rate of convergence for the nonincreasing sequence $(\|T^{n+k}x-T^nx\|)_{n\ge0}$. Now, given $\varepsilon>0$, utilizing Theorem 6.2, we have

$$\forall n,m\ge M_0:=\Omega(\varepsilon/4,(\rho_k),\varphi)\forall k\big(\|\frac{1}{n}\sum_{i=0}^{n-1}T^{i+k}x-\frac{1}{m}\sum_{i=0}^{m-1}T^{i+k}x\|\le\frac{\varepsilon}{4}\big). \tag{6.12}$$

Moreover, by the property of ρ, we have

$$\forall k\ge\rho(\frac{\varepsilon}{2(M_0+1)})\big(\|T^kx-T^{k+1}x\|\le\frac{\varepsilon}{2(M_0+1)}\big). \tag{6.13}$$

In view of (6.12), we may assume $\lim_n\frac{1}{n}\sum_{i=0}^{n-1}T^{i+k}x=y$, uniformly in $k\in\mathbb{N}$. Using (6.12) and (6.13), we have thus for any $k\ge\rho(\frac{\varepsilon}{2(M_0+1)})$,

$$
\begin{aligned}
\|T^kx-y\| &\le \|T^kx-\frac{1}{M_0}\sum_{i=0}^{M_0-1}T^{i+k}x\|+\|\frac{1}{M_0}\sum_{i=0}^{M_0-1}T^{i+k}x-y\| \\
&\le \frac{1}{M_0}\sum_{i=0}^{M_0-1}\|T^kx-T^{i+k}x\|+\frac{\varepsilon}{4}\le\frac{1}{M_0}\sum_{i=0}^{M_0-1}i\frac{\varepsilon}{2(M_0+1)}+\frac{\varepsilon}{4}\le\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\frac{\varepsilon}{2}.
\end{aligned}
$$

Therefore, we have proved that

$$\forall n,m\ge\rho(\frac{\varepsilon}{2(M_0+1)})\big(\|T^nx-T^mx\|\le\varepsilon\big).$$

□

Qualitatively, Theorem 6.3 is a witness for this: for a regular map T, $(T^n x)$ is asymptotically regular iff $(T^n x)$ is convergent!

Remark 6.4. The concept of asymptotic regularity for a mapping $T : C \to C$ appeared in Browder and Petryshyn [8]: T is said to be asymptotically regular at $x \in C$, if $T^{n+1}x - T^n x \to 0$ as $n \to \infty$. They showed [8, Theorem 5] that if E is a uniformly convex Banach space and T is an averaged nonexpansive mapping on a closed convex subset of E with $Fix(T) \neq \emptyset$, then T is asymptotically regular at each point of C.

Remark 6.5. Strongly nonexpansive (SNE) mappings [11] on a subset of a Banach space that possess a fixed point represent an interesting class of asymptotically regular mappings. For SNE-mappings, a rate $\rho : \mathbb{R}^*_+ \to \mathbb{N}$ as in Theorem 6.3 was computed in [23] as $\rho(d, \varepsilon) := \lceil d/\omega(d, \varepsilon)\rceil$ depending on an SNE-modulus ω and $d \geq \|x - p\|$ for some $p \in Fix(T)$. In particular, for λ-firmly nonexpansive mappings in uniformly convex spaces, ω is computed in [23, Prop. 2.17] as $\omega(d, \varepsilon) := \lambda(1 - \lambda)\eta(\varepsilon/c) \cdot \varepsilon$, where η is a modulus of uniform convexity. Note that ρ does not depend on x other than via d.

We recall that a mapping $T : C \to E$ is said to be affine if $T(\alpha x + (1 - \alpha)y) = \alpha Tx + (1 - \alpha)Ty$ for all $x, y \in C$ and $\alpha \in [0, 1]$.

Borzdyński and Wiśnicki [7, Lemma 4.1] showed that an averaged affine self-map T defined on a convex and bounded subset C of a Banach space is uniformly asymptotic regular; i.e., $\lim_n \sup_{x\in C} \|T^{n+1}x - T^n x\| = 0$. More precisely, they showed that if $T = \frac{1}{2}(I + S)$, where S is affine self-mapping on C, then $\sup_{x\in C} \|T^n x - T^{n+1}x\| \leq a_n \cdot \mathrm{diam}C$, where

$$a_n := \begin{cases} \frac{1}{2^{n+1}}\binom{n}{k}, & n = 2k, \\ \frac{1}{2^{n+1}}\frac{(2k)!}{(k!)^2}, & n = 2k - 1. \end{cases} \tag{6.14}$$

By the Stirling's approximation, it follows that $a_n \to 0$. Let $\alpha : \mathbb{R}^*_+ \to \mathbb{N}$ be a rate of convergence for $a_n \to 0$. Then $\rho(\varepsilon) := \alpha(\varepsilon/D)$, for $D \geq \mathrm{diam}C$, is a rate of asymptotic regularity for $(T^n x)$ for any $x \in C$. Consequently, the rate ρ in Theorem 6.3 can be explicitly computed for $T = \frac{1}{2}(I + S)$.

Example 6.6. Define the closed and convex subset $C = \prod_{i=1}^{\infty}[0, 1/2]$ in $E = (\ell^\infty, \|\cdot\|_+)$, where $\|(x_i)\|_+ := \sup_{i\in\mathbb{N}^*} |x_{2i-1}| + \sup_{i\in\mathbb{N}^*} |x_{2i}|$. Then, C is not compact in the Banach space E. Define $T : C \to C$ by

$$T(x_1, x_2, \dots, x_{2k-1}, x_{2k}, \dots) = (x_1, \frac{1}{2}x_2^2, \dots, x_{2k-1}, \frac{1}{2}x_{2k}^2, \dots).$$

Then, for $0 < \delta \leq 1/2$, we have

$$F_\delta(T) = [0, \frac{1}{2}] \times [0, 1 - (1 - 2\delta)^{\frac{1}{2}}] \times [0, \frac{1}{2}] \times [0, 1 - (1 - 2\delta)^{\frac{1}{2}}] \times \cdots, \tag{6.15}$$

and

$$Fix(T) = [0, \frac{1}{2}] \times \{0\} \times [0, \frac{1}{2}] \times \{0\} \times \cdots .$$

It is straightforward to show that the function $\phi : (0,\infty) \to (0,\infty)$ defined as

$$\phi(\varepsilon) := \frac{1}{2}(1 - ((1-\varepsilon)^+)^2),$$

where $(1-\varepsilon)^+ := \max\{0, (1-\varepsilon)\}$, is a modulus of regularity for T with respect to $Fix(T)$. In fact, it suffices to note that, in view of (6.15), $\|x - Tx\| < \phi(\varepsilon)$ implies that $\mathrm{dist}(x, Fix(T)) < 1 - (1-2\phi(\varepsilon))^{\frac{1}{2}}$, and the inequality $1 - (1-2\phi(\varepsilon))^{\frac{1}{2}} \leq \varepsilon$ is equivalent to $\phi(\varepsilon) \leq \frac{1}{2}(1 - ((1-\varepsilon)^+)^2)$.

Furthermore, T^n is of type (γ) for all n, where γ is the identity map. Let $x = (x_i), y = (y_i) \in C$ and $0 \leq \lambda \leq 1$. Then, the equality

$$\begin{aligned}
&\|\lambda Tx + (1-\lambda)Ty - T(\lambda x + (1-\lambda)y)\| \\
&\quad = \frac{1}{2}\sup_{i\in\mathbb{N}^*} |\lambda x_{2i}^2 + (1-\lambda)y_{2i}^2 - (\lambda x_{2i} + (1-\lambda)y_{2i})^2| = \sup_{i\in\mathbb{N}^*} \frac{\lambda(1-\lambda)}{2}(x_{2i} - y_{2i})^2
\end{aligned}$$

and the inequality

$$\|Tx - Ty\| = \frac{1}{2}\sup_{i\in\mathbb{N}^*} |x_{2i}^2 - y_{2i}^2| + \sup_{i\in\mathbb{N}^*} |x_{2i-1} - y_{2i-1}| \leq \frac{1}{2}\sup_{i\in\mathbb{N}^*} |x_{2i} - y_{2i}| + \sup_{i\in\mathbb{N}^*} |x_{2i-1} - y_{2i-1}|$$

imply

$$\begin{aligned}
&\|\lambda Tx + (1-\lambda)Ty - T(\lambda x + (1-\lambda)y)\| + \|Tx - Ty\| \\
&\quad \leq \sup_{i\in\mathbb{N}^*} \frac{\lambda(1-\lambda)}{2}(x_{2i} - y_{2i})^2 + \frac{1}{2}\sup_{i\in\mathbb{N}^*} |x_{2i} - y_{2i}| + \sup_{i\in\mathbb{N}^*} |x_{2i-1} - y_{2i-1}| \\
&\quad \leq \sup_{i\in\mathbb{N}^*} |x_{2i} - y_{2i}| + \sup_{i\in\mathbb{N}^*} |x_{2i-1} - y_{2i-1}| = \|x - y\|.
\end{aligned}$$

That is, T is of type (id), and therefore T^n is of type (id) for each n, by [34, Corollary 2.4] stating that if T is a mapping of type (γ), then T^n is of type $(n\gamma(\frac{t}{n}))$.

Finally, we obtain a rate of convergence for Cesàro means of affine mappings in terms of a rate of regularity. While this result can be obtained directly as a corollary of Theorem 6.2, we provide a direct proof to avoid reliance on other rates.

Proposition 6.7. *Suppose that $C \subseteq B_{b/2}(0)$ is a nonempty, closed and convex subset of a Banach space E, and let $T : C \to C$ be an affine nonexpansive mapping with $Fix(T) \neq \emptyset$. Let $\phi : (0,\infty) \to (0,\infty)$ be a modulus of regularity for T. Set $S_n := \frac{1}{n}\sum_{i=0}^{n-1} T^i$. Then, we have*

$$\forall \varepsilon > 0 \forall x \in C \forall m, n \geq \left\lceil \frac{2(\lceil \frac{b}{\phi(\varepsilon/8)} \rceil - 1)b}{\varepsilon} \right\rceil \big(\|S_n(x) - S_m(x)\| \leq \varepsilon\big).$$

Moreover, $\lim S_n(x) \in Fix(T)$.

Proof. Given $\varepsilon > 0$, let $n_0 := \lceil \frac{b}{\phi(\varepsilon/8)} \rceil$. Since T is affine, it follows that $T(\frac{1}{n}\sum_{i=0}^{n-1} T^i x) = \frac{1}{n}\sum_{i=0}^{n-1} T^i x + \frac{1}{n}(T^n x - x)$. Thus, we have, for all $x \in C$ and $n \geq n_0$,

$$\|T(S_n(x)) - S_n(x)\| \leq \frac{b}{n} \leq \phi(\frac{\varepsilon}{8}).$$

and consequently,

$$\mathrm{dist}(S_n(x), f(T)) \leq \frac{\varepsilon}{8},$$

Let $x \in C$, and pick $f \in F(T)$ with $\|S_{n_0}(x) - f\| \leq \varepsilon/4$. From Lemma 5.2, we may write for any $n \geq 1$,

$$S_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} T^i x = \frac{1}{n}\sum_{i=0}^{n-1}\frac{1}{n_0}\sum_{j=0}^{n_0-1} T^{j+i}x + \frac{1}{nn_0}\sum_{i=1}^{n_0-1}(n_0 - i)(T^{i-1}x - T^{n+i-1}x).$$

Using the latter equation and the affiness of T, we have for $n \geq \lceil \frac{2(n_0-1)b}{\varepsilon} \rceil$,

$$\begin{aligned}\|S_n(x) - f\| &\leq \frac{1}{n}\sum_{i=0}^{n-1}\|T^i(\frac{1}{n_0}\sum_{j=0}^{n_0-1} T^j x) - f\| + \frac{(n_0-1)b}{2n} \\ &\leq \|\frac{1}{n_0}\sum_{j=0}^{n_0-1} T^j x - f\| + \frac{(n_0-1)b}{2n} \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.\end{aligned}$$

Therefore, for all $n, m \geq \lceil \frac{2(n_0-1)b}{\varepsilon} \rceil$, $\|S_n(x) - S_m(x)\| \leq \|S_n(x) - f\| + \|S_m(x) - f\| \leq \varepsilon$. ☐

Remark 6.8. The existence of a modulus of regularity ϕ for T in the above results is not redundant. In fact, for $C := \prod_{i\in\mathbb{N}}[0,1] \subseteq \ell^\infty$, which is not compact in ℓ^∞, we define $T : C \to C$ by

$$T(x_1, x_2, \cdots) := (\lambda_1 x_1, \lambda_2 x_2, \cdots),$$

where (λ_i) is a sequence with $0 < \lambda_i < 1$ and $\lim_{i\to\infty} \lambda_i = 1$. It is easy to check that T is affine (and hence of type (γ)), $Fix(T) = \{0\}$ and

$$F_{\frac{1}{n}}(T) = \prod_{i=1}^{\infty}([0, \frac{1}{n(1-\lambda_i)}] \cap [0,1]).$$

Hence T does not admit a modulus of regularity. For $x = (1, 1, \cdots) \in C$, since $\lim_{i\to\infty} \lambda_i = 1$, we see that

$$\|\frac{1}{n}\sum_{i=0}^{n-1} T^i x\| = \sup_k(\frac{1}{n}\sum_{i=0}^{n-1}\lambda_k^i) = 1.$$

Consequently, $(S_n(x))$ fails to converge since a limit would have to be a fixed point of T while $Fix(T) = \{0\}$.

7. Effective rates for averaged Mann type iterations

7.1. Rates of asymptotic regularity and convergence.

Definition 7.1. A function $D : \mathbb{N} \to \mathbb{N}$ is a rate of divergence for a series $\sum_{i=1}^{\infty} a_i$, where $(a_i) \subset \mathbb{R}_+$, if

$$\forall n \in \mathbb{N}\left(\sum_{i=0}^{D(n)} a_i \geq n\right).$$

Remark 7.2. Let $(\alpha_i) \subset [0,1]$ and $\sum_{i=0}^{\infty}(1-\alpha_i) = \infty$. It is known that $1 - x \leq e^{-x}$ for all $x \in \mathbb{R}$. Thus $\alpha_i \leq e^{-(1-\alpha_i)}$, and hence

$$\prod_{i=m}^{m+n} \alpha_i = \alpha_m \alpha_{m+1} \cdots \alpha_{m+n} \leq e^{-\sum_{i=m}^{m+n}(1-\alpha_i)}.$$

Now suppose that D is a rate of divergence for $\sum_{i=0}^{\infty}(1-\alpha_i) = \infty$. Let $\varepsilon > 0$ and $m \in \mathbb{N}$. Then, for each $n \in \mathbb{N}$ with $n + m \geq D(|\lceil \ln(\varepsilon^{-1}) + \sum_{i=0}^{m-1}(1-\alpha_i)\rceil|)$, we have

$$\prod_{i=m}^{m+n} \alpha_i \leq e^{-\sum_{i=0}^{m+n}(1-\alpha_i)} e^{\sum_{i=0}^{m-1}(1-\alpha_i)} \leq e^{-|\lceil \ln(\varepsilon^{-1}) + \sum_{i=0}^{m-1}(1-\alpha_i)\rceil|} e^{\sum_{i=0}^{m-1}(1-\alpha_i)}$$

$$\leq e^{-(\ln(\varepsilon^{-1}) + \sum_{i=0}^{m-1}(1-\alpha_i))} e^{\sum_{i=0}^{m-1}(1-\alpha_i)} = e^{-\ln(\varepsilon^{-1})} = \varepsilon.$$

Thus, $\tilde{D} : \mathbb{N} \times (0,\infty) \to \mathbb{N}$ defined by

$$\tilde{D}(m,\varepsilon) := |D(|\lceil \ln(\varepsilon^{-1}) + \sum_{i=0}^{m-1}(1-\alpha_i)\rceil|) - m|$$

is a rate of convergence for $\prod_{i=0}^{\infty} \alpha_i = 0$ in the sense that:

$$\forall \varepsilon > 0 \forall m \in \mathbb{N}\Big(\prod_{i=m}^{m+\tilde{D}(m,\varepsilon)} \alpha_i \leq \varepsilon \Big). \tag{7.1}$$

C.f. [26, Lemma 5.2].

We will need the following lemma.

Lemma 7.3. *Suppose that $\gamma \in \Gamma$ and $C \subset B_{b/2}(0)$ is a nonempty, closed and convex subset of a Banach space E. Let $\mathcal{F}$ be a family of mappings from C to C of type (γ) having a modulus $\theta : (0,\infty) \to (0,\infty)$ of convex regularity. Let $(\alpha_n) \subset [0,1)$ be such that $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ with a rate of divergence D. Given $\varepsilon > 0$, set $\tilde{\tau}(\varepsilon) := \min\{\theta(\frac{\varepsilon}{3}), \frac{\varepsilon}{6b}, \varepsilon\}$, $p(\varepsilon) := \lceil \frac{2b}{\tilde{\tau}(\varepsilon)^2} \rceil$, and $\Delta(\varepsilon) = \min\{\frac{\varepsilon}{3}, \gamma^{p(\varepsilon)-1}(\frac{2\tilde{\tau}(\varepsilon)^2}{4^{p(\varepsilon)}})\}$. Then for any $x_1 \in C$ and $T : C \to C$ in $\mathcal{F}$, defining*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) S_n(x_n), \ \ (S_n := \frac{1}{n}\sum_{i=0}^{n-1} T^i)$$

we have

$$\|Tx_n - x_n\| \leq \varepsilon,$$

$\forall n \geq \omega(b,\gamma,\theta,D,\varepsilon) := |D(|\lceil \ln(\frac{4b}{\varepsilon}) + \sum_{i=0}^{\lceil \frac{b}{\Delta(\theta(\frac{\varepsilon}{2}))}\rceil - 1}(1-\alpha_i)\rceil|) - \lceil \frac{b}{\Delta(\theta(\frac{\varepsilon}{2}))}\rceil| + 1.$

Proof. Given $\varepsilon > 0$, let $N := \lceil \frac{b}{\Delta(\theta(\frac{\varepsilon}{2}))} \rceil$. In view of Theorem 5.7 and Lemma 5.10 applied to $y_i := T^i x$, we have

$$\forall n \geq N \forall x \in C\Big(\|S_n(x) - T(S_n(x))\| \leq \theta(\frac{\varepsilon}{2})\Big).$$

Then, picking

$$k > \tilde{D}(N, \frac{\varepsilon}{4b}) = |D(|\lceil \ln(\frac{4b}{\varepsilon}) + \sum_{i=0}^{N-1}(1-\alpha_i)\rceil|) - N|,$$

we have by Remark 7.2,

$$\prod_{i=N}^{N+k-1} \alpha_i \leq \frac{\varepsilon}{4b}.$$

We write

$$x_{N+k} = (\prod_{i=N}^{N+k-1} \alpha_i) x_N + (1 - \prod_{i=N}^{N+k-1} \alpha_i) y_k,$$

where

$$y_k = \frac{1}{1 - \prod_{i=N}^{N+k-1} \alpha_i} (\sum_{j=N}^{N+k-2} (\prod_{i=j+1}^{N+k-1} \alpha_i)(1-\alpha_j) S_j(x_j) + (1 - \alpha_{N+k-1}) S_{N+k-1}(x_{N+k-1})).$$

From

$$\sum_{j=N}^{N+k-2} ((\prod_{i=j+1}^{N+k-1} \alpha_i)(1-\alpha_j)) + (1-\alpha_{N+k-1}) = 1 - \prod_{i=N}^{N+k-1} \alpha_i,$$

we have

$$y_k \in co\{S_n(x_n) : \ n \geq N\} \subseteq coF_{\theta(\frac{\varepsilon}{2})}(T) \subseteq F_{\frac{\varepsilon}{2}}(T).$$

Morover,

$$\|x_{N+k} - y_k\| = (\prod_{i=N}^{N+k-1} \alpha_i) \|x_N - y_k\| \leq \frac{\varepsilon}{4b} b = \frac{\varepsilon}{4}.$$

Consequently,

$$\begin{aligned} \|Tx_{N+k} - x_{N+k}\| &\leq \|Tx_{N+k} - Ty_k\| + \|Ty_k - y_k\| + \|y_k - x_{N+k}\| \\ &\leq 2\|x_{N+k} - y_k\| + \|Ty_k - y_k\| \leq 2\frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

Theorem 7.4. *Suppose that $\gamma \in \Gamma$, and $C \subseteq B_{b/2}(0)$ is a nonempty, closed and convex subset of a Banach space E. Let $(\alpha_n) \subseteq [0,1)$ be such that $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ with a rate of divergence D. Let $T : C \to C$ be a mapping of type (γ) with $Fix(T) \neq \emptyset$. Let $\phi : (0,\infty) \to (0,\infty)$ be a modulus of regularity for T, and set $\theta(t) := \phi(\frac{t}{2})$. Then, defining $x_{n+1} = \alpha_n x_n + (1-\alpha_n) S_n x_n$, (x_n) is asymptotically regular with a rate of $\omega(b,\gamma,\theta,D,\varepsilon)$, as well as (x_n) converges to a fixed point of T with a rate of convergence $\omega(b,\gamma,\theta,D,\phi(\frac{\varepsilon}{2}))$.*

Proof. By Lemma 4.2, θ is a modulus of convex regularity for $\mathcal{F} = \{T\}$. Thus the rate ω defined in Lemma 7.3 is also a rate for asymptotic regularity of (x_n). Now, it suffices to apply [28, Theorem 4.1] to deduce the second part of the theorem, since (x_n) is Fejér monotone w.r.t. $Fix(T)$. □

Open question: Without the presence of a rate of divergence D, is it possible to obtain a rate of metastability from the above result?

By Lemma 2.3, for the case where C is additionally compact and E is strictly convex, every nonexpansive mapping $T : C \to C$ is of type (γ), for some $\gamma \in \Gamma$. Hence, we conclude the following corollary of Theorem 7.4:

Corollary 7.5. *Let $C \subseteq B_{b/2}(0)$ be a nonempty, compact and convex subset of a strictly convex Banach space E. Let $(\alpha_n) \subseteq [0,1)$ be such that $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ with a rate of divergence D. Let $T : C \to C$ be a nonexpansive mapping with a modulus of regularity $\phi : (0,\infty) \to (0,\infty)$. Define $\gamma \in \Gamma$ as in Lemma 2.3, and set $\theta(t) := \phi(\frac{t}{2})$. Then, defining $x_{n+1} = \alpha_n x_n + (1-\alpha_n)S_n x_n$, (x_n) is asymptotically regular with a rate of $\omega(b,\gamma,\theta,D,\varepsilon)$, as well as (x_n) converges to a fixed point of T with a rate of convergence $\omega(b,\gamma,\theta,D,\phi(\frac{\varepsilon}{2}))$.*

7.2. **Rates of metastability per uniform Fejér monotonicity.**

Definition 7.6. [27, Definition 4.6]. Let $T : C \to C$, $(x_n) \subseteq C$, $AF_k := F_{1/k}(T)$ and $F := Fix(T)$. Then (x_n) is said to be uniformly Fejér monotone w.r.t. F if, for all $r, n, m \in \mathbb{N}$,

$$\exists k \in \mathbb{N} \forall p \in C \left(p \in AF_k \to \forall l \leq m (\|x_{n+l} - p\| < \|x_n - p\| + \frac{1}{r+1}) \right)$$

and any upper bound $\chi(n,m,r)$ of " $\exists k \in \mathbb{N}$ " is called a modulus of (x_n) being uniformly Fejér monotone w.r.t. F.

Lemma 7.7. *Let $T : C \to C$ be nonexpansive, $(\alpha_n) \subseteq [0,1]$, and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)S_n x_n, \ \ (S_n := \frac{1}{n}\sum_{i=0}^{n-1} T^i).$$

Then, $\chi(n,m,r) = m(n+m)(r+1)$ is a modulus of (x_n) being uniformly Fejér monotone w.r.t. $Fix(T)$.

Proof. Let $p \in C$. Since $\|T^i p - p\| \leq \|T^i p - T^{i-1}p\| + \cdots + \|Tp - p\| \leq i\|Tp-p\|$, we get

$$\|S_n(p) - p\| = \|\frac{1}{n}\sum_{i=0}^{n-1}(T^i p - p)\| \leq \frac{1}{n}\sum_{i=0}^{n-1}\|T^i p - p\| \leq \frac{1}{n}\sum_{i=0}^{n-1} i\|Tp-p\| = \frac{1}{2}(n-1)\|Tp-p\|,$$

and thus

$$\begin{aligned}\|x_{n+1} - p\| &\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|S_n(x_n) - S_n(p)\| + (1-\alpha_n)\|S_n(p) - p\| \\ &\leq \|x_n - p\| + \|S_n(p) - p\| \leq \|x_n - p\| + \frac{1}{2}(n-1)\|Tp - p\|.\end{aligned}$$

Now, by induction, we get

$$\begin{aligned}\|x_{n+m} - p\| &\leq \|x_n - p\| + \frac{1}{2}((n-1) + n + \cdots + (n+m-2))\|Tp-p\| \\ &= \|x_n - p\| + \frac{1}{2}(mn + \frac{(m-2)(m-1)}{2} - 1)\|Tp - p\| \\ &< \|x_n - p\| + m(n+m)\|Tp-p\|.\end{aligned}$$

This completes the proof. ◻

As mentioned, for a sequence (x_n) in E, any bound $\varphi : (0,\infty) \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ such that

$$\forall k \in \mathbb{N} \ \forall g : \mathbb{N} \to \mathbb{N} \ \exists N \leq \varphi(\varepsilon, g) \ \forall i,j \in [N, N+g(N)](\|x_i - x_j\| \leq \frac{1}{k+1})$$

is called a rate of metastability (see, e.g., [24]).

In the following we study the metastability based on [27].

Definition 7.8. [27] An approximate F-point bound for (x_n) is any $\Phi : \mathbb{N} \to \mathbb{N}$ satisfying

$$\forall k \in \mathbb{N}\, \exists N \leq \Phi(k)\ (x_N \in AF_k).$$

If Φ is an approximate F-point bound for (x_n), then

$$\Phi^M : \mathbb{N} \to \mathbb{N},\ \ \Phi^M(k) := \max\{\Phi(m) |\ m \leq k\},$$

is monotone nondecreasing and again an approximate F-point bound for (x_n).

In our situation, we deal with an asymptotically regular sequence (x_n), $AF_k = F_{1/k}(T)$, and Φ is even a rate of asymptotic regularity for (x_n).

Remark 7.9. Under the assumptions of Lemma 7.3, $\Phi(k) := \omega(b, \gamma, \theta, D, 1/k)$ is a rate of asymptotic regularity (or, an approximate $Fix(T)$-bound) for (x_n) which is independent of T.

Theorem 7.10. *Suppose that E is a Banach space, and $C \subseteq B_{b/2}(0)$ is a nonempty convex compact subset of E with a I-modulus of total boundedness α. Let $\gamma \in \Gamma$ and define $\theta : (0, \infty) \to (0, \infty)$ by $\theta(t) := \gamma^{\tilde{p}}(t/3^{\tilde{p}+1})$, where $\tilde{p} \in \mathbb{N}$ is such that $2^{\tilde{p}} \geq \alpha(2\lceil 3/t \rceil) + 1$. Let $(\alpha_n) \subseteq [0, 1)$ and $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$ with a rate of divergence D. Let $k \in \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$. Define*

$$\overline{\Phi}(k) := \max\{\omega(b, \gamma, \theta, D, 1/m) |\ m \leq k\}$$

with ω defined in Lemma 7.3. Define

$$\chi(n, m, k) := m(n + m)(k + 1),\ \ \chi_g(n, k) := \chi(n, g(n), k),$$

$$\begin{aligned}
\chi_g^M(n, k) &:= \max\{\chi_g(i, k) |\ i \leq n\},\\
P &:= \beta(4k + 3),\\
\Psi_0(0, k, g, \overline{\Phi}) &:= 0,\\
\Psi_0(n + 1, k, g, \overline{\Phi}) &:= \overline{\Phi}(\chi_g^M(\Psi_0(n, k, g, \overline{\Phi}), 4k + 3)).
\end{aligned}$$

Now define $\Psi(k, g, b, \gamma, \beta) := \Psi_0(P, k, g, \overline{\Phi})$. Then

$$\exists N \leq \Psi(k, g, b, \gamma, \beta)\ \forall i, j \in [N, N + g(N)]\ (\|x_i - x_j\| \leq \frac{1}{k + 1}),$$

for any sequence (x_n) defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S_n x_n, \tag{7.2}$$

where $T : C \to C$ is a mapping of type (γ).

Proof. The proof is a direct consequence of [27, Theorem 5.1], in view of Remark 7.9, and Lemmas 4.11, 7.3 and 7.7. In fact, by Lemma 4.11, $\theta(t) := \gamma^{\tilde{p}}(t/3^{\tilde{p}+1})$, where $\tilde{p} \in \mathbb{N}$ is such that $2^{\tilde{p}} \geq \alpha(2\lceil 3/t \rceil) + 1$, is a modulus of convex regularity for the family of mappings on C of type (γ). Now, we may apply Lemma 7.3 to get $\omega(b, \gamma, \theta, D, 1/k)$ with the property $\|Tx_n - x_n\| \le 1/k$ for all $n \ge \omega(b, \gamma, \theta, D, 1/k)$, hence for all $n \ge \overline{\Phi}(k)$, and for any (x_n) defined in (7.2) with any $T : C \to C$ of type (γ). On the other hand, $\chi(n, m, k)$ is a modulus of (x_n) being uniformly Fejér monotone as shown in Lemma 7.7. The result follows now as a consequence of [27, Theorem 5.1]. □

Remark 7.11. If E is additionally strictly convex, we may replace $\gamma(t)$ in Theorem 7.10 with the one defined in Lemma 2.3, and in this case the result is valid for every nonexpansive mapping.

7.3. **Rates of metastability per a modulus of uniqueness.** Let $T : C \to C$ be a mapping having at most one fixed point; i.e.,

$$\forall p_1, p_2 \in C\big(p_1 = Tp_1 \wedge p_2 = Tp_2 \to p_1 = p_2\big).$$

Then, in view of [22], T is said to have uniformly at most one fixed point with modulus of uniqueness $\tilde{\omega} : (0, \infty) \to (0, \infty)$, if

$$\forall \varepsilon > 0 \forall p_1, p_2 \in C\big(\|p_1 - Tp_1\|, \|p_2 - Tp_2\| \le \tilde{\omega}(\varepsilon) \to \|p_1 - p_2\| \le \varepsilon\big). \tag{7.3}$$

If T is continuous and C is compact there always exists such a modulus $\tilde{\omega}$. See [22] for more details. In [29], a class of nonexpansive operators for which a modulus of uniqueness can be computed is described.

Using a modulus of uniqueness, we obtain the following result without any assumption on the coefficients (α_n).

Theorem 7.12. *Suppose that $\gamma \in \Gamma$ and $C \subset B_{b/2}(0)$ is a nonempty, closed and convex subset of a Banach space E. Let $\mathcal{F}$ be a family of mappings from C to C of type (γ) having a modulus of convex regularity $\theta : (0, \infty) \to (0, \infty)$ and a modulus of uniqueness $\tilde{\omega} : (0, \infty) \to (0, \infty)$. Let $(\alpha_n) \subset [0, 1)$ be arbitrary. Given $\varepsilon > 0$, set $\tilde{\tau}(\varepsilon) := \min\{\theta(\frac{\varepsilon}{3}), \frac{\varepsilon}{6b}, \varepsilon\}$, $p(\varepsilon) := \lceil \frac{2b}{\tilde{\tau}(\varepsilon)^2} \rceil$, and $\Delta(\varepsilon) = \min\{\frac{\varepsilon}{3}, \gamma^{p(\varepsilon)-1}(\frac{2\tilde{\tau}(\varepsilon)^2}{4^{p(\varepsilon)}})\}$. Then for any $x_0 \in C$ and T in $\mathcal{F}$, defining*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S_n(x_n),$$

we have

$$\forall \varepsilon > 0 \forall g : \mathbb{N} \to \mathbb{N} \exists N \le \Psi(\varepsilon, g, b, \gamma, \theta, \beta)\ \forall i, j \in [N, N + g(N)]\ \big(\|x_i - x_j\| \le \varepsilon\big),$$

where $\Psi(\varepsilon, g, b, \gamma, \theta, \beta) = \lceil \frac{b}{\Delta(\theta(\tilde{\omega}(\frac{\varepsilon}{4})))} \rceil + \tilde{h}^{(2^{\tilde{k}}+1)}(0)$ for $\tilde{k} \in N$ with $2^{-\tilde{k}} < \frac{\varepsilon}{2b}$, $\tilde{h}(n) := n + h(n)$, and $h(n) := \Delta(\theta(\tilde{\omega}(\frac{\varepsilon}{4}))) + \max\{g(n), g(n + \Delta(\theta(\tilde{\omega}(\frac{\varepsilon}{4}))))\}$.

Remark 7.13. When T has a fixed point, the modulus of uniqueness $\tilde{\omega}$ becomes also a modulus of regularity for T. Therefore, according to Lemma 4.2, $\theta(t) := \tilde{\omega}(t/2)$ is a modulus of convex regularity.

Proof. Given $\varepsilon > 0$, let $\tilde{\varepsilon} := \tilde{\omega}(\frac{\varepsilon}{4})$ and $M := \lceil \frac{b}{\Delta(\theta(\tilde{\varepsilon}))} \rceil$. In view of Theorem 5.7 and Lemma 5.10, we have

$$\forall n \ge M \forall x \in C\Bigg(\|S_n(x) - T(S_n(x))\| \le \theta(\tilde{\varepsilon})\Bigg).$$

We write

$$x_{M+k} = (\prod_{i=M}^{M+k-1} \alpha_i) x_M + (1 - \prod_{i=M}^{M+k-1} \alpha_i) y_k,$$

where

$$y_k = \frac{1}{1 - \prod_{i=M}^{M+k-1} \alpha_i} (\sum_{j=M}^{M+k-2} ((\prod_{i=j+1}^{M+k-1} \alpha_i)(1-\alpha_j) S_j(x_j)) + (1 - \alpha_{M+k-1}) S_{M+k-1}(x_{M+k-1})).$$

We have then

$$y_k \in co\{S_n(x_n) : \ n \geq M\} \subseteq coF_{\theta(\tilde{\varepsilon})}(T) \subseteq F_{\tilde{\varepsilon}}(T) = F_{\tilde{\omega}(\frac{\varepsilon}{4})}(T).$$

Hence, $\|y_i - y_j\| \leq \frac{\varepsilon}{4}$, for all i, j. Now, choose an arbitrary $g : \mathbb{N} \to \mathbb{N}$. Choose $\tilde{k} \in N$ such that $2^{-\tilde{k}} \leq \frac{\varepsilon}{2b}$. Define $h(n) := M + \max\{g(n), g(n+M)\}$, $\tilde{h}(n) := n + h(n)$, and

$$\Phi(h, \tilde{k}) := \tilde{h}^{(2^{\tilde{k}}+1)}(0).$$

Let $a_k := \prod_{i=M}^{M+k-1} \alpha_i$ for $k \geq 1$. At this stage, using the metastability of bounded monotone sequences (see, e.g., [21, corollary 2.28] for details), we can choose some $n_0 \in \mathbb{N}$ such that

$$n_0 \leq \Phi(h, \tilde{k}) \wedge \forall i, j \in [n_0; n_0 + h(n_0)] \big(|a_i - a_j| < 2^{-\tilde{k}}\big). \tag{7.4}$$

Consequently, for $i, j \in [n_0; n_0 + h(n_0)]$, we have

$$\begin{aligned} \|x_{M+i} - x_{M+j}\| &\leq |a_i - a_j| \|x_M\| + \|(1-a_i) y_i - (1 - a_j) y_j\| \\ &\leq |a_i - a_j| \|x_M\| + \|y_i - y_j\| + a_i \|y_i - y_j\| + |a_i - a_j| \|y_j\| \\ &< b 2^{-\tilde{k}} + 2\|y_i - y_j\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{7.5}$$

Note that $[M + n_0; M + n_0 + g(n_0 + M)] \subset [n_0; n_0 + h(n_0)]$. Thus for all $i, j \in [M + n_0; M + n_0 + g(M + n_0)]$ we have $\|x_i - x_j\| < \varepsilon$. Therefore, by taking $\Psi(\varepsilon, g, b, \gamma, \theta, \beta) := M + \Phi(h, \tilde{k})$, we obtain the desired result. ☐

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