QUANTITATIVE ASYMPTOTIC BEHAVIOR OF A SECOND-ORDER ACCRETIVE DIFFERENCE INCLUSION

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Abstract. In this paper we present new qualitative and quantitative results on the asymptotic behavior of solutions to a second order difference inclusion of accretive type in Banach spaces. We also discuss variants of Pazy's convergence condition, aiming at generalizing that notion without requiring projections. Our results represent the first applications of the proof mining paradigm to difference inclusions, and the idea has the potential to extend to their continuous counterparts.

Keywords: Accretive operator; Proof mining; Rates of convergence; Second-order difference inclusion.
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1. Introduction

By a well-known result of Crandall and Liggett [15], if X is a Banach space, and $A \subset X \times X$ is m-accretive, then for any $x \in \overline{D(A)}$ and $t \geq 0$, the limit

$$S(t)x := \lim_{n\to\infty} (I + \frac{t}{n}A)^{-n}x \; (= \lim_{n\to\infty} J^n_{t/n}x) \tag{1.1}$$

exists, and generates a continuous semigroup of nonexpansive mappings on $\overline{D(A)}$. Moreover, for $x \in D(A)$, the function $S(t)x$ is Lipschitz continuous in t (on bounded subsets) by the proof of [15, Theorem I].

The asymptotic behavior of nonexpansive semigroups is closely connected to the asymptotic behavior of the solutions of particular differential equations. In fact, in a Banach space X, the possible solution to the first order Cauchy problem

$$\begin{cases} u'(t) \in -Au(t), & \text{a.e. on } \mathbb{R}^+, \\ u(0) = x \in D(A), \end{cases} \tag{1.2}$$

where $A \subseteq X \times X$ is accretive, forms a nonexpansive semigroup $S(t)$ of mappings on $\overline{D(A)}$ by the accretivity of A. If A is m-accretive, it is known from [15, Theorem II] and [12, Theorem 2.1] that (1.2) has a strong solution if and only if $S(t)x$ in (1.1) is differentiable almost everywhere, and in this case it is the unique solution to (1.2). Moreover, if X is reflexive, then Lipschitz continuous functions of a real variable with values in X are differentiable almost everywhere by Kōmura's theorem. In particular, when A is m-accretive and X is reflexive, the function $S(t)x$, defined in (1.1) for $x \in D(A)$, is differentiable almost everywhere, so $u(t) := S(t)x$ solves problem (1.2).

It is known ([16]) that in the case in which C is a closed convex subset of a Hilbert space H, the family nonexpansive semigroups $S(t)$ can be put in one to one correspondence in the sense of (1.1) with the family of maximal monotone operators $A \subseteq H \times H$ with $\overline{D(A)} = C$. In Hilbert spaces maximal monotone operators coincide with m-accretive operators. For an extension in a Banach space X, we refer to [49, Theorem 3.4], where C is a nonexpansive retract of X, and X is uniformly convex with a uniformly Gâteaux differentiable norm.

Similarly, second order differential equations and inclusions of the form

$$\begin{cases} p(t)u''(t) + r(t)u'(t) \in Au(t) + f(t), & \text{a.e. on } \mathbb{R}^+, \\ u(0) = x, \quad \sup\{\|u(t)\| : t \geqslant 0\} < \infty \end{cases} \tag{1.3}$$

have been investigated for the existence and asymptotic behavior of solutions by many authors in Hilbert and Banach spaces. The first results in this direction were proved by Barbu [7, 8, 10] in Hilbert spaces for the case where $p \equiv 1$ and $r, f \equiv 0$. In particular, he derived in view of [16, Theorem A2] a definition for the square root of a maximal monotone operator A, identifying it as the unique maximal monotone operator corresponding, via the exponential formula, to the semigroup generated by the solutions of (1.3). The solutions of (1.3) in a more general setting provide a better definition for the square root of A. Poffald and Reich [48] studied the same problem for the existence as well as the asymptotic behavior of solutions in the form of generated semigroups in the Banach space setting. Such problems together with some generalizations were investigated by many authors. We refer the reader in particular to the books by Barbu [9, 10], as well as to the references [11, 7, 8, 13, 14, 53, 54, 13, 39, 48, 1, 2, 3, 6, 40, 41, 19, 20, 26]. In particular, for the existence and uniqueness of bounded solutions to the general differential equation (1.3), we refer to [40] for the Hilbert space and to [26] for the Banach space case.

Additionally, second-order difference inclusions of the form

$$\begin{cases} u_{i+1} - (1+\theta_i)u_i + \theta_i u_{i-1} \in c_i A u_i + f_i, & i \in \mathbb{N}^*, \\ u_0 = x, \quad \sup\{\|u_i\| : i \geqslant 0\} < \infty, \end{cases} \tag{1.4}$$

where A is a nonlinear accretive (m-accretive) operator in a Banach space X, $c_i > 0$ and $\theta_i > 0$, correspond to the discrete version of the second-order evolution equation (1.3). Roughly speaking, using the forward and backward Euler method to approximate the first and second derivatives of u, we may use

$$u'(t) \approx \frac{u(t) - u(t-h)}{h} \approx \frac{u_n - u_{n-1}}{h},$$

$$u''(t) \approx \frac{u(t+h) - u(t) - (u(t) - u(t-h))}{h^2} \approx \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2},$$

and discrete versions of the coefficients in (1.3) to get

$$\frac{1}{h^2}(u_{n+1} - (1 + (1 + h\widetilde{r_n}))u_n + (1 + h\widetilde{r_n})u_{n-1})$$
$$= \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} - \widetilde{r_n}\frac{u_n - u_{n-1}}{h} \in \widetilde{c_n} A u_n + \widetilde{f_n},$$

which is equivalent to $u_{n+1} - (1+\theta_n)u_n + \theta_n u_{n-1} \in c_n A u_n + f_n$, for $\theta_n = 1 + h\widetilde{r_n}$.

Moro??anu [42] investigated the difference inclusion (1.4) for the existence and asymptotic behavior of solutions, and obtained the convergence of $\{u_i\}$ to an element of $A^{-1}(0)$, whenever A is a maximal monotone operator in a Hilbert space, $0 \in R(A)$, $\theta_i \equiv 1$ and $f_i \equiv 0$ (the homogeneous case). Investigations on the existence and asymptotic behavior of solutions to (1.4) were followed by many authors; see e.g., [48, 50, 4, 21, 23, 25, 27, 28, 17, 18]. In general (1.4) has no solution even if $A = 0$, $\theta_i \equiv 1$ and $(f_i)_{i\geq 1} \in \ell^1(X)$; see [48]. Pazy [44] presented the notion of 'convergence condition' for a maximal monotone operator A in a Hilbert space H to assure the strong convergence of the semigroup generated by A via the exponential formula to a zero of A. The strong convergence of the semigroup generated by A via the exponential formula (1.1) was extended to Banach spaces which are both uniformly convex and uniformly smooth by Nevanlinna and Reich in [43], by adapting the Pazy's convergence condition to such classes of Banach spaces. Since Pazy introduced his 'convergence condition', this approach has been frequently used (1.3) and (1.4) to investigate the strong convergence behavior of solutions. Very recently, Pinto and Pischke [45] (see also [34]) provided quantitative information on the Pazy convergence condition and extracted quantitative information on the results of Nevanlinna and Reich [43] (and Xu [56] as well) for the strong convergence of the semigroup generated by A via the exponential formula in uniformly convex and uniformly smooth Banach spaces. In particular, in the general spirit of [31], they introduced a modulus for the convergence condition of Pazy (and its extension by Nevanlinna and Reich) and obtained rates of convergence which depend on this modulus. Moreover, Pischke [47] provided a quantitative version of some result due to Poffald and Reich [48] for the second-order evolution equation (1.3), for the case where $p \equiv 1$ and $r, f \equiv 0$, in uniformly convex and uniformly smooth Banach spaces with a strongly monotone duality map, in the form of an effective rate of convergence depending on a modulus of convergence condition. Quantitative versions of some asymptotic behavior results of almost-orbits of the solution semigroups are also obtained in [45, 47], where in [47] even the qualitative convergence result is new.

The above-mentioned quantitative results were obtained within the proof-mining paradigm [30], where tools from mathematical logic are used to convert prima facie non-quantitative proofs in such a way that new quantitative information can be extracted. We note that, in general, computable rates of convergence are unattainable even for a bounded monotone sequence in $\mathbb{R}$. Considering this situation, Kohlenbach suggested in [32] the following (noneffectively) equivalent but constructively weakened reformulation of the Cauchy property of a sequence (x_i) in normed spaces:

$$\forall \varepsilon > 0\, \forall g : \mathbb{N} \to \mathbb{N}\, \exists n \in \mathbb{N}\, \forall i, j \in [n; n + g(n)] \left(\|x_i - x_j\| < \varepsilon \right), \tag{1.5}$$

with the aim of efficiently transforming other bounds in the premises of a specific case study into a bound on $\exists n \in \mathbb{N}$. Such a bound, which is a bound for Kreisel's no-counterexample interpretation [37, 38] of (1.5), is called a rate of metastability, since Tao

[51, 52] calls an interval $[n; n+g(n)]$ with the property in (1.5) an interval of metastability. Interestingly, $\Phi(g, \varepsilon, K) := \tilde{g}^{(K\lceil 1/\varepsilon \rceil)}(0)$, where $\tilde{g}(i) := i + g(i)$ and $\tilde{g}^{(i)}(0)$ denotes the i-th iteration of $\tilde{g}$ starting with 0, is a rate of metastability for monotone sequences in $[0, K] \subset \mathbb{R}$ (see, [30, Proposition 2.27]). The concept of metastability has been studied within the proof mining program, based on variants of G??del's functional interpretation and transformation of moduli between different settings.

In this paper, we study and analyze the problem (1.4) in Banach spaces, presenting new qualitative and quantitative results on the asymptotic behavior. We provide several quantitative results concerning the strong convergence of the solutions to problem (1.4), and discuss variants of the convergence condition - based on logical techniques from proof mining - aiming at generalizing the notion without requiring the presence of a projection map while ensuring their validity to the Yosida approximation. In the nonhomogeneous case, by providing a quantitative estimate for the monotonicity of the duality map in uniformly convex Banach spaces, we obtain a rate of convergence for the solution (u_n) of the difference inclusion (1.4), depending on a Cauchy rate for the series $\sum_{i=1}^{\infty} h_i \|f_i\| < \infty$, and in the absence of a rate of convergence for this series, we obtain a rate of metastability. Here, we focus on difference inclusions, as our central ideas are more clearly articulated in the discrete setting. Nevertheless, the underlying principles have the potential to extend to their continuous counterparts.

2. PRELIMINARIES

In this section, we recall notations, definitions, and preliminary facts from multi-valued analysis which are used throughout the paper. Let X be a Banach space, X^* be the dual space of X and $(\cdot, \cdot)$ the pairing between X and X^*. X is called uniformly convex, if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|(x+y)/2\| \leq 1 - \delta$, for each $x, y \in X$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$. In this case, a function $\delta : (0, 2] \to (0, 1]$ is a modulus of uniform convexity for X, if for all $\varepsilon \in (0, 2]$ and $x, y \in X$,

$$\|x\|, \|y\| \leq 1 \text{ and } \|x - y\| \geq \varepsilon \quad \Rightarrow \|\frac{x+y}{2}\| \leq 1 - \delta(\varepsilon). \tag{2.1}$$

Let X and Y be two real Banach spaces. A multi-valued operator is a mapping $A : D(A) \subseteq X \to 2^Y$ (or a subset of $X \times Y$), where $D(A) := \{x \in X : Ax \neq \emptyset\}$, $R(A) := \cup\{Ax : x \in D(A)\}$ and $G(A) := \{(x, y) : x \in D(A),\ y \in Ax\}$. Sometimes, we identify an operator with its graph and write $(x, y) \in A$ instead of $(x, y) \in G(A)$. The duality mapping J from X into 2^{X^*} is defined by $J(x) = \{x^* \in X^* : (x, x^*) = \|x\|^2 = \|x^*\|^2\}$, for every $x \in X$. From the Hahn-Banach theorem, we get that $J(x) \neq \emptyset$ for each $x \in X$. A Banach space X is said to be smooth if J is single-valued. In this case, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} = (y, J(x))$$

exists, for each $x, y \in S(X) = \{x \in X : \ \|x\| = 1\}$. The space X is said to be uniformly smooth if the limit is attained uniformly for $x, y \in S(X)$.

Lemma 2.1. (See [9]) *Let X be a Banach space and let $J : X \to 2^{X^*}$ be the normalized duality mapping. Then:*

(1) $(x - y, j_x - j_y) \geq (\|x\| - \|y\|)^2$, for all $x, y \in X, j_x \in J(x)$ and $j_y \in J(y)$, and consequently J is monotone;

(2) $\|x\|^2 - \|y\|^2 \geq 2(x - y, j_y)$, for all $x, y \in X$ and $j_y \in J(y)$;

(3) $(x, j_y) \leq \|x\|\|y\| \leq \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2$, for all $x, y \in X$ and $j_y \in J(y)$.

The following lemma is well known.

Lemma 2.2. *Let X be a smooth Banach space, and let $C \subseteq X$ be a nonempty, closed, and convex subset. Let $x, z \in C$. Then*

$$\|x - z\| = \min_{y \in C} \|x - y\| \quad \Leftrightarrow \quad (y - z, J(x - z)) \leq 0, \quad \forall y \in C.$$

An operator $A \subseteq X \times X$ is called accretive if $\forall y_i \in Ax_i$, $i = 1, 2, \exists j \in J(x_1 - x_2)$ such that $(y_1 - y_2, j) \geq 0$. The accretive operator $A \subseteq X \times X$ is m-accretive if $R(I + A) = X$, where I is the identity operator of X. It then follows that $R(I + \lambda A) = X, \forall \lambda > 0$ (see, e.g., [9]).

For an accretive operator A, the resolvent and the Yosida approximation of A are defined by

$$J_\lambda x = (I + \lambda A)^{-1} x$$

and

$$A_\lambda x = \frac{I - J_\lambda}{\lambda} x,$$

respectively, where $x \in R(I + \lambda A)$. Obviously,

$$A_\lambda x = \lambda^{-1}(x - J_\lambda x) \in \lambda^{-1}((I + \lambda A)J_\lambda x - J_\lambda x) = AJ_\lambda x. \tag{2.2}$$

Moreover, J_λ is nonexpansive in the sense that $\|J_\lambda x - J_\lambda y\| \leq \|x - y\|$, for all $x, y \in R(I + \lambda A)$.

Lemma 2.3. (See [10]). Let A be an m-accretive operator in $X \times X$. Then

(1) $\|J_\lambda x - J_\lambda y\| \leq \|x - y\|$, for all $x, y \in X$;

(2) $\|J_\lambda x - x\| = \lambda \|A_\lambda x\| \leq \lambda \inf\{\|y\|;\ y \in Ax\}$, for all $x \in D(A)$;

(3) A_λ is $m-$accretive on X and $\|A_\lambda x - A_\lambda y\| \leq (2/\lambda)\|x - y\|$, for all $\lambda > 0$ and $x, y \in X$;

Lemma 2.4. ([28]) *Let $\{a_i\}$ be a sequence of positive real numbers with $\Sigma_{i=1}^{\infty} a_i^{-1} = \infty$. If $\{b_i\}$ is a bounded sequence, then $\liminf_{i \to \infty} a_i(b_{i+1} - b_i) \leq 0$.*

Let us consider the second order difference equation (1.4), as well as the auxiliary sequence $(a_i)_{i \geqslant 1}$ given by

$$a_0 = 1 \quad , \quad a_i = \frac{1}{\theta_1 \theta_2 ... \theta_i}, \quad i \geq 1. \tag{2.3}$$

Observe that

$$a_i\theta_i = a_{i-1}, \quad i \geq 1. \tag{2.4}$$

We denote

$$h_k := \Sigma_{i=1}^k \frac{1}{\theta_k\theta_{k-1}...\theta_i}, \quad \forall k \geq 1. \tag{2.5}$$

In this paper, we assume that the difference inclusion (1.4) has a solution for an initial value $u_0 = x$ in X. For an existence result on the solution for (1.4) in Banach spaces, we mention the following:

Theorem 2.5. [27, Theorem 4.4] *Let X be a uniformly smooth and uniformly convex Banach space. Let $A \subseteq X \times X$ be $m-$accretive with $A^{-1}0 \neq \emptyset$ and $c_i, \theta_i > 0, \forall i \geq 1$, such that $\sum_{i=1}^\infty \frac{1}{h_i} = \infty$ holds. If $(f_i)_{i\geq 1}$ is a sequence in X satisfying $\Sigma_{i=1}^\infty h_i\|f_i\| < \infty$, then* (1.4) *has a unique solution for every initial point $x \in X$.*

Notation. To simplify the presentation of formulas, we will occasionally adopt the following notational conventions throughout the paper.

- $\mathbb{N} := \{0, 1, 2, \dots\}$ and $\mathbb{N}^* := \{1, 2, \dots\}$.
- Expressions such as $n^{\mathbb{N}}$ and x^X indicate that $n \in \mathbb{N}$ and $x \in X$, respectively.
- $f^{\mathbb{N}\to\mathbb{N}}$ denotes a function $f : \mathbb{N} \to \mathbb{N}$.
- $(x_i)^{\mathbb{N}\to X}$ denotes a sequence in X, i.e., a function from $\mathbb{N}$ to X.

3. A quantitative estimate for the monotonicity of the duality map

We recall the following interesting result that was proved by H.K. Xu [55]:

Proposition 3.1. (See [55, Corollary 3]) *Let $r > 0$ and let X be a Banach space. Then X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$, such that $(x - y, j_x - j_y) \geqslant g(\|x - y\|)$, for all $x, y \in \{z \in X : \|z\| \leq r\}, j_x \in J(x)$ and $j_y \in J(y)$.*

Since the proof given in [55] for Proposition 3.1 is nonconstructive, we present an effective proof to compute such a function g in terms of a given modulus of uniform convexity for X.

Lemma 3.2. *Let X be a uniformly convex Banach space with a modulus of uniform convexity δ. Then, for all $x, y \in X$ with $x \neq y$ and $\|x\| \geq \|y\|$, we have*

$$2(\|x\|^2 + \|x\|\|y\|)\delta(\frac{\|x - y\|}{\|x\|}) - 2\|y\|(\|x\| - \|y\|) \leq (x - y, j_x - j_y). \tag{3.1}$$

Proof. Let $x, y \in X$ with $x \neq y$ and $\|x\| \geq \|y\|$. Obviously, $\|x\| > 0$. Defining $\varepsilon := \|x - y\|/\|x\|$, we have $0 < \varepsilon \leq 2$. Since $-(x-y, j_x-j_y)+2\|x\|^2+2\|y\|^2 = (x+y, j_x+j_y)$, we obtain

$$\frac{-1}{\|x\|^2}(x-y,j_x-j_y)+2+2\frac{\|y\|^2}{\|x\|^2} = \frac{1}{\|x\|^2}(x+y,j_x+j_y)$$
$$\leq (\frac{\|x+y\|}{\|x\|})(\frac{\|x\|+\|y\|}{\|x\|})$$
$$\leq 2(1-\delta(\varepsilon))(1+\frac{\|y\|}{\|x\|}).$$

Consequently,

$$2\delta(\varepsilon)(1+\frac{\|y\|}{\|x\|})+2(\frac{\|y\|^2}{\|x\|^2}-\frac{\|y\|}{\|x\|}) \leq \frac{1}{\|x\|^2}(x-y,j_x-j_y).$$

That is,

$$2(\|x\|^2+\|x\|\|y\|)\delta(\varepsilon)+2(\|y\|^2-\|x\|\|y\|) \leq (x-y,j_x-j_y),$$

which is the desired inequality. □

Suppose that X is uniformly convex with a modulus $\delta : (0,2] \to (0,1]$. Then $\delta_1 : [0,\infty) \to [0,1]$ defined as

$$\delta_1(\varepsilon) := \sup\{\delta(\varepsilon')|0<\varepsilon' \leq \min\{2,\varepsilon\}\}, \ \ \delta_1(0) := 0,$$

is an increasing modulus of uniform convexity. Thus, we may define

$$\tilde{\delta}(\varepsilon) := \frac{1}{2}\int_0^{\varepsilon} \delta_1(t)dt, \tag{3.2}$$

obtaining a continuous, strictly increasing, and convex function $\tilde{\delta} : [0,\infty) \to [0,\infty)$ such that

$$\forall \varepsilon \leq 2 \ \big(\tilde{\delta}(\varepsilon) \leq \delta_1(\varepsilon)\big). \tag{3.3}$$

It is notable that, when restricted to $(0,2]$, both δ_1 and $\tilde{\delta}$ act as moduli of uniform convexity for X. In fact, given $\varepsilon \in (0,2]$ and $x,y \in X$ such that $\|x\|,\|y\| \leq 1$ and $\|x-y\| \geq \varepsilon$, we have, for any $0<\varepsilon' \leq \varepsilon$, that $\|(x+y)/2\| \leq 1-\delta(\varepsilon')$ by (2.1). Taking the infimum over $0<\varepsilon' \leq \varepsilon$, we then obtain $\|(x+y)/2\| \leq 1-\delta_1(\varepsilon)$. That is, δ_1 restricted to $(0,2]$ is a modulus of uniform convexity for X. Based on this and in view of (3.3), it is clear that $\tilde{\delta}$, when restricted to $(0,2]$, is also a modulus of uniform convexity for X.

It is worth noting that

$$\tilde{\delta}(\varepsilon) \geq \frac{1}{2}\int_{\frac{\varepsilon}{3}}^{\varepsilon} \delta_1(t)dt \geq \frac{\varepsilon}{3}\delta_1(\frac{\varepsilon}{3}) \geq \frac{\varepsilon}{3}\delta(\min\{2,\frac{\varepsilon}{3}\}).$$

Proposition 3.3. *Let X be a uniformly convex Banach space with a modulus of uniform convexity δ and let $b>0$. Define g_1 and g_2 on $\mathbb{R}_+$ by $g_1(\varepsilon) = (\frac{\varepsilon}{2}\delta_1(\frac{\varepsilon}{b}))^2$ and $g_2(\varepsilon) = b^2\tilde{\delta}(\frac{\varepsilon}{b})^2$. Then, for all $x,y \in B_b(0)$, $j_x \in Jx$ and $j_y \in Jy$, we have*

$$g_i(\|x-y\|) \leq (x-y,j_x-j_y), \ \ i=1,2. \tag{3.4}$$

Remark 3.4. Throughout the paper, for the applications of Proposition 3.3, the convexity of g_2 does not need to be used, and we can use either g_1 or g_2.

Proof. Let $x, y \in B_b(0)$. Without loss of generality, we may assume $x \neq y$ and $\|x\| \geq \|y\|$. Let $\varepsilon = \|x - y\|$. First, we show that

$$(\|x\|\delta(\varepsilon/\|x\|))^2 \leq (x - y, j_x - j_y). \tag{3.5}$$

We consider two cases:

Case 1: If $(\|x\|^2 + \|x\|\|y\|)\delta(\varepsilon/\|x\|) \geq 2(\|x\|\|y\| - \|y\|^2)$, then in view of (3.1) we have

$$\|x\|^2\delta(\varepsilon/\|x\|)^2 \leq (\|x\|^2 + \|x\|\|y\|)\delta(\varepsilon/\|x\|) \leq (x - y, j_x - j_y).$$

Case 2: If $(\|x\|^2 + \|x\|\|y\|)\delta(\varepsilon/\|x\|) < 2(\|x\|\|y\| - \|y\|^2)$, then $\|y\| > 0$, and that

$$\begin{aligned}(x - y, j_x - j_y) &\geq (\|x\| - \|y\|)^2 \\ &> (\frac{1}{2\|y\|}(\|x\|^2 + \|x\|\|y\|)\delta(\varepsilon/\|x\|))^2 \\ &\geq (\|x\|\delta(\varepsilon/\|x\|))^2.\end{aligned}$$

This completes the proof of (3.5). Now, replacing δ with δ_1 in (3.5), since $\|x\| \geq \|y\|$ and δ_1 is increasing, we have

$$\begin{aligned}(x - y, j_x - j_y) &\geq (\|x\|\delta_1(\varepsilon/\|x\|))^2 \\ &\geq (\frac{\|x - y\|}{2}\delta_1(\varepsilon/\|x\|))^2 \geq (\frac{\varepsilon}{2}\delta_1(\varepsilon/b))^2.\end{aligned}$$

Replacing δ with $\tilde{\delta}$ in (3.5), and using the convexity of $\tilde{\delta}$, we obtain

$$(x - y, j_x - j_y) \geq (\|x\|\tilde{\delta}(\varepsilon/\|x\|))^2 = (b\frac{\|x\|}{b}\tilde{\delta}(\varepsilon/\|x\|))^2 \geq (b\tilde{\delta}(\varepsilon/b))^2.$$

□

4. Convergence rates for the homogeneous case

The original formulation of the so-called 'convergence condition' is due to Pazy [44]: A maximal monotone operator $A \subseteq H \times H$ with $C = A^{-1}0 \neq \emptyset$ satisfies the convergence condition if, for all bounded sequences $(x_i, y_i) \in A$, the condition $\lim_{i\to\infty}(y_i, J(x_i - P_C x_i)) = 0$ implies that $\liminf_{i\to\infty} \mathrm{dist}(x_i, A^{-1}0) = 0$ (or, $\lim_{i\to\infty} \mathrm{dist}(x_i, A^{-1}0) = 0$). For a maximal monotone operator A in a Hilbert space, the zero set $A^{-1}0$ is a closed and convex set, and hence the projection onto $A^{-1}0$ is well-defined. Pazy presented this notion for a maximal monotone operator A in a Hilbert space H to assure the strong convergence of the semigroup generated by A via the exponential formula to a zero of A. The convergence condition is satisfied, for example, if A is the subdifferential of a l.s.c. convex function $\varphi \geq 0$ whose level sets are compact and $\min_{x\in H}\varphi(x) = 0$ (see Pazy [44]). It is obvious that every strongly monotone operator A satisfies the convergence condition.

We know that if C is a nonempty, closed and convex subset of a uniformly convex Banach space X, and $x \in X$, then there exists a unique element $z \in C$ such that

$\operatorname{dist}(x, C) = \|x - z\|$. Denoting $z = P(x)$, P is called the (nearest point) projection map of the Banach space X onto C.

Moreover, by Lemma 2.2, if X is smooth, then $z \in C$ is the nearest point projection of $x \in X$ onto C, if and only if

$$(y - z, J(x - z)) \leq 0, \quad \forall y \in C. \tag{4.1}$$

It is known that if $A \subseteq X \times X$ is m-accretive, then A is closed, and hence $A^{-1}0$ is closed (see, e.g., [9]). Furthermore, if X is uniformly convex, then $A^{-1}0$ is closed and convex since it is the fixed point set of any resolvent J_λ of A. It is worth pointing out that in the case where X is a uniformly convex Banach space and $A \subseteq X \times X$ is m-accretive, $A^{-1}0 \neq \emptyset$ holds if and only if $\liminf_{\lambda\to\infty} \|J_\lambda x\| < \infty$ for some $x \in X$ (see [29, Theorem 1]).

Definition 4.1. [43] Let X be smooth and uniformly convex, and A be m-accretive, and assume that $A^{-1}0 \neq \emptyset$. Let $P : X \to A^{-1}0$ be the nearest point projection map onto the (closed and convex) zero set of A. Then, A satisfies the convergence condition if, for all bounded sequences $(x_i, y_i) \in A$, the condition $\lim_{i\to\infty}(y_i, J(x_i - Px_i)) = 0$ implies that $\lim_{i\to\infty} \|x_i - Px_i\| = 0$.

Let X be a smooth and uniformly convex Banach space, and assume that $A \subseteq X \times X$ is m-accretive such that $A^{-1}0 \neq \emptyset$. As mentioned, the nearest point projection map $P : X \to A^{-1}0$ of X onto $A^{-1}0$ is well-defined in this case, and the convergence condition is equivalent to have:

$$\begin{aligned} \forall K^{\mathbb{N}}\, \forall (x_i)^{\mathbb{N}\to X}, (y_i)^{\mathbb{N}\to X} \Big(& \forall i \left((x_i, y_i) \in A \wedge \|x_i\|, \|y_i\| \leq K\right) \\ & \wedge \lim_{i\to\infty}(y_i, J(x_i - Px_i)) = 0 \to \lim_{i\to\infty} \|x_i - Px_i\| = 0\Big). \end{aligned} \tag{4.2}$$

Pinto and Pischke [45] showed that A satisfies the convergence condition if, and only if,

$$\begin{aligned} \forall k^{\mathbb{N}}, K^{\mathbb{N}}\, \exists n^{\mathbb{N}}\, \forall x^X, y^X \Big(& (x, y) \in A \wedge \|x\|, \|y\| \leq K \\ & \wedge |(y, J(x - Px))| \leq \frac{1}{n+1} \to \|x - Px\| \leq \frac{1}{k+1}\Big). \end{aligned} \tag{4.3}$$

The same authors also introduced the following definition for a modulus for the convergence condition (as well as, based on [24, 33], discussed logical metatheorems to guarantee the extractability of a computable modulus):

Definition 4.2. A modulus for the convergence condition is a functional $\Omega : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ satisfying that for any $k, K \in \mathbb{N}$: if $(x, y) \in A$ are such that $\|x\|, \|y\| \leq K$, then

$$|(y, J(x - Px))| \leq \frac{1}{\Omega(K, k) + 1} \Rightarrow \|x - Px\| \leq \frac{1}{k+1}. \tag{4.4}$$

We prove a new quantitative result that incorporates the main idea of some previous convergence results (e.g., [48, 23, 17, 18]) under the convergence condition. Additionally, we determine how the modulus for the convergence condition is reflected in the proof.

Theorem 4.3. *Let X be a smooth and uniformly convex Banach space, and assume that $A \subseteq X \times X$ is m-accretive such that $A^{-1}0 \neq \emptyset$ and satisfies the convergence condition with a modulus Ω. Let (c_i) be bounded away from zero, (θ_i) be bounded with $\sum_{i=1}^{\infty} \theta_1\theta_2...\theta_i = \infty$, and choose $k_0 \in \mathbb{N}$ such that $(c_i) \subset [1/k_0, \infty)$ and $(\theta_i) \subset (0, k_0]$. Let $b_0, b_1 > 0$, and let $p_0 \in A^{-1}0$, $x \in X$ with $\|p_0\| \leq b_0$ and $\|x\| \leq b_1$. If (u_i) is a solution for the homogeneous form of* (1.4) *(i.e., $f_i \equiv 0$) for initial point x, then we have:*

$$\forall k \in \mathbb{N} \, \forall n, m \geq \mu(\Omega(K, 2k+1)+1) \left(\|u_n - u_m\| \leq \frac{1}{k+1} \right)$$

with $\mu(l) := \lceil \tilde{c} \cdot l \rceil$ for $l \in \mathbb{N}$, $\tilde{c} \geq \frac{k_0^2}{2}\|x - p_0\|^2$ (e.g., $\tilde{c} := \frac{k_0^2}{2}(b_0 + b_1)^2$), and $K := 2k_0(1 + k_0)\lceil b_0 + b_1 \rceil$.
Note that $\tilde{c}$ can be defined in terms of K, e.g. by $\tilde{c} := K^2$.

Proof. Let $P : X \to A^{-1}0$ be the nearest point projection map of X onto $A^{-1}0$. We write

$$v_i := \frac{1}{c_i}((u_{i+1} - u_i) - \theta_i(u_i - u_{i-1})) \in Au_i, \qquad \forall i \geq 1. \tag{4.5}$$

Note that, using (4.1), we have

$$\begin{aligned} &(Pu_{i+1} - (1+\theta_i)Pu_i + \theta_i Pu_{i-1}, J(u_i - Pu_i)) \\ &\quad = (Pu_{i+1} - Pu_i, J(u_i - Pu_i)) + \theta_i(Pu_{i-1} - Pu_i, J(u_i - Pu_i)) \leq 0. \end{aligned} \tag{4.6}$$

From the accretivity of A, (4.5), (1.4), (4.6), and Lemma 2.1, for all $i \geq 1$, we have

$$\begin{aligned} 0 \leq & c_i(v_i, J(u_i - Pu_i)) = (u_{i+1} - (1+\theta_i)u_i + \theta_i u_{i-1}, J(u_i - Pu_i)) \\ =& (u_{i+1} - Pu_{i+1}, J(u_i - Pu_i)) - (1+\theta_i)(u_i - Pu_i, J(u_i - Pu_i)) \\ & + \theta_i(u_{i-1} - Pu_{i-1}, J(u_i - Pu_i)) \\ & + (Pu_{i+1} - (1+\theta_i)Pu_i + \theta_i Pu_{i-1}, J(u_i - Pu_i)) \\ \leq & \frac{1}{2}\|u_{i+1} - Pu_{i+1}\|^2 + \frac{1}{2}\|u_i - Pu_i\|^2 - (1+\theta_i)\|u_i - Pu_i\|^2 \\ & + \frac{\theta_i}{2}\|u_{i-1} - Pu_{i-1}\|^2 + \frac{\theta_i}{2}\|u_i - Pu_i\|^2 \\ =& \frac{1}{2}(\|u_{i+1} - Pu_{i+1}\|^2 - \|u_i - Pu_i\|^2) - \frac{\theta_i}{2}(\|u_i - Pu_i\|^2 - \|u_{i-1} - Pu_{i-1}\|^2). \end{aligned}$$

Multiplying both sides of the above inequality by a_i and summing up from $i = n$ to m, we have

$$\begin{aligned} 0 \leq \sum_{i=n}^{m} c_i a_i(v_i, J(u_i - Pu_i)) \leq & \frac{a_m}{2}(\|u_{m+1} - Pu_{m+1}\|^2 - \|u_m - Pu_m\|^2) \\ & - \frac{a_{n-1}}{2}(\|u_n - Pu_n\|^2 - \|u_{n-1} - Pu_{n-1}\|^2). \end{aligned}$$

Taking liminf as $m \to \infty$, by using our assumption and Lemma 2.4, we get, for all n,

$$0 \leq \sum_{i=n}^{\infty} c_i a_i (v_i, J(u_i - Pu_i)) \leq \frac{a_{n-1}}{2}(\|u_{n-1} - Pu_{n-1}\|^2 - \|u_n - Pu_n\|^2). \tag{4.7}$$

This implies that $\|u_i - Pu_i\|$ is non-increasing. That is,

$$\forall i \big(\|u_{i+1} - Pu_{i+1}\| \leq \|u_i - Pu_i\|\big). \tag{4.8}$$

Furthermore, by repeating the above argument with an arbitrary $p \in A^{-1}0$ replacing Pu_i for every i, we obtain

$$0 \leq \sum_{i=n}^{\infty} c_i a_i (v_i, J(u_i - p)) \leq \frac{a_{n-1}}{2}(\|u_{n-1} - p\|^2 - \|u_n - p\|^2), \tag{4.9}$$

which implies in turn that, for all $p \in A^{-1}0$,

$$\forall i \big(\|u_{i+1} - p\| \leq \|u_i - p\|\big) \tag{4.10}$$

In particular, we obtain

$$\forall i \big(\|u_{i+1} - p_0\| \leq \|x - p_0\|\big). \tag{4.11}$$

Thus, we have $\|u_i - p_0\| \leq \|x\| + \|p_0\|$, leading to the following bound for (u_i):

$$\forall i \big(\|u_i\| \leq \|x\| + 2\|p_0\|\big). \tag{4.12}$$

Combining (4.5) and (4.11), and using the assumptions, we also have

$$\forall i \big(\|v_i\| \leq 2k_0(1 + k_0)(\|x\| + \|p_0\|)\big). \tag{4.13}$$

At this stage, by dividing both sides of the inequality (4.7) by a_{n-1} and summing up from $n = 1$ to ∞, we have

$$\sum_{n=1}^{\infty} \sum_{i=n}^{\infty} c_i \frac{1}{\theta_i \theta_{i-1} \ldots \theta_n} (v_i, J(u_i - Pu_i)) \leq \frac{1}{2}\|u_0 - Pu_0\|^2 = \frac{1}{2}\|x - Px\|^2. \tag{4.14}$$

We also know that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{i=n}^{\infty} c_i \frac{1}{\theta_i \theta_{i-1} \ldots \theta_n} (v_i, J(u_i - Pu_i)) \\
&\quad = \sum_{i=1}^{\infty} \frac{c_i (v_i, J(u_i - Pu_i))}{\theta_i \theta_{i-1} \ldots \theta_1} + \sum_{i=2}^{\infty} \frac{c_i (v_i, J(u_i - Pu_i))}{\theta_i \theta_{i-1} \ldots \theta_2} + \cdots + \sum_{i=m}^{\infty} \frac{c_i (v_i, J(u_i - Pu_i))}{\theta_i \theta_{i-1} \ldots \theta_m} + \cdots \\
&\quad = \frac{1}{\theta_1} c_1 (v_1, J(u_1 - Pu_1)) + (\frac{1}{\theta_2} + \frac{1}{\theta_2 \theta_1}) c_2 (v_2, J(u_2 - Pu_2)) \\
&\quad + \cdots + (\frac{1}{\theta_m} + \frac{1}{\theta_m \theta_{m-1}} + \cdots + \frac{1}{\theta_m \cdots \theta_2 \theta_1}) c_m (v_m, J(u_m - Pu_m)) + \cdots \\
&\quad = \sum_{m=1}^{\infty} h_m c_m (v_m, J(u_m - Pu_m)).
\end{aligned} \tag{4.15}$$

The latter equality, along with (4.14) and the assumption $(c_i) \subset [1/k_0, \infty)$, yields

$$\sum_{m=1}^{\infty} h_m(v_m, J(u_m - Pu_m)) \leq \frac{k_0}{2}\|x - Px\|^2. \tag{4.16}$$

Thus

$$\sum_{m=1}^{\infty}(v_m, J(u_m - Pu_m)) \leq \frac{k_0^2}{2}\|x - Px\|^2, \tag{4.17}$$

since $h_m \geq 1/\theta_m \geq 1/k_0$.

Claim. Taking $\mu(l) := \lceil \tilde{c} \cdot l \rceil$, where

$$\tilde{c} \geq \frac{k_0^2}{2}\|x - p_0\|^2 \geq \frac{k_0^2}{2}\|x - Px\|^2,$$

we have:

$$\forall l \in \mathbb{N}^* \exists n \leq \mu(l)\big((v_n, J(u_n - Pu_n)) \leq \frac{1}{l}\big). \tag{4.18}$$

The proof of the claim easily follows from (4.17), by contradiction.

Now, let $K \geq 2k_0(1+k_0)\lceil(\|x\|+\|p_0\|)\rceil$. Then, for any k, by taking $l := \Omega(K, 2k+1) + 1$ in (4.18), we may choose some natural number n_0 with $0 < n_0 \leq \mu(\Omega(K, 2k+1) + 1)$ such that $(v_{n_0}, J(u_{n_0} - Pu_{n_0})) \leq \frac{1}{\Omega(K,2k+1)+1}$. In view of (4.12) and (4.13), and considering the definition of Ω, we now deduce that $\|u_{n_0} - Pu_{n_0}\| \leq \frac{1}{2k+2}$. From this and (4.8), it follows that, for all $n \geq \mu(\Omega(K, 2k+1) + 1)$, $\|u_n - Pu_n\| \leq \frac{1}{2k+2}$. Consequently, by (4.10),

$$\|u_{m+n} - u_n\| \leq \|u_{m+n} - Pu_n\| + \|u_n - Pu_n\| \leq 2\|u_n - Pu_n\| \leq \frac{1}{k+1},$$

for all $n \geq \mu(\Omega(K, 2k+1) + 1)$ and $m \in \mathbb{N}$. □

Remark 4.4. *(For logicians).* The rate in Theorem 4.3 does not depend on a rate of divergence for $\sum_{i=1}^{\infty} \theta_1\theta_2...\theta_i = \infty$. This fact can be logically explained as follows: the divergence is only used to proof the purely universal version of (4.7) where instead of (4.7) one states

$$\forall k\big(\sum_{i=n}^{k} c_i a_i(v_i, J(u_i - Pu_i)) \leq \frac{a_{n-1}}{2}(\|u_{n-1} - Pu_{n-1}\|^2 - \|u_n - Pu_n\|^2)\big),$$

which could be added as an axiom for the extraction of the rate. Note that also in (4.16) and (4.17) one never needs these sums to actually converge but only that their partial sums are bounded by the quantities given.

It is also noteworthy that the smoothness and uniform convexity of X are not explicitly manifested in the extracted rate presented in Theorem 4.3 in the sense that the rate extracted does not depend on moduli of uniform convexity or uniform smoothness for X. This observation is logically discussed in Remark 4.16 at the end of this section. This also raises the question of whether the result can be further improved in this aspect. Through inspection of the classic proof presented above, we realize that the nearest point projection map $P : X \to A^{-1}0$ should be well-defined, since it exists in the premise of the convergence condition. So, it is natural to maintaining these assumptions: A is m-accretive, X is smooth and uniformly convex; these are the standard assumptions to define a projection on the zeros of an operator in literature. Apart from being used in the convergence condition, we also applied the particular property of the projections

$$\forall x, z \in X\big(z = Px \leftrightarrow \forall y \in A^{-1}0,\ (y - z, J(x - z)) \leq 0\big), \tag{4.19}$$

which holds for closed convex sets in smooth and uniformly convex Banach spaces. This property has been applied to guarantee that

$$(Pu_{i+1} - (1 + \theta_i)Pu_i + \theta_i Pu_{i-1}, J(u_i - Pu_i)) \leq 0. \quad \text{(See (4.6))}$$

Based on the above discussion, at first glance, it seems that there is not much possibility of further generalizing the result to more general Banach spaces, even under different assumptions on the coefficients. Despite this, by inspecting the proof presented above, we are able to extract additional constructive data:

It is worth noting first that the proof relies on particular selections Pu_i of the elements of $A^{-1}0$. For (Pu_i), there is a bound depending on a bound $\widetilde{K} \geq 2\|x\| + \|p_0\|$ for (u_i) (see 4.12). In fact,

$$\|Pu_i\| \leq \|u_i - p_0\| + \|u_i\| \leq 2\|u_i\| + \|p_0\| \leq 2\widetilde{K} + \|p_0\| \leq 3\widetilde{K}. \tag{4.20}$$

One of the key goals in the proof of Theorem 4.3 is to show (4.18), which involves finding a computable bound for n in the sentence

$$\forall l \in \mathbb{N}^* \,\exists n\big((v_n, J(u_n - Pu_n)) \leq \frac{1}{l}\big). \tag{4.21}$$

By analyzing the above proof, we will see (in Theorem 4.14 and Corollary 4.15) that, without using any property of the projection P, it is possible, under slightly different assumptions on the coefficients, to show that for each $l \in \mathbb{N}^*$ there is some n with a computable bound such that for all $p \in A^{-1}0$ with $\|p\| \leq 3\widetilde{K}$,

$$(v_n, J(u_n - p)) \leq \frac{1}{l}, \tag{4.22}$$

which includes (4.21) in view of (4.20). If (4.20) is proven, it follows that, in this case, the projection P serves exclusively to define the formal convergence condition and does not play a significant role in calculations.

Note that in the definition of the convergence condition the assumptions of the uniform convexity of X and of the m-accretivity (instead of only 'accretivity') of $A \subseteq X \times X$ are used just to define the nearest point projection map P over $A^{-1}0$. On the other hand, the computability of the projection P depends on the complexity of $A^{-1}0$.

Here, we suggest a more general variant of the convergence condition, without using a projection:

Definition 4.5. Let X be a smooth Banach space and $A \subseteq X \times X$ be an (accretive) operator with $A^{-1}0 \neq \emptyset$. Then, A satisfies the generalized convergence condition if, for all bounded sequences $(x_i, y_i) \in A$, the condition $\lim_{i\to\infty}(y_i, J(x_i - p)) = 0$, uniformly on bounded subsets of $A^{-1}0$, implies that $\lim_{i\to\infty} \mathrm{dist}(x_i, A^{-1}0) = 0$.

For each $\ell \in \mathbb{N}$, denote $\mathcal{Z}_\ell := \{p \in A^{-1}0 : \ \|p\| \leq \ell\}$. The generalized convergence condition is equivalent to have:

$$\begin{aligned} &\forall K^{\mathbb{N}}\, \forall (x_i)^{\mathbb{N}\to X}, (y_i)^{\mathbb{N}\to X} \Big(\forall i (y_i \in Ax_i \wedge \|x_i\|, \|y_i\| \leq K) \\ &\wedge \forall \ell^{\mathbb{N}} \big(\lim_{i\to\infty} \sup_{p\in\mathcal{Z}_\ell} |(y_i, J(x_i - p))| = 0 \big) \to \lim_{i\to\infty} \mathrm{dist}(x_i, A^{-1}0) = 0 \Big), \end{aligned} \tag{4.23}$$

which obviously is implied by the convergence condition; i.e.,

$$\text{The convergence condition} \quad \Rightarrow \quad \text{The generalized convergence condition.}$$

The main point is that the generalized convergence condition can be formulated in situations where the standard convergence condition cannot, and the implication above applies only to the case in which P_C exists.

We prove the following lemma.

Lemma 4.6. *Let X be a smooth Banach space and $A \subseteq X \times X$ be an operator with $A^{-1}0 \neq \emptyset$. For each $\ell \in \mathbb{N}$, let $\mathcal{Z}_\ell := \{p \in A^{-1}0 : \ \|p\| \leq \ell\}$. Then, the following statements are equivalent:*

(i) A satisfies the generalized convergence condition;

(ii)

$$\begin{aligned} &\forall k^{\mathbb{N}}, K^{\mathbb{N}}\, \exists n^{\mathbb{N}}\, \forall g^{\mathbb{N}\to\mathbb{N}}\, \exists m^{\mathbb{N}}\, \forall (x_i)^{\mathbb{N}\to X}, (y_i)^{\mathbb{N}\to X} \Big(\forall i \big((x_i, y_i) \in A \wedge \|x_i\|, \|y_i\| \leq K \big) \\ &\wedge \forall \ell^{\mathbb{N}} \big(\sup_{p\in\mathcal{Z}_\ell} |(y_{g(\ell)}, J(x_{g(\ell)} - p))| \leq \frac{1}{n+1} \big) \to \mathit{dist}(x_m, A^{-1}0) \leq \frac{1}{k+1} \Big); \end{aligned} \tag{4.24}$$

(iii)

$$\begin{aligned} &\forall k^{\mathbb{N}}, K^{\mathbb{N}}\, \exists n^{\mathbb{N}}, \ell^{\mathbb{N}}\, \forall x^X, y^X \Big((x, y) \in A \wedge \|x\|, \|y\| \leq K \\ &\wedge \big(\sup_{p\in\mathcal{Z}_\ell} |(y, J(x - p))| \leq \frac{1}{n+1} \big) \to \mathit{dist}(x, A^{-1}0) \leq \frac{1}{k+1} \Big). \end{aligned} \tag{4.25}$$

Proof. To establish (i) $\Rightarrow$ (ii), assume by contradiction that (4.24) is not true. Then for some $k, K \in \mathbb{N}$, we have

$$\begin{aligned} &\forall n^{\mathbb{N}}\, \exists g^{\mathbb{N}\to\mathbb{N}}\, \forall m^{\mathbb{N}}\, \exists (x_i)^{\mathbb{N}\to X}, (y_i)^{\mathbb{N}\to X} \Big(\forall i \big((x_i, y_i) \in A \wedge \|x_i\|, \|y_i\| \leq K \big) \\ &\wedge \forall \ell^{\mathbb{N}} \big(\sup_{p\in\mathcal{Z}_\ell} |(y_{g(\ell)}, J(x_{g(\ell)} - p))| \leq \frac{1}{n+1} \big) \wedge \mathrm{dist}(x_m, A^{-1}0) > \frac{1}{k+1} \Big). \end{aligned} \tag{4.26}$$

From the above assertion, for any $n \in \mathbb{N}$ we first choose $g_n : \mathbb{N} \to \mathbb{N}$, and then for $m = g_n(n)$, we choose sequences $(x_i^n, y_i^n) \in A$ with $\|x_i^n\|, \|y_i^n\| \leq K$ such that

$$\forall \ell^{\mathbb{N}} \big(\sup_{p\in\mathcal{Z}_\ell} |(y^n_{g_n(\ell)}, J(x^n_{g_n(\ell)} - p))| \leq \frac{1}{n+1} \big),$$

and

$$\operatorname{dist}(x^n_{g_n(n)}, A^{-1}0) > \frac{1}{k+1}.$$

Consequently,

$$\lim_n \sup_{p\in\mathcal{Z}_\ell} |(y^n_{g_n(\ell)}, J(x^n_{g_n(\ell)} - p))| = 0,$$

uniformly in $\ell \in \mathbb{N}$. However, the sequence $(\operatorname{dist}(x^n_{g_n(n)}, A^{-1}0))$ is bounded away from zero. This contradicts the generalized convergence condition. To show that (ii) $\Rightarrow$ (i), suppose that (x_i) and (y_i) are sequences in X such that, for all i, $(x_i, y_i) \in A$, $\|x_i\|, \|y_i\| \leq K$ for some $K \in \mathbb{N}$, and

$$\lim_{i\to\infty} \sup_{p\in\mathcal{Z}_\ell} |(y_i, J(x_i - p))| = 0, \tag{4.27}$$

for all $\ell \in \mathbb{N}$. For a given $k \in \mathbb{N}$, choose n as defined in (4.24). Then in view of (4.27) we may define a function $g : \mathbb{N} \to \mathbb{N}$ such that, for all $\ell \in \mathbb{N}$,

$$\sup_{p\in\mathcal{Z}_\ell} |(y_{g(\ell)}, J(x_{g(\ell)} - p))| \leq \frac{1}{n+1}.$$

Now, applying (4.24), we deduce that there exists some $m \in \mathbb{N}$ such that

$$\operatorname{dist}(x_m, A^{-1}0) \leq \frac{1}{k+1}. \tag{4.28}$$

For any $m_0 \in \mathbb{N}$, repeating the above argument for the shifted sequences (x_{i+m_0}) and (y_{i+m_0}) instead of (x_i) and (y_i), we conclude the existence of some $m \geq m_0$ with property (4.28). That is, $\liminf_m \operatorname{dist}(x_m, A^{-1}0) = 0$. This concludes that A satisfies the generalized convergence condition. We now prove that (iii) $\Rightarrow$ (i): Let $K \in \mathbb{N}$, and assume that (x_i) and (y_i) are sequences in X with $\|x_i\|, \|y_i\| \leq K$, $(x_i, y_i) \in A$, and that (4.27) holds for all $\ell \in \mathbb{N}$. Let $k \in \mathbb{N}$ and take n_0 and ℓ_0 as in (4.25). Let i_0 be so large that

$$\forall i \geq i_0 \big(\sup_{p\in\mathcal{Z}_{\ell_0}} |(y_i, J(x_i - p))| < \frac{1}{n_0+1} \big). \tag{4.29}$$

Then, (4.25) gives $\operatorname{dist}(x_i, A^{-1}0) \leq 1/(k+1)$ for all $i \geq i_0$. That is, $\lim_i \operatorname{dist}(x_i, A^{-1}0) = 0$. To establish (i) $\Rightarrow$ (iii), suppose (4.25) would be false; i.e.,

$$\begin{aligned}\exists k, K \in \mathbb{N}\, \forall n, \ell \in \mathbb{N}\, \exists x_{n,\ell}, y_{n,\ell} \in X \Big(&(x_{n,\ell}, y_{n,\ell}) \in A \wedge \|x_{n,\ell}\|, \|y_{n,\ell}\| \leq K \\ &\wedge \sup_{p\in\mathcal{Z}_\ell} |(y_{n,\ell}, J(x_{n,\ell} - p))| \leq \frac{1}{n+1} \wedge \operatorname{dist}(x_{n,\ell}, A^{-1}0) > \frac{1}{k+1} \Big).\end{aligned} \tag{4.30}$$

Define $\tilde{x}_n := x_{n,n}$ and $\tilde{y}_n := y_{n,n}$. Then,

$$\forall \ell \in \mathbb{N} \big(\lim_{n\to\infty} \sup_{p\in\mathcal{Z}_\ell} |(\tilde{y}_n, J(\tilde{x}_n - p))| = 0 \big). \tag{4.31}$$

Indeed, given $m \in \mathbb{N}$, using (4.30), we have that for any $n \geq \max\{m, \ell\}$,

$$\sup_{p\in\mathcal{Z}_\ell} |(\tilde{y}_n, J(\tilde{x}_n - p))| \leq \sup_{p\in\mathcal{Z}_n} |(\tilde{y}_n, J(\tilde{x}_n - p))| = \sup_{p\in\mathcal{Z}_n} |(y_{n,n}, J(x_{n,n} - p))| \leq \frac{1}{n+1} \leq \frac{1}{m+1}. \tag{4.32}$$

However, assumption (i), together with (4.31), implies that $\lim_{n\to\infty} \operatorname{dist}(x_{n,n}, A^{-1}0) = 0$, which contradicts (4.30). $\square$

Remark 4.7. *(For logicians).* Note, by Lemma 4.6, that (4.25) is essentially a uniform version of the property

$$\forall x^X, y^X\ \forall k^{\mathbb{N}}\ \exists n^{\mathbb{N}}, \ell^{\mathbb{N}}\Big((x,y)\in A \wedge \big(\sup_{p\in\mathcal{Z}_\ell}|(y, J(x-p))| \le \frac{1}{n+1}\big) \to \mathrm{dist}(x, A^{-1}0) < \frac{1}{k+1}\Big), \tag{4.33}$$

which can readily be seen to be equivalent to

$$\forall x^X, y^X\Big((x,y)\in A \wedge \forall p\in A^{-1}0\big((y, J(x-p))=0\big) \to x\in \overline{A^{-1}0}\Big). \tag{4.34}$$

The premise of (4.34) holds true, for instance, when $x\in A^{-1}0$ and $y=0$.

The nonuniform version (4.33) of (4.25) takes the form of a universal/existential formula. In fact, $\sup_{p\in\mathcal{Z}_\ell}|(y,J(x-p))| \le \frac{1}{n+1}$ can be replaced by

$$\forall p^X\big(\chi_A(p,0) =_{\mathbb{N}} 0 \wedge \|p\| <_{\mathbb{R}} l \to |(y, J(x-p))| \le_{\mathbb{R}} \frac{1}{n+1}\big),$$

where it is rewritten in prenex normal form as a universally quantified formula (taking into account the hidden quantifiers in $<_{\mathbb{R}}$ and $\le_{\mathbb{R}}$). Moreover, $\mathrm{dist}(x, A^{-1}0) < 1/(k+1)$ can be replaced by

$$\exists z^X(\chi_A(z,0) =_{\mathbb{N}} 0 \wedge \|x-z\| <_{\mathbb{R}} 1/(k+1)),$$

which is a purely existential formula. Formalizing them in the language of the systems used in the logical metatheorems due to the first author in [24, 33], and also in [46], one may use the metatheorems in [24, 33, 46] to extract a uniform bound on '$\exists n^{\mathbb{N}}, \ell^{\mathbb{N}}$', depending on a bound for $(x,y)\in A$ and suitable majorizing data of the other parameters involved from proofs of (4.33) that can be carried out in these systems.

Motivated by the preceding discussion and in light of the previous lemma (and the monotonicity of the quantifiers '$\exists n, l$' by which bounds are actually witnesses), we propose the following moduli for the generalized convergence condition.

Definition 4.8. A I-modulus for the generalized convergence condition is a function $\omega : \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ such that

$$\forall k^{\mathbb{N}}, K^{\mathbb{N}}, x^X, y^X\Big((x,y)\in A \wedge \|x\|, \|y\| \le K \wedge \big(\sup_{p\in\mathcal{Z}_{\omega(K,k)}}|(y, J(x-p))| \le \frac{1}{\omega(K,k)+1}\big) \to \mathrm{dist}(x, A^{-1}0) \le \frac{1}{k+1}\Big). \tag{4.35}$$

Remark 4.9. Note also that "$\forall k, K$" are monotone in (4.25) and so it would be equivalent to formulate the modulus as a unary function $\omega(N)$ satisfying (4.25) for all $k, K \le N$.

In light of (4.24), we propose an alternative modulus for the generalized convergence condition, to be carried out comparable to (4.35):

Definition 4.10. A II-modulus for the generalized convergence condition is a pair (Ω, Φ) of functionals of the forms $\Omega : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $\Phi : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ satisfying

$$\begin{aligned} &\forall k^{\mathbb{N}}, K^{\mathbb{N}}, g^{\mathbb{N}\to\mathbb{N}}, (x_i)^{\mathbb{N}\to X}, (y_i)^{\mathbb{N}\to X} \\ &\quad \Big(\forall i \big((x_i, y_i) \in A \wedge \|x_i\|, \|y_i\| \leq K \big) \wedge \forall \ell^{\mathbb{N}} \big(\sup_{p \in \mathcal{Z}_\ell} |(y_{g(\ell)}, J(x_{g(\ell)} - p))| \leq \frac{1}{\Omega(K,k)+1} \big) \\ &\quad \to \exists m \leq \Phi(\Omega(K,k), g) \big(\mathrm{dist}(x_m, A^{-1}0) \leq \frac{1}{k+1} \big) \Big). \end{aligned} \tag{4.36}$$

Proposition 4.11. *Let ω satisfy (4.35). Then, $\Omega := \omega$, $\Phi(r, g) := g(r)$ define a II-modulus satisfying (4.36).*

Proof. Let k, K, g, (x_i) and (y_i) be as in (4.36), and assume the premise of (4.36). Then for $\ell := \omega(K, k)$ one in particular has

$$\sup_{p \in \mathcal{Z}_\ell} |(y_{g(\ell)}, J(x_{g(\ell)} - p))| \leq \frac{1}{\Omega(K,k)+1} = \frac{1}{\omega(K,k)+1}.$$

Hence by (4.35), applied to $y := y_{g(\ell)}$ and $x := x_{g(\ell)}$, one gets

$$\mathrm{dist}(x_{g(\ell)}, A^{-1}0) \leq \frac{1}{k+1}.$$

So (4.36) is satisfied with

$$m := g(\ell) = g(\omega(K,k)) = g(\Omega(K,k)) = \Phi(\Omega(K,k), g).$$

□

Remark 4.12. In the proof of "(ii) ⇒ (i)" in Lemma 4.6, one can assume that g is strictly increasing. Hence, we can weaken "$\forall g$" in (4.24) and (4.36) to such functions g. For simplicity, we denote such functions by $g_{\nearrow}$.

The converse of proposition 4.11 also holds. More precisely, we show that from a pair (Ω, Φ) of functionals satisfying (4.36), one can construct a modulus ω satisfying (4.35), utilizing a bound $b \geq \|z\|$ for some $z \in A^{-1}0$.

Proposition 4.13. *Let (Ω, Φ) be a pair of functionals satisfying (4.36), and let $b \geq \|z\|$ for some $z \in A^{-1}0$, then*

$$\omega(K,k) := \max\{\Omega(\max\{K,b\},k), \Phi(\Omega(\max\{K,b\},k), id)\}$$

defines a I-modulus for the generalized convergence condition in the sense of (4.35).

Proof. Consider $b \in \mathbb{N}$ such that $b \geq \|z\|$ for some $z \in A^{-1}0$. For any $K \in \mathbb{N}$, we define $K_b := \max\{K, b\}$. Let (Ω, Φ) satisfy (4.36).

Claim. (4.36) implies the seemingly stronger form

$$\begin{aligned}&\forall k^{\mathbb{N}},K^{\mathbb{N}},\ g_{\nearrow}^{\mathbb{N}\to\mathbb{N}},\ (x_i)^{\mathbb{N}\to X},(y_i)^{\mathbb{N}\to X}\,\exists \ell\le \Phi(\Omega(K_b,k),g_{\nearrow})\\ &\quad\Big(\forall i\big((x_i,y_i)\in A\wedge \|x_i\|,\|y_i\|\le K_b\big)\wedge\big(\sup_{p\in\mathcal{Z}_\ell}|(y_{g_{\nearrow}(\ell)},J(x_{g_{\nearrow}(\ell)}-p))|\le\frac{1}{\Omega(K_b,k)+1}\big)\\ &\quad\to\exists m\le\Phi(\Omega(K_b,k),g_{\nearrow})\big(\mathrm{dist}(x_m,A^{-1}0)\le\frac{1}{k+1}\big)\Big).\end{aligned}\tag{4.37}$$

To prove the claim, let $k, K \in \mathbb{N}$, let $g : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function, and let $(x_i), (y_i)$ be given such that

$$\forall i\big((x_i,y_i)\in A\wedge \|x_i\|,\|y_i\|\le K_b\big),$$

and

$$\forall \ell\le N:=\Phi(\Omega(K_b,k),g)\big(\sup_{p\in\mathcal{Z}_\ell}|(y_{g(\ell)},J(x_{g(\ell)}-p))|\le\frac{1}{\Omega(K_b,k)+1}\big).\tag{4.38}$$

Define

$$\tilde{x}_n:=\begin{cases}x_n, & \text{if } n\le g(N),\\ z, & \text{otherwise,}\end{cases}$$

and

$$\tilde{y}_n:=\begin{cases}y_n, & \text{if } n\le g(N),\\ 0, & \text{otherwise.}\end{cases}$$

Note that $(\tilde{x}_i,\tilde{y}_i)\in A$ and $\|\tilde{x}_i\|,\|\tilde{y}_i\|\le K_b$. Then, (4.38) implies that

$$\forall \ell\in\mathbb{N}\big(\sup_{p\in\mathcal{Z}_\ell}|(\tilde{y}_{g(\ell)},J(\tilde{x}_{g(\ell)}-p))|\le\frac{1}{\Omega(K_b,k)+1}\big),$$

since for $\ell\le N$, we have $g(l)\le g(N)$ and so $\tilde{x}_{g(\ell)}=x_{g(\ell)}$, $\tilde{y}_{g(\ell)}=y_{g(\ell)}$, while for $\ell>N$, we have $g(\ell)>g(N)$, and thus

$$\sup_{p\in\mathcal{Z}_\ell}|(\tilde{y}_{g(\ell)},J(\tilde{x}_{g(\ell)}-p))|=\sup_{p\in\mathcal{Z}_\ell}|(0,J(z-p))|=0\le\frac{1}{\Omega(K_b,k)+1}.$$

Hence, by (4.36), and using that $g(N)\ge N$ as g is strictly increasing,

$$\exists m\le\underbrace{\Phi(\Omega(K_b,k),g)}_{N}\big(\mathrm{dist}(x_m,A^{-1}0)=\mathrm{dist}(\tilde{x}_m,A^{-1}0)\le\frac{1}{k+1}\big).$$

The proof of the claim is now complete.

Now, let $k, K \in \mathbb{N}$, and define $\omega(K,k):=\max\{\Omega(K_b,k),\Phi(\Omega(K_b,k),id)\}$. Let $(x,y)\in A$ with $\|x\|,\|y\|\le K$ and

$$\sup_{p\in\mathcal{Z}_{\omega(K,k)}}|(y,J(x-p))|\le\frac{1}{\omega(K,k)+1}.$$

Thus, we have for $\ell\le\Phi(\Omega(K_b,k),id)$,

$$\sup_{p\in\mathcal{Z}_\ell}|(y,J(x-p))|\le\frac{1}{\omega(K,k)+1}\le\frac{1}{\Omega(K_b,k)+1}.\tag{4.39}$$

Then, (4.39) is the premise of (4.37) with $x_i := x$, $y_i := y$, for all i, and $g := id$. Thus, (4.37) implies that $\mathrm{dist}(x, A^{-1}0) \le \frac{1}{k+1}$. Therefore, (4.35) holds for such ω and we have thus completed the proof. $\square$

Theorem 4.14. *Let X be a smooth Banach space, and assume that $A \subseteq X \times X$ is accretive such that $A^{-1}0 \neq \emptyset$. Let (Ω, Φ) be a II-modulus of the generalized convergence condition for A as in (4.36). Let $\sum_{i=1}^{\infty} \theta_1\theta_2 ... \theta_i = \infty$. Suppose that for some $k_0 \in \mathbb{N}^*$, $(c_i a_i) \subset [1/k_0, \infty)$ and $(a_i) \subset (0, k_0]$. Let $b_0, b_1 > 0$, and let $p_0 \in A^{-1}0$, $x \in X$ with $\|p_0\| \le b_0$ and $\|x\| \le b_1$. If (u_i) is a solution for the homogeneous form of* (1.4) *for initial point x, then we have:*

$$\forall k \in \mathbb{N}\, \forall n, m \ge \Phi(\Omega(K, 4k+3), g)\left(\|u_n - u_m\| \le \frac{1}{k+1}\right)$$

with $K := \lceil 4k_0^2(b_1 + b_0)\rceil + 1$, $g : \mathbb{N} \to \mathbb{N}$ defined by $g(\ell) := \mu_\ell(\Omega(K, 4k+3))$, and $\mu_\ell : \mathbb{N} \to \mathbb{N}$ defined by $\mu_\ell(\tilde{k}) := (1 + \tilde{k})\lceil \frac{k_0^2}{2}(b_1 + \ell)^2 \rceil$.

Proof. In view of (1.4), we may write

$$\begin{aligned} v_i :=& \frac{1}{c_i}((u_{i+1} - u_i) - \theta_i(u_i - u_{i-1})) \\ =& \frac{1}{c_i a_i}(a_i(u_{i+1} - u_i) - a_{i-1}(u_i - u_{i-1})) \in Au_i, \qquad \forall i \ge 1. \end{aligned} \tag{4.40}$$

From the accretivity of A and Lemma 2.1, for all $p \in A^{-1}0$ and $i \ge 1$, we have

$$\begin{aligned} 0 \le& c_i(v_i, J(u_i - p)) = (u_{i+1} - (1 + \theta_i)u_i + \theta_i u_{i-1}, J(u_i - p)) \\ =& (u_{i+1} - p, J(u_i - p)) - (1 + \theta_i)\|u_i - p\|^2 + \theta_i(u_{i-1} - p, J(u_i - p)) \\ \le& \frac{1}{2}\|u_{i+1} - p\|^2 + \frac{1}{2}\|u_i - p\|^2 - (1 + \theta_i)\|u_i - p\|^2 \\ &+ \frac{\theta_i}{2}\|u_{i-1} - p\|^2 + \frac{\theta_i}{2}\|u_i - p\|^2 \\ =& \frac{1}{2}(\|u_{i+1} - p\|^2 - \|u_i - p\|^2) - \frac{\theta_i}{2}(\|u_i - p\|^2 - \|u_{i-1} - p\|^2). \end{aligned}$$

Multiplying both sides of the above inequality by a_i and summing up from $i = n$ to m, we have

$$\begin{aligned} 0 \le \sum_{i=n}^{m} c_i a_i(v_i, J(u_i - p)) \le& \frac{a_m}{2}(\|u_{m+1} - p\|^2 - \|u_m - p\|^2) \\ &- \frac{a_{n-1}}{2}(\|u_n - p\|^2 - \|u_{n-1} - p\|^2). \end{aligned}$$

Taking liminf as $m \to \infty$, by using our assumption and Lemma 2.4, we get, for all n,

$$0 \le \sum_{i=n}^{\infty} c_i a_i(v_i, J(u_i - p)) \le \frac{a_{n-1}}{2}(\|u_{n-1} - p\|^2 - \|u_n - p\|^2). \tag{4.41}$$

Thus, by the assumptions, we deduce

$$0 \leq \sum_{i=n}^{\infty}(v_i, J(u_i - p)) \leq \frac{k_0^2}{2}(\|u_{n-1} - p\|^2 - \|u_n - p\|^2). \tag{4.42}$$

Thus, for any $p \in A^{-1}0$,

$$\forall i\big(\|u_{i+1} - p\| \leq \|u_i - p\|\big). \tag{4.43}$$

In particular, for the fixed $p_0 \in A^{-1}0$, we have $\|u_i - p_0\| \leq \|x\| + \|p_0\|$, leading to the following bounds for (u_i) and (v_i):

$$\forall i\big(\|u_i\| \leq \|x\| + 2\|p_0\| \text{ and } \|v_i\| \leq 4k_0^2(\|x\| + \|p_0\|\big). \tag{4.44}$$

Taking $K := \lceil 4k_0^2(b_1 + b_0)\rceil + 1$, we have

$$\forall i\big(\|u_i\|, \|v_i\| < K\big). \tag{4.45}$$

Now, choose an arbitrary $\ell \in \mathbb{N}$. For any $p \in \mathcal{Z}_\ell = \{p \in A^{-1}0 : \ \|p\| \leq \ell\}$ and $i \in \mathbb{N}$, let

$$\alpha_{i,p} := \frac{k_0^2}{2}\|u_i - p\|^2 \text{ and } L_\ell := \lceil \frac{k_0^2}{2}(\|x\| + \ell)^2\rceil. \tag{4.46}$$

Then, for all $p \in \mathcal{Z}_\ell$ and $i \in \mathbb{N}$, we have

$$\alpha_{i+1,p} \leq \alpha_{i,p} \leq L_\ell. \tag{4.47}$$

In view of (4.42), we know that

$$0 \leq \sum_{i=n}^{\infty}(v_i, J(u_i - p)) \leq \alpha_{n-1,p} - \alpha_{n,p}. \tag{4.48}$$

Claim. Let $\mu_\ell : \mathbb{N} \to \mathbb{N}$ be defined by $\mu_\ell(\tilde{k}) := (1+\tilde{k})L_\ell$, and let $\beta_{i,p} := (v_i, J(u_i-p)) \geq 0$. Then, μ_ℓ defines a rate of uniform convergence on $\mathcal{Z}_\ell$ for the sequence $(\beta_{i,p})_i$ towards 0:

In view of (4.42), we know that

$$0 \leq \sum_{i=n}^{\infty}\beta_{i,p} \leq \alpha_{n-1,p} - \alpha_{n,p}. \tag{4.49}$$

On the other hand, from $\alpha_{i+1,p} \leq \alpha_{i,p} \leq L_\ell$, it is easy (by contradiction) to prove that for all $p \in \mathcal{Z}_\ell$ and $\tilde{k} \in \mathbb{N}$,

$$\exists n \leq \mu_\ell(\tilde{k})\big(\alpha_{n-1,p} - \alpha_{n,p} \leq \frac{1}{1+\tilde{k}}\big). \tag{4.50}$$

This along with (4.49) implies that for such n,

$$\forall \tilde{k} \in \mathbb{N}\, \forall p \in \mathcal{Z}_\ell\big(\sum_{i=\mu_\ell(\tilde{k})}^{\infty}\beta_{i,p} \leq \sum_{i=n}^{\infty}\beta_{i,p} \leq \alpha_{n-1,p} - \alpha_{n,p} \leq \frac{1}{1+\tilde{k}}\big). \tag{4.51}$$

Consequently,

$$\forall \tilde{k}, \ell \in \mathbb{N}\, \forall i \geq \mu_\ell(\tilde{k})\, \forall p \in \mathcal{Z}_\ell\big(\beta_{i,p} \leq \frac{1}{1+\tilde{k}}\big). \tag{4.52}$$

This completes the proof of the claim.

Now, let $k \in \mathbb{N}$, and define $\tilde{k} := \Omega(K, 4k+3)$ with K as in (4.45). Then, by (4.52), we have, for all $\ell \in \mathbb{N}$, $p \in \mathcal{Z}_\ell$ and $i \geq \mu_\ell(\tilde{k})$,

$$(v_i, J(u_i - p)) = \beta_{i,p} \leq \frac{1}{\tilde{k}+1} = \frac{1}{\Omega(K, 4k+3)+1}. \tag{4.53}$$

At this stage, defining $g : \mathbb{N} \to \mathbb{N}$ by $g(\ell) := \mu_\ell(\Omega(K, 4k+3))$, we conclude in view of (4.53) and definition of Φ that there exists $m \leq \Phi(\Omega(K, 4k+3), g)$ such that

$$\mathrm{dist}(u_m, A^{-1}0) \leq \frac{1}{4k+4}.$$

Pick some $\tilde{p} \in A^{-1}0$ with

$$\mathrm{dist}(u_m, A^{-1}0) \leq \|u_m - \tilde{p}\| < \frac{1}{2k+2}.$$

This, along with (4.43), implies that for all $i \geq \Phi(\Omega(K, 4k+3), g)$:

$$\|u_i - \tilde{p}\| < \frac{1}{2k+2}.$$

Consequently, for all $m, n \geq \Phi(\Omega(K, 4k+3), g)$,

$$\|u_m - u_n\| \leq \|u_m - \tilde{p}\| + \|u_n - \tilde{p}\| < \frac{1}{k+1}.$$

□

Note that compared to Theorem 4.3, the above theorem neither uses that X is uniformly convex nor that A is m-accretive (but only that it is accretive).

Corollary 4.15. *With the same assumptions as in Theorem 4.14, if $\omega : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a I-modulus of the generalized convergence condition for A as in (4.35), then we have:*

$$\forall k \in \mathbb{N}\, \forall n, m \geq (1 + \omega(K, 4k+3))\lceil \frac{k_0^2}{2}(\|x\| + \omega(K, 4k+3))^2 \rceil \left(\|u_n - u_m\| \leq \frac{1}{k+1} \right)$$

with $K = \lceil 4k_0^2(b_1 + b_0) \rceil + 1$.

Proof. Let $\omega : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a modulus satisfying (4.35). Utilizing Proposition 4.11, $\Omega := \omega$, $\Phi(r, g) := g(r)$ define a II-modulus (Ω, Φ) for the generalized convergence condition. Then, in view of Theorem 4.14, we obtain the rate $\Phi(\Omega(K, 4k+3), g)$ with $K = \lceil 4k_0^2(b_1 + b_0) \rceil + 1$, $g : \mathbb{N} \to \mathbb{N}$ defined by $g(\ell) := \mu_\ell(\Omega(K, 4k+3))$, and $\mu_\ell : \mathbb{N} \to \mathbb{N}$ defined by $\mu_\ell(\tilde{k}) := (1 + \tilde{k})\lceil \frac{k_0^2}{2}(b_1 + \ell)^2 \rceil$. Then,

$$\begin{aligned} \Phi(\Omega(K, 4k+3), g) &= g(\Omega(K, 4k+3)) \\ &= (1 + \Omega(K, 4k+3))\lceil \frac{k_0^2}{2}(b_1 + \Omega(K, 4k+3))^2 \rceil \\ &= (1 + \omega(K, 4k+3))\lceil \frac{k_0^2}{2}(b_1 + \omega(K, 4k+3))^2 \rceil. \end{aligned}$$

□

Remark 4.16. *(For logicians).* As mentioned, the uniform convexity and smoothness of the Banach space X are not directly reflected in the extracted rate obtained in Theorem 4.3. More precisely, the rate does not depend on a modulus of convexity, nor does it depend on a modulus of uniform smoothness???or, equivalently, on a modulus for the norm-to-norm continuity of the duality map J on bounded sets. Logical metatheorems typically upgrade smoothness to uniform smoothness together with a modulus. However, J can also by hardwired into logical metatheorems by adding a constant J_X of type $1(X)(X)$, where 1 is the type $\mathbb{N} \to \mathbb{N}$, and the purely universal axiom

$$(J_X) :\equiv \forall x^X, y^X \big(J_X xx =_{\mathbb{R}} \|x\|_X^2 \wedge |J_X xx|_{\mathbb{R}} \leq_{\mathbb{R}} \|x\|_X \cdot_{\mathbb{R}} \|y\|_X \\ \wedge\ \forall \alpha^1, \beta^1, u^X, v^X \big(J_X x(\alpha \cdot_{\mathbb{R}} u +_X \beta \cdot_{\mathbb{R}} v) =_{\mathbb{R}} \alpha \cdot_{\mathbb{R}} J_X xu +_{\mathbb{R}} \beta \cdot_{\mathbb{R}} J_X xv\big)\big)$$

to the formal framework from Kohlenbach [30] (see Kohlenbach and Leuştean [35] for details).

This axiom clearly holds when J_X is interpreted in a smooth Banach space as the single-valued normalized duality map J. The proof above only uses (J_X) but not the axiom (J_X, ω_X) (also found in [35]) which states that ω_X is a modulus of uniform continuity on bounded sets of J_X.

Moreover, the uniform convexity in Theorem 4.3 is only used to guarantee that the nearest point projection map $P : X \to A^{-1}0$ be well-defined. Such a function, however, can be directly hardwired as a new constant P of type $X(X)$ with the following universal axiom (see also [46]):

$$\forall x^X, y^X \big((\chi_A(y, 0) =_0 0 \to (y - Px, J_X(x - Px)) \leq_{\mathbb{R}} 0) \wedge \chi_A(Px, 0) =_0 0\big).$$

The premise formalizes that y is a zero of A. Similarly, the second conjunct formalizes that Px is a zero of A. This axiom clearly holds when in a uniformly convex Banach space the constant P is interpreted as the metric projection onto $A^{-1}(0)$ while χ_A is interpreted as the characteristic function of (the graph of) A.

Both P and J_X are easily majorizable. For P, we have $\|Px\| \leq \|x\| + \|x - p_0\|$, since $\|x - Px\| \leq \|x - p_0\|$. The majorizability of J_X is discussed in detail on page 3454 of [35]. Hence the bound guaranteed to be extractable from a proof formalized in a logical framework based on the above axioms will be true in any Banach space which is smooth and uniformly convex but will not depend on any moduli for (uniform) smoothness or uniform convexity.

This treatment suffices, as neither the extensionality of P (i.e., $x =_X y \to Px =_X Py$) nor that of J is invoked in proof of Theorem 4.3. If those had been used, this would require a quantitative treatment of extensionality for P, respectively J, and hence modululi of (uniform) continuity on bounded sets for P and J, respectively. For P, this would require a modulus of uniform convexity for X. In the case of J, one would either need to stipulate this directly via axiom (J_X, ω_X) as presented in [35], or alternatively, axiomatize the uniform smoothness of X with a modulus τ (as on page 3456 in [35]), from which then a modulus ω_X can be computed according to [35, Proposition 2.5].

5. Moduli of the Convergence Condition: Further Generalization and Comparisons

The Yosida approximation is a powerful tool for studying the existence and asymptotic behavior of solutions to difference and differential inclusions governed by monotone (accretive) operators. For a Hilbert space H, if $A \subseteq H \times H$ is maximal monotone, $A^{-1}0 \neq \emptyset$, and A satisfies the convergence condition, then the Yosida approximation A_λ satisfies the convergence condition. Pazy [44] proved this result as follows:

let (x_i) be a bounded sequence in H and P be the projection onto the closed convex subset $A^{-1}0$. Let $\lim_{i\to\infty}(A_\lambda x_i, x_i - Px_i) = 0$. Then,

$$(A_\lambda x_i, x_i - Px_i) = (A_\lambda x_i, J_\lambda x_i - Px_i) + \lambda\|A_\lambda x_i\|^2 \tag{5.1}$$

implies that $\lim_{i\to\infty}\|x_i - J_\lambda x_i\| = \lim_{i\to\infty}\lambda\|A_\lambda x_i\| = 0$ and $\lim_{i\to\infty}(A_\lambda x_i, J_\lambda x_i - Px_i) = 0$ since $(A_\lambda x_i, J_\lambda x_i - Px_i) \geq 0$. By the convergence condition assumption on A, the continuity of P, and the property $A_\lambda x_i \in AJ_\lambda x_i$, we deduce that $\lim_{i\to\infty}\|x_i - Px_i\| = \lim_{i\to\infty}\|J_\lambda x_i - PJ_\lambda x_i\| = 0$.

Question: If X is a smooth and uniformly convex Banach space, and $A \subseteq X \times X$ is m-accretive, satisfying the 'convergence condition' and $A^{-1}0 \neq \emptyset$, does the Yosida approximation A_λ satisfy the 'convergence condition'?

The above-mentioned argument lacks the capacity for generalization to Banach spaces, since the equality corresponding to (5.1) in Banach spaces requires the linearity of the duality mapping. Therefore, the above question remains open. However, for the case of the 'generalized convergence condition' (Definition 4.5), we obtain the following result.

Proposition 5.1. *Let X be a smooth Banach space, and assume that $A \subseteq X \times X$ is an accretive operator such that $A^{-1}0 \neq \emptyset$. If A satisfies the generalized convergence condition, then its Yosida approximation A_λ does as well.*

Proof. Let $\lambda > 0$. For all sequences (x_i) in $R(I + \lambda A)$ and $(p_i) \subset A^{-1}0$, we have (using Lemma 2.1(1))

$$\begin{aligned}(A_\lambda x_i, J(x_i - p_i)) &= (A_\lambda x_i, J(J_\lambda x_i - p_i)) \\ &\quad + (A_\lambda x_i, J(J_\lambda x_i - p_i + \lambda A_\lambda x_i) - J(J_\lambda x_i - p_i)) \\ &\geq (A_\lambda x_i, J(J_\lambda x_i - p_i)) + \lambda^{-1}(\|J_\lambda x_i - p_i + \lambda A_\lambda x_i\| - \|J_\lambda x_i - p_i\|)^2,\end{aligned} \tag{5.2}$$

where J_λ is the resolvent of A. Now, let (x_i) be a bounded sequence in $R(I+\lambda A)$ such that $(A_\lambda x_i)$ is bounded, and also assume that for all bounded sequences (p_i) in $A_\lambda^{-1}0 = A^{-1}0$,

$$\lim_{i\to\infty}(A_\lambda x_i, J(x_i - p_i)) = 0. \tag{5.3}$$

We need to show that $\lim_{i\to\infty}\operatorname{dist}(x_i, A^{-1}0) = 0$.

We note that, for any $x \in R(I + \lambda A)$, we have $A_\lambda x \in AJ_\lambda x$ (see (2.2)). Thus $(A_\lambda x_i, J(J_\lambda x_i - p_i)) \geq 0$, and hence in view of (5.2) and (5.3) we deduce

$$\lim_{i\to\infty} (A_\lambda x_i, J(J_\lambda x_i - p_i)) = 0. \tag{5.4}$$

Since the sequences $(J_\lambda x_i)$ and $(A_\lambda x_i)$ with $(J_\lambda x_i, A_\lambda x_i) \in A$ are bounded and $\{p_i\}$ is an arbitrary bounded sequence in $A^{-1}0$, the generalized convergence condition for A implies that

$$\lim_{i\to\infty} \operatorname{dist}(J_\lambda x_i, A^{-1}0) = 0.$$

Now, for any $n \in \mathbb{N}$, we choose some $\tilde{p}_n \in A^{-1}0$ with

$$\|J_\lambda x_n - \tilde{p}_n\| < \operatorname{dist}(J_\lambda x_n, A^{-1}0) + \frac{1}{n+1}.$$

Thus, for the bounded sequence $(\tilde{p}_i)$ in $A^{-1}0$, we have

$$\lim_{i\to\infty} \|J_\lambda x_i - \tilde{p}_i\| \leq \lim_{i\to\infty} \operatorname{dist}(J_\lambda x_i, A^{-1}0) = 0. \tag{5.5}$$

From this, and by considering $\tilde{p}_i$ instead of p_i in (5.3) and (5.2), we conclude that

$$\lim_{i\to\infty} \|A_\lambda x_i\| = 0. \tag{5.6}$$

Therefore, by (5.5) and (5.6), we have

$$\lim_{i\to\infty} \operatorname{dist}(x_i, A^{-1}0) \leq \lim_{i\to\infty} \|x_i - \tilde{p}_i\| = \lim_{i\to\infty} \|J_\lambda x_i + \lambda A_\lambda x_i - \tilde{p}_i\| = 0,$$

as desired. □

Remark 5.2. If $A \subseteq H \times H$ is maximal monotone and $A^{-1}0 \neq \emptyset$, where H is a Hilbert space, then the following are equivalent:

(i) A_λ satisfies the convergence condition;
(ii) A_λ satisfies the generalized convergence condition;
(iii) for any bounded sequence (x_i), $\lim_{i\to\infty} \|A_\lambda x_i\| = 0 \to \lim_{i\to\infty} \operatorname{dist}(x_i, A^{-1}0) = 0$.

It suffices to note that for all bounded sequences $(x_i) \subset H$ and $(p_i) \subset A^{-1}0$, since $(A_\lambda x_i, J_\lambda x_i - p_i) \geq 0$ and

$$(A_\lambda x_i, x_i - p_i) = (A_\lambda x_i, J_\lambda x_i - p_i) + \lambda \|A_\lambda x_i\|^2, \tag{5.7}$$

we have $\lim_{i\to\infty}(A_\lambda x_i, x_i - p_i) = 0$ if and only if $\lim_{i\to\infty} \|A_\lambda x_i\| = 0$.

Motivated by Remark 5.2 (iii), and for reasons that will become clear in the next section, we introduce a weaker notion of the convergence condition.

Definition 5.3. Let A be a nonlinear set-valued operator in a Banach space X with $0 \in R(A)$ (i.e., $A^{-1}0 \neq \emptyset$). We say that A satisfies the 'convergence condition type' if for any bounded sequence (x_n) in $D(A)$, we have:

$$\lim_n \operatorname{dist}(0, Ax_n) = 0 \to \lim_n \operatorname{dist}(x_n, A^{-1}0) = 0. \tag{5.8}$$

From the definitions, it follows for smooth Banach spaces that:

Generalized convergence condition (Def. 4.5) $\Rightarrow$ Convergence condition type (Def. 5.3).

Remark 5.4. If the nearest point projection P onto $A^{-1}0$ is well-defined (e.g., when X is smooth and uniformly convex, and A is m-accretive), then (5.8) is easily seen to be equivalent to

$$\lim_n \|A^0 x_n\| = 0 \to \lim_n \|x_n - Px_n\| = 0, \tag{5.9}$$

where $A^0 x$ is the unique element in Ax with minimum norm.

Remark 5.5. *(For logicians).* It is easy to show that the convergence condition type (5.8) is equivalent to the following property:

$$\forall k^{\mathbb{N}}, K^{\mathbb{N}}\, \exists n^{\mathbb{N}} \forall x^X, y^X \big((x,y) \in A \wedge \|x\| \le K \wedge \|y\| \le \frac{1}{n+1} \to \mathrm{dist}(x, A^{-1}0) \le \frac{1}{k+1}\big). \tag{5.10}$$

In fact, (5.10) is the uniform version of the property

$$\forall x^X, y^X\, \forall k^{\mathbb{N}}\, \exists n^{\mathbb{N}} \big((x,y) \in A \wedge \|y\| \le \frac{1}{n+1} \to \mathrm{dist}(x, A^{-1}0) \le \frac{1}{k+1}\big), \tag{5.11}$$

which is equivalent (using extensionality) to the trivial implication

$$\forall x \in D(A)\big(0 \in Ax \to x \in \overline{A^{-1}0}\big). \tag{5.12}$$

Without extensionality (5.11) is equivalent to

$$\forall x^X, y^X\, \forall k^{\mathbb{N}}\, \exists n^{\mathbb{N}} \big((x,y) \in A \wedge \|y\| \le \frac{1}{n+1} \to \exists p^X ((p,0) \in A \wedge \|x-p\| \le \frac{1}{k+1})\big), \tag{5.13}$$

where '$\big(\dots\big)$' is an $\exists$-formula. Hence if (5.13) is provable in formal systems which have a bound extraction metatheorem, one can extract from the proof a bound on (and hence a witness for) '$\exists n^{\mathbb{N}}$' which only depends on k and a norm bound $K \ge \|x\|$ (note that w.l.o.g. we may assume that $\|y\| \le 1$). This suggests the next definition.

We know define a modulus for the convergence condition type:

Definition 5.6. A modulus for the convergence condition type of A is a function $\Omega : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$\forall k^{\mathbb{N}}, K^{\mathbb{N}}, x^X, y^X \big((x,y) \in A \wedge \|x\| \le K \wedge \|y\| \le \frac{1}{\Omega(k,K)+1} \to \mathrm{dist}(x, A^{-1}0) \le \frac{1}{k+1}\big). \tag{5.14}$$

Here, we examine the consistency of the Yosida approximation with respect to the notion of convergence condition type (Definition 5.3).

Proposition 5.7. *Let A be an accretive operator in a Banach space X with $A^{-1}0 \neq \emptyset$. If A satisfies the convergence condition type, then so does its Yosida approximation A_λ. The converse also holds, provided that A is additionally m-accretive.*

Proof. Let $\lambda > 0$, and let (x_n) be a bounded sequence in $R(I+\lambda A)$ such that $\lim_n \|A_\lambda x_n\| = 0$. From $A_\lambda x_n \in AJ_\lambda x_n$, we the have $\lim_n \mathrm{dist}(0, AJ_\lambda x_n) = 0$. Since $(J_\lambda x_n)$ is bounded and A satisfies the convergence condition type, we deduce that $\lim_n \mathrm{dist}(J_\lambda x_n, A^{-1}0) = 0$. Now, we may choose a sequence (p_n) in $A^{-1}0 = A_\lambda^{-1}0$ with $\lim_n \|J_\lambda x_n - p_n\| = 0$. Then, $\lim_n \|x_n - p_n\| = 0$ since $\|x_n - p_n\| \leq \|\lambda A_\lambda x_n\| + \|J_\lambda x_n - p_n\|$. Thus, $\lim_n \mathrm{dist}(x_n, A^{-1}0) = 0$, and therefore A_λ satisfies the convergence condition type. For the converse, assume that A is m-accretive and that its Yosida approximation A_λ satisfies the convergence condition type. We show that A does as well. Let (x_n) be a bounded sequence in $D(A)$ such that $\lim_n \mathrm{dist}(0, Ax_n) = 0$. By Lemma 2.3 (2), $\|A_\lambda x_n\| \leq \mathrm{dist}(0, Ax_n)$. Thus, $\lim_n \|A_\lambda x_n\| = 0$, and by assumption, $\lim_n \mathrm{dist}(x_n, A^{-1}0) = \lim_n \mathrm{dist}(x_n, A_\lambda^{-1}0) = 0$, which completes the proof.

□

We recall the notion of the modulus of regularity, which was originally introduced in [36].

Definition 5.8. Let (M, d) be a metric space and $F : M \to \mathbb{R}$ be a mapping with $zerF = \{x \in M : F(x) = 0\} \neq \emptyset$. Fixing $z \in zerF$ and $r > 0$, we say that $\phi : (0, \infty) \to (0, \infty)$ is a modulus of regularity for F w.r.t. $zerF$ and $\overline{B}(z, r)$, if for all $\varepsilon > 0$ and $x \in \overline{B}(z, r)$ we have the following:

$$|F(x)| < \phi(\varepsilon) \Rightarrow \mathrm{dist}(x, zerF) < \varepsilon.$$

In the following, we show that a modulus $\Omega : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ for the convergence condition type of an accretive operator A transfers to a modulus of regularity $\phi : (0, \infty) \to (0, \infty)$ for the absolute value of its Yosida approximation $|A_\lambda|$, where $|A_\lambda|(x) := \|A_\lambda(x)\|$ for $x \in R(I + \lambda A)$. Conversely, if A is additionally m-accretive, a modulus of regularity for $|A_\lambda|$ yields a modulus for the convergence condition type of A.

Theorem 5.9. *Let A be an accretive operator in a Banach space X with $A^{-1}0 \neq \emptyset$. Then:*

1. *(1) If $\Omega : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ is a modulus for the convergence condition type of A, then for any $z \in A^{-1}0$ and $r > 0$, $\phi_{z,r} : (0, \infty) \to (0, \infty)$ defined by $\phi_{z,r}(\varepsilon) := (1 + \Omega_\lambda(\lceil \varepsilon^{-1} \rceil + 1, \|z\| + r))^{-1}$, where $\Omega_\lambda(k, K) := \max\{(2k + 2)\lceil \lambda \rceil, 1 + \Omega(2k + 1, K + \lceil \lambda \rceil)\} - 1$, is a modulus of regularity for $|A_\lambda| : R(I + \lambda A) \to \mathbb{R}$ w.r.t. $zer|A_\lambda| = A^{-1}0$ and $\overline{B}(z, r) \cap R(I + \lambda A)$.*
2. *(2) Conversely, if A is additionally m-accretive and $|A_\lambda|$ admits a modulus of regularity $\phi_{z,r}$ for any $z \in zer|A_\lambda|$ and $r > 0$, then one can extract a modulus Ω for the convergence condition type of A by defining $\Omega(k, K) := \lceil \phi_{z,K+\lceil \|z\| \rceil}(\frac{1}{k+1}) \rceil^{-1}$.*

The proof of Theorem 5.9 is preceded by two supporting lemmas. The first lemma presents a quantitative refinement of Proposition 5.7:

Lemma 5.10. *Let A be an accretive operator in a Banach space X with $A^{-1}0 \neq \emptyset$. Then:*

(1) If Ω is a modulus for the convergence condition type of A, then $\Omega_\lambda : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by

$$\Omega_\lambda(k, K) := \max\{(2k+2)\lceil\lambda\rceil, 1 + \Omega(2k+1, K + \lceil\lambda\rceil)\} - 1, \tag{5.15}$$

is a modulus for the convergence condition type of the Yosida approximation A_λ.

(2) Conversely, if A is additionally m-accretive and Ω is a modulus for the convergence condition type of A_λ, it follows that Ω is likewise a modulus for the convergence condition type of A.

Proof. Let Ω be a modulus for the convergence condition type of A. Suppose that Ω_λ is defined as in (5.15). Consider $k, K \in \mathbb{N}$, and let $x \in R(I + \lambda A)$ such that $\|x\| \leq K$ and

$$\|A_\lambda x\| \leq \frac{1}{1 + \Omega_\lambda(k, K)}. \tag{5.16}$$

Thus,

$$\|J_\lambda x\| \leq \|x\| + \lambda\|A_\lambda x\| \leq K + \frac{\lambda}{1 + \Omega_\lambda(k, K)} \leq K + \lceil\lambda\rceil. \tag{5.17}$$

In view of (5.16) and (5.15), we have

$$\|A_\lambda x\| \leq \frac{1}{1 + \Omega(2k+1, K + \lceil\lambda\rceil)}. \tag{5.18}$$

Since $A_\lambda x \in AJ_\lambda x$, it follows from (5.17), (5.18) and the definition of Ω, that

$$\operatorname{dist}(J_\lambda x, A^{-1}0) \leq \frac{1}{2k+2}. \tag{5.19}$$

From (5.16) and (5.15), we also have

$$\|A_\lambda x\| \leq \frac{1}{(2k+2)\lceil\lambda\rceil}. \tag{5.20}$$

Now, combining (5.19) and (5.20), we obtain

$$\operatorname{dist}(x, A^{-1}0) = \operatorname{dist}(\lambda A_\lambda x + J_\lambda x, A^{-1}0) \leq \lambda\|A_\lambda x\| + \operatorname{dist}(J_\lambda x, A^{-1}0) \leq \frac{1}{k+1}.$$

This completes the proof of (1). To prove (2), assume that A is m-accretive. Then, by Lemma 2.3, $\|A_\lambda x\| \leq \operatorname{dist}(0, Ax)$ for each $x \in D(A)$. This inequality implies that any modulus Ω for the convergence condition type of A_λ also serves as a modulus for the convergence condition type of A. □

The proof of the following lemma is straightforward and is therefore omitted.

Lemma 5.11. *Let $\tilde{A} : D(\tilde{A}) \to X$ be a single-valued operator such that $\tilde{A}^{-1}0 \neq \emptyset$. Then:*

(1) Let Ω be a modulus of convergence condition type for $\tilde{A}$. Fixing $z \in zer\tilde{A}$ and $r > 0$, the function $\phi_{z,r}(\varepsilon) := (1 + \Omega(\lceil\varepsilon^{-1}\rceil + 1, \|z\| + r))^{-1}$ defines a modulus of regularity for $|\tilde{A}| : D(\tilde{A}) \to \mathbb{R}$ w.r.t. $zer\tilde{A}$ and $\overline{B}(z, r) \cap D(\tilde{A})$, where $|\tilde{A}|(x) := \|\tilde{A}(x)\|$.

(2) *Conversely, if* $\phi_{z,r} : (0,\infty) \to (0,\infty)$ *is a modulus of regularity for* $|\tilde{A}|$ *w.r.t.* $zer|\tilde{A}|$ *and* $\overline{B}(z,r) \cap D(\tilde{A})$*, where* $z \in zerF$ *and* $r > 0$*, then the functional* $\Omega : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$*, defined by*

$$\Omega(k,K) := \lceil \phi_{z,K+\lceil \|z\| \rceil}(\frac{1}{k+1}) \rceil^{-1}, \tag{5.21}$$

is a modulus of convergence condition type for $\tilde{A}$*.*

Here is the proof of Theorem 5.9 using Lemmas 5.10 and 5.11:

Proof of Theorem 5.9.

Let $\Omega : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a modulus for the convergence condition type of A. By Lemma 5.10, Ω_λ defined in (5.15) is a modulus for the convergence condition type of the Yosida approximation A_λ. Since A_λ is single-valued, Lemma 5.11 implies that for any $z \in zerA$ and $r > 0$, the function $\phi_{z,r}(\varepsilon) := (1 + \Omega_\lambda(\lceil \varepsilon^{-1} \rceil + 1, \|z\| + r))^{-1}$ defines a modulus of regularity for $|A_\lambda| : R(I + \lambda A) \to \mathbb{R}$ w.r.t. $zerA$ and $\overline{B}(z,r) \cap R(I + \lambda A)$. This completes the proof of (1). Now, assume that A is m-accretive and $|A_\lambda|$ admits a modulus of regularity $\phi_{z,r}$ for any $z \in zer|A_\lambda|$ and $r > 0$. Then, by Lemma 5.11, the functional Ω defined by (5.21) is a modulus of convergence condition type for A_λ. It then follows from Lemma 5.10 that Ω is also a modulus for the convergence condition type of A, thereby completing the proof of (2). □

Let C be a closed convex subset of a Banach space X. Consider a mapping $T : C \to X$ with $Fix(T) \neq \emptyset$, and define $F : C \to \mathbb{R}$ by $F(x) = \|x - Tx\|$. Let $z \in Fix(T)$ and $r > 0$. Similarly, a modulus of regularity for T with respect to $Fix(T)$ and $\overline{B}(z,r)$ is defined as a modulus of regularity for F with respect to $zerF$ and $\overline{B}(z,r)$.

Remark 5.12. If C is additionally locally compact (e.g., if $\dim X < \infty$), T is continuous, $z \in Fix(T)$, and $r > 0$, then T has a modulus of regularity with respect to $Fix(T)$ and $\overline{B}(z,r)$ (see [36, Corollary 3.5]).

Remark 5.13. Let A be an accretive operator in a Banach space X with $A^{-1}0 \neq \emptyset$. Since $J_\lambda : R(I + \lambda A) \to X$ is nonexpansive, $Fix(J_\lambda) = A^{-1}0 = A_\lambda^{-1}0$, and $\lambda A_\lambda = I - J_\lambda$, it follows that for $z \in Fix(J_\lambda)$ and $r > 0$, ϕ is a modulus of regularity for J_λ w.r.t $Fix(T)$ and $\overline{B}(z,r)$ if and only if $\lambda^{-1}\phi$ is a modulus of regularity for $|A_\lambda|$ w.r.t $A^{-1}0$ and $\overline{B}(z,r) \cap R(I + \lambda A)$. Therefore, we may apply Theorem 5.9 to transfer a modulus of regularity for J_λ into a modulus of convergence condition type for A, and vice versa. Specifically, when $\dim X < \infty$ and A is m-accretive, Remark 5.12 ensures that A possesses a modulus of convergence condition type.

6. Rates of Metastability and Convergence for Nonhomogeneous Difference Inclusions

In this section, we establish quantitative results on the strong convergence of solutions to the nonhomogeneous problem (1.4) for an accretive A in Banach spaces which are uniformly convex. We also assume either $\forall i(0 < \theta_i \leq 1)$ or $\forall i(1 \leq \theta_i)$. It is worth noting that the strong convergence of solutions to (1.4) in the nonhomogeneous case has been investigated in the literature under the two scenarios for (θ_i) mentioned above (see, e.g., [5, 23, 22, 27]). In fact, unlike the homogeneous case, constructing monotone sequences using the products of coefficients θ_i is essential due to the absence of the monotone property $\|u_{i+1} - p\| \leq \|u_i - p\|$, $p \in A^{-1}0$, in the nonhomogeneous case.

In this case, the proof procedure relies on establishing the property

$$\forall k \in \mathbb{N}\, \exists n \in \mathbb{N}\, \exists v \in Au_n(\|v\| \leq \frac{1}{k+1}),$$

which serves as the premise of the 'convergence condition type' (Definition 5.3). Therefore, considering these separate cases for θ_i allows us to apply the 'convergence condition type,' which is the most general condition, encompassing both the standard convergence condition and the generalized convergence condition.

To proceed, we first establish the following lemmas.

Lemma 6.1. *Let X be a Banach space, and let (f_i) be a sequence in X. Then, for any $n \geq 1$,*

$$\sum_{i=n}^{\infty}\sum_{k=i}^{\infty} \frac{\|f_k\|}{\theta_k\theta_{k-1}...\theta_i} \leq \sum_{i=n}^{\infty} h_i\|f_i\|, \tag{6.1}$$

where $h_k := \sum_{i=1}^{k} \frac{1}{\theta_k\theta_{k-1}...\theta_i}$ and $\theta_i > 0$.

Proof. By the proof of Lemma 3.3 in [18], we have

$$\sum_{i=n}^{\infty}\sum_{k=i}^{\infty} \frac{\|f_k\|}{\theta_k\theta_{k-1}\cdots\theta_i} = \frac{1}{\theta_n}\|f_n\| + (\frac{1}{\theta_{n+1}} + \frac{1}{\theta_{n+1}\theta_n})\|f_{n+1}\|$$

$$+ (\frac{1}{\theta_{n+2}} + \frac{1}{\theta_{n+2}\theta_{n+1}} + \frac{1}{\theta_{n+2}\theta_{n+1}\theta_n})\|f_{n+2}\| + \cdots + (\sum_{i=n}^{n+m} \frac{1}{\theta_{n+m}...\theta_i})\|f_{n+m}\| + \cdots .$$

Then, (6.1) follows immediately from $\sum_{i=n}^{n+m} \frac{1}{\theta_{n+m}...\theta_i} \leq h_{n+m}$. □

Lemma 6.2. *Let X be a Banach space and let $A \subseteq X \times X$ be an accretive operator such that $A^{-1}0 \neq \emptyset$. Let $p_0 \in A^{-1}0$ and $x \in X$. Let $\sum_{i=1}^{\infty} \theta_1\theta_2...\theta_i = \infty$. Suppose that $\sum_{i=1}^{\infty} h_i\|f_i\| < C$, and let (u_i) be a solution for (1.4) with the initial point x. Then $(u_i) \subset B_b(0)$, for $b \geq \|x\| + 2\|p_0\| + C$.*

Proof. Since $p_0 \in A^{-1}0$, by the accretivity of A and (1.4), we have

$$(u_{i+1} - (1+\theta_i)u_i + \theta_i u_{i-1} - f_i, j(u_i - p_0)) \geq 0, \qquad \forall i \geq 1. \tag{6.2}$$

Then, it is easy to check that we have

$$(\|u_{i+1} - p_0\| - \|u_i - p_0\|) - \theta_i(\|u_i - p_0\| - \|u_{i-1} - p_0\|) + \|f_i\| \geq 0,$$

for all $i \geq 1$. Therefore

$$\begin{aligned}
&\|u_i - p_0\| - \|u_{i-1} - p_0\| \\
&\quad \leq \frac{1}{\theta_i}(\|u_{i+1} - p_0\| - \|u_i - p_0\|) + \frac{\|f_i\|}{\theta_i} \\
&\quad \leq \frac{1}{\theta_{i+1}\theta_i}(\|u_{i+2} - p_0\| - \|u_{i+1} - p_0\|) + \frac{\|f_{i+1}\|}{\theta_{i+1}\theta_i} + \frac{\|f_i\|}{\theta_i} \\
&\quad \vdots \\
&\quad \leq \frac{1}{\theta_{i+j}\cdots\theta_{i+1}\theta_i}(\|u_{i+j+1} - p_0\| - \|u_{i+j} - p_0\|) + \sum_{k=i}^{i+j} \frac{\|f_k\|}{\theta_k\theta_{k-1}\cdots\theta_i} \\
&\quad = a_{i-1}^{-1}a_{i+j}(\|u_{i+j+1} - p_0\| - \|u_{i+j} - p_0\|) + \sum_{k=i}^{i+j} \frac{\|f_k\|}{\theta_k\theta_{k-1}\cdots\theta_i},
\end{aligned}$$

for all $i \geq 1, j \geq 0$. Taking liminf as $j \to \infty$, by $\sum_{i=1}^{\infty} a_i^{-1} = \infty$, (1.4) and Lemma 2.4, we obtain for all $i \geq 1$,

$$\|u_i - p_0\| - \|u_{i-1} - p_0\| \leq \sum_{k=i}^{\infty} \frac{\|f_k\|}{\theta_k\theta_{k-1}...\theta_i}. \tag{6.3}$$

Thus, by (6.1), we have

$$\|u_m - p_0\| \leq \|u_0 - p_0\| + \sum_{i=1}^{m}\sum_{k=i}^{\infty} \frac{\|f_k\|}{\theta_k\theta_{k-1}...\theta_i} < \|x - p_0\| + C, \quad \forall m \geq 1. \tag{6.4}$$

This completes the proof. □

Lemma 6.3. *Let X be a uniformly convex Banach space with a modulus of uniform convexity δ, and let $A \subseteq X \times X$ be accretive such that $A^{-1}0 \neq \emptyset$. Let $\sum_{i=1}^{\infty} \theta_1\theta_2...\theta_i = \infty$. Let $p_0 \in A^{-1}0$ and $x \in X$. Suppose that $\sum_{i=1}^{\infty} h_i\|f_i\| < C$, and let (u_i) be a solution for (1.4) with the initial point x. Let $b \geq \|x\| + 2\|p_0\| + C$ and $\tilde{g}(\varepsilon) = b^2\tilde{\delta}(\frac{\varepsilon}{b})^2$ on $\mathbb{R}_+$. Then*

(1) If $\forall i(0 < \theta_i \leq 1)$, then $\Sigma_{i=1}^{\infty}\tilde{g}(\|u_i - u_{i-1}\|) < 3b^2$.
(2) If $\forall i(1 \leq \theta_i)$, then $\Sigma_{i=1}^{\infty}\tilde{g}(a_{i-1}\|u_i - u_{i-1}\|) < 3b^2$.

Proof. First note that using (6.2) we have

$$(u_{i+1} - u_i, j(u_i - p_0)) - \theta_i(u_i - u_{i-1}, j(u_i - p_0)) - (f_i, j(u_i - p_0)) \geq 0, \quad \forall i. \tag{6.5}$$

We prove (1). As in the proof of [18, Lemma 3.2], one shows, using Proposition 3.3 and Lemma 6.2, that

$$\Sigma_{i=k}^{\infty}\tilde{g}(\|u_i - u_{i-1}\|) \leq -(u_k - u_{k-1}, j(u_{k-1} - p_0)) + \Sigma_{i=k}^{\infty}\frac{a_i}{a_{k-1}}\|f_i\|\|u_i - p_0\|.$$

Therefore, from (6.4) and since $a_{k-1} \geq 1$, we obtain

$$\Sigma_{i=k}^{\infty}\tilde{g}(\|u_i - u_{i-1}\|) < 2b^2 + b\Sigma_{i=k}^{\infty}h_i\|f_i\| < 2b^2 + bC \leq 3b^2.$$

We now prove (2). We assume $\forall i(1 \leq \theta_i)$. Thus $\forall i(a_i \leq 1)$ and so for all i, j, we have $a_i u_j \in B_b(0)$. Thus, applying Proposition 3.3, we have

$$(a_{i-1}u_i - a_{i-1}u_{i-1}, j(a_{i-1}(u_i - p_0)) - j(a_{i-1}(u_{i-1} - p_0))) \geq \tilde{g}(a_{i-1}\|u_i - u_{i-1}\|),$$

for all i. That is,

$$a_{i-1}(u_i - u_{i-1}, j(u_i - p_0)) - a_{i-1}(u_i - u_{i-1}, j(u_{i-1} - p_0)) \geq a_{i-1}^{-1}\tilde{g}(a_{i-1}\|u_i - u_{i-1}\|), \quad (6.6)$$

for all i. On the other hand, multiplying both sides of (6.5) by a_i, we have

$$a_i(u_{i+1} - u_i, j(u_i - p_0)) - a_{i-1}(u_i - u_{i-1}, j(u_i - p_0)) - a_i(f_i, j(u_i - p_0)) \geq 0, \quad (6.7)$$

for all i. Combining (6.7) and (6.6), we deduce

$$\begin{aligned} a_{i-1}^{-1}\tilde{g}(a_{i-1}\|u_i - u_{i-1}\|) &\leq a_i(u_{i+1} - u_i, j(u_i - p_0)) \\ &\quad - a_{i-1}(u_i - u_{i-1}, j(u_{i-1} - p_0)) - a_i(f_i, j(u_i - p_0)). \end{aligned} \quad (6.8)$$

Now, summing up from $i = k$ to m and using again Lemma 2.1 (2), we arrive at

$$\begin{aligned} \Sigma_{i=k}^m a_{i-1}^{-1}\tilde{g}(a_{i-1}\|u_i - u_{i-1}\|) &\leq \frac{a_m}{2}(\|u_{m+1} - p_0\|^2 - \|u_m - p_0\|^2) \\ &\quad - a_{k-1}(u_k - u_{k-1}, j(u_{k-1} - p_0)) + \Sigma_{i=k}^m a_i\|f_i\|\|u_i - p_0\|. \end{aligned} \quad (6.9)$$

Taking liminf, as $m \to \infty$, and using Lemma 2.4, we obtain

$$\Sigma_{i=k}^\infty a_{i-1}^{-1}\tilde{g}(a_{i-1}\|u_i - u_{i-1}\|) \leq -a_{k-1}(u_k - u_{k-1}, j(u_{k-1} - p_0)) + \Sigma_{i=k}^\infty a_i\|f_i\|\|u_i - p_0\|.$$

Since $\forall i(a_i \leq 1)$ in this case, we deduce

$$\begin{aligned} &\Sigma_{i=k}^\infty \tilde{g}(a_{i-1}\|u_i - u_{i-1}\|) \leq \Sigma_{i=k}^\infty a_{i-1}^{-1}\tilde{g}(a_{i-1}\|u_i - u_{i-1}\|) \\ &\quad \leq -a_{k-1}(u_k - u_{k-1}, j(u_{k-1} - p_0)) + \Sigma_{i=k}^\infty a_i\|f_i\|\|u_i - p_0\| < 2b^2 + b\Sigma_{i=k}^\infty h_i\|f_i\| < 3b^2. \end{aligned}$$

□

Theorem 6.4. *Let X be a uniformly convex Banach space with a modulus of uniform convexity δ, and assume that $A \subseteq X \times X$ is accretive such that $A^{-1}0 \neq \emptyset$ and satisfies the convergence condition type with a modulus Ω. Let $p_0 \in A^{-1}0$, $x \in X$ and $k_0 \in \mathbb{N}$. Let $\sum_{i=1}^\infty \theta_1\theta_2...\theta_i = \infty$, $(a_ic_i), (c_i) \subseteq [1/k_0, \infty)$, and let (u_i) be a solution for (1.4) with the initial point x. Assume that $\forall i(0 < \theta_i \leq 1)$ or $\forall i(1 \leq \theta_i)$. If $\sum_{i=1}^{\infty} h_i\|f_i\| < C \in \mathbb{N}$, then the following holds:*

$$\forall k^{\mathbb{N}}\, \forall f^{\mathbb{N}\to\mathbb{N}}\, \exists n \leq \Psi(k, f, C, \delta, \Omega, x, p_0)\, \forall i, j \in [n; n + f(n)] \left(\|u_i - u_j\| < \frac{1}{k+1} \right),$$

where $\Psi(k, f, C, \delta, \Omega, x, p_0) := \Phi(h, k, C) + \Gamma(k)$, $\Phi(h, k, C) := \tilde{h}^{(C(4k+4))}(0)$, $\tilde{h}(n) := n + h(n)$, $h(n) := \Gamma(k) + \max\{f(i);\ n < i \leq n + \Gamma(k)\}$, $\tilde{g}(t) = b^2\tilde{\delta}(\frac{t}{b})^2$ on $\mathbb{R}_+$, $b \geq \|x\| + 2\|p_0\| + C$, $\tilde{\mu}(\epsilon) := \lceil \frac{7b^2}{\epsilon} \rceil$ on $\mathbb{R}_+ \setminus \{0\}$, and

$$\Gamma(k) := \tilde{\mu}(\min\{\frac{1}{3k_0(1 + \Omega(4k+4, \lceil b \rceil))}, \tilde{g}(\frac{1}{3k_0(1 + \Omega(4k+4, \lceil b \rceil))})\}). \quad (6.10)$$

Moreover,

$$\forall k^{\mathbb{N}}\,\forall f^{\mathbb{N}\to\mathbb{N}}\,\exists n \le \Psi(k, f, C, \delta, \Omega, x, p_0)\,\forall i \in [n; n + f(n)]\Big(dist(u_i, A^{-1}0) < \frac{1}{2k+2}\Big). \tag{6.11}$$

Consequently, (u_i) *converges to an element of* $\overline{A^{-1}0}$*.*

Proof. From (1.4), we have

$$v_i := \frac{1}{c_i}((u_{i+1} - u_i) - \theta_i(u_i - u_{i-1}) - f_i) \in Au_i, \qquad \forall i \ge 1. \tag{6.12}$$

In view of (2.4), we get

$$v_i = \frac{1}{c_i a_i}(a_i(u_{i+1} - u_i) - a_{i-1}(u_i - u_{i-1}) - a_i f_i). \tag{6.13}$$

For $\epsilon > 0$, define $\tilde{\mu}(\epsilon) := \lceil \frac{7b^2}{\epsilon} \rceil$.

Claim 1: If $\forall i(0 < \theta_i \le 1)$, then

$$\forall \tilde{k} \in \mathbb{N}\,\forall \epsilon > 0 \exists n\big(\tilde{k} < n \le \tilde{k} + \tilde{\mu}(\epsilon) \wedge \tilde{g}(\|u_{n+1} - u_n\|) + \tilde{g}(\|u_n - u_{n-1}\|) + \|f_n\| < \epsilon\big). \tag{6.14}$$

Suppose that $\forall i(0 < \theta_i \le 1)$. Then $\sum\limits_{i=1}^{\infty} \|f_i\| \le \sum\limits_{i=1}^{\infty} h_i\|f_i\| < C$. We know from Lemma 6.3 (1) that

$$\Sigma_{i=1}^{\infty}\big(\tilde{g}(\|u_{i+1} - u_i\|) + \tilde{g}(\|u_i - u_{i-1}\|) + \|f_i\|\big) < 3b^2 + 3b^2 + C \le 7b^2.$$

This implies the claim, since otherwise there exists $\epsilon > 0$ such that

$$\epsilon\lceil\frac{7b^2}{\epsilon}\rceil = \epsilon\tilde{\mu}(\epsilon) \le \Sigma_{i=1}^{\infty}\big(\tilde{g}(\|u_{i+1} - u_i\|) + \tilde{g}(\|u_i - u_{i-1}\|) + \|f_i\|\big) < 7b^2,$$

which is a contradiction.

Claim 2: If $\forall i(1 \le \theta_i)$, then

$$\forall \tilde{k} \in \mathbb{N}\,\forall \epsilon > 0\,\exists n \\ \big(\tilde{k} < n \le \tilde{k} + \tilde{\mu}(\epsilon) \wedge \tilde{g}(a_n\|u_{n+1} - u_n\|) + \tilde{g}(a_{n-1}\|u_n - u_{n-1}\|) + a_n\|f_n\| < \epsilon\big). \tag{6.15}$$

Suppose that $\forall i(1 \le \theta_i)$. Then $\sum_{i=1}^{\infty} a_i\|f_i\| \le \sum_{i=1}^{\infty} h_i\|f_i\| < C$. We know from Lemma 6.3 (2) that $\Sigma_{i=1}^{\infty}\big(\tilde{g}(a_i\|u_{i+1}-u_i\|)+\tilde{g}(a_{i-1}\|u_i-u_{i-1}\|)+a_i\|f_i\|\big) < 7b^2$. Now we can repeat the argument used for Claim 1 to complete the proof of Claim 2.

At this stage, let $\xi > 0$ and let $\epsilon := \min\{\frac{\xi}{3k_0}, \tilde{g}(\frac{\xi}{3k_0})\}$ in (6.14) and (6.15). By straight calculations, we have from (6.14) and (6.15):

$$\forall i(0 < \theta_i \le 1) \to \forall \tilde{k} \in \mathbb{N}\,\forall \xi > 0\,\exists \tilde{k} < n \le \tilde{k} + \tilde{\mu}(\min\{\frac{\xi}{3k_0}, \tilde{g}(\frac{\xi}{3k_0})\}) \\ \Big(\|u_{n+1} - u_n\| + \|u_n - u_{n-1}\| + \|f_n\| < \frac{\xi}{k_0}\Big), \tag{6.16}$$

and

$$\forall i(1 \leq \theta_i) \to \forall \tilde{k} \in \mathbb{N}\,\forall \xi > 0\,\exists \tilde{k} < n \leq \tilde{k} + \tilde{\mu}(\min\{\frac{\xi}{3k_0}, \tilde{g}(\frac{\xi}{k_0})\})$$
$$\Big(a_n\|u_{n+1} - u_n\| + a_{n-1}\|u_n - u_{n-1}\| + a_n\|f_n\| < \frac{\xi}{k_0}\Big). \tag{6.17}$$

Now assume either $\forall i(0 < \theta_i \leq 1)$, or $\forall i(1 \leq \theta_i)$. Combining (6.12) and (6.16) for the case $\forall i(0 < \theta_i \leq 1)$, as well as combining (6.13) and (6.17) when $\forall i(1 \leq \theta_i)$, we have

$$\forall \tilde{k} \in \mathbb{N}\,\forall \xi > 0\,\exists \tilde{k} < n \leq \tilde{k} + \tilde{\mu}(\min\{\frac{\xi}{3k_0}, \tilde{g}(\frac{\xi}{3k_0})\})\big(\|v_n\| < \xi\big). \tag{6.18}$$

Now, choose arbitrary $k \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$. Taking

$$\xi := \frac{1}{1 + \Omega(4k+4, \lceil b \rceil)}$$

in (6.18), and denoting

$$\Gamma(k) := \tilde{\mu}(\min\{\frac{1}{3k_0(1 + \Omega(4k+4, \lceil b \rceil))}, \tilde{g}(\frac{1}{3k_0(1 + \Omega(4k+4, \lceil b \rceil))})\}),$$

we have

$$\forall \tilde{k} \in \mathbb{N}\,\exists \tilde{k} < n \leq \tilde{k} + \Gamma(k)\left(\|v_n\| < \frac{1}{1 + \Omega(4k+4, \lceil b \rceil)}\right). \tag{6.19}$$

Define $h(n) := \Gamma(k) + \max\{f(i);\ n < i \leq n + \Gamma(k)\}$, $\tilde{h}(n) := n + h(n)$, and

$$\Phi(h, k, C) := \tilde{h}^{(C(4k+4))}(0).$$

On the other hand, defining $R_i := \sum_{k=i}^{\infty} \frac{\|f_k\|}{\theta_k\theta_{k-1}...\theta_i}$ and using Lemma 6.1, we have:

$$\sum_{i=1}^{\infty} R_i = \sum_{i=1}^{\infty}\sum_{k=i}^{\infty} \frac{\|f_k\|}{\theta_k\theta_{k-1}...\theta_i} \leq \sum_{i=1}^{\infty} h_i\|f_i\| < C. \tag{6.20}$$

At this stage, using the metastability of bounded monotone sequences (see, e.g., [30, Proposition 2.27] for details), we can choose some $n_0 \in \mathbb{N}$ such that

$$n_0 \leq \Phi(h, k, C) \wedge \forall i, j \in [n_0; n_0 + h(n_0)]\big(i < j \to \sum_{s=i+1}^{j} R_s < \frac{1}{4k+4}\big), \tag{6.21}$$

and then using (6.19), we may choose $\tilde{n}_0 \in \mathbb{N}$ such that $n_0 < \tilde{n}_0 \leq n_0 + \Gamma(k)$ and

$$\|v_{\tilde{n}_0}\| < \frac{1}{1 + \Omega(4k+4, \lceil b \rceil)}. \tag{6.22}$$

Note that, defining $\Psi(k, f, C, \delta, \Omega, x, p_0) := \Phi(h, k, C) + \Gamma(k)$,

$$n_0 < \tilde{n}_0 \leq n_0 + \Gamma(k) \leq \Psi(k, f, C, \delta, \Omega, x, p_0),$$

and since

$$\tilde{n}_0 + f(\tilde{n}_0) \leq n_0 + \Gamma(k) + f(\tilde{n}_0) \leq n_0 + \Gamma(k) + \max\{f(i);\ n_0 < i \leq n_0 + \Gamma(k)\} = n_0 + h(n_0),$$

we deduce

$$[\tilde{n}_0; \tilde{n}_0 + f(\tilde{n}_0)] \subset [n_0; n_0 + h(n_0)]. \tag{6.23}$$

Moreover, since $v_{\tilde{n}_0} \in A(u_{\tilde{n}_0})$ and $\|u_{\tilde{n}_0}\| \leq \lceil b \rceil$, we conclude by (6.22) and the property of Ω,

$$\mathrm{dist}(u_{\tilde{n}_0}, A^{-1}0) < \frac{1}{4k+4}. \tag{6.24}$$

Thus, we may choose some $\tilde{p}_0 \in A^{-1}0$ such that

$$\|u_{\tilde{n}_0} - \tilde{p}_0\| < \frac{1}{4k+4}. \tag{6.25}$$

By the same line of reasoning in (6.3) and (6.4), we derive

$$\|u_{n+m} - \tilde{p}_0\| \leq \|u_n - \tilde{p}_0\| + \sum_{i=n+1}^{n+m} R_i. \tag{6.26}$$

Thus, in view of (6.21), (6.23) and (6.25), we obtain for all $i \in [\tilde{n}_0; \tilde{n}_0 + f(\tilde{n}_0)]$,

$$\|u_i - \tilde{p}_0\| < \|u_{\tilde{n}_0} - \tilde{p}_0\| + \frac{1}{4k+4} < \frac{1}{2k+2}.$$

This proves (6.11). Moreover, for any $k \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$, we have found some $\tilde{n}_0 \leq \Psi(k, f, C, \delta, \Omega, x, p_0)$ such that for all $i, j \in [\tilde{n}_0; \tilde{n}_0 + f(\tilde{n}_0)]$,

$$\|u_i - u_j\| \leq \|u_i - \tilde{p}_0\| + \|u_j - \tilde{p}_0\| < \frac{1}{k+1}.$$

This completes the proof. □

Definition 6.5. We say that $\beta : \mathbb{N} \to \mathbb{N}$ is a Cauchy rate for a series $\sum_{n=1}^{\infty} \alpha_n < \infty$ with $\alpha_n \geq 0$, if

$$\forall k, m \Big(\sum_{n=\beta(k)}^{\beta(k)+m} \alpha_n \leq \frac{1}{k+1} \Big).$$

In the presence of a Cauchy rate β for the series $\sum_{i=1}^{\infty} h_i \|f_i\| < \infty$, we obtain, as a corollary of Theorem 6.4, a Cauchy rate for the solution (u_i) of the difference inclusion (1.4), depending on β.

Corollary 6.6. *Under the same assumptions as in Theorem 6.4, if* $\sum_{i=1}^{\infty} h_i \|f_i\|$ *converges with a Cauchy rate* $\beta : \mathbb{N} \to \mathbb{N}$*, then the following holds:*

$$\forall k^{\mathbb{N}} \, \forall n, m \geq \beta(4k+3) + \Gamma(k) \big(\|u_n - u_m\| < \frac{1}{k+1} \big),$$

where $\Gamma(k)$ *is defined as in (6.10).*

Proof. Choose an arbitrary $k \in \mathbb{N}$, and define $\Gamma(k)$ as in (6.10). In view of (6.19), there exists some $\beta(4k+3) < n_0 \leq \beta(4k+3) + \Gamma(k)$ such that

$$\|v_{n_0}\| < \frac{1}{1 + \Omega(4k+4, \lceil b \rceil)}. \tag{6.27}$$

In veiw of (6.27), the property of Ω implies that

$$\operatorname{dist}(u_{n_0}, A^{-1}0) < \frac{1}{4k+4}.$$

In particular, we may choose some $p_{n_0} \in A^{-1}0$ such that

$$\|u_{n_0} - p_{n_0}\| < \frac{1}{4k+4}. \tag{6.28}$$

From (6.28), and using (6.3), (6.1) and definition of β, we have for each $m \in \mathbb{N}^*$,

$$\begin{aligned}\|u_{n_0+m} - p_{n_0}\| &\leq \|u_{n_0} - p_{n_0}\| + \sum_{i=n_0+1}^{n_0+m} \sum_{k=i}^{\infty} \frac{\|f_k\|}{\theta_k \theta_{k-1} \cdots \theta_i} \\ &< \frac{1}{4k+4} + \sum_{i=n_0+1}^{\infty} h_i \|f_i\| \leq \frac{1}{4k+4} + \sum_{i=\beta(4k+3)}^{\infty} h_i \|f_i\| \leq \frac{1}{2k+2}.\end{aligned}$$

Consequently, for all $n \geq \beta(4k+3) + \Gamma(k)$, we have

$$\|u_n - p_{n_0}\| < \frac{1}{2k+2},$$

and, in turn, for such n and for all $m \in \mathbb{N}$, it follows that

$$\|u_{n+m} - u_n\| \leq \|u_{n+m} - p_{n_0}\| + \|u_n - p_{n_0}\| < \frac{1}{k+1}.$$

□

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